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On Iverson's law of similarity

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ABSTRACT

Iverson (2006b) proposed the law of similarity

$\xi_s(\lambda x) = \gamma(\lambda, s)\xi_{\eta(\lambda,s)}(x)$

for the sensitivity functions ξ_s ($s \in S$). Compared to the former models, the generality of this one lies in that here γ and η can also depend on the variables λ and s. In the literature, this model (or its special cases) is usually considered together with a given psychophysical representation (e.g. Fechnerian, subtractive, or affine). Our goal, however, is to study at first Iverson's law of similarity on its own. We show that if certain mild assumptions are fulfilled, then ξ can be written in a rather simple form containing only one-variable functions. The obtained form proves to be very useful when we assume some kind of representation.

Motivated by Hsu and Iverson (2016), we then study the above model assuming that the mapping η is multiplicatively translational. First, we show how these mappings can be characterized. Later we turn to the examination of Falmagne's power law. According to our results, the corresponding function ξ can have a Fechnerian representation, and also it can have a subtractive representation. We close the paper with the study of the shift invariance property.

1. Introduction

Iverson (2006b) proposed the similarity model

$$\xi_s(\lambda x) = \gamma(\lambda, s)\xi_{\eta(\lambda,s)}(x), \tag{1}$$

which we call *Iverson's law of similarity*. As described shortly, this model may be applied to a number of phenomena in psychophysics and experimental psychology. The meaning of the notation in the model depends on the particular experimental context, examples of which follow. The purpose of the present work is to further the study of Iverson's law of similarity on theoretical grounds. Models in psychology and other empirical sciences are often formalized by equations containing unknown functions, so-called functional equations. Due to this, many results in psychophysical theory can be obtained through applications of functional equations, see Aczél (1966, 1987), Aczél et al. (2000), Falmagne (1985). A seminal example of this is the work in Luce and Edwards (1958) in uncovering the possible 'sensory scales' relating physical stimuli with the sensation strengths they elicit (see also Iverson, 2006a). Another illustrative example is the so-called Plateau (1872)'s midgray experiment (briefly described below) (see also Heller, 2006). Regarding the theory of functional equations, we will primarily rely on J. Aczél's monographs (Aczél, 1966, 1987) and J.-C. Falmagne's monograph (Falmagne, 1985). In addition, we will use results from the articles Aczél (2005) and Lundberg (1977). These statements (see Lemmas 1 and 2) are presented at the end of the second section.

The reader may wish to keep in mind the following experimental contexts for Iverson's law of similarity (1) when considering the functional equation results in this paper. In one experimental context, the participant is asked to judge which of two stimuli has the greater sensory impact (i.e., is louder, is heavier, is brighter, etc.). The value x is a non-negative number representing an intensity, and the value $\xi_s(x)$ indicates the intensity that is judged to be greater than intensity x according to the measure of discriminability s. Depending on the experimental paradigm used, s may be, for example, a probability, or

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a value proportional to d' in signal detection theory (Green & Swets, 1966), or some other measure of likelihood.[4] The value λ is a non-negative real number, and the subscript $\eta(\lambda, s)$, like s, measures the discriminability between stimuli; the function η may depend on s or λ (or both). The multiplier $\gamma(\lambda, s)$ similarly may depend on one or both of s and λ. In this experimental context, Iverson's law of similarity generalizes 'Weber's law', for which the equation

$$\xi_s(\lambda x) = \lambda\, \xi_s(x)$$

holds (Luce & Galanter, 1963), and it also generalizes 'Falmagne's power law', for which

$$\xi_s(\lambda x) = \lambda^{\phi(s)} \xi_s(x)$$

holds. (We use the label 'Falmagne's power law' following Iverson (2006b).) Falmagne's power law gives the so-called 'near-miss to Weber's law' (McGill & Goldberg, 1968) when the exponent $\phi(s)$ is close to but different from 1 for some or all values of s (see also Augustin, 2008; Doble et al., 2006). The near-miss to Weber's law model has been applied to data from a number of experimental situations, including pure-tone intensity discrimination (Doble et al., 2006; Florentine, 1986; Florentine et al., 1987; Jesteadt et al., 1977; Osman et al., 1980, and many others), visual area discrimination (Augustin & Roscher, 2008), and even sugar concentration discrimination in nectar-feeding animals (Nachev et al., 2013).

In another experimental context, an auditory tone is embedded in a broadband noise background. The participant's task is to match the perceived loudness of this tone/noise pairing with the loudness of an unmasked tone (a tone presented in quiet). This is the partial masking experimental context of Pavel and Iverson (1981). If we write $\xi_s(x)$ to represent the intensity of the unmasked tone that matches the loudness of a tone of intensity x embedded in a background of intensity s, Pavel and Iverson (1981) found that the data suggest the 'shift invariance' relationship

$$\xi_{\lambda^\theta s}(\lambda x) = \lambda\, \xi_s(x),$$

where x and λ vary in respective non-negative intervals, and θ takes values in the interval $]0, 1[$. This equation is another specialization of Iverson's law of similarity.

There are still other experimental contexts for which specializations of Iverson's law of similarity may be applied. For example, in the midgray experiment by Plateau (1872) alluded to above, participants were asked to paint a gray disk midway between a given white disk and a given black disk. As described in Falmagne (1985) and Heller (2006), the data suggest the relationship $\xi_{\lambda s}(\lambda x) = \lambda\, \xi_s(x)$, where $\xi_s(x)$ is the luminance of a disk judged to be midway between a disk of luminance x and another of luminance s, and λ is a positive number. Further examples of the application of specializations of Iverson's law of similarity are described in Hsu et al. (2010) and Hsu and Iverson (2016).

Given the above phenomena and their associated specializations of Iverson's law of similarity, there is the natural question of the usefulness of generalizing to Eq. (1); after all, if the specializations of Eq. (1) are appropriate models for the phenomena, why would generalizations be of interest? For the case of the near-miss to Weber's law for pure-tone intensity discrimination, generalizing to Eq. (1) is of practical use because models such as

$$\xi_s(\lambda x) = \lambda^{\phi(s)} \xi_s(x)$$

are known to fail at low intensities (Schroder et al., 1994; Viemeister & Bacon, 1988; Wakefield & Viemeister, 1990). More general candidate models are needed, and the study of Eq. (1) can guide experimental work in distinguishing competing models; see page 289 of Iverson (2006b) in this regard. In fact, deviations from Weber's law are so common across sensory modalities (see, for example, Carriot et al., 2021) that similar comments may be made for a number of other intensity discrimination contexts beside those involving auditory tones. Studying generalizations of models that have been successfully applied is a theme of the present work; for instance, in Section 3, we study a generalization of the transformation $\eta(\lambda, s) = \lambda^{-\theta} s$ appearing in the 'shift invariance' relationship above, namely, we study the generalization of a 'multiplicatively translational' η,

$$\eta(\lambda\tilde{\lambda}, s) = \eta(\tilde{\lambda}, \eta(\lambda, s)).$$

Previously in the study of the possible functional forms in Eq. (1) and its specializations, researchers assumed additional representations for the function ξ_s. This approach stems historically from the original work by Fechner (1860/1966), but it also has the purpose of helping to narrow down the possible solutions to the functional equations; without further assumptions about the functions ξ_s, η, and γ, Eq. (1) allows for too many solutions for these functions. As examples of studies of (specializations of) Eq. (1) in conjunction with additional representations for ξ_s, we find Falmagne (1994), who studied the near-miss model $\xi_s(\lambda x) = \lambda^{\phi(s)} \xi_s(x)$ while also assuming a 'subtractive representation'

$$s = u(\xi_s(x)) - w(x);$$

Iverson (2006b), who studied Eq. (1) in the context of a 'Fechnerian representation'

$$s = u(\xi_s(x)) - u(x);$$

Falmagne and Lundberg (2000), who studied the near-miss model assuming a 'gain-control representation'

$$s = \frac{u(\xi_s(x)) - w(x)}{\sigma(x)};$$

Iverson and Pavel (1981), who studied the shift invariance relationship

$$\xi_{\lambda^\theta s}(\lambda x) = \lambda\, \xi_s(x)$$

under a similar gain-control representation; Hsu and Iverson (2016), who studied the model

$$\xi_s(\lambda x) = \lambda^{\phi(s)} \xi_{\eta(\lambda,s)}(x)$$

under a 'a gain-control representation' of the form

$$s = \frac{u(\xi_s(x)) - u(x)}{\sigma(x)};$$

and Doble and Hsu (2020), who studied Eq. (1) assuming a subtractive representation. As will be seen, in the present paper we examine representations such as these (Fechnerian, subtractive, and gain-control), but we also consider Eq. (1) on its own, especially with assumptions about the function η.

The present work is arranged as follows. Following Falmagne (1985), Section 2 contains the most basic concepts such as 'psychometric families', 'sensitivity functions' and some of their properties (e.g. 'anchored', 'balanced' and 'parallel families'). Here we also present some representations known in psychophysics and their interpretations (e.g. Fechnerian, subtractive, affine, gain-control). Our results can be found in Section 3, which we divide into two subsections. First, we study Eq. (1), purely from the point of view of the theory of functional equations. In Remark 1, we show that if there is at least one s for which the function $\lambda \longmapsto \eta(\lambda, s)$ is invertible, then ξ can be written with the aid of one-variable functions in the form $\xi_s(x) = \frac{\Phi(f(s)x)}{g(s)}$. As a supplement to this, in Theorem 1 we prove that if there exists a subset $\tilde{S} \subset S$ such that for all $s \in \tilde{S}$, the mapping $J \ni \lambda \longmapsto \eta(\lambda, s)$ is constant, then ξ also has a special form, namely $\xi_s(x) = \kappa(s)x^{\rho(s)}$.

[4] In, for example, Iverson (2006b), s has the following meaning. Let $P(x, y)$ be the probability that intensity y is judged to be greater than intensity x. If such probabilities are assumed to follow a model such as $P(x, y) = F[u(y) - u(x)]$, where F is strictly monotonic, then fixing $P(x, y) = \pi$, we can write that model as $F^{-1}(\pi) = u(y) - u(x)$. But note that with π fixed, we have that y in this equation depends on both x and π, so we write $y = \xi_\pi(x)$, that is, $\xi_\pi(x)$ is the intensity judged greater than x with probability π. It is convenient to define $F^{-1}(\pi) = s$ and also $\xi_\pi(x) = \xi_s(x)$.

The form obtained for the function ξ in Remark 1 proves to be very useful when we assume some kind of representation. In line with this, we first describe those functions ξ that have a certain gain-control representation (see Proposition 1), and then we examine Eq. (1) again in the case where ξ admits a representation

$$\xi_s(x) = u(x) + v(s)$$

(see Proposition 2). This latter representation is closely related to the so-called parallel families. In Proposition 3, we provide a complete description of anchored, balanced and parallel families. These results are summarized in Part A of the graphical abstract.

Motivated by Hsu and Iverson (2016), in the second part of the third section we study (1) assuming that the mapping η is multiplicatively translational. First, we show that multiplicatively translational mappings can be written in the form

$$\eta(\lambda, s) = H(\lambda \cdot H^{-1}(s))$$

if certain mild conditions are met, see Part B of the graphical abstract. Using this form, we examine what special form the function γ in (1) can then take. Later we turn to the examination of Falmagne's power law, i.e.,

$$\xi_s(\lambda x) = \lambda^{\phi(s)} \xi_{\eta(\lambda,s)}(x)$$

which is an important special case of (1). According to Remark 10,

$$\xi_s(x) = x^{\phi(s)} F(x \cdot H^{-1}(s))$$

is satisfied. After that, we show that if the function ϕ in the exponent is monotonic, then ϕ cannot be strictly monotonic on any subinterval of positive length. According to Proposition 5, the corresponding function ξ then has a subtractive representation. As an application of the results of the subsection, we close the paper with the study of the shift invariance

$$\xi_s(\lambda x) = \gamma(\lambda, s) \xi_{\lambda^\theta s}(x).$$

These latter results are summarized in Part C of the graphical abstract.

2. Preliminaries

Our mathematical results are applicable to each of the experimental contexts mentioned in the Introduction, but for concreteness and uniformity, we will use terminology and notation from the intensity discrimination context. (Other contexts, especially the partial masking context of Pavel and Iverson (1981), are referred to when especially relevant.) To this end, suppose a participant must compare a stimulus of intensity x with one of intensity y (both measured in ratio scale units[5]) and judge which has the greater sensory impact (i.e., is louder, is heavier, is brighter, etc.).[6] Let $P(x, y)$ be the probability that intensity y is judged greater than intensity x. A simple model for these probabilities is

$$P(x, y) = F(u(y) - u(x)), \qquad (\mathscr{F})$$

in which u and F are continuous and strictly increasing functions.[7] This model expresses that the stimulus intensities x and y are scaled by the sensory mechanism to the values $u(x)$ and $u(y)$, resp., and the probabilities $P(x, y)$ are determined by the differences $u(y) - u(x)$. Equation $(\mathscr{F})$ is called a *Fechnerian representation*, and u is termed to be a *scale*. The model is closely related to the problem of Fechner (1860/1966); for more details, see especially Dzhafarov and Colonius (1999) and Falmagne (1985).

This representation has been extended and generalized in several ways. There may be asymmetries between the stimuli, e.g. biases based on the order or position of stimulus presentations can occur. In this case, it is more appropriate to consider the following model

$$P(x, y) = F(u(y) - w(x)), \qquad (\mathscr{S})$$

that is called a *subtractive representation*.

A still more general representation is

$$P(x, y) = F(u(y) + h(y)g(x)) \quad \text{or} \quad P(x, y) = F(u(y)h(x) + g(x)),$$

which is called in the literature an *affine representation*; see Hsu et al. (2010). Affine representations include so-called *gain-control representations*, one of which we examine in this paper, namely, the representation

$$P(x, y) = F\left(\frac{u(y) - u(x)}{\sigma(x)}\right). \qquad (\mathscr{G})$$

This representation allows for the fact that the sensory mechanism may adjust its "gain" via the normalizing factor $\sigma(x)$. See Hsu et al. (2010) for further description.

For all fixed x, introducing the function p_x as

$$p_x(y) = P(x, y),$$

a family of functions $\mathcal{F} = \{p_x \,|\, x \in I\}$ arises. Due to the properties of the probability, this family has (among others) the following properties

- for all fixed x, the range of the function p_x is contained in the interval $]0, 1[$;
- for all fixed x, the function p_x is strictly increasing and continuous.

In what follows, our goal is to determine such families of functions under certain conditions. Therefore, following Falmagne (1985), we introduce the following notion.

A family of functions $\mathcal{F} = \{p_x \,|\, x \in I\}$ is called a *psychometric family*, if

(i) for all fixed x, the domain of the function p_x is an open interval[8];
(ii) for all fixed x, the range of the function p_x is contained in the interval $]0, 1[$;
(iii) for all fixed x, the function p_x is strictly increasing and continuous.

As we will see, instead of psychometric families, it will be much more convenient to work with sensitivity functions.

Let $\mathcal{F} = \{p_a \mid a \in I\}$ be a psychometric family. The *sensitivity function* of $\mathcal{F}$ is a function ξ defined for all backgrounds (or standards) a, and all probabilities π in the range of a psychometric function p_a, by the equation

$$\xi_\pi(a) = p_a^{-1}(\pi), \qquad (2)$$

in other words,

$$p_a\left(\xi_\pi(a)\right) = \pi.$$

In words: $\xi_\pi(a)$ is the intensity of the stimulus yielding a response probability π, for the background a. As J.-C. Falmagne in Falmagne (1985) writes: *"The change of notation, from $p_a^{-1}(\pi)$ to $\xi_\pi(a)$, symbolizes an important shift of focus in our analysis. The quantity π, the probability of*

[5] For applications of (specializations of) Iverson's law of similarity to contexts in which the stimuli are measured in log-intensity scales, see Doble et al. (2003) and Doble et al. (2006).

[6] In fact, for the discrimination task, there are some subtle differences regarding whether one is concerned about the probability of 'subjective responses' or is concerned about the probability of 'correct responses'. We have presented discussion in an earlier paper (Hsu & Doble, 2015); in that paper, we also provided a mechanism to reconcile the two psychometric functions (pages 165–166).

[7] In this article, we confine our discussion to Fechnerian models that assume strictly-increasing internal scales. Regarding the issue of subliminality, see, for example, Hsu and Doble (2015) for a brief discussion (page 159).

[8] In this work every interval has positive length.

the response, ceases to be the variable of interest and becomes the parameter. Typically, at most a couple of values of π are considered in experimental plots of Weber functions or sensitivity functions. By contrast, the effect on $\xi_\pi(a)$ of the variable a is investigated in minute detail. This is in line with a long tradition in psychophysical research in which the sensory scales uncovered by the analysis of the data are deemed of central importance".

In terms of the sensitivity functions, ($\mathscr{F}$), ($\mathscr{S}$), and ($\mathscr{G}$) resp. can be re-formulated as

$$s = u(\xi_s(x)) - u(x),$$

$$s = u(\xi_s(x)) - w(x),$$

and

$$s\sigma(x) = u(\xi_s(x)) - u(x).$$

We note that in each of these re-formulations, the switch from π to s occurs through the process described in Footnote 4. We will use the notation with s in the sequel.

If we do not assume anything about the sensitivity functions, then an infinite number of scale functions can be considered. In this paper we consider in addition a *similarity law* initiated by G. J. Iverson in Iverson (2006b). In this case, we assume that the sensitivities fulfill identity

$$\xi_s(\lambda x) = \gamma(\lambda, s)\xi_{\eta(\lambda,s)}(x)$$

for all possible values of the variables λ, x and s and with appropriate two-variable functions γ and η.

In accordance with Falmagne (1985), a psychometric family $\mathcal{F} = \{p_a \mid a \in I\}$ is called *anchored* if there exists a number $\alpha \in]0,1[$ such that

(i) for all $a \in I$, there exists an x satisfying $p_a(x) = \alpha$;
(ii) for all x, there exists an $a \in I$ such that $p_a(x) = \alpha$.

The above conditions mean in words that for every background a there is a stimulus x, and for every stimulus x there is a background a, such that $p_a(x) = \alpha$. A number $\alpha \in]0,1[$ satisfying these conditions will be called an *anchor* of $\mathcal{F}$.

A psychometric family $\mathcal{F}$ is called *parallel* if for all $p_a, p_b \in \mathcal{F}$, we have

$$p_a^{-1}(\pi) - p_a^{-1}(\pi^*) = p_b^{-1}(\pi) - p_b^{-1}(\pi^*)$$

whenever all four terms are defined.

If a parallel psychometric family $\mathcal{F} = \{p_a \mid a \in I\}$ is given, then and only then

$$\xi_\pi(a) - \xi_{\pi^*}(a) = \xi_\pi(b) - \xi_{\pi^*}(b)$$

is fulfilled by the sensitivity functions. In view of Falmagne (1985, Theorem 8.5), an anchored psychometric family $\mathcal{F} = \{p_a \mid a \in I\}$ is parallel if and only if there exist strictly increasing and continuous functions u and v such that

$$\xi_s(x) = u(x) + v(s)$$

for all possible x and s.

Intuitively, a psychometric family is parallel if any two psychometric functions can be made to coincide by a horizontal shift of one toward the other. This suggests that given one psychometric function, say p_a, any other psychometric function p_b is completely characterized by the value of one parameter depending on b that is denoted by $g(b)$, expressing the length and direction of the above-mentioned shift ($g(b)$ may be negative).

A psychometric family $\mathcal{F} = \{p_a \mid a \in I\}$ is called *balanced* if

$$p_a(b) + p_b(a) = 1$$

holds for all possible a and b.

The assumption that the psychometric family $\mathcal{F} = \{p_a \mid a \in I\}$ is balanced holds if and only if for the sensitivity function we have

$$\xi_{1-\pi}\left(\xi_\pi(a)\right) = a$$

for all π and a.

As J.-C. Falmagne in Falmagne (1985) writes, there are cases (i.e. the brightness discrimination experiment) where the balance condition naturally holds by the paradigm or of the method of data collection. At the same time, observe that the order of the stimuli in the notation (a, b) suggests the fact that stimulus a is presented first, followed by stimulus b. The balance condition states, in effect, that the order of presentation does not affect the result of a comparison. This may not be true, for example, for auditory stimuli presented successively.

These properties may seem natural due to the aforementioned, but we will see later that these notions are far too restrictive, even without Iverson's law of similarity, see Propositions 2 and 3.

To prove Proposition 1, the lemma below will be utilized from Aczél (2005). Recall that we say that a function is *philandering* if it is not constant on any interval of positive length.

Lemma 1. *Let $D \subset]0,+\infty[^2$ be a nonempty, open and connected set. Assume that the real-valued functions f, g, k, h fulfill*

$$f(st) = g(s) + h(s) \cdot k(t)$$

for all $(s,t) \in D$. If the function f is measurable and philandering, then

(A) either

$$\begin{aligned} f(r) &= \beta_0 \ln(r) + \beta_1 + \beta_2\beta_3 \\ g(s) &= \beta_0 \ln(s) + \beta_1 \\ h(s) &= \beta_3 \\ k(t) &= \frac{\beta_0}{\beta_3} \ln(t) + \beta_2 \end{aligned}$$

(B) or

$$\begin{aligned} f(r) &= \beta_3\beta_4 r^{\beta_0} + \beta_1 \\ g(s) &= \beta_2 s^{\beta_0} + \beta_1 \\ h(s) &= \beta_3 s^{\beta_0} \\ k(t) &= \beta_4 t^{\beta_0} - \frac{\beta_2}{\beta_3} \end{aligned}$$

for all $s \in \{s \mid \text{there exists } t \text{ such that } (s,t) \in D\}$, $t \in \{t \mid \text{there exists } s \text{ such that } (s,t) \in D\}$ and $r \in \{st \mid (s,t) \in D\}$, where β_i are real constants for $i = 1,2,3,4$ such that $\beta_0, \beta_3, \beta_4 \neq 0$.

Further, in the next section we also use a result of Lundberg (1977) on the functional equation

$$f(\ell(y) + g(s)) = m(y) + h(y + s)$$

which is the statement below.

Lemma 2. *Suppose the equation*

$$f(\ell(x) + g(y)) = m(x) + h(x + y) \tag{3}$$

holds for all (x, y) in a body in $\mathbb{R}^2$, where f, g, h, ℓ and m are real-valued, continuous functions, each defined on an interval (possibly different intervals for different functions). Suppose also that h, ℓ and m are philandering. Then the solutions are exactly those given in Cases I-V below, where the parameters can take any real values as long as the resulting expressions are real numbers.

Case I:

$$f(x) = \alpha + \rho x$$

$$g(x) = \beta + bx$$

$$h(x) = -\tau + \rho b x$$

$\ell(x)$ is arbitrary

$$m(x) = \rho\ell(x) - \rho b x + \alpha + \rho\beta + \tau$$

Case II:

$$f(x) = \alpha + \rho \log(c + e^{\kappa x})$$
$$g(x) = \frac{1}{\kappa} \log(-\beta c + d e^{\delta x})$$
$$h(x) = -\tau + \alpha + \rho \log(bc + d e^{\delta x})$$
$$\ell(x) = -\frac{1}{\kappa} \log(\beta + b e^{-\delta x})$$
$$m(x) = \tau - \rho \log(b + \beta e^{\delta x})$$

Case III:

$$f(x) = \rho \log(\alpha - b e^{\kappa x})$$
$$g(x) = \frac{1}{\kappa} \log(\beta - d\alpha x)$$
$$h(x) = -\tau + \rho \log(bd\alpha x + \alpha\epsilon - b\beta)$$
$$\ell(x) = -\frac{1}{\kappa} \log(\epsilon + bdx)$$
$$m(x) = \tau - \rho \log(\epsilon + bdx)$$

Case IV:

$$f(x) = \alpha + \rho e^{\kappa x}$$
$$g(x) = \beta + \frac{1}{\kappa} \log(b + c e^{\delta x})$$
$$h(x) = -\tau + \alpha + \rho c e^{\delta x}$$
$$\ell(x) = -\beta + \frac{\delta}{\kappa} x$$
$$m(x) = \tau + \rho b e^{\delta x}$$

Case V:

$$f(x) = \alpha + \frac{\rho}{\delta} \log(\beta + x)$$
$$g(x) = -\beta - \epsilon + c e^{\delta x}$$
$$h(x) = -\tau + \alpha + \frac{\rho}{\delta} \log(b + c e^{\delta x})$$
$$\ell(x) = \epsilon + b e^{-\delta x}$$
$$m(x) = \tau - \rho x$$

We end this section with some short comments. Recall from Footnote 8 that all intervals in this paper have positive length. Another is that in several results below, we assume that the value $s = 0$ is attainable. Note that $s = 0$ is feasible in all of the experimental contexts mentioned in the Introduction. In the case of intensity discrimination experiments, the index $s = 0$ may be assumed to correspond to the probability of $1/2$ (or to the d' value of 0), and for other experiments such as partial masking, the intensity $s = 0$ would also make sense for the model. The third short comment is that even though we are using terminology adopted from an intensity discrimination context, we will continue to use that terminology when referring to other experimental contexts. For example, we use the term 'sensitivity function' even in a partial masking context.

3. Results

3.1. Iverson's law of similarity

According to our primary aim, now we study Iverson's law of similarity, i.e., identity

$$\xi_s(\lambda x) = \gamma(\lambda, s)\xi_{\eta(\lambda,s)}(x) \qquad (x \in I, \lambda \in J, s \in S). \tag{4}$$

In the following statement, we show that if there exists an $s^* \in S$ such that the mapping $\lambda \longmapsto \eta(\lambda, s^*)$ is invertible, then the function ξ can be expressed with the aid of one-variable functions in a rather special form.

Before this, however, we would like to clarify under which conditions we will be able to prove this statement. Additionally, we would like to compare our assumptions here with the standard assumptions found in the literature. In what follows (until stated otherwise), $\left\{\xi_s : I \to \mathbb{R} \mid s \in S\right\}$ will always denote a one-parameter family that is defined on the Cartesian product $I \times S$ of some real intervals I and S. This means that we do not assume that $\left\{\xi_s : I \to \mathbb{R} \mid s \in S\right\}$ is a family of sensitivity functions, which among other things would imply that for all fixed s, the mapping ξ_s is continuous and strictly increasing. Further, $\gamma : J \times S \to \mathbb{R}$ and $\eta : J \times S \to S$ will be functions. It is important to emphasize that apart from these assumptions, we do not impose any a priori regularity assumptions on the involved functions, but in several cases, we get them as a consequence. Finally, the assumption $I \cdot J \subset I$ helps Eq. (4) to hold for all $x \in I$, $\lambda \in J$ and $s \in S$. Thus, from one point of view, in this paper we will work under more general conditions, since in most papers the authors assume that the functions γ and η are continuous in each of their variables. On the other hand, the domain of ξ is assumed to be $I \times S$, which, compared to other works, is a bit more special. As an example, we mention that in Doble and Hsu (2020), s is taken from an open interval S; for all $s \in S$, the mapping ξ_s is defined on an open interval $I_s \subset]0, +\infty[$ such that the set

$$B = \left\{(x, s) \mid s \in S, x \in I_s\right\} \subset \mathbb{R}^2$$

is an open and connected set and Eq. (4) is supposed to hold for all triples (x, λ, s) such that $\lambda \in J$, $(x, s) \in B$ and $(\lambda x, s) \in B$.

Remark 1. Let $I, J, S \subset \mathbb{R}$ be intervals with $I \cdot J = \{\lambda x \mid x \in I, \lambda \in J\} \subset I$. Let further $\gamma : J \times S \to \mathbb{R}$ and $\eta : J \times S \to S$ be functions and assume that the one-parameter family of functions $\xi_s : I \to \mathbb{R}\, (s \in S)$ fulfills

$$\xi_s(\lambda x) = \gamma(\lambda, s)\xi_{\eta(\lambda,s)}(x) \qquad (x \in I, \lambda \in J, s \in S).$$

If there exists an $s^* \in S$ such that the mapping $\lambda \longmapsto \eta(\lambda, s^*)$ is invertible, then there exist functions $f : \tilde{S} \to J$, $g : \tilde{S} \to \mathbb{R}$ and $\Phi : I \to \mathbb{R}$ such that

$$g(s) \cdot \xi_s(x) = \Phi(f(s) \cdot x) \qquad \left(x \in I, s \in \tilde{S}\right).$$

Here $\tilde{S}$ denotes the range of the mapping $\lambda \longmapsto \eta(\lambda, s^*)$, that is, $\tilde{S} = \eta(J, s^*)$. Indeed, by our assumptions, there exists an $s^* \in J$ such that the mapping $\lambda \longmapsto \eta(\lambda, s^*)$ is invertible. Then the inverse of this mapping is defined on the range of $\lambda \longmapsto \eta(\lambda, s^*)$, that is, on $\tilde{S} = \eta(J, s^*)$. Therefore, the mapping

$$J \ni \lambda \longmapsto \eta(\lambda, s^*) \in \tilde{S}$$

is a *bijection* between the sets J and $\tilde{S}$. In other words, for all $s \in \tilde{S}$, there exists a uniquely determined $\lambda \in J$ such that $\eta(\lambda, s^*) = s$, or equivalently $\lambda = \eta^{-1}(s, s^*)$.

The above identity with s^* instead of s is just

$$\xi_{s^*}(\lambda x) = \gamma(\lambda, s^*)\xi_{\eta(\lambda,s^*)}(x) \qquad (x \in I, \lambda \in J). \tag{$*$}$$

Let now $s \in \tilde{S}$ be arbitrary. From this, with the substitution $s = \eta^{-1}(\lambda, s^*)$ we get that

$$\xi_{s^*}(\eta^{-1}(s, s^*)x) = \gamma(\eta^{-1}(s, s^*), s^*)\xi_s(x) \qquad \left(x \in I, s \in \tilde{S}\right).$$

Let now

$$f(s) = \eta^{-1}(s, s^*) \qquad g(s) = \gamma(\eta^{-1}(s, s^*), s^*)$$

and

$$\Phi(x) = \xi_{s^*}(x) \qquad \left(x \in I, s \in \tilde{S}\right)$$

to deduce the statement.

Remark 2. Using the notations and the assumptions of Remark 1, if the one-parameter family of functions $\xi_s : I \to \mathbb{R}\, (s \in J)$ fulfills

$$\xi_s(\lambda x) = \gamma(\lambda, s)\xi_{\eta(\lambda,s)}(x) \qquad \left(x \in I, \lambda \in J \setminus \{0\}, s \in \tilde{S}\right),$$

then $\gamma(\lambda^*, s) = 0$ implies that $\xi_s(x) = 0$ on some subinterval of I. Indeed, let $s \in \tilde{S}$ be fixed and assume that there exists $\lambda^* \in J \setminus \{0\}$ such that $\gamma(\lambda^*, s) = 0$, then

$$\xi_s(\lambda^* x) = \gamma(\lambda^*, s)\xi_{\eta(\lambda^*,s)}(x) = 0$$

for all $x \in I$, yielding that ξ_s is identically zero on the interval $\lambda^* I = \{\lambda^* x \mid x \in I\}$.

As mentioned in the Introduction, in many places in the literature $\xi_s(x)$ represents a stimulus intensity, in which case it may be assumed that $\xi_s(x) > 0$ for all possible s and all possible x. Note that in such a case, it follows that we have

$$\xi_s(x) = \frac{\Phi(f(s)x)}{g(s)}$$

for all $x \in I$ and $s \in \tilde{S}$ with a nowhere zero function g.

Remark 3. The assumption that there exists $s^* \in S$ such that the mapping

$$J \ni \lambda \longmapsto \eta(\lambda, s^*)$$

is invertible, is essential while proving Remark 1. However, from the point of view of applications, this condition may seem artificial. Instead of invertibility, for example, monotonicity is a much more reasonable assumption. Note, however, that then the following alternative is fulfilled. Accordingly, suppose that there exists an $s^* \in S$ such that the mapping $\lambda \longmapsto \eta(\lambda, s^*)$ is monotonic on the interval J. Then the points of discontinuity of this function form a countable set. These points of discontinuity induce a (countable) disjoint partition $(J_\alpha)_{\alpha\in A}$ of the interval J. Then for all $\alpha \in A$ the mapping

$$J_\alpha^\circ \ni \lambda \longmapsto \eta(\lambda, s^*)$$

is continuous (and monotonic), where J_α° denotes the interior of the interval J_α. If for some $\alpha \in A$ the mapping $J_\alpha^\circ \ni \lambda \mapsto \eta(\lambda, s^*)$ is not strictly monotonic, then there is a subinterval $J_{\alpha,c}$ of J_α° on which this mapping is constant. At the same time, if for some $\alpha \in A$ there is a subinterval $J_{\alpha,s} \subset J_\alpha^\circ$ such that the mapping $\lambda \longmapsto \eta(\lambda, s^*)$ is strictly monotonic on $J_{\alpha,s}$, then this mapping, while restricted to $J_{\alpha,s}$, is invertible. Thus the assumptions of Remark 1 are fulfilled on $J_{\alpha,s}$. This motivates the following theorem, in which Iverson's law of similarity is examined in the case where the mapping η is constant for certain values s.

Theorem 1. *Let $S, J \subset \mathbb{R}$ be open intervals, $I \subset]0, +\infty[$ be an interval such that $J \times I \subset \mathbb{R}^2$ is open and connected, and suppose we also have $I \cdot J \subset I$. Suppose further that*

(i) there exists $s^* \in S$ such that the mapping

$$J \ni \lambda \longmapsto \eta(\lambda, s^*)$$

is constant, while the mapping

$$I_{s^*} \ni x \longmapsto \xi_{s^*}(x)$$

is philandering and measurable;

(ii) we have

$$\xi_{s^*}(\lambda x) = \gamma(\lambda, s^*)\xi_{\eta(\lambda,s^*)}(x)$$

for all $\lambda \in J$ and $x \in I$.

Then there exist positive constants $\kappa(s^*)$ and $\rho(s^*)$ such that

$$\xi_{s^*}(x) = \kappa(s^*)x^{\rho(s^*)} \qquad (x \in I).$$

Proof. Since the mapping

$$J \ni \lambda \longmapsto \eta(\lambda, s^*)$$

is constant, Iverson's law of similarity with $s = s^*$, i.e.

$$\xi_{s^*}(\lambda x) = \gamma(\lambda, s^*)\xi_{\eta(\lambda,s^*)}(x) \qquad (x \in I, \lambda \in J),$$

becomes a multiplicative Pexider equation. This equation is satisfied on the open and connected set $J \times I \subset \mathbb{R}^2$. Further, the function ξ_{s^*} is philandering and measurable on the interval I. Thus, from Lemma 1 (or directly from Aczél (2005, Corollary 6)) we obtain that there exist real constants (depending on s^*), say $\kappa(s^*)$ and $\rho(s^*)$ such that

$$\xi_{s^*}(x) = \kappa(s^*)x^{\rho(s^*)} \qquad (x \in I). \quad \square \tag{5}$$

Remark 4. Notice that the representation obtained in the above theorem satisfies an identity much more specific than Iverson's law of similarity. Indeed, if

$$\xi_s(x) = \kappa(s)x^{\rho(s)}$$

is fulfilled for all $s \in S$ and $x \in I$, then

$$\xi_s(\lambda x) = \lambda^{\rho(s)}\xi_s(x)$$

for all $\lambda \in J$, $x \in I$ and $s \in S$, that is, Falmagne's power law model holds. As mentioned earlier, in the case of intensity discrimination, this model is called the near-miss to Weber's law when the exponent $\rho(s)$ is close to but different from 1 for some or all s, and it gives Weber's law when $\rho(s) = 1$ for all s. This law (in the form $\xi_s(x) = \kappa(s)x^{\rho(s)}$ or $\xi_s(\lambda x) = \lambda^{\rho(s)}\xi_s(x)$) is studied in some detail in Doble et al. (2006) and Augustin (2008). In the latter, it is shown that the law is compatible with common psychometric functions such as the Weibull, logistic, Gaussian, and Gumbel distributions, with possible restrictions on the functions $\kappa(s)$ and $\rho(s)$. But note that this law is not general enough to cover some empirical situations, such as the partial masking experiment mentioned in the Introduction, for which the shift invariance model $\xi_{\lambda^\theta s}(\lambda x) = \lambda\xi_s(x)$ has been shown to be appropriate; see Iverson and Pavel (1981). (This shift invariance model is distinct from the model $\xi_s(\lambda x) = \lambda^{\rho(s)}\xi_s(x)$, as can be seen by the presence of $\xi_{\lambda^\theta s}$ in the shift invariance model, which is indexed by $\lambda^\theta s$ rather than by s.)

So we see from Remark 1 and Theorem 1 that assumptions about η give helpful information about Iverson's law of similarity without assuming a further representation for ξ. Theorem 1 says that for s for which the mapping $\lambda \longmapsto \eta(\lambda, s)$ is constant, Iverson's law of similarity simplifies to Falmagne's power law model. In the extreme case that this mapping is constant for all s (that is, $\eta(\lambda, s) = \nu(s)$ for some function ν, for all (λ, s)), Falmagne's power law model holds for all s. Remark 1 looks at a case for which there is some s^* for which this does not hold, and more specifically, that there is some s^* such that $\lambda \longmapsto \eta(\lambda, s^*)$ is invertible. In such a case, Remark 1 shows that the function ξ can be expressed with the aid of one-variable functions.

In the following, we determine those mappings which have the form that appears in Remark 1 and that also have a representation of the form

$$\xi_s(x) = u^{-1}(s\sigma(x) + u(x))$$

for all $x \in I$ and $s \in S$, with a continuous and strictly monotonic function $u : I \to \mathbb{R}$ and with a function $\sigma : I \to \mathbb{R}$.

Proposition 1. *Let $I, J, S \subset [0, +\infty[$ be intervals such that $I \cdot J \subset I$ and suppose that the one-parameter family of functions $\xi_s : I \to [0, +\infty[$ $(s \in S)$ admits a representation*

$$\xi_s(x) = \frac{\Phi(f(s)x)}{g(s)} \qquad (x \in I, s \in S)$$

with some functions $\Phi : I \to \mathbb{R}$, $f : S \to J$ and $g : S \to]0, +\infty[$. Assume further, that the family $\{\xi_s \mid s \in S\}$ fulfills the following representation

$$\xi_s(x) = u^{-1}(s\sigma(x) + u(x))$$

for all $x \in I$ and $s \in S$, with a continuous and strictly monotonic function $u : I \to \mathbb{R}$ and with a function $\sigma : I \to \mathbb{R}$. Assume that $0 \in S$, and let $\tilde{I} = f(0) \cdot I$ and denote by $\tilde{I}^\circ$ the interior of the interval $\tilde{I}$. Then

(A) *either there exist nonzero real numbers* β_0, β_3 *and a real number* β_1 *such that*

$$\sigma(x) = \beta_3 \qquad \text{and} \qquad u(x) = \beta_0 \ln(x) + \beta_1$$

and

$$\xi_s(x) = \exp\left(\frac{\beta_3}{\beta_0} s\right) x$$

holds for all $x \in \tilde{I}^\circ$ *and* $s \in S$;

(B) *or there exist nonzero real numbers* $\beta_0, \beta_2, \beta_3$ *and a real number* β_1 *such that*

$$\sigma(x) = \beta_3 x^{\beta_0} \qquad \text{and} \qquad u(x) = \beta_2 x^{\beta_0} + \beta_1$$

and

$$\xi_s(x) = \sqrt[\beta_0]{\frac{\beta_3}{\beta_2} s + 1} \cdot x$$

holds for all $x \in \tilde{I}^\circ$ *and* $s \in S$;

(C) *or the function* σ *is the identically zero function,* u *is any strictly monotonic, continuous function, and we have*

$$\xi_s(x) = x \qquad \left(x \in \tilde{I}, s \in S\right).$$

Proof. Under the assumptions of the proposition, we have

$$\frac{\Phi(f(s)x)}{g(s)} = u^{-1}\left(s\sigma(x) + u(x)\right) \qquad (x \in I, s \in S).$$

Let $s = 0$ in this identity to deduce that

$$\frac{\Phi(f(0)x)}{g(0)} = u^{-1}(u(x)) = x \qquad (x \in I),$$

that is,

$$\Phi(x) = \alpha x \qquad (x \in f(0) \cdot I),$$

where $\alpha = g(0)$. Note that $f(0) \neq 0$, because otherwise $\frac{\Phi(0)}{g(0)} = x$ would hold for every $x \in I$, which is impossible. Observe that this already yields that

$$\xi_s(x) = \frac{\Phi(f(s)x)}{g(s)} = \beta(s) \cdot x \qquad \left(x \in \tilde{I}, s \in S\right)$$

with the function

$$\beta(s) = \frac{\alpha f(s)}{g(s)} \qquad (s \in S)$$

where $\tilde{I} = f(0) \cdot I$. Note that the assumption $J \cdot I \subset I$ guarantees that $\tilde{I} \subset I$.

Therefore

$$\beta(s)x = u^{-1}(s\sigma(x) + u(x)) \qquad \left(x \in \tilde{I}, s \in S\right),$$

or equivalently

$$u(\beta(s)x) = s\sigma(x) + u(x) \qquad \left(x \in \tilde{I}, s \in S\right).$$

If σ is the identically zero function on $\tilde{I}$, then we obtain

$$\beta(s)x = x \qquad \left(x \in \tilde{I}, s \in S\right),$$

from which $\beta(s) = 1$ follows for all $s \in S$. This leads to alternative (C).

If σ is not the identically zero function on $\tilde{I}$, then there exists an $x^* \in \tilde{I}$ such that $\sigma(x^*) \neq 0$. Note that if such a point x^* exists, then we necessarily have $x^* \neq 0$. Indeed, if we would have $x^* = 0$, then the above identity would take the following form

$$u(0) = s\sigma(0) + u(0) \qquad (s \in S),$$

which is impossible. Further, the above identity with x^* instead of x is

$$\beta(s)x^* = u^{-1}(s\sigma(x^*) + u(x^*)) \qquad (s \in S).$$

Since the function u is strictly monotonic and continuous, the right-hand side, as a function of the variable s, i.e., the mapping

$$s \longmapsto u^{-1}(s\sigma(x^*) + u(x^*))$$

is also strictly monotonic and continuous. From this, we obtain that the function β is strictly monotonic and continuous. Thus the set $\beta(S) = \{\beta(s) \mid s \in S\} \subset [0, +\infty[$ is an interval, and the function β is invertible. Therefore we have

$$u(tx) = \beta^{-1}(t)\sigma(x) + u(x) \qquad \left(x \in \tilde{I}^\circ, t \in \beta(S) \setminus \{0\}\right).$$

In view of Lemma 1,[9] the functions β^{-1}, u and σ, and thus ξ can be determined.

(a) Either

$$\begin{aligned} u(x) &= \beta_0 \ln(x) + \beta_1 + \beta_2\beta_3 \\ u(x) &= \beta_0 \ln(x) + \beta_1 \\ \sigma(x) &= \beta_3 \\ \beta^{-1}(t) &= \tfrac{\beta_0}{\beta_3} \ln(t) + \beta_2 \end{aligned}$$

(b) or

$$\begin{aligned} u(x) &= \beta_3\beta_4 x^{\beta_0} + \beta_1 \\ u(x) &= \beta_2 x^{\beta_0} + \beta_1 \\ \sigma(x) &= \beta_3 x^{\beta_0} \\ \beta^{-1}(t) &= \beta_4 t^{\beta_0} - \tfrac{\beta_2}{\beta_3}. \end{aligned}$$

In case of alternative (a) we have

$$\sigma(x) = \beta_3 \qquad \text{and} \qquad u(x) = \beta_0 \ln(x) + \beta_1$$

and

$$u^{-1}(x) = \exp\left(\frac{x - \beta_1}{\beta_0}\right)$$

for all $x \in \tilde{I}^\circ$, from which

$$\begin{aligned} \xi_s(x) &= \exp\left(\frac{\left(\beta_3 s + \beta_0 \ln(x) + \beta_1\right) - \beta_1}{\beta_0}\right) \\ &= \exp\left(\frac{\beta_3}{\beta_0} s\right) x \end{aligned}$$

follows for all $x \in \tilde{I}^\circ$ and $s \in S$. Or, in case of alternative (b) we have

$$\sigma(x) = \beta_3 x^{\beta_0} \qquad \text{and} \qquad u(x) = \beta_2 x^{\beta_0} + \beta_1$$

and

$$u^{-1}(x) = \sqrt[\beta_0]{\frac{x - \beta_1}{\beta_2}}$$

for all $x \in \tilde{I}^\circ$, from which

$$\begin{aligned} \xi_s(x) &= \sqrt[\beta_0]{\frac{\left(\beta_3 x^{\beta_0} \cdot s + \beta_2 x^{\beta_0} + \beta_1\right) - \beta_1}{\beta_2}} \\ &= \sqrt[\beta_0]{\frac{\beta_3}{\beta_2} s + 1} \cdot x \end{aligned}$$

follows for all $x \in \tilde{I}^\circ$ and $s \in S$. □

Remark 5. If the conditions of the above proposition are fulfilled, then we deduce that in all of alternatives (A)–(C), the mapping ξ_s must satisfy Weber's law, i.e. $\xi_s(\lambda x) = \lambda \xi_s(x)$ must hold for all $x \in \tilde{I}$ and $s \in S$. As Weber's law is too restrictive for many experimental situations, we conclude that the assumptions of the proposition (the forms $\xi_s(x) = \frac{\Phi(f(s)x)}{g(s)}$ and $\xi_s(x) = u^{-1}\left(s\sigma(x) + u(x)\right)$, and the fact that

[9] To apply Lemma 1, the set $\beta(S) \setminus \{0\}$ should be an open interval of $]0, +\infty[$. This may not be fulfilled. At the same time, if this is the case, then we work on the interior of this interval. The theorem of Aczél is then applicable to this open intervals. Note, however, that when we move to the interior of the mentioned interval, we "lose" at most two points, the endpoints. However, since the functions in the statement are continuous, the function values at the points in question can be obtained with taking the limits at the end of the proof.

$0 \in S$) are quite limiting. (Especially, we see that Alternative (C), for which ξ_s does not depend on s, would not occur in any applications under consideration.) However, see Remark 6 and the representation considered in Proposition 2 below.

Remark 6. Let $I, S =]0, +\infty[$ and consider the sensitivity functions ξ_s defined on I for all $s \in S$ by

$$\xi_s(x) = x + s \qquad (x \in I, s \in S).$$

These functions fulfill Iverson's law of similarity since we have

$$\xi_s(\lambda x) = s + \lambda x = \lambda\left(\frac{s}{\lambda} + x\right) = \lambda \xi_{\frac{s}{\lambda}}(x)$$

for all $\lambda, x \in I$ and $s \in S$, that is, in this case

$$\gamma(\lambda, s) = \lambda \qquad \text{and} \qquad \eta(\lambda, s) = \frac{s}{\lambda} \qquad (\lambda \in I, s \in S).$$

Moreover, in accordance with Remark 1 we have

$$\xi_s(x) = \frac{\Phi(f(s)x)}{g(s)} \qquad (x \in I, s \in S),$$

since

$$\xi_s(x) = s + x = s\left(1 + \frac{x}{s}\right) = \frac{\left(1 + \frac{x}{s}\right)}{\frac{1}{s}} = \frac{\left(1 + \frac{1}{s}x\right)}{\frac{1}{s}} \qquad (x \in I, s \in S),$$

i.e.,

$$f(s) = \frac{1}{s} \qquad g(s) = \frac{1}{s} \qquad \text{and} \qquad \Phi(x) = x + 1 \qquad (x \in I, s \in S).$$

Finally, we have

$$\xi_s(x) = u^{-1}(s + u(x)) \qquad (x \in I, s \in S),$$

i.e., a gain-control type affine representation with

$$u(x) = x \qquad \text{and} \qquad \sigma(x) = 1 \qquad (x \in I).$$

Nevertheless, the sensitivity functions ξ_s cannot be written in any of the forms (A)–(C) in Proposition 1. This is caused by the fact that in this example the assumption $0 \in S$ does not hold.

We now examine the form $\xi_s(x) = \frac{\Phi(f(s)x)}{g(s)}$ along with a different representation than in Proposition 1, namely, we examine the representation $\xi_s(x) = u(x) + v(s)$. Recall that this is the representation associated with an anchored and parallel psychometric family, although here we do not assume that u and v are continuous.

Proposition 2. *Let $I, J, S \subset [0, +\infty[$ be intervals such that $I \cdot J \subset I$ and suppose that the one-parameter family of functions $\xi_s : I \to [0, +\infty[$ $(s \in S)$ admits a representation*

$$\xi_s(x) = \frac{\Phi(f(s)x)}{g(s)} \qquad (x \in I, s \in S)$$

with some functions $\Phi : I \to \mathbb{R}$, $f : S \to J$ and $g : S \to]0, +\infty[$. Suppose further that the mapping ξ admits a representation of the form

$$\xi_s(x) = u(x) + v(s) \qquad (x \in I, s \in S) \qquad (6)$$

with the functions $u : I \to \mathbb{R}$ and $v : S \to \mathbb{R}$. If the function Φ is measurable and philandering and f is strictly monotonic, positive and continuous, then

(a) *either there exist real constants α, β, γ with $\alpha \neq 0$ such that*

$$u(x) = \alpha \ln(x) + \beta \qquad v(s) = \alpha \ln(s) + \gamma \qquad (x \in I, s \in S)$$

and

$$\xi_s(x) = \alpha \ln(f(s)x) + \alpha + \beta \qquad (x \in I, s \in S)$$

(b) *or there exist nonzero real constants α, γ, ρ and a real constant β such that*

$$u(x) = \alpha x^{\rho} - \beta \qquad v(s) = \frac{\gamma}{f(s)^{\rho}} + \beta \qquad (x \in I, s \in S)$$

and

$$\xi_s(x) = \alpha x^{\rho} + \frac{\gamma}{f(s)^{\rho}} \qquad (x \in I, s \in S).$$

Proof. Combining the assumptions of the proposition,

$$\Phi(f(s) \cdot x) = g(s)u(x) + g(s)v(s) \qquad (x \in I, s \in S).$$

Since the function f is strictly monotonic and continuous, the set $f(S)$ is an interval and f is invertible. Thus we get that

$$\Phi(sx) = g(f^{-1}(s))u(x) + g(f^{-1}(s))v(f^{-1}(s))$$

for all $s \in f(S)$ and $x \in I$. Further, the function Φ is assumed to be measurable and philandering, therefore Lemma 1 applies. This means that

(a) either

$$\begin{aligned} \Phi(x) &= \beta_0 \ln(x) + \beta_1 + \beta_2\beta_3 \\ g(f^{-1}(s))v(f^{-1}(s)) &= \beta_1 \ln(s) + \beta_1 \\ g(f^{-1}(s)) &= \beta_3 \\ u(x) &= \frac{\beta_0}{\beta_3}\ln(x) + \beta_2 \end{aligned}$$

(b) or

$$\begin{aligned} \Phi(x) &= \beta_3\beta_4 x^{\beta_0} + \beta_1 \\ g(f^{-1}(s))v(f^{-1}(s)) &= \beta_2 s^{\beta_0} + \beta_1 \\ g(f^{-1}(s)) &= \beta_3 s^{\beta_0} \\ u(x) &= \beta_4 x^{\beta_0} - \frac{\beta_2}{\beta_3}, \end{aligned}$$

where β_i are real constants for $i = 1, 2, 3, 4$ such that $\beta_0, \beta_3, \beta_4 \neq 0$.

Let us consider alternative (a) first. The third identity shows that the function g is constant. Using this in the second identity,

$$\beta_3 v(f^{-1}(s)) = \beta_1 \ln(s) + \beta_1,$$

i.e.,

$$v(s) = \frac{\beta_1}{\beta_3}\ln(f(s)) + \frac{\beta_1}{\beta_3}$$

follows for all $s \in S$. So

$$\xi_s(x) = u(x) + v(s) = \frac{\beta_0}{\beta_3}\ln(x) + \beta_2 + \frac{\beta_1}{\beta_3}\ln(f(s)) + \frac{\beta_1}{\beta_3}.$$

At the same time, we also have

$$g(s)\xi_s(x) = \Phi(f(s)x).$$

Comparing the coefficients, this is possible only if $\beta_0 = \beta_1$. Therefore

$$\begin{aligned} u(x) &= \frac{\beta_1 \ln(x)}{\beta_3} + \beta_2 \\ v(s) &= \frac{\beta_1 \ln(f(s))}{\beta_3} + \frac{\beta_1}{\beta_3} \end{aligned} \qquad (x \in I, s \in S)$$

and

$$\begin{aligned} \xi_s(x) &= \frac{\beta_1 \ln(x)}{\beta_3} + \frac{\beta_1 \ln(f(s))}{\beta_3} + \frac{\beta_1}{\beta_3} + \beta_2 \\ &= \frac{\beta_1}{\beta_3}\ln(f(s)x) + \frac{\beta_1}{\beta_3} + \beta_2 \end{aligned}$$

for all $x \in I$ and $s \in S$. So assertion (a) holds with $\alpha = \frac{\beta_1}{\beta_3}$ and $\beta = \beta_2$.

Finally, consider alternative (b). The third identity yields that

$$g(s) = \beta_3 f(s)^{\beta_0} \qquad (s \in S)$$

and the second identity implies that

$$v(x) = \frac{\beta_2 f(s)^{\beta_0} + \beta_1}{\beta_3 f(s)^{\beta_0}} = \frac{\beta_2}{\beta_3} + \frac{\beta_1}{\beta_3} f(s)^{-\beta_0}.$$

Since

$$u(x) = \beta_4 x^{\beta_0} - \frac{\beta_2}{\beta_3} \qquad (x \in I),$$

we obtain that

$$\xi_s(x) = \left(\beta_4 x^{\beta_0} - \frac{\beta_2}{\beta_3}\right) + \left(\frac{\beta_2}{\beta_3} + \frac{\beta_1}{\beta_3} f(s)^{-\beta_0}\right)$$

$$= \beta_4\, x^{\beta_0} + \frac{\beta_1}{\beta_3} f(s)^{-\beta_0} \qquad (x \in I, s \in S).$$

Observe that this is assertion (b) if we consider $\alpha = \beta_4$, $\beta = \frac{\beta_2}{\beta_3}$, $\gamma = \frac{\beta_1}{\beta_3}$ and $\rho = \beta_0$. □

Remark 7. Observe that in contrast to Proposition 1, in Proposition 2 there was no need to assume that $0 \in S$. Thus, the sensitivity function that was considered in Remark 6, appears here as a possible solution. Indeed, let us take in (b)

$$\alpha = \gamma = \rho = 1 \quad \text{and} \quad \beta = 0,$$

further

$$f(s) = \frac{1}{s}$$

to get that

$$\xi_s(x) = x + s.$$

If instead of general one-parameter family of functions, we consider a sensitivity function ξ, then identity (6) expresses the fact that ξ is the sensitivity function of an anchored and parallel psychometric family $\mathcal{F} = \{p_a \mid a \in I\}$.

As the following statement shows this assumption is quite restrictive even *without* Iverson's law of similarity. Below we describe those sensitivity functions that stem from an anchored, parallel, and balanced psychometric family.

Proposition 3. *Let $\mathcal{F} = \{p_a \mid a \in I\}$ be a psychometric family that is anchored, parallel and balanced, and denote by ξ the sensitivity function of $\mathcal{F}$. Then there exists a function $\nu : S \to \mathbb{R}$ which is antisymmetric with respect to the point $s_0 = \frac{1}{2}$, i.e.*

$$\nu(1-s) = -\nu(s) \qquad (s \in S)$$

such that

$$\xi_s(x) = x + \nu(s)$$

holds for all $x \in I$ and $s \in S$.

Proof. Due to the anchored and parallel properties of $\mathcal{F}$, there exist strictly increasing, continuous functions u and v such that

$$\xi_s(x) = u(x) + v(s) \qquad (x \in I, s \in S)$$

holds. Since $\mathcal{F}$ was assumed to be balanced as well, we have

$$\xi_{1-s}\left(\xi_s(x)\right) = x$$

for all possible $x \in I$ and $s \in S$. Using the above representation, this means that the functions u and v have to satisfy

$$u(u(x) + v(s)) = x - v(1-s) \qquad (x \in I, s \in S).$$

The functions u and v are continuous and strictly increasing, thus the latter identity implies that

$$u(x+s) = u^{-1}(x) - v(1 - v^{-1}(s)) \qquad (x \in u(I), s \in v(S)),$$

which is a Pexider equation. Thus, we have especially that

$$u(x) = \alpha_1 x + \beta_1 \quad (x \in I) \qquad \text{and} \qquad u^{-1}(x) = \alpha_1 x + \beta_2 \qquad (x \in u(I))$$

with some appropriate constants $\alpha_1, \beta_1, \beta_2$. This is possible however only if $\alpha_1 = 1$ and $\beta_2 = -\beta_1$. So

$$v(s) + 2\beta_1 = -v(1-s) \qquad (s \in S).$$

This means that the function $\nu : S \to \mathbb{R}$ defined by

$$\nu(s) = v(s) + \beta_1 \qquad (s \in S)$$

is antisymmetric with respect to the point $s_0 = \frac{1}{2}$. Therefore,

$$\xi_s(x) = u(x) + v(s) = \left[x + \beta_1\right] + \left[\nu(s) - \beta_1\right] = x + \nu(s)$$

for all $x \in I$ and $s \in S$, as stated. □

3.2. Multiplicatively translational mappings

In this subsection, we investigate Iverson's law of similarity

$$\xi_s(\lambda x) = \gamma(\lambda, s)\xi_{\eta(\lambda,s)}(x) \qquad (x \in I, \lambda \in J, s \in S)$$

assuming that the two-variable mapping η is multiplicatively translational.

Definition 1. Let $S, J \subset \mathbb{R}$ be intervals such that $J \cdot J \subset J$, $0 \in S$ and $1 \in J$. A mapping $\eta : J \times S \to S$ is called *multiplicatively translational* if

$$\eta(\lambda\tilde{\lambda}, s) = \eta(\tilde{\lambda}, \eta(\lambda, s))$$

and also the boundary conditions

$$\eta(\lambda, 0) = 0 \qquad \eta(1, s) = s$$

hold for all $\lambda, \tilde{\lambda} \in J$ and $s \in S$.

A source of motivation for studying multiplicatively translational η comes from the shift invariance relationship

$$\xi_{\lambda^\theta s}(\lambda x) = \lambda\, \xi_s(x)$$

appearing in the work of Pavel and Iverson (1981). In this context, $\xi_s(x)$ represents the intensity of an unmasked tone that matches the loudness of a tone of intensity x embedded in a background of intensity s. We could write this shift invariance relationship as

$$\xi_s(\lambda x) = \lambda\, \xi_{\lambda^{-\theta} s}(x),$$

which is clearly a case of Iverson's law of similarity with $\eta(\lambda, s) = \lambda^{-\theta} s$. Note that the multiplicatively translational property generalizes the transformation $\eta(\lambda, s) = \lambda^{-\theta} s$, since we have for this transformation that

$$\eta(\tilde{\lambda}, \eta(\lambda, s)) = \eta(\tilde{\lambda}, \lambda^{-\theta} s) = \tilde{\lambda}^{-\theta} \lambda^{-\theta} s = \eta(\lambda\tilde{\lambda}, s).$$

We also point out that in this experimental context, it is reasonable that the shift invariance relationship would hold for $s = 0$.

Note the following relationship between Iverson's law of similarity and the multiplicatively translational property, also pointed out in Hsu and Iverson (2016). If we write $\tilde{\lambda} x$ in place of x in Iverson's law of similarity, then by a successive application of the similarity law,

$$\begin{aligned}\xi_s(\lambda\tilde{\lambda} x) &= \gamma(\lambda, s)\xi_{\eta(\lambda,s)}(\tilde{\lambda} x)\\ &= \gamma(\lambda, s)\gamma(\tilde{\lambda}, \eta(\lambda, s))\xi_{\eta(\tilde{\lambda}, \eta(\lambda,s))}(x) \qquad \left(\lambda, \tilde{\lambda} \in J, s \in S\right)\end{aligned}$$

follows. On the other hand, if we use Iverson's law of similarity only once, but for $\lambda\tilde{\lambda}$, we obtain that

$$\xi_s(\lambda\tilde{\lambda} x) = \gamma(\lambda\tilde{\lambda}, s)\xi_{\eta(\lambda\tilde{\lambda},s)}(x) \qquad \left(\lambda, \tilde{\lambda} \in J, s \in S\right).$$

Thus necessarily,

$$\gamma(\lambda, s)\gamma(\tilde{\lambda}, \eta(\lambda, s))\xi_{\eta(\tilde{\lambda}, \eta(\lambda,s))}(x) = \gamma(\lambda\tilde{\lambda}, s)\xi_{\eta(\lambda\tilde{\lambda},s)}(x) \qquad \left(\lambda, \tilde{\lambda} \in J, s \in S\right).$$

Therefore, the assumption that η is multiplicatively translational expresses in some sense a kind of consistency in the above computations.

Lemma 3. *Let $S, J \subset \mathbb{R}$ be intervals such that $J \cdot J \subset J$, $0 \in S$ and $1 \in J$. Assume the mapping $\eta : J \times S \to S$ is multiplicatively translational and there exists an $s^* \in S$ such that the mapping*

$$J \ni \lambda \longmapsto \eta(\lambda, s^*)$$

is bijective. Then either $J = \{1\}$, $S = \{0\}$ and $\eta(1, 0) = 0$, or $J \neq \{1\}$, but then $0 \in J$ and there exists a bijective function $H : J \to S$ with $H(0) = 0$ such that

$$\eta(\lambda, s) = H(\lambda \cdot H^{-1}(s)) \qquad (\lambda \in J, s \in S).$$

Proof. Define the function $H : J \to S$ through

$$H(\lambda) = \eta(\lambda, s^*) \qquad (\lambda \in J).$$

Then H is a bijection from J onto S. Further, since η is multiplicatively translational,

$$H(\lambda\tilde{\lambda}) = \eta(\lambda, H(\tilde{\lambda}))$$

holds for all $\lambda, \tilde{\lambda} \in J$, from which

$$\eta(\lambda, s) = H(\lambda \cdot H^{-1}(s)) \qquad (\lambda \in J, s \in S)$$

follows. Moreover, we also have

$$0 = \eta(\lambda, 0) = H(\lambda \cdot H^{-1}(0)) \qquad (\lambda \in J),$$

from which we deduce

$$H^{-1}(0) = \lambda H^{-1}(0)$$

for all $\lambda \in J$. One possibility is that $J = \{1\}$ and in this case $S = \{0\}$. The other possibility, however, is that $J \neq \{1\}$. This means that from the above identity it necessarily follows that $H^{-1}(0) = 0$. That is, on one hand, $0 = H^{-1}(0) \in J$, and on the other hand, $H(0) = 0$. □

Remark 8. We further note that in the above proof, the boundary conditions only play a role in that with their help we can show that $H(0) = 0$. This means that if we omit these boundary conditions, the representation

$$\eta(\lambda, s) = H(\lambda \cdot H^{-1}(s))$$

is still true.

Remark 9. Multiplicatively translational mappings were previously studied in Hsu and Iverson (2016) and even in Aczél (1966, 1987). However, we would like to emphasize that the conditions there are different from the ones we use. Indeed, according to Aczél (1987, Section 6, Theorem 3), if S is an interval and $\eta :]0, +\infty[\times S \to S$ is a function that is *continuous in both variables* and for which the mapping

$$\lambda \to \eta(\lambda, s)$$

is *nonconstant for every* $s \in S$, and further

$$\eta(\tilde{\lambda}\lambda, s) = \eta(\lambda, \eta(\lambda, s))$$

holds for all $\tilde{\lambda}, \lambda$ and $s \in S$, then there exists a continuous and strictly monotonic function $\psi :]0, +\infty[\to S$ such that

$$\eta(\lambda, s) = \psi(\lambda \cdot \psi^{-1}(s)) \qquad (\lambda \in]0, +\infty[, s \in S).$$

Although the representations are the same, in this case it is assumed that $J =]0, +\infty[$ and the regularity conditions are different.

In what follows, we will consider only those multiplicatively translational mappings η for which there exists a bijective function $H : J \to S$ with $H(0) = 0$ such that

$$\eta(\lambda, s) = H(\lambda \cdot H^{-1}(s)) \qquad (\lambda \in J, s \in S).$$

For the sake of simplicity and easier distinction, such mappings will be termed *regular multiplicatively translational* mappings.

Observe that in this case the assumptions of Remark 1 are fulfilled. Indeed, a careful adaption of that proof yields that we have

$$\xi_{s^*}\left(\frac{H^{-1}(s)}{H^{-1}(s^*)}x\right) = \gamma\left(\frac{H^{-1}(s)}{H^{-1}(s^*)}, s^*\right)\xi_s(x) \qquad (x \in I, s \in S).$$

Proposition 4. *Let $I, J, S \subset \mathbb{R}$ be intervals with $J \cdot J \subset J$, $I \cdot J \subset I$ and $0, 1 \in J$, $0, 1 \in S$. Let further $\gamma, \eta : J \times S \to S$ be functions and assume that the one-parameter family of functions $\xi_s : I \to \mathbb{R}\setminus\{0\}\ (s \in S)$ fulfills*

$$\xi_s(\lambda x) = \gamma(\lambda, s)\xi_{\eta(\lambda,s)}(x) \qquad (x \in I, \lambda \in J, s \in S).$$

If η is a regular multiplicatively translational mapping, then there exist functions $\kappa : J \to \mathbb{R}\setminus\{0\}$ and $h : S \to \mathbb{R}$ such that

$$\gamma(\lambda, s) = \frac{\kappa(h(s)\lambda)}{\kappa(h(s))} \qquad (\lambda, s \in S)$$

holds.

Proof. Under the assumptions of the proposition, let $\tilde{\lambda} \in J$ be arbitrary and let us substitute $\tilde{\lambda}x$ in place of x in Iverson's law of similarity to get

$$\begin{aligned}\xi_s(\lambda\tilde{\lambda}x) &= \gamma(\lambda, s)\xi_{\eta(\lambda,s)}(\tilde{\lambda}x)\\ &= \gamma(\lambda, s)\gamma(\tilde{\lambda}, \eta(\lambda, s))\xi_{\eta(\lambda,\eta(\tilde{\lambda},s))}(x) \qquad \left(x \in I, \lambda, \tilde{\lambda} \in J, s \in S\right).\end{aligned}$$

On the other hand, Iverson's law of similarity with $\lambda\tilde{\lambda}$ instead of λ is

$$\xi_s(\lambda\tilde{\lambda}x) = \gamma(\lambda\tilde{\lambda}, s)\xi_{\eta(\lambda\tilde{\lambda},s)}(x) \qquad \left(x \in I, \lambda, \tilde{\lambda} \in J, s \in S\right).$$

Using that the mapping η is multiplicatively translational, and that for all $s \in S$, the function ξ_s is nowhere zero, we get that

$$\gamma(\lambda\tilde{\lambda}, s) = \gamma(\lambda, s) \cdot \gamma(\tilde{\lambda}, \eta(\lambda, s))$$

holds for all $\lambda, \tilde{\lambda} \in J, s \in S$. Further, since η is not only multiplicatively translational but also regular multiplicatively translational, we have

$$\gamma(\lambda\tilde{\lambda}, s) = \gamma(\lambda, s) \cdot \gamma(\tilde{\lambda}, H(\lambda \cdot H^{-1}(s))) \qquad \left(\lambda, \tilde{\lambda} \in J, s \in S\right).$$

Let us substitute $s = 1$ in this equation to obtain that

$$\gamma(\lambda\tilde{\lambda}, 1) = \gamma(\lambda, 1) \cdot \gamma(\tilde{\lambda}, H(\lambda \cdot H^{-1}(1))) \qquad \left(\lambda, \tilde{\lambda} \in J\right).$$

Further, this identity with $\tilde{\lambda} = \lambda$ and $\lambda = \frac{H^{-1}(s)}{H^{-1}(1)}$ becomes

$$\begin{aligned}\gamma\left(\lambda \cdot \frac{H^{-1}(s)}{H^{-1}(1)}, 1\right) &=\\ &= \gamma\left(\frac{H^{-1}(s)}{H^{-1}(1)}, 1\right) \cdot \gamma\left(\lambda, H\left(\frac{H^{-1}(s)}{H^{-1}(1)} \cdot H^{-1}(1)\right)\right)\\ &= \gamma\left(\frac{H^{-1}(s)}{H^{-1}(1)}, 1\right) \cdot \gamma(\lambda, s) \qquad \left(\lambda, \tilde{\lambda} \in J\right).\end{aligned}$$

Define the functions $h : S \to \mathbb{R}$ and $\kappa : J \to \mathbb{R}$ by

$$h(s) = \frac{H^{-1}(s)}{H^{-1}(1)} \qquad \text{and} \qquad \kappa(\lambda) = \gamma(\lambda, 1) \qquad (\lambda \in J, s \in S)$$

to deduce that

$$\kappa(h(s)\lambda) = \kappa(h(s)) \cdot \gamma(\lambda, s) \qquad (\lambda \in J, s \in S).$$

Since, for all $s \in S$ the mapping ξ_s has values in $\mathbb{R}\setminus\{0\}$, Iverson's law of similarity implies that $\gamma(\lambda, s) \neq 0$ for all $\lambda \in J$ and $s \in S$. Thus the function κ is nowhere zero, and from the above identity, we get that

$$\gamma(\lambda, s) = \frac{\kappa(h(s)\lambda)}{\kappa(h(s))}$$

holds for all $\lambda \in J$ and $s \in S$. □

Remark 10. Let $I, J, S \subset \mathbb{R}$ be intervals with $1 \in I$, $0, 1 \in J$, $0, 1 \in S$, and $J \cdot J \subset J$, $I \cdot J \subset I$. Let further $\eta : J \times S \to S$ and $\phi : S \to \mathbb{R}$ be functions and assume that the one-parameter family of functions $\xi_s : I \to \mathbb{R}\ (s \in S)$ fulfills the following power law

$$\xi_s(\lambda x) = \lambda^{\phi(s)}\xi_{\eta(\lambda,s)}(x) \qquad (x \in I, \lambda \in J, s \in S).$$

If η is a regular multiplicatively translational mapping, then there exists a function $F : I \to \mathbb{R}$ such that

$$\xi_s(x) = x^{\phi(s)}F(x \cdot H^{-1}(s)) \qquad (x \in I, s \in S).$$

Indeed, in this case we have that

$$\eta(\lambda, s) = H(\lambda \cdot H^{-1}(s)) \qquad \text{and} \qquad \gamma(\lambda, s) = \lambda^{\phi(s)} \qquad (\lambda \in J, s \in S)$$

with some function $H : J \to S$ such that $H(0) = 0$. Further,

$$\xi_s(\lambda x) = \lambda^{\phi(s)}\xi_{H(\lambda \cdot H^{-1}(s))}(x) \qquad (x \in I, \lambda \in J, s \in S).$$

With $x = 1$, we get that

$$\xi_s(\lambda) = \lambda^{\phi(s)}\xi_{H(\lambda\cdot H^{-1}(s))}(1) \qquad (x \in I, \lambda \in J, s \in S).$$

Define the function $F : I \to \mathbb{R}$ by

$$F(x) = \xi_{H(x)}(1) \qquad (x \in I)$$

to get that

$$\xi_s(x) = x^{\phi(s)}F(x \cdot H^{-1}(s)) \qquad (x \in I, s \in S).$$

In the statement below we consider the power law

$$\xi_s(\lambda x) = \lambda^{\phi(s)}\xi_{\eta(\lambda,s)}(x) \qquad (x \in I, \lambda \in J, s \in S)$$

with a *monotonic* function ϕ.

Corollary 1. *Let $I, J, S \subset \mathbb{R}$ be intervals with $1 \in I$, $0, 1 \in J$, $0, 1 \in S$, and $J \cdot J \subset J$, $I \cdot J \subset I$. Let further $\phi : S \to S$ be a monotonic function on the interval S. Suppose the one-parameter family of functions $\xi_s : I \to \mathbb{R}\setminus\{0\}$ $(s \in S)$ fulfills the following power law*

$$\xi_s(\lambda x) = \lambda^{\phi(s)}\xi_{\eta(\lambda,s)}(x) \qquad (x \in I, \lambda \in J, s \in S)$$

with a regular multiplicatively translational mapping $\eta : J \times S \to S$. Then there exists a countable partition $(S_\alpha)_{\alpha\in A}$ of subintervals of S such that for all $\alpha \in A$, the interval S_α cannot have an open subinterval $\tilde{S}$ with positive length such that ϕ is strictly monotonic on $\tilde{S}$.

Proof. If the function ϕ is monotonic on the interval S, then the points of discontinuity of the function ϕ form a countable subset in S. Further this countable set induces a countable partition $(S_\alpha)_{\alpha\in A}$ of the interval S such that for all $\alpha \in A$, the function ϕ is continuous on S_α° (on the interior of the interval S_α).

Assume to the contrary that there exists an $\alpha \in A$ such that the interval S_α has a proper open subinterval $\tilde{S}$ with positive length such that ϕ is strictly monotonic (and due to the construction, continuous) on $\tilde{S}$.

Due to Iverson's law of similarity, we have

$$\xi_s(\lambda\tilde{\lambda}x) = (\lambda\tilde{\lambda})^{\phi(s)}\xi_{\eta(\lambda\tilde{\lambda},s)}(x) \qquad \left(x \in I, \lambda, \tilde{\lambda} \in J, s \in \tilde{S}\right).$$

On the other hand, we also have

$$\begin{aligned}\xi_s(\lambda\tilde{\lambda}x) &= \xi_s(\lambda(\tilde{\lambda}x)) = \lambda^{\phi(s)} \cdot \xi_{\eta(\lambda,s)}(\tilde{\lambda}x)\\ &= \lambda^{\phi(s)} \cdot \tilde{\lambda}^{\phi(\eta(\lambda,s))}\xi_{\eta(\tilde{\lambda},\eta(\lambda,s))}(x) \qquad \left(x \in I, \lambda, \tilde{\lambda} \in J, s \in \tilde{S}\right).\end{aligned}$$

Since η satisfies the regular multiplicative translation property, we have

$$(\lambda\tilde{\lambda})^{\phi(s)} = \lambda^{\phi(s)} \cdot \tilde{\lambda}^{\phi(\eta(\lambda,s))},$$

that is,

$$\phi(s) = \phi(\eta(\lambda, s))$$

for all $\lambda \in J, s \in \tilde{S}$. At the same time, the function ϕ was assumed to be continuous and strictly monotonic on $\tilde{S}$. Thus

$$s = \eta(\lambda, s)$$

holds for all $\lambda \in J, s \in \tilde{S}$. Since η is a regular multiplicatively translational mapping,

$$\eta(\lambda, s) = H(\lambda \cdot H^{-1}(s))$$

holds for all $\lambda \in J, s \in S$ with an appropriate function H. This however means that we have

$$s = H(\lambda \cdot H^{-1}(s)),$$

i.e.,

$$H^{-1}(s) = \lambda \cdot H^{-1}(s)$$

for all $\lambda \in J, s \in \tilde{S}$ implying that $\lambda = 1$ holds for all $\lambda \in J$. This is a contradiction. Thus ϕ cannot be strictly monotonic on $\tilde{S}$. □

Remark 11. Notice that the assumption that the function η is a *regularly* multiplicative mapping was used only at the end of the proof of Remark 10. Indeed, to deduce that we have

$$\phi(s) = \phi(\eta(\lambda, s)),$$

we used only that η fulfills $\eta(\lambda\tilde{\lambda}, s) = \eta(\lambda, \eta(\tilde{\lambda}, s))$. In case $\eta(\lambda, s) = s$ for all possible values λ and s, we have Falmagne's power law for the sensitivity functions ξ_s $(s \in S)$, i.e.,

$$\xi_s(\lambda x) = \lambda^{\phi(s)}\xi_s(x) \qquad (x \in I, s \in S)$$

holds. Recall that ξ is termed to be *weakly balanced* if for all $x \in I$ we have $\xi_{\frac{1}{2}}(x) = x$. According to Hsu and Iverson (2016, Theorem 6), if, in addition to the above, ξ admits a weakly balanced affine representation and ϕ is continuous, strictly monotonic and nowhere zero, then ξ has a Fechnerian representation. The mentioned Hsu and Iverson (2016) paper also contains results about the power law

$$\xi_s(\lambda x) = \lambda^{\phi(s)}\xi_{\eta(\lambda,s)}(x) \qquad (x \in I, s \in S)$$

with a multiplicatively translational mapping η. Recall that in the case where η is regularly multiplicatively translational, the conditions of Remark 1 hold. Thus, with the help of Proposition 1, we can specially obtain those sensitivities that satisfy the above power law and have an affine representation. In such a way we deduce Corollary 7 of Hsu and Iverson (2016) without assuming continuous differentiability about the scales u and σ and twice differentiability about ξ.

According to Remark 10, if the one-parameter family of functions ξ satisfies the power law

$$\xi_s(\lambda x) = \lambda^{\phi(s)}\xi_{\eta(\lambda,s)}(x)$$

with some regular multiplicatively translational mapping η, then ξ can be written in the form

$$\xi_s(x) = x^{\phi(s)}F(xH^{-1}(s)).$$

Afterwards, we showed that if ϕ is *monotone*, then the interval S cannot have a proper open subinterval on which the function ϕ is strictly monotone. This motivates the study of one-parameter families of the form

$$\xi_s(x) = x^r F(xH^{-1}(s)).$$

In the following, we examine whether one-parameter families of this form can have a subtractive representation. As we shall see, the answer is yes. Note that one-parameter families of this form automatically satisfy the above power law, with the choice $\eta(\lambda, s) = H(\lambda \cdot H^{-1}(s))$ and $\phi(s) = r$. In this case η is obviously multiplicatively translational, but not necessarily *regular* multiplicatively translational.

Proposition 5. *Let $I, J, S \subset]0, +\infty[$ be intervals of positive length such that $I \cdot J \subset I$, and let r be a nonzero real number. Let further $u, w : I \to]0, +\infty[$ and $H : J \to S$ be continuous and strictly increasing functions and $F : I \to]0, +\infty[$ be a continuous function. Suppose that the one-parameter family of functions $\xi_s : I \to \mathbb{R}$ $(s \in S)$ can be represented as*

$$\xi_s(x) = x^r F(x \cdot H^{-1}(s)) \qquad (x \in I, s \in S) \tag{7}$$

and also as

$$\xi_s(x) = u^{-1}(s + w(x)) \qquad (x \in I, s \in S) \tag{8}$$

Then the following cases are possible.

Case I

$$\xi_s(x) = ae^{b\rho s}x^{r+\rho} \qquad (x \in I, s \in S)$$

with appropriate constants a, b and ρ.

Case II

$$\xi_s(x) = a\left(cx^{\frac{r}{\rho}} + s - \varepsilon\right)^{\rho} \qquad (x \in I, s \in S)$$

with appropriate constants a, ε, ρ and c.

Proof. The assumption that the domain of the functions u, w and H and the range of the function F is contained in the set of positive reals guarantees that we can take the logarithm of both the sides of Eq. (7). In this case

$$\begin{aligned}\log(\xi_s(x)) &= r\log(x) + \log(F(x\cdot H^{-1}(s)))\\ &= r\log(x) + \log\circ F\circ\exp(\log(x) + \log\circ H^{-1}(s))\end{aligned}$$

follows for all $x \in I$ and $s \in S$. Further, if we take the logarithm of both the sides of (8), we deduce

$$\log(\xi_s(x)) = \log\circ u^{-1}(s + w(x)) \qquad (x \in I, s \in S).$$

Therefore we have

$$\log\circ u^{-1}(s + w(x)) = r\cdot\log(x) + \log\circ F\circ\exp(\log(x) + \log\circ H^{-1}(s)),$$

that is,

$$\log\circ u^{-1}(s + y) = r\cdot\log\circ w^{-1}(y) + \log\circ F\circ\exp(\log\circ w^{-1}(y) + \log\circ H^{-1}(s))$$

for all $y \in W$ and $s \in S$, where the interval W denotes the range of the function w. Let us introduce the functions h, m, f, ℓ and g by

$$\begin{aligned}h &= \log\circ u^{-1}\\ m &= -r\cdot\log w^{-1}\\ f &= \log\circ F\circ\exp\\ \ell &= \log\circ w^{-1}\\ g &= \log\circ H^{-1}\end{aligned}$$

to obtain that the latter equation takes the form

$$f(\ell(y) + g(s)) = m(y) + h(y + s) \qquad (y \in W, s \in S).$$

Since h, ℓ and m are philandering, due to the respective properties of the functions u and w, Lemma 2 can be applied. However, let us note that in our case with the above notations, we have $r\cdot\ell + m \equiv 0$. This shows that Cases IV and V are impossible in our case.

Case I In this case

$$m(x) = \rho\ell(x) - \rho bx + \alpha + \rho\beta + \tau,$$

at the same time, due to $r\cdot l + m \equiv 0$, so

$$-r\ell(x) = \rho\ell(x) - \rho bx + \alpha + \rho\beta + \tau$$

holds. From this

$$\ell(x) = \frac{\rho b}{r+\rho}x - \frac{\alpha+\rho\beta+\tau}{r+\rho},$$

and

$$w(x) = \frac{r+\rho}{\rho b}\log(x) + \frac{\alpha+\rho\beta+\tau}{\rho b}$$

follows. Further,

$$h(x) = -\tau + \rho bx,$$

so

$$u^{-1}(x) = e^{-\tau}e^{\rho bx}.$$

Therefore,

$$\xi_s(x) = \exp(\beta\rho + \log(x)\rho + bs\rho + \alpha + r\log(x)) = e^{\beta\rho+\alpha}e^{b\rho s}x^{r+\rho}$$

holds.

Case II In this case

$$h(x) = -\tau + \alpha + \rho\log(bc + de^{\delta x}),$$

yielding that

$$u^{-1}(x) = e^{-\tau+\alpha}\left(bc + de^{\delta x}\right)^{\rho}$$

and

$$\ell(x) = -\frac{1}{\kappa}\log\left(\beta + be^{-\delta x}\right).$$

From this,

$$w^{-1}(x) = \left(\beta + be^{-\delta x}\right)^{-\frac{1}{\kappa}},$$

that is,

$$w(x) = -\frac{1}{\delta}\log\left(\frac{x^{-\kappa}-\beta}{b}\right)$$

follows. From these, we obtain

$$\xi_s(x) = e^{-\tau+\alpha}\left(bc + de^{\delta s}\frac{b}{x^{-\kappa}-\beta}\right)^{\rho} = e^{\alpha}\left(e^{-\frac{\tau}{\rho}}bc + de^{-\frac{\tau}{\rho}}e^{\delta s}\frac{b}{x^{-\kappa}-\beta}\right)^{\rho}.$$

At the same time, since

$$g(x) = \frac{1}{\kappa}\log(-\beta c + de^{\delta x}),$$

we get that

$$H^{-1}(s) = \left(-\beta c + de^{\delta s}\right)^{\frac{1}{\kappa}}.$$

Further,

$$f(x) = \alpha + \rho\log(c + e^{\kappa x}),$$

we obtain that

$$F(x) = e^{\alpha}(c + x^{\kappa})^{\rho}.$$

Therefore

$$\begin{aligned}\xi_s(x) &= x^r e^{\alpha}((x(de^{\delta s} - c\beta)^{1/\kappa})^{\kappa} + c)^{\rho} = x^r e^{\alpha}(x^{\kappa}(de^{\delta s} - c\beta) + c)^{\rho}\\ &= e^{\alpha}(x^{\kappa+\frac{r}{\rho}}(de^{\delta s} - c\beta) + cx^{\frac{1}{\rho}})^{\rho}\end{aligned}$$

Again, the obtained representations for ξ have to agree everywhere, so

$$e^{\alpha}\left(e^{-\frac{\tau}{\rho}}bc + de^{-\frac{\tau}{\rho}}e^{\delta s}\frac{b}{x^{-\kappa}-\beta}\right)^{\rho} = e^{\alpha}(x^{\kappa+\frac{r}{\rho}}(de^{\delta s} - c\beta) + cx^{\frac{1}{\rho}})^{\rho},$$

from which we obtain that

$$e^{-\frac{\tau}{\rho}}bc + de^{-\frac{\tau}{\rho}}e^{\delta s}\frac{b}{x^{-\kappa}-\beta} = x^{\kappa+\frac{r}{\rho}}(de^{\delta s} - c\beta) + cx^{\frac{1}{\rho}}$$

should hold for all possible x and s. Differentiating both sides with respect to the variable s,

$$e^{-\frac{\tau}{\rho}}\frac{b}{x^{-\kappa}-\beta} = x^{\kappa+\frac{r}{\rho}}$$

follows, after simplification with $d\delta$. Transforming this further, we get that

$$e^{-\frac{\tau}{\rho}}b = x^{\frac{r}{\rho}} - \beta x^{\kappa+\frac{r}{\rho}}.$$

In other words,

$$1\cdot x^{\frac{r}{\rho}} - \beta\cdot x^{\kappa+\frac{r}{\rho}} - b\cdot e^{-\frac{\tau}{\rho}} = 0.$$

As monomials corresponding to different powers are linearly independent, this is possible only if some powers are equal, or all the coefficients are zero. Since none of the coefficients can be zero, some powers should be equal. At the same time, $\kappa \neq 0$ and $r \neq 0$ by our assumptions. So this case is impossible.

Case III Note that in our case $r\cdot m + \ell \equiv 0$, implying that $\tau = 0$ and $\frac{r}{\rho} + \kappa = 0$. Thus, the last possibility is

$$h(x) = \rho\log(bd\alpha x + \alpha\varepsilon - b\beta)$$

so

$$u^{-1}(x) = (bd\alpha x + \alpha\varepsilon - b)^{\rho}$$

and since

$$\ell(x) = -\frac{1}{\kappa}\log(\varepsilon + bdx),$$

we have

$$w(x) = \frac{x^{-\kappa} - \varepsilon}{bd}.$$

Therefore,

$$\xi_s(x) = (bd\alpha s + \alpha x^{-\kappa} - b\beta)^{\rho}.$$

At the same time, since

$$g(x) = \frac{1}{\kappa}\log(\beta - bdx),$$

we get that

$$H^{-1}(s) = (\beta - bds)^{\frac{1}{\kappa}}$$

and

$$f(x) = \rho\log(\alpha - be^{\kappa x}),$$

thus

$$F(x) = (\alpha - bx^{\kappa})^{\rho}.$$

So

$$\begin{aligned}\xi_s(x) &= x^r(\alpha - bx^{\kappa}(\beta - bds))^{\rho}\\ &= \left(\alpha x^{\frac{r}{\rho}} - bx^{\frac{r}{\rho}+\kappa}(\beta - bds)\right)^{\rho}\\ &= \left(\alpha x^{\frac{r}{\rho}} - b\beta + b^2ds\right)^{\rho}\\ &= (b^2d)^{\rho}\cdot\left(\frac{\alpha}{\beta^2 d}\cdot x^{\frac{r}{\rho}} + s - \frac{\beta}{bd}\right)^{\rho}\end{aligned}$$

where we also used that $\frac{r}{\rho} + \kappa = 0$. Therefore,

$$\xi_s(x) = a\left(cx^{\frac{r}{\rho}} + s - \varepsilon\right)^{\rho}$$

holds with $a = (b^2d)^{\rho}$, $c = \frac{\alpha}{\beta^2 d}$ and $\varepsilon = \frac{\beta}{bd}$. □

Regarding the assumptions of Proposition 5, note that the domain of the functions u, w and H and the range of the function F will be contained in the set of positive reals if the sets I and J are contained in the set of positive reals and the outputs of ξ_s are positive. These are reasonable assumptions in models of intensity discrimination.

Applying the above proof to the function $w = u$, we obtain the sensitivity functions that have a Fechnerian representation.

Corollary 2. *Let $I, J, S \subset]0, +\infty[$ be intervals of positive length such that $I \cdot J \subset I$, and let r be a nonzero real number. Let further $u : I \to]0, +\infty[$ and $H : J \to S$ be continuous and strictly increasing functions and $F : I \to]0, +\infty[$ be a continuous function. Suppose that the one-parameter family of functions $\xi_s : I \to \mathbb{R}\,(s \in S)$ can be represented as*

$$\xi_s(x) = x^r F(x\cdot H^{-1}(s)) \qquad (x \in I, s \in S) \tag{9}$$

and also as

$$\xi_s(x) = u^{-1}(s + u(x)) \qquad (x \in I, s \in S) \tag{10}$$

Then

$$\xi_s(x) = e^{\rho s}\cdot x \qquad (x \in I, s \in S)$$

holds with an appropriate constant ρ.

Proof. If the one-parameter family of functions $\xi_s : I \to \mathbb{R}\,(s \in S)$ admits a Fechnerian representation (10), then it has a subtractive representation (8) with the function $w = u$. Thus, in view of the proof of Proposition 5 we have following cases.

Case I

$$w(x) = \frac{r+\rho}{\rho b}\log(x) + \frac{\alpha + \rho\beta + \tau}{\rho b} \quad\text{and}\quad u(x) = \frac{\log(x)}{\rho b} + \frac{\tau}{\rho b}$$

At the same time $w = u$, which in this case is possible only if $\alpha + \rho b = 0$ and $r + \rho = 1$ hold. This means however that

$$\xi_s(x) = e^{b\rho s}\cdot x \qquad (x \in I, s \in S)$$

Letting $\bar{\rho} = b\rho$ we obtain the statement.

Case II The other possibility is that we have Case III in the proof of Proposition 5, that is,

$$u^{-1}(x) = (bd\alpha x + \alpha\varepsilon - b)^{\rho}$$

and

$$w(x) = \frac{x^{-\kappa} - \varepsilon}{bd}.$$

In this case however

$$u(x) = \frac{x^{\frac{1}{\rho}} + b - \alpha\varepsilon}{bd\alpha}.$$

Therefore $w \equiv u$ would mean that

$$\begin{aligned}\frac{x^{-\kappa} - \varepsilon}{bd} &= \frac{x^{\frac{1}{\rho}} + b - \alpha\varepsilon}{bd\alpha}\\ \alpha\cdot(x^{-\kappa} - \varepsilon) &= x^{\frac{1}{\rho}} + b - \alpha\varepsilon\\ \alpha\cdot x^{-\kappa} - x^{\frac{1}{\rho}} &= b.\end{aligned}$$

From this, it would follow that $\alpha = 1$, $-\kappa = \frac{1}{\rho}$ and $b = 0$, which is impossible, as $b \neq 0$. □

Remark 12. Observe that the mapping ξ obtained in Case I of Proposition 5 corresponds to Case I of Theorem 10 in Doble and Hsu (2020) if we choose there the function η to be

$$\eta(\lambda, s) = s + \frac{1}{b}\log(\lambda).$$

Case II agrees with Case V of Theorem 10 in Doble and Hsu (2020). The constants are denoted differently here than in the mentioned paper, but the mentioned representations are the same.

Regarding Proposition 5, it is very important to emphasize that while regular multiplicatively translational mappings are considered in the former, we only assume that the mapping in question is multiplicatively translational mapping in the latter. As we saw in the previous remark, Case II of Proposition 5 corresponds to Case V of Theorem 10 in Doble and Hsu (2020). In that case, we have

$$\eta(\lambda, s) = \lambda^{-\delta}(s - \varepsilon) + \varepsilon.$$

Observe that we have

$$\begin{aligned}\eta(\lambda, \eta(\tilde{\lambda}, s)) &= \lambda^{-\delta}(\eta(\tilde{\lambda}, s) - \varepsilon) + \varepsilon\\ &= \lambda^{-\delta}((\tilde{\lambda}^{-\delta}(s - \varepsilon) + \varepsilon) - \varepsilon) + \varepsilon = \lambda^{-\delta}\tilde{\lambda}^{-\delta}(s - \varepsilon) + \varepsilon\\ &= \left(\lambda\tilde{\lambda}\right)^{-\delta}\cdot(s - \varepsilon) + \varepsilon = \eta(\lambda\tilde{\lambda}, s)\end{aligned}$$

for all possible $\lambda, \tilde{\lambda}$ and s. So η is multiplicatively translational. Further, we have

$$\eta(1, s) = 1^{-\delta}(s - \varepsilon) + \varepsilon = s \qquad (s \in S).$$

At the same time, the boundary condition $\eta(\lambda, 0) = 0$ holds if and only if

$$0 = \eta(\lambda, 0) = \lambda^{-\delta}(0 - \varepsilon) + \varepsilon = \varepsilon(1 - \lambda^{-\delta}),$$

that is, if and only if $\varepsilon = 0$ or $\delta = 0$. Assuming $\varepsilon = 0$, we have

$$\eta(\lambda, s) = \lambda^{-\delta}s.$$

The case when δ is negative (i.e. when $-\delta$ is positive) is studied in Corollary 4 with $\theta = -\delta$ (see below).

If δ is positive (i.e. when $-\delta$ is negative), then η is although multiplicatively translational, but not *regular* multiplicatively translational. Indeed, in this case $J =]0, +\infty[$ and $S = [0, +\infty[$ and for any fixed $0 \neq s^* \in S$, the range of the mapping

$$\lambda \longmapsto \eta(\lambda, s^*)$$

is $]0,+\infty[\subsetneq[0,+\infty[$, since 0 is not contained in the range. So the above mapping is not surjective, indicating that it cannot be bijective either. Nevertheless, we have

$$\eta(\lambda,s)=H(\lambda\cdot H^{-1}(s))$$

for all $\lambda,s\in]0,+\infty[$ with the function $H(s)=s^{-\delta}\ (s\in]0,+\infty[)$.

Finally, assuming $\delta=0$, we have

$$\eta(\lambda,s)=s$$

for all possible λ and s. In this case for all $s^*\in S$, the mapping

$$\lambda\longmapsto\eta(\lambda,s^*)$$

is constant, i.e., cannot be bijective. Further, there is no function $H: J\to S$ such that we would have

$$\eta(\lambda,s)=H(\lambda\cdot H^{-1}(s))\qquad(\lambda\in J,s\in S).$$

Thus, in this case it makes no sense to talk about a representation of the form (7). Note, however, that the power model used as a starting point in this part reduces in this case to the model

$$\xi_s(\lambda x)=\lambda^{\phi(s)}\xi_s(x)\qquad(x\in I,s\in S).$$

Corollary 3. *Let $I,J,S\subset[0,+\infty[$ be intervals with $1\in I$, $0,1\in J$, $0,1\in S$, and $J\cdot J\subset J$, $I\cdot J\subset I$. Suppose that the one-parameter family of functions $\xi_s: I\to\mathbb{R}\setminus\{0\}\,(s\in S)$ fulfills*

$$\xi_s(\lambda x)=g(\lambda)\xi_{\eta(\lambda,s)}(x)\qquad(x\in I,\lambda\in J,s\in S).\qquad(11)$$

Assume that the function $g: J\to\mathbb{R}$ is measurable on some subinterval $\tilde{J}$ of J with positive length. Suppose further that η is a regular multiplicatively translational mapping, that is

$$\eta(\lambda,s)=H(\lambda\cdot H^{-1}(s))\qquad(\lambda\in J,s\in S)\qquad(12)$$

holds with some bijection $H: J\to S$ satisfying $H(0)=0$.

Then there exist a real number α, a nonnegative real number μ and a function $F: I\to\mathbb{R}$ such that

$$g(\lambda)=\lambda^{\mu}\quad\text{and}\quad\xi_s(x)=x^{\mu}F(x\cdot H^{-1}(s))\qquad(x\in I,s\in S).\qquad(13)$$

Conversely, if we consider the mappings $g: J\to\mathbb{R}$, $\eta: J\times S\to S$ and $\xi_s: I\to\mathbb{R}\,(s\in S)$ defined by (12) and (13), where μ is a nonnegative real number, $H: J\to S$ is a bijection with $H(0)=0$ and $F: I\to\mathbb{R}$, then identity (11) is fulfilled for all $x\in I,\lambda\in J$ and $s\in S$.

Proof. Let us substitute $s=0$ in Eq. (11) to get that

$$\xi_0(\lambda x)=g(\lambda)\cdot\xi_0(x)\qquad(x\in I,\lambda\in J).$$

Now let $\lambda,\tilde{\lambda}\in J$. Then the above identity with $\lambda\tilde{\lambda}$ is

$$\xi_0(\lambda\tilde{\lambda}x)=g(\lambda\tilde{\lambda})\cdot\xi_0(x)\qquad(x\in I,\lambda\in J).$$

At the same time, a successive application of the above identity leads to

$$\xi_0(\lambda\tilde{\lambda}x)=\xi_0(\lambda(\tilde{\lambda}x))=g(\lambda)\xi_0(\tilde{\lambda}x)=g(\lambda)g(\tilde{\lambda})\cdot\xi_0(x)$$

for all $x\in I$ and $\lambda,\tilde{\lambda}\in J$. As the function ξ_0 is assumed to be nowhere zero, we deduce that

$$g(\lambda\tilde{\lambda})=g(\lambda)\cdot g(\tilde{\lambda})\qquad\left(\lambda,\tilde{\lambda}\in J\right).$$

As $0,1\in J$ holds for the interval J, we especially find that g fulfills a multiplicative Cauchy equation for all $x,y\in]0,1[$. Thus, in view of Theorem 13.1.6 of Kuczma (2009), there exists a nonnegative real number μ such that

$$g(\lambda)=\lambda^{\mu}\qquad(\lambda\in J).$$

In such a case however, due to Remark 10 we have

$$\xi_s(x)=x^{\mu}F(x\cdot H^{-1}(s))\qquad(x\in I,s\in S).$$

Conversely, let us consider the mappings $g: J\to\mathbb{R}$, $\eta: J\times S\to S$ and $\xi_s: I\to\mathbb{R}\,(s\in S)$ defined by (13), where μ is a positive real number, $H: J\to S$ is a bijection with $H(0)=0$ and $F: I\to\mathbb{R}$. Suppose that identity (11) is fulfilled for all $x\in I,\lambda\in J$ and $s\in S$. Then

$$\xi_s(\lambda x)=(\lambda x)^{\mu}\cdot F(\lambda x\cdot H^{-1}(s))\qquad(x\in I,\lambda\in J,s\in S)$$

and

$$\begin{aligned}g(\lambda)\xi_{\eta(\lambda,s)}(x)&=\lambda^{\mu}\cdot x^{\mu}\cdot F(x\cdot H^{-1}(H(\lambda\cdot H^{-1}(s))))\\&=(\lambda x)^{\mu}\cdot F(\lambda x\cdot H^{-1}(s))\qquad(x\in I,\lambda\in J,s\in S).\end{aligned}\qquad\square$$

The cases when η is not regular multiplicatively translational, but only multiplicatively translational, can be important from the point of view of applications, see Pavel and Iverson (1981) and Hsu and Iverson (2016).

We would also like to illustrate this case with an example below. Before all this, however, we examine the case when

$$\eta(\lambda,s)=\lambda^{\theta}s\qquad(\lambda\in J,s\in S).$$

with a positive θ.

Corollary 4. *Let $I,J,S\subset\mathbb{R}$ be intervals with $J\cdot J\subset J$, $I\cdot J\subset I$ and $0,1\in J$, $0,1\in S$, $1\in I$. Let further $\theta>0$ be arbitrarily fixed, $\gamma: J\times S\to\mathbb{R}$ be a function. Then the one-parameter family of functions $\xi_s: I\to\mathbb{R}\setminus\{0\}\,(s\in S)$ fulfills the following shift invariance*

$$\xi_s(\lambda x)=\gamma(\lambda,s)\xi_{\lambda^{\theta}s}(x)\qquad(x\in I,\lambda\in J,s\in S)$$

if and only if exist nowhere zero functions $f,F: I\to\mathbb{R}$ such that

$$\gamma(\lambda,s)=\frac{f(\lambda s^{\frac{1}{\theta}})}{f(s^{\frac{1}{\theta}})}\quad\text{and}\quad\xi_s(x)=\frac{f(xs^{\frac{1}{\theta}})}{f(s^{\frac{1}{\theta}})}\cdot F(x^{\theta}s)\qquad(x,\lambda\in J,s\in S).\qquad(14)$$

Proof. Consider the function H defined on J by

$$H(s)=s^{\theta}\qquad(s\in S).$$

Then

$$H(\lambda\cdot H^{-1}(s))=\lambda^{\theta}s$$

holds for all $\lambda\in J,s\in S$. This means that Proposition 4 applies with $h(s)=s^{\frac{1}{\theta}}$ and we have

$$\gamma(\lambda,s)=\frac{f(\lambda s^{\frac{1}{\theta}})}{f(s^{\frac{1}{\theta}})}$$

for all $\lambda\in J,s\in S$. Writing this representation back to the shift-invariance property, we get that

$$\xi_s(\lambda x)=\frac{f(\lambda s^{\frac{1}{\theta}})}{f(s^{\frac{1}{\theta}})}\xi_{\lambda^{\theta}s}(x)$$

for all $x\in I$ and $\lambda\in J,s\in S$, yielding that

$$\xi_s(x)=\frac{f(xs^{\frac{1}{\theta}})}{f(s^{\frac{1}{\theta}})}\cdot F(x^{\theta}s)\qquad(x\in J,s\in S)$$

with an appropriate function F. Conversely, if $f,F: I\to\mathbb{R}$ are nowhere zero functions and the functions $\gamma: J\times S\to\mathbb{R}$ and $\xi_s: I\to\mathbb{R}\setminus\{0\}$ are defined by (14), then

$$\begin{aligned}\xi_s(\lambda x)&=\frac{f((\lambda x)s^{\frac{1}{\theta}})}{f(s^{\frac{1}{\theta}})}\cdot F((\lambda x)^{\theta}s)\\&=\frac{f(\lambda s^{\frac{1}{\theta}})}{f(s^{\frac{1}{\theta}})}\cdot\frac{f(x(\lambda^{\theta}s)^{\frac{1}{\theta}})}{f((\lambda^{\theta}s)^{\frac{1}{\theta}})}\cdot F(x^{\theta}(\lambda^{\theta}s))\\&=\gamma(\lambda,s)\xi_{\lambda^{\theta}s}(x)\end{aligned}$$

holds for all $x\in I,\lambda\in J,s\in S$. $\square$

In the following statement, we will consider the special shift-invariance corresponding to $\theta = -1$ in the work Iverson and Pavel (1981). Note that, due to the comments above, Corollary 4 cannot be applied in this case, since the mapping

$$\eta(\lambda, s) = \frac{s}{\lambda}$$

is not *regular* multiplicatively translational, even though it has the multiplicatively translational property. Accordingly, other ideas and tools are needed to investigate this case.

Proposition 6. *Let $S, I \subset [0, +\infty[$ and $J \subset]0, +\infty[$ be intervals such that $0 \in S$, $I \cdot J \subset I$ and $\tilde{S} = \left\{ \frac{s}{\lambda} \mid s \in S, \lambda \in J \right\} \subset S$. If the one-parameter family of functions $\xi_s : I \to \mathbb{R}\,(s \in S)$ fulfills*

$$\xi_s(\lambda x) = \lambda \xi_{\frac{s}{\lambda}}(x) \qquad (x \in I, \lambda \in J, s \in S),$$

then there exists a real constant c and a function $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$\xi_s(x) = \begin{cases} cx, & \text{if } s = 0 \\ s \cdot \varphi\left(\frac{x}{s}\right), & \text{if } s \neq 0 \end{cases} \qquad \left(x \in I, s \in \tilde{S}\right).$$

Proof. With $s = 0$ we obtain the multiplicative Pexider equation

$$\xi_0(\lambda x) = \lambda \xi_0(x) \qquad (x \in I, \lambda \in J).$$

Thus

$$\xi_0(x) = cx \qquad (x \in I)$$

with an appropriate real constant c.

If $s \in \tilde{S} \setminus \{0\}$, then there exist $s^* \in S \setminus \{0\}$ and $\lambda \in J$ such that $s = \frac{s^*}{\lambda}$. Then, with the substitution $s = s^*$, we obtain

$$\xi_{s^*}(\lambda x) = \lambda \xi_{\frac{s^*}{\lambda}}(x) \qquad (x \in I).$$

As $s = \frac{s^*}{\lambda}$, we also have $\lambda = \frac{s^*}{s}$, so the above equation becomes

$$\xi_s(x) = \frac{s}{s^*} \xi_{s^*}\left(\frac{s^*}{s} x\right) \qquad (x \in I).$$

Thus, if we consider the function φ defined on I by

$$\varphi(x) = \frac{1}{s^*} \xi_{s^*}(s^* x) \qquad (x \in I),$$

the statement of the proposition follows. □

In summary, in this subsection we examined the following. Suppose that there is a one-parameter family of functions ξ that satisfies Iverson's law of similarity. In addition, suppose that the function η is regular multiplicative translational. The statements above tell us that in this case both the functions ξ and γ have some special form. The question of which of these one-parameter families ξ satisfy some psychophysical representations will be answered in one of our future papers.

Finally, we also compare the current results with the results of Doble and Hsu (2020). In that paper those sensitivity functions that satisfy Iverson's law of similarity and also admit a subtractive representation were determined. Note that this automatically entails the fulfillment of certain regularity conditions, which we did not assume in this paper.

Nevertheless, it may be interesting to raise the question of which of the functions η in Doble and Hsu (2020) have the regular multiplicatively translational property. As we will see below, the boundary conditions appearing in the definition of this notion are quite restrictive.

Indeed, an easy computation shows that all the functions (these appear in Cases II, IV, and V in Doble and Hsu (2020))

$$\begin{aligned} \eta_1(\lambda, s) &= -\frac{1}{\kappa} \log\left(\lambda^{-\delta}(e^{-\kappa s} - \beta) + \beta\right) \\ \eta_2(\lambda, s) &= s + \frac{\delta}{\kappa} \log(\lambda) \\ \eta_3(\lambda, s) &= \lambda^{-\delta}(s - \varepsilon) + \varepsilon \end{aligned}$$

fulfill the equation

$$\eta(\lambda \tilde{\lambda}, s) = \eta(\lambda, \eta(\tilde{\lambda}, s)) \qquad (\lambda \in J, s \in J).$$

However, the boundary conditions in the definition are satisfied only if

- $\beta = 1$ or $\delta = 0$ in case of the function η_1, i.e.,
$$\eta_1(\lambda, s) = -\frac{1}{\kappa} \log\left(\frac{e^{-\kappa s} - 1}{\lambda^\delta} + 1\right) \text{ or } \eta_1(\lambda, s) = s \qquad (\lambda \in J, s \in S).$$
- $\delta = 0$ in case of the function η_2, i.e.,
$$\eta_2(\lambda, s) = s \qquad (\lambda \in J, s \in S).$$
- $\delta = 0$ or $\varepsilon = 0$ in case of the function η_3, i.e.,
$$\eta_3(\lambda, s) = s \quad \text{or} \quad \eta_3(\lambda, s) = s\lambda^{-\delta} \qquad (\lambda \in J, s \in S).$$

Note that these are the multiplicatively translational mappings, of which the function

$$\eta(\lambda, s) = s$$

is *not* regular. In this case, however, Iverson's law of similarity is a multiplicative type equation. Thus if we assume e.g. that for all fixed $s \in S$, η is monotonic on I, then we obtain that

$$\xi_s(x) = c(s) x^{\varphi(s)} \qquad (x \in I, s \in S)$$

with some appropriate functions $c, \varphi : S \to \mathbb{R}$.

Further, the mapping

$$\eta(\lambda, s) = s \cdot \lambda^{-\delta}$$

is regular multiplicatively translational, and this case corresponds to the shift-invariance dealt with in Corollary 4.

Finally, if we consider only regular multiplicatively translational mappings, then in the first case of η_1 we have

$$H(s) = -\frac{1}{\kappa} \log\left(\frac{\alpha}{s^\delta} + 1\right) \qquad (s \in S).$$

Since now we have $[0, 1] \subset J$, the parameter δ must be negative. So we may write $-\delta$ in place of δ (with some positive δ). Further, we set $\kappa = -\log(\alpha + 1)$, which guarantees $H(1) = 1$. The latter is only a technical condition that makes calculations easier. This brings us to the last case, i.e., when

$$H(s) = \frac{\log(\alpha s^\delta + 1)}{\log(\alpha + 1)} \qquad (s \in S),$$

in other words,

$$\eta(\lambda, s) = \frac{\log\left(\lambda^\delta \left(e^{\log(\alpha+1)s} - 1\right) + 1\right)}{\log(\alpha + 1)} \qquad (\lambda \in J, s \in S).$$

Note that in this case, we have

$$H^{-1}(s) = \left(\frac{\exp(\log(\alpha + 1)s) - 1}{\alpha}\right)^{\frac{1}{\rho}} \qquad (s \in S).$$

Given the argument presented at the beginning of this section,

$$\xi_{s^*}\left(\frac{H^{-1}(s)}{H^{-1}(s^*)} x\right) = \gamma\left(\frac{H^{-1}(s)}{H^{-1}(s^*)}, s^*\right) \xi_s(x) \qquad (x \in I, s \in S)$$

with some fixed $s^* \in S$. Assume now that there exists $\bar{s} \in S$ and a subinterval $\bar{S} \subset S$ such that $\gamma(\lambda, \bar{s}) \neq 0$ for all $\lambda \in J$. Then

$$\xi_s(x) = \frac{\Phi(H^{-1}(s)x)}{g(H^{-1}(s))} \qquad (x \in I, s \in S)$$

holds with some appropriate functions Φ and g. The question of which one-parameter functions of this form satisfy a psychophysical representation may also be answered in a future paper.

CRediT authorship contribution statement

Eszter Gselmann: Writing – review & editing, Writing – original draft, Visualization, Validation, Project administration, Methodology, Investigation, Conceptualization. **Christopher W. Doble:** Writing – review & editing, Writing – original draft. **Yung-Fong Hsu:** Writing – review & editing.

Consent for publication

We hereby provide consent for the publication of the manuscript detailed above, including any accompanying images or data contained within the manuscript that may directly or indirectly disclose our identity.

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Declaration of generative AI

We declare that have not used any AI tools or technologies to prepare this manuscript.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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