Integrals of weighted maximal kernels with respect to Vilenkin systems

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Abstract. The integrals of maximal Dirichlet- and Fejér kernels are infinite, so we have to use some weight function to "pull them back" to the finite. In this paper we give necessary and sufficient conditions for weight function to get finite integral on arbitrary Vilenkin groups. Especially some equivalence to the finitness of integral norm of weighted maximal kernels follows in the so-called bounded case. We investigate also the role of the bounded structure of Vilenkin groups in this connection. Similar results are known with respect to the Walsh-Kaczmarz-Dirichlet kernels proved by Gy. Gát [1].

1. In this section we introduce the most important definitions and notations and formulate some known results with respect to the Vilenkin systems. For details we refeer to the book SCHIPP-WADE-SIMON and PÁL [3] and to VILENKIN [5].

If $m = (m_0, m_1, ..., m_k, ...)$ is a sequence of natural numbers such that $m_k \geq 2$ $(k \in \mathbf{N} := \{0, 1, ...\})$ then for all $k \in \mathbf{N}$ we shall denote by Z_{m_k} the m_k-th discrete cyclic group. Let Z_{m_k} be represented by $\{0, 1, ..., m_k - 1\}$. The group operation in Z_{m_k}, i.e. the addition modulo m_k will be denoted by $\oplus$.

G_m will denote the complete direct product of Z_{m_k}'s, then G_m forms a compact Abelian group with Haar measure 1. The usual symbol L^1 denotes the Lebesgue space of complex-valued functions f defined on G_m with the norm $\|f\|_1 := \int_{G_m} |f|$.

The elements of G_m are sequences of the form $x = (x_k, k \in \mathbf{N})$, where $x_k \in Z_{m_k}$ for every $k \in \mathbf{N}$. If $y = (y_k, k \in \mathbf{N}) \in G_m$, then $x \dot{+} y := (x_k \oplus y_k, k \in \mathbf{N})$ is the sum of x, y in G_m.

The topology of the group G_m is completely determined by the sets

$$I_n := \{(x_0, x_1, ..., x_k, ...) \in G_m : x_j = 0 \quad (j = 0, ..., n-1)\}$$

$(0 \neq n \in \mathbf{N}, I_0 := G_m)$.

It is well-known that the characters of G_m (the so-called Vilenkin system) form a complete orthonormal system $\widehat{G}_m$ in L^1. If

$$r_n(x) := \exp \frac{2\pi \imath x_n}{m_n}$$

Mathematics Subject Classification: 42C10.

Key words and phrases: Vilenkin system, Dirichlet- and Fejér kernels, weighted maximal kernels.

$(n \in \mathbf{N}, x = (x_0, x_1, ...) \in G_m, \imath := \sqrt{-1})$, then these functions and their finite products are evidently characters. Let these products be ordered in Paley's sense, which means the following enumeration of the elements of $\widehat{G}_m$. We write each $n \in \mathbf{N}$ uniquely in the form

$$n = \sum_{k=0}^{\infty} n_k M_k,$$

where $M_0 := 1, M_k \quad (k \geq 1)$ are defined above and $n_k \in Z_{m_k} \quad (k \in \mathbf{N})$. It can easily be seen that the elements of $\widehat{G}_m$ are nothing else but the functions

$$\Psi_n := \prod_{k=0}^{\infty} r_k^{n_k}.$$

If $m_n = 2$ for all $n \in \mathbf{N}$, then $\widehat{G}_m$ is the well-known Walsh-Paley system.

Let D_n and K_n be the n-th Dirichlet and Fejér kernel, respectively, defined by

$$D_n := \sum_{k=0}^{n-1} \Psi_k \quad , \quad K_n := \frac{1}{n} \sum_{k=1}^{n} D_k \qquad (0 < n \in \mathbf{N}).$$

We need the following well-known results with respect to the kernels from the Vilenkin-Fourier analysis (see e.g. PÁL-SIMON [2], SIMON [4]):

(1) $$D_n = \Psi_n \sum_{k=0}^{\infty} \sum_{j=m_k-n_k}^{m_k-1} r_k^j D_{M_k} \qquad (n = \sum_{k=0}^{\infty} n_k M_k \in \mathbf{N});$$

(2) $$D_{M_n} = \chi_{I_n} M_n \qquad (n \in \mathbf{N}),$$

where χ_A denotes the characteristic function of a set A;

if $n \in \mathbf{N}$ and for some $s \in \mathbf{N}$ we have $M_{s-1} \leq n < M_s$, then

(3) $$|K_n(x)| = \frac{1}{n} \left| \sum_{\nu=0}^{s-1} \sum_{i=\nu}^{s-1} \sum_{j=m_i-n_i}^{m_i-1} r_i(x)^j c_{ij}^{\nu}(x) \right| \qquad (x \in G_m),$$

where

$$c_{ij}^{\nu}(x) := n_\nu D_{M_i}(x) - \sum_{k=0}^{m_\nu-1} k m_\nu^{-1} \sum_{l=0}^{m_\nu-1} r_\nu^{n_\nu-k} r_i(le_\nu)^j D_{M_i}(x \dot{+} le_\nu),$$

$le_\nu := (0, 0, ..., 0, l, 0, ...) \in G_m$ and l is the $(\nu+1)$-th coordinate of the element in question.

From now on C will denote positive absolute constant not always the same at different occurences.

2. Let $\alpha : [0,+\infty) \to (0,+\infty)$ be a monotone increasing function and define the weighted maximal functions D_α^*, K_α^* as follows:

$$D_\alpha^* := \sup_n \frac{|D_n|}{\alpha(n)} \quad , \quad K_\alpha^* := \sup_n \frac{|K_n|}{\alpha(n)}.$$

Then the next statements are true.

Theorem 1. *There are positive absolute contsants C_1, C_2 such that*

$$C_1 \sum_{k=0}^{\infty} \frac{\log m_k}{\alpha(M_{k+1})} \le \|R_\alpha^*\|_1 \le C_2 \sum_{k=0}^{\infty} \frac{\log m_k}{\alpha(M_k)},$$

where $R_\alpha^* := D_\alpha^*$ or $R_\alpha^* := K_\alpha^*$.

PROOF. First we deal with D_α^* and with the right hand side inequlity of the theorem. To this end write $\|D_\alpha^*\|_1$ as $\|D_\alpha^*\|_1 = \sum_{A=0}^{\infty} \int_{I_A \setminus I_{A+1}} D_{\alpha^*}$. If $A \in \mathbf{N}$ and $x \in I_A \setminus I_{A+1}$, then we get by (1) and (2) that

$$|D_n(x)| = \left| \sum_{k=0}^{A-1} n_k M_k + M_A \sum_{j=m_A-n_A}^{m_A-1} r_A(x)^j \right| = |D_{\tilde{n}}(x)|,$$

if $\tilde{n} := \sum_{k=0}^{A} n_k M_k$. Therefore

$$D_\alpha^*(x) = \sup_{n<M_{A+1}} \frac{|D_n(x)|}{\alpha(n)} \le \sum_{k=0}^{A} \sup_{M_k \le n < M_{k+1}} \frac{|D_n(x)|}{\alpha(n)} \le$$

$$\le \sum_{k=0}^{A} \sup_{M_k \le n < M_{k+1}} \frac{|D_n(x)|}{\alpha(M_k)}.$$

When here $n < M_{k+1} \le M_A$ (i.e. $k = 0, ..., A-1$), then (see above) $|D_n(x)| \le \sum_{l=0}^{k} n_l M_l < M_{k+1}$. Furthermore, for $M_A \le n < M_{A+1}$ we get

$$|D_n(x)| = \sum_{k=0}^{A-1} n_k M_k + M_A \left| \sum_{j=1}^{n_A} \exp \frac{2\pi \imath j x_A}{m_A} \right| \le$$

$$M_A + \left| M_A \frac{\exp \frac{2\pi \imath n_A x_A}{m_A} - 1}{\exp \frac{2\pi \imath x_A}{m_A} - 1} \right| = M_A \left(1 + \frac{|\sin \frac{\pi n_A x_A}{m_A}|}{\sin \frac{\pi x_A}{m_A}} \right) \le C M_A \left(1 + \frac{m_A}{\tilde{x}_A} \right),$$

where $\tilde{x}_A := x_A$ if $x_A \le m_A/2$, while $x_A := m_A - x_A$ in the case $x_A > m_A/2$. (We remember that $x \in I_A \setminus I_{A+1}$, i.e. $x_A \neq 0$.) Summarized above facts we have

$$\|D_\alpha^*\|_1 \le \sum_{A=0}^{\infty} \int_{I_A \setminus I_{A+1}} \sum_{k=0}^{A} \frac{\sup_{M_k \le n < M_{k+1}} |D_n(x)|}{\alpha(M_k)} \le$$

$$\sum_{A=0}^{\infty}\int_{I_A\setminus I_{A+1}} M_{k+1}\sum_{k=0}^{A-1}\frac{1}{\alpha(M_k)} + C\sum_{A=0}^{\infty}\frac{M_A}{\alpha(M_A)}\int_{I_A\setminus I_{A+1}}\left(1+\frac{m_A}{\tilde{x}_A}\right)dx \le$$

$$\sum_{k=0}^{\infty}\frac{M_{k+1}}{\alpha(M_k)}\sum_{A=k+1}^{\infty}\left(\frac{1}{M_A}-\frac{1}{M_{A+1}}\right) + C\sum_{A=0}^{\infty}\frac{M_A}{\alpha(M_A)}\sum_{x_A=1}^{m_A-1}\frac{1}{M_{A+1}}(1+\frac{m_A}{\tilde{x}_A}) \le$$

$$\sum_{k=0}^{\infty}\frac{1}{\alpha(M_k)} + C\sum_{A=0}^{\infty}\frac{M_A}{\alpha(M_A)}\sum_{1\le l\le m_A/2}\frac{1}{M_{A+1}}(1+\frac{m_A}{l}) \le$$

$$\sum_{k=0}^{\infty}\frac{1}{\alpha(M_k)} + C\sum_{A=0}^{\infty}\frac{M_A}{\alpha(M_A)}\frac{m_A\log m_A}{M_{A+1}} \le C\sum_{k=0}^{\infty}\frac{\log m_k}{\alpha(M_k)}.$$

To the proof of the lower estimation for $\|D_\alpha^*\|_1$ in Theorem 1 let Δ_n be the entire part of $m_n/2 \quad (n\in\mathbf{N})$. Then

$$\|D_\alpha^*\|_1 \ge \sum_{A=0}^{\infty}\int_{I_A\setminus I_{A+1}}\frac{|D_{\Delta_A M_A}|}{\alpha(\Delta_A M_A)} \ge \sum_{A=0}^{\infty}\frac{1}{\alpha(M_{A+1})}\int_{I_A\setminus A_{A+1}}|D_{\Delta_A M_A}|.$$

The integrals $\int_{I_A\setminus I_{A+1}}|D_{\Delta_A M_A}|$ can be estimated from below as follows. If $A\in\mathbf{N}$ and $x\in I_A\setminus I_{A+1}$, then taking into consideration (1) and (2) we get (as above)

$$|D_{\Delta_A M_A}(x)| = M_A\frac{\left|\sin\dfrac{\pi\Delta_A x_A}{m_A}\right|}{\sin\dfrac{\pi x_A}{m_A}}.$$

Therefore

$$\|D_\alpha^*\|_1 \ge \sum_{A=0}^{\infty}\frac{1}{\alpha(M_{A+1})}\frac{1}{M_{A+1}}\sum_{l=1}^{m_A-1}M_A\frac{\left|\sin\dfrac{\pi\Delta_A l}{m_A}\right|}{\sin\dfrac{\pi l}{m_A}} \ge$$

$$C\sum_{A=0}^{\infty}\frac{1}{\alpha(M_{A+1})}\sum_{1\le l\le\Delta_A}\frac{\left|\sin\dfrac{\pi\Delta_A l}{m_A}\right|}{l} \ge C\sum_{A=0}^{\infty}\frac{\log m_A}{\alpha(M_{A+1})}.$$

The right hand side inequality for $\|K_\alpha^*\|_1$ follows trivially from the case $R_\alpha^* = D_\alpha^*$, since

$$K_\alpha^* = \sup_n\frac{|\sum_{k=1}^n D_k|}{n\alpha(n)} \le \sup_n\frac{\sum_{k=1}^n|D_k|}{n\alpha(n)} \le \sup_n\frac{1}{n}\sum_{k=1}^{n}\frac{|D_k|}{\alpha(k)} \le \sup_n\frac{\sum_{k=1}^n D_\alpha^*}{n} = D_\alpha^*.$$

To the proof of the estimation of $\|K_\alpha^*\|_1$ from below we compute first $|K_{qM_p}(x)|$ $(x \in G_m)$, if $p \in \mathbf{N}$ and $q := \Delta_p$. It is clear that $(qM_p)_i = 0 \quad (\mathbf{N} \ni i \neq p)$ and $(qM_p)_p = q$. Applying (3) we get

$$\left|K_{qM_p}(x)\right| = \frac{1}{qM_p}\Big|\sum_{\nu=0}^{p} M_\nu \sum_{j=m_p-q}^{m_p-1} r_p^j(x)(qM_p)_\nu D_{M_p}(x) -$$

$$\sum_{\nu=0}^{p} M_\nu \sum_{j=m_p-q}^{m_p-1} r_p^j(x) \sum_{k=0}^{m_\nu-1} \frac{k}{m_\nu} \sum_{l=0}^{m_\nu-1} (r_\nu(le_\nu))^{(qM_p)_\nu-k} r_p^j(le_\nu) D_{M_p}(x \dot{+} le_\nu)\Big| =$$

$$\frac{1}{qM_p}\Big|M_p \sum_{j=m_p-q}^{m_p-1} r_p^j(x) q D_{M_p}(x) -$$

$$\sum_{\nu=0}^{p} M_\nu \sum_{j=m_p-q}^{m_p-1} r_p^j(x) \sum_{k=0}^{m_\nu-1} \frac{k}{m_\nu} \sum_{l=0}^{m_\nu-1} (r_\nu(le_\nu))^{(qM_p)_\nu-k} r_p^j(le_\nu) D_{M_p}(x \dot{+} le_\nu)\Big|.$$

If $x \in I_p \setminus I_{p+1}$, then by (1) $D_{M_p}(x \dot{+} le_\nu) = 0$ for all $\nu = 0, ..., p-1$ and $l = 1, ..., m_\nu - 1$. Furthermore, $D_{M_p}(x \dot{+} le_p) = D_{M_p}(x) = M_p \quad (l = 0, ..., m_p - 1)$. Therefore

$$\left|K_{qM_p}(x)\right| = \frac{1}{qM_p}\Big|qM_p^2 \sum_{j=m_p-q}^{m_p-1} r_p^j(x) -$$

$$M_p^2 \sum_{j=m_p-q}^{m_p-1} r_p^j(x) \sum_{k=0}^{m_p-1} \frac{k}{m_p} \sum_{l=0}^{m_p-1} (r_p(le_p))^{j+q-k} - M_p \sum_{\nu=0}^{p-1} M_\nu \sum_{j=m_p-q}^{m_p-1} r_p^j(x) \sum_{k=0}^{m_\nu-1} \frac{k}{m_\nu}\Big|.$$

Here the next equalities hold:

$$M_p \sum_{\nu=0}^{p-1} M_\nu \sum_{j=m_p-q}^{m_p-1} r_p^j(x) \sum_{k=0}^{m_\nu-1} \frac{k}{m_\nu} = M_p \sum_{\nu=0}^{p-1} M_\nu \sum_{j=m_p-q}^{m_p-1} r_p^j(x) \frac{m_\nu - 1}{2} =$$

$$\frac{M_p(M_p-1)}{2} \sum_{j=m_p-q}^{m_p-1} r_p^j(x),$$

and

$$M_p^2 \sum_{j=m_p-q}^{m_p-1} r_p^j(x) \sum_{k=0}^{m_p-1} \frac{k}{m_p} \sum_{l=0}^{m_p-1} (r_p(le_p))^{j+q-k} =$$

$$M_p^2 \sum_{s=0}^{q-1} r_p(x)^{s-q} \sum_{k=0}^{m_p-1} \frac{k}{m_p} \sum_{l=0}^{m_p-1} (r_p(le_p))^{s-k} = M_p^2 \sum_{s=0}^{q-1} (r_p(x))^{s-q} \frac{s}{m_p} m_p.$$

(We recall that $\sum_{l=0}^{m_p-1}(r_p(le_p))^{s-k}=0$, when $s\neq k$.) Thus it follows that

$$\left|K_{qM_p}(x)\right| = \frac{1}{qM_p}\left|\left(qM_p^2-\frac{M_p(M_p-1)}{2}\right)\sum_{s=0}^{q-1}r_p^s(x)-M_p^2\sum_{s=0}^{q-1}sr_p^s(x)\right| =$$

$$\frac{M_p}{q}\left|\left(q-\frac{1}{2}+\frac{1}{2M_p}\right)\frac{r_p^q(x)-1}{r_p(x)-1}-\frac{qr_p^q(x)}{r_p(x)-1}-\frac{r_p(x)(r_p^q(x)-1)}{(r_p(x)-1)^2}\right| \geq$$

$$\frac{M_p}{|r_p(x)-1|}\left(1-\left(1-\frac{1}{2q}+\frac{1}{2qM_p}\right)\left|r_p^q(x)-1\right|-\frac{\left|r_p^q(x)-1\right|}{q|r_p(x)-1|}\right).$$

It is not hard to see that

$$\left(1-\frac{1}{2q}+\frac{1}{2qM_p}\right)\left|r_p^q(x)-1\right|\leq 1/4 \quad \text{and} \quad \frac{\left|r_p^q(x)-1\right|}{q|r_p(x)-1|}\leq 1/4,$$

if $x_p\leq \frac{m_p}{3\pi}$ is even and m_p is large enough, say $m_p>6\pi$. Indeed,

$$|r_p^q(x)-1| = \left|\exp\frac{2\pi\imath qx_p}{m_p}-1\right| = \left|\sin\frac{\pi qx_p}{m_p}\right| = 0$$

for all $x_p=1,...,m_p-1$, if m_p and x_p are even. Assume that $m_p=2l+1$ for some $0<l\in\mathbf{N}$ which implies $q=l$ and $|r_p^q(x)-1|=\left|\sin\frac{\pi lx_p}{2l+1}\right|$. Let $x_p=2k \quad (k=1,...,l)$, then

$$|r_p^q(x)-1| = \left|\sin\frac{2\pi lk}{2l+1}\right| = \sin\frac{\pi k}{2l+1}\leq\frac{\pi k}{2l+1},$$

i.e.

$$\left(1-\frac{1}{2q}+\frac{1}{2qM_p}\right)|r_p^q(x)-1|\leq\frac{3}{2}|r_p^q(x)-1|\leq\frac{3\pi k}{2(2l+1)}\leq\frac{1}{4}$$

for all $k\leq\frac{2l+1}{6\pi}$. This last inequality is equivalent to $x_p\leq\frac{m_p}{3\pi}$. Here $x_p\geq 2$, therefore we assume that $m_p>6\pi$.

On the other hand (see above)

$$\frac{|r_p^q(x)-1|}{q|r_p^q(x)-1|}=0,$$

when $x_p=1,...,m_p-1$ and m_p,x_p are even. For $m_p=2l+1 \quad (0<l\in\mathbf{N})$, $x_p=2k$ $(1\leq k\leq l/2)$ we get

$$\frac{|r_p^q(x)-1|}{q|r_p^q(x)-1|} = \frac{|\sin\frac{\pi lx_p}{2l+1}|}{l\sin\frac{\pi x_p}{2l+1}} = \frac{|\sin\frac{\pi k}{2l+1}|}{\frac{l\pi 2k}{2l+1}} \leq \frac{\pi}{2}\frac{\pi k/(2l+1)}{l\pi 2k/(2l+1}=\frac{\pi}{4l}\leq\frac{1}{4},$$

if $l \geq \pi$, i.e. when $m_p \geq 2\pi + 1$. Hence in this case

$$\left|K_{qM_p}(x)\right| \geq \frac{1}{2}\frac{M_p}{|r_p(x)-1|}$$

and so

$$\int_{I_p\setminus I_{p+1}} \left|K_{qM_p}\right| \geq \frac{M_p}{2}\int_{I_p\setminus I_{p+1}} \frac{dx}{|r_p(x)-1|} \geq \frac{1}{2m_p}\sum_{1\leq x_p\leq m_p/(3\pi),x_p \text{ is even}} \frac{1}{\sin\frac{\pi x_p}{m_p}} \geq$$

$$C\sum_{1\leq x_p\leq m_p/(3\pi),x_p \text{ is even}} \frac{1}{x_p} \geq C\log m_p.$$

If m_p is "small", i.e. $m_p < 6\pi$ and $p > 0$, then

$$\int_{I_p\setminus I_{p+1}} |K_{M_p}| = \left(\frac{1}{M_p} - \frac{1}{M_{p+1}}\right)\frac{M_p - 1}{2} \geq \frac{1}{8} \geq C\log m_p,$$

since by (1) and (2) $K_{M_p}(x) = (M_p - 1)/2 \quad (x \in I_p \setminus I_{p+1})$ follows immediately.

Thus

$$\|K_\alpha^*\|_1 \geq \sum_{p=1,m_p<6\pi}^{\infty}\int_{I_p\setminus I_{p+1}} \frac{|K_{M_p}|}{\alpha(M_p)} + \sum_{p=0,m_p>6\pi}^{\infty}\int_{I_p\setminus I_{p+1}} \frac{|K_{\Delta_p M_p}|}{\alpha(\Delta_p M_p)} \geq$$

$$C\sum_{p=1}^{\infty}\frac{\log m_p}{\alpha(M_p)} + C\sum_{p=1}^{\infty}\frac{\log m_p}{\alpha(M_{p+1})} \geq C\sum_{p=0}^{\infty}\frac{\log m_p}{\alpha(M_{p+1})}.$$

This proves Theorem 1.

From Theorem 1 some corollaries follow immediately. Namely,

Corollary 1. *If* $\sum_{k=0}^{\infty}\frac{\log m_k}{\alpha(M_k)} < +\infty$, *then* $R_\alpha^* \in L^1(G_m)$. *Furthermore, if* $R_\alpha^* \in L^1(G_m)$, *then* $\sum_{k=0}^{\infty}\frac{\log m_k}{\alpha(M_{k+1})} < +\infty$.

Corollary 2. *Assume that the generating sequence* m *is bounded. Then* $R_\alpha^* \in L^1(G_m)$ *if and only if* $\sum_{k=0}^{\infty}\frac{1}{\alpha(M_k)} < +\infty$.

Simple examples show that the boundedness of m in the previous corollary cannot be omitted, although this boundedness is also not necessary. Namely, the next theorem will be proved.

Theorem 2. *There exist* m *and* α *such that* $\sum_{k=0}^{\infty}\frac{1}{\alpha(M_k)} < +\infty$ *and* $R_\alpha^* \notin L^1(G_m)$. *Furthermore, for some unbounded* m *the equivalence in Corollary 2 holds.*

PROOF. We give details only for D_α^*. Let $m_k := 2^{(k+1)^2}$ and $\alpha(M_k) := (k+1)^2$ $(k \in \mathbf{N})$. Then $\sum_{k=0}^{\infty} 1/\alpha(M_k) < +\infty$ holds trivially. Furthermore,

$$\sum_{k=0}^{\infty} \frac{\log m_k}{\alpha(M_{k+1})} \geq C \sum_{k=0}^{\infty} \frac{(k+1)^2}{(k+2)^2} = +\infty,$$

i.e. by Theorem 1 we get $\|D_\alpha^*\|_1 = +\infty$.

Now, let $m_{2^l} := 2^{2^l}$ for $l \in \mathbf{N}$ and $m_k := 2$, when $\mathbf{N} \ni k \neq 2^l \quad (l \in \mathbf{N})$. Then by means of simple considerations the next equivalences follow for all α :

$$\sum_{k=0}^{\infty} \frac{\log m_k}{\alpha(M_k}) < +\infty \iff \sum_{k=0}^{\infty} \frac{\log m_k}{\alpha(M_{k+1})} < +\infty \iff \sum_{k=0}^{\infty} \frac{1}{\alpha(M_k)} < +\infty,$$

which completes the proof of Theorem 2.

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