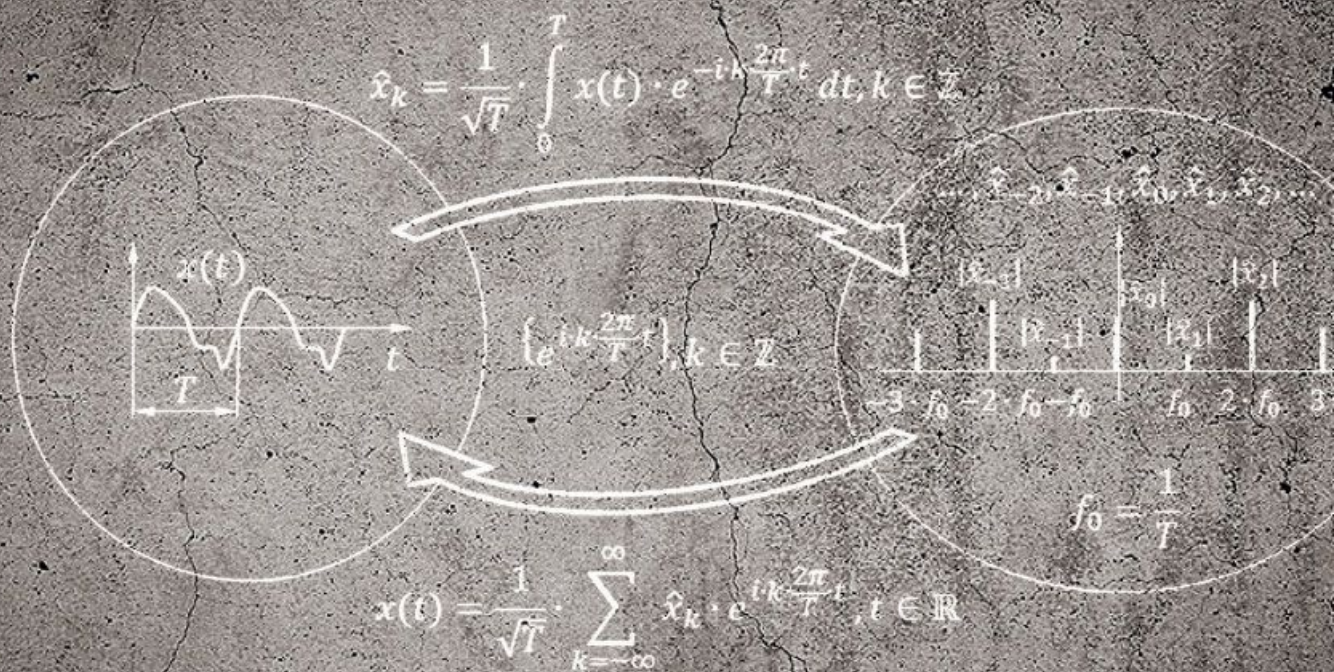


Dr. Imre Kocsis

Integral Transforms^{with} Applications in Engineering



Debreceni Egyetem Műszaki Kar
Műszaki Alaptárgyi Tanszék

DEBRECENI EGYETEM
MŰSZAKI KAR
MŰSZAKI ALAPTÁRGYI TANSZÉK

Dr. Imre Kocsis

INTEGRAL TRANSFORMS
WITH APPLICATIONS IN ENGINEERING



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Introduction

Topics of this problem book appear in different courses in engineering study programmes of the Faculty of Engineering University of Debrecen.

The aim of this problem book is to collect selected problems related to Fourier and Laplace theory using a “practical approach” that can assist students to meet the learning outcomes of different engineering courses based on this part of mathematics. The level of problems is varying from the basics (e.g. properties of sine and cosine functions) to advanced investigations in abstract spaces.

The reader can find some references to technical areas where the methods can be applied, for instance signal processing, control theory, mechanics. Chapter 4.4 presents a detailed outlook to vibration analysis which is a special area of technical diagnostics in the field of mechanical engineering. This is an important part of industrial engineering where the Fourier transform and other integral transforms are basic tools of the analysis.

The modern methods of technical diagnostics are based on digital measuring and data acquisition systems, this is why discrete transformations are in the focus when practical industrial problems are to be solved. As an example, some functions of the SPM Condition Monitoring System related to the frequency spectrum are mentioned.

Chapters start with brief theoretical summaries concentrating only on basic definitions and formulas that are necessary for calculations. They do not serve as complete discussions of the related mathematical theory.

1 Basic Concepts

1.1 Metric Spaces

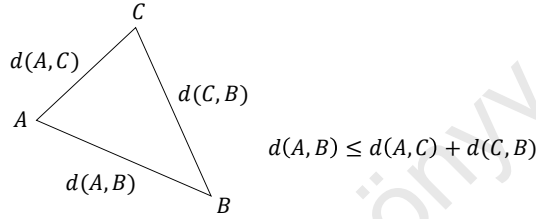
A non-empty set X with the so-called *distance function* (or simply *distance*)

$$d: X \times X \rightarrow \mathbb{R}$$

is called *metric space* if

- $d(x, y) \geq 0$, for all $x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$;
- $d(x, y) = d(y, x)$ for all $x, y \in X$;
- $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$ (*triangle inequality*).

In the space of spatial vectors the triangle inequality is a simple geometrical statement about the sides of a triangle as the figure below illustrates.



Hilbert spaces which are in the focus of this problem book (see Chapter 2) are metric spaces (and normed spaces as well, see Chapter 1.3), where the distance is generated by the inner product as

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y \rangle}.$$

In Hilbert spaces the triangle inequality can be written as

$$\|x + y\| \leq \|x\| + \|y\|, \quad x, y \in X$$

Remark: To see the equivalence write $u = x - z$ and $v = z - y$ in

$$d(x, y) = \|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\| = d(x, z) + d(z, y)$$

The substitution results $\|u + v\| \leq \|u\| + \|v\|$, $u, v \in X$.

Remark: Another form of the triangle inequality is

$$\|x - y\| \geq |\|x\| - \|y\||, \quad x, y \in X$$

A simple calculation proves it:

$$\|x\| = \|y + x - y\| \leq \|y\| + \|x - y\| \Rightarrow \|x\| - \|y\| \leq \|x - y\|$$

$$\|y\| = \|x + y - x\| \leq \|x\| + \|x - y\| \Rightarrow \|y\| - \|x\| \leq \|x - y\| \Rightarrow \|x\| - \|y\| \geq -\|x - y\|$$

that is

$$-\|x - y\| \leq \|x\| - \|y\| \leq \|x - y\|$$

which is equivalent with the statement $\|x - y\| \geq |\|x\| - \|y\||$.

In Hilbert spaces the properties of the distance function can be derived from the properties of the inner product. For instance, the triangle inequality can be proved as follows. Let H be a Hilbert space and $x, y \in H$. Then, using the Cauchy-Schwarz inequality

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \cdot \langle y, y \rangle = \|x\|^2 \cdot \|y\|^2, \text{ we have}$$

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle = \|x\|^2 + 2 \cdot \langle x, y \rangle + \|y\|^2 \leq \\ &\leq \|x\|^2 + 2 \cdot \|x\| \cdot \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2, \end{aligned}$$

that is,

$$\|x + y\| \leq \|x\| + \|y\|, \quad x, y \in H$$

A sequence $(x_k): \mathbb{N} \rightarrow X$ is called *convergent* in a metric space X if there exists $x \in X$ for which $\lim_{k \rightarrow \infty} d(x_k, x) = 0$.

A sequence $(x_k): \mathbb{N} \rightarrow X$ is called *Cauchy sequence* in a metric space X if for any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that $n, m > N(\varepsilon) \Rightarrow d(x_n, x_m) < \varepsilon$.

Theorem: In a metric space all convergent sequences are also Cauchy sequences.

Definition: A metric space X is called *complete* if every Cauchy sequence in X converges in X .

>>>Exercise 1.1 – 1

Show that the following sets (X) are metric spaces with the given distance functions (d) .

1.1 – 1 – a $X = \mathbb{R}, d(x, y) = |x - y|, x, y \in \mathbb{R}$

1.1 – 1 – b $X = \mathbb{R}^n, d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}, x, y \in \mathbb{R}^n$

1.1 – 1 – c $X = \mathbb{C}, d(x, y) = |x - y|, x, y \in \mathbb{C}$

1.1 – 1 – d $X = \mathbb{C}^n, d(x, y) = \sqrt{\sum_{i=1}^n |x_i - y_i|^2}, x, y \in \mathbb{C}^n$

Exercise 1.1 – 1 – a-Solution

The required properties follow from the analogous properties of the absolute value function.

The first two properties are easy to see:

$$d(x, y) = |x - y| \geq 0, x, y \in \mathbb{R}$$

$$d(x, y) = |x - y| = 0 \Leftrightarrow x = y, x, y \in \mathbb{R}$$

$$d(x, y) = |x - y| = |y - x| = d(y, x), x, y \in \mathbb{R}$$

To show the triangle inequality we need the inequality $|x + y| \leq |x| + |y|, x, y \in \mathbb{R}$ which can be proved as follows: $(|x| + |y|)^2 = |x|^2 + 2 \cdot |x| \cdot |y| + |y|^2 \geq x^2 + 2xy + y^2 = (x + y)^2$.

Using it we have $d(x, y) = |x - y| = |x - z + z - y| \leq |x - z| + |z - y|$ for all $x, y, z \in \mathbb{R}$

>>>Exercise 1.1 – 1 – b-Solution

$$d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} \geq 0, x, y \in \mathbb{R}^n,$$

$$d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} = 0 \Leftrightarrow \sum_{i=1}^n (x_i - y_i)^2 = 0 \Leftrightarrow$$

$$\Leftrightarrow x_i - y_i = 0, i = 1, \dots, n \Leftrightarrow x = y, x, y \in \mathbb{R}^n$$

$$d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} = \sqrt{\sum_{i=1}^n (y_i - x_i)^2} = d(y, x), x, y \in \mathbb{R}^n$$

Since $\mathbb{R}^n$ is a Hilbert space with inner product $\langle x, y \rangle = \sum_{i=1}^n x_i \cdot y_i$, (see Chapter 2) the triangle inequality holds.

>>>Exercise 1.1 – 1 – c-Solution

The required properties follow from the related properties of the magnitude (norm) of complex numbers (see Appendix 1). The first two properties are easy to see:

$$d(x, y) = |x - y| \geq 0, x, y \in \mathbb{C}$$

$$d(x, y) = |x - y| = 0 \Leftrightarrow x = y, x, y \in \mathbb{C}$$

$$d(x, y) = |x - y| = |y - x| = d(y, x) \text{ for all } x, y \in \mathbb{C}$$

To show the triangle inequality let $z = a + b \cdot i$ and $w = c + d \cdot i$. Then

$$|z| = \sqrt{a^2 + b^2}, |w| = \sqrt{c^2 + d^2}, |z + w| = \sqrt{(a + c)^2 + (b + d)^2}$$

We have to check inequality

$$\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} \geq \sqrt{(a+c)^2 + (b+d)^2}$$

$$\left(\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2}\right)^2 \geq (a+c)^2 + (b+d)^2$$

$$a^2 + b^2 + c^2 + d^2 + 2 \cdot \sqrt{a^2 + b^2} \cdot \sqrt{c^2 + d^2} \geq a^2 + b^2 + c^2 + d^2 + 2ac + 2bd$$

$$\sqrt{a^2 + b^2} \cdot \sqrt{c^2 + d^2} \geq ac + bd$$

$$(a^2 + b^2) \cdot (c^2 + d^2) \geq (ac + bd)^2$$

$$a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \geq a^2c^2 + 2abcd + b^2d^2$$

$$a^2d^2 + b^2c^2 \geq 2abcd$$

$$(ad - bc)^2 \geq 0 \quad \text{TRUE}$$

>>> **Exercise 1.1 – 1 – d-Solution**

$$d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} \geq 0, \quad x, y \in \mathbb{C}^n$$

$$d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} = 0 \Leftrightarrow \sum_{i=1}^n (x_i - y_i)^2 = 0 \Leftrightarrow$$

$$\Leftrightarrow x_i - y_i = 0, i = 1, \dots, n \Leftrightarrow x = y, \quad x, y \in \mathbb{C}^n$$

$$d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} = \sqrt{\sum_{i=1}^n (y_i - x_i)^2} = d(y, x), \quad x, y \in \mathbb{C}^n$$

Since $\mathbb{C}^n$ is a Hilbert space with inner product $\langle x, y \rangle = \sum_{i=1}^n x_i \cdot y_i^*$, the triangle inequality holds.

1.2 Linear Spaces

Let $\mathbb{K}$ denote the scalar field of real numbers ($\mathbb{R}$) or complex numbers ($\mathbb{C}$).

A non-empty set X with the operations *addition*

$$+ : X \times X \rightarrow X$$

and *scalar multiplication*

$$\cdot : \mathbb{K} \times X \rightarrow X$$

is called *linear space* (or *vector space*) over $\mathbb{K}$ if

- the addition is *commutative*, that is, $x + y = y + x$;
- the addition is *associative*, that is, $x + (y + z) = (x + y) + z$;
- there exists an element $0 \in X$, the *identity element of addition* (also called zero element or zero vector), such that $x + 0 = x$ for all $x \in X$;
- for every $x \in X$, there exists an element $-x \in X$, called the *additive inverse* of x , such that $x + (-x) = 0$;
- the scalar multiplication is *compatible* with field multiplication, that is, $a \cdot (b \cdot x) = (a \cdot b) \cdot x$ for all $a, b \in \mathbb{K}, x \in X$;
- 1 is the *identity element of scalar multiplication*, that is, $1 \cdot x = x$ for all $x \in X$;
- the scalar multiplication is *distributive* with respect to vector addition, that is $a \cdot (x + y) = a \cdot x + a \cdot y$ for all $a \in \mathbb{K}, x, y \in X$;
- the scalar multiplication is *distributive* with respect to field addition, that is, $(a + b) \cdot x = a \cdot x + b \cdot x$ for all $a, b \in \mathbb{K}, x \in X$.

Elements of X are called *vectors*.

In the case $\mathbb{K} = \mathbb{R}$ the linear space is called *real*, if $\mathbb{K} = \mathbb{C}$ it is called *complex*.

If $Y \subset X$ is a vector space over the same field and with the same operations, then Y is called the *subspace* of X .

Let X be a linear space and n be a positive integer.

If $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ and $x_1, \dots, x_n \in X$ then

$$\alpha_1 \cdot x_1 + \dots + \alpha_n \cdot x_n \in X$$

is called the *linear combination* of vectors $x_1, \dots, x_n$ with scalars $\alpha_1, \dots, \alpha_n$.

A finite system of vectors $\{x_1, \dots, x_n\} \subset X$ is called *linearly independent* if

$$\alpha_1 \cdot x_1 + \dots + \alpha_n \cdot x_n = 0 \implies \alpha_1 = \dots = \alpha_n = 0.$$

A system of vectors $\{x_k\}_{k=1}^{\infty} \subset X$ is called *linearly independent* if all of its finite subsystems are linearly independent.

The *span of system of vectors* $\{x_1, \dots, x_n\} \subset X$ is defined as

$$\text{span}\{x_1, \dots, x_n\} = \{\alpha_1 \cdot x_1 + \dots + \alpha_n \cdot x_n \mid \alpha_1, \dots, \alpha_n \in \mathbb{R}\}.$$

The set $\text{span}\{x_1, \dots, x_n\}$ is a subspace of X .

If a finite system of vectors $\{b_1, \dots, b_n\} \subset X$ is linearly independent and $\text{span}\{b_1, \dots, b_n\} = X$, then

- X is called a *finite dimensional linear space*;
- $\{b_1, \dots, b_n\}$ is called a *basis* of X ;
- n is called the *dimension* of X .

If a linear space X is not finite dimensional (that is, it has no finite basis), it is called *infinite dimensional linear space*.

Remark: For instance, the linear space $C(I) = \{x: I \rightarrow \mathbb{R} \mid x \text{ is continuous on } I\}$ is infinite dimensional.

Theorem: If X is an n -dimensional linear space, then a system of vectors $\{x_1, \dots, x_n\} \subset X$ is a basis of X , if and only if the system is linearly independent.

The *rank of system of vectors* $\{x_1, \dots, x_n\} \subset X$ is the dimension of $\text{span}\{x_1, \dots, x_n\}$. It is denoted by $\text{rank}\{x_1, \dots, x_n\}$.

Remark: Let $\{x_1, \dots, x_n\} \subset \mathbb{R}^k$. The rank of matrix $(x_1 \dots x_n) \in \mathbb{R}^{k \times n}$, by definition, is the rank of system of vectors $\{x_1, \dots, x_n\} \subset \mathbb{R}^k$.

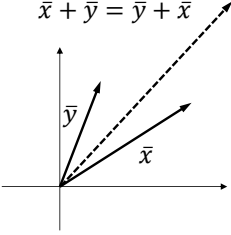
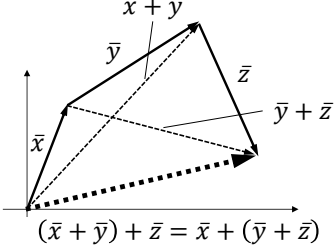
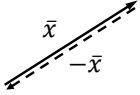
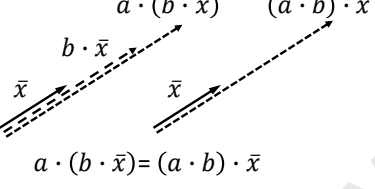
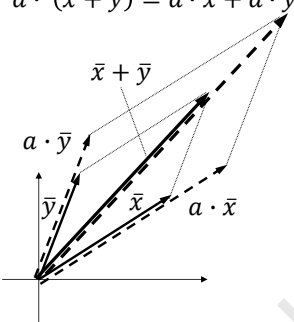
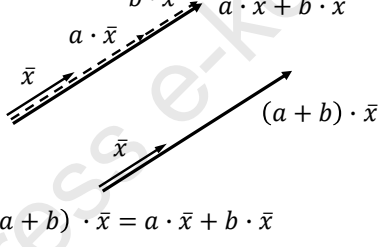
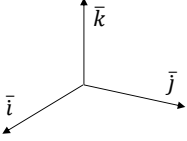
Remark: Let $A \in \mathbb{R}^{n \times n}$. Then $\text{rank} A = n$ if and only if $\det A \neq 0$.

Exercise 1.2 – 1

Show that the following sets are linear spaces with the commonly used addition and scalar multiplication. Give the dimension of the spaces. Give a basis of the finite dimensional ones.

- 1.2 – 1 – a spatial vectors (over $\mathbb{R}$)
- 1.2 – 1 – b $\mathbb{R}^n$ (over $\mathbb{R}$)
- 1.2 – 1 – c $\mathbb{C}^n$ (over $\mathbb{C}$ and over $\mathbb{R}$)
- 1.2 – 1 – d $\mathbb{R}^{n \times m}$, the set of real matrices of type $n \times m$ (over $\mathbb{R}$)
- 1.2 – 1 – e $\mathbb{C}^{n \times m}$, the set of complex matrices of type $n \times m$ (over $\mathbb{C}$ and over $\mathbb{R}$)
- 1.2 – 1 – f $\{x_k\}_{k=1}^{\infty}: \mathbb{N} \rightarrow \mathbb{R}$, the set of real sequences (over $\mathbb{R}$)
- 1.2 – 1 – g $\{x_k\}_{k=1}^{\infty}: \mathbb{N} \rightarrow \mathbb{C}$, the set of real sequences (over $\mathbb{R}$ and over $\mathbb{C}$)
- 1.2 – 1 – h $x: I \rightarrow \mathbb{R}$, I is an interval (over $\mathbb{R}$)
- 1.2 – 1 – i $x: I \rightarrow \mathbb{C}$, I is an interval (over $\mathbb{R}$ and over $\mathbb{C}$)

>>>Exercise 1.2 – 1 – a-Solution

commutativity $\bar{x} + \bar{y} = \bar{y} + \bar{x}$ 	associativity  $(\bar{x} + \bar{y}) + \bar{z} = \bar{x} + (\bar{y} + \bar{z})$	zero vector If the starting point and the endpoint are the same, we have a zero vector $\bar{0}$
additive inverse $-1 \cdot \bar{x}$ is the additive inverse of $\bar{x}$  $\bar{x} + (-\bar{x}) = \bar{0}$	compatibility  $a \cdot (b \cdot \bar{x}) = (a \cdot b) \cdot \bar{x}$	1 is the identity element of the scalar multiplication: $1 \cdot \bar{x} = \bar{x}$
distributivity $a \cdot (\bar{x} + \bar{y}) = a \cdot \bar{x} + a \cdot \bar{y}$ 	distributivity  $(a + b) \cdot \bar{x} = a \cdot \bar{x} + b \cdot \bar{x}$	Basis 

$\{\bar{i}, \bar{j}, \bar{k}\}$ is a basis in the space of spatial vectors (three unit vectors which are pairwise perpendicular), thus the dimension of the space is 3.

>>>Exercise 1.2 – 1 – b-Solution

Because of the definition of addition in $\mathbb{R}^n$ the commutativity and the associativity are consequences of the analogous properties of addition in $\mathbb{R}$: if $x, y, z \in \mathbb{R}^n$ then $x + y = y + x$ and $x + (y + z) = (x + y) + z$, respectively.

The zero vector $\mathcal{O} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$ is the identity element of addition: $x + \mathcal{O} = x$ for all $x \in \mathbb{R}^n$.

For every $x \in \mathbb{R}^n$, $-x$ is the additive inverse of x : $x + (-x) = \mathcal{O}$.

Because of the definition of the scalar multiplication in $\mathbb{R}^n$ the further properties (compatibility, 1 is the identity element of scalar multiplication, distributivity) are the consequences of the analogous properties in $\mathbb{R}$.

System of unit vectors $\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^n$ is a basis in $\mathbb{R}^n$, thus the dimension of the space is n .

>>>Exercise 1.2 – 1 – c-Solution

Because of the definition of addition in $\mathbb{C}^n$ the commutativity and the associativity are consequences of the analogous properties of addition in $\mathbb{C}$ (see Appendix 1): if $x, y, z \in \mathbb{C}^n$ then $x + y = y + x$ and $x + (y + z) = (x + y) + z$, respectively.

The zero vector $\mathcal{O} = \begin{pmatrix} 0 + 0 \cdot i \\ 0 + 0 \cdot i \\ \vdots \\ 0 + 0 \cdot i \end{pmatrix} \in \mathbb{R}^n$ is the identity element of addition: $x + \mathcal{O} = x$ for all $x \in \mathbb{C}^n$.

For every $z = \begin{pmatrix} a_1 + b_1 \cdot i \\ a_2 + b_2 \cdot i \\ \vdots \\ a_n + b_n \cdot i \end{pmatrix} \in \mathbb{C}^n$, $-z = \begin{pmatrix} -a_1 - b_1 \cdot i \\ -a_2 - b_2 \cdot i \\ \vdots \\ -a_n - b_n \cdot i \end{pmatrix}$ is the additive inverse of z :

$$\begin{pmatrix} a_1 + b_1 \cdot i \\ a_2 + b_2 \cdot i \\ \vdots \\ a_n + b_n \cdot i \end{pmatrix} + \begin{pmatrix} -a_1 - b_1 \cdot i \\ -a_2 - b_2 \cdot i \\ \vdots \\ -a_n - b_n \cdot i \end{pmatrix} = \begin{pmatrix} 0 + 0 \cdot i \\ 0 + 0 \cdot i \\ \vdots \\ 0 + 0 \cdot i \end{pmatrix}$$

Let us consider the scalar field $\mathbb{C}$ with the properties of addition and multiplication. Because of the definition of the scalar multiplication in $\mathbb{C}^n$ (over field $\mathbb{C}$) the further properties (compatibility, distributivity) are simple consequences of the analogous properties of multiplication in $\mathbb{C}$.

$1 + 0 \cdot i \in \mathbb{C}$ is the identity element of scalar multiplication: $(1 + 0 \cdot i) \cdot z = z$ for all $z \in \mathbb{C}$.

System of unit vectors $\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\} \subset \mathbb{C}^n$ is a basis in $\mathbb{C}^n$, thus the dimension of the space over field $\mathbb{C}$ is n .

We can consider $\mathbb{C}^n$ over field $\mathbb{R}$ as well. Then the identity element of scalar multiplication is $1 \in \mathbb{R}$, and it is easy to see that all the properties of linear spaces hold similarly. But the dimension is different.

System of unit vectors $\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ i \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ i \end{pmatrix} \right\} \subset \mathbb{C}^n$ is a basis in $\mathbb{C}^n$, thus the dimension of the space over $\mathbb{R}$ is $2n$.

>>>Exercise 1.2 – 1 – d-Solution

Because of the definition of the matrix addition in $\mathbb{R}^{n \times m}$ the commutativity and the associativity are consequences of the analogous properties of addition in $\mathbb{R}$: if $A, B, C \in \mathbb{R}^{n \times m}$ then $A + B = B + A$ and $A + (B + C) = (A + B) + C$, respectively.

The zero matrix $\mathcal{O} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}^T \in \mathbb{R}^{n \times m}$ is the identity element of addition: $A + \mathcal{O} = A$ for all $A \in \mathbb{R}^{n \times m}$.

For every $A = (a_{jk}) \in \mathbb{R}^{n \times m}$, $-A = (-a_{jk})$ is the additive inverse of A : $A + (-A) = \mathcal{O}$.

Because of the definition of the scalar multiplication in $\mathbb{R}^{n \times m}$ the further properties (compatibility, $1 \in \mathbb{R}$ is the identity element of scalar multiplication, distributivity) are the consequences of the analogous properties in $\mathbb{R}$.

Using the notation $I_{j,k}$ for matrices whose element in j -th row and k -th column is 1 and all other elements are 0, system of matrices $\{I_{j,k}\}_{j=1,\dots,n,k=1,\dots,m} \subset \mathbb{R}^{n \times m}$ is a basis in $\mathbb{R}^{n \times m}$, thus the dimension of the space is $n \cdot m$.

>>>Exercise 1.2 – 1 – e-Solution

Because of the definition of the matrix addition in $\mathbb{C}^{n \times m}$ the commutativity and the associativity are consequences of the commutativity and associativity properties of addition in $\mathbb{C}$: if $A, B, C \in \mathbb{C}^{n \times m}$ then $A + B = B + A$ and $A + (B + C) = (A + B) + C$, respectively.

The zero matrix $\mathcal{O} = \begin{pmatrix} 0 + 0 \cdot i & \dots & 0 + 0 \cdot i \\ \vdots & & \vdots \\ 0 + 0 \cdot i & \dots & 0 + 0 \cdot i \end{pmatrix} \in \mathbb{C}^{n \times m}$ is the identity element of addition:

$A + \mathcal{O} = A$ for all $A \in \mathbb{C}^{n \times m}$.

For every $A = (a_{jk} + b_{jk} \cdot i)_{j=1,\dots,n,k=1,\dots,m} \in \mathbb{C}^{n \times m}$, $-A = (-a_{jk} - b_{jk} \cdot i)_{j=1,\dots,n,k=1,\dots,m}$ is the additive inverse of A : $A + (-A) = \mathcal{O}$.

Let us consider the scalar field $\mathbb{C}$ with the properties of addition and multiplication. Because of the definition of the scalar multiplication in $\mathbb{C}^{n \times m}$ (over field $\mathbb{C}$) the further properties (compatibility, distributivity) are the consequences of the analogous properties of multiplication in $\mathbb{C}$.

$1 + 0 \cdot i \in \mathbb{C}$ is the identity element of scalar multiplication: $(1 + 0 \cdot i) \cdot A = A$ for all $A \in \mathbb{C}^{n \times m}$.

Using the notation $I_{j,k}$ for matrices whose element in j -th row and k -th column is $1 + 0 \cdot i$ and all other elements are $0 + 0 \cdot i$, system of matrices $\{I_{j,k}\}_{j=1,\dots,n,k=1,\dots,m} \subset \mathbb{C}^{n \times m}$ is a basis in $\mathbb{C}^{n \times m}$,

thus the dimension of the space is $n \cdot m$.

We can consider $\mathbb{C}^{n \times m}$ over field $\mathbb{R}$ as well. Then the identity element of scalar multiplication is $1 \in \mathbb{R}$, and it is easy to see that all the properties of linear spaces hold similarly. But the dimension is different.

Using the notation $I_{j,k}$ for matrices whose element in j -th row and k -th column is 1 and all other elements are 0, and the notation $J_{j,k}$ for matrices whose element in j -th row and k -th column is i and all other elements are 0, system of matrices $\{I_{j,k}\}_{j=1,\dots,n,k=1,\dots,m} \cup \{J_{j,k}\}_{j=1,\dots,n,k=1,\dots,m} \subset \mathbb{C}^{n \times m}$

is a basis in $\mathbb{C}^{n \times m}$, thus the dimension of the space is $2 \cdot n \cdot m$.

>>>Exercise 1.2 – 1 – f-Solution

In the set of real sequences $\{x_k\}_{k=1}^{\infty}: \mathbb{N} \rightarrow \mathbb{R}$ addition and scalar multiplication are defined elementwise:

$$\begin{aligned} \{x_k\}_{k=1}^{\infty} + \{y_k\}_{k=1}^{\infty} &= \{x_k + y_k\}_{k=1}^{\infty} \\ a \cdot \{x_k\}_{k=1}^{\infty} &= \{a \cdot x_k\}_{k=1}^{\infty}, \quad a \in \mathbb{R}. \end{aligned}$$

Thus commutativity and the associativity are consequences of the analogous properties of addition in $\mathbb{R}$: if $\{x_k\}_{k=1}^{\infty}$, $\{y_k\}_{k=1}^{\infty}$, and $\{z_k\}_{k=1}^{\infty}$ are real sequences then

$$\{x_k\}_{k=1}^{\infty} + \{y_k\}_{k=1}^{\infty} = \{y_k\}_{k=1}^{\infty} + \{x_k\}_{k=1}^{\infty}$$

and

$$\{x_k\}_{k=1}^{\infty} + (\{y_k\}_{k=1}^{\infty} + \{z_k\}_{k=1}^{\infty}) = (\{x_k\}_{k=1}^{\infty} + \{y_k\}_{k=1}^{\infty}) + \{z_k\}_{k=1}^{\infty},$$

respectively.

The zero sequence $\mathcal{O} = \{c_k = 0\}_{k=1}^{\infty}$ is the identity element of addition: $\{x_k\}_{k=1}^{\infty} + \mathcal{O} = \{x_k\}_{k=1}^{\infty}$ for all real sequences $\{x_k\}_{k=1}^{\infty}$.

For every real sequence $\{x_k\}_{k=1}^{\infty}$, $\{-x_k\}_{k=1}^{\infty}$ is the additive inverse of $\{x_k\}_{k=1}^{\infty}$: $\{x_k\}_{k=1}^{\infty} + \{-x_k\}_{k=1}^{\infty} = \mathcal{O}$.

Because of the definition of the scalar multiplication of real sequences the further properties (compatibility, 1 is the identity element of scalar multiplication, distributivity) are simple consequences of the analogous properties in $\mathbb{R}$.

The linear space of real sequences is not finite dimensional.

>>>Exercise 1.2 – 1 – g-Solution

In the set of complex sequences $\{x_k\}_{k=1}^{\infty}: \mathbb{N} \rightarrow \mathbb{C}$ addition and scalar multiplication (over $\mathbb{R}$ or $\mathbb{C}$) are defined elementwise:

$$\begin{aligned}\{x_k\}_{k=1}^{\infty} + \{y_k\}_{k=1}^{\infty} &= \{x_k + y_k\}_{k=1}^{\infty} \\ a \cdot \{x_k\}_{k=1}^{\infty} &= \{a \cdot x_k\}_{k=1}^{\infty}, \quad a \in \mathbb{R} \text{ or } \mathbb{C}.\end{aligned}$$

Thus commutativity and the associativity are consequences of the analogous properties of addition in $\mathbb{C}$: if $\{x_k\}_{k=1}^{\infty}$, $\{y_k\}_{k=1}^{\infty}$, and $\{z_k\}_{k=1}^{\infty}$ are complex sequences then

$$\{x_k\}_{k=1}^{\infty} + \{y_k\}_{k=1}^{\infty} = \{y_k\}_{k=1}^{\infty} + \{x_k\}_{k=1}^{\infty}$$

and

$$\{x_k\}_{k=1}^{\infty} + (\{y_k\}_{k=1}^{\infty} + \{z_k\}_{k=1}^{\infty}) = (\{x_k\}_{k=1}^{\infty} + \{y_k\}_{k=1}^{\infty}) + \{z_k\}_{k=1}^{\infty},$$

respectively.

The zero sequence $\mathcal{O} = \{c_k = 0 + 0 \cdot i\}_{k=1}^{\infty}$ is the identity element of addition: $\{x_k\}_{k=1}^{\infty} + \mathcal{O} = \{x_k\}_{k=1}^{\infty}$ for all complex sequences $\{x_k\}_{k=1}^{\infty}$.

For every complex sequence $\{x_k\}_{k=1}^{\infty}$, $\{-x_k\}_{k=1}^{\infty}$ is the additive inverse of $\{x_k\}_{k=1}^{\infty}$: $\{x_k\}_{k=1}^{\infty} + \{-x_k\}_{k=1}^{\infty} = \mathcal{O}$.

Because of the definition of the scalar multiplication of complex sequences the further properties (compatibility, distributivity) are simple consequences of the analogous properties in $\mathbb{C}$.

$1 + 0 \cdot i \in \mathbb{C}$ is the identity element of scalar multiplication: $(1 + 0 \cdot i) \cdot \{x_k\}_{k=1}^{\infty} = \{x_k\}_{k=1}^{\infty}$ for all complex sequences $\{x_k\}_{k=1}^{\infty}$.

We can consider the linear space of complex sequences over field $\mathbb{R}$ as well. Then the identity element of scalar multiplication is $1 \in \mathbb{R}$, and it is easy to see that all the properties of linear spaces hold similarly.

The linear space of complex sequences is not finite dimensional.

>>>Exercise 1.2 – 1 – h-Solution

In the set of functions $x: I \rightarrow \mathbb{R}$ addition and scalar multiplication are defined elementwise:

$$\begin{aligned}(x + y)(t) &= x(t) + y(t), \quad t \in I \\ (a \cdot x)(t) &= a \cdot x(t), \quad t \in I\end{aligned}$$

Thus commutativity and the associativity are consequences of the analogous properties of addition in $\mathbb{R}$: if we have functions $x: I \rightarrow \mathbb{R}$, $y: I \rightarrow \mathbb{R}$, $z: I \rightarrow \mathbb{R}$ then

$$x + y = y + x$$

and

$$x + (y + z) = (x + y) + z,$$

respectively.

The zero function $\mathcal{O}(t) = 0, t \in I$ is the identity element of addition: $x + \mathcal{O} = x$ for all functions $x: I \rightarrow \mathbb{R}$.

For every function $x: I \rightarrow \mathbb{R}$, $(-x)(t) = -x(t)$, $t \in I$ is the additive inverse of x : $x + (-x) = \mathcal{O}$. Because of the definition of the scalar multiplication of real sequences the further properties (compatibility, 1 is the identity element of scalar multiplication, distributivity) are simple consequences of the analogous properties in $\mathbb{R}$.

The linear space of functions $I \rightarrow \mathbb{R}$ is not finite dimensional.

>>>Exercise 1.2 – 1 – i-Solution

In the set of functions $x: I \rightarrow \mathbb{C}$ addition and scalar multiplication (over $\mathbb{R}$ or $\mathbb{C}$) are defined elementwise:

$$(x + y)(t) = x(t) + y(t), \quad t \in I$$

$$(a \cdot x)(t) = a \cdot x(t), \quad t \in I$$

Thus commutativity and the associativity are consequences of the analogous properties of addition in $\mathbb{C}$: if we have functions $x: I \rightarrow \mathbb{C}$, $y: I \rightarrow \mathbb{C}$, $z: I \rightarrow \mathbb{C}$ then

$$x + y = y + x$$

and

$$x + (y + z) = (x + y) + z,$$

respectively.

The zero function $\mathcal{O}(t) = 0 + 0 \cdot i$, $t \in I$ is the identity element of addition: $x + \mathcal{O} = x$ for all functions $x: I \rightarrow \mathbb{C}$.

For every function $x: I \rightarrow \mathbb{C}$, $(-x)(t) = -x(t)$, $t \in I$ is the additive inverse of x : $x + (-x) = \mathcal{O}$.

Because of the definition of the scalar multiplication of real sequences the further properties (compatibility, distributivity) are simple consequences of the analogous properties in $\mathbb{C}$.

$1 + 0 \cdot i \in \mathbb{C}$ is the identity element of scalar multiplication: $(1 + 0 \cdot i) \cdot x = x$ for all functions $x: I \rightarrow \mathbb{C}$.

The linear space of functions $I \rightarrow \mathbb{C}$ is not finite dimensional.

>>>Exercise 1.2 – 2

Consider the linear space $X = \mathbb{R}^3$. Write the vector y as a linear combination of the vectors in the given system $\{x_1, \dots, x_n\}$.

	y	$\{x_1, \dots, x_n\}$
1.2 – 2 – a	$\begin{pmatrix} 10 \\ 17 \\ 1 \end{pmatrix}$	$\left\{ \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \right\}$
1.2 – 2 – b	$\begin{pmatrix} -9 \\ 11 \\ -8 \end{pmatrix}$	$\left\{ \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$
1.2 – 2 – c	$\begin{pmatrix} 10 \\ 17 \\ 1 \end{pmatrix}$	$\left\{ \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} \right\}$
1.2 – 2 – d	$\begin{pmatrix} 7 \\ 2 \\ 9 \end{pmatrix}$	$\left\{ \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$

>>>Exercise 1.2 – 2 – a – Solution

Solution of the system $\alpha_1 \cdot \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ 17 \\ 1 \end{pmatrix}$ gives the answer to the

question. The matrix form of the system is $\begin{pmatrix} -1 & 1 & 3 \\ 3 & 1 & 4 \\ 0 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 17 \\ 1 \end{pmatrix}$. Since

$\det \begin{pmatrix} -1 & 1 & 3 \\ 3 & 1 & 4 \\ 0 & 2 & -1 \end{pmatrix} = 30 \neq 0$ the system is determined, so it can be solved e.g. with Cramer's rule:

$$\alpha_1 = \frac{\det \begin{pmatrix} 10 & 1 & 3 \\ 17 & 1 & 4 \\ 1 & 2 & -1 \end{pmatrix}}{\det \begin{pmatrix} -1 & 1 & 3 \\ 3 & 1 & 4 \\ 0 & 2 & -1 \end{pmatrix}} = 1, \quad \alpha_2 = \frac{\det \begin{pmatrix} -1 & 10 & 3 \\ 3 & 17 & 4 \\ 0 & 1 & -1 \end{pmatrix}}{\det \begin{pmatrix} -1 & 1 & 3 \\ 3 & 1 & 4 \\ 0 & 2 & -1 \end{pmatrix}} = 2, \quad \alpha_3 = \frac{\det \begin{pmatrix} -1 & 1 & 10 \\ 3 & 1 & 17 \\ 0 & 2 & 1 \end{pmatrix}}{\det \begin{pmatrix} -1 & 1 & 3 \\ 3 & 1 & 4 \\ 0 & 2 & -1 \end{pmatrix}} = 3$$

The solution is $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, thus we have $1 \cdot \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + 3 \cdot \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ 17 \\ 1 \end{pmatrix}$

>>>Exercise 1.2 – 2 – b – Solution

Solution of the system $\alpha_1 \cdot \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -9 \\ 11 \\ -8 \end{pmatrix}$ gives the answer to the question.

This system contains three equations but only two unknowns

$$\begin{aligned} \text{I.} \quad & -\alpha_1 + \alpha_2 = -9 \\ \text{II.} \quad & 3\alpha_1 + \alpha_2 = 11 \\ \text{III.} \quad & 2\alpha_2 = -8 \end{aligned}$$

Considering equations I and III we have $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$, and this is a solution of equation II as well. Thus we have $5 \cdot \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} - 4 \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -9 \\ 11 \\ -8 \end{pmatrix}$.

>>>Exercise 1.2 – 2 – c – Solution

Solution of the system $\alpha_1 \cdot \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ 17 \\ 1 \end{pmatrix}$ gives the answer to the question. The matrix form of the system is $\begin{pmatrix} -1 & 1 & 0 \\ 3 & 1 & 4 \\ 0 & 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 17 \\ 1 \end{pmatrix}$. Since

$\det \begin{pmatrix} -1 & 1 & 0 \\ 3 & 1 & 4 \\ 0 & 2 & 2 \end{pmatrix} = 0$ the system is undetermined, and it can be solved e.g. with Gaussian elimination.

-1	1	0	10	
3	1	4	17	
0	2	2	1	
1	-1	0	-10	
0	4	4	47	
0	2	2	1	
1	-1	0	-10	
0	4	4	47	
0	0	0	-22.5	$\Rightarrow$ inconsistent system

The results shows that $\begin{pmatrix} 10 \\ 17 \\ 1 \end{pmatrix}$ cannot be written as a linear combination of vectors

$$\left\{ \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} \right\}.$$

>>>Exercise 1.2 – 2 – d – Solution

Solution of the system $\alpha_1 \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 9 \end{pmatrix}$ gives the answer to the

question. The matrix form of the system is $\begin{pmatrix} 2 & 3 & 1 \\ -1 & 1 & 1 \\ 5 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 9 \end{pmatrix}$. Since

$\det \begin{pmatrix} 2 & 3 & 1 \\ -1 & 1 & 1 \\ 5 & 0 & 2 \end{pmatrix} = 20 \neq 0$ the system is determined, so it can be solved e.g. with Cramer's rule:

$$\alpha_1 = \frac{\det \begin{pmatrix} 7 & 3 & 1 \\ 2 & 1 & 1 \\ 9 & 0 & 2 \end{pmatrix}}{\det \begin{pmatrix} 2 & 3 & 1 \\ -1 & 1 & 1 \\ 5 & 0 & 2 \end{pmatrix}} = 1, \quad \alpha_2 = \frac{\det \begin{pmatrix} 2 & 7 & 1 \\ -1 & 2 & 1 \\ 5 & 9 & 2 \end{pmatrix}}{\det \begin{pmatrix} 2 & 3 & 1 \\ -1 & 1 & 1 \\ 5 & 0 & 2 \end{pmatrix}} = 2, \quad \alpha_3 = \frac{\det \begin{pmatrix} 2 & 3 & 7 \\ -1 & 1 & 2 \\ 5 & 0 & 9 \end{pmatrix}}{\det \begin{pmatrix} 2 & 3 & 1 \\ -1 & 1 & 1 \\ 5 & 0 & 2 \end{pmatrix}} = 2$$

The solution is $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, thus we have $1 \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + 2 \cdot \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 9 \end{pmatrix}$

>>>Exercise 1.2 – 3

Check the linear independency of the given systems.

1.2 – 3 – a $\left\{ x_1 = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, x_3 = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \right\}$

1.2 – 3 – b $\left\{ x_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, x_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, x_3 = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} \right\}$

>>>Exercise 1.2 – 3 – a – Solution

First solution: Since a 3-element system of vectors is given in the 3-dimensional space $\mathbb{R}^3$, it is

enough to check the determinant $\det \begin{pmatrix} -1 & 1 & 3 \\ 3 & 1 & 4 \\ 0 & -2 & 1 \end{pmatrix} = -30$. It is different from zero, so the system is independent.

Second solution: We solve the system $\alpha_1 \cdot \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The matrix

form of the system is $\begin{pmatrix} -1 & 1 & 3 \\ 3 & 1 & 4 \\ 0 & -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Since $\det \begin{pmatrix} -1 & 1 & 3 \\ 3 & 1 & 4 \\ 0 & -2 & 1 \end{pmatrix} = 20 \neq 0$ the

system is determined and the unique solution is $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Thus the system of vectors is independent.

>>>Exercise 1.2 – 3 – b – Solution

Since $\det \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & 7 \end{pmatrix} = 0$, the system is dependent.

>>>Exercise 1.2 – 4

Check whether the system $\left\{ a_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, a_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, a_3 = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} \right\}$ is a basis of $\mathbb{R}^3$.

>>>Exercise 1.2 – 4 – Solution

Since a 3-element system of vectors is given in the 3-dimensional space $\mathbb{R}^3$, it is enough to check the independency of the system. The determinant is $\det \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & 7 \end{pmatrix} = 0$, thus the system is dependent, and consequently, it is not a basis.

1.3 Normed Spaces

A linear space X with function $\| \cdot \|: X \rightarrow \mathbb{K}$ satisfying

- $\|x\| \geq 0$ for all $x \in X$
- $\|x\| = 0 \Leftrightarrow x = 0$
- $\|a \cdot x\| = |a| \cdot \|x\|$ for all $a \in \mathbb{R}, x \in X$
- $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (triangle inequality)

is called (linear) normed space.

If $x \in X$, $\|x\|$ is called the *norm* of x .

Remark: A linear space can be a normed space with different norms.

Hilbert spaces (see in Chapter 2) are also normed spaces with the norm $\|x - y\| = \sqrt{\langle x - y, x - y \rangle}$. It was shown In Chapter 1.1 that the triangle inequality holds when the norm is generated by an inner product.

A (linear) normed space is called Banach space if it is complete.

>>>Exercise 1.3 – 1

Show that $\mathbb{R}^n$ is a normed space with the following norm functions:

$$\mathbf{1.3 - 1 - a} \quad \|x\| = \|x_1, \dots, x_n\| = \sqrt{x_1^2 + \dots + x_n^2}, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n$$

$$\mathbf{1.3 - 1 - b} \quad \|x\| = \|x_1, \dots, x_n\| = \max\{|x_1|, \dots, |x_n|\}, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n$$

$$\mathbf{1.3 - 1 - c} \quad \|x\| = \|x_1, \dots, x_n\| = |x_1| + \dots + |x_n|, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n$$

>>>Exercise 1.3 – 1 – a- Solution

Since the square and the square root functions are non-negative $\|x\| = \sqrt{x_1^2 + \dots + x_n^2} \geq 0$ is evident, furthermore $\|x\| = \sqrt{x_1^2 + \dots + x_n^2} = 0 \Leftrightarrow x_1 = \dots = x_n = 0 \Leftrightarrow x = 0$.

$$\|a \cdot x\| = \sqrt{(a \cdot x_1)^2 + \dots + (a \cdot x_n)^2} = |a| \cdot \sqrt{x_1^2 + \dots + x_n^2} = |a| \cdot \|x\|$$

$\mathbb{R}^n$ is a Hilbert space with inner product $\langle x, y \rangle = \sum_{i=1}^n x_i \cdot y_i$, thus the triangle inequality holds.

>>>Exercise 1.3 – 1 – b- Solution

Since the absolute value function is non-negative $\|x\| = \max\{|x_1|, \dots, |x_n|\} \geq 0$ is evident, furthermore $\|x\| = \max\{|x_1|, \dots, |x_n|\} = 0 \Leftrightarrow x_1 = \dots = x_n = 0 \Leftrightarrow x = 0$.

$$\|a \cdot x\| = \max\{|a \cdot x_1|, \dots, |a \cdot x_n|\} = |a| \cdot \max\{|x_1|, \dots, |x_n|\} = |a| \cdot \|x\|$$

$$\begin{aligned} \|x + y\| &= \max\{|x_1 + y_1|, \dots, |x_n + y_n|\} \leq \max\{|x_1| + |y_1|, \dots, |x_n| + |y_n|\} \leq \\ &\leq \max\{|x_1|, \dots, |x_n|\} + \max\{|y_1|, \dots, |y_n|\} = \|x\| + \|y\| \end{aligned}$$

>>>Exercise 1.3 – 1 – c- Solution

Since the absolute value function is non-negative $\|x\| = |x_1| + \dots + |x_n| \geq 0$ is evident, furthermore $\|x\| = |x_1| + \dots + |x_n| = 0 \Leftrightarrow x_1 = \dots = x_n = 0 \Leftrightarrow x = 0$.

$$\|a \cdot x\| = |a \cdot x_1| + \dots + |a \cdot x_n| = |a| \cdot (|x_1| + \dots + |x_n|) = |a| \cdot \|x\|$$

$$\|x + y\| = |x_1 + y_1| + \dots + |x_n + y_n| \leq |x_1| + |y_1| + \dots + |x_n| + |y_n| = \|x\| + \|y\|$$

>>>Exercise 1.3 – 2

Write the distance function in $\mathbb{R}^n$ belonging to the norms given in Exercise 1.3-1.

>>>Exercise 1.3 – 2 – a- Solution

$$\|x\| = \|(x_1, \dots, x_n)\| = \sqrt{x_1^2 + \dots + x_n^2} \Rightarrow d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

>>>Exercise 1.3 – 2 – b- Solution

$$\|x\| = \|(x_1, \dots, x_n)\| = \max\{|x_1|, \dots, |x_n|\} \Rightarrow d(x, y) = \|x - y\| = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$$

>>>Exercise 1.3 – 2 – c- Solution

$$\|x\| = \|(x_1, \dots, x_n)\| = |x_1| + \dots + |x_n| \Rightarrow d(x, y) = \|x - y\| = |x_1 - y_1| + \dots + |x_n - y_n|$$

1.4 Integrals

Let I be an interval of positive length. Integrals of functions $x: I \rightarrow \mathbb{R}$ (or $x: I \rightarrow \mathbb{C}$) play central role in the further chapters: orthonormal systems, Fourier series, integral transforms. The concept of the integrals is based on the measure theory, one of the fundamental areas of mathematic, which is out of the scope of this problem book. The well-known Riemann integral is a special case in the integral theory but, from the point of view of the practice, it is ‘the integral’: when we determine an integral, ultimately, we calculate a Riemann integral. In this problem book we always suppose that the functions appearing in integral formulas are Riemann integrable.

An interpretation of the *Riemann integral* is the following. Let $f: [a, b] \rightarrow \mathbb{R}$ be bounded function. A *partition* of $[a, b]$ is the set of sub-intervals

$$P(n, [a, b]) = \{[x_{i-1}, x_i] \mid i = 1, \dots, n\},$$

where

$$a = x_0 < x_1 < \dots < x_n = b, \quad n \in \mathbb{N}.$$

The *mesh* of the partition is $M(P) = \max_{1 \leq i \leq n} (x_i - x_{i-1})$.

The partition is called *tagged* if $t_i \in [x_{i-1}, x_i]$, $i = 1, \dots, n$ are given.

The *Riemann sum* belonging to f and a tagged partition $P(n, [a, b])$ is

$$I(f, P) = \sum_{i=1}^n f(t_i) \cdot (x_i - x_{i-1}).$$

$f: [a, b] \rightarrow \mathbb{R}$ is *Riemann integrable* if there exists $S \in \mathbb{R}$ such that for all sequences of tagged partitions $P(n, [a, b])$, $n \in \mathbb{N}$ satisfying $\lim_{n \rightarrow \infty} M(P(n, [a, b])) = 0$ we have

$$\lim_{n \rightarrow \infty} I(f, P(n, [a, b])) = S.$$

If f is Riemann integrable then we use notation $S = \int_a^b f$

The geometrical interpretation of $\int_a^b f$ is the area “under” the curve of f on interval $[a, b]$ if f is non-negative.

Some basic properties of the Riemann integral are the following. Supposing that the functions appearing in the following formulas are integrable we have

linearity

$$f, g: I \rightarrow \mathbb{R}, \alpha, \beta \in \mathbb{R} \Rightarrow \int_I (\alpha \cdot f + \beta \cdot g) = \alpha \cdot \int_I f + \beta \cdot \int_I g$$

monotonicity

$$f, g: I \rightarrow \mathbb{R}, f \leq g \Rightarrow \int_I f \leq \int_I g$$

$$f: I \rightarrow \mathbb{R} \Rightarrow \left| \int_I f \right| \leq \int_I |f|$$

additivity on intervals

$$f: I \rightarrow \mathbb{R}, a, b, c \in I \Rightarrow \int_a^b f = \int_a^c f + \int_c^b f,$$

where, by definition, $\int_b^a f = -\int_a^b f$.

In our formulas we frequently have improper integral when $I = \mathbb{R}$ or $I = [0, \infty[$. By definition these integrals can be given as limits of integrals considering the lower or the upper limit of the integral as a variable, e.g. $\int_0^\infty f = \lim_{b \rightarrow \infty} \int_0^b f$. The improper integrals are convergent only for certain functions. When using an improper integral we always consider the largest domain where the integrals are convergent.

In the theory of integral transformations and orthogonal systems spaces L_1 and L_2 defined as

$$L_1 = \left\{ x: I \rightarrow \mathbb{R} \mid x \text{ is measurable, } \int_I |x| < \infty \right\},$$

$$L_2 = \left\{ x: I \rightarrow \mathbb{R} \mid x \text{ is measurable, } \int_I |x|^2 < \infty \right\}$$

play central role.

2 Hilbert Spaces, Orthogonality

2.1 The Concept of Hilbert Spaces

Let X be a real or complex linear space. A function $\langle \cdot \rangle: X \times X \rightarrow \mathbb{C}$ is called *inner product* (or *scalar product*) if

$$\begin{aligned}\mathbb{R} \ni \langle x, x \rangle &\geq 0, \quad \langle x, x \rangle = 0 \iff x = 0 \\ \langle x, y \rangle &= \langle y, x \rangle^* \\ \langle \lambda \cdot x, y \rangle &= \lambda \cdot \langle x, y \rangle \\ \langle x + y, z \rangle &= \langle x, z \rangle + \langle y, z \rangle\end{aligned}$$

hold for all $x, y, z \in X$ and $\lambda \in \mathbb{C}$.

x^* denotes the complex conjugate of $x \in \mathbb{C}$.

Remark: $\langle x, \lambda \cdot y \rangle = \langle \lambda \cdot y, x \rangle^* = (\lambda \cdot \langle y, x \rangle)^* = \lambda^* \cdot \langle y, x \rangle^* = \lambda^* \cdot \langle x, y \rangle$

Remark: $\langle x, y + z \rangle = \langle y + z, x \rangle^* = \langle y, x \rangle^* + \langle z, x \rangle^* = \langle x, y \rangle + \langle x, z \rangle$

Remark: In the special case when the inner product is a real-valued function $\langle \cdot \rangle: X \times X \rightarrow \mathbb{R}$ the second property can be simply written as $\langle x, y \rangle = \langle y, x \rangle$.

The pair $(X, \langle \cdot \rangle)$ is called *inner product space*.

An inner product space $(X, \langle \cdot \rangle)$ is called *Hilbert space* if X is a Banach space with the norm function $\| \cdot \|: X \rightarrow \mathbb{R}$ defined as $\|x\| = \sqrt{\langle x, x \rangle}$.

Remark: Each Hilbert space $(X, \langle \cdot \rangle)$ is a normed space with the norm

$$\|x\| = \sqrt{\langle x, x \rangle}, \quad x \in X$$

and is also a metric space with the distance function

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y \rangle}, \quad x, y \in X.$$

The value of inner product characterizes the 'similarity' of elements in a Hilbert space. The higher the value of the inner product is, the more 'similar' the two elements are. (See similarity of functions in Section 2.3).

Remark: In the space of spatial vectors the inner product two vectors of given length is zero if they are orthogonal ('not similar'), and it reaches its maximum if they are parallel ('similar').

The Cauchy-Schwarz inequality

$$\langle x, y \rangle \leq \|x\| \cdot \|y\|, \quad x, y \in X$$

or in the equivalent form

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \cdot \langle y, y \rangle, \quad x, y \in X$$

is an important property of Hilbert spaces.

>>>Exercise 2.1 – 1

Let $(X, \langle \cdot \rangle)$ be a Hilbert space and let function $\| \cdot \|: X \rightarrow \mathbb{C}$ be defined as $\|x\| = \sqrt{\langle x, x \rangle}$. Show that $(X, \| \cdot \|)$ is a normed space.

>>>Exercise 2.1 – 1 – Solution

$$\|x\| = \sqrt{\langle x, x \rangle} \geq 0 \quad \text{for all } x \in X$$

$$\|x\| = \sqrt{\langle x, x \rangle} = 0 \iff x = 0$$

$$\|\lambda \cdot x\| = \sqrt{\langle \lambda \cdot x, \lambda \cdot x \rangle} = \sqrt{\lambda \cdot \lambda^*} \cdot \sqrt{\langle x, x \rangle} = |\lambda| \cdot \|x\| \quad \text{for all } \lambda \in \mathbb{C}, x \in X$$

To show the triangle inequality we have to use the Cauchy-Schwarz inequality $|\langle x, y \rangle|^2 \leq \langle x, x \rangle \cdot \langle y, y \rangle = \|x\| \cdot \|y\|$:

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle = \|x\|^2 + 2 \cdot \langle x, y \rangle + \|y\|^2 \leq \\ &\leq \|x\|^2 + 2 \cdot \|x\| \cdot \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2, \end{aligned}$$

that is, $\|x + y\| \leq \|x\| + \|y\|$, $x, y \in X$.

2.2 Finite Dimensional Hilbert spaces

A Hilbert space $(X, \langle \cdot, \cdot \rangle)$ is finite dimensional if X is a finite dimensional linear space.

Let n be a positive integer and suppose that X is an n -dimensional Hilbert space.

A system of vectors $\{b_1, \dots, b_k\} \subset X, k \in \mathbb{N}$ is *orthogonal* if its elements are pairwise orthogonal.

Remark: It is easy to see that each orthogonal system of non-zero vectors is linearly independent.

The system is *orthonormal* if orthogonal and normed, that is, the elements are unit vectors (the norm of each vector in the system is equal to 1).

If an orthogonal (orthonormal) system $\{b_1, \dots, b_n\} \subset X$ is a basis of X , it is called *orthogonal (orthonormal) basis* of X .

If $\{b_1, \dots, b_n\} \subset X$ is an orthonormal basis of X and $x \in X$ then

$$x = \sum_{i=1}^n \langle x, b_i \rangle \cdot b_i.$$

This sum is also called the *decomposition* $x \in X$ with respect to the orthonormal basis $\{b_1, \dots, b_n\}$. The coefficients

$$\langle x, b_i \rangle, \quad i = 1, \dots, n$$

are the *coordinates* of $x \in X$ with respect to the orthonormal basis $\{b_1, \dots, b_n\}$.

Remark: Coordinates of a vector $x \in X$ with respect to an orthonormal basis $\{b_1, \dots, b_n\}$ can be calculated as the inner product of x and the basis vectors b_i . We will use the generalization of this statement in function spaces when we are talking about the decomposition of functions with respect to an orthonormal system of functions.

The space of spatial (planar) vectors

The space of *spatial (planar) vectors* is a 3-dimensional (2-dimensional) Hilbert space with the inner product

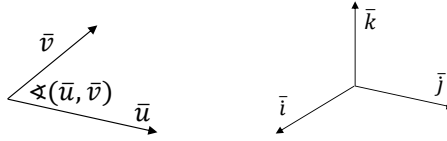
$$\langle \vec{u}, \vec{v} \rangle = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos \angle(\vec{u}, \vec{v})$$

Remark: The inner product of spatial (planar) vectors is generally called scalar product and denoted simply by $\vec{u} \cdot \vec{v}$ or $\vec{u}\vec{v}$.

Remark: Using the definition we have

$$\angle(\vec{u}, \vec{v}) = \arccos \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \cdot \|\vec{v}\|},$$

that is, the angle of spatial (planar) vectors can be calculated from the scalar product and the magnitude (norm) of the vectors.



Spatial (planar) vectors $\bar{u}$ and $\bar{v}$ are *perpendicular* in geometry iff $\bar{u} \cdot \bar{v} = 0$. Perpendicular vectors are also called *orthogonal*.

In applications (for instance in mechanics) the orthonormal basis is generally denoted by $\{\bar{i}, \bar{j}, \bar{k}\}$. Coordinates of a spatial vector $\bar{v}$ with respect to the orthonormal basis $\{\bar{i}, \bar{j}, \bar{k}\}$ are

$$v_x = \langle \bar{v}, \bar{i} \rangle, \quad v_y = \langle \bar{v}, \bar{j} \rangle, \quad v_z = \langle \bar{v}, \bar{k} \rangle$$

and the decomposition $\bar{v}$ with respect to the orthonormal basis $\{\bar{i}, \bar{j}, \bar{k}\}$ is

$$\bar{v} = v_x \cdot \bar{i} + v_y \cdot \bar{j} + v_z \cdot \bar{k} = \langle \bar{v}, \bar{i} \rangle \cdot \bar{i} + \langle \bar{v}, \bar{j} \rangle \cdot \bar{j} + \langle \bar{v}, \bar{k} \rangle \cdot \bar{k}$$

The space of real n -tuples

$\mathbb{R}^n$ is a n -dimensional Hilbert space with the inner product

$$\langle x, y \rangle = \sum_{i=1}^n x_i \cdot y_i, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n, y = (y_1, \dots, y_n) \in \mathbb{R}^n.$$

$x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ are called *orthogonal* if

$$\langle x, y \rangle = \sum_{i=1}^n x_i \cdot y_i = 0.$$

The “natural” orthonormal basis in $\mathbb{R}^n$ is $\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$ which is

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{R}^3$ and $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{R}^2$.

The space of complex n -tuples

$\mathbb{C}^n$ (which is an n -dimensional linear space over $\mathbb{C}$) is a Hilbert space with the inner product

$$\langle x, y \rangle = \sum_{i=1}^n x_i \cdot y_i^*, \quad x = (x_1, \dots, x_n) \in \mathbb{C}^n, y = (y_1, \dots, y_n) \in \mathbb{C}^n.$$

$x = (x_1, \dots, x_n) \in \mathbb{C}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{C}^n$ are called *orthogonal* if

$$\langle x, y \rangle = \sum_{i=1}^n x_i \cdot y_i^* = 0.$$

>>>Exercise 2.2 – 1

2.2 – 1 – a Show that

$$\langle x, y \rangle = \sum_{i=1}^n x_i \cdot y_i, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n, y = (y_1, \dots, y_n) \in \mathbb{R}^n.$$

is inner product in $\mathbb{R}^n$.

2.2 – 1 – b Give the norm function belonging to this inner product.

2.2 – 1 – c Write and prove the Cauchy-Schwarz inequality in $\mathbb{R}^n$.

>>>Exercise 2.2 – 1 – a- Solution

It is evident that $\langle x, x \rangle = \sum_{i=1}^n x_i^2 \geq 0, x = (x_1, \dots, x_n) \in \mathbb{R}^n$, furthermore $\langle x, x \rangle = \sum_{i=1}^n x_i^2 = 0 \Leftrightarrow x_1 = \dots = x_n = 0$.

Since $\langle x, y \rangle = \sum_{i=1}^n x_i \cdot y_i \in \mathbb{R}$ we have that $\langle x, y \rangle = \langle y, x \rangle^*$ is equivalent with $\langle x, y \rangle = \langle y, x \rangle$. It holds because $\sum_{i=1}^n x_i \cdot y_i = \sum_{i=1}^n y_i \cdot x_i$.

$$\langle \lambda \cdot x, y \rangle = \sum_{i=1}^n (\lambda \cdot x_i) \cdot y_i = \lambda \cdot \sum_{i=1}^n x_i \cdot y_i = \lambda \cdot \langle x, y \rangle$$

$$\langle x + y, z \rangle = \sum_{i=1}^n (x_i + y_i) \cdot z_i = \sum_{i=1}^n x_i \cdot z_i + \sum_{i=1}^n y_i \cdot z_i = \langle x, z \rangle + \langle y, z \rangle$$

>>>Exercise 2.2 – 1 – b- Solution

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^n x_i^2}, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n$$

>>>Exercise 2.2 – 1 – c- Solution

The Cauchy-Schwarz inequality in $\mathbb{R}^n$ is

$$\left(\sum_{i=1}^n x_i \cdot y_i \right)^2 \leq \sum_{i=1}^n x_i^2 \cdot \sum_{i=1}^n y_i^2$$

To see it let us consider the following calculation

$$\begin{aligned} 0 &\leq \sum_{i=1}^n \sum_{j=1}^n (x_i \cdot y_j - x_j \cdot y_i)^2 = \sum_{i=1}^n x_i^2 \cdot \sum_{j=1}^n y_j^2 + \sum_{i=1}^n y_i^2 \cdot \sum_{j=1}^n x_j^2 - 2 \cdot \sum_{i=1}^n x_i \cdot y_i \cdot \sum_{j=1}^n y_j \cdot x_j = \\ &= 2 \cdot \sum_{i=1}^n x_i^2 \cdot \sum_{i=1}^n y_i^2 - 2 \cdot \left(\sum_{i=1}^n x_i \cdot y_i \right)^2 \end{aligned}$$

>>>Exercise 2.2 – 2

2.2 – 2 – a Show that

$$\langle x, y \rangle = \sum_{i=1}^n x_i \cdot y_i^*, \quad x = (x_1, \dots, x_n) \in \mathbb{C}^n, y = (y_1, \dots, y_n) \in \mathbb{C}^n.$$

is inner product in $\mathbb{C}^n$.

2.2 – 2 – b Give the norm function belonging to this inner product.

>>>Exercise 2.2 – 2 – a- Solution

Since $x \cdot x^* = |x|^2 \geq 0, x \in \mathbb{C}$ it is evident that $\langle x, x \rangle = \sum_{i=1}^n x_i \cdot x_i^* = \sum_{i=1}^n |x_i|^2 \geq 0, x = (x_1, \dots, x_n) \in \mathbb{C}^n$, furthermore $\langle x, x \rangle = \sum_{i=1}^n |x_i|^2 = 0 \Leftrightarrow x_1 = \dots = x_n = 0 + 0 \cdot i \Leftrightarrow x = 0 + 0 \cdot i$.

$$\langle x, y \rangle = \sum_{i=1}^n x_i \cdot y_i^* = \sum_{i=1}^n y_i^* \cdot x_i = \sum_{i=1}^n (y_i \cdot x_i^*)^* = \langle y, x \rangle^*$$

$$\langle \lambda \cdot x, y \rangle = \sum_{i=1}^n (\lambda \cdot x_i) \cdot y_i^* = \lambda \cdot \sum_{i=1}^n x_i \cdot y_i^* = \lambda \cdot \langle x, y \rangle$$

Remark: $\langle x, \lambda \cdot y \rangle = \sum_{i=1}^n x_i \cdot (\lambda \cdot y_i)^* = \lambda^* \cdot \sum_{i=1}^n x_i \cdot y_i^* = \lambda^* \cdot \langle x, y \rangle$

$$\langle x + y, z \rangle = \sum_{i=1}^n (x_i + y_i) \cdot z_i^* = \sum_{i=1}^n x_i \cdot z_i^* + \sum_{i=1}^n y_i \cdot z_i^* = \langle x, z \rangle + \langle y, z \rangle$$

Remark:

$$\langle x, y + z \rangle = \sum_{i=1}^n x_i \cdot (y_i + z_i)^* = \sum_{i=1}^n x_i \cdot (y_i^* + z_i^*) = \sum_{i=1}^n x_i \cdot y_i^* + \sum_{i=1}^n x_i \cdot z_i^* = \langle x, y \rangle + \langle x, z \rangle$$

>>>Exercise 2.2 – 2 – b- Solution

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^n x_i \cdot x_i^*} = \sqrt{\sum_{i=1}^n |x_i|^2}, \quad x = (x_1, \dots, x_n) \in \mathbb{C}^n$$

>>>Exercise 2.2 – 3

Calculate the inner product of the given vectors.

2.2. – 3 – a $a_1 = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$

2.2. – 3 – b $r_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

2.2. – 3 – c $m = \begin{pmatrix} 2 \\ 1 \\ 4 \\ -2 \end{pmatrix}, \quad n = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 4 \end{pmatrix}$

>>>Exercise 2.2 – 3- Solution

a, $\langle a_1, a_2 \rangle = -2 \cdot 1 + 3 \cdot (-4) + 2 \cdot 5 = -4$

b, $\langle r_1, r_2 \rangle = 1 \cdot 3 + (-2) \cdot (-4) = 11$

c, $\langle m, n \rangle = 2 \cdot 1 + 1 \cdot 0 + 4 \cdot 3 + (-2) \cdot 4 = 6$

>>>Exercise 2.2 – 4

Find real number x if vectors $p = \begin{pmatrix} 2 \\ 1 \\ 5 \\ x \end{pmatrix}$ and $q = \begin{pmatrix} -1 \\ x \\ 0 \\ 3 \end{pmatrix}$ are known to be orthogonal.

>>>Exercise 2.2 – 4- Solution

$$0 = p \cdot q = 2 \cdot (-1) + 1 \cdot x + 5 \cdot 0 + x \cdot 3 = 4x - 2 \quad \Rightarrow \quad x = 0.5$$

>>>Exercise 2.2 – 5

Check whether the given systems are orthogonal or orthonormal.

2.2. – 5 – a $\left\{ a_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, a_2 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}, a_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\}$

2.2. – 5 – b $\left\{ u = \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}, v = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, w = \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix} \right\}$

>>>Exercise 2.2 – 5- Solution

a,

$$\|a_1\| = 3$$

$$\|a_2\| = 3$$

$$\|a_3\| = 3$$

$$\langle a_1, a_2 \rangle = 0$$

$$\langle a_1, a_3 \rangle = 0$$

$$\langle a_2, a_3 \rangle = 0$$

System $\{a_1, a_2, a_3\}$ is orthogonal but not orthonormal.

b,

$$\begin{aligned}\|u\| &= 1 \\ \langle u, v \rangle &= 0\end{aligned}$$

$$\begin{aligned}\|v\| &= 1 \\ \langle u, w \rangle &= 0\end{aligned}$$

$$\begin{aligned}\|w\| &= 1 \\ \langle v, w \rangle &= 0\end{aligned}$$

System $\{u, v, w\}$ is orthonormal.

>>> **Exercise 2.2 – 6**

Give the decomposition of vector $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ with respect to the orthonormal system

$$\left\{ a_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}, a_2 = \begin{pmatrix} -2/3 \\ 2/3 \\ 1/3 \end{pmatrix}, a_3 = \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix} \right\}$$

>>> **Exercise 2.2 – 6- Solution**

$$v_1 = \langle v, a_1 \rangle = -\frac{1}{3}$$

$$v_2 = \langle v, a_2 \rangle = \frac{5}{3}$$

$$v_3 = \langle v, a_3 \rangle = \frac{10}{3}$$

$$v = -\frac{1}{3} \cdot a_1 + \frac{5}{3} \cdot a_2 + \frac{10}{3} \cdot a_3$$

>>> **Exercise 2.2 – 7**

Let

$$x = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}, \quad y = \begin{pmatrix} -5 \\ 1 \\ -1 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix}, \quad b_3 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

Give $\langle x, y \rangle$, $\|x\|$, $\|y\|$, and the angle of x and y .

Is the system $\{b_1, b_2, b_3\}$ orthonormal?

If yes, give the decomposition of x with respect to system $\{b_1, b_2, b_3\}$.

>>> **Exercise 2.2 – 7- Solution**

$$\langle x, y \rangle = 2 \cdot (-5) + 3 \cdot 1 + (-4) \cdot (-1) = -3$$

$$\|x\| = \sqrt{2^2 + 3^2 + (-4)^2} = \sqrt{29}$$

$$\|y\| = \sqrt{(-5)^2 + 1^2 + (-1)^2} = \sqrt{27}$$

For the angle φ between x and y we have

$$\varphi = \arccos \frac{\langle x, y \rangle}{\|x\| \cdot \|y\|} = \arccos \frac{-3}{\sqrt{29} \cdot \sqrt{27}} = \arccos(-0.107) = 1.678 \quad (= 96,15^\circ)$$

System $\{b_1, b_2, b_3\}$ is orthonormal, since on the one hand

$$\|b_1\| = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 1, \quad \|b_2\| = \sqrt{\frac{1}{6} + \frac{1}{6} + \frac{4}{6}} = 1, \quad \|b_3\| = \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = 1,$$

on the other hand

$$\langle b_1, b_2 \rangle = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}} \cdot \left(-\frac{2}{\sqrt{6}}\right) = 0.$$

$$\langle b_1, b_3 \rangle = \frac{1}{\sqrt{3}} \cdot \left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \cdot 0 = 0.$$

$$\langle b_2, b_3 \rangle = \frac{1}{\sqrt{6}} \cdot \left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{2}} + \left(-\frac{2}{\sqrt{6}}\right) \cdot 0 = 0.$$

that is, vectors b_i are pairwise orthogonal.

The inner products of x and the elements of the system are

$$\langle x, b_1 \rangle = \frac{1}{\sqrt{3}}, \quad \langle x, b_2 \rangle = \frac{13}{\sqrt{6}}, \quad \langle x, b_3 \rangle = \frac{1}{\sqrt{2}}$$

so the decomposition of x with respect to the orthonormal system $\{b_1, b_2, b_3\}$ is

$$x = \frac{1}{\sqrt{3}} \cdot b_1 + \frac{13}{\sqrt{6}} \cdot b_2 + \frac{1}{\sqrt{2}} \cdot b_3$$

>>>Exercise 2.2 – 8

Check whether the system $\{\gamma_1, \gamma_2, \gamma_3\}$ is orthonormal. Give the decomposition of ρ with respect to the system $\{\gamma_1, \gamma_2, \gamma_3\}$.

2.2. – 8 – a

$$\rho = \begin{pmatrix} 11 \\ 2 \\ -7 \end{pmatrix} \quad \left\{ \gamma_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

2.2. – 8 – b

$$\rho = \begin{pmatrix} 11 \\ 2 \\ -7 \end{pmatrix} \quad \left\{ \gamma_1 = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} \right\}$$

2.2. – 8 – c

$$\rho = \begin{pmatrix} -22 \\ 7 \\ -9 \end{pmatrix} \quad \left\{ \gamma_1 = \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 1 \\ 16 \\ 10 \end{pmatrix} \right\}$$

2.2. – 8 – d

$$\rho = \begin{pmatrix} 13 \\ 4 \\ -23 \end{pmatrix} \quad \left\{ \gamma_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} \right\}$$

>>>Exercise 2.2 – 8 – a- Solution

System $\{\gamma_1, \gamma_2, \gamma_3\}$ is orthonormal, this is the natural basis in $\mathbb{R}^3$. In this basis the coordinates are equal to the components of the vector:

$$\begin{pmatrix} 11 \\ 2 \\ -7 \end{pmatrix} = 11 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 7 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

>>>Exercise 2.2 – 8 – b- Solution

$$\|\gamma_1\| = 3, \quad \|\gamma_2\| = 4, \quad \|\gamma_3\| = 6$$

$$\langle \gamma_1, \gamma_2 \rangle = 0, \quad \langle \gamma_2, \gamma_3 \rangle = 0, \quad \langle \gamma_1, \gamma_3 \rangle = 0$$

System $\{\gamma_1, \gamma_2, \gamma_3\}$ is orthogonal but not orthonormal. So we have to solve the system of equations

$$\alpha_1 \cdot \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 11 \\ 2 \\ -7 \end{pmatrix}$$

An easy calculation gives the solution: $\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \begin{pmatrix} 11/3 \\ -2/4 \\ -7/6 \end{pmatrix}$, thus

$$\begin{pmatrix} 11 \\ 2 \\ -7 \end{pmatrix} = \frac{11}{3} \cdot \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} - \frac{2}{4} \cdot \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix} - \frac{7}{6} \cdot \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$$

>>>Exercise 2.2 – 8 – c- Solution

$$\|\gamma_1\| = \sqrt{21}, \quad \|\gamma_2\| = \sqrt{17}, \quad \|\gamma_3\| = \sqrt{357}$$

$$\langle \gamma_1, \gamma_2 \rangle = 0, \quad \langle \gamma_2, \gamma_3 \rangle = 0, \quad \langle \gamma_1, \gamma_3 \rangle = 0$$

System $\{\gamma_1, \gamma_2, \gamma_3\}$ is orthogonal but not orthonormal. So we have to solve the system of equations

$$\alpha_1 \cdot \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} 1 \\ 16 \\ 10 \end{pmatrix} = \begin{pmatrix} -22 \\ 7 \\ -9 \end{pmatrix}$$

Since $\det \begin{pmatrix} -4 & 2 & 1 \\ -1 & -2 & 16 \\ 2 & 3 & 10 \end{pmatrix} = 357 \neq 0$ the system can be solved e.g. with Cramer's rule:

$$\alpha_1 = \frac{\det \begin{pmatrix} -22 & 2 & 1 \\ 7 & -2 & 16 \\ -9 & 3 & 10 \end{pmatrix}}{\det \begin{pmatrix} -4 & 2 & 1 \\ -1 & -2 & 16 \\ 2 & 3 & 10 \end{pmatrix}} = 3, \alpha_2 = \frac{\det \begin{pmatrix} -4 & -22 & 1 \\ -1 & 7 & 16 \\ 2 & -9 & 10 \end{pmatrix}}{\det \begin{pmatrix} -4 & 2 & 1 \\ -1 & -2 & 16 \\ 2 & 3 & 10 \end{pmatrix}} = -5, \alpha_3 = \frac{\det \begin{pmatrix} -4 & 2 & -22 \\ -1 & -2 & 7 \\ 2 & 3 & -9 \end{pmatrix}}{\det \begin{pmatrix} -4 & 2 & 1 \\ -1 & -2 & 16 \\ 2 & 3 & 10 \end{pmatrix}} = 0$$

$$\text{Thus } \begin{pmatrix} -22 \\ 7 \\ -9 \end{pmatrix} = 3 \cdot \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} + 2 \cdot \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ 16 \\ 10 \end{pmatrix}$$

>>>Exercise 2.2 – 8 – d- Solution

$$\|\gamma_1\| = \sqrt{3}, \|\gamma_2\| = \sqrt{6}, \|\gamma_3\| = \sqrt{18}$$

$$\langle \gamma_1, \gamma_2 \rangle = 0, \langle \gamma_2, \gamma_3 \rangle = 0, \langle \gamma_1, \gamma_3 \rangle = 0$$

System $\{\gamma_1, \gamma_2, \gamma_3\}$ is orthogonal but not orthonormal. So we have to solve the system of equations

$$\alpha_1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 13 \\ 4 \\ -23 \end{pmatrix}$$

Since $\det \begin{pmatrix} 1 & 1 & 3 \\ 1 & -2 & 0 \\ 1 & 1 & -3 \end{pmatrix} = 18$ the system can be solved e.g. with Cramer's rule:

$$\alpha_1 = \frac{\det \begin{pmatrix} 13 & 1 & 3 \\ 4 & -2 & 0 \\ -23 & 1 & -3 \end{pmatrix}}{\det \begin{pmatrix} 1 & 1 & 3 \\ 1 & -2 & 0 \\ 1 & 1 & -3 \end{pmatrix}} = -2, \alpha_2 = \frac{\det \begin{pmatrix} 1 & 13 & 3 \\ 1 & 4 & 0 \\ 1 & -23 & -3 \end{pmatrix}}{\det \begin{pmatrix} 1 & 1 & 3 \\ 1 & -2 & 0 \\ 1 & 1 & -3 \end{pmatrix}} = -3, \alpha_3 = \frac{\det \begin{pmatrix} 1 & 1 & 13 \\ 1 & -2 & 4 \\ 1 & 1 & -23 \end{pmatrix}}{\det \begin{pmatrix} 1 & 1 & 3 \\ 1 & -2 & 0 \\ 1 & 1 & -3 \end{pmatrix}} = 6$$

$$\text{Thus } \begin{pmatrix} 13 \\ 4 \\ -23 \end{pmatrix} = -2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 3 \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + 6 \cdot \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}$$

>>>Exercise 2.2 – 9

Calculate the angle of the given vectors

$$\mathbf{2.2 - 9 - a} \quad a_1 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{2.2 - 9 - b} \quad x = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix}, \quad y = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}$$

>>>Exercise 2.2 – 9- Solution

a,

$$\varphi = \arccos \frac{\langle a_1, a_2 \rangle}{\|a_1\| \cdot \|a_2\|} = \arccos \frac{1 \cdot 2 - 2 \cdot 1 + 4 \cdot 1}{\sqrt{21} \cdot \sqrt{6}} = \arccos 0.356 = 0.976 \text{ (69.1°)}$$

b,

$$\varphi = \arccos \frac{\langle x, y \rangle}{\|x\| \cdot \|y\|} = \arccos \frac{0 \cdot 4 + 3 \cdot 2 + (-4) \cdot 4}{\sqrt{25} \cdot \sqrt{36}} = \arccos \left(-\frac{1}{3}\right) = 1.91 \text{ (109.5°)}$$

>>>Exercise 2.2 – 10

Find real number z if the angle of vectors $a_1 = \begin{pmatrix} 3 \\ z \\ 1 \end{pmatrix}$ and $a_2 = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix}$ is known to be 60° .

>>>Exercise 2.2 – 10- Solution

$$\begin{aligned}\frac{1}{2} = \cos 60^\circ &= \frac{\langle a_1, a_2 \rangle}{\|a_1\| \cdot \|a_2\|} = \frac{3 \cdot (-4) + z \cdot 3 + 1 \cdot 0}{\sqrt{z^2 + 10} \cdot \sqrt{25}} = \frac{1}{5} \cdot \frac{3z - 12}{\sqrt{z^2 + 10}} \Leftrightarrow \\ 5 \cdot \sqrt{z^2 + 10} &= 6z - 24 \Leftrightarrow 25z^2 + 250 = 36z^2 - 288z + 576 \Leftrightarrow \\ 11z^2 - 288z + 326 &= 0 \Leftrightarrow z = \frac{288 \pm \sqrt{68600}}{22} \Leftrightarrow z = 13.04 \text{ or } z = 274.96\end{aligned}$$

2.3 Orthogonality and Similarity of Functions

Let I be an interval. A function $x: I \rightarrow \mathbb{C}$ is *square integrable* if

$$\int_I |x(t)|^2 < \infty.$$

$|\cdot|$ denotes the magnitude (norm) of a complex number. The space of the square integrable functions defined on I is denoted by $L_2(I)$.

Remark: A real valued function $x: I \rightarrow \mathbb{R}$ is square integrable if $\int_I x^2 < \infty$.

Remark: Piecewise continuous functions are square integrable.

Remark: Square integrable functions are mathematical representations of finite energy signals.

The *inner product* of functions $x \in L_2(I)$ and $\psi \in L_2(I)$ is

$$\langle x, \psi \rangle = \int_I (x \cdot \psi^*) = \int_I x(t) \cdot \psi^*(t) dt.$$

Remark: If $x \in L_2(I)$ and $\psi \in L_2(I)$ are real-valued functions then $\psi^* = \psi$, and we can write

$$\langle x, \psi \rangle = \int_I (x \cdot \psi) = \int_I x(t) \cdot \psi(t) dt.$$

Remark: In the definition of the inner product in $L_2(I)$ the so-called Lebesgue integral is used, which is more general than the Riemann integral and more suitable for the general Fourier theory. If a function is Riemann integrable then it is also Lebesgue integrable, and the two integrals are equal. In engineering models, principally, the so-called piecewise continuous functions appear which are Riemann integrable. In our examples we have piecewise continuous functions, thus we have to calculate Riemann integrals.

Remark: In the definition the inner product the Lebesgue integrability of functions is supposed. If we want to introduce the concept of inner product $\langle x, \psi \rangle = \int_I (x \cdot \psi^*)$ in the class of Riemann integrable functions we also have to suppose the Riemann integrability of $x \cdot \psi^*$.

The *norm* of function $x \in L_2(I)$ is

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\int_I |x|^2}.$$

Remark: If $x \in L_2(I)$ is a real-valued function we can write $\|x\| = \sqrt{\int_I x^2}$.

Remark: In technical textbooks $\int_I |x|^2$ is frequently referred as the “energy content” of signal $x \in L_2(I)$.

Functions $x \in L_2(I)$ and $\psi \in L_2(I)$ are *orthogonal* if

$$\langle x, \psi \rangle = \int_I (x \cdot \psi^*) = 0.$$

Remark: The real-valued functions $x \in L_2(I)$ and $\psi \in L_2(I)$ are orthogonal if $\langle x, \psi \rangle = \int_I (x \cdot \psi) = 0$.

Remark: The value of the inner product characterizes the “similarity” of the functions.

>>>Exercise 2.3 – 1

Consider the following functions in $L_2([0, 2\pi])$:

$$\begin{aligned} x_1(t) &= \sin t, & x_2(t) &= \cos t, \\ x_3(t) &= \sin 2t, & x_4(t) &= \cos 2t, \end{aligned} \quad t \in [0, 2\pi]$$

Check the orthogonality of each pair of these functions and calculate their norm.

>>>Exercise 2.3 – 1– Solution

$\langle x_i, x_j \rangle = 0$ if $i \neq j$, that is, functions x_i are pairwise orthogonal. For example, the calculation of $\langle x_1, x_4 \rangle$ and $\langle x_3, x_4 \rangle$ is as follows

$$\langle x_1, x_4 \rangle = \int_0^{2\pi} \sin t \cdot \cos 2t \, dt = \left[-\frac{2}{3} \cdot \cos^3 t - \cos t \right]_0^{2\pi} = 0$$

Details of the integration:

$$\begin{aligned} \int \sin t \cdot \cos 2t \, dt &= \int \sin t \cdot (\cos^2 t - \sin^2 t) \, dt = \int \sin t \cdot (2\cos^2 t - 1) \, dt = \\ &= -2 \int -\sin t \cdot \cos^2 t \, dt - \int \sin t \, dt = -\frac{2}{3} \cdot \cos^3 t - \cos t \end{aligned}$$

$$\langle x_3, x_4 \rangle = \int_0^{2\pi} \sin 2t \cdot \cos 2t \, dt = \left[-\frac{1}{8} \cdot \cos 4t \right]_0^{2\pi} = 0$$

Details of the integration:

$$\int \sin 2t \cdot \cos 2t \, dt = \frac{1}{2} \int \sin 4t \, dt = -\frac{1}{8} \cdot \cos 4t$$

Calculation of square norms $\|x_i\|^2$, $i = 1, 2, 3, 4$ is as follows

$$\|x_1\|^2 = \int_0^{2\pi} \sin^2 t \, dt = \frac{1}{2} \cdot \int_0^{2\pi} (1 - \cos 2t) \, dt = \frac{1}{2} \cdot \left[t - \frac{1}{2} \cdot \sin 2t \right]_0^{2\pi} = \pi$$

$$\|x_2\|^2 = \int_0^{2\pi} \cos^2 t \, dt = \frac{1}{2} \cdot \int_0^{2\pi} (1 + \cos 2t) \, dt = \frac{1}{2} \cdot \left[t + \frac{1}{2} \cdot \sin 2t \right]_0^{2\pi} = \pi$$

$$\|x_3\|^2 = \int_0^{2\pi} \sin^2 2t \, dt = \frac{1}{2} \cdot \int_0^{2\pi} (1 - \cos 4t) \, dt = \frac{1}{2} \cdot \left[t - \frac{1}{4} \cdot \sin 4t \right]_0^{2\pi} = \pi$$

$$\|x_4\|^2 = \int_0^{2\pi} \cos^2 2t \, dt = \frac{1}{2} \cdot \int_0^{2\pi} (1 + \cos 4t) \, dt = \frac{1}{2} \cdot \left[t + \frac{1}{4} \cdot \sin 4t \right]_0^{2\pi} = \pi$$

That is, $\|x_i\| = \sqrt{\pi}$, $i = 1, 2, 3, 4$.

>>>Exercise 2.3 – 2

Consider the following functions in $L_2([0, \pi])$:

$$\psi(t) = \sin 2t, t \in [0, \pi]$$

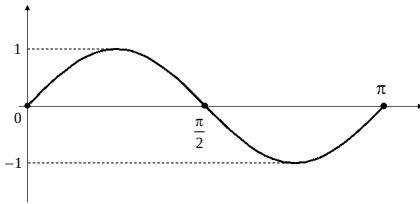
$$x_1(t) = \begin{cases} 1 & \text{if } t \in [0, \frac{\pi}{2}] \\ -1 & \text{if } t \in [\frac{\pi}{2}, \pi] \end{cases}$$

$$x_2(t) = \begin{cases} 1 & \text{if } t \in [0, \frac{\pi}{4}] \text{ or } t \in [\frac{\pi}{2}, \frac{3\pi}{4}] \\ -1 & \text{if } t \in [\frac{\pi}{4}, \frac{\pi}{2}] \text{ or } t \in [\frac{3\pi}{4}, \pi] \end{cases}$$

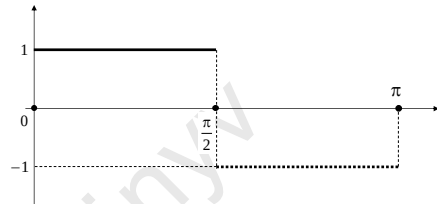
$$x_3(t) = \begin{cases} 1 & \text{if } t \in [0, \frac{\pi}{3}] \text{ or } t \in [\frac{2\pi}{3}, \pi] \\ -1 & \text{if } t \in [\frac{\pi}{3}, \frac{2\pi}{3}] \end{cases}$$

Calculate the inner product of ψ with x_1, x_2 and x_3 and compare the similarity of ψ with the three functions.

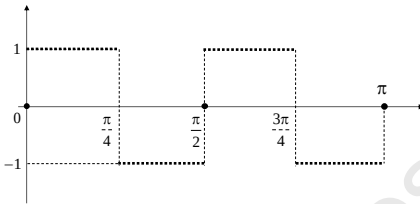
ψ



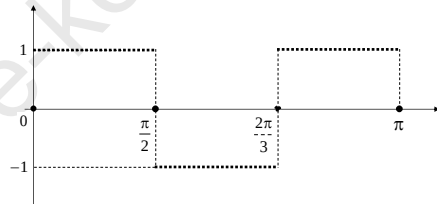
x_1



x_2



x_3



>>>Exercise 2.3 – 2- Solution

The inner product of ψ with x_1, x_2 and x_3 are

$$\langle x_1, \psi \rangle = \int_0^{\pi} x_1(t) \cdot \psi(t) dt = \int_0^{\frac{\pi}{2}} \sin 2t dt - \int_{\frac{\pi}{2}}^{\pi} \sin 2t dt = -\frac{1}{2} \cdot [\cos 2t]_0^{\frac{\pi}{2}} + \frac{1}{2} \cdot [\cos 2t]_{\frac{\pi}{2}}^{\pi} = 2$$

$$\langle x_2, \psi \rangle = \int_0^{\pi} x_2(t) \cdot \psi(t) dt = \int_0^{\frac{\pi}{4}} \sin 2t dt - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin 2t dt + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \sin 2t dt - \int_{\frac{3\pi}{4}}^{\pi} \sin 2t dt = 0$$

$$\langle x_3, \psi \rangle = \int_0^{\pi} x_3(t) \cdot \psi(t) dt = \int_0^{\frac{\pi}{3}} \sin 2t dt - \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin 2t dt + \int_{\frac{2\pi}{3}}^{\pi} \sin 2t dt = 1$$

That is, $0 = \langle x_2, \psi \rangle < \langle x_3, \psi \rangle < \langle x_1, \psi \rangle$.

This result implies that the similarity is the highest between x_1 and ψ , while x_2 and ψ are not similar (actually, they are orthogonal).

>>>Exercise 2.3 – 3

Calculate the inner product of the given pairs of functions

2.3. – 3 – a $f(t) = 200$ $g(t) = \cos(0.001 \cdot t)$ $t \in [0, 1]$

$$\mathbf{2.3.-3-b} \quad f(t) = e^{10 \cdot t} \quad g(t) = \sin(100\pi \cdot t) \quad t \in [0,1]$$

$$\mathbf{2.3.-3-c} \quad f(t) = (15 + i) \cdot t \quad g(t) = e^{4\pi \cdot i \cdot t} \quad t \in [0,1]$$

>>> **Exercise 2.3 – 3 – a- Solution**

$$\langle f, g \rangle = \int_0^1 200 \cdot \cos(0.001 \cdot t) dt = 200000 \cdot [\sin(0.001 \cdot t)]_0^1 = 200000 \cdot \sin(0.001) \approx 100$$

>>> **Exercise 2.3 – 3 – b- Solution**

$$\begin{aligned} \langle f, g \rangle &= \int_0^1 e^{10 \cdot t} \cdot \sin(100\pi \cdot t) dt = \\ &= \frac{1}{1 + 100\pi^2} \cdot \left[\frac{1}{10} \cdot e^{10 \cdot t} \cdot \sin(100\pi \cdot t) - \pi \cdot e^{10 \cdot t} \cdot \cos(100\pi \cdot t) \right]_0^1 = \frac{\pi \cdot (1 - e^{10})}{1 + 100\pi^2} \end{aligned}$$

Details of the integration:

$$\begin{aligned} \int e^{10 \cdot t} \cdot \sin(100\pi \cdot t) dt &= \frac{1}{10} \cdot e^{10 \cdot t} \cdot \sin(100\pi \cdot t) - 10\pi \cdot \int e^{10 \cdot t} \cdot \cos(100\pi \cdot t) dt = \\ &\quad \left[\begin{array}{ll} g(t) = \sin(100\pi \cdot t) & \Rightarrow \quad g'(t) = 100\pi \cdot \cos(100\pi \cdot t) \\ f'(t) = e^{10 \cdot t} & \Rightarrow \quad f(t) = \frac{1}{10} \cdot e^{10 \cdot t} \end{array} \right] \\ &\quad \left[\begin{array}{ll} g(t) = \cos(100\pi \cdot t) & \Rightarrow \quad g'(t) = -100\pi \cdot \sin(100\pi \cdot t) \\ f'(t) = e^{10 \cdot t} & \Rightarrow \quad f(t) = \frac{1}{10} \cdot e^{10 \cdot t} \end{array} \right] \\ &= \frac{1}{10} \cdot e^{10 \cdot t} \cdot \sin(100\pi \cdot t) - 10\pi \cdot \left(\frac{1}{10} \cdot e^{10 \cdot t} \cdot \cos(100\pi \cdot t) + 10\pi \cdot \int e^{10 \cdot t} \cdot \sin(100\pi \cdot t) dt \right) = \\ &= \frac{1}{10} \cdot e^{10 \cdot t} \cdot \sin(100\pi \cdot t) - \pi \cdot e^{10 \cdot t} \cdot \cos(100\pi \cdot t) - 100\pi^2 \cdot \int e^{10 \cdot t} \cdot \sin(100\pi \cdot t) dt \\ &\quad \int e^{10 \cdot t} \cdot \sin(100\pi \cdot t) dt = \\ &= \frac{1}{10} \cdot e^{10 \cdot t} \cdot \sin(100\pi \cdot t) - \pi \cdot e^{10 \cdot t} \cdot \cos(100\pi \cdot t) - 100\pi^2 \cdot \int e^{10 \cdot t} \cdot \sin(100\pi \cdot t) dt \\ &(1 + 100\pi^2) \cdot \int e^{10 \cdot t} \cdot \sin(100\pi \cdot t) dt = \frac{1}{10} \cdot e^{10 \cdot t} \cdot \sin(100\pi \cdot t) - \pi \cdot e^{10 \cdot t} \cdot \cos(100\pi \cdot t) \\ &\int e^{10 \cdot t} \cdot \sin(100\pi \cdot t) dt = \frac{1}{1 + 100\pi^2} \cdot \left(\frac{1}{10} \cdot e^{10 \cdot t} \cdot \sin(100\pi \cdot t) - \pi \cdot e^{10 \cdot t} \cdot \cos(100\pi \cdot t) \right) \end{aligned}$$

>>> **Exercise 2.3 – 3 – c- Solution**

$$\begin{aligned} \langle f, g \rangle &= \int_0^1 (15 + i) \cdot t \cdot e^{4\pi \cdot i \cdot t} dt = \left[\left(\frac{1 - 15 \cdot i}{4\pi} \cdot t + \frac{15 + i}{16\pi^2} \right) \cdot e^{4\pi \cdot i \cdot t} \right]_0^1 = \\ &= \left(\frac{1 - 15 \cdot i}{4\pi} + \frac{15 + i}{16\pi^2} \right) \cdot e^{4\pi \cdot i} - \frac{15 + i}{16\pi^2} = \frac{1 - 15 \cdot i}{4\pi} \end{aligned}$$

Details of the integration:

$$\begin{aligned} \int (15 + i) \cdot t \cdot e^{4\pi \cdot i \cdot t} dt &= (15 + i) \cdot \int t \cdot e^{4\pi \cdot i \cdot t} dt = (15 + i) \cdot \left(\frac{1}{4\pi \cdot i} \cdot t \cdot e^{4\pi \cdot i \cdot t} - \frac{1}{4\pi \cdot i} \cdot \int e^{4\pi \cdot i \cdot t} dt \right) = \\ &\quad \left[\begin{array}{ll} g(t) = t & \Rightarrow \quad g'(t) = 1 \\ f'(t) = e^{4\pi \cdot i \cdot t} & \Rightarrow \quad f(t) = \frac{1}{4\pi \cdot i} \cdot e^{4\pi \cdot i \cdot t} \end{array} \right] \\ &= (15 + i) \cdot \left(\frac{1}{4\pi \cdot i} \cdot t \cdot e^{4\pi \cdot i \cdot t} - \frac{1}{(4\pi \cdot i)^2} \cdot e^{4\pi \cdot i \cdot t} \right) = \left(\frac{1 - 15 \cdot i}{4\pi} \cdot t + \frac{15 + i}{16\pi^2} \right) \cdot e^{4\pi \cdot i \cdot t} \end{aligned}$$

>>>Exercise 2.3 – 4

Show that functions

$$x_1(t) = \sin\left(\frac{6\pi}{T} \cdot t\right) \quad \text{and} \quad x_2(t) = \cos\left(\frac{6\pi}{T} \cdot t\right)$$

are orthogonal in $L_2([0, T])$ space. Give the norm of x_2 .

>>>Exercise 2.3 – 4– Solution

$$\langle x_1, x_2 \rangle = \int_0^T \left(\sin\left(\frac{6\pi}{T} \cdot t\right) \cdot \cos\left(\frac{6\pi}{T} \cdot t\right) \right) dt = \frac{1}{2} \cdot \int_0^T \sin\left(\frac{12\pi}{T} \cdot t\right) dt$$

$$= -\frac{1}{2} \cdot \frac{T}{12\pi} \cdot \left[\cos\left(\frac{12\pi}{T} \cdot t\right) \right]_0^T = -\frac{T}{24\pi} \cdot (1 - 1) = 0$$

$$\|x_2\|^2 = \int_0^T \cos^2\left(\frac{6\pi}{T} \cdot t\right) dt = \frac{1}{2} \cdot \int_0^T 1 + \cos\left(\frac{12\pi}{T} \cdot t\right) dt = \frac{1}{2} \cdot \left[t + \frac{T}{12\pi} \cdot \sin\left(\frac{12\pi}{T} \cdot t\right) \right]_0^T = \frac{T}{2}$$

$$\|x_2\| = \sqrt{\frac{T}{2}}$$

>>>Exercise 2.3 – 5– Solution

Show that functions

$$x_1(t) = \frac{1}{\sqrt{T}} \cdot e^{i\frac{6\pi}{T}t} \quad \text{and} \quad x_2(t) = \frac{1}{\sqrt{T}} \cdot e^{i\frac{10\pi}{T}t}$$

are orthogonal in $L_2([0, T])$ space. Give the norm of x_2 .

>>>Exercise 2.3 – 5– Solution

$$\langle x_1, x_2 \rangle = \int_0^T \left(\frac{1}{\sqrt{T}} \cdot e^{i\frac{6\pi}{T}t} \cdot \frac{1}{\sqrt{T}} \cdot e^{-i\frac{10\pi}{T}t} \right) dt = \frac{1}{T} \cdot \int_0^T e^{i\frac{-4\pi}{T}t} dt =$$

$$= \frac{1}{T} \cdot \frac{1}{i \cdot \frac{-4\pi}{T}} \cdot \left[e^{i\frac{-4\pi}{T}t} \right]_0^T = \frac{-1}{4\pi \cdot i} \cdot (e^{-4\pi \cdot i} - 1) = 0$$

$$\|x_2\|^2 = \int_0^T \left(\frac{1}{\sqrt{T}} \cdot e^{i\frac{10\pi}{T}t} \cdot \frac{1}{\sqrt{T}} \cdot e^{-i\frac{10\pi}{T}t} \right) dt = \int_0^T \frac{1}{T} dt = 1$$

3 Fourier Series

3.1 Orthonormal Systems

Let I be an interval. A sequence of functions $\{\varphi_j\}_{j \in \mathbb{N}} \subset L_2(I)$ is *orthonormal* if its elements are pairwise orthogonal and the norm of each element is 1.

Remark: An orthonormal sequence is also called *orthonormal system*.

Remark: In the definition of orthonormal sequences (systems) we can write $\mathbb{Z}$ or $\mathbb{N} \cup \{0\}$ instead of $\mathbb{N}$. For example, in the case of the exponential system we have $j \in \mathbb{Z}$ (see Chapter 3.3), while in the case of the trigonometric system we write $j \in \mathbb{N}$ or $j \in \mathbb{N} \cup \{0\}$ (see Chapter 3.2).

The *Fourier coefficients* of a function $x \in L_2(I)$ with respect to the orthonormal system $\{\varphi_j\}_{j \in \mathbb{N}} \subset L_2(I)$ are

$$\hat{x}_k = \langle x, \varphi_k \rangle = \int_I (x \cdot \varphi_k^*), \quad k \in \mathbb{N}$$

The series of functions

$$\mathcal{FS}(x) = \sum_{k=1}^{\infty} (\hat{x}_k \cdot \varphi_k) = \sum_{k=1}^{\infty} (\langle x, \varphi_k \rangle \cdot \varphi_k)$$

is called the *Fourier series* of x with respect to the orthonormal system $\{\varphi_j\}_{j \in \mathbb{N}}$.

The connection between $x \in L_2(I)$ and $\mathcal{FS}(x)$ is important question in the Fourier theory. In L_2 we say that $x = \sum_{k=1}^{\infty} (\hat{x}_k \cdot \varphi_k)$ if $\lim_{n \rightarrow \infty} \|x - \sum_{k=1}^n (\hat{x}_k \cdot \varphi_k)\| = 0$. An orthonormal system $\{\varphi_j\}_{j \in \mathbb{N}} \subset L_2(I)$ is called *complete* if $x = \mathcal{FS}(x)$ for all $x \in L_2(I)$.

From the point of view of engineering practice it is generally enough to know that the Fourier series of a piecewise continuous function converges to the value of the function at every point t where the function is continuous ($\lim_{n \rightarrow \infty} \sum_{k=1}^n \hat{x}_k \cdot \varphi_k(t) = x(t)$) and converges to the midpoint of the discontinuity (the average of the left- and right-hand limits) wherever the function is discontinuous.

The *Parseval equality*

$$\|x\|^2 = \sum_{k=-\infty}^{\infty} |\hat{x}_k|^2$$

states that the square norm of a function (energy content of a signal) can be calculated directly from its Fourier coefficients.

3.2 The Trigonometric System

3.2.1 The Orthonormal Trigonometric System

Let $T > 0$. System of functions

$$\left\{ \text{CONST}(t) = \frac{1}{\sqrt{T}}, \text{COS}_k(t) = \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), \text{SIN}_k(t) = \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \right\}_{k \in \mathbb{N}}$$

is orthonormal in $L_2([0, T])$.

T -periodic functions

$$t \rightarrow \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(\frac{2\pi}{T} \cdot t\right) \quad \text{and} \quad t \rightarrow \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(\frac{2\pi}{T} \cdot t\right)$$

of 'frequency' $f_0 = \frac{1}{T}$ are called *basic functions* of the system, while T/k -periodic functions

$$t \rightarrow \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), \quad t \rightarrow \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right), \quad k = 2, 3, \dots$$

of frequency $k \cdot f_0 = k/T$ are the *harmonics*.

The *Fourier series* of a function $x \in L_2([0, T])$ with respect to the orthonormal trigonometric system is

$$\mathcal{FS}(x)(t) = \hat{A}_0 \cdot \text{CONST} + \sum_{k=1}^{\infty} \hat{A}_k \cdot \text{COS}_k(t) + \sum_{k=1}^{\infty} \hat{B}_k \cdot \text{SIN}_k(t),$$

where

$$\begin{aligned} \hat{A}_0 &= \langle x, \text{CONST} \rangle = \int_0^T x(t) \cdot \text{CONST}(t) dt = \int_0^T x(t) \cdot \frac{1}{\sqrt{T}} dt \\ \hat{A}_k &= \langle x, \text{COS}_k \rangle = \int_0^T x(t) \cdot \text{COS}_k(t) dt = \int_0^T x(t) \cdot \left(\frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \right) dt, \quad k = 1, 2, \dots \\ \hat{B}_k &= \langle x, \text{SIN}_k \rangle = \int_0^T x(t) \cdot \text{SIN}_k(t) dt = \int_0^T x(t) \cdot \left(\frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \right) dt, \quad k = 1, 2, \dots \end{aligned}$$

$\hat{A}_0, \hat{A}_k$ and $\hat{B}_k, k = 1, 2, \dots$ are the *Fourier coefficients* of x with respect to the orthonormal trigonometric system.

In the special case $T = 2\pi$ the orthonormal trigonometric system is

$$\left\{ \text{CONST}(t) = \frac{1}{\sqrt{2\pi}}, \text{COS}_k(t) = \frac{1}{\sqrt{\pi}} \cdot \cos(k \cdot t), \text{SIN}_k(t) = \frac{1}{\sqrt{\pi}} \cdot \sin(k \cdot t) \right\}_{k \in \mathbb{N}}$$

and the Fourier coefficients of x are

$$\begin{aligned} \hat{A}_0 &= \int_0^{2\pi} x(t) \cdot \frac{1}{\sqrt{2\pi}} dt \\ \hat{A}_k &= \int_0^{2\pi} x(t) \cdot \left(\frac{1}{\sqrt{\pi}} \cdot \cos(k \cdot t) \right) dt, \quad k = 1, 2, \dots \\ \hat{B}_k &= \int_0^{2\pi} x(t) \cdot \left(\frac{1}{\sqrt{\pi}} \cdot \sin(k \cdot t) \right) dt, \quad k = 1, 2, \dots \end{aligned}$$

If function x is odd, then $\hat{A}_k = 0, k = 0, 1, 2, \dots$, if x is even, then $\hat{B}_k = 0, k = 1, 2, \dots$

The Parseval's equality in the case of the orthonormal trigonometric system is

$$\|x\|^2 = \int_0^T x^2 = \hat{A}_0^2 + \sum_{k=1}^{\infty} \hat{A}_k^2 + \sum_{k=1}^{\infty} \hat{B}_k^2$$

Remark: When calculating the Fourier coefficients of the T -periodic functions with respect to the orthonormal trigonometric system we can

take the integrals on any interval of length T . E.g. we often do the calculations on interval $\left[-\frac{T}{2}, \frac{T}{2}\right]$.

>>>Exercise 3.2 – 1

Let $T > 0$. Show that the system of functions

$$\left\{ \text{CONST}(t) = \frac{1}{\sqrt{T}}, \text{COS}_k(t) = \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), \text{SIN}_k(t) = \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \right\}_{k \in \mathbb{N}}$$

is orthonormal in $L_2([0, T])$.

>>>Exercise 3.2 – 1 – Solution

$$\int_0^T \text{CONST}^2(t) dt = \int_0^T \frac{1}{T} dt = 1$$

$$\begin{aligned} \int_0^T \text{COS}_k^2(t) dt &= \int_0^T \frac{2}{T} \cdot \cos^2\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt = \frac{1}{T} \cdot \int_0^T \left(1 + \cos\left(k \cdot \frac{4\pi}{T} \cdot t\right)\right) dt = \\ &= \frac{1}{T} \cdot \left[t + \frac{T}{4\pi \cdot k} \cdot \sin\left(k \cdot \frac{4\pi}{T} \cdot t\right)\right]_0^T = 1 \end{aligned}$$

$$\begin{aligned} \int_0^T \text{SIN}_k^2(t) dt &= \int_0^T \frac{2}{T} \cdot \sin^2\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt = \frac{1}{T} \cdot \int_0^T \left(1 - \cos\left(k \cdot \frac{4\pi}{T} \cdot t\right)\right) dt = \\ &= \frac{1}{T} \cdot \left[t - \frac{T}{4\pi \cdot k} \cdot \sin\left(k \cdot \frac{4\pi}{T} \cdot t\right)\right]_0^T = 1 \end{aligned}$$

If $k \neq n$

$$\begin{aligned} \int_0^T \text{COS}_k(t) \cdot \text{SIN}_n(t) dt &= \frac{2}{T} \cdot \int_0^T \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) dt = \\ &= \frac{k}{k^2 - n^2} \cdot \frac{T}{2\pi} \cdot \left[\sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right)\right]_0^T + \\ &+ \frac{1}{k^2 - n^2} \cdot \frac{n \cdot T}{2\pi} \cdot \left[\cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \cos\left(n \cdot \frac{2\pi}{T} \cdot t\right)\right]_0^T = \\ &= \frac{k}{k^2 - n^2} \cdot \frac{T}{2\pi} \cdot (\sin(k \cdot 2\pi) \cdot \sin(n \cdot 2\pi) - \sin 0 \cdot \sin 0) + \\ &+ \frac{1}{k^2 - n^2} \cdot \frac{n \cdot T}{2\pi} \cdot (\cos(k \cdot 2\pi) \cdot \cos(n \cdot 2\pi) - \cos 0 \cdot \cos 0) = 0 \end{aligned}$$

Details of the integration:

$$\begin{aligned} \int \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) dt &= \\ \left[\begin{aligned} g(t) &= \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) \Rightarrow g'(t) = n \cdot \frac{2\pi}{T} \cdot \cos\left(n \cdot \frac{2\pi}{T} \cdot t\right) \\ f'(t) &= \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \Rightarrow f(t) = \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \end{aligned} \right] \\ &= \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) - \frac{n}{k} \cdot \int \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \cos\left(n \cdot \frac{2\pi}{T} \cdot t\right) dt = \end{aligned}$$

$$\begin{aligned}
& \left[g(t) = \cos\left(n \cdot \frac{2\pi}{T} \cdot t\right) \Rightarrow g'(t) = -n \cdot \frac{2\pi}{T} \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) \right. \\
& \left. f'(t) = \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \Rightarrow f(t) = -\frac{1}{k \cdot \frac{2\pi}{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \right] \\
& = \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) - \\
& \quad - \frac{n}{k} \cdot \left(-\frac{1}{k \cdot \frac{2\pi}{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \cos\left(n \cdot \frac{2\pi}{T} \cdot t\right) - \frac{n}{k} \cdot \int \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) dt \right) = \\
& = \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) + \\
& \quad + \frac{n}{k^2 \cdot \frac{2\pi}{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \cos\left(n \cdot \frac{2\pi}{T} \cdot t\right) + \frac{n^2}{k^2} \cdot \int \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) dt \\
& \left(1 - \frac{n^2}{k^2} \right) \cdot \int \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) dt = \\
& \quad = \frac{1}{k \cdot \frac{2\pi}{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) + \frac{n}{k^2 \cdot \frac{2\pi}{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \cos\left(n \cdot \frac{2\pi}{T} \cdot t\right) \\
& \int \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) dt = \\
& = \frac{k}{k^2 - n^2} \cdot \frac{T}{2\pi} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \sin\left(n \cdot \frac{2\pi}{T} \cdot t\right) + \frac{1}{k^2 - n^2} \cdot \frac{n \cdot T}{2\pi} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \cdot \cos\left(n \cdot \frac{2\pi}{T} \cdot t\right)
\end{aligned}$$

We can show similarly that

$$\int_0^T \text{SIN}_k(t) \cdot \text{SIN}_n(t) dt = 0 \quad \text{and} \quad \int_0^T \text{COS}_k(t) \cdot \text{COS}_n(t) dt = 0$$

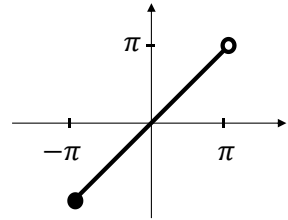
>>>Exercise 3.2 – 2

Calculate the Fourier coefficients of the 2π -periodic function x defined as

$$x(t) = t, \quad -\pi \leq t < \pi$$

with respect to the orthonormal trigonometric system.

Use the Parseval's equality to give the sum $\sum_{k=1}^{\infty} \frac{1}{k^2}$.



>>>Exercise 3.2 – 2- Solution

Since function x is odd, $\hat{A}_k = 0, k = 0, 1, 2, \dots$

$$\begin{aligned}
\hat{B}_k &= \int_{-\pi}^{\pi} t \cdot \left(\frac{1}{\sqrt{\pi}} \cdot \sin(k \cdot t) \right) dt = \frac{1}{\sqrt{\pi}} \cdot \left[-\frac{1}{k} \cdot t \cdot \cos(k \cdot t) + \frac{1}{k^2} \cdot \sin(k \cdot t) \right]_{-\pi}^{\pi} = \\
&= \frac{1}{\sqrt{\pi}} \cdot \left(\left(-\frac{1}{k} \cdot \pi \cdot \cos(k \cdot \pi) + \frac{1}{k^2} \cdot \sin(k \cdot \pi) \right) - \left(-\frac{1}{k} \cdot \pi \cdot \cos(k \cdot \pi) - \frac{1}{k^2} \cdot \sin(k \cdot \pi) \right) \right) = \\
&= \frac{1}{\sqrt{\pi}} \cdot \left(-\frac{2}{k} \cdot \pi \cdot \cos(k \cdot \pi) \right) = 2 \cdot \sqrt{\pi} \cdot (-1)^{k+1} \cdot \frac{1}{k}
\end{aligned}$$

Details of the calculation (integration by parts):

$$\int t \cdot \sin(k \cdot t) dt = -\frac{1}{k} \cdot t \cdot \cos(k \cdot t) + \frac{1}{k} \cdot \int \cos(k \cdot t) dt = -\frac{1}{k} \cdot t \cdot \cos(k \cdot t) + \frac{1}{k^2} \cdot \sin(k \cdot t)$$

$$\left[\begin{array}{ll} g(t) = t & \Rightarrow g'(t) = 1 \\ f'(t) = \sin(k \cdot t) & \Rightarrow f(t) = -\frac{1}{k} \cdot \cos(k \cdot t) \end{array} \right]$$

According to the Parseval's equality

$$\|x\|^2 = \int_{-\pi}^{\pi} t^2 dt = \sum_{k=1}^{\infty} \hat{B}_k^2 = 4\pi \cdot \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Since $\int_{-\pi}^{\pi} t^2 dt = \frac{1}{3} \cdot [t^3]_{-\pi}^{\pi} = \frac{2}{3} \cdot \pi^3$ we have

$$\frac{2}{3} \cdot \pi^3 = 4\pi \cdot \sum_{k=1}^{\infty} \frac{1}{k^2}$$

that is

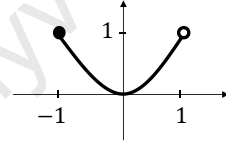
$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

>>>Exercise 3.2 – 3

Calculate the Fourier coefficients of the 2-periodic function x defined as

$$x(t) = t^2, \quad -1 \leq t < 1$$

with respect to the orthonormal trigonometric system.



>>>Exercise 3.2 – 3– Solution

Since function x is even, $\hat{B}_k = 0, k = 1, 2, \dots$

$$\hat{A}_0 = \langle x, \text{CONST} \rangle = \int_{-1}^1 t^2 \cdot \frac{1}{\sqrt{2}} dt = \frac{1}{\sqrt{2}} \cdot \frac{1}{3} \cdot [t^3]_{-1}^1 = \frac{\sqrt{2}}{3}$$

$$\begin{aligned} \hat{A}_k &= \langle x, \text{COS}_k \rangle = \int_{-1}^1 t^2 \cdot (\cos(k \cdot \pi \cdot t)) dt = \\ &= \left[\frac{1}{k \cdot \pi} \cdot t^2 \cdot \sin(k \cdot \pi \cdot t) + \frac{2}{k^2 \cdot \pi^2} \cdot t \cdot \cos(k \cdot \pi \cdot t) - \frac{2}{k^3 \cdot \pi^3} \cdot \sin(k \cdot \pi \cdot t) \right]_{-1}^1 = \\ &= \left(\frac{1}{k \cdot \pi} \cdot \sin(k \cdot \pi) + \frac{2}{k^2 \cdot \pi^2} \cdot \cos(k \cdot \pi) - \frac{2}{k^3 \cdot \pi^3} \cdot \sin(k \cdot \pi) \right) - \\ &\quad - \left(-\frac{1}{k \cdot \pi} \cdot \sin(k \cdot \pi) - \frac{2}{k^2 \cdot \pi^2} \cdot \cos(k \cdot \pi) + \frac{2}{k^3 \cdot \pi^3} \cdot \sin(k \cdot \pi) \right) = \\ &= \frac{2}{k \cdot \pi} \cdot \sin(k \cdot \pi) + \frac{4}{k^2 \cdot \pi^2} \cdot \cos(k \cdot \pi) - \frac{4}{k^3 \cdot \pi^3} \cdot \sin(k \cdot \pi) = \frac{4}{k^2 \cdot \pi^2} \cdot \cos(k \cdot \pi) \\ \hat{A}_k &= \begin{cases} \frac{4}{k^2 \cdot \pi^2} & \text{if } k \text{ is even} \\ -\frac{4}{k^2 \cdot \pi^2} & \text{if } k \text{ is odd} \end{cases} \end{aligned}$$

Details of the calculation (integration by parts):

$$\begin{aligned} \int t^2 \cdot (\cos(k \cdot \pi \cdot t)) dt &= \frac{1}{k\pi} \cdot t^2 \cdot \sin(k \cdot \pi \cdot t) - \frac{2}{k\pi} \cdot \int t \cdot \sin(k \cdot \pi \cdot t) dt = \\ &\left[\begin{array}{ll} g(t) = t^2 & \Rightarrow g'(t) = 2t \\ f'(t) = \cos(k \cdot \pi \cdot t) & \Rightarrow f(t) = \frac{1}{k\pi} \cdot \sin(k \cdot \pi \cdot t) \end{array} \right] \end{aligned}$$

$$\begin{aligned}
& \left[\begin{array}{ll} g(t) = t & \Rightarrow g'(t) = 1 \\ f'(t) = \sin(k \cdot \pi \cdot t) & \Rightarrow f(t) = -\frac{1}{k\pi} \cdot \cos(k \cdot \pi \cdot t) \end{array} \right] \\
& = \frac{1}{k \cdot \pi} \cdot t^2 \cdot \sin(k \cdot \pi \cdot t) - \frac{2}{k \cdot \pi} \cdot \left(-\frac{1}{k \cdot \pi} \cdot t \cdot \cos(k \cdot \pi \cdot t) + \frac{1}{k \cdot \pi} \cdot \int \cos(k \cdot \pi \cdot t) dt \right) = \\
& = \frac{1}{k \cdot \pi} \cdot t^2 \cdot \sin(k \cdot \pi \cdot t) + \frac{2}{k^2 \cdot \pi^2} \cdot t \cdot \cos(k \cdot \pi \cdot t) - \frac{2}{k^3 \cdot \pi^3} \cdot \sin(k \cdot \pi \cdot t)
\end{aligned}$$

3.2.2 The Trigonometric System

Let $T > 0$. System of functions

$$\left\{ 1, \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \right\}_{k \in \mathbb{N}}$$

is orthogonal (but not orthonormal) in $L_2([0, T])$. It is called the *trigonometric system*.

The Fourier coefficients of a function $x \in L_2([0, T])$ with respect to the trigonometric system are

$$\hat{a}_0 = \frac{1}{T} \cdot \int_0^T x(t) dt$$

$$\hat{a}_k = \frac{2}{T} \cdot \int_0^T x(t) \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt, \quad k = 1, 2, \dots$$

$$\hat{b}_k = \frac{2}{T} \cdot \int_0^T x(t) \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt, \quad k = 1, 2, \dots$$

and the *Fourier series* of x with respect to the trigonometric system is

$$\mathcal{FS}(x)(t) = \hat{a}_0 + \sum_{k=1}^{\infty} \hat{a}_k \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) + \sum_{k=1}^{\infty} \hat{b}_k \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)$$

If function x is odd, then $\hat{a}_k = 0, k = 0, 1, 2, \dots$, if x is even, then $\hat{b}_k = 0, k = 1, 2, \dots$

In the special case $T = 2\pi$ the trigonometric system is

$$\{1, \cos(k \cdot t), \sin(k \cdot t)\}_{k \in \mathbb{N}}$$

and the Fourier coefficients are

$$\hat{a}_0 = \frac{1}{2\pi} \cdot \int_0^{2\pi} x(t) dt$$

$$\hat{a}_k = \frac{1}{\pi} \cdot \int_0^{2\pi} x(t) \cdot \cos(k \cdot t) dt, \quad k = 1, 2, \dots$$

$$\hat{b}_k = \frac{1}{\pi} \cdot \int_0^{2\pi} x(t) \cdot \sin(k \cdot t) dt, \quad k = 1, 2, \dots$$

Using the trigonometric equality

$$A \cdot \sin x + B \cdot \cos x = \sqrt{A^2 + B^2} \cdot \sin(x + \varphi), \text{ where } \varphi = \begin{cases} \arctg \frac{b}{a}, & \text{if } a \geq 0 \\ \arctg \frac{b}{a} + \pi, & \text{if } a < 0 \end{cases}$$

an alternative form of the Fourier series is

$$\mathcal{FS}(x)(t) = \hat{c}_0 + \sum_{k=1}^{\infty} \hat{c}_k \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t + \varphi_k\right),$$

is obtained, where $\hat{c}_0 = \hat{a}_0$, $\hat{c}_k = \sqrt{\hat{a}_k^2 + \hat{b}_k^2}$, $k = 1, 2, \dots$ and

$$\varphi_k = \begin{cases} \operatorname{arctg} \frac{\hat{b}_k}{\hat{a}_k}, & \text{if } \hat{a}_k \geq 0 \\ \operatorname{arctg} \frac{\hat{b}_k}{\hat{a}_k} + \pi, & \text{if } \hat{a}_k < 0 \end{cases}$$

is the *phase* of the harmonic belonging to index k .

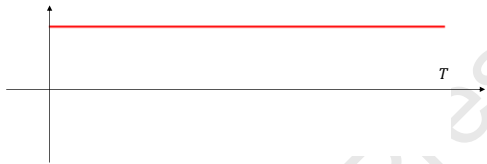
Remark: An advantage of this formula is that it contains at most one term (a sine function) of a certain frequency, while in the original formula two functions (a sine and a cosine) of the same frequency can appear. This fact is important in certain applications when the frequencies of the harmonic components (and the related amplitude) are in the focus. We have to note that the decompositions with respect to the exponential system also contains at most one term corresponding to a certain frequency.

>>>Exercise 3.2 – 4

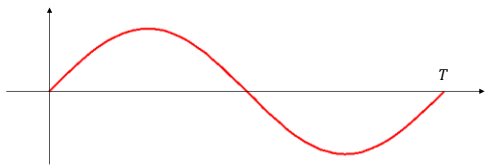
Sketch the constant function and the first four cosine and the first four sine functions of the trigonometric system belonging to the period T on the interval $[0, T]$.

>>>Exercise 3.2 – 4- Solution

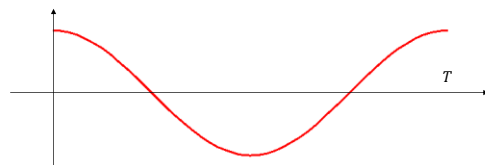
$t \rightarrow 1$



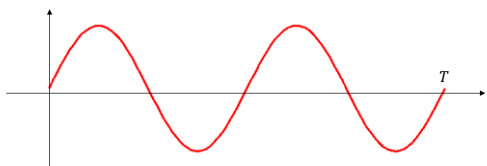
$t \rightarrow \sin\left(\frac{2\pi}{T} \cdot t\right)$



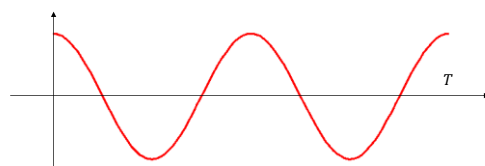
$t \rightarrow \cos\left(\frac{2\pi}{T} \cdot t\right)$



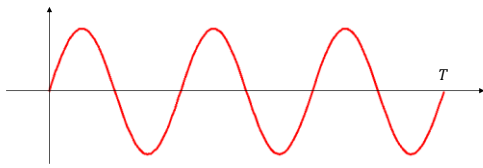
$t \rightarrow \sin\left(2 \cdot \frac{2\pi}{T} \cdot t\right)$



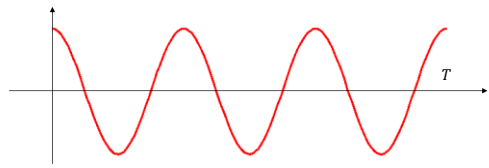
$t \rightarrow \cos\left(2 \cdot \frac{2\pi}{T} \cdot t\right)$



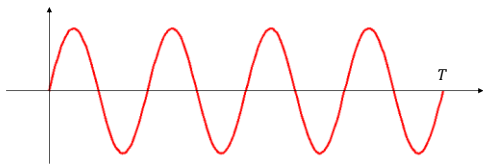
$$t \rightarrow \sin\left(3 \cdot \frac{2\pi}{T} \cdot t\right)$$



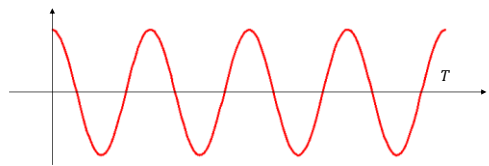
$$t \rightarrow \cos\left(3 \cdot \frac{2\pi}{T} \cdot t\right)$$



$$t \rightarrow \sin\left(4 \cdot \frac{2\pi}{T} \cdot t\right)$$



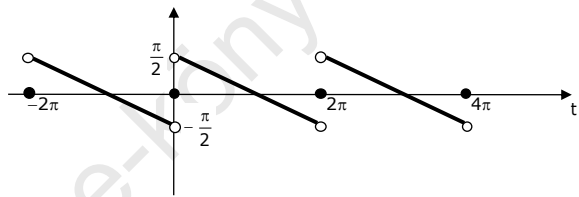
$$t \rightarrow \cos\left(4 \cdot \frac{2\pi}{T} \cdot t\right)$$



>>>Exercise 3.2 – 5

Determine the Fourier coefficients of the 2π -periodic function x defined as

$$x(t) = \begin{cases} 0, & \text{if } t = 0 \\ -\frac{1}{2}t + \frac{\pi}{2}, & \text{if } 0 < t < 2\pi \end{cases}$$



with respect to the trigonometric system.

Give the sum of the first 5 terms and the sum of the first 10 terms in the Fourier series.

>>>Exercise 3.2 – 5– Solution

Function x is odd, so $\hat{a}_k = 0, k = 0, 1, 2, \dots$

We can get the coefficients $\hat{b}_k$ by integration by parts

$$\begin{aligned} \hat{b}_k &= \frac{1}{\pi} \cdot \int_0^{2\pi} \left(-\frac{1}{2}t + \frac{\pi}{2}\right) \cdot \sin(k \cdot t) dt = \\ &= \frac{1}{\pi} \cdot \left[\left(\frac{1}{2k}t - \frac{\pi}{2k}\right) \cdot \cos(k \cdot t) - \frac{1}{2k^2} \cdot \sin(k \cdot t) \right]_0^{2\pi} = \\ &= \frac{1}{\pi} \cdot \left(\left(\left(\frac{\pi}{k} - \frac{\pi}{2k}\right) \cdot \cos(k \cdot 2\pi) - \frac{1}{2k^2} \cdot \sin(k \cdot 2\pi)\right) - \left(-\frac{\pi}{2k} \cdot \cos 0 - \frac{1}{2k^2} \cdot \sin 0\right) \right) = \frac{1}{k} \end{aligned}$$

Details of the calculation (integration by parts):

$$\begin{aligned} \int \left(-\frac{1}{2}t + \frac{\pi}{2}\right) \cdot \sin(k \cdot t) dt &= -\frac{1}{k} \cdot \left(-\frac{1}{2}t + \frac{\pi}{2}\right) \cdot \cos(k \cdot t) - \frac{1}{2k} \cdot \int \cos(k \cdot t) dt = \\ &\left[g(t) = -\frac{1}{2}t + \frac{\pi}{2} \Rightarrow g'(t) = -\frac{1}{2} \right. \\ &\left. f'(t) = \sin(k \cdot t) \Rightarrow f(t) = -\frac{1}{k} \cdot \cos(k \cdot t) \right] \\ &= -\frac{1}{k} \cdot \left(-\frac{1}{2}t + \frac{\pi}{2}\right) \cdot \cos(k \cdot t) - \frac{1}{2k^2} \cdot \sin(k \cdot t) = \left(\frac{1}{2k}t - \frac{\pi}{2k}\right) \cdot \cos(k \cdot t) - \frac{1}{2k^2} \cdot \sin(k \cdot t) \end{aligned}$$

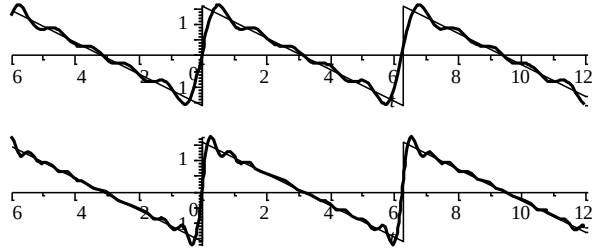
Since $\hat{a}_k = 0, k = 0, 1, 2, \dots$ and $\hat{b}_k = \frac{1}{k}, k = 1, 2, \dots$ the Fourier series of x is

$$\mathcal{FS}x(t) = \sum_{k=1}^{\infty} \frac{\sin(k \cdot t)}{k}.$$

The two partial sums are

$$t \rightarrow \sum_{k=1}^5 \frac{\sin(k \cdot t)}{k}$$

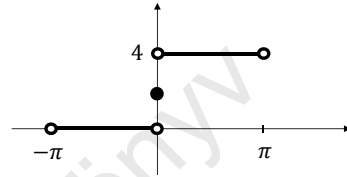
$$t \rightarrow \sum_{k=1}^{10} \frac{\sin(k \cdot t)}{k}$$



>>>Exercise 3.2 – 6

Determine the Fourier coefficients of the 2π -periodic function x defined as

$$x(t) = \begin{cases} 0, & \text{if } -\pi < t < 0 \\ 2, & \text{if } t = 0 \\ 4, & \text{if } 0 < t < \pi \\ 2, & \text{if } t = \pi \end{cases}$$



with respect to the trigonometric system.

Give the sum of the first 4 terms and the sum of the first 8 terms in the Fourier series.

>>>Exercise 3.2 – 6- Solution

$$\hat{a}_0 = \frac{1}{2\pi} \cdot \int_0^{\pi} 4 \, dt = 2$$

$$\hat{a}_k = \frac{1}{\pi} \cdot \int_0^{\pi} 4 \cdot \cos(k \cdot t) \, dt = \frac{4}{k \cdot \pi} \cdot [\sin(k \cdot t)]_0^{\pi} = 0$$

$$\hat{b}_k = \frac{1}{\pi} \cdot \int_0^{\pi} 4 \cdot \sin(k \cdot t) \, dt = \frac{-4}{k \cdot \pi} \cdot [\cos(k \cdot t)]_0^{\pi} = \frac{4}{k \cdot \pi} \cdot (1 - \cos(k \cdot \pi))$$

We have that

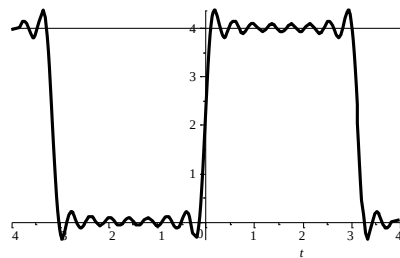
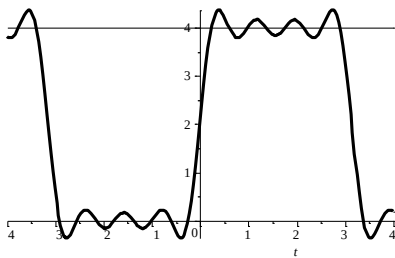
$$\hat{b}_k = \begin{cases} \frac{8}{k \cdot \pi} & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

writing the odd numbers k in the form $k = 2n - 1$ the Fourier series of x is

$$x(t) = 2 + \frac{8}{\pi} \cdot \sum_{n=1}^{\infty} \frac{\sin((2n - 1) \cdot t)}{2n - 1}$$

The two partial sums are

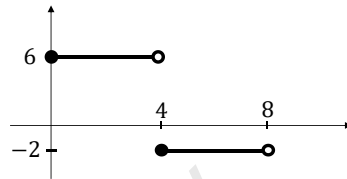
$$t \rightarrow 2 + \sum_{k=1}^4 \frac{8}{(2k - 1) \cdot \pi} \cdot \sin((2k - 1) \cdot t) \quad \left| \quad t \rightarrow 2 + \sum_{k=1}^8 \frac{8}{(2k - 1) \cdot \pi} \cdot \sin((2k - 1) \cdot t) \right.$$



>>>Exercise 3.2 – 7

Determine the Fourier coefficients of the 8-periodic function x defined as

$$x(t) = \begin{cases} 6 & \text{if } 0 \leq t < 4 \\ -2 & \text{if } 4 \leq t < 8 \end{cases}$$



>>>Exercise 3.2 – 7- Solution

$$\hat{a}_0 = \frac{1}{8} \cdot \int_0^4 6 \, dt + \frac{1}{8} \cdot \int_4^8 -2 \, dt = 2$$

$$\begin{aligned} \hat{a}_k &= \frac{1}{4} \cdot \int_0^4 6 \cdot \cos\left(k \cdot \frac{\pi}{4} \cdot t\right) dt + \frac{1}{4} \cdot \int_4^8 -2 \cdot \cos\left(k \cdot \frac{\pi}{4} \cdot t\right) dt = \\ &= \frac{6}{k \cdot \pi} \cdot \left[\sin\left(k \cdot \frac{\pi}{4} \cdot t\right)\right]_0^4 - \frac{2}{k \cdot \pi} \cdot \left[\sin\left(k \cdot \frac{\pi}{4} \cdot t\right)\right]_4^8 = 0 \end{aligned}$$

$$\begin{aligned} \hat{b}_k &= \frac{1}{4} \cdot \int_0^4 6 \cdot \sin\left(k \cdot \frac{\pi}{4} \cdot t\right) dt + \frac{1}{4} \cdot \int_4^8 -2 \cdot \sin\left(k \cdot \frac{\pi}{4} \cdot t\right) dt = \\ &= \frac{-6}{k \cdot \pi} \cdot \left[\cos\left(k \cdot \frac{\pi}{4} \cdot t\right)\right]_0^4 + \frac{2}{k \cdot \pi} \cdot \left[\cos\left(k \cdot \frac{\pi}{4} \cdot t\right)\right]_4^8 = \\ &= \frac{-6}{k \cdot \pi} \cdot (\cos(k \cdot \pi) - 1) + \frac{2}{k \cdot \pi} \cdot (\cos(2k \cdot \pi) - \cos(k \cdot \pi)) \end{aligned}$$

$$\hat{b}_k = \begin{cases} \frac{16}{k \cdot \pi} & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

writing the odd numbers k in the form $k = 2n - 1$ the Fourier series of x is

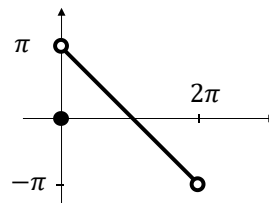
$$x(t) = 2 + \frac{16}{\pi} \cdot \sum_{n=1}^{\infty} \frac{\sin\left((2n-1) \cdot \frac{\pi}{4} \cdot t\right)}{2n-1}$$

>>>Exercise 3.2 – 8

Calculate the Fourier coefficient $\hat{b}_4$ of the 2π -periodic function x defined as

$$x(t) = \begin{cases} 0, & \text{if } t = 0 \\ \pi - t, & \text{if } 0 < t < 2\pi \end{cases}$$

with respect to the trigonometric system.



>>>Exercise 3.2 – 8– Solution

$$\begin{aligned}\hat{b}_4 &= \frac{1}{\pi} \cdot \int_0^{2\pi} (\pi - t) \cdot \sin(4 \cdot t) dt = \\ &= \frac{1}{\pi} \cdot \left[-\frac{1}{4} \cdot (\pi - t) \cdot \cos(4 \cdot t) - \frac{1}{16} \cdot \sin(4 \cdot t) \right]_0^{2\pi} = \frac{1}{\pi} \cdot \left(\frac{\pi}{4} + \frac{\pi}{4} \right) = \frac{1}{2}\end{aligned}$$

Details of the calculation (integration by parts):

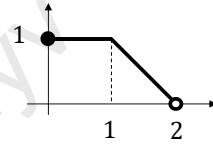
$$\begin{aligned}\int (\pi - t) \cdot \sin(4 \cdot t) dt &= -\frac{1}{4} \cdot (\pi - t) \cdot \cos(4 \cdot t) - \frac{1}{4} \cdot \int \cos(4 \cdot t) dt = \\ &= -\frac{1}{4} \cdot (\pi - t) \cdot \cos(4 \cdot t) - \frac{1}{16} \cdot \sin(4 \cdot t) \\ \left[\begin{array}{ll} g(t) = \pi - t & \Rightarrow \quad g'(t) = -1 \\ f'(t) = \sin(4 \cdot t) & \Rightarrow \quad f(t) = -\frac{1}{4} \cdot \cos(4 \cdot t) \end{array} \right]\end{aligned}$$

>>>Exercise 3.2 – 9

Calculate the Fourier coefficient $\hat{b}_{10}$ of the 2-periodic function x defined as

$$x(t) = \begin{cases} 1, & \text{if } 0 \leq t < 1 \\ 2 - t, & \text{if } 1 < t < 2 \end{cases}$$

with respect to the trigonometric system.



Exercise 3.2 – 9– Solution

$$\begin{aligned}\hat{b}_{10} &= \int_0^2 x(t) \cdot \sin(10 \cdot \pi \cdot t) dt = \int_0^1 \sin(10\pi \cdot t) dt + \int_1^2 (2 - t) \cdot \sin(10\pi \cdot t) dt = \\ &= -\frac{1}{10\pi} \cdot [\cos(10\pi \cdot t)]_0^1 + \left[\frac{t-2}{10\pi} \cdot \cos(10\pi \cdot t) - \frac{1}{100\pi^2} \cdot \sin(10\pi \cdot t) \right]_1^2 = \frac{1}{10\pi}\end{aligned}$$

Details of the calculation (integration by parts):

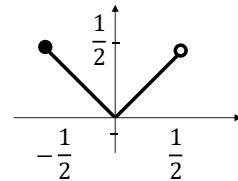
$$\begin{aligned}\int (2 - t) \cdot \sin(10\pi \cdot t) dt &= -\frac{1}{10\pi} \cdot (2 - t) \cdot \cos(10\pi \cdot t) - \frac{1}{10\pi} \cdot \int \cos(10\pi \cdot t) dt = \\ &= -\frac{1}{10\pi} \cdot (2 - t) \cdot \cos(10\pi \cdot t) - \frac{1}{100\pi^2} \cdot \sin(10\pi \cdot t) \\ \left[\begin{array}{ll} g(t) = 2 - t & \Rightarrow \quad g'(t) = -1 \\ f'(t) = \sin(10\pi \cdot t) & \Rightarrow \quad f(t) = -\frac{1}{10\pi} \cdot \cos(10\pi \cdot t) \end{array} \right]\end{aligned}$$

>>>Exercise 3.2 – 10

Calculate the Fourier coefficient $\hat{a}_2$ of the 1-periodic function x defined as

$$x(t) = |t|, \quad -\frac{1}{2} \leq t < \frac{1}{2}$$

with respect to the trigonometric system.



>>>Exercise 3.2 – 10– Solution

$$\hat{a}_2 = 2 \cdot \int_{-\frac{1}{2}}^{\frac{1}{2}} |t| \cdot \cos(2 \cdot 2\pi \cdot t) dt = 4 \cdot \int_0^{\frac{1}{2}} t \cdot \cos(4\pi \cdot t) dt =$$

$$\begin{aligned}
&= 4 \cdot \left[\frac{1}{4\pi} \cdot t \cdot \sin(4\pi \cdot t) + \frac{1}{16\pi^2} \cdot \cos(4\pi \cdot t) \right]_0^{1/2} = \\
&= 4 \cdot \left(\frac{1}{4\pi} \cdot \frac{1}{2} \cdot \sin(2\pi) + \frac{1}{16\pi^2} \cdot \cos(2\pi) - \frac{1}{16\pi^2} \right) = 0
\end{aligned}$$

Details of the calculation (integration by parts):

$$\begin{aligned}
\int t \cdot \cos(4\pi \cdot t) dt &= \frac{1}{4\pi} \cdot t \cdot \sin(4\pi \cdot t) - \frac{1}{4\pi} \cdot \int \sin(4\pi \cdot t) dt = \\
&= \frac{1}{4\pi} \cdot t \cdot \sin(4\pi \cdot t) + \frac{1}{16\pi^2} \cdot \cos(4\pi \cdot t)
\end{aligned}$$

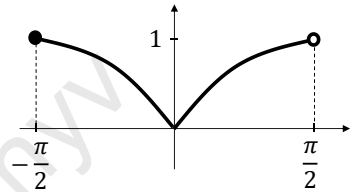
$$\left[\begin{array}{ll} g(t) = t & \Rightarrow \quad g'(t) = 1 \\ f'(t) = \cos(4\pi \cdot t) & \Rightarrow \quad f(t) = \frac{1}{4\pi} \cdot \sin(4\pi \cdot t) \end{array} \right]$$

>>>Exercise 3.2 – 11

Calculate the Fourier coefficient $\hat{a}_9$ of the π -periodic function x defined as

$$x(t) = |\sin t|, \quad -\frac{\pi}{2} \leq t < \frac{\pi}{2}$$

with respect to the trigonometric system.



>>>Exercise 3.2 – 11- Solution

$$\begin{aligned}
\hat{a}_k &= \frac{2}{T} \cdot \int_0^T x(t) \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt, \\
\hat{a}_9 &= \frac{2}{\pi} \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin t| \cdot \cos(9 \cdot 2 \cdot t) dt = \frac{4}{\pi} \cdot \int_0^{\frac{\pi}{2}} \sin t \cdot \cos(18 \cdot t) dt = \\
&= \frac{4}{\pi} \cdot \left[\frac{18}{323} \cdot \sin t \cdot \sin(18 \cdot t) + \frac{1}{323} \cdot \cos t \cdot \cos(18 \cdot t) \right]_0^{\pi/2} = \\
&= \frac{4}{\pi} \cdot \left(\frac{18}{323} \cdot \sin \frac{\pi}{2} \cdot \sin(9\pi) + \frac{1}{323} \cdot \cos \frac{\pi}{2} \cdot \cos(9\pi) - \frac{1}{323} \right) = \frac{-4}{323 \cdot \pi}
\end{aligned}$$

Details of the calculation (integration by parts):

$$\begin{aligned}
\int \sin t \cdot \cos(18 \cdot t) dt &= \frac{1}{18} \cdot \sin t \cdot \sin(18 \cdot t) - \frac{1}{18} \cdot \int \cos t \cdot \sin(18 \cdot t) dt = \\
&= \frac{1}{18} \cdot \sin t \cdot \sin(18 \cdot t) - \frac{1}{18} \cdot \left(-\frac{1}{18} \cdot \cos t \cdot \cos(18 \cdot t) - \frac{1}{18} \cdot \int \sin t \cdot \cos(18 \cdot t) dt \right) = \\
&= \frac{1}{18} \cdot \sin t \cdot \sin(18 \cdot t) + \frac{1}{324} \cdot \cos t \cdot \cos(18 \cdot t) + \frac{1}{324} \cdot \int \sin t \cdot \cos(18 \cdot t) dt \\
&\left[\begin{array}{ll} g(t) = \sin t & \Rightarrow \quad g'(t) = \cos t \\ f'(t) = \cos(18 \cdot t) & \Rightarrow \quad f(t) = \frac{1}{18} \cdot \sin(18 \cdot t) \end{array} \right] \\
&\left[\begin{array}{ll} g(t) = \cos t & \Rightarrow \quad g'(t) = -\sin t \\ f'(t) = \sin(18 \cdot t) & \Rightarrow \quad f(t) = -\frac{1}{18} \cdot \cos(18 \cdot t) \end{array} \right]
\end{aligned}$$

$$\int \sin t \cdot \cos(18 \cdot t) dt = \frac{1}{18} \cdot \sin t \cdot \sin(18 \cdot t) + \frac{1}{324} \cdot \cos t \cdot \cos(18 \cdot t) + \frac{1}{324} \cdot \int \sin t \cdot \cos(18 \cdot t) dt$$

$$\left(1 - \frac{1}{324}\right) \cdot \int \sin t \cdot \cos(18 \cdot t) dt = \frac{1}{18} \cdot \sin t \cdot \sin(18 \cdot t) + \frac{1}{324} \cdot \cos t \cdot \cos(18 \cdot t)$$

$$\int \sin t \cdot \cos(18 \cdot t) dt = \frac{18}{323} \cdot \sin t \cdot \sin(18 \cdot t) + \frac{1}{323} \cdot \cos t \cdot \cos(18 \cdot t)$$

>>>Exercise 3.2 – 12

Determine the period of the signal

$$x(t) = 6 \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) + 12 \cdot \cos\left(\frac{2\pi}{30} \cdot t\right)$$

and give the Fourier coefficients $\hat{a}_0, \hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{b}_1, \hat{b}_2, \hat{b}_3$.

>>>Exercise 3.2 – 12– Solution

Period of function $t \rightarrow 6 \cdot \sin\left(\frac{2\pi}{20} \cdot t\right)$ is 20, period of function $t \rightarrow 12 \cdot \cos\left(\frac{2\pi}{30} \cdot t\right)$ is 30. It is easy to see, that period of their sum is equal to the smallest common multiple of 20 and 30, that is $T = 60$.

Now it is evident that signal x contains two harmonic components, namely

$$t \rightarrow 6 \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) = 6 \cdot \sin\left(3 \cdot \frac{2\pi}{60} \cdot t\right)$$

and

$$t \rightarrow 12 \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) = 12 \cdot \cos\left(2 \cdot \frac{2\pi}{60} \cdot t\right)$$

thus $\hat{b}_3 = 6$ and $\hat{a}_2 = 12$. All other Fourier coefficients are equal to zero.

We can calculate the Fourier coefficients according to the formulas. $T = 60$ thus

$$\begin{aligned} \hat{b}_3 &= \frac{2}{60} \cdot \int_0^{60} \left(6 \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) + 12 \cdot \cos\left(\frac{2\pi}{30} \cdot t\right)\right) \cdot \sin\left(3 \cdot \frac{2\pi}{60} \cdot t\right) dt = \\ &= \frac{12}{60} \cdot \int_0^{60} \sin^2\left(\frac{2\pi}{20} \cdot t\right) dt + \frac{24}{60} \cdot \int_0^{60} \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) dt = \\ &= \frac{12}{60} \cdot \left[\frac{1}{2} \cdot \left(t - \frac{10}{2\pi} \cdot \sin\left(\frac{2\pi}{10} \cdot t\right) \right) \right]_0^{60} + \\ &\quad + \frac{24}{60} \cdot \left[-\frac{18}{\pi} \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \cos\left(\frac{2\pi}{20} \cdot t\right) - \frac{6}{5} \cdot \sin\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) \right]_0^{60} = 6 \end{aligned}$$

Details of the calculation

$$\begin{aligned} \int \sin^2\left(\frac{2\pi}{20} \cdot t\right) dt &= \frac{1}{2} \cdot \int 1 - \cos\left(\frac{2\pi}{10} \cdot t\right) dt = \frac{1}{2} \cdot \left(t - \frac{10}{2\pi} \cdot \sin\left(\frac{2\pi}{10} \cdot t\right) \right) \\ \int \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) dt &= \\ &= -\frac{20}{2\pi} \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \cos\left(\frac{2\pi}{20} \cdot t\right) - \frac{2}{3} \cdot \int \sin\left(\frac{2\pi}{30} \cdot t\right) \cdot \cos\left(\frac{2\pi}{20} \cdot t\right) dt = \\ &\quad \left[\begin{aligned} g(t) &= \cos\left(\frac{2\pi}{30} \cdot t\right) \Rightarrow g'(t) = -\frac{2\pi}{30} \cdot \sin\left(\frac{2\pi}{30} \cdot t\right) \\ f'(t) &= \sin\left(\frac{2\pi}{20} \cdot t\right) \Rightarrow f(t) = -\frac{20}{2\pi} \cdot \cos\left(\frac{2\pi}{20} \cdot t\right) \end{aligned} \right] \\ &\quad \left[\begin{aligned} g(t) &= \sin\left(\frac{2\pi}{30} \cdot t\right) \Rightarrow g'(t) = \frac{2\pi}{30} \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) \\ f'(t) &= \cos\left(\frac{2\pi}{20} \cdot t\right) \Rightarrow f(t) = \frac{20}{2\pi} \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) \end{aligned} \right] \\ &= -\frac{20}{2\pi} \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \cos\left(\frac{2\pi}{20} \cdot t\right) - \end{aligned}$$

$$\begin{aligned}
& -\frac{2}{3} \cdot \left(\sin\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) - \frac{2}{3} \cdot \int \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) dt \right) = \\
& = -\frac{20}{2\pi} \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \cos\left(\frac{2\pi}{20} \cdot t\right) - \frac{2}{3} \cdot \sin\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) + \frac{4}{9} \cdot \int \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) dt \\
& \frac{5}{9} \cdot \int \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) dt = -\frac{20}{2\pi} \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \cos\left(\frac{2\pi}{20} \cdot t\right) - \frac{2}{3} \cdot \sin\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) \\
& \int \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) dt = -\frac{18}{\pi} \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \cos\left(\frac{2\pi}{20} \cdot t\right) - \frac{6}{5} \cdot \sin\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) \\
& \hat{a}_2 = \frac{2}{60} \cdot \int_0^{60} \left(6 \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) + 12 \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) \right) \cdot \cos\left(2 \cdot \frac{2\pi}{60} \cdot t\right) dt = \\
& = \frac{12}{60} \cdot \int_0^{60} \sin\left(\frac{2\pi}{20} \cdot t\right) \cdot \cos\left(2 \cdot \frac{2\pi}{60} \cdot t\right) dt + \frac{24}{60} \cdot \int_0^{60} \cos^2\left(\frac{2\pi}{30} \cdot t\right) dt = \\
& = \frac{12}{60} \cdot \left[\frac{18}{\pi} \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \cos\left(\frac{2\pi}{20} \cdot t\right) - \frac{6}{5} \cdot \sin\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) \right]_0^{60} + \\
& \quad + \frac{24}{60} \cdot \left[\frac{1}{2} \cdot \left(t + \frac{15}{2\pi} \cdot \sin\left(\frac{2\pi}{15} \cdot t\right) \right) \right]_0^{60} = 12
\end{aligned}$$

Details of the calculation

$$\begin{aligned}
& \int \cos^2\left(\frac{2\pi}{30} \cdot t\right) dt = \frac{1}{2} \cdot \int 1 + \cos\left(\frac{2\pi}{15} \cdot t\right) dt = \frac{1}{2} \cdot \left(t + \frac{15}{2\pi} \cdot \sin\left(\frac{2\pi}{15} \cdot t\right) \right) \\
& \int \sin\left(\frac{2\pi}{20} \cdot t\right) \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) dt = \frac{18}{\pi} \cdot \cos\left(\frac{2\pi}{30} \cdot t\right) \cdot \cos\left(\frac{2\pi}{20} \cdot t\right) - \frac{6}{5} \cdot \sin\left(\frac{2\pi}{30} \cdot t\right) \cdot \sin\left(\frac{2\pi}{20} \cdot t\right) \\
& \text{(see the same calculation above)}
\end{aligned}$$

>>>Exercise 3.2 – 13

Give the spectrum of the following signals

3.2 – 13 – a $x(t) = 0.2 \cdot \sin(250 \cdot t - 5.6) -$

$-4.52 \cdot \sin(1250 \cdot t - 3.2) + 2.87 \cdot \sin(800 \cdot t)$

3.2 – 13 – b $x(t) = 100 \cdot \sin(5.48 \cdot t - 0.6) + 55 \cdot \sin(6.28 \cdot t - 3) +$

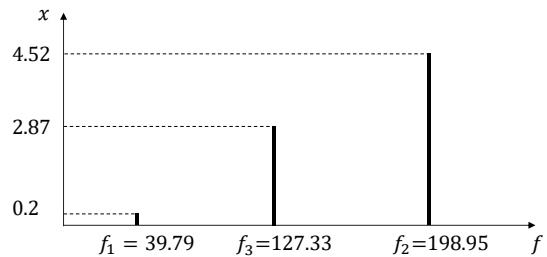
$+21 \cdot \sin(7.27 \cdot t + 1) + 66 \cdot \sin(t - 1.9)$

>>>Exercise 3.2 – 13 – a- Solution

$$\omega_1 = 250 \left[\frac{\text{rad}}{\text{s}} \right] \Rightarrow f_1 = \frac{\omega_1}{2\pi} = 39.79 [\text{Hz}]$$

$$\omega_2 = 1250 \left[\frac{\text{rad}}{\text{s}} \right] \Rightarrow f_2 = \frac{\omega_2}{2\pi} = 198.95 [\text{Hz}]$$

$$\omega_3 = 800 \left[\frac{\text{rad}}{\text{s}} \right] \Rightarrow f_3 = \frac{\omega_3}{2\pi} = 127.33 [\text{Hz}]$$



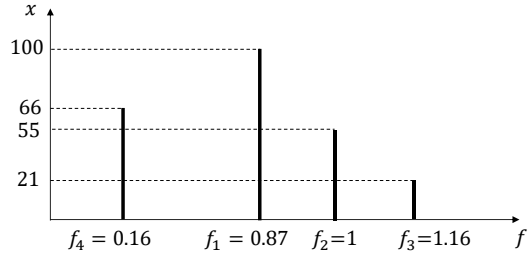
>>>Exercise 3.2 – 13 – b- Solution

$$\omega_1 = 5.48 \left[\frac{\text{rad}}{\text{s}} \right] \Rightarrow f_1 = \frac{\omega_1}{2\pi} = 0.87 [\text{Hz}]$$

$$\omega_2 = 6.28 \left[\frac{\text{rad}}{\text{s}} \right] \Rightarrow f_2 = \frac{\omega_2}{2\pi} = 1 [\text{Hz}]$$

$$\omega_3 = 7.27 \left[\frac{\text{rad}}{\text{s}} \right] \Rightarrow f_3 = \frac{\omega_3}{2\pi} = 1.16 [\text{Hz}]$$

$$\omega_4 = 1 \left[\frac{\text{rad}}{\text{s}} \right] \Rightarrow f_4 = \frac{\omega_4}{2\pi} = 0.16 [\text{Hz}]$$



3.3 The Exponential System

3.3.1 The Orthonormal Exponential System

Let $T > 0$. System of functions

$$\left\{ \text{EXP}_k(t) = \frac{1}{\sqrt{T}} \cdot e^{i \cdot k \cdot \frac{2\pi}{T} t} \right\}_{k \in \mathbb{Z}}$$

is orthonormal in $L_2([0, T])$. This system is called *orthonormal exponential system*.

Remark: In the exponential system index k is from $\mathbb{Z}$, that is, there are negative indices as well.

Remark: Because of the periodic nature of the complex exponential function in the imaginary part of the variable the coefficients $k \cdot \frac{2\pi}{T}$ in the exponents have "frequency" meaning similarly to the elements of the trigonometric system (see in Appendix 1).

The Fourier coefficients of a function $x \in L_2([0, T])$ with respect to the orthonormal exponential system are

$$\hat{X}_k = \langle x, \text{EXP}_k \rangle = \int_0^T x(t) \cdot \left(\frac{1}{\sqrt{T}} \cdot e^{-i \cdot k \cdot \frac{2\pi}{T} t} \right) dt, \quad k \in \mathbb{Z},$$

and the Fourier series (decomposition) of x is

$$\mathcal{FS}(x) = \sum_{k=-\infty}^{\infty} (\hat{X}_k \cdot \text{EXP}_k) = \sum_{k=-\infty}^{\infty} (\langle x, \text{EXP}_k \rangle \cdot \text{EXP}_k).$$

Remark: When calculating the Fourier coefficients of the T -periodic functions with respect to the orthonormal exponential system we can take the integrals on any interval of length T . E.g. we often do the calculations on interval $\left[-\frac{T}{2}, \frac{T}{2}\right]$.

>>>Exercise 3.3 – 1

Show that system of functions

$$\left\{ \text{EXP}_k(t) = \frac{1}{\sqrt{T}} \cdot e^{i \cdot k \cdot \frac{2\pi}{T} t} \right\}_{k \in \mathbb{Z}}$$

is orthonormal in $L_2([0, T])$.

>>>Exercise 3.3 – 1- Solution

For arbitrary $k \in \mathbb{Z}$ we have

$$\|\text{EXP}_k\|^2 = \int_0^T \left(\frac{1}{\sqrt{T}} \cdot e^{i \cdot k \cdot \frac{2\pi}{T} t} \cdot \frac{1}{\sqrt{T}} \cdot e^{-i \cdot k \cdot \frac{2\pi}{T} t} \right) dt = \int_0^T \frac{1}{T} dt = 1$$

(we used that $(e^{i \cdot \alpha})^* = e^{-i \cdot \alpha}$, $\alpha \in \mathbb{R}$)

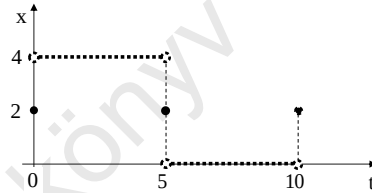
For arbitrary $k, l \in \mathbb{Z}, k \neq l$ we have

$$\begin{aligned} \langle \text{EXP}_k, \text{EXP}_l \rangle &= \int_0^T \left(\frac{1}{\sqrt{T}} \cdot e^{i \cdot k \cdot \frac{2\pi}{T} t} \cdot \frac{1}{\sqrt{T}} \cdot e^{-i \cdot l \cdot \frac{2\pi}{T} t} \right) dt = \frac{1}{T} \cdot \int_0^T e^{i \cdot (k-l) \cdot \frac{2\pi}{T} t} dt = \\ &= \frac{1}{T} \cdot \frac{1}{i \cdot (k-l) \cdot \frac{2\pi}{T}} \cdot \left[e^{i \cdot (k-l) \cdot \frac{2\pi}{T} t} \right]_0^T = \frac{1}{2\pi \cdot i \cdot (k-l)} \cdot (e^{2\pi \cdot i \cdot (k-l)} - 1) = 0 \end{aligned}$$

>>>Exercise 3.3 – 2

Give the Fourier series of 10-periodic function x defined as

$$x(t) = \begin{cases} 4, & \text{if } 0 < t < 5 \\ 0, & \text{if } 5 < t < 10 \\ 2, & \text{if } x \in \{0, 5, 10\} \end{cases}$$



with respect to the orthonormal exponential system.

>>>Exercise 3.3 – 2- Solution

$$\hat{X}_0 = \langle x, \text{EXP}_0 \rangle = \int_0^5 4 \cdot \frac{1}{\sqrt{10}} dt = \frac{20}{\sqrt{10}}$$

If $k \neq 0$

$$\begin{aligned} \hat{X}_k &= \langle x, \text{EXP}_k \rangle = \int_0^5 4 \cdot \left(\frac{1}{\sqrt{10}} \cdot e^{-i \cdot k \cdot \frac{2\pi}{10} t} \right) dt = \frac{4}{\sqrt{10}} \cdot \frac{-10}{2\pi \cdot i \cdot k} \cdot \left[e^{-i \cdot k \cdot \frac{2\pi}{10} t} \right]_0^5 = \\ &= \frac{20 \cdot i}{\sqrt{10} \cdot \pi \cdot k} \cdot (e^{-i \cdot k \cdot \pi} - 1) = \begin{cases} \frac{-40 \cdot i}{\sqrt{10} \cdot \pi \cdot k}, & \text{if } k \text{ is odd} \\ 0, & \text{if } k \text{ is even, } k \neq 0 \end{cases} \end{aligned}$$

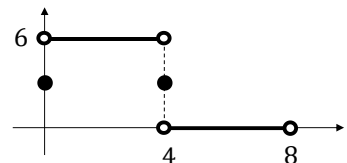
Using the notation $k = 2l - 1, l \in \mathbb{Z}$ the Fourier series of x is

$$\begin{aligned} \mathcal{FS}f(x)(t) &= 2 + \sum_{l=-\infty}^{\infty} \left(\frac{-40 \cdot i}{\sqrt{10} \cdot \pi \cdot (2l-1)} \cdot \frac{1}{\sqrt{10}} \cdot e^{i \cdot (2l-1) \cdot \frac{2\pi}{10} t} \right) = \\ &= 2 + \sum_{l=-\infty}^{\infty} \left(\frac{-4 \cdot i}{\pi \cdot (2l-1)} \cdot e^{i \cdot (2l-1) \cdot \frac{2\pi}{10} t} \right) \end{aligned}$$

>>>Exercise 3.3 – 3

Calculate the Fourier coefficient $\hat{x}_3$ of the 8-periodic function x defined as

$$x(t) = \begin{cases} 6, & \text{if } 0 < t < 4 \\ 0, & \text{if } 4 < t < 8 \\ 3, & \text{if } x \in \{0, 4\} \end{cases}$$



with respect to the orthonormal exponential system.

>>>Exercise 3.3 – 3– Solution

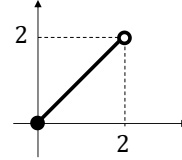
$$\begin{aligned}\hat{X}_3 &= \langle x(t), \frac{1}{\sqrt{8}} \cdot e^{-i \cdot 3 \cdot \frac{2\pi}{8} t} \rangle = \int_0^4 6 \cdot \frac{1}{\sqrt{8}} \cdot e^{-i \cdot \frac{3\pi}{4} t} dt = \frac{6}{\sqrt{8}} \cdot \frac{-4}{3\pi \cdot i} \cdot \left[e^{-i \cdot \frac{3\pi}{4} t} \right]_0^4 = \\ &= \frac{-\sqrt{8}}{\pi \cdot i} \cdot (e^{-i \cdot 3\pi} - 1) = \frac{-\sqrt{8}}{\pi \cdot i} \cdot (\cos(-3\pi) + i \cdot \sin(-3\pi) - 1) = \frac{2 \cdot \sqrt{8}}{\pi \cdot i}\end{aligned}$$

>>>Exercise 3.3 – 4

Calculate the Fourier coefficient $\hat{x}_4$ of the 2-periodic function x defined as

$$x(t) = t, \quad 0 \leq t < 2$$

with respect to the orthonormal exponential system.



>>>Exercise 3.3 – 4– Solution

$$\begin{aligned}\hat{X}_4 &= \langle x(t), \frac{1}{\sqrt{2}} \cdot e^{-i \cdot 4 \cdot \frac{2\pi}{2} t} \rangle = \int_0^2 t \cdot \frac{1}{\sqrt{2}} \cdot e^{-i \cdot 4\pi t} dt = \frac{1}{\sqrt{2}} \cdot \frac{-1}{4\pi \cdot i} \cdot \left[\left(t + \frac{1}{4\pi i} \right) \cdot e^{-i \cdot 4\pi t} \right]_0^2 = \\ &= \frac{1}{\sqrt{2}} \cdot \frac{-1}{4\pi \cdot i} \cdot \left(\left(2 + \frac{1}{4\pi i} \right) \cdot e^{-i \cdot 8\pi} - \frac{1}{4\pi i} \right)\end{aligned}$$

Details of the calculation

$$\begin{aligned}\int t \cdot e^{-i \cdot 4\pi t} dt &= \frac{-1}{4\pi \cdot i} \cdot t \cdot e^{-i \cdot 4\pi t} + \frac{1}{4\pi \cdot i} \cdot \int e^{-i \cdot 4\pi t} dt = \frac{-1}{4\pi \cdot i} \cdot t \cdot e^{-i \cdot 4\pi t} + \frac{1}{16\pi^2} \cdot e^{-i \cdot 4\pi t} = \\ \left[\begin{array}{l} g(t) = t \\ f'(t) = e^{-i \cdot 4\pi t} \end{array} \Rightarrow \begin{array}{l} g'(t) = 1 \\ f(t) = \frac{1}{-4\pi \cdot i} \cdot e^{-i \cdot 4\pi t} \end{array} \right] &= \frac{-1}{4\pi \cdot i} \cdot \left(t + \frac{1}{4\pi i} \right) \cdot e^{-i \cdot 4\pi t}\end{aligned}$$

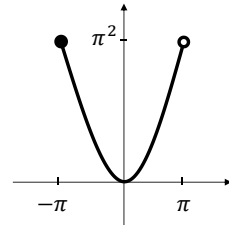
>>>Exercise 3.3 – 5

Calculate the Fourier coefficients of the 2π -periodic function x defined as

$$x(t) = t^2, \quad -\pi \leq t < \pi$$

with respect to the orthonormal exponential system.

Use the Parseval equality to give the sum $\sum_{k=1}^{\infty} \frac{1}{k^4}$.



>>>Exercise 3.3 – 5– Solution

$$\hat{X}_0 = \int_{-\pi}^{\pi} t^2 \cdot \frac{1}{\sqrt{2\pi}} dt = \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{3} \cdot \pi^3$$

If $k \neq 0$

$$\begin{aligned}\hat{X}_k &= \int_{-\pi}^{\pi} t^2 \cdot \left(\frac{1}{\sqrt{2\pi}} \cdot e^{-i \cdot k \cdot t} \right) dt = \frac{1}{\sqrt{2\pi}} \cdot \left[\left(\frac{1}{k} \cdot i \cdot t^2 + \frac{2}{k^2} \cdot t - \frac{2}{k^3} \cdot i \right) \cdot e^{-i \cdot k \cdot t} \right]_{-\pi}^{\pi} = \\ &= \frac{1}{\sqrt{2\pi}} \cdot \left(\left(\frac{1}{k} \cdot i \cdot \pi^2 + \frac{2}{k^2} \cdot \pi - \frac{2}{k^3} \cdot i \right) \cdot e^{-i \cdot k \cdot \pi} - \left(\frac{1}{k} \cdot i \cdot \pi^2 - \frac{2}{k^2} \cdot \pi - \frac{2}{k^3} \cdot i \right) \cdot e^{-i \cdot k \cdot (-\pi)} \right) = \\ &= \frac{4\pi}{\sqrt{2\pi}} \cdot (-1)^k \cdot \frac{1}{k^2}\end{aligned}$$

Details of the calculation (integration by parts):

$$\int t^2 \cdot e^{-i \cdot k \cdot t} dt = \frac{1}{-k \cdot i} \cdot t^2 \cdot e^{-i \cdot k \cdot t} + \frac{2}{k \cdot i} \cdot \int t \cdot e^{-i \cdot k \cdot t} dt =$$

$$\begin{aligned} \left[\begin{array}{l} g(t) = t^2 \\ f'(t) = e^{-i \cdot k \cdot t} \end{array} \Rightarrow \begin{array}{l} g'(t) = 2t \\ f(t) = \frac{1}{-k \cdot i} \cdot e^{-i \cdot k \cdot t} \end{array} \right] & \left[\begin{array}{l} g(t) = t \\ f'(t) = e^{-i \cdot k \cdot t} \end{array} \Rightarrow \begin{array}{l} g'(t) = 1 \\ f(t) = \frac{1}{-k \cdot i} \cdot e^{-i \cdot k \cdot t} \end{array} \right] \\ &= \frac{1}{-k \cdot i} \cdot t^2 \cdot e^{-i \cdot k \cdot t} + \frac{2}{k \cdot i} \cdot \left(\frac{1}{-k \cdot i} \cdot t \cdot e^{-i \cdot k \cdot t} + \frac{1}{k \cdot i} \cdot \int e^{-i \cdot k \cdot t} dt \right) = \\ &= \frac{1}{-k \cdot i} \cdot t^2 \cdot e^{-i \cdot k \cdot t} + \frac{2}{k^2} \cdot t \cdot e^{-i \cdot k \cdot t} + \frac{2}{k^3 \cdot i} \cdot e^{-i \cdot k \cdot t} = \left(\frac{1}{k} \cdot i \cdot t^2 + \frac{2}{k^2} \cdot t - \frac{2}{k^3} \cdot i \right) \cdot e^{-i \cdot k \cdot t} \end{aligned}$$

We used that

$$e^{-i \cdot k \cdot \pi} = \cos(-k \cdot \pi) + i \cdot \sin(-k \cdot \pi) = \cos(k \cdot \pi) = (-1)^k$$

$$e^{i \cdot k \cdot \pi} = \cos(k \cdot \pi) + i \cdot \sin(k \cdot \pi) = \cos(k \cdot \pi) = (-1)^k$$

According to the Parseval's equality

$$\|x\|^2 = \int_{-\pi}^{\pi} t^4 dt = \sum_{k=-\infty}^{\infty} \hat{X}_k^2 = \hat{X}_0^2 + 2 \cdot \sum_{k=1}^{\infty} \hat{X}_k^2 = \frac{2}{9} \cdot \pi^5 + 2 \cdot 8\pi \cdot \sum_{k=1}^{\infty} \frac{1}{k^4}$$

Since $\int_{-\pi}^{\pi} t^4 dt = \frac{1}{5} \cdot [t^5]_{-\pi}^{\pi} = \frac{2}{5} \cdot \pi^5$ we have

$$\frac{2}{5} \cdot \pi^5 = \frac{2}{9} \cdot \pi^5 + 16\pi \cdot \sum_{k=1}^{\infty} \frac{1}{k^4} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$$

3.3.2 The Exponential System

System of functions

$$\left\{ e^{i \cdot k \cdot \frac{2\pi}{T} \cdot t} \right\}_{k \in \mathbb{Z}}$$

is orthogonal (but not orthonormal) in $L_2([0, T])$. It is called *exponential system*.

The (complex) Fourier coefficients of function $x \in L_2([0, T])$ with respect to the exponential system are

$$\hat{x}_k = \frac{1}{T} \cdot \int_0^T x(t) \cdot e^{-i \cdot k \cdot \frac{2\pi}{T} \cdot t} dt, \quad k \in \mathbb{Z},$$

the Fourier series of x is

$$\mathcal{FS}(x)(t) = \sum_{k=-\infty}^{\infty} \left(\hat{x}_k \cdot e^{i \cdot k \cdot \frac{2\pi}{T} \cdot t} \right).$$

Functions $k \rightarrow |\hat{x}_k|$, $k \rightarrow |\hat{x}_k|^2$, and $k \rightarrow \arg \hat{x}_k$ are called *amplitude spectrum*, *energy spectrum* and *phase spectrum*, respectively.

>>>Exercise 3.3 – 6

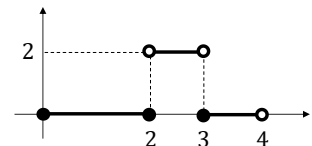
Calculate the Fourier coefficient $\hat{x}_5$ of the 4-periodic function x defined as

$$x(t) = \begin{cases} 2, & \text{if } 2 < t < 3 \\ 0, & \text{otherwise on } [0, 4[\end{cases}$$

with respect to the exponential system.

>>>Exercise 3.3 – 6– Solution

$$\hat{x}_5 = \frac{1}{4} \cdot \int_2^3 2 \cdot e^{-i \cdot 5 \cdot \frac{2\pi}{4} \cdot t} dt = \frac{1}{4} \cdot \frac{-2}{5\pi \cdot i} \cdot \left[e^{-i \cdot \frac{5\pi}{2} \cdot t} \right]_0^3 =$$



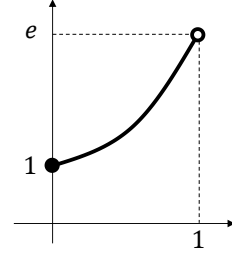
$$= \frac{-1}{10\pi \cdot i} \cdot (e^{-i \cdot 10\pi} - 1) = \frac{-1}{10\pi \cdot i} \cdot (\cos(-10\pi) + i \cdot \sin(-10\pi) - 1) = 0$$

>>>Exercise 3.3 – 7

Calculate the Fourier coefficient $\hat{x}_3$ of the 1-periodic function x defined as

$$x(t) = e^t, \quad 0 \leq t < 1$$

with respect to the exponential system.



>>>Exercise 3.3 – 7– Solution

$$\begin{aligned} \hat{x}_3 &= \int_0^1 e^t \cdot e^{-i \cdot 3 \cdot 2\pi \cdot t} dt = \int_0^1 e^{(1-6\pi i) \cdot t} dt = \frac{1}{1-6\pi i} \cdot [e^{(1-6\pi i) \cdot t}]_0^1 = \frac{1}{1-6\pi i} \cdot (e^{1-6\pi i} - 1) = \\ &= \frac{1}{1-6\pi i} \cdot (e \cdot e^{-6\pi i} - 1) = \frac{e}{1-6\pi i} \cdot (\cos(-6\pi) + i \cdot \sin(-6\pi)) - \frac{1}{1-6\pi i} = \frac{e-1}{1-6\pi i} \end{aligned}$$

3.4 Real and Complex Fourier Coefficients

If $x \in L_2([0, T])$ is a real-valued function, we have

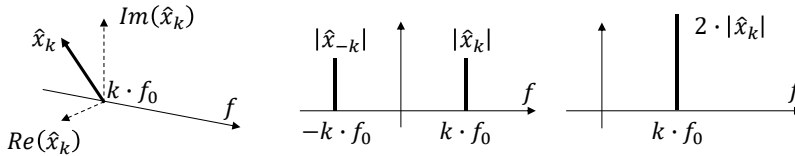
$$\hat{x}_{-k} = \hat{x}_k^* \quad k \in \mathbb{Z}$$

and, consequently

$$|\hat{x}_{-k}| = |\hat{x}_k|, \quad k \in \mathbb{Z}$$

showing that the complex spectrum has symmetric nature and the fact that the Fourier coefficients of a real-valued function belonging to ‘negative frequencies’ have not independent meaning.

The complex spectrum is displayed in different ways. We can draw a “3D” diagram showing the complex values (the real and the imaginary part of the coefficients), or we can plot only the values $|\hat{x}_k|$, and finally we can plot values $2 \cdot |\hat{x}_k|$ on the non-negative frequency axis.



Consider the orthonormal trigonometric system

$$\left\{ \text{CONST}(t) = \frac{1}{\sqrt{T}}, \text{COS}_k(t) = \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), \text{SIN}_k(t) = \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \right\}_{k \in \mathbb{N}}$$

and the orthonormal exponential system

$$\left\{ \text{EXP}_k(t) = \frac{1}{\sqrt{T}} \cdot e^{i \cdot k \cdot \frac{2\pi}{T} \cdot t} \right\}_{k \in \mathbb{Z}}$$

in $L_2([0, T])$. Since both the real and the complex Fourier coefficients $(\hat{A}_k, \hat{B}_k, \hat{X}_k)$ belong to frequency $k \cdot \frac{2\pi}{T}$, they are expected to be connected. In fact

$$\hat{X}_0 = \hat{A}_0,$$

furthermore the properties of sine, cosine and exponential functions imply that for $k \in \mathbb{Z}, k > 0$ we have

$$\hat{X}_k = \frac{1}{\sqrt{2}} \cdot (\hat{A}_k - \hat{B}_k \cdot i), \quad \hat{X}_{-k} = \frac{1}{\sqrt{2}} \cdot (\hat{A}_k + \hat{B}_k \cdot i), \quad \text{and} \quad |\hat{X}_k| = \frac{1}{\sqrt{2}} \cdot \sqrt{\hat{A}_k^2 + \hat{B}_k^2}.$$

Considering the trigonometric system

$$\left\{1, \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)\right\}_{k \in \mathbb{N}}$$

and the exponential system

$$\left\{e^{i \cdot k \cdot \frac{2\pi}{T} \cdot t}\right\}_{k \in \mathbb{Z}}$$

the connection between the real and complex Fourier coefficients $(\hat{a}_k, \hat{b}_k, \hat{x}_k)$ is as follows:

$$\hat{x}_0 = \hat{a}_0,$$

furthermore, for $k \in \mathbb{Z}, k > 0$, we have

$$\hat{x}_k = \frac{1}{2} \cdot (\hat{a}_k - \hat{b}_k \cdot i), \quad \hat{x}_{-k} = \frac{1}{2} \cdot (\hat{a}_k + \hat{b}_k \cdot i), \quad |\hat{x}_k| = \frac{1}{2} \cdot \sqrt{\hat{a}_k^2 + \hat{b}_k^2}.$$

>>>Exercise 3.4 – 1

Using the Euler formula $e^{i \cdot t} = \cos t + i \cdot \sin t, t \in \mathbb{R}$ show that

$$\hat{X}_k = \frac{1}{\sqrt{2}} \cdot (\hat{A}_k - \hat{B}_k \cdot i), \quad \hat{X}_{-k} = \frac{1}{\sqrt{2}} \cdot (\hat{A}_k + \hat{B}_k \cdot i), \quad \text{and} \quad |\hat{X}_k| = \frac{1}{\sqrt{2}} \cdot \sqrt{\hat{A}_k^2 + \hat{B}_k^2}, \quad k \in \mathbb{Z}, k > 0$$

Then express $\hat{X}_k + \hat{X}_{-k}$ and $i \cdot (\hat{X}_k - \hat{X}_{-k}), k \in \mathbb{Z}, k > 0$.

>>>Exercise 3.4 – 1 – Solution

Let $k \in \mathbb{Z}, k > 0$.

$$\begin{aligned} \hat{X}_k &= \int_0^T x(t) \cdot \left(\frac{1}{\sqrt{T}} \cdot e^{-i \cdot k \cdot \frac{2\pi}{T} \cdot t}\right) dt = \\ &= \int_0^T x(t) \cdot \frac{1}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt - i \cdot \int_0^T x(t) \cdot \frac{1}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt = \\ &= \frac{1}{\sqrt{2}} \cdot \int_0^T x(t) \cdot \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt - \frac{1}{\sqrt{2}} \cdot i \cdot \int_0^T x(t) \cdot \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt = \frac{1}{\sqrt{2}} \cdot \hat{A}_k - \frac{1}{\sqrt{2}} \cdot i \cdot \hat{B}_k \end{aligned}$$

The sine function is odd while the cosine function is even thus

$$\begin{aligned} \hat{X}_{-k} &= \int_0^T x(t) \cdot \frac{1}{\sqrt{T}} \cdot \cos\left(-k \cdot \frac{2\pi}{T} \cdot t\right) dt - i \cdot \int_0^T x(t) \cdot \frac{1}{\sqrt{T}} \cdot \sin\left(-k \cdot \frac{2\pi}{T} \cdot t\right) dt = \\ &= \int_0^T x(t) \cdot \frac{1}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt + i \cdot \int_0^T x(t) \cdot \frac{1}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt = \\ &= \frac{1}{\sqrt{2}} \cdot \int_0^T x(t) \cdot \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt + \frac{1}{\sqrt{2}} \cdot i \cdot \int_0^T x(t) \cdot \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt = \frac{1}{\sqrt{2}} \cdot \hat{A}_k + \frac{1}{\sqrt{2}} \cdot i \cdot \hat{B}_k \end{aligned}$$

$$\hat{X}_k + \hat{X}_{-k} = \sqrt{2} \cdot \hat{A}_k, \quad k \in \mathbb{Z}, k > 0$$

$$i \cdot (\hat{X}_k - \hat{X}_{-k}) = \sqrt{2} \cdot \hat{B}_k, \quad k \in \mathbb{Z}, k > 0$$

In the formula $\hat{X}_k = \frac{1}{\sqrt{2}} \cdot \hat{A}_k - \frac{1}{\sqrt{2}} \cdot i \cdot \hat{B}_k$ we can see that $\text{Re}(\hat{X}_k) = \frac{1}{\sqrt{2}} \cdot \hat{A}_k$ and $\text{Im}(\hat{X}_k) = -\frac{1}{\sqrt{2}} \cdot \hat{B}_k$ thus

$$|\hat{X}_k| = \sqrt{\frac{1}{2} \cdot \hat{A}_k^2 + \frac{1}{2} \cdot \hat{B}_k^2} = \frac{1}{\sqrt{2}} \cdot \sqrt{\hat{A}_k^2 + \hat{B}_k^2}$$

>>>Exercise 3.4 – 2

Using the formulas obtained in the previous exercise, manipulate the Fourier series of a function $x \in L_2([0, T])$ with respect to the orthonormal exponential system to get the Fourier series of x with respect to the orthonormal trigonometric system.

>>>Exercise 3.4 – 2- Solution

$$\begin{aligned} \mathcal{FS}(x) &= \sum_{k=-\infty}^{\infty} (\hat{X}_k \cdot \text{EXP}_k) = \sum_{k=-\infty}^{\infty} \hat{X}_k \cdot \left(\frac{1}{\sqrt{T}} \cdot e^{i \cdot k \cdot \frac{2\pi}{T} \cdot t} \right) = \\ &= \sum_{k=-\infty}^{\infty} \left(\hat{X}_k \cdot \frac{1}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \right) + i \cdot \sum_{k=-\infty}^{\infty} \left(\hat{X}_k \cdot \frac{1}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \right) = \\ &= \hat{X}_0 \cdot \frac{1}{\sqrt{T}} + \sum_{k=1}^{\infty} \left((\hat{X}_k + \hat{X}_{-k}) \cdot \frac{1}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \right) + \sum_{k=1}^{\infty} \left(i \cdot (\hat{X}_k - \hat{X}_{-k}) \cdot \frac{1}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \right) = \\ &= \hat{A}_0 \cdot \frac{1}{\sqrt{T}} + \sum_{k=1}^{\infty} \left(\hat{A}_k \cdot \frac{\sqrt{2}}{\sqrt{T}} \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) \right) + \sum_{k=1}^{\infty} \left(\hat{B}_k \cdot \frac{\sqrt{2}}{\sqrt{T}} \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \right) = \\ &= \hat{A}_0 \cdot \text{CONST} + \sum_{k=1}^{\infty} \hat{A}_k \cdot \text{COS}_k(t) + \sum_{k=1}^{\infty} \hat{B}_k \cdot \text{SIN}_k(t) \end{aligned}$$

>>>Exercise 3.4 – 3

Using the Euler formula $e^{i \cdot t} = \cos t + i \cdot \sin t, t \in \mathbb{R}$ show that

$$\hat{x}_k = \frac{1}{2} \cdot (\hat{a}_k - \hat{b}_k \cdot i), \quad \hat{X}_{-k} = \frac{1}{2} \cdot (\hat{a}_k + \hat{b}_k \cdot i), \quad |\hat{X}_k| = \frac{1}{2} \cdot \sqrt{\hat{a}_k^2 + \hat{b}_k^2}.$$

>>>Exercise 3.4 – 3- Solution

Let $k \in \mathbb{Z}, k > 0$.

$$\begin{aligned} \hat{x}_k &= \frac{1}{T} \cdot \int_0^T x(t) \cdot e^{-i \cdot k \cdot \frac{2\pi}{T} \cdot t} dt = \frac{1}{T} \cdot \int_0^T x(t) \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt - i \cdot \frac{1}{T} \cdot \int_0^T x(t) \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt = \\ &= \frac{1}{2} \cdot \frac{2}{T} \cdot \int_0^T x(t) \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt - \frac{1}{2} \cdot i \cdot \frac{2}{T} \cdot \int_0^T x(t) \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt = \frac{1}{2} \cdot \hat{a}_k - \frac{1}{2} \cdot i \cdot \hat{b}_k. \end{aligned}$$

The sine function is odd while the cosine function is even thus

$$\begin{aligned} \hat{x}_{-k} &= \frac{1}{T} \cdot \int_0^T x(t) \cdot \cos\left(-k \cdot \frac{2\pi}{T} \cdot t\right) dt - i \cdot \frac{1}{T} \cdot \int_0^T x(t) \cdot \sin\left(-k \cdot \frac{2\pi}{T} \cdot t\right) dt = \\ &= \frac{1}{T} \cdot \int_0^T x(t) \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt + i \cdot \frac{1}{T} \cdot \int_0^T x(t) \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt = \\ &= \frac{1}{2} \cdot \frac{2}{T} \cdot \int_0^T x(t) \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt + \frac{1}{2} \cdot i \cdot \frac{2}{T} \cdot \int_0^T x(t) \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) dt = \frac{1}{2} \cdot \hat{a}_k + \frac{1}{2} \cdot i \cdot \hat{b}_k. \end{aligned}$$

In the formula $\hat{x}_k = \frac{1}{2} \cdot \hat{a}_k - \frac{1}{2} \cdot i \cdot \hat{b}_k$ we can see that $\text{Re}(\hat{x}_k) = \frac{1}{2} \cdot \hat{a}_k$ and $\text{Im}(\hat{x}_k) = -\frac{1}{2} \cdot \hat{b}_k$ thus

$$|\hat{x}_k| = \sqrt{\frac{1}{4} \cdot \hat{a}_k^2 + \frac{1}{4} \cdot \hat{b}_k^2} = \frac{1}{2} \cdot \sqrt{\hat{a}_k^2 + \hat{b}_k^2}.$$

4 Fourier Transform

4.1 Integral Transforms, Convolution

Let $K: \mathbb{C} \times \mathbb{R} \rightarrow \mathbb{C}$ be a given integrable function. Function

$$F(s) = \int_a^b f(t) \cdot K(s, t) dt, \quad s \in \mathbb{C}$$

is called the *integral transform* of function $f: [a, b] \rightarrow \mathbb{C}$ if the integral is convergent.

The *Fourier transform* and the *Laplace transform* are two well-known integral transforms, which are frequently used in different fields of engineering and sciences. Some special transformations appear in special applications, e.g. the wavelet transform is important tool, for example in technical diagnostics.

Some transformations (e.g. Fourier and wavelet) have so-called continuous and discrete forms. The discrete transformations are related to the discrete signal processing where only a sampled signal is available rather than the formula of the function (signal) analysed.

A basic concept in signal analysis is the convolution. It has different forms depending on the field of the application. *Convolution* of integrable functions $x: \mathbb{R} \rightarrow \mathbb{R}$ and $h: \mathbb{R} \rightarrow \mathbb{R}$ is defined as

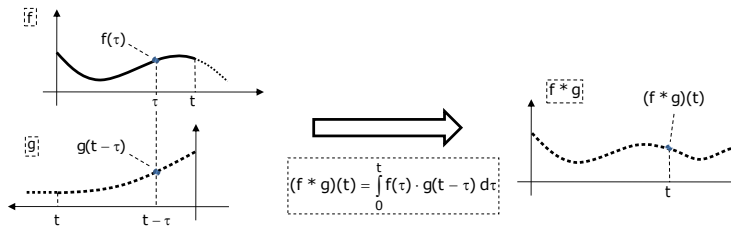
$$(x * h)(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t - \tau) d\tau, \quad t \in \mathbb{R}.$$

The convolution formula can be considered as a weighted average of the function $x(\tau)$ at the moment t where the weighting is given by $h(t - \tau)$. As t changes, the weighting function emphasizes different parts of the input function.

For functions x and h supported only on the non-negative real line a modified formula is used. *Convolution* of integrable functions $x: [0, \infty[\rightarrow \mathbb{R}$ and $h: [0, \infty[\rightarrow \mathbb{R}$ is

$$(x * h)(t) = \int_0^t x(\tau) \cdot h(t - \tau) d\tau, \quad t \in [0, \infty[.$$

Remark: If $h(t), t \in \mathbb{R}$ is the impulse response (response to the delta function input) of a linear time-invariant system, then the response of the system for an input $x(t), t \in \mathbb{R}$ is $y(t) = (x * h)(t), t \in \mathbb{R}$.



Convolution of two discrete-time signals $\{x[n]\}_{n \in \mathbb{N}}$ and $\{h[n]\}_{n \in \mathbb{N}}$ is

$$(x * h)[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n - k], \quad n \in \mathbb{N}.$$

Remark: If $h[n]$ is the impulse response (response to the delta function input) of a linear time-invariant system, then the response of the system for a discrete input $\{x[n]\}_{n \in \mathbb{N}}$ is

$$y[n] = (x * h)[n], \quad n \in \mathbb{N}.$$

>>>Exercise 4.1 – 1

Determine the convolution of functions $f: [0, \infty[\rightarrow \mathbb{R}$ and $g: [0, \infty[\rightarrow \mathbb{R}$ defined as

$$f(t) = e^{-2 \cdot t}, \quad g(t) = e^{5 \cdot t}.$$

>>>Exercise 4.1 – 1- Solution

$$\begin{aligned} (f * g)(t) &= e^{-2 \cdot t} * e^{5 \cdot t} = \int_0^t e^{-2 \cdot \tau} \cdot e^{5 \cdot (t - \tau)} d\tau = e^{5 \cdot t} \cdot \int_0^t e^{-7 \cdot \tau} d\tau = \\ &= \frac{-1}{7} \cdot e^{5 \cdot t} \cdot [e^{-7 \cdot \tau}]_0^t = \frac{-1}{7} \cdot e^{5 \cdot t} \cdot (e^{-7 \cdot t} - 1) = \frac{-1}{7} \cdot e^{-2 \cdot t} + \frac{1}{7} e^{5 \cdot t} \end{aligned}$$

>>>Exercise 4.1 – 2

Determine the convolution of functions $x: [0, \infty[\rightarrow \mathbb{R}$ and $h: [0, \infty[\rightarrow \mathbb{R}$ defined as

$$x(t) = 5t - 3, \quad h(t) = e^{-\frac{1}{2}t}.$$

>>>Exercise 4.1 – 2- Solution

$$\begin{aligned} (x * h)(t) &= (5t - 3) * e^{-\frac{1}{2}t} = \int_0^t (5\tau - 3) \cdot e^{-(t - \tau)} d\tau = \\ &= [(5\tau - 8) \cdot e^{-(t - \tau)}]_{\tau=0}^t = 5t - 8 + 8 \cdot e^{-t} \end{aligned}$$

Details of the calculation (integration by parts):

$$\begin{aligned} \int (5\tau - 3) \cdot e^{-(t - \tau)} d\tau &= (5\tau - 3) \cdot e^{-(t - \tau)} - 5 \cdot \int e^{-(t - \tau)} d\tau = \\ \begin{bmatrix} g(\tau) = 5\tau - 3 \\ f'(\tau) = e^{-(t - \tau)} \end{bmatrix} &\Rightarrow \begin{bmatrix} g'(\tau) = 5 \\ f(\tau) = e^{-(t - \tau)} \end{bmatrix} \\ &= (5\tau - 3) \cdot e^{-(t - \tau)} - 5 \cdot e^{-(t - \tau)} = (5\tau - 8) \cdot e^{-(t - \tau)} \end{aligned}$$

>>>Exercise 4.1 – 3

Determine the convolution $\Pi * \Pi$ where $\Pi(t) = \begin{cases} 1 & \text{if } |t| \leq 1 \\ 0 & \text{if } |t| > 1 \end{cases}$ is the rectangular pulse function on $[-1, 1]$.

>>>Exercise 4.1 – 3- Solution

$$\Pi * \Pi(t) = \int_{-\infty}^{\infty} \Pi(\tau) \cdot \Pi(t - \tau) d\tau$$

The integral is different from zero when both $\Pi(\tau)$ and $\Pi(t - \tau)$ are different from zero, that is, when $-1 \leq \tau \leq 1$ and $-1 \leq t - \tau \leq 1$. The latter system of inequalities can be written as $t - 1 \leq \tau \leq t + 1$.

The two requirements imply that the limits of the integration are $\max\{-1, t - 1\}$ and $\min\{1, t + 1\}$ assuming that $\max\{-1, t - 1\} \leq \min\{1, t + 1\}$, that is, when $-2 \leq t \leq 2$.

	-2	0	2		
	-1	t - 1			
max{-1, t - 1}					
	t + 1	1			
min{1, t + 1}					

$$\Pi * \Pi(t) = \int_{\max\{-1, t - 1\}}^{\min\{1, t + 1\}} \Pi(\tau) \cdot \Pi(t - \tau) d\tau$$

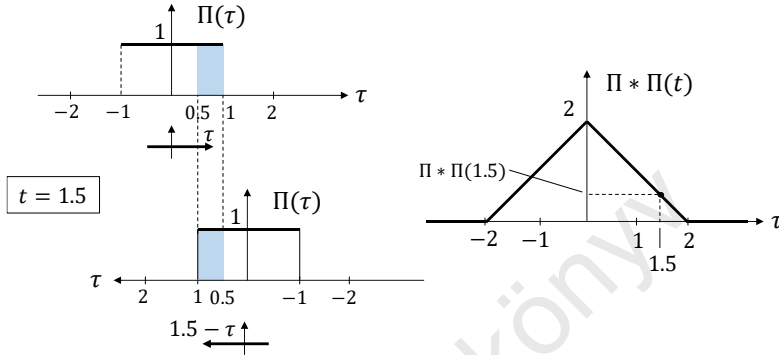
If $-2 \leq t \leq 0$, then

$$\Pi * \Pi(t) = \int_{\tau=-1}^{t+1} 1 d\tau = [\tau]_{\tau=-1}^{t+1} = t + 2$$

If $0 \leq t \leq 2$, then

$$\Pi * \Pi(t) = \int_{\tau=t-1}^1 1 d\tau = [\tau]_{\tau=t-1}^1 = -t + 2$$

The following figure, as an example, shows the calculation of $\Pi(1.5)$:



The convolution is the triangle function $\text{tri } t = \begin{cases} t + 2 & \text{if } -2 \leq t \leq 0 \\ -t + 2 & \text{if } 0 \leq t \leq 2 \\ 0 & \text{if } |t| > 2 \end{cases}$.

>>>Exercise 4.1 – 4

Determine the convolution of functions $x: \mathbb{N} \rightarrow \mathbb{R}$ and $h: \mathbb{N} \rightarrow \mathbb{R}$ defined as

n	0	1	2	3
$x[n]$	4	1	2	5

n	0	1	2
$h[n]$	1	2	-1

$x[n] = 0, h[n] = 0$ otherwise.

>>>Exercise 4.1 – 4– Solution

$$(x * h)[0] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[-k] = x[0] \cdot h[0] = 4 \cdot 1 = 4$$

$$(x * h)[1] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[1-k] = x[0] \cdot h[1] + x[1] \cdot h[0] = 4 \cdot 2 + 1 \cdot 1 = 9$$

$$(x * h)[2] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[2-k] = x[0] \cdot h[2] + x[1] \cdot h[1] + x[2] \cdot h[0] =$$

$$= 4 \cdot (-1) + 1 \cdot 2 + 2 \cdot 1 = 0$$

$$(x * h)[3] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[3-k] = x[1] \cdot h[2] + x[2] \cdot h[1] + x[3] \cdot h[0] =$$

$$= 1 \cdot (-1) + 2 \cdot 2 + 5 \cdot 1 = 8$$

$$(x * h)[4] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[4-k] = x[2] \cdot h[2] + x[3] \cdot h[1] = 2 \cdot (-1) + 5 \cdot 2 = 8$$

$$(x * h)[5] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[5 - k] = x[3] \cdot h[2] = 5 \cdot (-1) = -5$$

$(x * h)[n] = 0$ otherwise.

n	...	-1	0	1	2	3	4	5	6	...
$(x * h)[n]$	...	0	4	9	0	8	8	-5	0	...

4.2 Continuous Fourier Transform

4.2.1 Fourier Transform, Fourier Integral

Function

$$\mathcal{FT}(x)(\omega) = \hat{x}(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-i\omega t} dt, \quad \omega \in \mathbb{R}$$

is the *Fourier transform* of function $x: \mathbb{R} \rightarrow \mathbb{R}$ if the integral is convergent.

The *Fourier integral* of $x: \mathbb{R} \rightarrow \mathbb{R}$ is

$$\mathcal{FI}(x)(t) = \mathcal{FT}^{-1}(\hat{x})(t) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \hat{x}(\omega) \cdot e^{i\omega t} d\omega, \quad t \in \mathbb{R}.$$

Functions

$$\omega \rightarrow |\hat{x}(\omega)|, \quad \omega \rightarrow |\hat{x}(\omega)|^2, \quad \text{and} \quad \omega \rightarrow \angle \hat{x}(\omega)$$

are called *amplitude spectrum*, *energy spectrum* and *phase spectrum*, respectively, in technical literature.

Remark: Different formulas can be found in different textbooks for the Fourier transform and for the Fourier integral. Here we use the 'angular frequency' ω as a variable and we write coefficient $\frac{1}{2\pi}$ in the Fourier integral rather than in the Fourier transform to harmonize it with the formula of the Laplace transform. Instead of ω the 'frequency' f can be used as a variable and the coefficients in the two formulas can be written in different way.

Remark: The Fourier coefficients of periodic functions have discrete nature, values of $\hat{x}_k$ are linked to the 'discrete angular frequencies' $k \cdot \omega_0 = k \cdot \frac{2\pi}{T}$, this is why this spectrum is called 'discrete'. Furthermore, the Fourier series gives the function in the form of a (countable) infinite summation.

The spectrum of non-periodic functions provided by the Fourier transform is called continuous since the domain of the Fourier transform is the real line, that is, any real 'angular frequencies' can appear in the spectrum. Instead of summation there is an integration in the Fourier integral.

Using the Euler's formula $e^{i \cdot t} = \cos t + i \cdot \sin t, t \in \mathbb{R}$ we can write the Fourier transform of $x: \mathbb{R} \rightarrow \mathbb{R}$ as

$$\hat{x}(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-i\omega t} dt =$$

$$\begin{aligned}
&= \int_{t=-\infty}^{\infty} x(t) \cdot \cos(-\omega \cdot t) dt + i \cdot \int_{t=-\infty}^{\infty} x(t) \cdot \sin(-\omega \cdot t) dt = \\
&= \int_{t=-\infty}^{\infty} x(t) \cdot \cos(\omega \cdot t) dt - i \cdot \int_{t=-\infty}^{\infty} x(t) \cdot \sin(\omega \cdot t) dt = \hat{a}(\omega) - i \cdot \hat{b}(\omega), \quad \omega \in \mathbb{R}.
\end{aligned}$$

When x is even, then $\hat{b}_x = 0$ and we have

$$\hat{x}(\omega) = \int_{t=-\infty}^{\infty} x(t) \cdot \cos(\omega \cdot t) dt = 2 \cdot \int_{t=0}^{\infty} x(t) \cdot \cos(\omega \cdot t) dt, \quad \omega \in \mathbb{R}.$$

When x is odd, then $\hat{a}_x = 0$ and we have

$$\hat{x}(\omega) = -i \cdot \int_{t=-\infty}^{\infty} x(t) \cdot \sin(\omega \cdot t) dt = -i \cdot 2 \cdot \int_{t=0}^{\infty} x(t) \cdot \sin(\omega \cdot t) dt, \quad \omega \in \mathbb{R}.$$

Integrals

$$\mathcal{FT}_{\cos}(x)(\omega) = 2 \cdot \int_{t=0}^{\infty} x(t) \cdot \cos(\omega \cdot t) dt, \quad \omega \in \mathbb{R}, \omega \geq 0$$

and

$$\mathcal{FT}_{\sin}(x)(\omega) = 2 \cdot \int_{t=0}^{\infty} x(t) \cdot \sin(\omega \cdot t) dt, \quad \omega \in \mathbb{R}, \omega \geq 0$$

are called the *cosine Fourier transform* and the *sine Fourier transform* of function $x: [0, \infty[\rightarrow \mathbb{R}$.

The *Fourier cosine integral* of x is

$$\mathcal{FI}_{\cos}(x)(t) = \frac{1}{\pi} \cdot \int_{\omega=0}^{\infty} \mathcal{FT}_{\cos}(x)(\omega) \cdot \cos(\omega \cdot t) d\omega, \quad t \in \mathbb{R}, t \geq 0,$$

while the *Fourier sine integral* of x is

$$\mathcal{FI}_{\sin}(x)(t) = \frac{1}{\pi} \cdot \int_{\omega=0}^{\infty} \mathcal{FT}_{\sin}(x)(\omega) \cdot \sin(\omega \cdot t) d\omega, \quad t \in \mathbb{R}, t \geq 0.$$

Each real function $x: \mathbb{R} \rightarrow \mathbb{R}$ (having Fourier transform) can be analysed with its cosine and sine Fourier transform since x can be written as

$$x(t) = \frac{x(t) + x(-t)}{2} + \frac{x(t) - x(-t)}{2} = g(t) + h(t), \quad t \in \mathbb{R}.$$

where g is even and h is odd. Thus

$$\mathcal{FT}(x) = \mathcal{FT}(g) + \mathcal{FT}(h) = \mathcal{FT}_{\cos}(g) - i \cdot \mathcal{FT}_{\sin}(h)$$

If function x is piecewise continuous then $\mathcal{FI}(x)$ is equal to x wherever x is continuous, and $\mathcal{FI}(x)$ is the average the left- and right-hand limits wherever x is discontinuous.

Remark: Since a piecewise continuous function (signal) can be reconstructed from its Fourier transform (through its Fourier integral) we can say that the Fourier transform contains all information about the function, and can be considered as an alternative representation. For instance, a vibration process can be described in the 'time domain' (e.g. vibration velocity vs. time function) and also in 'frequency domain' (e.g. vibration frequency spectrum).

Parseval's equality (energy of a signal):

$$\int_{t=-\infty}^{\infty} x^2(t) dt = \frac{1}{2\pi} \cdot \int_{\omega=-\infty}^{\infty} |\hat{x}(\omega)|^2 d\omega.$$

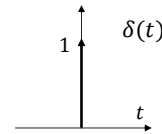
The following table shows some functions with their Fourier transforms. We can consider these pairs as properties of the Fourier transform showing how will change the Fourier transform (in the frequency domain) when the function is changed in the time domain, and vice versa. For $\alpha, \beta, T, \omega_0 \in \mathbb{R}$

	time domain	frequency domain
	$t \rightarrow x(t) = \mathcal{FT}^{-1}(\hat{x})(t)$	$\omega \rightarrow \hat{x}(\omega) = \mathcal{FT}(x)(\omega)$
linearity	$t \rightarrow \alpha \cdot x(t) + \beta \cdot y(t)$	$\omega \rightarrow \alpha \cdot \hat{x}(\omega) + \beta \cdot \hat{y}(\omega)$
shift in the time domain	$t \rightarrow x(t - T)$	$\omega \rightarrow \hat{x}(\omega) \cdot e^{-i \cdot T \cdot \omega}$
shift in the frequency domain (modulation)	$t \rightarrow x(t) \cdot e^{i \cdot \omega_0 \cdot t}$	$\omega \rightarrow \hat{x}(\omega - \omega_0)$
scaling	$t \rightarrow x(\alpha \cdot t)$	$\omega \rightarrow \frac{1}{ \alpha } \cdot \hat{x}\left(\frac{\omega}{\alpha}\right)$
convolution	$t \rightarrow (x * y)(t)$	$\omega \rightarrow \hat{x}(\omega) \cdot \hat{y}(\omega)$

In the theory of integral transforms *unit impulse function* (or *Dirac delta function*) has important role. Dirac delta is not a common function, its value is different from zero only at 0, and its integral on $\mathbb{R}$ is 1:

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$



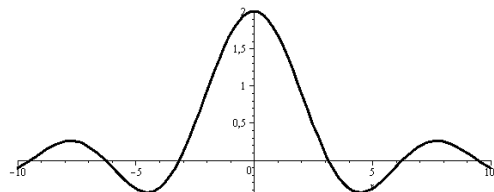
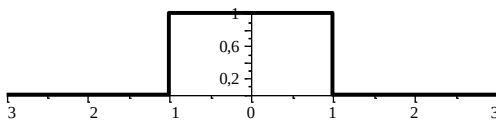
>>>Exercise 4.2 – 1

Determine the complex Fourier transform and the Fourier integral of the rectangular pulse function

$$x(t) = \Pi(t) = \begin{cases} 1 & \text{if } |t| \leq 1 \\ 0 & \text{if } |t| > 1 \end{cases}$$

>>>Exercise 4.2 – 1- Solution

$$\begin{aligned} \hat{x}(\omega) &= \int_{t=-\infty}^{\infty} x(t) \cdot e^{-i \cdot \omega \cdot t} dt = \int_{t=-1}^1 e^{-i \cdot \omega \cdot t} dt = \left[\frac{e^{-i \cdot \omega \cdot t}}{-i \cdot \omega} \right]_{t=-1}^1 = \\ &= \frac{1}{-i \cdot \omega} \cdot (e^{-i \cdot \omega} - e^{i \cdot \omega}) = \frac{2}{\omega} \cdot \frac{e^{i \cdot \omega} - e^{-i \cdot \omega}}{2i} = 2 \cdot \frac{\sin \omega}{\omega} = 2 \cdot \text{sinc } \omega \\ t \rightarrow x(t) &= \Pi(t) \qquad \qquad \qquad \omega \rightarrow \hat{x}(\omega) = 2 \cdot \frac{\sin \omega}{\omega} \end{aligned}$$



The Fourier integral of x is

$$\mathcal{FI}(x)(t) = \frac{1}{2\pi} \cdot \int_{\omega=-\infty}^{\infty} \hat{x}(\omega) \cdot e^{i \cdot \omega \cdot t} d\omega = \frac{1}{\pi} \cdot \int_{\omega=-\infty}^{\infty} \frac{\sin(\omega)}{\omega} \cdot e^{i \cdot \omega \cdot t} d\omega$$

>>>Exercise 4.2 – 2

Determine the sine and cosine Fourier transform of the rectangular pulse function

$$x(t) = \Pi(t) = \begin{cases} 1 & \text{if } |t| \leq 1 \\ 0 & \text{if } |t| > 1 \end{cases}$$

>>>Exercise 4.2 – 2- Solution

Since x is even $\mathcal{FT}_{\sin}(x) = 0$.

$$\mathcal{FT}_{\cos}(x)(\omega) = 2 \cdot \int_{t=0}^{\infty} x(t) \cdot \cos(\omega \cdot t) dt = 2 \cdot \int_{t=0}^1 \cos(\omega \cdot t) dt = 2 \cdot \frac{\sin \omega}{\omega}$$

The Fourier cosine integral of x is

$$\mathcal{FI}(x)(t) = \frac{2}{\pi} \cdot \int_{\omega=0}^{\infty} \frac{\sin \omega}{\omega} \cdot \cos(\omega \cdot t) d\omega$$

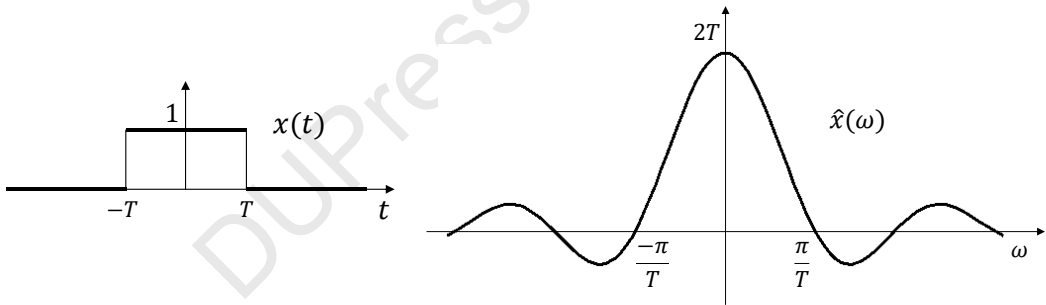
>>>Exercise 4.2 – 3

Determine the Fourier transform of the rectangular pulse function

$$x(t) = \begin{cases} 1 & \text{if } |t| \leq T \\ 0 & \text{if } |t| > T \end{cases}$$

>>>Exercise 4.2 – 3- Solution

$$\begin{aligned} \hat{x}(\omega) &= \int_{t=-T}^T e^{-i\omega \cdot t} dt = \frac{-1}{i \cdot \omega} \cdot [e^{-i\omega \cdot t}]_{t=-T}^T = \frac{-1}{i \cdot \omega} \cdot (e^{-i\omega \cdot T} - e^{i\omega \cdot T}) = \\ &= 2 \cdot \frac{1}{\omega} \cdot \frac{e^{i\omega \cdot T} - e^{-i\omega \cdot T}}{2i} = 2 \cdot \frac{\sin(\omega \cdot T)}{\omega} \end{aligned}$$



>>>Exercise 4.2 – 4

Determine the Fourier transform of the unit impulse (Dirac delta) function

>>>Exercise 4.2 – 4- Solution

$$\mathcal{FT}(\delta)(\omega) = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-i\omega \cdot t} dt = e^{-i \cdot 0 \cdot t} \cdot \int_{-\infty}^{\infty} \delta(t) \cdot dt = 1$$

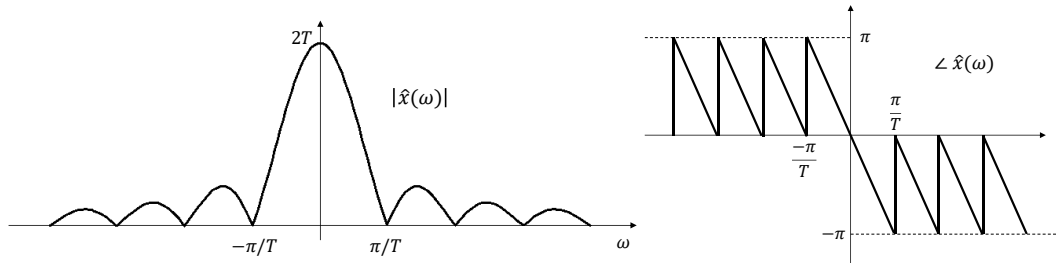
>>>Exercise 4.2 – 5

Determine the Fourier transform of the shifted rectangular pulse function

$$x(t) = \begin{cases} 1 & \text{if } 1 - T \leq t \leq 1 + T \\ 0 & \text{if } t < 1 - T \text{ or } t > 1 + T \end{cases}$$

>>>Exercise 4.2 – 5- Solution

$$\begin{aligned}\hat{x}(\omega) &= \int_{t=1-T}^{1+T} e^{-i\omega t} dt = \frac{-1}{i \cdot \omega} \cdot [e^{-i\omega t}]_{t=1-T}^{1+T} = \frac{-1}{i \cdot \omega} \cdot (e^{-i\omega(1+T)} - e^{-i\omega(1-T)}) = \\ &= \frac{-1}{i \cdot \omega} \cdot e^{-i\omega} \cdot (e^{-i\omega T} - e^{i\omega T}) = 2 \cdot e^{-i\omega} \cdot \frac{1}{\omega} \cdot \frac{e^{i\omega T} - e^{-i\omega T}}{2i} = 2 \cdot \frac{\sin(\omega \cdot T)}{\omega} \cdot e^{-i\omega}\end{aligned}$$



>>>Exercise 4.2 – 6

Determine the Fourier transform of the one-sided decaying exponential function

$$x(t) = \begin{cases} 0 & \text{if } t < 0 \\ e^{-a \cdot t} & \text{if } t \geq 0 \end{cases}, \quad a > 0$$

>>>Exercise 4.2 – 6- Solution

$$\begin{aligned}\hat{x}(\omega) &= \int_0^{\infty} e^{-a \cdot t} \cdot e^{-i\omega t} dt = \int_0^{\infty} e^{-(a+i\omega) \cdot t} dt = \lim_{b \rightarrow \infty} \int_0^b e^{-(a+i\omega) \cdot t} dt = \\ &= \frac{-1}{a+i\omega} \cdot \lim_{b \rightarrow \infty} [e^{-(a+i\omega) \cdot t}]_{t=0}^b = \frac{-1}{a+i\omega} \cdot \lim_{b \rightarrow \infty} (e^{-(a+i\omega) \cdot b} - 1) = \frac{1}{a+i\omega} = \frac{a-i\omega}{a^2+\omega^2}\end{aligned}$$

Remark: The Laplace transform of the one-sided decaying exponential function

$$x(t) = \begin{cases} 0 & \text{if } t < 0 \\ e^{-a \cdot t} & \text{if } t \geq 0 \end{cases}, \quad a > 0 \text{ is}$$

$$\begin{aligned}\mathcal{L}(x)(s) &= \int_0^{\infty} e^{-a \cdot t} \cdot e^{-s \cdot t} dt = \int_0^{\infty} e^{-(a+s) \cdot t} dt = \lim_{b \rightarrow \infty} \int_0^b e^{-(a+s) \cdot t} dt = \\ &= \frac{-1}{a+s} \cdot \lim_{b \rightarrow \infty} [e^{-(a+s) \cdot t}]_{t=0}^b = \frac{-1}{a+s} \cdot \lim_{b \rightarrow \infty} (e^{-(a+s) \cdot b} - 1) = \frac{1}{a+s}, \quad \text{Re } s > -a\end{aligned}$$

>>>Exercise 4.2 – 7- Solution

Determine the Fourier transform of the one-sided growing exponential function

$$x(t) = \begin{cases} 0 & \text{if } t < 0 \\ e^{a \cdot t} & \text{if } t \geq 0 \end{cases}, \quad a > 0$$

>>>Exercise 4.2 – 7- Solution

The Fourier transform doesn't exist.

Remark: The Laplace transform of the one-sided growing exponential function

$$x(t) = \begin{cases} 0 & \text{if } t < 0 \\ e^{a \cdot t} & \text{if } t \geq 0 \end{cases}, \quad a > 0 \text{ (see in Chapter 5) is}$$

$$\mathcal{L}(x)(s) = \int_0^{\infty} e^{a \cdot t} \cdot e^{-s \cdot t} dt = \frac{1}{2\pi} \cdot \int_0^{\infty} e^{(a-s) \cdot t} dt = \lim_{b \rightarrow \infty} \int_0^b e^{(a-s) \cdot t} dt =$$

$$= \frac{1}{a-s} \cdot \lim_{b \rightarrow \infty} [e^{(a-s) \cdot t}]_{t=0}^b = \frac{1}{a-s} \cdot \lim_{b \rightarrow \infty} (e^{(a-s) \cdot b} - 1) = \frac{1}{a-s}, \quad \text{Re } s > a$$

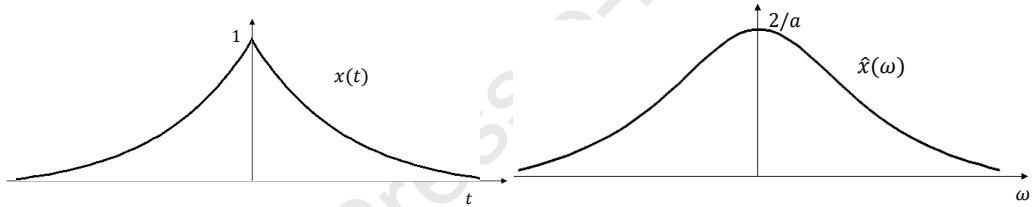
>>>Exercise 4.2 – 8

Determine the Fourier transform of function

$$x(t) = e^{-a \cdot |t|}, \quad a > 0$$

>>>Exercise 4.2 – 8- Solution

$$\begin{aligned} \hat{x}(\omega) &= \int_0^{\infty} e^{-a \cdot |t|} \cdot e^{-i \cdot \omega \cdot t} dt = \int_{-\infty}^0 e^{a \cdot t} \cdot e^{-i \cdot \omega \cdot t} dt + \int_0^{\infty} e^{-a \cdot t} \cdot e^{-i \cdot \omega \cdot t} dt = \\ &= \lim_{b \rightarrow -\infty} \int_b^0 e^{(a-i \cdot \omega) \cdot t} dt + \lim_{b \rightarrow \infty} \int_0^b e^{-(a+i \cdot \omega) \cdot t} dt = \\ &= \frac{1}{a-i \cdot \omega} \cdot \lim_{b \rightarrow -\infty} [e^{(a-i \cdot \omega) \cdot t}]_{t=b}^0 + \frac{-1}{a+i \cdot \omega} \cdot \lim_{b \rightarrow \infty} [e^{-(a+i \cdot \omega) \cdot t}]_{t=0}^b = \\ &= \frac{1}{a-i \cdot \omega} \cdot \lim_{b \rightarrow -\infty} (1 - e^{(a-i \cdot \omega) \cdot b}) - \frac{1}{a+i \cdot \omega} \cdot \lim_{b \rightarrow \infty} (1 - e^{-(a+i \cdot \omega) \cdot b}) = \\ &= \frac{1}{a-i \cdot \omega} + \frac{1}{a+i \cdot \omega} = \frac{2a}{a^2 + \omega^2} \end{aligned}$$



>>>Exercise 4.2 – 9

Calculate $\lim_{a \rightarrow 0+0} \hat{x}(\omega)$ when $x(t) = \begin{cases} 0 & \text{if } t < 0 \\ e^{-a \cdot t} & \text{if } t \geq 0 \end{cases}, \quad a > 0$

>>>Exercise 4.2 – 9- Solution

$$\lim_{a \rightarrow 0+0} \hat{x}(\omega) = \lim_{a \rightarrow 0+0} \frac{a}{a^2 + \omega^2} + \lim_{a \rightarrow 0+0} \frac{-i \cdot \omega}{a^2 + \omega^2} = \pi \cdot \delta(\omega) + \frac{1}{i \cdot \omega}$$

The unit step function $x(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$ has not Fourier transform since the integral

$$\int_{t=0}^{\infty} e^{-i \cdot \omega \cdot t} dt = \int_{t=0}^{\infty} \cos(\omega \cdot t) dt - i \cdot \int_{t=0}^{\infty} \sin(\omega \cdot t) dt$$

is not convergent.

The unit step function can be considered as the limit of $x(t) = \begin{cases} 0 & \text{if } t < 0 \\ e^{-a \cdot t} & \text{if } t \geq 0 \end{cases}, a > 0$ as $a \rightarrow 0+0$, thus we can define, symbolically, the 'Fourier transform' of unit step function as $\pi \cdot \delta(\omega) + \frac{1}{i \cdot \omega}$.

This definition yields Fourier transform of further important functions.

>>>Exercise 4.2 – 10

Determine symbolically the Fourier transform of the negative unit step function $x(t) = \begin{cases} 1 & \text{if } t < 0 \\ 0 & \text{if } t \geq 0 \end{cases}$ using the Fourier transform of the unit step function.

>>>Exercise 4.2 – 10– Solution

$$\int_{t=-\infty}^0 e^{-i \cdot \omega \cdot t} dt = \int_{t=0}^{\infty} e^{-i \cdot (-\omega) \cdot t} dt = \pi \cdot \delta(\omega) - \frac{1}{i \cdot \omega}$$

>>>Exercise 4.2 – 11

Determine symbolically the Fourier transform of constant function $x(t) = 1$ using the Fourier transform of the unit step function.

>>>Exercise 4.2 – 11– Solution

The constant function $x(t) = 1$ is the sum of the unit step function and the negative unit step function, thus

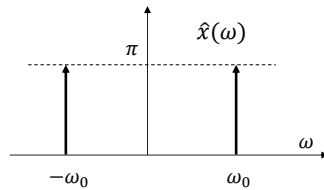
$$\hat{x}(\omega) = \int_{t=-\infty}^{\infty} e^{-i \cdot \omega \cdot t} dt = \int_{t=-\infty}^0 e^{-i \cdot \omega \cdot t} dt + \int_{t=0}^{\infty} e^{-i \cdot \omega \cdot t} dt = 2\pi \cdot \delta(\omega)$$

>>>Exercise 4.2 – 12

Determine symbolically the Fourier transform of periodic function $x(t) = \cos(\omega_0 \cdot t)$ using that $\int_{t=-\infty}^{\infty} e^{-i \cdot \omega \cdot t} dt = 2\pi \cdot \delta(\omega)$.

>>>Exercise 4.2 – 12– Solution

$$\begin{aligned} \hat{x}(\omega) &= \int_{t=-\infty}^{\infty} \cos(\omega_0 \cdot t) \cdot e^{-i \cdot \omega \cdot t} dt = \frac{1}{2} \cdot \int_{t=-\infty}^{\infty} (e^{i \cdot \omega_0 \cdot t} + e^{-i \cdot \omega_0 \cdot t}) \cdot e^{-i \cdot \omega \cdot t} dt = \\ &= \frac{1}{2} \cdot \int_{t=-\infty}^{\infty} (e^{-i \cdot (\omega - \omega_0) \cdot t} + e^{-i \cdot (\omega + \omega_0) \cdot t}) dt = \pi \cdot \delta(\omega - \omega_0) + \pi \cdot \delta(\omega + \omega_0) \end{aligned}$$



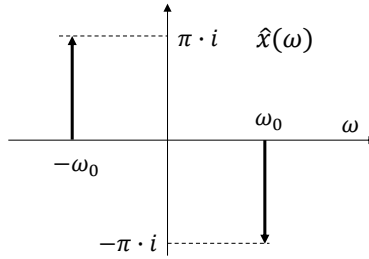
>>>Exercise 4.2 – 13

Determine symbolically the Fourier transform of periodic function $x(t) = \sin(\omega_0 \cdot t)$ using that $\int_{t=-\infty}^{\infty} e^{-i \cdot \omega \cdot t} dt = 2\pi \cdot \delta(\omega)$.

>>>Exercise 4.2 – 13– Solution

$$\hat{x}(\omega) = \int_{t=-\infty}^{\infty} \sin(\omega_0 \cdot t) \cdot e^{-i \cdot \omega \cdot t} dt = \frac{1}{2i} \cdot \int_{t=-\infty}^{\infty} (e^{i \cdot \omega_0 \cdot t} - e^{-i \cdot \omega_0 \cdot t}) \cdot e^{-i \cdot \omega \cdot t} dt =$$

$$= \frac{1}{2i} \cdot \int_{t=-\infty}^{\infty} (e^{-i \cdot (\omega - \omega_0) \cdot t} - e^{-i \cdot (\omega + \omega_0) \cdot t}) dt = -i \cdot \pi \cdot \delta(\omega - \omega_0) + i \cdot \pi \cdot \delta(\omega + \omega_0)$$



>>>Exercise 4.2 – 14

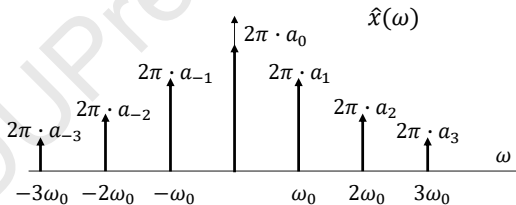
Determine symbolically the Fourier transform of periodic function (with basic frequency ω_0)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{i \cdot k \cdot \omega_0 \cdot t}$$

using that $\int_{t=-\infty}^{\infty} e^{-i \cdot \omega \cdot t} dt = 2\pi \cdot \delta(\omega)$.

>>>Exercise 4.2 – 14- Solution

$$\begin{aligned} \hat{x}(\omega) &= \sum_{k=-\infty}^{\infty} a_k \cdot \int_{t=-\infty}^{\infty} e^{i \cdot k \cdot \omega_0 \cdot t} \cdot e^{-i \cdot \omega \cdot t} dt = \sum_{k=-\infty}^{\infty} a_k \cdot \int_{t=-\infty}^{\infty} e^{i \cdot (k \cdot \omega_0 - \omega) \cdot t} dt = \\ &= 2\pi \cdot \sum_{k=-\infty}^{\infty} a_k \cdot \delta(\omega - k \cdot \omega_0) \end{aligned}$$



>>>Exercise 4.2 – 15

Use time shift property of Fourier transform $\mathcal{FT}(x(t - T)) = e^{-i \cdot \omega \cdot T} \cdot \mathcal{FT}(x(t))$ to determine the Fourier transform of function $x(t) = \cos(\omega_0 \cdot t + \varphi)$.

>>>Exercise 4.2 – 15- Solution

$x(t) = \cos(\omega_0 \cdot t + \varphi) = \cos\left(\omega_0 \cdot \left(t + \frac{\varphi}{\omega_0}\right)\right)$. Using the transform of $x(t) = \cos(\omega_0 \cdot t)$ and the time shift property we have

$$\hat{x}(\omega) = e^{i \cdot \omega \cdot \frac{\varphi}{\omega_0}} \cdot \pi \cdot (\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$$

>>>Exercise 4.2 – 16

Determine the Fourier transform of triangle function $x(t) = \text{tri}(t) =$

$$\begin{cases} t+2 & \text{if } -2 \leq t \leq 0 \\ -t+2 & \text{if } 0 \leq t \leq 2 \\ 0 & \text{if } |t| > 2 \end{cases}$$
 using the convolution theorem $\mathcal{FT}(x * h) = \mathcal{FT}(x) \cdot \mathcal{FT}(h)$.

>>>Exercise 4.2 – 16– Solution

According to the Exercise 4.2.1 $x(t) = \text{tri}(t) = \Pi * \Pi(t)$ where $\Pi(t) = \begin{cases} 1 & \text{if } |t| \leq 1 \\ 0 & \text{if } |t| > 1 \end{cases}$, is the rectangular impulse function.

Using the convolution theorem we get

$$\hat{x}(\omega) = \mathcal{FT}(\Pi)(\omega) \cdot \mathcal{FT}(\Pi)(\omega) = 4 \cdot \frac{\sin^2 \omega}{\omega^2}$$

4.3 Discrete Fourier Transform

4.3.1 The Concept of the Discrete Fourier Transform

Since digital measuring systems are used in practice and the data are stored and processed with digital computers, only sampled signals are available for signal processing. Consequently, the continuous transforms requiring the formula of the signal are unusable, we need algorithms (discrete transforms) that provide (or approximate) the spectrum. Since the sampled signal contains limited measure of information about the signal, it is generally impossible to determine the exact spectrum from the sample.

Let $T > 0$ be a fixed real number and N be a fixed positive integer and suppose that values

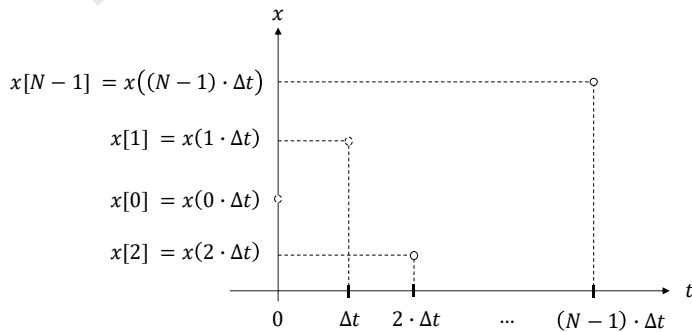
$$x[n] = x[n \times \Delta T], \quad n = 0, 1, \dots, N-1$$

of signal x are provided by a sampling process.

The discrete Fourier transform of sampled signal $x[0], \dots, x[N-1]$ is

$$X[k] = \frac{1}{N} \cdot \sum_{n=0}^{N-1} x[n] \cdot e^{-i \cdot k \cdot n \cdot \frac{2\pi}{N}}, \quad k = 0, 1, \dots, N-1$$

Remark: $e^{-i \cdot k \cdot n \cdot \frac{2\pi}{N}}$ type values of the complex exponential function are discussed in Appendix 1.



Mathematically, both the input and the output of the discrete Fourier transform consist of N pure numbers. In practice, when the circumstances of sampling are known, these values have physical meaning, and values $X[0], \dots, X[N-1]$ provide a 'discrete spectrum'.

Considering a sampling process, T is the *sampling time*, N is the *sample size* (number of elements in the sample), ΔT is time between two measurements.

Further quantities are the *sampling frequency* $f_s = N/T = 1/\Delta T$, the *frequency resolution* $\Delta f = 1/T = f_s/N$. The potential frequency values in the discrete spectrum are $k \times \Delta f$, $k = 0, 1, \dots, N - 1$.

The discrete Fourier transform can also be calculated as matrix multiplication.

Introducing the notation $W = e^{-i\frac{2\pi}{N}}$ (then $e^{-i\cdot k\cdot n\cdot\frac{2\pi}{N}} = W^{k\cdot n}$), the transformation matrix is

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & W & W^2 & \dots & W^{N-1} \\ 1 & W^2 & W^4 & \dots & W^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W^{N-1} & W^{2(N-1)} & \dots & W^{(N-1)^2} \end{pmatrix}$$

If $N = 2$ the transformation matrix is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

If $N = 4$ the transformation matrix is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

Calculation with the matrix:

$$\begin{pmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{pmatrix} = \frac{1}{N} \cdot \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & W & W^2 & \dots & W^{N-1} \\ 1 & W^2 & W^4 & \dots & W^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W^{N-1} & W^{2(N-1)} & \dots & W^{(N-1)^2} \end{pmatrix} \cdot \begin{pmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{pmatrix}$$

The inverse transformation is

$$x[n] = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X[k] \cdot e^{i\cdot k\cdot n\cdot\frac{2\pi}{N}}, \quad n = 0, 1, \dots, N - 1$$

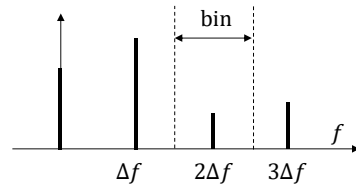
Computing DFT directly from the definition is often too slow to be practical in real time applications. A Fast Fourier Transform (FFT) is a way to compute the same result more quickly. The difference in speed can be enormous, especially for long data sets where N may be in the thousands or millions. FFT is a common built-in function in measuring systems (e.g. vibration monitoring systems).

>>>Exercise 4.3 – 1

Determine the frequency resolution if the sampling frequency is $f_s = 1000 \text{ Hz}$, and the sample size is $N = 1024$.

>>>Exercise 4.3 – 1- Solution

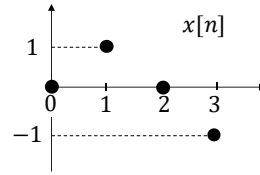
$$\Delta f = \frac{1000}{1024} = 0.9766 \frac{\text{Hz}}{\text{bin}}$$



>>>Exercise 4.3 – 2

Determine the discrete Fourier transform of the sampled signal.

n	0	1	2	3
$x[n]$	0	1	0	-1

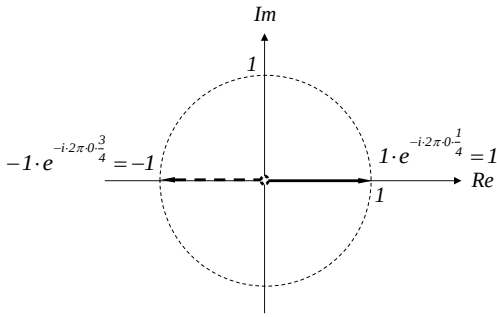


Plot the complex numbers in the complex plane appearing in the sums.

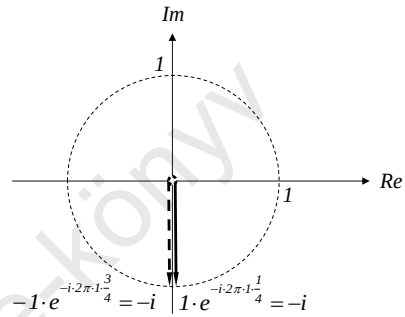
>>>Exercise 4.3 – 2- Solution

$$X[0] = \frac{1}{4} \cdot \sum_{n=0}^3 x[n] \cdot e^{-i \cdot 0 \cdot n \frac{2\pi}{4}} = \frac{1}{4} \cdot \sum_{n=0}^3 x[n] = 0 + 1 + 0 - 1 = 0$$

Values in the sum giving $X[0]$



Values in the sum giving $X[1]$



$$X[1] = \frac{1}{4} \cdot \sum_{n=0}^3 x[n] \cdot e^{-i \cdot 1 \cdot n \frac{2\pi}{4}} = \frac{1}{4} \cdot \sum_{n=0}^3 x[n] \cdot e^{-i \cdot n \frac{\pi}{2}} =$$

$$= \frac{1}{4} \cdot \left(0 \cdot e^{-i \cdot 0 \cdot \frac{\pi}{2}} + 1 \cdot e^{-i \cdot 1 \cdot \frac{\pi}{2}} + 0 \cdot e^{-i \cdot 2 \cdot \frac{\pi}{2}} - 1 \cdot e^{-i \cdot 3 \cdot \frac{\pi}{2}} \right) = \frac{1}{4} \cdot \left(e^{-i \frac{\pi}{2}} - e^{-i \cdot 3 \frac{\pi}{2}} \right) =$$

$$= \frac{1}{4} \cdot \left(\cos\left(-\frac{\pi}{2}\right) + i \cdot \sin\left(-\frac{\pi}{2}\right) \right) - \left(\cos\left(-\frac{3\pi}{2}\right) + i \cdot \sin\left(-\frac{3\pi}{2}\right) \right) =$$

$$= \frac{1}{4} \cdot (0 - i + 0 - i) = -\frac{1}{2} \cdot i$$

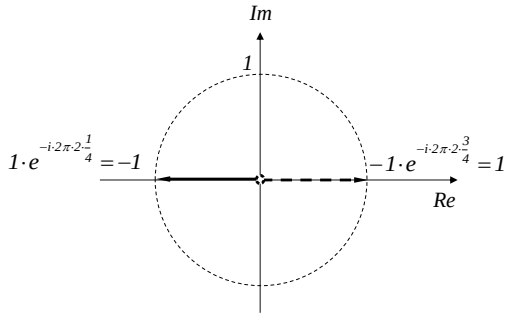
$$X[2] = \frac{1}{4} \cdot \sum_{n=0}^3 x[n] \cdot e^{-i \cdot 2 \cdot n \frac{2\pi}{4}} = \frac{1}{4} \cdot \sum_{n=0}^3 x[n] \cdot e^{-i \cdot n \pi} =$$

$$= \frac{1}{4} \cdot \left(0 \cdot e^{-i \cdot 0 \cdot \pi} + 1 \cdot e^{-i \cdot 1 \cdot \pi} + 0 \cdot e^{-i \cdot 2 \cdot \pi} - 1 \cdot e^{-i \cdot 3 \cdot \pi} \right) = \frac{1}{4} \cdot (e^{-i \pi} - e^{-i \cdot 3 \pi}) =$$

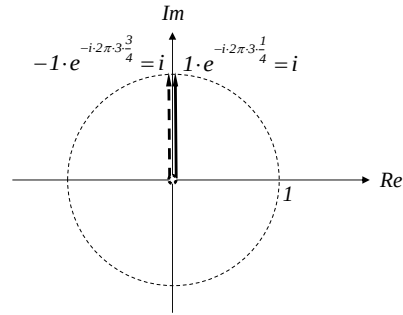
$$= \frac{1}{4} \cdot \left((\cos(-\pi) + i \cdot \sin(-\pi)) - (\cos(-3\pi) + i \cdot \sin(-3\pi)) \right) =$$

$$= \frac{1}{4} \cdot (0 - 1 + 0 + 1) = 0$$

Values in the sum giving $X[2]$

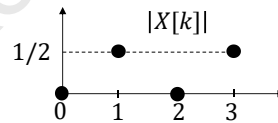


Values in the sum giving $X[3]$



$$\begin{aligned}
 X[3] &= \frac{1}{4} \cdot \sum_{n=0}^3 x[n] \cdot e^{-i \cdot 3 \cdot n \cdot \frac{2\pi}{4}} = \frac{1}{4} \cdot \sum_{n=0}^3 x[n] \cdot e^{-i \cdot n \cdot \frac{3\pi}{2}} = \\
 &= \frac{1}{4} \cdot \left(0 \cdot e^{-i \cdot 0 \cdot \frac{3\pi}{2}} + 1 \cdot e^{-i \cdot 1 \cdot \frac{3\pi}{2}} + 0 \cdot e^{-i \cdot 2 \cdot \frac{3\pi}{2}} - 1 \cdot e^{-i \cdot 3 \cdot \frac{3\pi}{2}} \right) = \frac{1}{4} \cdot \left(e^{-i \cdot \frac{3\pi}{2}} - e^{-i \cdot 3 \cdot \frac{3\pi}{2}} \right) = \\
 &= \frac{1}{4} \cdot \left(\left(\cos\left(-\frac{3\pi}{2}\right) + i \cdot \sin\left(-\frac{3\pi}{2}\right) \right) - \left(\cos\left(-\frac{9\pi}{2}\right) + i \cdot \sin\left(-\frac{9\pi}{2}\right) \right) \right) = \\
 &= \frac{1}{4} \cdot (0 + i + 0 + i) = \frac{1}{2} \cdot i
 \end{aligned}$$

k	0	1	2	3
$X[k]$	0	$-\frac{1}{2} \cdot i$	0	$\frac{1}{2} \cdot i$
$ X[k] $	0	$\frac{1}{2}$	0	$\frac{1}{2}$



>>> **Exercise 4.3 – 3**

Determine the discrete Fourier transform of the sampled signal

n	0	1	2	3
$x[n]$	0	1	0	-1

using the transformation matrix.

>>> **Exercise 4.3 – 3– Solution**

$$\frac{1}{4} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} 0 \\ -2 \cdot i \\ 0 \\ 2 \cdot i \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2} \cdot i \\ 0 \\ \frac{1}{2} \cdot i \end{pmatrix} = \begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{pmatrix}$$

>>> **Exercise 4.3 – 4– Solution**

Determine the discrete Fourier transform of the sampled signal

n	0	1	2	3
$x[n]$	8	4	8	0

using the transformation matrix.

>>> **Exercise 4.3 – 4– Solution**

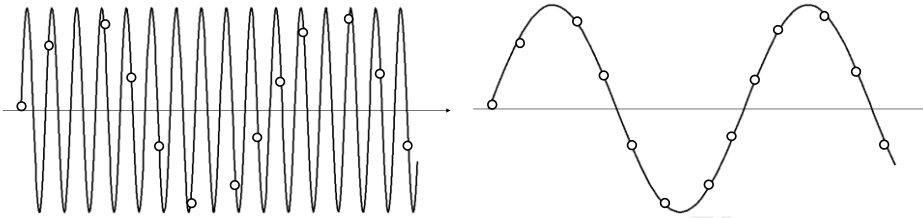
$$\frac{1}{4} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 4 \\ 8 \\ 0 \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} 20 \\ -4 \cdot i \\ 12 \\ 4 \cdot i \end{pmatrix} = \begin{pmatrix} 5 \\ -i \\ 3 \\ i \end{pmatrix} = \begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{pmatrix}$$

4.3.2 Aliasing and Leakage in the Spectrum

The physical parameters of technical systems can be considered as continuous functions of time. Calculating the spectrum from a sampled signal causes some 'problems'. Two of them are discussed below.

Aliasing

If the signal contains components of frequency higher than the half of the sampling frequency, then *false frequencies* appear in the spectrum. The left-hand figure below shows the result of a sampling when the sampling frequency is too low relative to the frequency of the signal. Calculations by the discrete Fourier transform result a false frequency, namely the frequency of the signal in the right-hand figure below.

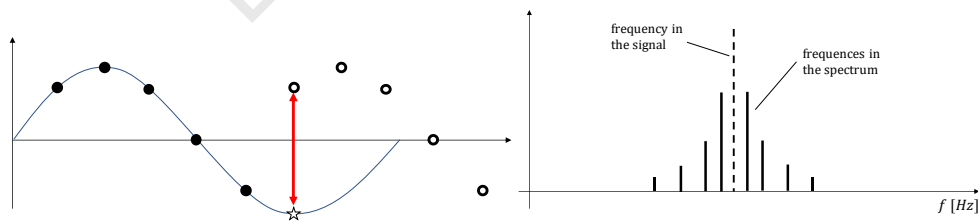


To avoid the phenomena of aliasing the sampling frequency has to be at least two times higher than the maximum frequency in the signal.

Remark: In practice the maximum sampling frequency is limited by the measuring device. The high frequencies causing aliasing are filtered from the signal before the transformation (low-pass filtering).

Leakage

The investigated signals are generally not periodic, and even if they are periodic the sampling time is generally cannot be harmonized with the period. It is the nature of the discrete Fourier transformation that the sampling time T acts as a period of the signal, but it is generally not a real period. Consequently, the signal seems to be discontinuous because of the 'virtual jump' appearing when the virtual periods are connected, as the left-hand side figure shows below. The 'virtual discontinuity' causes the phenomenon called *leakage* in the spectrum when several false frequencies appear around the real frequency in the spectrum (right-hand side figure below).

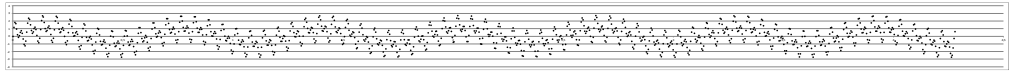


The 'virtual jump' can be removed (or reduced) by applying the so-called window functions. The window functions are defined on finite intervals, and they have zero values at the ends of their domain. To avoid leakage phenomena we multiply the original signal with a window function first to reduce the 'virtual jump', and we apply the transform to the product.

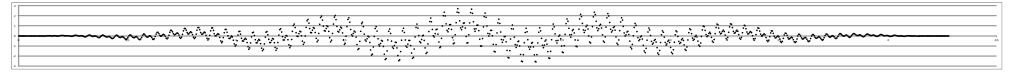
A typical discrete window function is the *Hanning function* defined as $w(n) = \sin^2\left(\frac{n\pi}{N-1}\right)$

Figures below show a sampled signal, the windowed signal, and their discrete Fourier transforms.

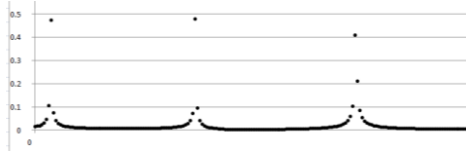
The original sampled signal (sum of three sine functions of different frequencies, $N = 200$)



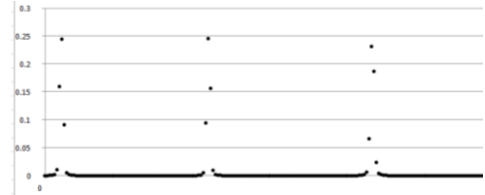
The original sampled signal multiplied by the Hanning function



a part of the transform of the original signal showing the three frequencies (the measure of leakage is relatively high)



a part of the transform of the windowed signal showing the three frequencies (the measure of leakage is reduced by windowing)



4.4 Vibration Monitoring

4.4.1 Goals of Vibration Monitoring

The word 'diagnostics' originally referred to the identification of human health problems resulting the 'diagnosis', the evaluation of the actual health status of a human. Technical diagnostics aims the diagnosis of technical objects (machines, parts) to support engineering decisions (reparation or change of a part, process improvement).

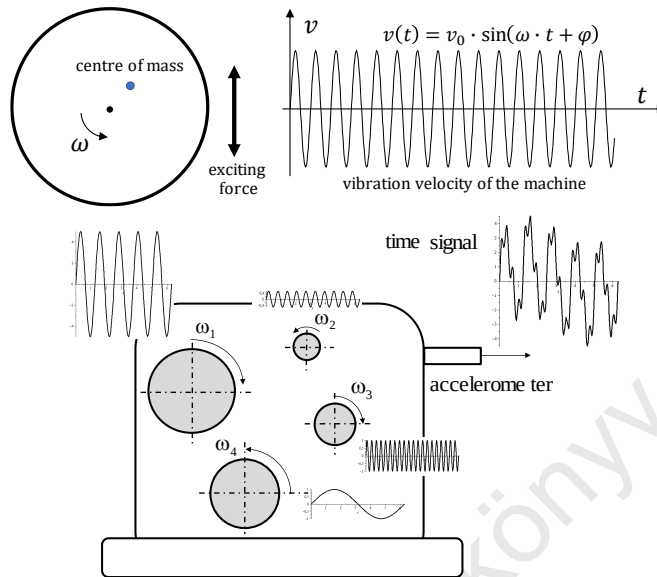
Fault is the abnormal behaviour of a machine (state), *failure* is the termination of the ability to perform a required function (event), *failure mode* is an observable phenomenon caused by a failure. *Root cause failure analysis* is an examination aiming to identify the causes of a failure mode.

The first step of condition monitoring is to find connection between failures and measurable symptoms generated by the failures of interest. Since the symptoms are very different in appearance (vibration, sound, heat, wear particles in oil) technical diagnostics has several fields using special measuring systems: vibration monitoring, acoustics, thermography, oil analysis, etc.

In the field of vibration monitoring symptoms can be detected with the analysis of the sampled vibration signal. (In simple everyday situations the diagnosis can be based on the human senses – vision, hearing and touch first of all – but their accuracy and range is not enough for diagnosis of modern machineries.) Some symptoms appear in the so-called time-domain (e.g. vibration velocity vs. time function) others can be revealed from the frequency spectrum (frequency-domain analysis).

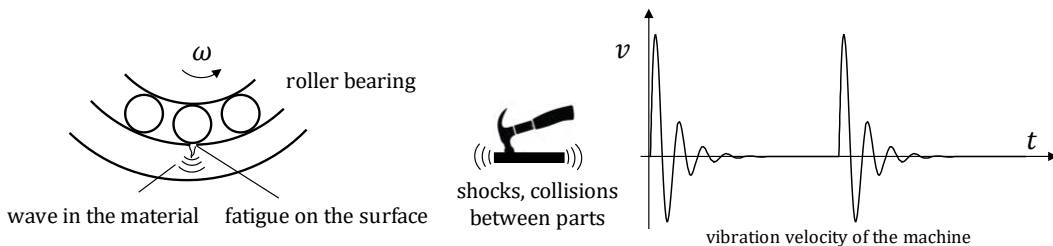
The basic tool for vibration analysis, from the beginning, is the Fourier analysis. Several mechanical failures generate nearly periodic (possibly) harmonic vibration of parts of a machine. Such failures, for example, are unbalance, angle of shift problem of couplings, looseness, misalignment of shafts, bended shafts. The generated vibrations have special frequencies depending on the rotational speed and the geometry of the rotating component.

To the majority of failures special patterns belong in the frequency spectrum containing several lines. Thus in some more complex cases pattern recognition is necessary not only a simple detection of certain frequency values.



Connecting an accelerometer we can measure the sum (superposition) of approximately harmonic vibrations generated by rotating parts of the machine and other vibrations (beats, noise). The vibration spectrum provided by the Fourier analysis shows the frequencies appearing in the vibration signal and the magnitudes belonging to them. Based on these data the problematic components and the severity of the failures can be identified.

A different type of failures causes so-called shock pulses instead of periodic vibrations. The most important examples are bearing and gear failures. Since these components are crucial in mechanical engineering, monitoring these parts are important. Shock pulses are non-periodic transient waves in the time signal, so Fourier analysis is not effective in detection of these parts of the signal. Transient components can be revealed using so-called simultaneous time-frequency methods, for instance short time Fourier transform or wavelet transform. Shock pulses repeat periodically with a certain magnitude and frequency which depend on the geometry and the rotational speed of the machine element and are characteristic to the failure.



4.4.2 Historical Aspects, Maintenance Strategies

As a part of a predictive maintenance system, technical diagnostics aims to improve reliability, availability and safety of technical systems, furthermore to ensure the desired level of product

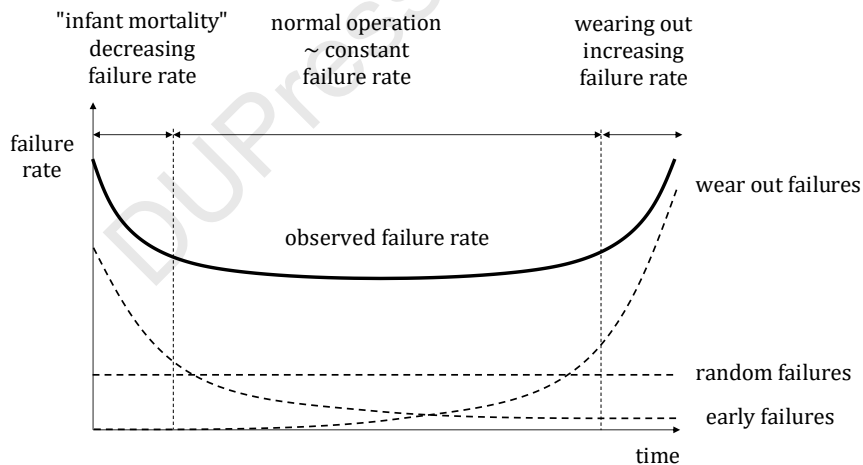
quality and process efficiency. *Condition monitoring* is the collection of data that indicate the state of an item, to avoid the consequences of failures, before they occur.

Prior to the Second World War machinery was generally quite rugged and relatively slow running, the instrumentation and control systems were very basic. The demands of production were not overly severe so that downtime was not usually a critical issue and it was adequate to maintain on a breakdown basis. This was the time of run-to-failure 'management' whose concept was that *"When a machine breaks down, fix it"* or *"If it ain't broke, don't fix it"*.

A plant using run-to-failure management does not spend any money on maintenance until a machine or system fails to operate. Actually, it is a no-maintenance approach. It is a reactive management technique that waits for machine or equipment failure before any maintenance action is taken.

From the 1950's with the rebuilding of industry after the war, there developed a much more competitive marketplace; there was increasing intolerance of downtime. The cost of labour became increasingly significant leading to more and more mechanisation and automation. Machinery was of lighter construction and ran at higher speeds. They wore out more rapidly and were seen as less reliable. Production demanded better maintenance which led to the development of Planned Preventative Maintenance.

Preventive maintenance management programs are time-driven, maintenance tasks are based on elapsed time or hours of operation. The bathtub curve (based on mean-time-to-failure statistics) indicates that a new machine has a high probability of failure because of installation problems during the first few weeks of operation. After this initial period, the probability of failure is relatively low for an extended period. After this normal machine life period, the probability of failure increases sharply with elapsed time.



All preventive maintenance management programs assume that machines will degrade within a time frame typical of their particular classification. Using preventive management techniques, the machine elements are to be removed from service and rebuilt before the end of the normal machine life period.

The problem with this approach is that the mode of operation and system or plant-specific variables directly affect the normal operating life of machinery. The normal result of using MTBF (Mean Time Between Failures) statistics to schedule maintenance is either unnecessary repairs or catastrophic failure.

From the 1980's plants and systems became increasingly complex, the demands of the competitive marketplace and intolerance of downtime increased, and maintenance costs continued to rise. Along with the demands for greater reliability at a lower cost came new awareness of failure processes, improved management techniques and new technologies to allow an understanding of machine and component health.

Predictive Maintenance is based on condition monitoring: data acquisition and data processing. The main sources of information about the actual condition of typical machinery are vibration signals, infrared images, acoustic signals, wear particle distribution in oil. The parallel development of information technology and digital measuring systems determined the development of technical diagnostics techniques. The enhancement of accuracy and the reduction of costs and measuring times resulted in effective data collection. The development of data processing methods (mathematical algorithms) enabled a precise analysis and provided high reliability failure detection.

The most frequently used method in manufacturing industry is vibration monitoring, since the majority of mechanical failures of machinery and some other important failures (e.g. electric motor problems) generate specific vibrations which can be detected in vibration spectrum.

The premise of predictive maintenance is that regular monitoring of the actual mechanical condition of machine-trains and operating efficiency of process systems will ensure the maximum interval between repairs; minimize the number and cost of unscheduled outages created by machine-train failures; and improve the overall availability of operating plants.

Condition monitoring systems ensure long-term, problem-free operation of machines and equipment working in the background. If things start to go wrong, they quickly raise alarm so that plant operators can take early action to prevent outages and production downtime.

That is because continuous condition monitoring, itself a form of factory intelligence, is an integral component of the Industry 4.0 vision. Condition monitoring has central role in Integrated Industry. Smart actuators and drive components with built-in sensors are key to integrating machines and equipment and controlling autonomous systems.

Customers now want total system health management solutions that capture and rapidly analyse all the relevant data. The data captured by condition monitoring systems provide the basis for plant integration and optimization in the sense of Industry 4.0.

4.4.3 Time-frequency Analysis

Vibration signals studied in diagnostics are non-stationary, that is, their frequency-domain representation (frequency spectrum) changes over time. The detection of transients in the signal requires information about frequencies and also about the time when a particular frequency component is present. Several time-frequency analysis methods are known which are able to provide both types of data, for instance the Windowed Fourier Transforms and the wavelet transforms. Because of their flexibility, the wavelet transforms (and the related multiresolution analysis) can be used more effectively in the condition monitoring of machine elements.

The Short Time Fourier Transform (STFT) was developed to address the limitation of the Fourier transform, which is rooted in its basis functions extending over an infinite period of time and, consequently, the Fourier transform is not well adapted to non-stationary transient signals with short durations. A solution to this problem is to perform a 'time localized' Fourier

transform within a sliding window. Popular choices for the window function include the Hamming, the Hanning, and the Gaussian functions. When a Gaussian window is chosen, the STFT is called a Gabor transform.

The disadvantage of the STFT is that its time resolution (the smallest separation in time of two signal components that can be discriminated) and bandwidth cannot be chosen to be simultaneously small. To overcome the resolution limitation of STFT, the wavelet transform was introduced for non-stationary signal analysis. In contrast to STFT with fixed windows, the wavelet transform uses short windows at high frequencies and long windows at low frequencies. Such a nature leads to the wavelet transform being called the constant relative-bandwidth frequency analysis. Unlike the Fourier transform, which expresses a signal as the sum of a series of single-frequency sine and cosine functions, the wavelet transform decomposes a signal into a set of basis functions.

These basis functions are obtained from a single base wavelet function by a two-step operation: scaling (through dilation and contraction of the base wavelet along the time axis), and time shift (i.e., translation along the time axis). Essentially, the wavelet transform process measures the ‘similarity’ between the signal being analysed and the base wavelet (see Chapter 2.3).

Through variations of the scales and time shifts of the base wavelet function, features hidden within the signal can be extracted, without requiring the signal to have a dominant frequency band. It can be concluded that the wavelet transform provides a powerful mathematical tool for the analysis, characterization, and classification of non-stationary signals typically seen in manufacturing.

The adaptive, multiresolution capability of the wavelet transform makes it well suited for decomposing signals of varying time and frequency resolutions that are characteristic of the underlying defect mechanisms associated with a machine, a dynamical structure, or a manufacturing process. Such capability makes the wavelet transform an enabling tool for advancing the science base of signal processing in manufacturing.

Wavelets are enabling the analysis on several timescales of the local properties of complex signals that can present nonstationary zones. A wavelet is a function oscillating as a wave but quickly damped. Being well localized simultaneously in time and frequency it makes it possible to define a family of analysing functions by translation in time and dilation in scale. Wavelets constitute a mathematical ‘zoom’ making it possible to simultaneously describe the properties of a signal on several timescales.

A function ψ will be called a wavelet if it verifies the admissibility condition

$$K_\psi = \int_0^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega = \int_{-\infty}^0 \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < +\infty$$

where $\hat{\psi}$ indicates the Fourier transform of ψ . The admissibility condition involves, that the wavelet integrates to zero, that is, $\int_{\mathbb{R}} \psi(x) dx = 0$. It is often reinforced by requiring that the wavelet has m vanishing moments, i.e.

$$\int_{\mathbb{R}} x^k \cdot \psi(x) dx = 0, \quad \text{for } k = 0, \dots, m$$

The oscillation of a wavelet is measured by the number of vanishing moments and its localization is evaluated by the interval where it takes values significantly different from zero.

Using translation and dilation a family of functions $\{\psi_{a,b}\}$ is defined by

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}} \psi\left(\frac{x-b}{a}\right)$$

for any scale $a \in \mathbb{R}^+$ and any position $b \in \mathbb{R}$. If ψ has norm 1 then all the functions $\psi_{a,b}$ have norm of 1.

A signal f of finite energy can be analysed by its wavelet coefficients

$$C_f(a, b) = \int_{\mathbb{R}} f(x) \cdot \psi_{a,b}^*(x) dx, \quad a \in \mathbb{R}^+, b \in \mathbb{R}.$$

The calculation of wavelet coefficients C_f is called continuous wavelet transform.

The $C_f(a, b)$ measures the fluctuation of function f at scale a , it depends on the values of f in a neighbourhood of b with a length proportional to a . Large values of $C_f(a, b)$ provide information on the local irregularity of f around position b and at scale a .

The squared magnitude of wavelet coefficients

$$|C_f(a, b)|^2 = \left| \int_{-\infty}^{\infty} f(t) \cdot \psi_{a,b}^*(x) \right|^2$$

constitutes the so-called scalogram, which is an important tool in evaluations based on wavelet transforms.

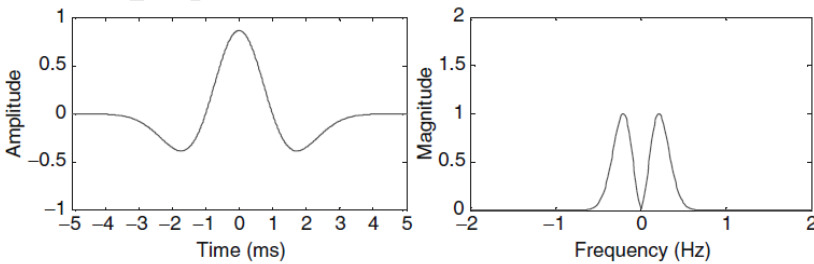
The coloured representation of scalogram in the time-frequency plane is a useful source of visual information e.g. in the identification of transients in vibration signal caused by surface defects of bearings.

Large number of wavelets are defined in the literature and many of them are used in vibration diagnostics as well. Three examples are the Mexican Hat wavelet, the Morlet Wavelet and the Gaussian wavelet.

The Mexican hat wavelet is a normalized, second derivative of a Gaussian function, which is mathematically defined as

$$\psi(t) = \frac{1}{\sqrt{2\pi} \cdot \sigma^3} \cdot \left(1 - \frac{\sigma^2}{t^2}\right) \cdot e^{-\frac{t^2}{2\sigma^2}}$$

Figure shows the Mexican hat wavelet and its associated magnitude spectrum.

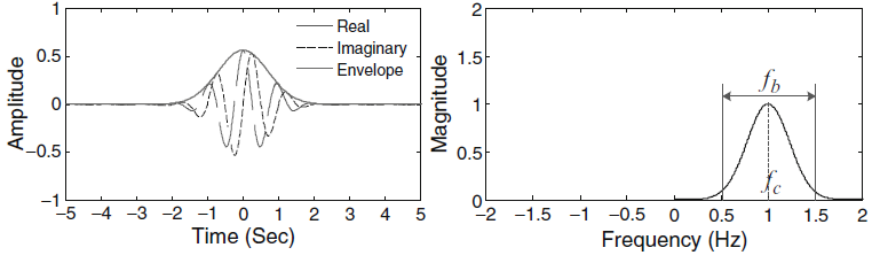


The Morlet wavelet is defined as

$$\psi_M(t) = \frac{1}{\sqrt{\pi \cdot f_b}} \cdot e^{j \cdot 2\pi f_c t} \cdot e^{-\frac{t^2}{T_b}}$$

where f_b is the bandwidth parameter and f_c denotes the wavelet center frequency.

Figure shows the Morlet wavelet and its associated magnitude spectrum.



The Gaussian function is expressed as

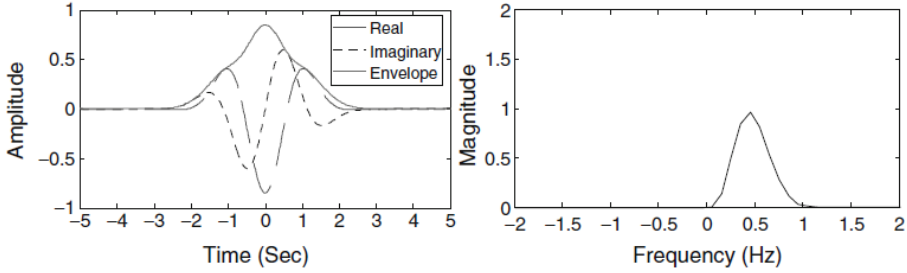
$$f(t) = e^{-j \cdot t} \cdot e^{-t^2}$$

Taking the N -th derivative of this function yields the Gaussian wavelet as

$$\psi_G(t) = c_N \cdot \frac{d^{(N)}f(t)}{dt^N}$$

where $N \geq 1$ is an integer parameter and denotes the order of the wavelet, and c_N is a constant introduced to ensure that $\|f^{(N)}(t)\|^2 = 1$.

Figure shows the Gaussian wavelet and its magnitude spectrum ($N = 2$).



Supposing certain stronger properties than merely the admissibility condition we can limit ourselves to the values

$$a = 2^j, b = k \cdot 2^j, \quad j, k \in \mathbb{Z}.$$

This idea leads to the discrete wavelet transform which is closely related to the so-called multi-resolution analysis.

For signals f the following three operations

$$\begin{aligned} (\tau_a f)(x) &= f(x + a), \quad a, x \in \mathbb{R} \\ (\delta_s f)(x) &= f(s \cdot x), \quad s, x \in \mathbb{R}, s > 0 \\ (E_1 f)(x) &= \sum_{k \in \mathbb{Z}} f(x + k) = \sum_{k \in \mathbb{Z}} (\tau_k f)(x), \quad x \in \mathbb{R} \end{aligned}$$

will be used.

The concept of multi-resolution is based on translation-invariant Riesz bases. A system $\{\varphi_k\}_{k \in \mathbb{Z}} \subset X$ is a Riesz basis of X if for all $x \in X$ there exists a unique $(c_k) \in \ell_2$ such that $x = \sum_{k \in \mathbb{Z}} c_k \varphi_k$.

It is important to note that all orthonormal bases are Riesz bases as well.

Let $\{\varphi_k\}_{k \in \mathbb{Z}} = \{\tau_k \varphi\}_{k \in \mathbb{Z}} \subset X$ be a Riesz basis with the generator function φ . Then for all $n \in \mathbb{Z}$

$$\varphi_k^n(x) = \frac{1}{\sqrt{2^n}} \cdot \varphi\left(\frac{1}{2^n} \cdot x - k\right), \quad x \in \mathbb{R}, k \in \mathbb{Z}$$

is Riesz basis for X_n , furthermore, if $\{\varphi_k\}_{k \in \mathbb{Z}}$ is orthonormal, then $\{\varphi_k^n\}_{k \in \mathbb{Z}}$ is orthonormal as well.

A sequence of closed subspaces $\{X_n\}_{n \in \mathbb{Z}} \subset X$ is called multi-resolution analysis of X if

1. $X_n \subset X_{n-1}, n \in \mathbb{Z}$ *monotonicity property*
2. $\overline{\bigcup_{n \in \mathbb{Z}} X_n} = X$ *density property*
3. $\bigcap_{n \in \mathbb{Z}} X_n = \{0\}$ *separability property*
4. $f \in X_n \Leftrightarrow \delta_2 f \in X_{n-1}, n \in \mathbb{Z}$ *scaling property*
5. X_0 is generated by a Riesz-basis given by

$$\varphi_k = \tau_k \varphi, \quad k \in \mathbb{Z}$$

(The basis is invariant to integer translations).

The *scaling* property characterizes the multi-resolution aspects: spaces X_j are obtained by dyadic dilation or contraction of the functions of the single space X_0 through the assumptions

$$(x \rightarrow f(x)) \in X_j \Leftrightarrow (x \rightarrow f(2 \cdot x)) \in X_{j-1}, \quad j \in \mathbb{Z}.$$

Property 5 states that there exists $\varphi \in X_0$ such that $\{\varphi(x - k)\}_{k \in \mathbb{Z}}$ is a Riesz basis of X_0 .

Function φ is also called scaling function of the MRA.

Let us denote the subspace generated by Riesz basis φ_k^n as

$$X_n^\varphi = \{\sum_{k \in \mathbb{Z}} c_k \cdot \varphi_k^n \mid (c_k) \in \ell_2\}$$

Each multi-resolution analysis generates an approximation process. If $P_n: X \rightarrow X_n, n \in \mathbb{Z}$ denote the orthogonal projection of f onto the closed subspace X_n then $\lim_{n \rightarrow \infty} \|P_n f - f\| = 0, f \in X$.

Let $\mathcal{M}_0$ be the set of non-negative, even, integrable functions decreasing on $[0, 1]$.

It can be proved that if an MRA is generated by a function φ majorated by an element of $\mathcal{M}_0$ then the equality

$$\sum_{k \in \mathbb{Z}} \varphi(x - k) = 1, \quad x \in \mathbb{R}$$

is equivalent with the density property.

Theorem: The following three statements are equivalent:

- i) $\{\varphi_k^n\}_{k \in \mathbb{Z}}$ is a Riesz basis of X_n for all $n \in \mathbb{Z}$.
- ii) $\{\varphi_k^0 = \tau_k \varphi\}_{k \in \mathbb{Z}}$ is a Riesz basis of X_0 .
- iii) There exist real numbers $0 < m \leq M < \infty$ such that $m \leq \sqrt{E_1(|\hat{\varphi}|^2)} \leq M$.

Let $\mathcal{R} = \{\varphi \in X \mid m \leq \sqrt{E_1(|\hat{\varphi}|^2)} \leq M\}$. According to the previous theorem $\mathcal{R}$ contains the functions $\varphi \in X$ for which system $\{\varphi_k^n\}_{k \in \mathbb{Z}}$ is a Riesz basis of X_n for all $n \in \mathbb{Z}$.

Theorem: Let $\varphi \in \mathcal{R}$. System $\{\tau_k \varphi\}_{k \in \mathbb{Z}}$ is orthonormal if and only if $E_1(|\hat{\varphi}|^2) = 1$.

Let $\varphi \in \mathcal{R}$ and let X_n^φ denote the Riesz basis generated by $\{\varphi_k^n\}_{k \in \mathbb{Z}}, n \in \mathbb{Z}$.

Theorem: The monotonicity property $X_n \subset X_{n-1}, n \in \mathbb{Z}$ is equivalent with the scaling equation

$$\hat{\varphi}(2x) = \alpha(x) \cdot \hat{\varphi}(x), \quad x \in \mathbb{R},$$

where $\alpha \in X$ is a suitable 1-periodic function. α is called the low-pass filter belonging to φ .

Furthermore, the scaling equation is equivalent with the equality

$$\frac{1}{2} \cdot \varphi\left(\frac{1}{2} \cdot x\right) = \sum_{k \in \mathbb{Z}} a_k \cdot \varphi(x + k), \quad x \in \mathbb{R}$$

where (a_k) is the sequence of the Fourier coefficients of α .

Let $\mathcal{M}$ denote the set of functions $\varphi \in \mathcal{R}$ having majorant in $\mathcal{M}_0$ and satisfying condition

$$\hat{\varphi}(0) = \int_{\mathbb{R}} \varphi(t) dt = 1.$$

From function φ generating MRA on orthonormal wavelet basis $\{X_n^\varphi\}_{n \in \mathbb{Z}}$ of X can be constructed as follows.

In the following we suppose that the normality conditions

$$E_1(|\hat{\varphi}|^2) = 1, \quad E_1(|\hat{\psi}|^2) = 1$$

hold, that is, systems $\{\varphi_k^n\}_{k \in \mathbb{Z}}$ and $\{\psi_k^n\}_{k \in \mathbb{Z}}$ are orthonormal for any $n \in \mathbb{Z}$.

It can be proved that there exists a 1-periodic filter $\beta \in X$ such that for ψ defined by

$$\delta_2(\hat{\psi}) = \beta \cdot \hat{\varphi}$$

we have the orthogonal decomposition $X_0^\varphi = X_1^\varphi \oplus X_1^\psi$.

Consequently, for all $n \in \mathbb{Z}$ we have $X_n^\varphi = X_{n+1}^\varphi \oplus X_{n+1}^\psi$, furthermore

$$X_j^\varphi = X_k^\varphi \oplus \left(\bigoplus_{i=k}^{j+1} X_i^\psi \right), \quad j < k,$$

$$X_j^\varphi = \bigoplus_{i=j+1}^{\infty} X_i^\psi$$

$$X = X_j^\varphi \oplus \left(\bigoplus_{i=-\infty}^j X_i^\psi \right)$$

$$X = \bigoplus_{i=-\infty}^{\infty} X_i^\psi$$

This result shows that system $\{\psi_k^n\}_{k \in \mathbb{Z}}$, where functions

$$\psi_k^n(x) = \frac{1}{\sqrt{2^n}} \cdot \psi\left(\frac{1}{2^n} \cdot x - k\right), \quad x \in \mathbb{R}, k \in \mathbb{Z}$$

are generated by function ψ satisfying $E_1(|\hat{\varphi}|^2) = 1, E_1(|\hat{\psi}|^2) = 1$ and $X_0^\varphi = X_1^\varphi \oplus X_1^\psi$ is orthonormal, that is $\langle \psi_k^n, \psi_l^m \rangle = \delta_{kl} \cdot \delta_{nm}$, $k, l, n, m \in \mathbb{Z}$. ψ is called the *mother wavelet* of the MRA.

Subspaces $X_j^\varphi, k \in \mathbb{Z}$ are called approximation spaces, while spaces $X_j^\psi, k \in \mathbb{Z}$ are called detail spaces.

The equality $X_n^\varphi = X_{n+1}^\varphi \oplus X_{n+1}^\psi$ says that an element of the approximation space of level n is decomposed into an approximation at level $n+1$, which is less accurate, and a detail at level $n+1$.

According to the equality $X = \bigoplus_{i=-\infty}^{\infty} X_i^\psi$ any signal is the sum of all its details, namely its orthogonal projections onto the spaces X_j^ψ .

The wavelet coefficients of a signal f are provided by

$$\alpha_{k,n} = C_f(k, n) = \int_{\mathbb{R}} f(x) \cdot \psi_k^n(x) dx, \quad k, n \in \mathbb{Z}$$

Considering the decomposition $X = X_j^\varphi \oplus \left(\bigoplus_{i=-\infty}^j X_i^\psi \right)$ the orthonormal system

$$\{\varphi_k^0, \psi_k^n\}_{k, n \in \mathbb{Z}, n < 0}$$

is used instead of $\{\psi_k^n\}_{k \in \mathbb{Z}}$.

4.4.4 Condition Monitoring Based on Vibration Measurement

With the terms '*shock*' and '*vibration*' we refer to certain types of mechanical excitation that may cause a dynamic response of a mechanical structure (machinery). Shock is a dynamic excitation with a relatively short duration, and a vibration is a dynamic (generally approximately periodic) excitation with a relatively long duration.

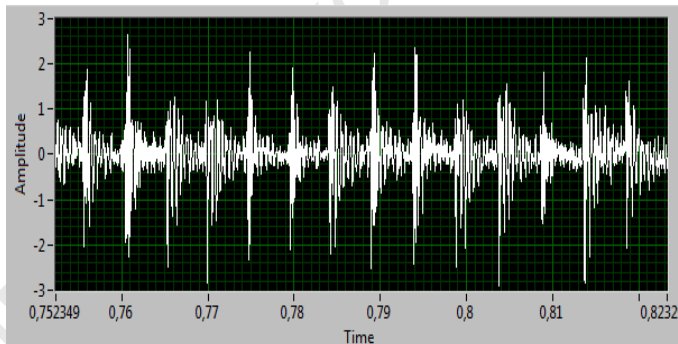
The two commonly used representation of vibration are the *waveform* (time-domain representation) and the *spectrum* (frequency-domain representation).

The waveform shows how the vibration level (e.g. displacement, velocity, acceleration) changes with time. Since the digital signal processing is based on sampled signals, the amount of information a waveform contains depends on the duration and resolution of the waveform. The duration of a waveform is the total time period over which information may be obtained from the waveform. In most cases, a few seconds are sufficient. The resolution of a waveform is a measure of the level of detail in the waveform and is determined by the sampling frequency. A spectrum is a graphical display of the frequencies at which a machine component is vibrating, together with the amplitudes of the component at these frequencies.

A waveform does not clearly display the individual frequencies at which vibration occurs. A waveform instead displays only the overall effect. Thus, the frequency spectrum is usually the primary tool for analysing machine vibration.

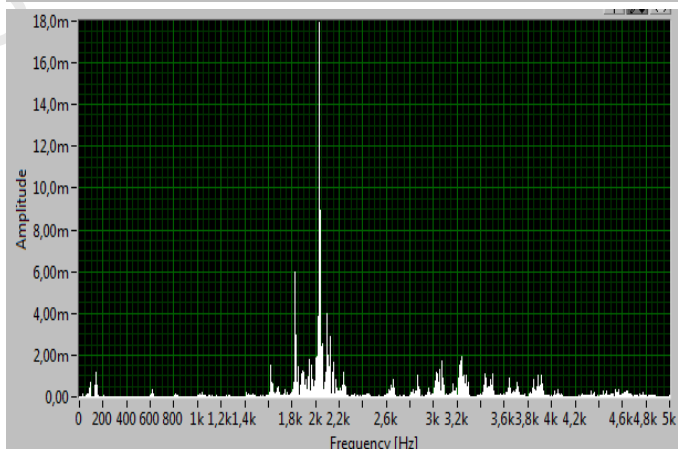
Waveform

a typical vibration signal
in the time domain
(amplitude vs. time)



Spectrum

a typical vibration signal
in the frequency domain
(frequency vs. time)



In technical diagnostics vibration is the physical movement of a machine containing rotating elements first of all.

Transducers connected to a structure (machine) convert the mechanical vibration into an electronic signal. With different mathematical calculations (e.g. statistical or Fourier analysis) information can be obtained about frequency and amplitude of the motion from the signal. Generally speaking, the amplitude describes relative severity of the problem, while the frequency describes what is wrong with the machine.

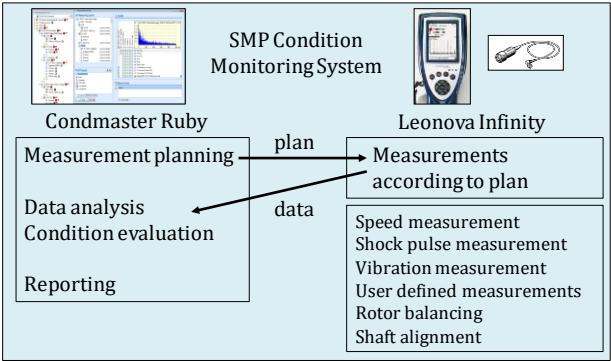
Some failures, for instance an unbalanced rotating equipment, generates (approximately) harmonic motion. Other failures, for example misaligned coupling, generates periodic but not harmonic motion. The third type of failures, for instance severe looseness, causes random motion of the structure.

The specific frequencies characteristic to a machine element can be calculated if the geometry and the operational parameters (e.g. rotational speed) are known. Some examples are imbalance, vane pass frequency (number of vanes times speed), gear-mesh frequency (number of teeth times speed), various frequencies generated by antifriction bearings, ball pass frequency of the outer ring (BPFO), ball pass frequency of inner ring (BPFI), ball spin frequency (BSF), and fundamental train frequency (FTF).

Vibration analysis of rotating machinery involves calculating the frequencies that a damaged machine can generate and measuring the frequencies the machine is actually generating. Then, relating the measured frequencies to the calculated frequencies the problems can be identified. The majority of problems generate complex symptoms, a group of lines (pattern) rather than a single line in the frequency spectrum. A typical symptom contains a fundamental frequency (related to a rotational speed of the connected shaft), and its second, third, etc. harmonics with different weights and so-called side-bands (lines in the neighbourhood of the characteristic frequencies).

The majority of mechanical failures of machine elements and also some electrical problems (e.g. problems of electric motors and variable speed drives) cause specific vibrations, professional vibration measuring systems with high accuracy are available at acceptable price, the installation of sensors (accelerometers first of all) is relatively easy. This is why the evaluation of the current condition of machines or machine elements is generally based on vibration measurement and vibration signal analysis in the industry. As an example, some functions of the SPM Condition Monitoring System related to the frequency spectrum are introduced in this part to show an industrial application of Fourier analysis.

SPM Condition Monitoring System contains a portable measuring device (Leonova Infinity) and a software (Condmaster Ruby).



The software is used for Measurement Planning (determination of measuring points and assigning measuring techniques) and for processing of (sampled) vibration signals downloaded from the portable measuring device (Data Analysis, Condition Evaluation, Reporting). The measuring device is used to accomplish the measuring plans uploaded to the device.

Measuring techniques are assigned to measuring points. Each selected technique requires measuring parameter settings, e.g. rotational speed, ISO number of bearing, number of fan blades or gear teeth.

The *Bearing Catalogue* (a built-in service of Condmaster Ruby) contains all data of bearings that are needed for the analysis. Figure shows two examples from the Bearing Catalogue (bearings of type ISO3205 and ISO6005).

Bearing number	3205	Bearing number	6005
Manufacturer	FAG	Manufacturer	KOY
TYPE no.	2	TYPE no.	3
Inner diameter	25 mm	Inner diameter	25 mm
Outer diameter	52 mm	Outer diameter	47 mm
Mean diameter	38,5 mm	Mean diameter	36 mm
Width	20,6 mm	Width	12 mm
	BPO 5,879		BPO 4,111
	BPI 8,121		BPI 5,889
	BS 2,634		BS 2,722
	FT 0,42		FT 0,411

The failure symptoms are enabled to the assignments by marking them in the *Symptom List*. We can determine what symptoms to be displayed in the graphics for each of the assignments. The default range for *SPM Spectrum* is 0 Hz to 1000 Hz, 1600 lines, full spectrum. This gives a good overview with room for several multiples of the bearing symptoms. In a *power spectrum*, line amplitudes are squared, which puts an emphasis on large amplitude lines while suppressing the smaller lines. The frequency range, the number of spectrum lines, spectrum type and number of peaks can be edited. Also the type of the window function can be set (rectangle, Hanning, Hamming or flat-top window).

Settings

Order tracking No ②
Upper frequency, Hz 200
Lower frequency, Hz 10
Window Hanning
Lines in spectrum 12800 ③
Envelope frequency (Hz) (None)

Advanced settings

FFT type Linear
Averaging type (None)
Zoom center 100 ①
Zoom factor 2 ⑤

0,46 RPM
50 -> 150 Hz (6521,74 -> 19565,22 Orders)
Resolution: 0,0078 Hz 0,47 CPM
Acquisition time: 128 second(s)
No. revolutions: 1

The appropriate frequency range setting depends on the operational parameters and the symptoms we want to check (number of orders). The software calculates the lowest available frequency range that accommodates all symptoms. The frequency range is limited by the transducer as well.

The frequency range for the spectrum should be large enough to contain the selected fault symptoms. The spectrum resolution is determined by the range divided by the number of lines in the spectrum. When lines from different symptoms are close together, we want a high

resolution to distinguish between them. For this, we select a large number of lines over a narrow frequency range.

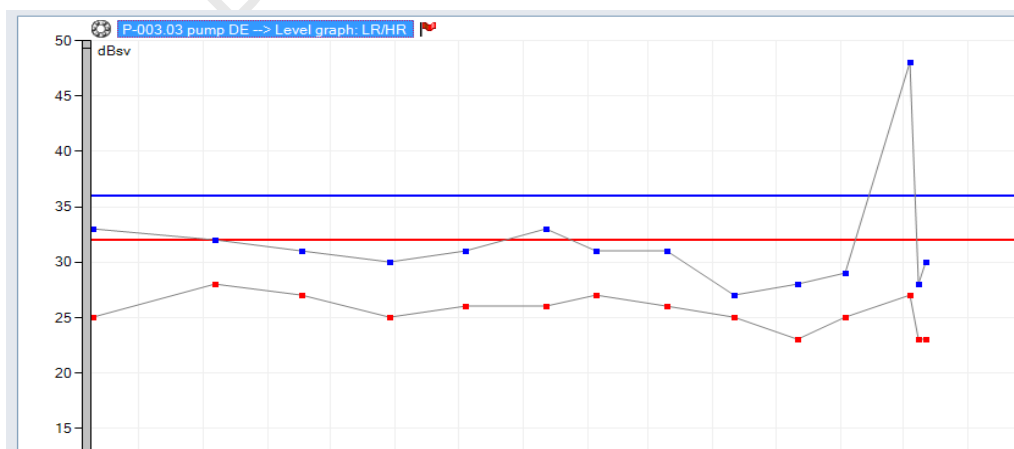
The setting of the required spectrum (frequency range, number of lines) will determine the frequency resolution of the spectrum and the data acquisition time as the lower right part of the figure shows. (Number of revolutions during the measurement is also displayed.)

Vibration velocity (VEL) is most useful and commonly measured on industrial machines in the normal speed range. Displacement (DISP) is used for low speed applications, while acceleration (ACC) is used for high speed applications. These physical quantities can be converted between each other easily.

A fault symptom highlights the spectrum lines affected by a certain type of machine fault. The simplest and most important is *Unbalance* which finds the line at $1\times$ shaft rotation in Hz. The symptom *Vane and impeller* is one of those you must adapt to your machine. The missing parameter is the number of vanes, e.g. the number of impeller blades in the pump. This data must be accurate, else the symptom is worthless.

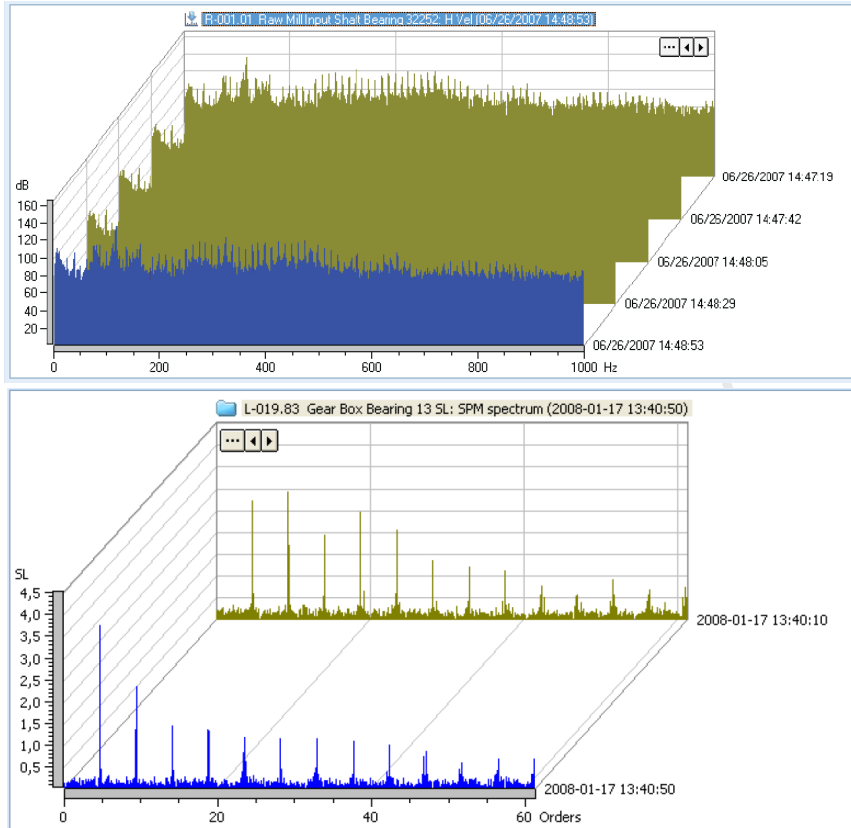
Cepstrum analysis, also called quefrency analysis (or cepstral analysis), is especially useful for detection and analysis of bearing vibrations and gear faults in gearboxes. Its strength is finding periodic components and repeated patterns in a time signal, which can be difficult to find in other types of spectra. Cepstrum is the forward Fourier transform of a logarithmic spectrum ('a spectrum of a spectrum'). Thus, while a frequency spectrum or FFT reveals the periodicity of a signal in the time domain, the cepstrum reveals the periodicity of a spectrum. In the cepstrum, overlapping sets of sidebands or harmonics will be separated, much like the spectrum separates repetitive time patterns in the waveform. The result of cepstrum analysis resembles a time signal, with the number of valid samples equal to the number of lines in the corresponding spectrum.

In predictive maintenance of machinery the control of the propagation of failures in time is even more important than the determination of the actual status. The SPM system contains many graphical functions providing information about changes in time. One type of diagram shows some numerical values as a function of time and also the related control limits. These diagrams enable the visual control, when a high value of a parameter appears relevant maintenance actions can be scheduled.

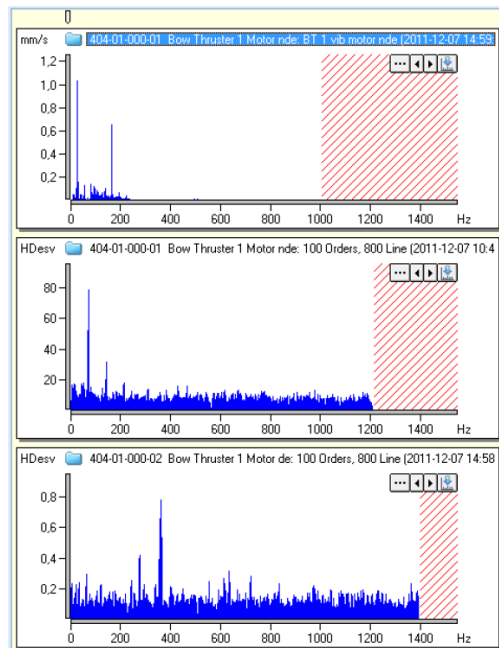


The other type of diagram shows the change of graphs, for instance the change of the spectrum. When the amplitudes belonging to critical frequencies increase or new frequencies appear in

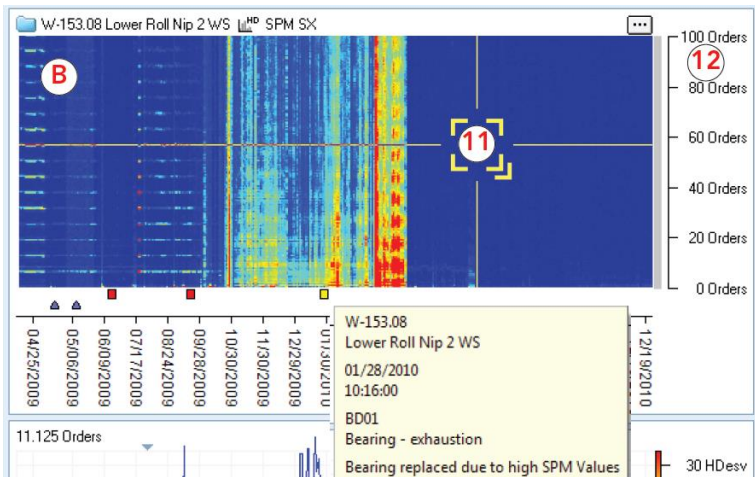
the spectrum the root cause of the change must be identified to avoid the further propagation of the failure. An important function is the so-called **Waterfall diagram** which is a three-dimensional display of up to 50 vibration spectra. The different readings are displayed along an axis, with the latest being the nearest the viewer.



With the **Compare spectrum** function we can view more than one frequency range and/or resolution at a time. This means that we can implement a variable frequency range from one measuring assignment to another and also between measuring points.



The **Coloured Spectrum Overview** is a three-dimensional view of all spectra under a particular measuring assignment. Its purpose is to simplify the process of identifying in spectra the patterns and trends which indicate damages. In the Coloured Spectrum Overview, signals which are always present in the machine are clearly distinguished from signals caused by developing damages. The Coloured Spectrum Overview provides a very good overall picture of machine condition development.

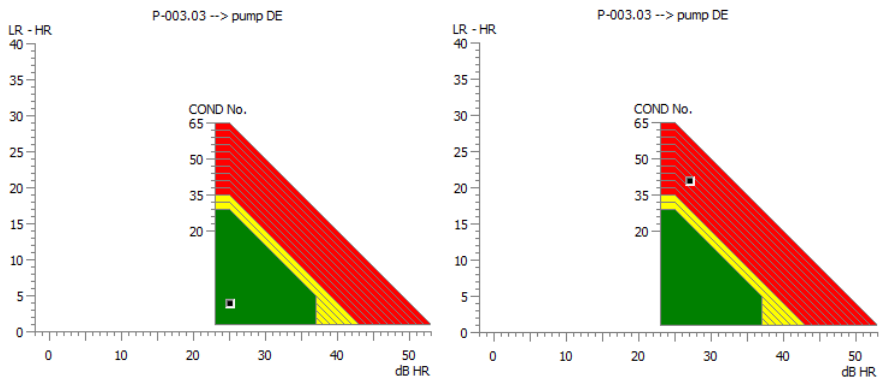


Important values related to shock pulse measurements are the so-called low-rate (LR) and high-rate (HR) values.

Also the healthy bearings generate low-energy shock pulses, their ‘frequency’ (number of pulses per second) is high. The magnitude belonging to the 1000 pulses per second ‘frequency’ (high rate of occurrence) is called *high-rate* (HR) *level* or *carpet level*. This level depends on the quality of lubrication first of all, thus the high value of high-rate level indicates the poor lubrication of the bearing.

When a fault appears on the surface of a bearing element then high-energy shock pulses appear with lower ‘frequency’ (number of pulses per second) depending on the type of the bearing, the geometry and the rotational speed of the shaft. The magnitude belonging to the 40 pulses per second ‘frequency’ (low rate of occurrence) is called *low-rate* (LR) *level*. The high value of high-rate level indicates a bad surface condition inside the bearing.

The LUBMASTER modul of the SPM system provides the following diagram showing the condition of bearings (LR-HR vs HR). The left-hand side diagram belongs to a healthy bearing while left-hand side diagram shows that the bearing has a surface fault.



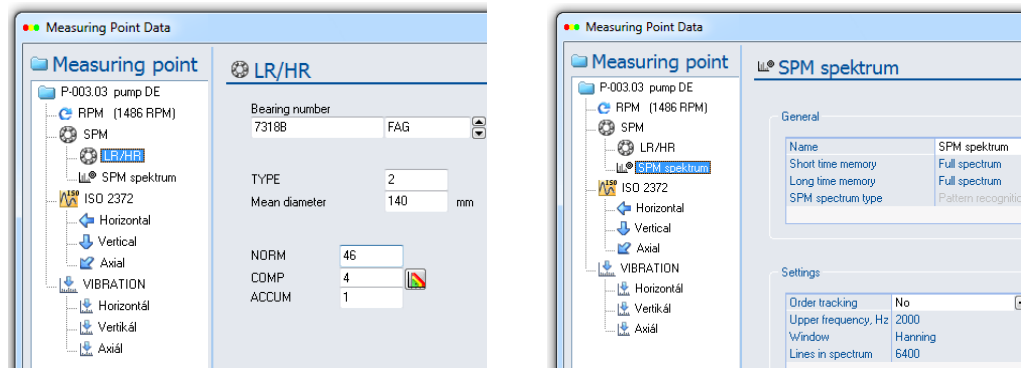
4.4.5 A Case Study: Condition Monitoring of a Pump Bearing

The drive-end bearing of a pump used in a chemical production process was investigated. Several measuring techniques were assigned to the measuring point, among others, the shock pulse measurement.

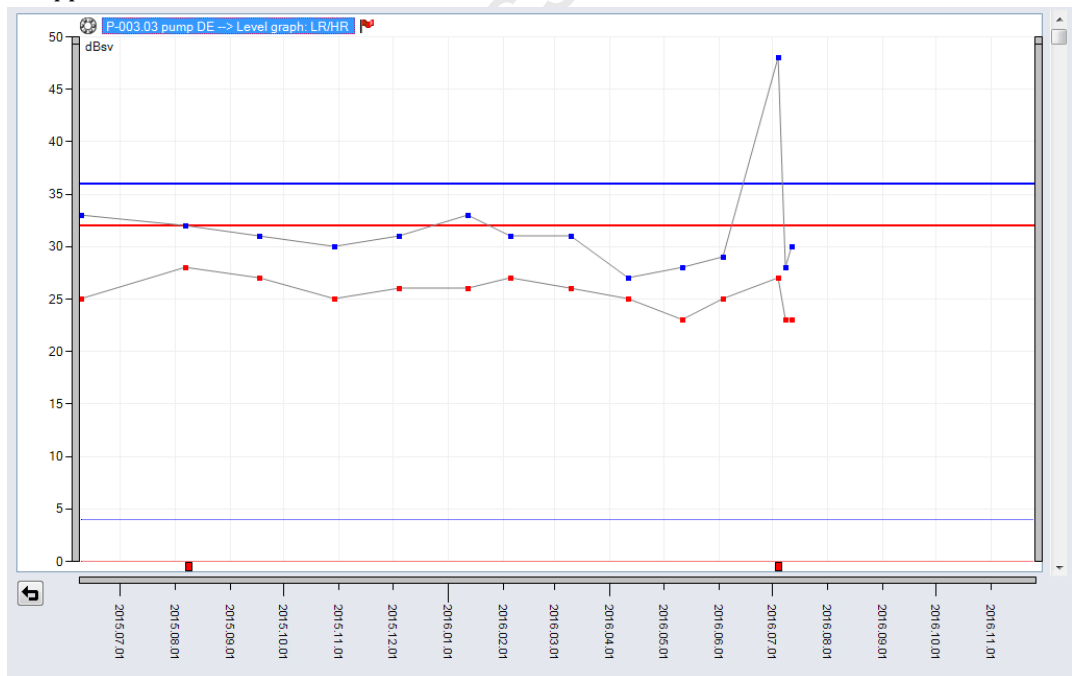
The type of the bearing was 7318B.



The figure shows the settings for shock pulse measurement and analysis.

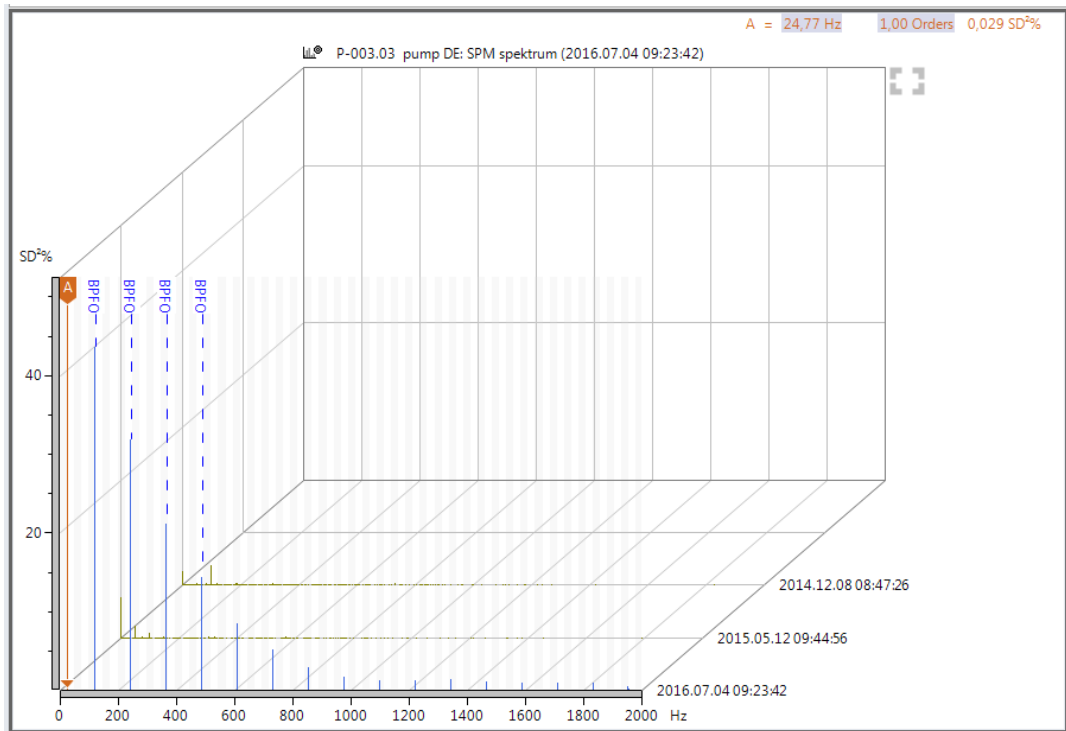


Studying the time series of LR/HR values we can see that on a certain date a very high value of LR appeared.

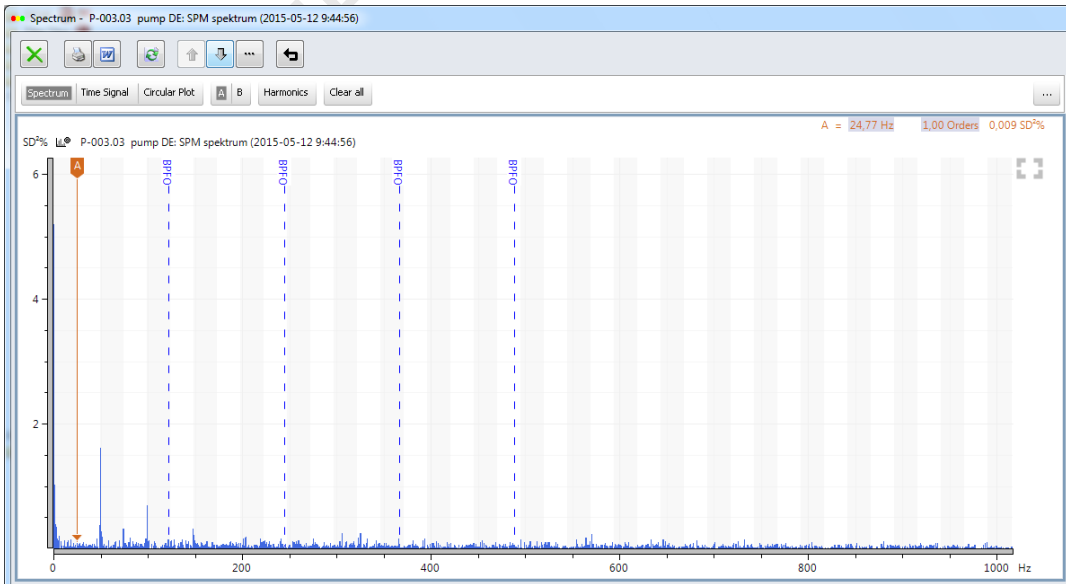


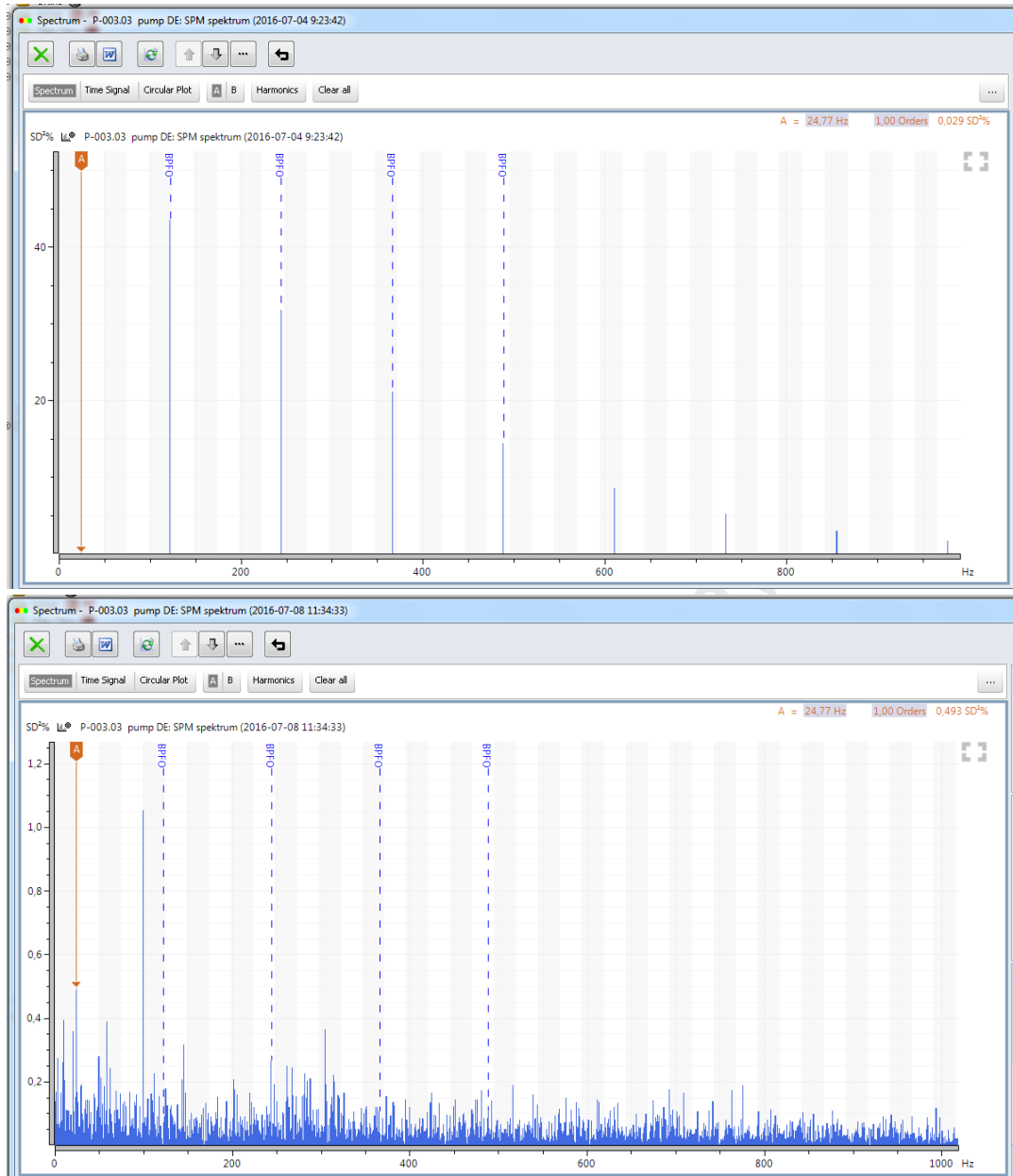
This peak triggered an alarm in the monitoring system, and then the experts checked all the data related to the measuring point (drive-end bearing of the pump). The waterfall diagram

shows clearly, that the measure of amplitude enhancement was significant at certain frequencies.



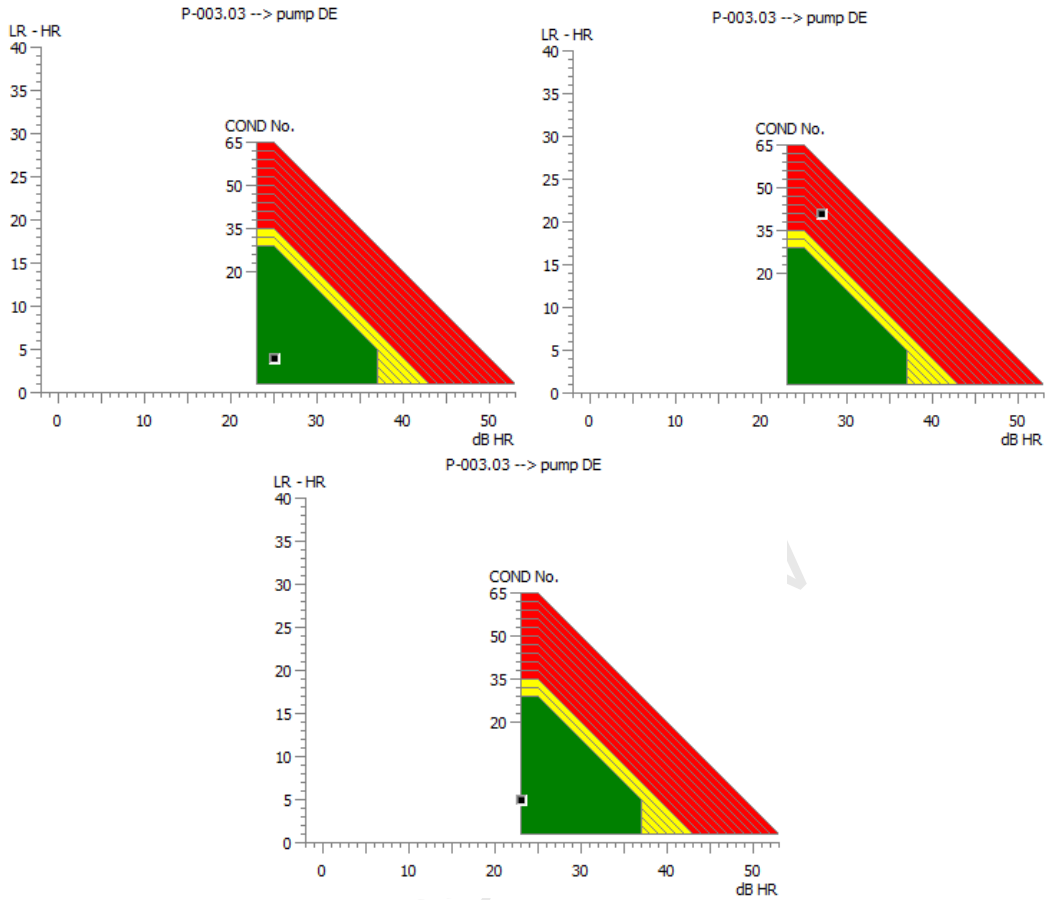
Further investigations showed that the high lines matched the symptom lines that belong to the outer ring fault (BPFO), that is, a failure of the outer ring was detected. The following three diagrams show the spectrum measured before outer ring fault appeared (good condition), when the problem has developed (presence of failure), and after installing a new bearing (good condition again).





The special bearing evaluation technique provided by SPM, namely the Lubmaster analysis indicated the problem clearly.

We can see in the following three diagrams that the black dot was in the red region (indicating a bad surface condition) when the fault developed, while the dot was in green region (indicating a good surface condition) before the appearance of the problem and after changing the bearing.



The picture shows the severe fault on the surface of the outer ring.



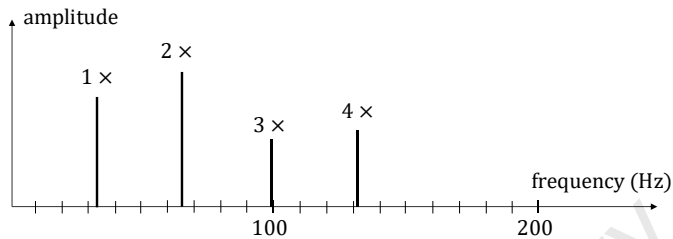
4.4.6 Exercises Related to the Vibration Spectrum

>>>Exercise 4.4 – 1

The bearing fault coefficients for the bearing of type ISO6302 are the following

Bearing number	6302		
Manufacturer	NSK		
TYPE no.	1		
Inner diameter	15	mm	BPO 2,558
Outer diameter	42	mm	BPI 4,442
Mean diameter	28,5	mm	BS 1,724
Width	13	mm	FT 0,365

The rotational speed of the shaft during the measurement is 1140 *rpm*. The sketch of the spectrum provided by SPM Ruby is



Which element of the bearing is damaged: outer ring, inner ring, ball, cage, or none of them?

>>> **Exercise 4.4 – 1 – Solution**

The bearing fault frequencies belonging to the rotational speed of 1140 *rpm* = 19 *rps* are

fault type	coefficient	rotational speed (rps)	fault frequency (Hz)
outer ring	2.558	19	48.60
inner ring	4.442	19	84.40
ball	1.724	19	32.76
cage	0.365	19	6.94

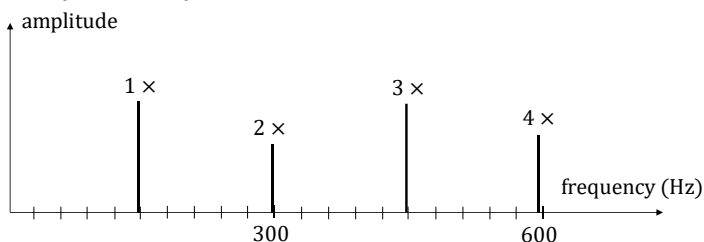
The spectrum contains a frequency near to 32.765 *Hz* (ball spin frequency) and its harmonics. It suggests that there is a fault on a ball.

>>> **Exercise 4.4 – 2**

The bearing fault coefficients for the bearing of type ISO30210 are the following

Bearing number	30210		
Manufacturer	SKF		
TYPE no.	6		
Inner diameter	50	mm	BPO 8,571
Outer diameter	90	mm	BPI 11,429
Mean diameter	70	mm	BS 3,301
Width	21,75	mm	FT 0,429

The rotational speed of the shaft during the measurement is 780 *rpm*. The sketch of the spectrum provided by SPM Ruby is



Which element the symptom lines in the figure belong to (outer ring, inner ring, ball, cage, or none of them)?

>>>**Exercise 4.4 – 2- Solution**

The bearing fault frequencies belonging to the rotational speed of 780 *rpm* = 13 *rps* are

fault type	coefficient	rotational speed (<i>rps</i>)	fault frequency (<i>Hz</i>)
outer ring	8.571	13	111.42
inner ring	11.429	13	148.58
ball	3.301	13	42.92
cage	0.429	13	5.577

The spectrum contains a frequency near to 148.58 *Hz* (inner ring frequency) and its harmonics. It suggests that there is a fault on the inner ring.

>>>**Exercise 4.4 – 3**

It is known that the specific symptom of a coupling problem is a line in the frequency spectrum at 2nd order. Determine the specific frequency belonging to the coupling problem if the rotational speed of the shaft is 1800 *rpm*.

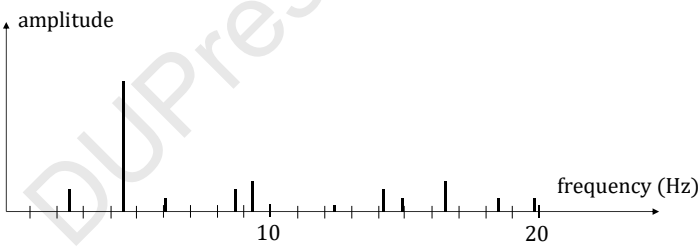
>>>**Exercise 4.4 – 3- Solution**

The rotational speed is 1800 *rpm* = 50 *rps*.

The line belonging to the coupling problem is at $2 \times 50 = 100$ *Hz* in the frequency spectrum.

>>>**Exercise 4.4 – 4**

Sometimes a possible way to determine the rotational speed of the shaft is to find the frequency of 1st order (frequency belonging to unbalance). Since this frequency generally belongs to the highest energy harmonic component of the signal, the shaft speed can be identified finding the highest line in the frequency spectrum. Give the probable value of the rotational speed of the shaft using the following spectrum.



>>>**Exercise 4.4 – 4- Solution**

The highest line in the frequency spectrum is near to 4.5 *Hz*. It suggests that the rotational speed of the shaft is $4.5 \text{ rps} = 4.5 \times 60 = 225 \text{ rpm}$.

5 The Laplace Transform

5.1 The Concept of the Laplace Transform

Let $f: [0, \infty[\rightarrow \mathbb{R}$ be an integrable function. Then function

$$\mathcal{L}(f)(s) = F(s) = \int_0^{\infty} f(t) \cdot e^{-s \cdot t} dt, \quad s \in \mathbb{C}$$

is called the *Laplace transform* of f if the integral is convergent.

Remark: Based on the calculation

$$F(s) = \int_0^{\infty} f(t) \cdot e^{-s \cdot t} dt = \int_0^{\infty} f(t) \cdot e^{-(\sigma + i \cdot \omega) \cdot t} dt = \int_0^{\infty} (f(t) \cdot e^{-\sigma \cdot t}) \cdot e^{-i \cdot \omega \cdot t} dt,$$

we can say that the Laplace transform of f is equal to the Fourier transform of function

$$t \rightarrow \begin{cases} f(t) \cdot e^{-\sigma \cdot t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Considering the properties of function $e^{-\sigma \cdot t}$ for positive σ , it is easy to see that, many Laplace transformable functions are not Fourier transformable. This advantage makes the Laplace transform important tool e.g. in control theory.

f is called a *function of exponential order* if there exist real numbers c, K and T such that

$$|f(t)| \leq K \cdot e^{c \cdot t}, \quad t \geq T.$$

If a piecewise continuous f is of exponential order, then $F(s)$ exists for values s satisfying $\text{Re}(s) \geq c$.

The inverse Laplace transform is given by

$$f(t) = \mathcal{L}^{-1}(F(s)) = \frac{1}{2\pi i} \cdot \int_{\gamma - i \cdot \infty}^{\gamma + i \cdot \infty} F(s) \cdot e^{s \cdot t} ds, \quad t \in \mathbb{R}$$

for some $0 < \gamma \in \mathbb{R}$.

Remark: In practice we rarely calculate Laplace transforms or inverse Laplace transforms, because the frequently used formulas are available in tables, and in mathematical software.

A table of the most frequently used Laplace transforms:

$f(t)$	$F(s)$	$f(t)$	$F(s)$
1	$\frac{1}{s}$	$\sin(\omega \cdot t)$	$\frac{\omega}{s^2 + \omega^2}$
t	$\frac{1}{s^2}$	$\cos(\omega \cdot t)$	$\frac{s}{s^2 + \omega^2}$
t^n	$\frac{n!}{s^{n+1}}$	$t \cdot \sin(\omega \cdot t)$	$\frac{2 \cdot \omega \cdot s}{(s^2 + \omega^2)^2}$
$e^{a \cdot t}$	$\frac{1}{s - a}$	$t \cdot \cos(\omega \cdot t)$	$\frac{s^2 - \omega^2}{s^2 + \omega^2}$
$t \cdot e^{a \cdot t}$	$\frac{1}{(s - a)^2}$	$e^{a \cdot t} \cdot \sin(\omega \cdot t)$	$\frac{\omega}{(s - a)^2 + \omega^2}$
$t^2 \cdot e^{a \cdot t}$	$\frac{2}{(s - a)^3}$	$e^{a \cdot t} \cdot \cos(\omega \cdot t)$	$\frac{s - a}{(s - a)^2 + \omega^2}$

$t^n \cdot e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$e^{at} \cdot \sinh(\omega \cdot t)$	$\frac{\omega}{(s-a)^2 - \omega^2}$
		$e^{at} \cdot \cosh(\omega \cdot t)$	$\frac{s-a}{(s-a)^2 - \omega^2}$

Some properties of Laplace transform

Linearity

$$\mathcal{L}(\alpha \cdot f + \beta \cdot g) = \alpha \cdot \mathcal{L}(f) + \beta \cdot \mathcal{L}(g)$$

Derivative in time domain

$$\mathcal{L}\left(\frac{df}{dt}\right) = s \cdot \mathcal{L}(f) - f(0)$$

$$\mathcal{L}\left(\frac{d^2f}{dt^2}\right) = s^2 \cdot \mathcal{L}(f) - s \cdot f(0) - \frac{df}{dt}(0)$$

$$\mathcal{L}\left(\frac{d^3f}{dt^3}\right) = s^3 \cdot \mathcal{L}(f) - s^2 \cdot f(0) - s \cdot \frac{df}{dt}(0) - \frac{d^2f}{dt^2}(0)$$

⋮

$$\mathcal{L}\left(\frac{d^nf}{dt^n}\right) = s^n \cdot \mathcal{L}(f) - s^{n-1} \cdot f(0) - \dots - s \cdot \frac{d^{n-2}f}{dt^{n-2}}(0) - \frac{d^{n-1}f}{dt^{n-1}}(0)$$

Integral in time domain

$$\mathcal{L}\left(\int_0^t f\right) = \frac{1}{s} \cdot \mathcal{L}(f)$$

Integral in frequency domain

$$\mathcal{L}\left(\frac{1}{t} \cdot f(t)\right) = \int_s^\infty F$$

Time domain shift

$$\mathcal{L}(f(t-b)) = e^{bs} \cdot \mathcal{L}(f)$$

Frequency domain shift

$$\mathcal{L}(e^{bt} \cdot f(t)) = F(s-b)$$

Time domain scaling

$$\mathcal{L}(f(a \cdot t)) = \frac{1}{a} \cdot F\left(\frac{s}{a}\right)$$

Convolution in time domain

$$\mathcal{L}(f * g) = \mathcal{L}\left(\int_0^t f(\tau) \cdot g(t-\tau) d\tau\right) = \mathcal{L}(f) \cdot \mathcal{L}(g)$$

Complex conjugation

$$\mathcal{L}(f^*(t)) = F^*(s^*)$$

>>>Exercise 5.1 – 1

Determine the Laplace transform of function $f: [0, \infty[\rightarrow \mathbb{R}$ defined as

$$f(t) = 1$$

>>>Exercise 5.1 – 1- Solution

$$F(s) = \int_0^\infty 1 \cdot e^{-st} dt = \lim_{b \rightarrow \infty} \int_0^b e^{-st} dt = \lim_{b \rightarrow \infty} \left[\frac{e^{-st}}{-s} \right]_0^b = -\frac{1}{s} \cdot \lim_{b \rightarrow \infty} (e^{-sb} - 1) = \frac{1}{s}$$

>>>Exercise 5.1 – 2

Determine the Laplace transform of function $f: [0, \infty[\rightarrow \mathbb{R}$ defined as

$$f(t) = e^{a \cdot t}$$

>>>Exercise 5.1 – 2- Solution

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{a \cdot t} \cdot e^{-s \cdot t} dt = \lim_{b \rightarrow \infty} \int_0^b e^{-(s-a) \cdot t} dt = \lim_{b \rightarrow \infty} \left[\frac{e^{-(s-a) \cdot t}}{-(s-a)} \right]_0^b = \\ &= -\frac{1}{s-a} \cdot \lim_{b \rightarrow \infty} (e^{-(s-a) \cdot b} - 1) = \frac{1}{s-a} \end{aligned}$$

>>>Exercise 5.1 – 3

Determine the inverse Laplace transform of function

$$X(s) = \frac{3}{s^2 + 3 \cdot s - 10}$$

>>>Exercise 5.1 – 3- Solution

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{3}{s^2 + 3 \cdot s - 10}\right) &= \mathcal{L}^{-1}\left(\frac{3}{7} \cdot \frac{1}{s-2} - \frac{3}{7} \cdot \frac{1}{s+5}\right) = \\ &= \frac{3}{7} \cdot \mathcal{L}^{-1}\left(\frac{1}{s-2}\right) - \frac{3}{7} \cdot \mathcal{L}^{-1}\left(\frac{1}{s+5}\right) = \frac{3}{7} \cdot e^{2 \cdot t} - \frac{3}{7} e^{-5 \cdot t} \end{aligned}$$

Second solution: using the convolution theorem

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{3}{s^2 + 3 \cdot s - 10}\right) &= \mathcal{L}^{-1}\left(3 \cdot \frac{1}{s-2} \cdot \frac{1}{s+5}\right) = \\ &= 3 \cdot \mathcal{L}^{-1}\left(\frac{1}{s-2}\right) * \mathcal{L}^{-1}\left(\frac{1}{s+5}\right) = 3 \cdot e^{2 \cdot t} * e^{-5 \cdot t} = \\ &= 3 \cdot \int_0^t e^{2 \cdot \tau} \cdot e^{-5 \cdot (t-\tau)} d\tau = 3 \cdot e^{-5 \cdot t} \cdot \int_0^t e^{7 \cdot \tau} d\tau = \\ &= \frac{3}{7} \cdot e^{-5 \cdot t} \cdot (e^{7 \cdot t} - 1) = \frac{3}{7} \cdot e^{2 \cdot t} - \frac{3}{7} e^{-5 \cdot t} = x(t) \end{aligned}$$

>>>Exercise 5.1 – 4

Determine the inverse Laplace transform of function

$$X(s) = \frac{4s^2 - s + 1}{s^3 - s^2}$$

>>>Exercise 5.1 – 4- Solution

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{4s^2 - s + 1}{s^3 - s^2}\right) &= \mathcal{L}^{-1}\left(-\frac{1}{s^2} + 4 \cdot \frac{1}{s-1}\right) = \\ &= -\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + 4 \cdot \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = -t + 4 \cdot e^t = x(t) \end{aligned}$$

>>>Exercise 5.1 – 5

Determine the inverse Laplace transform of function

$$X(s) = \frac{s^2 + 8s + 8}{s^3 + s^2}$$

>>>Exercise 5.1 – 5- Solution

$$\mathcal{L}^{-1}\left(\frac{s^2 + 8s + 8}{s^3 + s^2}\right) = \mathcal{L}^{-1}\left(\frac{8}{s^2} + \frac{1}{s+1}\right) = 8 \cdot \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = 8t + e^{-t} = x(t)$$

$$\frac{s^2 + 8s + 8}{s^3 + s^2} = \frac{s^2 + 8s + 8}{s^2 \cdot (s + 1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s + 1} = \frac{A \cdot (s + 1) + B \cdot s \cdot (s + 1) + C \cdot s^2}{s^2 \cdot (s + 1)}$$

$$= \frac{s^2 \cdot (B + C) + s \cdot (A + B) + A}{s^2 \cdot (s + 1)}$$

$$I. \quad B + C = 1$$

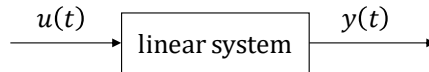
$$II. \quad A + B = 8 \quad \Rightarrow \quad A = 8, \quad B = 0, \quad C = 1$$

$$III. \quad A = 8$$

5.2 Transfer Function, Frequency Response

In this part the complex imaginary unit is denoted by j as usual in the technical literature, e.g. in control theory.

A linear time-invariant system



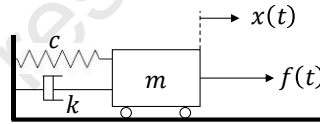
can be characterized by its *transfer function*, the quotient of the Laplace transform of the output function (Y) and the input function (U)

$$H(s) = \frac{Y(s)}{U(s)}, \quad s \in \mathbb{C}.$$

According to this definition, in the 'Laplace space', the output of a linear system can be given with a simple multiplication if the transfer function H is known:

$$Y(s) = H(s) \cdot U(s)$$

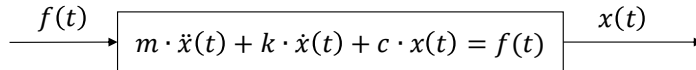
Find the transfer function for a translational mass-spring-damper system.



The differential equation of the system (Newton's second law) is

$$m \cdot \ddot{x}(t) + k \cdot \dot{x}(t) + c \cdot x(t) = f(t),$$

where m is the mass, k is the damping coefficient, c the spring stiffness, f is the external force acting on the body (input), x is the displacement of the body (output).



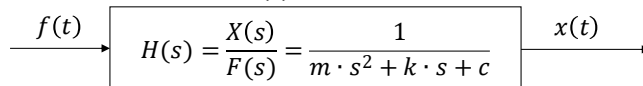
Supposing zero initial conditions

$$x(0) = \dot{x}(0) = 0$$

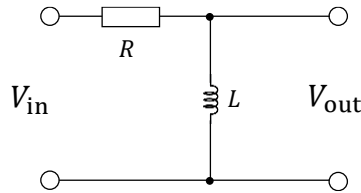
the Laplace transform of the equation is

$$m \cdot s^2 \cdot X(s) + k \cdot s \cdot X(s) + c \cdot X(s) = F(s)$$

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{m \cdot s^2 + k \cdot s + c}$$



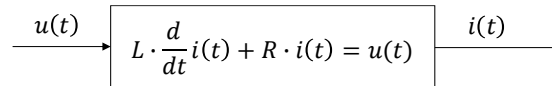
Find the transfer function of a series RL circuit connected to a continuous current voltage source.



The differential equation of the RL circuit is

$$L \cdot \frac{d}{dt} i(t) + R \cdot i(t) = u(t)$$

where R is the resistance, L is the inductance, $u = V_{in}$ is the voltage drop across the circuit (input), i is the electrical current through the circuit (output).



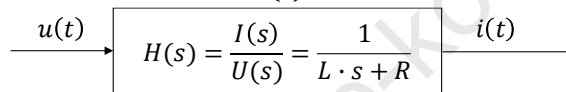
Supposing zero initial condition

$$i(0) = 0$$

the Laplace transform of the equation is

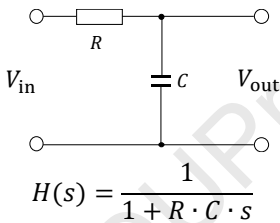
$$L \cdot s \cdot I(s) + R \cdot I(s) = U(s)$$

$$H(s) = \frac{I(s)}{U(s)} = \frac{1}{L \cdot s + R}$$

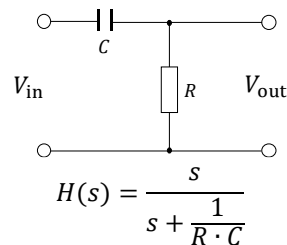


Some basic electrical circuits and their transfer functions

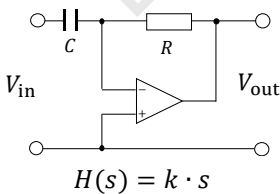
Low pass filter



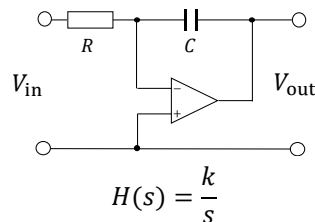
High pass filter



Differentiator



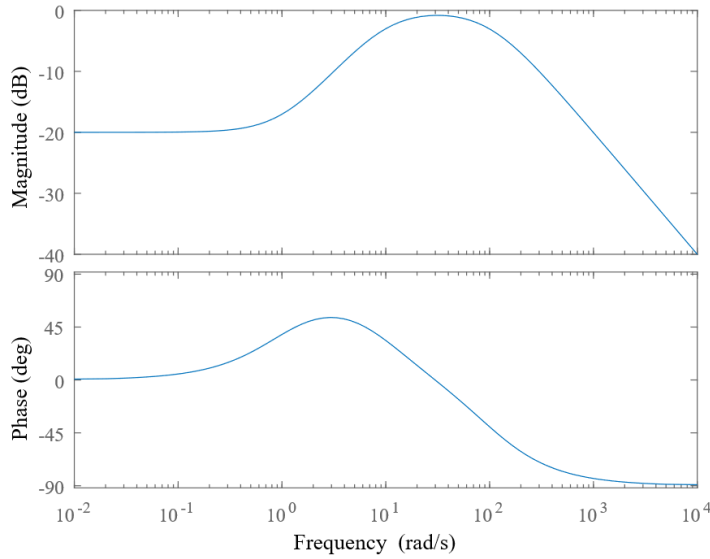
Integrator



As an example, consider a system with transfer function

$$H(s) = \frac{100 \cdot s + 100}{s^2 + 110 \cdot s + 1000}$$

The Bode diagram of the system (provided by software) is



Although mathematical software provides the exact Bode diagram, it is useful to sketch it to understand the role of the poles and zeros. To do it we need to rewrite the transfer function so that all the poles and zeros appear in the form $\left(1 + \frac{s}{\omega_0}\right)$:

$$H(s) = \frac{100 \cdot s + 100}{s^2 + 110 \cdot s + 1000} = 100 \cdot \frac{s + 1}{(s + 10) \cdot (s + 100)} = 0.1 \cdot \frac{1 + \frac{s}{1}}{\left(1 + \frac{s}{10}\right) \cdot \left(1 + \frac{s}{100}\right)}$$

This expression contains four terms of interest

a constant: 0.1, a zero: $s = -1$, and two poles: $s = -10$ and $s = -100$.

Using the phasor representation for the input and the transfer function, calculations we have to do are the multiplication of the magnitudes and the addition of the phases: the magnitude and phase of

$$H(j\omega) = 0.1 \cdot \frac{1 + \frac{j\omega}{1}}{\left(1 + \frac{j\omega}{10}\right) \cdot \left(1 + \frac{j\omega}{100}\right)}$$

are

$$|H(s)| = |0.1| \cdot \frac{\left|1 + \frac{j\omega}{1}\right|}{\left|1 + \frac{j\omega}{10}\right| \cdot \left|1 + \frac{j\omega}{100}\right|}$$

and

$$\angle(H(s)) = \angle(0.1) + \angle\left(1 + \frac{j\omega}{1}\right) - \angle\left(1 + \frac{j\omega}{10}\right) - \angle\left(1 + \frac{j\omega}{100}\right),$$

respectively.

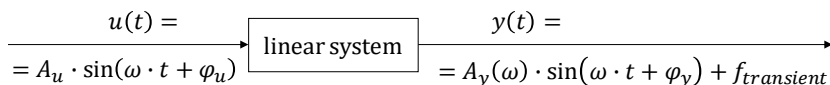
Addition of phases is quite simple. To avoid multiplication and division of magnitudes the decibel (dB) scale is commonly used by applying the following:

a quantity X in decibel scale is $20 \cdot \lg X$.

The magnitudes are simply added and subtracted in the decibel scale because the properties of the logarithm function:

$$\begin{aligned}
 |H(j\omega)|_{\text{dB}} &= 20 \cdot \lg(|H(s)|) = 20 \cdot \lg \left(|0.1| \cdot \frac{\left|1 + \frac{j\omega}{1}\right|}{\left|1 + \frac{j\omega}{10}\right| \cdot \left|1 + \frac{j\omega}{100}\right|} \right) = \\
 &= 20 \cdot \lg(|0.1|) + 20 \cdot \lg \left(\left|1 + \frac{j\omega}{1}\right| \right) - 20 \cdot \lg \left(\left|1 + \frac{j\omega}{10}\right| \right) - 20 \cdot \lg \left(\left|1 + \frac{j\omega}{100}\right| \right) = \\
 &= |0.1|_{\text{dB}} + \left|1 + \frac{j\omega}{1}\right|_{\text{dB}} - \left|1 + \frac{j\omega}{10}\right|_{\text{dB}} - \left|1 + \frac{j\omega}{100}\right|_{\text{dB}}
 \end{aligned}$$

The *frequency response (function)* plays important role in the analysis of linear time-invariant systems. The following diagram suggests a basic property of linear time-invariant systems, namely, the steady-state response to a sinusoidal input is sinusoidal output with the same frequency but with different amplitude and phase depending on ω .



Giving the change of the amplitude and phase as a function of ω we can describe a linear system properly.

Let $t \rightarrow u(t) = A_u \cdot \sin(\omega \cdot t + \varphi_u)$ and $t \rightarrow y(t) = A_y \cdot \sin(\omega \cdot t + \varphi_y)$ be the input and the steady-state output of a linear system, respectively. The *frequency response function* (or *frequency function*) of the system is the complex-valued function

$$H(j\omega) = a(\omega) \cdot e^{j\varphi(\omega)},$$

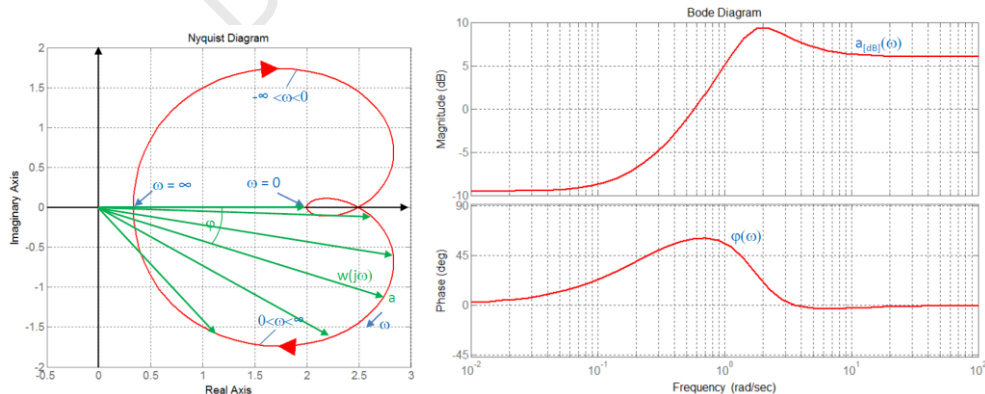
where

$$a(\omega) = \frac{A_y(\omega)}{A_u}, \quad \text{and} \quad \varphi(\omega) = \varphi_y - \varphi_u$$

are the *gain function* and the *phase shift function*.

The frequency response is closely related to the transfer function: if $s \rightarrow H(s)$ is the transfer function, then $\omega \rightarrow H(j\omega)$ is the frequency function.

Well-known representations of the transfer function are the *Nyquist diagram* and the *Bode diagram*. The Nyquist diagram shows the values of frequency function in the complex plane, each point belongs to a certain ω value. The Bode diagram contains the graphs of $t \rightarrow a(\omega)$ and $t \rightarrow \varphi(\omega)$ functions.



In the Bode diagram a logarithmic scale is used in the case of the frequency axis because of the wide range of frequencies generally need to be displayed.

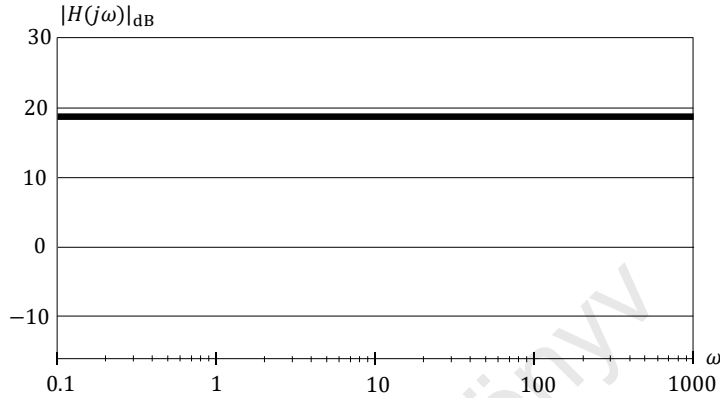
We also use logarithmic (decibel) scale when plotting the magnitude $a(\omega)$, because of the wide range of gains to be displayed, and additionally, the use of the decibel scale makes the sketch of the diagram easier in practice.

>>>Exercise 5.2 – 17

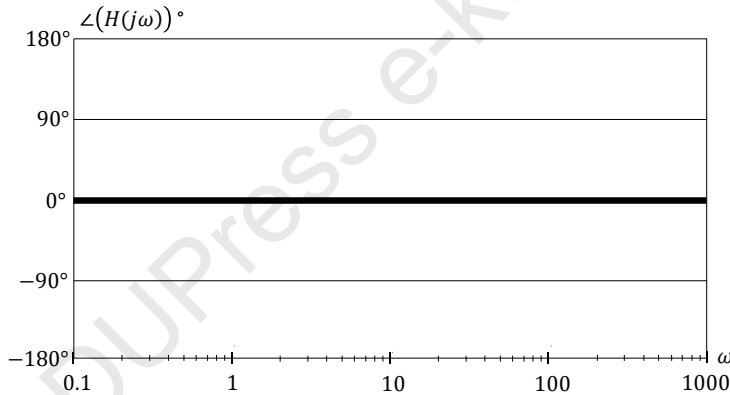
Sketch the Bode diagram of the system with transfer function $H(s) = 8.2$ (constant function).

>>>Exercise 5.2 – 17- Solution

$$|H(j\omega)|_{\text{dB}} = 20 \cdot \lg 8.2 = 18.28 \text{ dB (constant)}$$



$$\angle(H(s)) = 0^\circ \text{ (constant)}$$



>>>Exercise 5.2 – 18

Analyse the frequency response of the system with transfer function $H(s) = \frac{1}{1 + \frac{s}{\omega_0}}$ in the three special cases $\omega \ll \omega_0$ (low frequency), $\omega \gg \omega_0$ (high frequency), and $\omega \approx \omega_0$. (Simple real pole at $-\omega_0$).

>>>Exercise 5.2 – 18- Solution

Magnitude

$$|H(j\omega)|_{\text{dB}} = 20 \cdot \lg \left| \frac{1}{1 + j \cdot \frac{\omega}{\omega_0}} \right| = 20 \cdot \lg \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}}$$

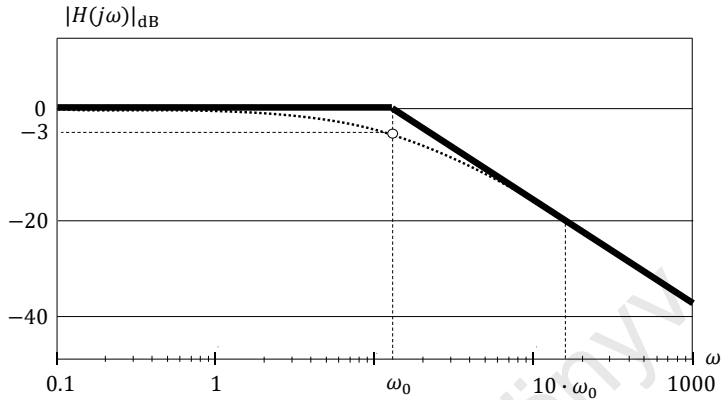
$$\text{If } \omega \ll \omega_0, \text{ then } \sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2} \approx 1 \Rightarrow |H(j\omega)|_{\text{dB}} \approx 0$$

$$\text{If } \omega \gg \omega_0, \text{ then } \sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2} \approx \frac{\omega}{\omega_0} \Rightarrow |H(j\omega)|_{\text{dB}} \approx 20 \cdot \lg \frac{\omega_0}{\omega} = -20 \cdot \lg \omega + 20 \cdot \lg \omega_0 = \ell(\omega)$$

This is a straight line with a slope of $-20 \frac{\text{dB}}{\text{decade}}$ going through the point $(\omega_0, 0 \text{ dB})$. For example, a couple of values of ℓ : $\ell(\omega_0) = 20 \cdot \lg 1 = 0 \text{ dB}$, $\ell(10 \cdot \omega_0) = 20 \cdot \lg 0.1 = -20 \text{ dB}$, $\ell(100 \cdot \omega_0) = 20 \cdot \lg 0.01 = -40 \text{ dB}$

If $\omega \approx \omega_0$, then $|H(j\omega_0)|_{\text{dB}} \approx 20 \cdot \lg \frac{1}{\sqrt{1 + \left(\frac{\omega_0}{\omega_0}\right)^2}} = 20 \cdot \lg \frac{1}{\sqrt{2}} \approx -3 \text{ dB}$

Remark: ω_0 is called break frequency (or corner frequency or 3 dB frequency)



Phase

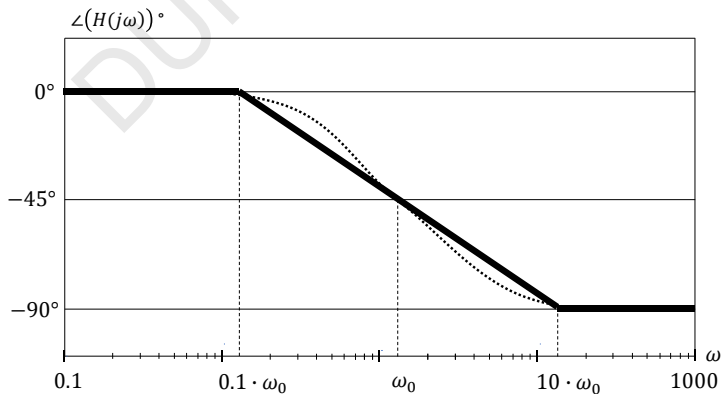
$$\angle(H(j\omega)) = \angle\left(\frac{1}{1 + j \cdot \frac{\omega}{\omega_0}}\right) = -\angle\left(1 + j \cdot \frac{\omega}{\omega_0}\right) = -\arctan \frac{\omega}{\omega_0}$$

If $\omega \ll \omega_0$, then $\angle H(j\omega) \approx -\arctan 0 = 0^\circ$

If $\omega \gg \omega_0$, then $\angle H(j\omega) \approx -\arctan \infty = -90^\circ$

If $\omega \approx \omega_0$, $\angle H(j\omega) \approx -\arctan 1 = -45^\circ$

In this case a possible plotting rule for the phase diagram is to draw straight line between points $(0.1 \cdot \omega_0, 0^\circ)$ and $(10 \cdot \omega_0, -90^\circ)$.



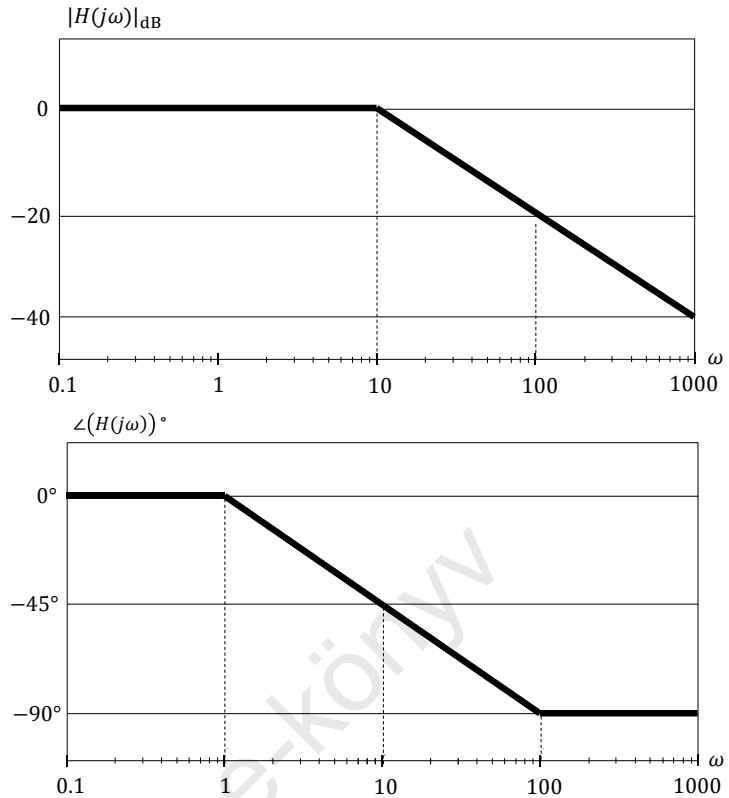
>>>Exercise 5.2 – 19

Sketch the Bode diagram of the system with transfer function $H(s) = \frac{1}{1 + \frac{s}{10}}$ (Simple real pole at $-10 \frac{\text{rad}}{\text{s}}$).

>>>Exercise 5.2 – 19– Solution

$$|H(j\omega)|_{dB} =$$

$$= 20 \cdot \lg \frac{1}{\sqrt{1 + \left(\frac{\omega}{10}\right)^2}}$$



>>>Exercise 5.2 – 20

Analyse the frequency response of the system with transfer function $H(s) = \frac{1}{\left(1 + \frac{s}{\omega_0}\right)^n}$ (n is a positive integer) in the three special cases $\omega \ll \omega_0$ (low frequency), $\omega \gg \omega_0$ (high frequency), and $\omega \approx \omega_0$ (n^{th} order real pole at $-\omega_0$).

>>>Exercise 5.2 – 20- Solution

Magnitude

$$|H(j\omega)| = \left| \frac{1}{\left(1 + j \cdot \frac{\omega}{\omega_0}\right)^n} \right| = \frac{1}{\left|1 + j \cdot \frac{\omega}{\omega_0}\right|^n} = \left(\frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}} \right)^n$$

$$|H(j\omega)|_{dB} = 20 \cdot \lg \left(\frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}} \right)^n = 20 \cdot n \cdot \lg \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}}$$

If $\omega \ll \omega_0$, then $|H(j\omega)|_{dB} \approx 0$

If $\omega \gg \omega_0$, then $|H(j\omega)|_{dB} \approx 20 \cdot n \cdot \lg \frac{\omega_0}{\omega} = -20 \cdot n \cdot \lg \omega + 20 \cdot n \cdot \lg \omega_0 = \ell(\omega)$

In the case of an n^{th} order pole the Bode plot for magnitude drops at $-20 \cdot n \frac{dB}{\text{decade}}$ after the break frequency ω_0 .

Phase

$$\angle(H(j\omega)) = \angle \frac{1}{\left(1 + j \cdot \frac{\omega}{\omega_0}\right)^n} = -n \cdot \angle \left(1 + j \cdot \frac{\omega}{\omega_0}\right) = -n \cdot \arctan \frac{\omega}{\omega_0}$$

If $\omega \ll \omega_0$, then $\angle H(j\omega) \approx -n \cdot \arctan 0 = 0^\circ$

If $\omega \gg \omega_0$, then $\angle H(j\omega) \approx -n \cdot \arctan \infty = -n \cdot 90^\circ$

If $\omega \approx \omega_0$, then $\angle H(j\omega) \approx -n \cdot \arctan 1 = -n \cdot 45^\circ$

The phase of an n^{th} order pole drops linearly from 0° to $-n \cdot 90^\circ$ after one tenth the break frequency.

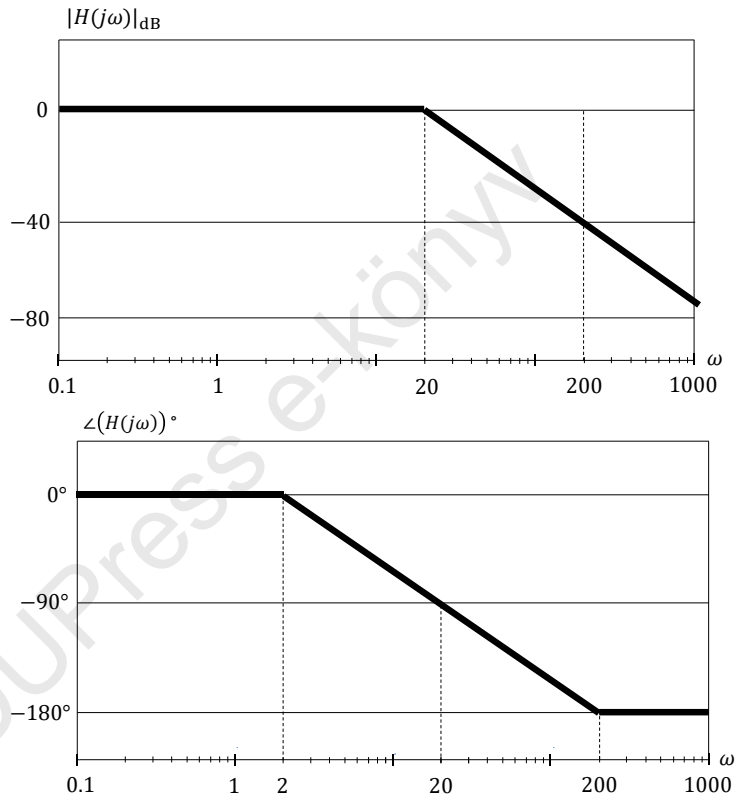
>>>Exercise 5.2 – 21

Sketch the Bode diagram of the system with transfer function $H(s) = \frac{1}{\left(1 + \frac{s}{20}\right)^2}$

(Second order real pole at $-20 \frac{\text{rad}}{\text{s}}$).

>>>Exercise 5.2 – 21- Solution

$$\begin{aligned} |H(j\omega)|_{\text{dB}} &= \\ &= 40 \cdot \lg \frac{1}{\sqrt{1 + \left(\frac{\omega}{20}\right)^2}} \end{aligned}$$



$$\begin{aligned} \angle(H(j\omega)) &= \\ &= -2 \cdot \arctan \frac{\omega}{20} \end{aligned}$$

>>>Exercise 5.2 – 22

Analyse the frequency response of the system with transfer function $H(s) = 1 + \frac{s}{\omega_0}$ in the three special cases $\omega \ll \omega_0$ (low frequency), $\omega \gg \omega_0$ (high frequency), and $\omega \approx \omega_0$. (A simple real zero at $-\omega_0$).

>>>Exercise 5.2 – 22- Solution

Magnitude

$$|H(j \cdot \omega)| = \left| 1 + j \cdot \frac{\omega}{\omega_0} \right| = \sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

$$|H(j \cdot \omega)|_{\text{dB}} = 20 \cdot \lg \sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

If $\omega \ll \omega_0$, then $|H(j\omega)|_{\text{dB}} \approx 0$.

If $\omega \gg \omega_0$, then $|H(j\omega)|_{\text{dB}} \approx 20 \cdot \lg \frac{\omega}{\omega_0} = 20 \cdot \lg \omega - 20 \cdot \lg \omega_0 = \ell(\omega)$

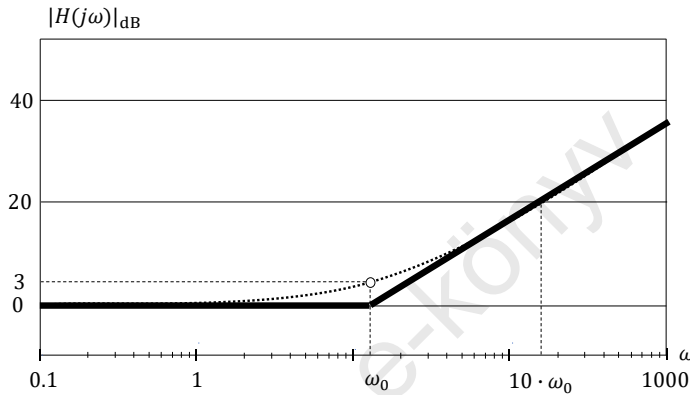
This is a straight line with a slope of $20 \frac{\text{dB}}{\text{decade}}$ going through the point $(\omega_0, 0 \text{ dB})$. Some values of ℓ :

$\ell(\omega_0) = 20 \cdot \lg 1 = 0 \text{ dB}$, $\ell(10 \cdot \omega_0) = 20 \cdot \lg 0.1 = 20 \text{ dB}$, $\ell(100 \cdot \omega_0) = 20 \cdot \lg 0.01 = 40 \text{ dB}$

If $\omega \approx \omega_0$, then $|H(j \cdot \omega)|_{\text{dB}} = 20 \cdot \lg \sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2} \approx 20 \cdot \lg \sqrt{2} = 3 \text{ dB}$

ω_0 is called the break frequency, the corner frequency or the 3 dB frequency.

For a simple real zero the piecewise linear asymptotic Bode plot for magnitude is at 0 dB until the break frequency and then increases at $20 \frac{\text{dB}}{\text{decade}}$.



Phase

The phase of a simple zero is given by:

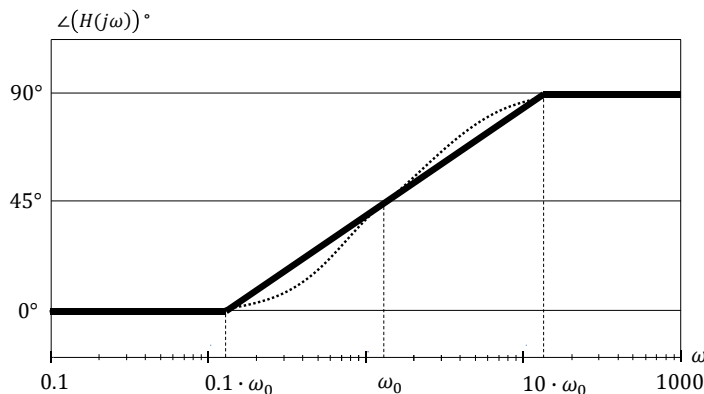
$$\angle H(j \cdot \omega) = \angle \left(1 + j \cdot \frac{\omega}{\omega_0}\right) = \arctan \frac{\omega}{\omega_0}$$

If $\omega \ll \omega_0$, then $\angle H(j\omega) \approx \arctan 0 = 0^\circ$

If $\omega \gg \omega_0$, then $\angle H(j\omega) \approx \arctan \infty = 90^\circ$

If $\omega \approx \omega_0$, then $\angle H(j\omega) \approx \arctan 1 = 45^\circ$

Follow the low frequency asymptote at 0° until one tenth the break frequency ($0.1 \cdot \omega_0$) then increase linearly to meet the high frequency asymptote at ten times the break frequency ($10 \cdot \omega_0$).



>>>Exercise 5.2 – 23

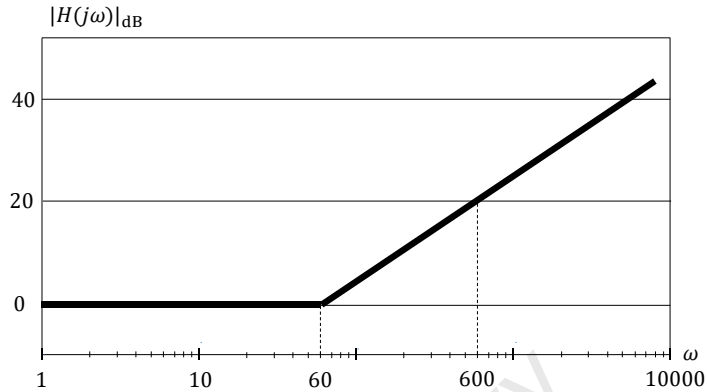
Sketch the Bode diagram of the system with transfer function $H(s) = 1 + \frac{s}{60}$

(Simple real zero at $\omega_0 = -60 \frac{\text{rad}}{\text{s}}$).

>>>Exercise 5.2 – 23- Solution

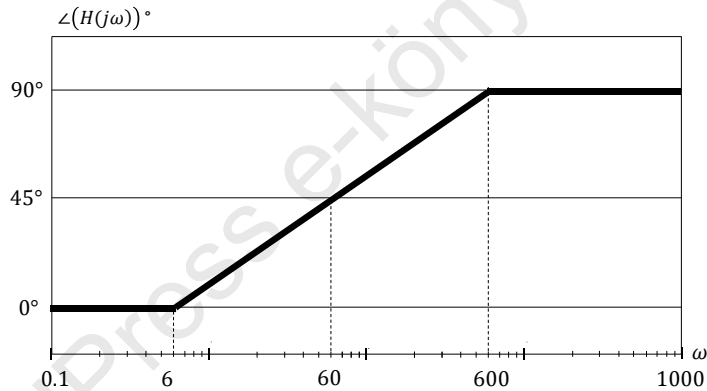
$$|H(j \cdot \omega)|_{\text{dB}} =$$

$$= 20 \cdot \lg \sqrt{1 + \left(\frac{\omega}{60}\right)^2}$$



$$\angle H(j \cdot \omega) =$$

$$= \arctan \frac{\omega}{60}$$



>>>Exercise 5.2 – 24

Analyse the frequency response of the system with transfer function $H(s) = \left(1 + \frac{s}{\omega_0}\right)^n$ (n is a positive integer) in the three special cases $\omega \ll \omega_0$ (low frequency), $\omega \gg \omega_0$ (high frequency), and $\omega \approx \omega_0$ (n^{th} order real zero at $-\omega_0$).

>>>Exercise 5.2 – 24- Solution

Magnitude

$$|H(j\omega)| = \left| \left(1 + j \cdot \frac{\omega}{\omega_0}\right)^n \right| = \left| 1 + j \cdot \frac{\omega}{\omega_0} \right|^n = \left(\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2} \right)^n$$

$$|H(j\omega)|_{\text{dB}} = 20 \cdot \lg \left(\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2} \right)^n = 20 \cdot n \cdot \lg \sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

$$\text{If } \omega \ll \omega_0, \text{ then } \sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2} \approx 1 \Rightarrow |H(j\omega)|_{\text{dB}} \approx 0$$

$$\text{If } \omega \gg \omega_0, \text{ then } \sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2} \approx \frac{\omega}{\omega_0} \Rightarrow |H(j\omega)|_{\text{dB}} \approx 20 \cdot n \cdot \lg \frac{\omega}{\omega_0} = 20 \cdot n \cdot \lg \omega - 20 \cdot n \cdot \lg \omega_0 = \ell(\omega)$$

In the case of an n^{th} order pole Bode plot for magnitude rises at $20 \cdot n \frac{\text{dB}}{\text{decade}}$ after the break frequency ω_0 .

$$\angle(H(j\omega)) = \angle\left(1 + j \cdot \frac{\omega}{\omega_0}\right)^n = n \cdot \angle\left(1 + j \cdot \frac{\omega}{\omega_0}\right) = n \cdot \arctan \frac{\omega}{\omega_0}$$

If $\omega \ll \omega_0$, then $\angle H(j\omega) \approx n \cdot \arctan 0 = 0^\circ$

If $\omega \gg \omega_0$, then $\angle H(j\omega) \approx n \cdot \arctan \infty = n \cdot 90^\circ$

If $\omega \approx \omega_0$, then $\angle H(j\omega) \approx n \cdot \arctan 1 = n \cdot 45^\circ$

The phase of an n^{th} order pole rises linearly from 0° to $n \cdot 90^\circ$ after one tenth the break frequency.

>>>Exercise 5.2 – 25

Analyse the frequency response of the system with transfer function $H(s) = \frac{1}{s}$

(Simple pole at the origin, integrator).

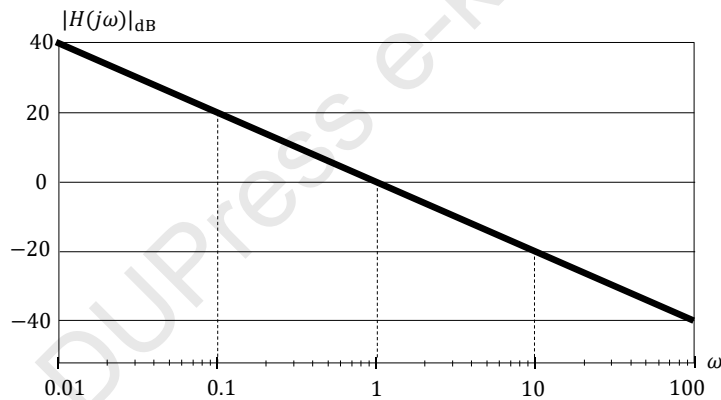
>>>Exercise 4.2 – 25- Solution

Magnitude

$$|H(j \cdot \omega)|_{\text{dB}} \approx 20 \cdot \lg \left| \frac{1}{j \cdot \omega} \right| = 20 \cdot \lg \left| -\frac{j}{\omega} \right| = 20 \cdot \lg \frac{1}{\omega} = -20 \cdot \lg \omega = \ell(\omega)$$

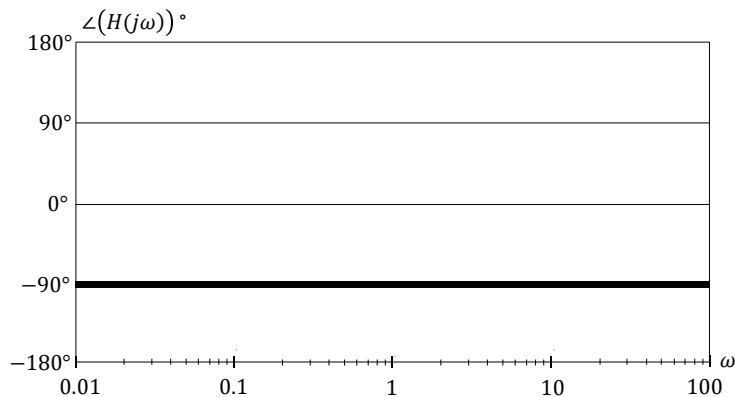
The function is represented by a straight line on a Bode plot with a slope of $-20 \frac{\text{dB}}{\text{decade}}$.

Some values of ℓ are $\ell\left(1 \frac{\text{rad}}{\text{s}}\right) = 0 \text{ dB}$, $\ell\left(10 \frac{\text{rad}}{\text{s}}\right) = -20 \text{ dB}$



Phase

$$\angle H(j \cdot \omega) = \angle\left(-\frac{j}{\omega}\right) = -90^\circ$$



>>>Exercise 5.2 – 26

Analyse the frequency response of the system with transfer function $H(s) = \frac{1}{s^n}$

(n^{th} order pole at the origin).

>>>Exercise 5.2 – 26- Solution

Magnitude

$$|H(j \cdot \omega)|_{dB} \approx 20 \cdot \lg \left| \frac{1}{(j \cdot \omega)^n} \right| = 20 \cdot \lg \frac{1}{\omega^n} = -20 \cdot n \cdot \lg \omega = \ell(\omega)$$

The function is represented by a straight line on a Bode plot with a slope of $-20 \cdot n \frac{\text{dB}}{\text{decade}}$.

Phase

$$\angle H(j \cdot \omega) = \angle \left(\frac{1}{(j \cdot \omega)^n} \right) = -n \cdot 90^\circ$$

>>>Exercise 5.2 – 27

Analyse the frequency response of the system with transfer function $H(s) = s$

(Simple zero at the origin, differentiator).

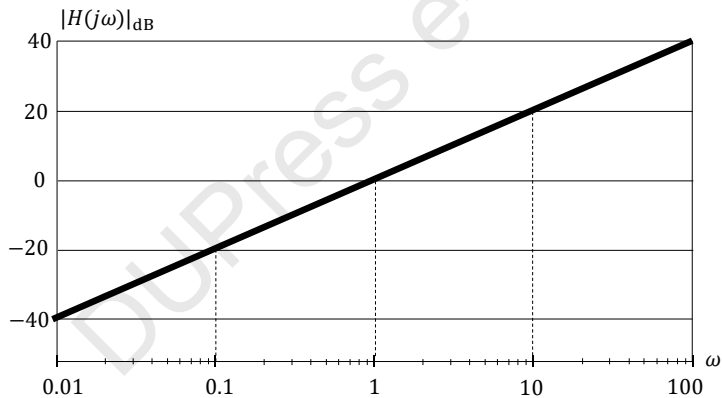
>>>Exercise 5.2 – 27- Solution

Magnitude

$$|H(j \cdot \omega)|_{dB} \approx 20 \cdot \lg |j \cdot \omega| = 20 \cdot \lg \omega = \ell(\omega)$$

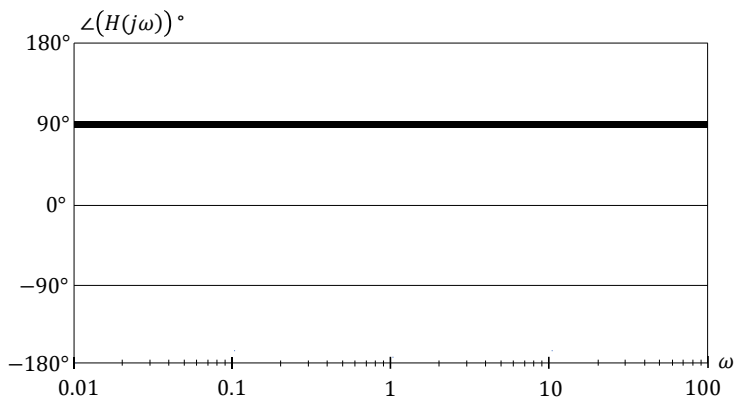
The function is represented by a straight line on a Bode plot with a slope of $20 \frac{\text{dB}}{\text{decade}}$.

Some values of ℓ : $\ell \left(1 \frac{\text{rad}}{\text{s}} \right) = 0 \text{ dB}$, $\ell \left(10 \frac{\text{rad}}{\text{s}} \right) = 20 \text{ dB}$



Phase

$$\angle H(j \cdot \omega) = \angle(j \cdot \omega) = 90^\circ$$



>>>Exercise 5.2 – 28

Analyse the frequency response of the system with transfer function $H(s) = s^n$ (n^{th} order zero at the origin).

>>>Exercise 5.2 – 28– Solution

Magnitude

$$|H(j \cdot \omega)|_{\text{dB}} \approx 20 \cdot \lg|(j \cdot \omega)^n| = 20 \cdot \lg(\omega^n) = 20 \cdot n \cdot \lg \omega = \ell(\omega)$$

The function is represented by a straight line on a Bode plot with a slope of $20 \cdot n \frac{\text{dB}}{\text{decade}}$.

Phase

$$\angle H(j \cdot \omega) = \angle (j \cdot \omega)^n = n \cdot 90^\circ$$

>>>Exercise 5.2 – 29

Analyse the frequency response of the system with transfer function

$$H(s) = \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1}, 0 < \zeta < 1$$

in the three special cases $\omega \ll \omega_0$ (low frequency), $\omega \gg \omega_0$ (high frequency), and $\omega \approx \omega_0$ (Complex conjugate pair of poles, underdamped system.)

>>>Exercise 5.2 – 29– Solution

Magnitude

$$|H(j \cdot \omega)| = \left| \frac{1}{\left(\frac{j \cdot \omega}{\omega_0}\right)^2 + 2\zeta\left(\frac{j \cdot \omega}{\omega_0}\right) + 1} \right| = \frac{1}{\left| \left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right) + j \cdot 2\zeta\left(\frac{\omega}{\omega_0}\right) \right|} = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\zeta\left(\frac{\omega}{\omega_0}\right)\right)^2}}$$

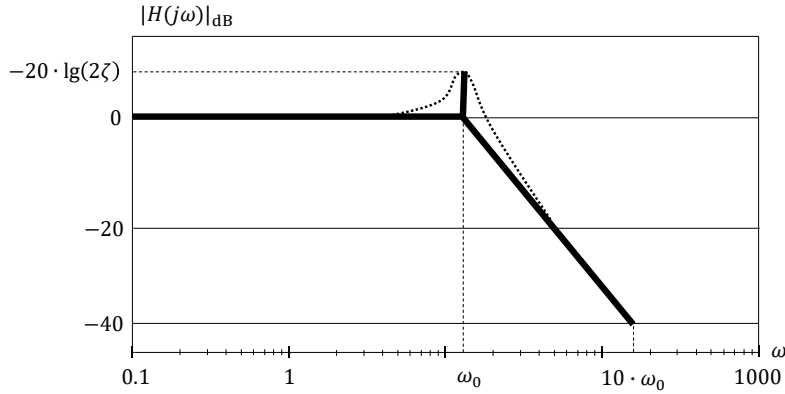
$$|H(j \cdot \omega)|_{\text{dB}} = -20 \cdot \lg \sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\zeta\left(\frac{\omega}{\omega_0}\right)\right)^2}$$

If $\omega \ll \omega_0$, then $|H(j \cdot \omega)|_{\text{dB}} \approx -20 \cdot \lg 1 = 0$

If $\omega \gg \omega_0$, then $|H(j \cdot \omega)|_{\text{dB}} \approx -20 \cdot \lg \left(\frac{\omega}{\omega_0}\right)^2 = -40 \cdot \lg \omega + 40 \cdot \lg \omega_0$

It is a straight line with a slope of $-40 \frac{\text{dB}}{\text{decade}}$ going through the point $(\omega_0, 0 \text{ dB})$.

Near $\omega \approx \omega_0$ a peak occurs with height $|H(j \cdot \omega_0)| = \frac{1}{2\zeta}$ or $|H(j \cdot \omega_0)|_{\text{dB}} \approx -20 \cdot \lg(2\zeta)$ which is considerable when $0 < \zeta < 0.5$.



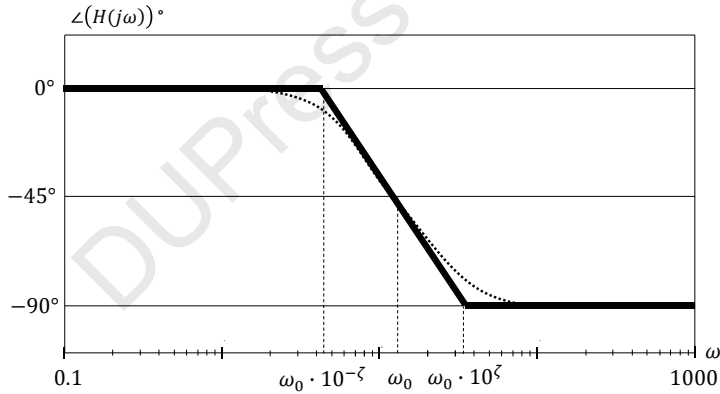
Phase

$$\begin{aligned}\angle H(j \cdot \omega) &= \angle \left(\frac{1}{\left(\frac{j \cdot \omega}{\omega_0}\right)^2 + 2\zeta \left(\frac{j \cdot \omega}{\omega_0}\right) + 1} \right) = -\angle \left(\left(\frac{j \cdot \omega}{\omega_0}\right)^2 + 2\zeta \left(\frac{j \cdot \omega}{\omega_0}\right) + 1 \right) = \\ &= -\angle \left(1 - \left(\frac{\omega}{\omega_0}\right)^2 + j \cdot 2\zeta \cdot \frac{\omega}{\omega_0} \right) = -\arctan \left(\frac{2\zeta \cdot \frac{\omega}{\omega_0}}{1 - \left(\frac{\omega}{\omega_0}\right)^2} \right)\end{aligned}$$

If $\omega \ll \omega_0$, then $\angle H(j \cdot \omega) \approx -\arctan \frac{\omega}{\omega_0} \approx 0^\circ$

If $\omega \gg \omega_0$, then $\angle H(j \cdot \omega) \approx -180^\circ$

If $\omega \approx \omega_0$, then $\angle H(j \cdot \omega) \approx -\arctan(2\zeta)$



>>>Exercise 5.2 – 30

Sketch the Bode diagram of the system with transfer function $H(s) = \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\zeta \left(\frac{s}{\omega_0}\right) + 1}$ when $\omega_0 =$

10, $\zeta = 0.3$.

>>>Exercise 5.2 – 30- Solution

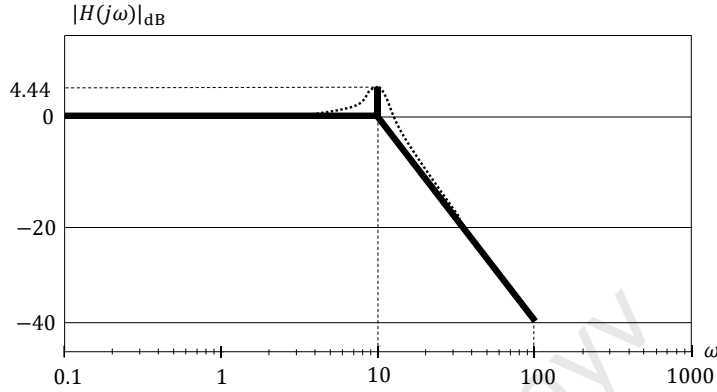
Magnitude

$$H(j \cdot \omega) = \frac{1}{\left(\frac{j \cdot \omega}{10}\right)^2 + 0.6 \cdot \left(\frac{j \cdot \omega}{10}\right) + 1}$$

$$|H(j \cdot \omega)|_{\text{dB}} = -20 \cdot \lg \sqrt{\left(1 - \left(\frac{\omega}{10}\right)^2\right)^2 + \left(0.6 \cdot \left(\frac{\omega}{10}\right)\right)^2}$$

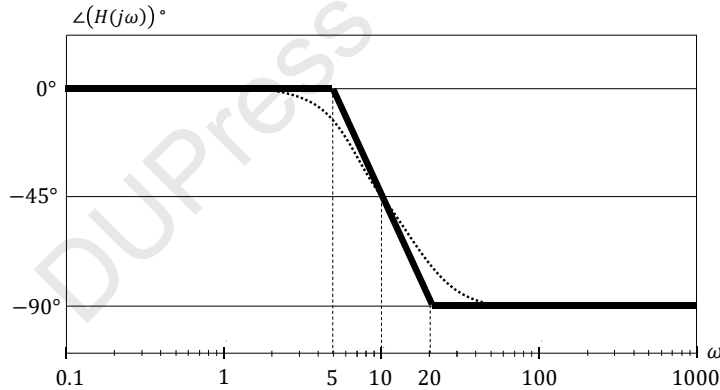
The peak will have an amplitude of $\frac{1}{2\zeta} = 1.67$ or 4.44 dB.

The approximation connects the low frequency asymptote to the high frequency asymptote starting at $\omega = \omega_0 \cdot 10^{-\zeta} = 10 \cdot 10^{-0.3} = 10^{0.7} \approx 5 \frac{\text{rad}}{\text{s}}$ and ending at $\omega = \omega_0 \cdot 10^{\zeta} = 10 \cdot 10^{0.3} = 10^{1.3} \approx 20 \frac{\text{rad}}{\text{s}}$.



Phase

Since $\zeta = 0.2$ in this case this means that the phase starts at 0° and then breaks downward at $\omega = \omega_0 \cdot 10^{-\zeta} = 10 \cdot 10^{-0.3} \approx 5 \frac{\text{rad}}{\text{s}}$. The phase reaches -180° at $\omega = \omega_0 \cdot 10^{\zeta} = 10 \cdot 10^{0.3} \approx 20 \frac{\text{rad}}{\text{s}}$.



>>>Exercise 5.2 – 31

Analyse the frequency response of the system with transfer function

$$H(s) = \left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1, \quad 0 < \zeta < 1$$

in the three special cases $\omega \ll \omega_0$ (low frequency), $\omega \gg \omega_0$ (high frequency), and $\omega \approx \omega_0$. (Complex conjugate pair of zeros).

>>>Exercise 5.2 – 31– Solution

Magnitude

$$|H(j \cdot \omega)| =$$

$$= \left| \left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_0}\right) + 1 \right| = \left| \left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right) + j \cdot 2\zeta\left(\frac{\omega}{\omega_0}\right) \right| = \sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\zeta\left(\frac{\omega}{\omega_0}\right)\right)^2}$$

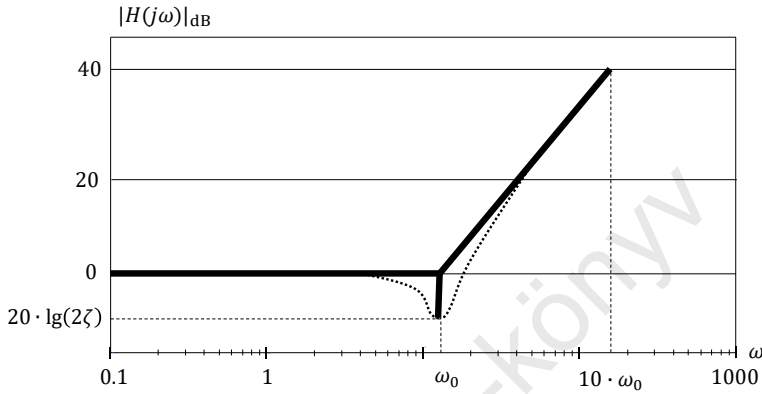
$$|H(j \cdot \omega)|_{dB} \approx 20 \cdot \lg \sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\zeta \left(\frac{\omega}{\omega_0}\right)\right)^2}$$

If $\omega \ll \omega_0$, then $|H(j \cdot \omega)|_{dB} \approx 20 \cdot \lg 1 = 0$

If $\omega \gg \omega_0$, then $|H(j \cdot \omega)|_{dB} \approx 20 \cdot \lg \left(\frac{\omega}{\omega_0}\right)^2 = 40 \cdot \lg \omega - 40 \cdot \lg \omega_0$

It is a straight line with a slope of $40 \frac{dB}{decade}$ going through the point $(\omega_0, 0 \text{ dB})$.

Near $\omega \approx \omega_0$ a peak occurs with height $|H(j \cdot \omega_0)| = 2\zeta$, $|H(j \cdot \omega_0)|_{dB} \approx 20 \cdot \lg 2\zeta$ which is considerable when $0 < \zeta < 0.5$.



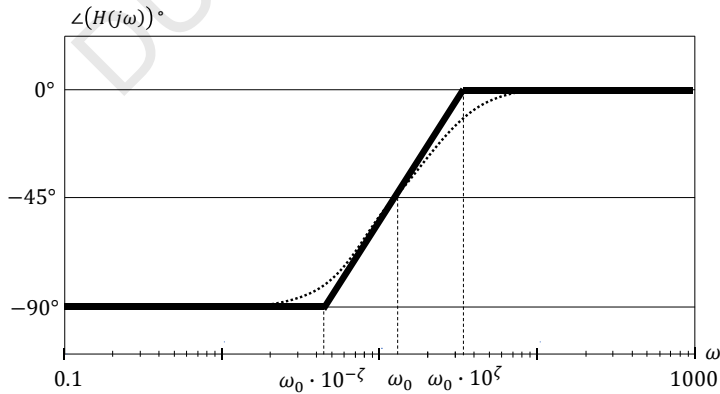
Phase

$$\angle H(j \cdot \omega) = \angle \left(\left(\frac{j \cdot \omega}{\omega_0} \right)^2 + 2\zeta \left(\frac{j \cdot \omega}{\omega_0} \right) + 1 \right) = \angle \left(1 - \left(\frac{\omega}{\omega_0} \right)^2 + j \cdot 2\zeta \cdot \frac{\omega}{\omega_0} \right) = \arctan \left(\frac{2\zeta \cdot \frac{\omega}{\omega_0}}{1 - \left(\frac{\omega}{\omega_0} \right)^2} \right)$$

If $\omega \ll \omega_0$, then $\angle H(j \cdot \omega) \approx \arctan \frac{\omega}{\omega_0} \approx 0^\circ$

If $\omega \gg \omega_0$, then $\angle H(j \cdot \omega) \approx 180^\circ$

If $\omega \approx \omega_0$, then $\angle H(j \cdot \omega) \approx -\arctan(2\zeta)$



>>>Exercise 5.2 – 32

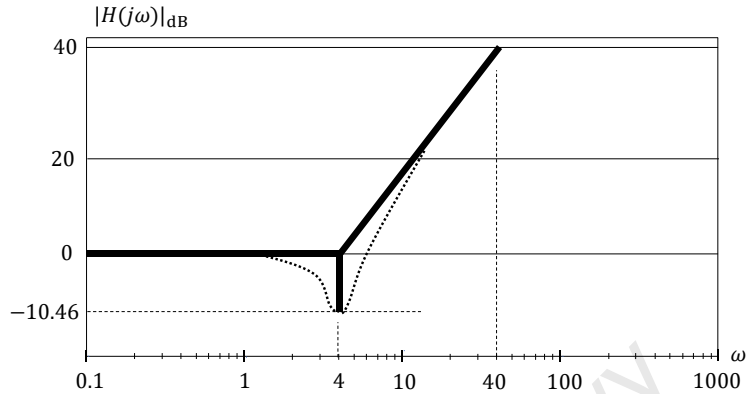
Sketch the Bode diagram of the system with transfer function $H(s) = \left(\frac{s}{\omega_0}\right)^2 + 2\zeta \left(\frac{s}{\omega_0}\right) + 1$ when $\omega_0 = 4$, $\zeta = 0.15$.

>>>Exercise 5.2 – 32- Solution

Magnitude

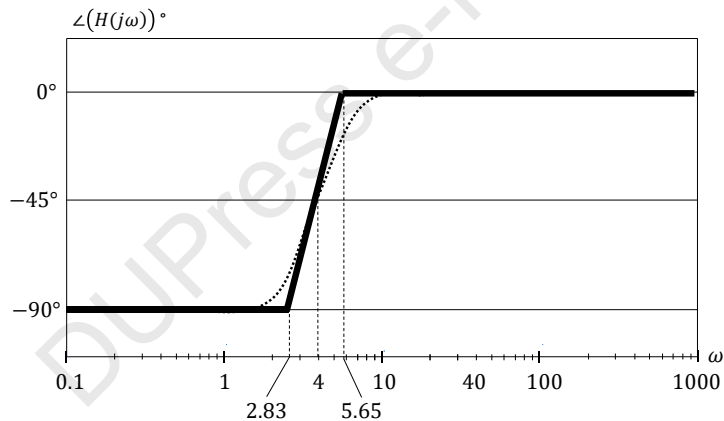
$$H(j \cdot \omega) = \left(\frac{j \cdot \omega}{4}\right)^2 + 0.3 \cdot \left(\frac{j \cdot \omega}{4}\right) + 1$$

The dip in the magnitude plot will have a magnitude of $2\zeta = 0.3$ or -10.46 dB.



Phase

The break frequencies for the phase are at $\omega = \omega_0 \cdot 10^{-\zeta} = 4 \cdot 10^{-0.15} = 2.83 \frac{\text{rad}}{\text{s}}$ and $\omega = \omega_0 \cdot 10^{\zeta} = 4 \cdot 10^{0.15} = 5.65 \frac{\text{rad}}{\text{s}}$



Remark: The shape of the graphs (magnitude peak height, steepness of phase transition) are determined solely by ζ , and the frequency at which the magnitude peak and phase transition occur are determined solely by ω_0 .

6 Solution of Linear Differential Equations with Laplace Transform

If functions $g_0, g_1, \dots, g_{n-1}, f: I \rightarrow \mathbb{R}$ are continuous, then an equation of the form

$$y^{(n)}(t) + g_{n-1}(t) \cdot y^{(n-1)}(t) + \dots + g_1(t) \cdot y'(t) + g_0(t) \cdot y(t) = f(t)$$

is called *n-th order linear differential equation*.

If $g_0, g_1, \dots, g_{n-1}$ are constant functions our equation

$$y^{(n)}(t) + g_{n-1} \cdot y^{(n-1)}(t) + \dots + g_1 \cdot y'(t) + g_0 \cdot y(t) = f(t)$$

is called *n-th order linear differential equation with constant coefficients*.

Remark: In the analysis of technical systems these coefficients are system parameters (e.g. mass, spring stiffness, damping coefficient, capacity, inductivity, resistance). In mechanical and electrical models these parameters are generally supposed to be time independent, thus the coefficients of the related differential equation are constants.

If we have additional information about the solution function, namely the value of functions $y, y', y'', \dots, y^{(n-1)}$ at a given place $t_0 \in I$ (at a specific instant of time)

$$y^{(k)}(t_0) = y_k, \quad k = 0, 1, \dots, n-1$$

we have an *n-th order linear initial value problem*.

If $t_0 = 0$, an *n-th order linear initial value problem with constant coefficients* can be solved with the help of Laplace transform as follows.

Using the Laplace transform an initial value problem

$$y^{(n)}(t) + g_{n-1} \cdot y^{(n-1)}(t) + \dots + g_1 \cdot y'(t) + g_0 \cdot y(t) = f(t)$$

$$y^{(k)}(0) = y_k, \quad k = 0, 1, \dots, n-1$$

transforms into the form

$$Y(s) = F(s) \cdot \frac{1}{Q(s)} + \frac{P(s)}{Q(s)},$$

where F and Y are Laplace transform of f and y , respectively, P and Q are polynomials depending on the coefficients of the equation and the initial values. Thus

$$y(t) = \mathcal{L}^{-1} \left(F(s) \cdot \frac{1}{Q(s)} + \frac{P(s)}{Q(s)} \right) = \mathcal{L}^{-1} \left(F(s) \cdot \frac{1}{Q(s)} \right) + \mathcal{L}^{-1} \left(\frac{P(s)}{Q(s)} \right)$$

Remark: The idea of the solution method is $f \xrightarrow{\mathcal{L}} F \xrightarrow{\text{algebraic equation}} Y \xrightarrow{\mathcal{L}^{-1}} y$

Applying the convolution theorem, we can avoid the calculation of $F = \mathcal{L}(f)$ and the inverse Laplace transformations are easier:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left(F(s) \cdot \frac{1}{Q(s)} \right) + \mathcal{L}^{-1} \left(\frac{P(s)}{Q(s)} \right) = \mathcal{L}^{-1}(F(s)) * \mathcal{L}^{-1} \left(\frac{1}{Q(s)} \right) + \\ &+ \mathcal{L}^{-1} \left(\frac{P(s)}{Q(s)} \right) = f(t) * v(t) + w(t) = \int_0^t f(\tau) \cdot v(t - \tau) d\tau + w(t) \end{aligned}$$

with the notations $v(t) = \mathcal{L}^{-1} \left(\frac{1}{Q(s)} \right), w(t) = \mathcal{L}^{-1} \left(\frac{P(s)}{Q(s)} \right)$.

Remark: Derivatives are denoted by dots in the following. So the Laplace transform formulas for the derivative functions are

$$\mathcal{L}(\dot{y}(t)) = s \cdot \mathcal{L}(y(t)) - y(0)$$

and

$$\mathcal{L}(\ddot{y}(t)) = s^2 \cdot \mathcal{L}(y(t)) - s \cdot y(0) - \dot{y}(0),$$

respectively.

Calculations in the case of first order linear initial value problems are as follows.

$$\begin{aligned}\dot{y}(t) + a \cdot y(t) &= f(t), \quad y(0) = y_0 \\ \mathcal{L}(\dot{y}(t)) + a \cdot \mathcal{L}(y(t)) &= \mathcal{L}(f(t)) \\ s \cdot \mathcal{L}(y(t)) - y(0) + a \cdot \mathcal{L}(y(t)) &= \mathcal{L}(f(t)) \\ s \cdot Y(s) - y_0 + a \cdot Y(s) &= F(s) \\ Y(s) \cdot (s + a) &= F(s) + y_0 \\ Y(s) &= F(s) \cdot \frac{1}{s + a} + \frac{y_0}{s + a}\end{aligned}$$

The further calculations without the convolution theorem

$$y(t) = \mathcal{L}^{-1}\left(F(s) \cdot \frac{1}{s + a}\right) + \mathcal{L}^{-1}\left(\frac{y_0}{s + a}\right)$$

The further calculations with the convolution theorem

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\left(F(s) \cdot \frac{1}{s + a}\right) + \mathcal{L}^{-1}\left(\frac{y_0}{s + a}\right) = \mathcal{L}^{-1}(F(s)) * \mathcal{L}^{-1}\left(\frac{1}{s + a}\right) + y_0 \cdot \mathcal{L}^{-1}\left(\frac{1}{s + a}\right) = \\ &= f(t) * e^{-a \cdot t} + y_0 \cdot e^{-a \cdot t} = \int_0^t f(\tau) \cdot e^{-a \cdot (t - \tau)} d\tau + y_0 \cdot e^{-a \cdot t}\end{aligned}$$

Remark: Equality $y(t) = \int_0^t f(\tau) \cdot e^{-a \cdot (t - \tau)} d\tau + y_0 \cdot e^{-a \cdot t}$ obtained from the previous derivation can be used as a calculation formula when a first order initial value problem is to be solved.

Calculations in the case of second order linear initial value problems are as follows.

$$\begin{aligned}\ddot{y}(t) + a \cdot \dot{y}(t) + b \cdot y(t) &= f(t), \quad y(0) = y_0, \dot{y}(0) = \dot{y}_0 \\ \mathcal{L}(\ddot{y}(t)) + a \cdot \mathcal{L}(\dot{y}(t)) + b \cdot \mathcal{L}(y(t)) &= \mathcal{L}(f(t)) \\ s^2 \cdot \mathcal{L}(y(t)) - s \cdot y(0) - \dot{y}(0) + a \cdot (s \cdot \mathcal{L}(y(t)) - y(0)) + b \cdot \mathcal{L}(y(t)) &= \mathcal{L}(f(t)) \\ s^2 \cdot Y(s) - s \cdot y_0 - \dot{y}_0 + a \cdot (s \cdot Y(s) - y_0) + b \cdot Y(s) &= F(s) \\ Y(s) \cdot (s^2 + a \cdot s + b) &= F(s) + (s + a) \cdot y_0 + \dot{y}_0 \\ Y(s) &= F(s) \cdot \frac{1}{s^2 + a \cdot s + b} + \frac{(s + a) \cdot y_0 + \dot{y}_0}{s^2 + a \cdot s + b}\end{aligned}$$

The further calculations without the convolution theorem

$$y(t) = \mathcal{L}^{-1}\left(F(s) \cdot \frac{1}{s^2 + a \cdot s + b}\right) + \mathcal{L}^{-1}\left(\frac{(s + a) \cdot y_0 + \dot{y}_0}{s^2 + a \cdot s + b}\right)$$

The further calculations with the convolution theorem

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\left(F(s) \cdot \frac{1}{s^2 + a \cdot s + b}\right) + \mathcal{L}^{-1}\left(\frac{(s + a) \cdot y_0 + \dot{y}_0}{s^2 + a \cdot s + b}\right) = \\ &= \mathcal{L}^{-1}(F(s)) * \mathcal{L}^{-1}\left(\frac{1}{s^2 + a \cdot s + b}\right) + \mathcal{L}^{-1}\left(\frac{(s + a) \cdot y_0 + \dot{y}_0}{s^2 + a \cdot s + b}\right) = \\ &= f(t) * \mathcal{L}^{-1}\left(\frac{1}{s^2 + a \cdot s + b}\right) + \mathcal{L}^{-1}\left(\frac{(s + a) \cdot y_0 + \dot{y}_0}{s^2 + a \cdot s + b}\right)\end{aligned}$$

>>>Exercise 6.1 – 1

Solve the following initial value problem with Laplace transform:

$$\dot{y}(t) + 2 \cdot y(t) = (t + 1) \cdot e^t, \quad y(0) = 1$$

>>>Exercise 6.1 – 1- Solution

>>>Solution with the convolution theorem

$$\begin{aligned}\mathcal{L}((t + 1) \cdot e^t) &= \mathcal{L}(t \cdot e^t) + \mathcal{L}(e^t) = \frac{1}{(s - 1)^2} + \frac{1}{s - 1} = \frac{s}{(s - 1)^2} \\ s \cdot Y(s) - 1 + 2 \cdot Y(s) &= \frac{s}{(s - 1)^2} \\ Y(s) \cdot (s + 2) &= \frac{s}{(s - 1)^2} + 1 = \frac{s^2 - s + 1}{(s - 1)^2} \\ Y(s) &= \frac{s^2 - s + 1}{(s - 1)^2 \cdot (s + 2)}\end{aligned}$$

The partial fraction decomposition of $Y(s)$ is

$$\begin{aligned}\frac{s^2 - s + 1}{(s - 1)^2 \cdot (s + 2)} &= \frac{A}{(s - 1)^2} + \frac{B}{s - 1} + \frac{C}{s + 2} = \frac{A \cdot (s + 2) + B \cdot (s - 1) \cdot (s + 2) + C \cdot (s - 1)^2}{(s - 1)^2 \cdot (s + 2)} = \\ &= \frac{(B + C) \cdot s^2 + (A + B - 2C) \cdot s + (2A - 2B + C)}{(s - 1)^2 \cdot (s + 2)} = \frac{1}{3} \cdot \frac{1}{(s - 1)^2} + \frac{2}{9} \cdot \frac{1}{s - 1} + \frac{7}{9} \cdot \frac{1}{s + 2} \\ \left[\begin{array}{lcl} \text{I.} & B + C & = 1 \\ \text{II.} & A + B - 2C & = -1 \\ \text{III.} & 2A - 2B + C & = 1 \end{array} \Rightarrow \begin{array}{l} A = 1/3 \\ B = 2/9 \\ C = 7/9 \end{array} \right]\end{aligned}$$

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\left(\frac{1}{(s - 1)^2 \cdot (s + 2)}\right) = \\ &= \frac{1}{3} \cdot \mathcal{L}^{-1}\left(\frac{1}{(s - 1)^2}\right) + \frac{2}{9} \cdot \mathcal{L}^{-1}\left(\frac{1}{s - 1}\right) + \frac{7}{9} \cdot \mathcal{L}^{-1}\left(\frac{1}{s + 2}\right) = \frac{1}{3} \cdot t \cdot e^t + \frac{2}{9} \cdot e^t + \frac{7}{9} \cdot e^{-2t}\end{aligned}$$

>>>Solution without the convolution theorem

$$\begin{aligned}s \cdot Y(s) - 1 + 2 \cdot Y(s) &= F(s) \\ Y(s) \cdot (s + 2) &= F(s) + 1 \\ Y(s) &= F(s) \cdot \frac{1}{s + 2} + \frac{1}{s + 2} \\ y(t) &= \mathcal{L}^{-1}\left(F(s) \cdot \frac{1}{s + 2}\right) + \mathcal{L}^{-1}\left(\frac{1}{s + 2}\right) = \mathcal{L}^{-1}(F(s)) * \mathcal{L}^{-1}\left(\frac{1}{s + 2}\right) + \mathcal{L}^{-1}\left(\frac{1}{s + 2}\right) = \\ &= ((t + 1) \cdot e^t) * e^{-2t} + e^{-2t} = \int_0^t (\tau + 1) \cdot e^\tau \cdot e^{-2 \cdot (t - \tau)} d\tau + e^{-2t} = \\ &= e^{-2t} \cdot \int_0^t (\tau + 1) \cdot e^{3\tau} \cdot d\tau + e^{-2t} = \\ &= e^{-2t} \cdot \left(-\frac{2}{9} + \frac{1}{3} \cdot t \cdot e^{3t} + \frac{2}{9} \cdot e^{3t}\right) + e^{-2t} = \frac{7}{9} \cdot e^{-2t} + \frac{1}{3} \cdot t \cdot e^t + \frac{2}{9} \cdot e^t\end{aligned}$$

>>>Exercise 6.1 – 2

Solve the following initial value problem with Laplace transform:

$$\dot{y}(t) - y(t) = 1, \quad y(0) = 2$$

>>>Exercise 6.1 – 2- Solution

>>>Solution without the convolution theorem

$$s \cdot Y(s) - 2 - Y(s) = \frac{1}{s}$$

$$Y(s) \cdot (s-1) = \frac{1}{s} + 2 = \frac{2s+1}{s}$$

$$Y(s) = \frac{2s+1}{s \cdot (s-1)}$$

The partial fraction decomposition of $Y(s)$ is

$$\frac{2s+1}{s \cdot (s-1)} = \frac{A}{s} + \frac{B}{s-1} = \frac{(A+B) \cdot s + (-A)}{s \cdot (s-1)} = -\frac{1}{s} + \frac{3}{s-1}$$

$$\begin{bmatrix} \text{I.} & A+B & = & 2 \\ \text{II.} & -A & = & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} A = -1 \\ B = 3 \end{bmatrix}$$

$$y(t) = \mathcal{L}^{-1}\left(-\frac{1}{s} + \frac{3}{s-1}\right) = -\mathcal{L}^{-1}\left(\frac{1}{s}\right) + 3 \cdot \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = -1 + 3 \cdot e^t$$

>>>Solution with the convolution theorem

$$s \cdot Y(s) - 2 - Y(s) = F(s)$$

$$Y(s) \cdot (s-1) = F(s) + 2$$

$$Y(s) = F(s) \cdot \frac{1}{s-1} + \frac{2}{s-1}$$

$$y(t) = f(t) * \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + 2 \cdot \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = \int_0^t 1 \cdot e^{t-\tau} d\tau + 2 \cdot e^t = [-e^{t-\tau}]_0^t + 2 \cdot e^t$$

$$= -1 + 3 \cdot e^t$$

>>>**Exercise 6.1 – 3**

Solve the following initial value problem with Laplace transform:

$$\dot{y}(t) + 3y(t) = e^{4t}, \quad y(0) = 1$$

>>>**Exercise 6.1 – 3- Solution**

>>>Solution without the convolution theorem

$$s \cdot Y(s) - 1 + 3 \cdot Y(s) = \frac{1}{s-4}$$

$$Y(s) \cdot (s+3) = \frac{1}{s-4} + 1 = \frac{s-3}{s-4}$$

$$Y(s) = \frac{s-3}{(s-4) \cdot (s+3)}$$

The partial fraction decomposition of $Y(s)$ is

$$\frac{s-3}{(s-4) \cdot (s+3)} = \frac{A}{s-4} + \frac{B}{s+3} = \frac{(A+B) \cdot s + (3A-4B)}{(s-4) \cdot (s+3)} = \frac{1}{7} \cdot \frac{1}{s-4} + \frac{6}{7} \cdot \frac{1}{s+3}$$

$$\begin{bmatrix} \text{I.} & A+B & = & 1 \\ \text{II.} & 3A-4B & = & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} A = 1/7 \\ B = 6/7 \end{bmatrix}$$

$$y(t) = \frac{1}{7} \cdot \mathcal{L}^{-1}\left(\frac{1}{s-4}\right) + \frac{6}{7} \cdot \mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = \frac{1}{7} \cdot e^{4t} + \frac{6}{7} \cdot e^{-3t}$$

>>>Solution with the convolution theorem

$$s \cdot Y(s) - 1 + 3 \cdot Y(s) = F(s)$$

$$Y(s) \cdot (s+3) = F(s) + 1$$

$$Y(s) = F(s) \cdot \frac{1}{s+3} + \frac{1}{s+3}$$

$$y(t) = f(t) * \mathcal{L}^{-1}\left(\frac{1}{s+3}\right) + \mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = \int_0^t e^{4\tau} \cdot e^{-3 \cdot (t-\tau)} d\tau + e^{-3t} =$$

$$\begin{aligned}
 &= e^{-3t} \cdot \int_0^t e^{7\tau} d\tau + e^{-3t} = e^{-3t} \cdot \left[\frac{1}{7} \cdot e^{7\tau} \right]_0^t + e^{-3t} = e^{-3t} \cdot \left[\frac{1}{7} \cdot e^{7\tau} \right]_0^t + e^{-3t} = \\
 &= e^{-3t} \cdot \left(\frac{1}{7} \cdot e^{7t} - \frac{1}{7} \right) + e^{-3t} = \frac{1}{7} \cdot e^{4t} + \frac{6}{7} \cdot e^{-3t}
 \end{aligned}$$

>>>Exercise 6.1 – 4

Solve the following initial value problem with Laplace transform:

$$\dot{y}(t) - 4y(t) = e^t - 1, \quad y(0) = 2$$

>>>Exercise 6.1 – 4– Solution

>>>Solution without the convolution theorem

$$\begin{aligned}
 s \cdot Y(s) - 2 - 4Y(s) &= \frac{1}{s-1} - \frac{1}{s} = \frac{1}{s \cdot (s-1)} \\
 Y(s) \cdot (s-4) &= \frac{1}{s \cdot (s-1)} + 2 = \frac{2s^2 - 2s + 1}{s \cdot (s-1)} \\
 Y(s) &= \frac{2s^2 - 2s + 1}{s \cdot (s-1) \cdot (s-4)}
 \end{aligned}$$

Details of the partial fraction decomposition for $Y(s)$:

$$\begin{aligned}
 \frac{2s^2 - 2s + 1}{s \cdot (s-1) \cdot (s-4)} &= \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-4} = \frac{A \cdot (s-1) \cdot (s-4) + B \cdot s \cdot (s-4) + C \cdot s \cdot (s-1)}{s \cdot (s-1) \cdot (s-4)} = \\
 &= \frac{A \cdot (s^2 - 5s + 4) + B \cdot (s^2 - 4s) + C \cdot (s^2 - s)}{s \cdot (s-1) \cdot (s-4)} = \frac{s^2 \cdot (A+B+C) + s \cdot (-5A-4B-C) + C \cdot 4A}{s \cdot (s-1) \cdot (s-4)} = \\
 &= \frac{1}{4} \cdot \frac{1}{s} - \frac{1}{3} \cdot \frac{1}{s-1} + \frac{25}{12} \cdot \frac{1}{s-4}
 \end{aligned}$$

$$\begin{aligned}
 \left[\begin{array}{lcl} \text{I.} & A+B+C & = 2 \\ \text{II.} & -5A-4B-C & = -2 \\ \text{III.} & 4A & = 1 \end{array} \Rightarrow \begin{array}{l} A=1/4 \\ B=-1/3 \\ C=25/12 \end{array} \right]
 \end{aligned}$$

$$\begin{aligned}
 y(t) &= \mathcal{L}^{-1} \left(\frac{2s^2 - 2s + 1}{s \cdot (s-1) \cdot (s-4)} \right) = \mathcal{L}^{-1} \left(\frac{1}{4} \cdot \frac{1}{s} - \frac{1}{3} \cdot \frac{1}{s-1} + \frac{25}{12} \cdot \frac{1}{s-4} \right) = \\
 &= \frac{1}{4} \cdot \mathcal{L}^{-1} \left(\frac{1}{s} \right) - \frac{1}{3} \cdot \mathcal{L}^{-1} \left(\frac{1}{s-1} \right) + \frac{25}{12} \cdot \mathcal{L}^{-1} \left(\frac{1}{s-4} \right) = \frac{1}{4} - \frac{1}{3} \cdot e^t + \frac{25}{12} \cdot e^{4t}
 \end{aligned}$$

Solution with the convolution theorem

$$\begin{aligned}
 s \cdot Y(s) - 2 - 4 \cdot Y(s) &= F(s) \\
 Y(s) \cdot (s-4) &= F(s) + 2 \\
 Y(s) &= F(s) \cdot \frac{1}{s-4} + \frac{2}{s-4}
 \end{aligned}$$

$$\begin{aligned}
 y(t) &= f(t) * \mathcal{L}^{-1} \left(\frac{1}{s-4} \right) + 2 \cdot \mathcal{L}^{-1} \left(\frac{1}{s-4} \right) = \int_0^t (e^\tau - 1) \cdot e^{4(t-\tau)} d\tau + 2 \cdot e^{4t} = \\
 &= e^{4t} \cdot \int_0^t (e^{-3\tau} - e^{-4\tau}) d\tau + 2 \cdot e^{4t} = e^{4t} \cdot \left[-\frac{1}{3} \cdot e^{-3\tau} + \frac{1}{4} \cdot e^{-4\tau} \right]_0^t + 2 \cdot e^{4t} = \\
 &= e^{4t} \cdot \left(-\frac{1}{3} \cdot e^{-3t} + \frac{1}{4} \cdot e^{-4t} + \frac{1}{3} - \frac{1}{4} \right) + 2 \cdot e^{4t} = -\frac{1}{3} \cdot e^t + \frac{1}{4} + \frac{25}{12} \cdot e^{4t}
 \end{aligned}$$

>>>Exercise 6.1 – 5

Solve the following initial value problem with Laplace transform

$$\dot{y}(t) + 7y(t) = e^{-3t}, \quad y(0) = 2$$

>>>Exercise 6.1 – 5– Solution

>>>Solution without the convolution theorem

$$\begin{aligned}s \cdot Y(s) - 2 + 7 \cdot Y(s) &= \frac{1}{s+3} \\ Y(s) \cdot (s+7) &= \frac{1}{s+3} + 2 = \frac{2s+7}{s+3} \\ Y(s) &= \frac{2s+7}{(s+7) \cdot (s+3)}\end{aligned}$$

The partial fraction decomposition of $Y(s)$ is

$$\begin{aligned}\frac{2s+7}{(s+7) \cdot (s+3)} &= \frac{A}{s+7} + \frac{B}{s+3} = \frac{(A+B) \cdot s + (3A+7B)}{(s+7) \cdot (s+3)} = \frac{7}{4} \cdot \frac{1}{s+7} + \frac{1}{4} \cdot \frac{1}{s+3} \\ \left[\begin{array}{lcl} \text{I.} & A+B & = 2 \\ \text{II.} & 3A+7B & = 7 \end{array} \Rightarrow \begin{array}{l} A = 7/4 \\ B = 1/4 \end{array} \right]\end{aligned}$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{7}{4} \cdot \frac{1}{s+7} + \frac{1}{4} \cdot \frac{1}{s+3}\right) = \frac{7}{4} \cdot \mathcal{L}^{-1}\left(\frac{1}{s+7}\right) + \frac{1}{4} \cdot \mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = \frac{7}{4} \cdot e^{-7t} + \frac{1}{4} \cdot e^{-3t}$$

>>>Solution with the convolution theorem

$$\begin{aligned}s \cdot Y(s) - 2 + 7 \cdot Y(s) &= F(s) \\ Y(s) \cdot (s+7) &= F(s) + 2 \\ Y(s) &= F(s) \cdot \frac{1}{s+7} + \frac{2}{s+7} \\ y(t) &= f(t) * \mathcal{L}^{-1}\left(\frac{1}{s+7}\right) + 2 \cdot \mathcal{L}^{-1}\left(\frac{1}{s+7}\right) = \int_0^t e^{-3\tau} \cdot e^{-7 \cdot (t-\tau)} d\tau + 2 \cdot e^{-7t} \\ &= e^{-7t} \cdot \int_0^t e^{4\tau} d\tau + 2 \cdot e^{-7t} = \\ &= e^{-7t} \cdot \left[\frac{1}{4} \cdot e^{4\tau}\right]_0^t + 2 \cdot e^{-7t} = e^{-7t} \cdot \left(\frac{1}{4} \cdot e^{4t} - \frac{1}{4}\right) + 2 \cdot e^{-7t} = \frac{7}{4} \cdot e^{-7t} + \frac{1}{4} \cdot e^{-3t}\end{aligned}$$

>>>Exercise 6.1 – 6

Solve the following initial value problem with Laplace transform

$$\ddot{y}(t) + 6 \cdot \dot{y}(t) + 8 \cdot y(t) = e^t + 1, \quad y(0) = 1, \dot{y}(0) = 2$$

>>>Exercise 6.1 – 6– Solution

>>>Solution without the convolution theorem

$$\begin{aligned}\mathcal{L}(e^t + 1) &= \mathcal{L}(e^t) + \mathcal{L}(1) = \frac{1}{s-1} + \frac{1}{s} = \frac{2s-1}{s \cdot (s-1)} \\ s^2 \cdot Y(s) - s - 2 + 6 \cdot (s \cdot Y(s) - 1) + 8 \cdot Y(s) &= \frac{2s-1}{s \cdot (s-1)} \\ Y(s) \cdot (s^2 + 6 \cdot s + 8) &= \frac{2s-1}{s \cdot (s-1)} + s + 6 + 2 \\ Y(s) \cdot (s+2) \cdot (s+4) &= \frac{2s-1}{s \cdot (s-1)} + s + 6 + 2 \\ Y(s) &= \frac{2s-1}{s \cdot (s-1) \cdot (s+2) \cdot (s+4)} + \frac{s+8}{(s+2) \cdot (s+4)} \\ Y(s) &= \frac{s^3 + 7s^2 - 6s - 1}{s \cdot (s-1) \cdot (s+2) \cdot (s+4)}\end{aligned}$$

The details of the partial fraction decomposition for $Y(s)$:

$$\begin{aligned}
Y(s) &= \frac{s^3 + 7s^2 - 6s - 1}{s \cdot (s-1) \cdot (s+2) \cdot (s+4)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+2} + \frac{D}{s+4} = \\
&= \frac{A \cdot (s-1)(s+2)(s+4) + B \cdot s(s+2)(s+4) + C \cdot s(s-1)(s+4) + D \cdot s(s-1)(s+2)}{s \cdot (s-1) \cdot (s+2) \cdot (s+4)} = \\
&= \frac{A \cdot (s^3 + 5s^2 - 2s - 8) + B \cdot (s^3 + 6s^2 + 8s) + C \cdot (s^3 + 3s^2 - 4s) + D \cdot (s^3 + s^2 - 2s)}{s \cdot (s-1) \cdot (s+2) \cdot (s+4)} = \\
&= \frac{(A+B+C+D) \cdot s^3 + (5A+6B+3C+D) \cdot s^2 + (2A+8B-4C-2D) \cdot s + (-8A)}{s \cdot (s-1) \cdot (s+2) \cdot (s+4)} = \\
&= \frac{1}{8} \cdot \frac{1}{s} + \frac{1}{15} \cdot \frac{1}{s-1} + \frac{31}{12} \cdot \frac{1}{s+2} - \frac{71}{40} \cdot \frac{1}{s+4} \\
&\quad \left[\begin{array}{lll} \text{I.} & A+B+C+D & = 1 \\ \text{II.} & 5A+6B+3C+D & = 7 \\ \text{III.} & 2A+8B-4C-2D & = -6 \\ \text{IV.} & -8A & = -1 \end{array} \Rightarrow \begin{array}{l} A = 1/8 \\ B = 1/15 \\ C = 31/12 \\ D = -71/40 \end{array} \right] \\
y(t) &= \mathcal{L}^{-1} \left(\frac{s^3 + 7s^2 - 6s - 1}{s \cdot (s-1) \cdot (s+2) \cdot (s+4)} \right) = \\
&= \frac{1}{8} \cdot \mathcal{L}^{-1} \left(\frac{1}{s} \right) + \frac{1}{15} \cdot \mathcal{L}^{-1} \left(\frac{1}{s-1} \right) + \frac{31}{12} \cdot \mathcal{L}^{-1} \left(\frac{1}{s+2} \right) - \frac{71}{40} \cdot \mathcal{L}^{-1} \left(\frac{1}{s+4} \right) = \\
&= \frac{1}{8} + \frac{1}{15} \cdot e^t + \frac{31}{12} \cdot e^{-2t} - \frac{71}{40} \cdot e^{-4t}
\end{aligned}$$

>>>Solution with the convolution theorem

$$\begin{aligned}
s^2 \cdot Y(s) - s - 2 + 6 \cdot (s \cdot Y(s) - 1) + 8 \cdot Y(s) &= F(s) \\
Y(s) \cdot (s^2 + 6 \cdot s + 8) &= F(s) + s + 6 + 2 \\
Y(s) &= F(s) \cdot \frac{1}{s^2 + 6 \cdot s + 8} + \frac{s+8}{s^2 + 6 \cdot s + 8} \\
y(t) &= \mathcal{L}^{-1} \left(F(s) \cdot \frac{1}{s^2 + 6 \cdot s + 8} \right) + \mathcal{L}^{-1} \left(\frac{s+8}{s^2 + 6 \cdot s + 8} \right) = \\
&= \mathcal{L}^{-1}(F(s)) * \mathcal{L}^{-1} \left(\frac{1}{s^2 + 6 \cdot s + 8} \right) + \mathcal{L}^{-1} \left(\frac{s+8}{s^2 + 6 \cdot s + 8} \right) \\
v(t) &= \mathcal{L}^{-1} \left(\frac{1}{s^2 + 6 \cdot s + 8} \right) = \mathcal{L}^{-1} \left(\frac{1}{2} \cdot \frac{1}{s+2} - \frac{1}{2} \cdot \frac{1}{s+4} \right) = \\
&= \frac{1}{2} \cdot \mathcal{L}^{-1} \left(\frac{1}{s+2} \right) - \frac{1}{2} \cdot \mathcal{L}^{-1} \left(\frac{1}{s+4} \right) = \frac{1}{2} \cdot e^{-2t} - \frac{1}{2} \cdot e^{-4t} \\
w(t) &= \mathcal{L}^{-1} \left(\frac{s+8}{s^2 + 6 \cdot s + 8} \right) = \mathcal{L}^{-1} \left(3 \cdot \frac{1}{s+2} - 2 \cdot \frac{1}{s+4} \right) = \\
&= 3 \cdot \mathcal{L}^{-1} \left(\frac{1}{s+2} \right) - 2 \cdot \mathcal{L}^{-1} \left(\frac{1}{s+4} \right) = 3 \cdot e^{-2t} - 2 \cdot e^{-4t} \\
y(t) &= (e^t + 1) * \left(\frac{1}{2} \cdot e^{-2t} - \frac{1}{2} \cdot e^{-4t} \right) + (3 \cdot e^{-2t} - 2 \cdot e^{-4t}) = \\
&= \int_0^t (e^\tau + 1) \cdot \left(\frac{1}{2} \cdot e^{-2 \cdot (t-\tau)} - \frac{1}{2} \cdot e^{-4 \cdot (t-\tau)} \right) d\tau + 3 \cdot e^{-2t} - 2 \cdot e^{-4t} = \\
&= \frac{1}{2} \cdot \int_0^t (e^{-2t+3\tau} - e^{-4t+5\tau} + e^{-2t+2\tau} - e^{-4t+4\tau}) d\tau + 3 \cdot e^{-2t} - 2 \cdot e^{-4t} = \\
&= \frac{1}{6} \cdot [e^{-2t+3\tau}]_{\tau=0}^t - \frac{1}{10} \cdot [e^{-4t+5\tau}]_{\tau=0}^t + \frac{1}{4} \cdot [e^{-2t+2\tau}]_{\tau=0}^t - \frac{1}{8} \cdot [e^{-4t+4\tau}]_{\tau=0}^t = \\
&= \frac{1}{6} \cdot e^t - \frac{1}{6} \cdot e^{-2t} - \frac{1}{10} \cdot e^t + \frac{1}{10} \cdot e^{-4t} + \frac{1}{4} - \frac{1}{4} \cdot e^{-2t} - \frac{1}{8} + \frac{1}{8} \cdot e^{-4t} + 3 \cdot e^{-2t} - 2 \cdot e^{-4t} =
\end{aligned}$$

$$= \frac{1}{15} \cdot e^t + \frac{31}{12} \cdot e^{-2 \cdot t} - \frac{71}{40} \cdot e^{-4 \cdot t} + \frac{1}{8}$$

>>>Exercise 6.1 – 7

Solve the following initial value problem with Laplace transform

$$\ddot{y}(t) - 10 \cdot \dot{y}(t) + 9 \cdot y(t) = 5t, \quad y(0) = -1, \dot{y}(0) = 2$$

>>>Exercise 6.1 – 7- Solution

>>>Solution without the convolution theorem

$$(s^2 \cdot Y(s) + s - 2) - 10 \cdot (s \cdot Y(s) + 1) + 9 \cdot Y(s) = \frac{5}{s^2}$$

$$Y(s) \cdot (s^2 - 10 \cdot s + 9) = \frac{5}{s^2} - s + 12$$

$$Y(s) \cdot (s - 9) \cdot (s - 1) = \frac{5 - s^3 + 12 \cdot s^2}{s^2}$$

$$Y(s) = \frac{-s^3 + 12 \cdot s^2 + 5}{s^2 \cdot (s - 9) \cdot (s - 1)}$$

The details of the partial fraction decomposition for $Y(s)$:

$$\begin{aligned} Y(s) &= \frac{-s^3 + 12 \cdot s^2 + 5}{s^2 \cdot (s - 9) \cdot (s - 1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s - 9} + \frac{D}{s - 1} = \\ &= \frac{A \cdot s(s - 9)(s - 1) + B \cdot (s - 9)(s - 1) + C \cdot s^2 \cdot (s - 1) + D \cdot s^2 \cdot (s - 9)}{s^2 \cdot (s - 9) \cdot (s - 1)} = \\ &= \frac{A \cdot (s^3 - 10s^2 + 9s) + B \cdot (s^2 - 10s + 9) + C \cdot (s^3 - s^2) + D \cdot (s^3 - 9s^2)}{s^2 \cdot (s - 9) \cdot (s - 1)} = \\ &= \frac{(A + C + D) \cdot s^3 + (-10A + B - C - 9D) \cdot s^2 + (9A - 10B) \cdot s + (9B)}{s^2 \cdot (s - 9) \cdot (s - 1)} = \\ &= \frac{50}{81} \cdot \frac{1}{s} + \frac{5}{9} \cdot \frac{1}{s^2} + \frac{31}{81} \cdot \frac{1}{s - 9} - 2 \cdot \frac{1}{s - 1} \end{aligned}$$

$$\left[\begin{array}{lcll} \text{I.} & A + C + D & = & -1 & A = 50/81 \\ \text{II.} & -10A + B - C - 9D & = & 12 & B = 5/9 \\ \text{III.} & 9A - 10B & = & 0 & C = 31/81 \\ \text{IV.} & 9B & = & 5 & D = -2 \end{array} \right] \Rightarrow$$

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left(\frac{-s^3 + 12 \cdot s^2 + 5}{s^2 \cdot (s - 9) \cdot (s - 1)} \right) = \\ &= \frac{50}{81} \cdot \mathcal{L}^{-1} \left(\frac{1}{s} \right) + \frac{5}{9} \cdot \mathcal{L}^{-1} \left(\frac{1}{s^2} \right) + \frac{31}{81} \cdot \mathcal{L}^{-1} \left(\frac{1}{s - 9} \right) - 2 \cdot \mathcal{L}^{-1} \left(\frac{1}{s - 1} \right) = \\ &= \frac{50}{81} + \frac{5}{9} \cdot t + \frac{31}{81} \cdot e^{9t} - 2 \cdot e^t \end{aligned}$$

>>>Solution with the convolution theorem

$$(s^2 \cdot Y(s) + s - 2) - 10 \cdot (s \cdot Y(s) + 1) + 9 \cdot Y(s) = F(s)$$

$$Y(s) \cdot (s^2 - 10 \cdot s + 9) = F(s) - s + 12$$

$$Y(s) = F(s) \cdot \frac{1}{s^2 - 10 \cdot s + 9} - \frac{s - 12}{s^2 - 10 \cdot s + 9}$$

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left(F(s) \cdot \frac{1}{s^2 - 10 \cdot s + 9} \right) - \mathcal{L}^{-1} \left(\frac{s - 12}{s^2 - 10 \cdot s + 9} \right) = \\ &= \mathcal{L}^{-1}(F(s)) * \mathcal{L}^{-1} \left(\frac{1}{s^2 - 10 \cdot s + 9} \right) - \mathcal{L}^{-1} \left(\frac{s - 12}{s^2 - 10 \cdot s + 9} \right) \end{aligned}$$

$$v(t) = \mathcal{L}^{-1} \left(\frac{1}{s^2 - 10 \cdot s + 9} \right) = \mathcal{L}^{-1} \left(\frac{1}{8} \cdot \frac{1}{s - 9} - \frac{1}{8} \cdot \frac{1}{s - 1} \right) =$$

$$\begin{aligned}
&= \frac{1}{8} \cdot \mathcal{L}^{-1}\left(\frac{1}{s-9}\right) - \frac{1}{8} \cdot \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = \frac{1}{8} \cdot e^{9t} - \frac{1}{8} \cdot e^t \\
w(t) &= -\mathcal{L}^{-1}\left(\frac{s-12}{s^2-10s+9}\right) = -\mathcal{L}^{-1}\left(-\frac{3}{8} \cdot \frac{1}{s-9} + \frac{11}{8} \cdot \frac{1}{s-1}\right) = \\
&= \frac{3}{8} \cdot \mathcal{L}^{-1}\left(\frac{1}{s-9}\right) - \frac{11}{8} \cdot \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = \frac{3}{8} \cdot e^{9t} - \frac{11}{8} \cdot e^t \\
y(t) &= (5t) * \left(\frac{1}{8} \cdot e^{9t} - \frac{1}{8} \cdot e^t\right) + \left(\frac{3}{8} \cdot e^{9t} - \frac{11}{8} \cdot e^t\right) = \\
&= \frac{5}{8} \cdot \int_0^t \tau \cdot (e^{9(t-\tau)} - e^{t-\tau}) d\tau + \frac{3}{8} \cdot e^{9t} - \frac{11}{8} \cdot e^t = \\
&= \frac{5}{8} \cdot \left(e^{9t} \cdot \int_0^t \tau \cdot e^{-9\tau} d\tau - e^t \cdot \int_0^t \tau \cdot e^{-\tau} d\tau \right) + \frac{3}{8} \cdot e^{9t} - \frac{11}{8} \cdot e^t = \\
&= \frac{5}{8} \cdot \left(-\frac{1}{81} \cdot e^{9t} \cdot [(1+9\tau) \cdot e^{-9\tau}]_{\tau=0}^t + e^t \cdot [(1+\tau) \cdot e^{-\tau}]_{\tau=0}^t \right) + \frac{3}{8} \cdot e^{9t} - \frac{11}{8} \cdot e^t = \\
&= -\frac{5}{648} \cdot e^{9t} \cdot ((1+9t) \cdot e^{-9t} - 1) + \frac{5}{8} \cdot e^t \cdot ((1+t) \cdot e^{-t} - 1) + \frac{3}{8} \cdot e^{9t} - \frac{11}{8} \cdot e^t = \\
&= -\frac{5}{648} - \frac{45}{648} \cdot t + \frac{5}{648} \cdot e^{9t} + \frac{5}{8} + \frac{5}{8} \cdot t - \frac{5}{8} \cdot e^t + \frac{3}{8} \cdot e^{9t} - \frac{11}{8} \cdot e^t = \\
&= \frac{50}{81} + \frac{5}{9} \cdot t + \frac{31}{81} \cdot e^{9t} - 2 \cdot e^t
\end{aligned}$$

>>>Exercise 6.1 – 8

Solve the following initial value problem with Laplace transform

$$2 \cdot \ddot{y}(t) + 3 \cdot \dot{y}(t) - 2 \cdot y(t) = t \cdot e^{-2t}, \quad y(0) = 0, \dot{y}(0) = -2$$

>>>Exercise 6.1 – 8- Solution

>>>Solution without the convolution theorem

$$\begin{aligned}
2 \cdot (s^2 \cdot Y(s) + 2) + 3 \cdot s \cdot Y(s) - 2 \cdot Y(s) &= \frac{1}{(s+2)^2} \\
Y(s) \cdot (2s^2 + 3 \cdot s - 2) &= \frac{1}{(s+2)^2} - 4 \\
Y(s) \cdot (2s-1) \cdot (s+2) &= \frac{1 - 4 \cdot s^2 - 16 \cdot s - 16}{(s+2)^2} \\
Y(s) &= \frac{-4 \cdot s^2 - 16 \cdot s - 15}{(2s-1) \cdot (s+2)^3}
\end{aligned}$$

The details of the partial fraction decomposition for $Y(s)$:

$$\begin{aligned}
Y(s) &= \frac{-4 \cdot s^2 - 16 \cdot s - 15}{(2s-1) \cdot (s+2)^3} = \frac{A}{2s-1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3} = \\
&= \frac{A \cdot (s+2)^3 + B \cdot (2s-1) \cdot (s+2)^2 + C \cdot (2s-1) \cdot (s+2) + D \cdot (2s-1)}{(2s-1) \cdot (s+2)^3} = \\
&= \frac{A \cdot (s^3 + 6s^2 + 12s + 8) + B \cdot (2s^3 + 7s^2 + 4s - 4) + C \cdot (2s^2 + 3s - 2) + D \cdot (2s - 1)}{(2s-1) \cdot (s+2)^3} = \\
&= \frac{(A+2B) \cdot s^3 + (6A+7B+2C) \cdot s^2 + (12A+4B+3C+2D) \cdot s + (8A-4B-2C-D)}{(2s-1) \cdot (s+2)^3} = \\
&= -\frac{192}{125} \cdot \frac{1}{2s-1} + \frac{96}{125} \cdot \frac{1}{s+2} - \frac{2}{25} \cdot \frac{1}{(s+2)^2} - \frac{1}{5} \cdot \frac{1}{(s+2)^3}
\end{aligned}$$

$$\begin{bmatrix} I. & A+2B & = & 0 & A = -192/125 \\ II. & 6A+7B+2C & = & -4 & B = 96/125 \\ III. & 12A+4B+3C+2D & = & -16 & C = -2/25 \\ IV. & 8A-4B-2C-D & = & -15 & D = -1/5 \end{bmatrix} \Rightarrow$$

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left(-\frac{96}{125} \cdot \frac{1}{s-\frac{1}{2}} + \frac{96}{125} \cdot \frac{1}{s+2} - \frac{2}{25} \cdot \frac{1}{(s+2)^2} - \frac{1}{5} \cdot \frac{1}{(s+2)^3} \right) = \\ &= -\frac{96}{125} \cdot \mathcal{L}^{-1} \left(\frac{1}{s-\frac{1}{2}} \right) + \frac{96}{125} \cdot \mathcal{L}^{-1} \left(\frac{1}{s+2} \right) - \frac{2}{25} \cdot \mathcal{L}^{-1} \left(\frac{1}{(s+2)^2} \right) - \frac{1}{10} \cdot \mathcal{L}^{-1} \left(\frac{2}{(s+2)^3} \right) = \\ &= -\frac{96}{125} \cdot e^{\frac{1}{2}t} + \frac{96}{125} \cdot e^{-2t} - \frac{2}{25} \cdot t \cdot e^{-2t} - \frac{1}{10} \cdot t^2 \cdot e^{-2t} \end{aligned}$$

>>>Solution with the convolution theorem

$$2 \cdot (s^2 \cdot Y(s) + 2) + 3 \cdot s \cdot Y(s) - 2 \cdot Y(s) = F(s)$$

$$Y(s) \cdot (2s^2 + 3 \cdot s - 2) = F(s) - 4$$

$$Y(s) = F(s) \cdot \frac{1}{2s^2 + 3 \cdot s - 2} - \frac{4}{2s^2 + 3 \cdot s - 2}$$

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left(F(s) \cdot \frac{1}{2s^2 + 3 \cdot s - 2} \right) + \mathcal{L}^{-1} \left(\frac{-4}{2s^2 + 3 \cdot s - 2} \right) = \\ &= \mathcal{L}^{-1}(F(s)) * \mathcal{L}^{-1} \left(\frac{1}{2s^2 + 3 \cdot s - 2} \right) + \mathcal{L}^{-1} \left(\frac{-4}{2s^2 + 3 \cdot s - 2} \right) \end{aligned}$$

$$\begin{aligned} v(t) &= \mathcal{L}^{-1} \left(\frac{1}{2s^2 + 3 \cdot s - 2} \right) = \mathcal{L}^{-1} \left(\frac{2}{5} \cdot \frac{1}{2s-1} - \frac{1}{5} \cdot \frac{1}{s+2} \right) = \\ &= \frac{1}{5} \cdot \mathcal{L}^{-1} \left(\frac{1}{s-\frac{1}{2}} \right) - \frac{1}{5} \cdot \mathcal{L}^{-1} \left(\frac{1}{s+2} \right) = \frac{1}{5} \cdot e^{\frac{1}{2}t} - \frac{1}{5} \cdot e^{-2t} \end{aligned}$$

$$\begin{aligned} w(t) &= \mathcal{L}^{-1} \left(\frac{-4}{2s^2 + 3 \cdot s - 2} \right) = \mathcal{L}^{-1} \left(-\frac{8}{5} \cdot \frac{1}{2s-1} + \frac{4}{5} \cdot \frac{1}{s+2} \right) = \\ &= -\frac{4}{5} \cdot \mathcal{L}^{-1} \left(\frac{1}{s-\frac{1}{2}} \right) + \frac{4}{5} \cdot \mathcal{L}^{-1} \left(\frac{1}{s+2} \right) = -\frac{4}{5} \cdot e^{\frac{1}{2}t} + \frac{4}{5} \cdot e^{-2t} \end{aligned}$$

$$\begin{aligned} y(t) &= (t \cdot e^{-2t}) * \left(\frac{1}{5} \cdot e^{\frac{1}{2}t} - \frac{1}{5} \cdot e^{-2t} \right) - \frac{4}{5} \cdot e^{\frac{1}{2}t} + \frac{4}{5} \cdot e^{-2t} = \\ &= \int_0^t (\tau \cdot e^{-2\tau}) \cdot \left(\frac{1}{5} \cdot e^{\frac{1}{2}(t-\tau)} - \frac{1}{5} \cdot e^{-2(t-\tau)} \right) d\tau - \frac{4}{5} \cdot e^{\frac{1}{2}t} + \frac{4}{5} \cdot e^{-2t} = \\ &= \frac{1}{5} \cdot e^{\frac{1}{2}t} \cdot \int_0^t \tau \cdot e^{-\frac{5}{2}\tau} d\tau - \frac{1}{5} \cdot e^{-2t} \cdot \int_0^t \tau d\tau - \frac{4}{5} \cdot e^{\frac{1}{2}t} + \frac{4}{5} \cdot e^{-2t} = \\ &= -\frac{2}{125} \cdot e^{\frac{1}{2}t} \cdot \left[(2+5\tau) \cdot e^{-\frac{5}{2}\tau} \right]_{\tau=0}^t - \frac{1}{10} \cdot e^{-2t} \cdot [\tau^2]_{\tau=0}^t - \frac{4}{5} \cdot e^{\frac{1}{2}t} + \frac{4}{5} \cdot e^{-2t} = \\ &= -\frac{2}{125} \cdot e^{\frac{1}{2}t} \cdot \left((2+5t) \cdot e^{-\frac{5}{2}t} - 2 \right) - \frac{1}{10} \cdot t^2 \cdot e^{-2t} - \frac{4}{5} \cdot e^{\frac{1}{2}t} + \frac{4}{5} \cdot e^{-2t} = \\ &= -\frac{4}{125} \cdot e^{-2t} - \frac{2}{25} \cdot t \cdot e^{-2t} + \frac{4}{125} \cdot e^{\frac{1}{2}t} - \frac{1}{10} \cdot t^2 \cdot e^{-2t} - \frac{4}{5} \cdot e^{\frac{1}{2}t} + \frac{4}{5} \cdot e^{-2t} = \\ &= -\frac{96}{125} \cdot e^{\frac{1}{2}t} + \frac{96}{125} \cdot e^{-2t} - \frac{2}{25} \cdot t \cdot e^{-2t} - \frac{1}{10} \cdot t^2 \cdot e^{-2t} \end{aligned}$$

>>>Exercise 6.1 – 9

Solve the following initial value problem with Laplace transform

$$\ddot{y}(t) - 6 \cdot \dot{y}(t) + 15 \cdot y(t) = 2 \cdot \sin 3t, \quad y(0) = -1, \dot{y}(0) = -4$$

>>>Exercise 6.1 – 9- Solution

>>>Solution without the convolution theorem

$$\begin{aligned} (s^2 \cdot Y(s) + s + 4) - 6 \cdot (s \cdot Y(s) + 1) + 15 \cdot Y(s) &= 2 \cdot \frac{3}{s^2 + 9} \\ Y(s) \cdot (s^2 - 6 \cdot s + 15) &= \frac{6}{s^2 + 9} - s + 2 = \frac{-s^3 + 2 \cdot s^2 - 9s + 24}{s^2 + 9} \\ Y(s) &= \frac{-s^3 + 2 \cdot s^2 - 9s + 24}{(s^2 + 9) \cdot (s^2 - 6 \cdot s + 15)} \end{aligned}$$

The partial fraction decomposition of $Y(s)$ is

$$\begin{aligned} Y(s) &= \frac{-s^3 + 2 \cdot s^2 - 9s + 24}{(s^2 + 9) \cdot (s^2 - 6 \cdot s + 15)} = \frac{A \cdot s + B}{s^2 + 9} + \frac{C \cdot s + D}{s^2 - 6 \cdot s + 15} = \\ &= \frac{(A \cdot s + B) \cdot (s^2 - 6 \cdot s + 15) + (C \cdot s + D) \cdot (s^2 + 9)}{(s^2 + 9) \cdot (s^2 - 6 \cdot s + 15)} = \\ &= \frac{(A + C) \cdot s^3 + (-6A + B + D) \cdot s^2 + (15A - 6B + 9C) \cdot s + (15B + 9D)}{(s^2 + 9) \cdot (s^2 - 6 \cdot s + 15)} = \\ &= \frac{1}{10} \cdot \frac{s + 1}{s^2 + 9} + \frac{1}{10} \cdot \frac{-11 \cdot s + 25}{s^2 - 6 \cdot s + 15} = \\ &\quad \left[\begin{array}{lll} I. & A + C & = -1 & A = 1/10 \\ II. & -6A + B + D & = 2 & B = 1/10 \\ III. & 15A - 6B + 9C & = -9 & C = -11/10 \\ IV. & 15B + 9D & = 24 & D = 25/10 \end{array} \right] \Rightarrow \end{aligned}$$

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left(\frac{1}{10} \cdot \frac{s + 1}{s^2 + 9} + \frac{1}{10} \cdot \frac{-11 \cdot s + 25}{s^2 - 6 \cdot s + 15} \right) = \\ &= \frac{1}{10} \cdot \mathcal{L}^{-1} \left(\frac{s + 1}{s^2 + 9} \right) + \frac{1}{10} \cdot \mathcal{L}^{-1} \left(\frac{-11 \cdot s + 25}{s^2 - 6 \cdot s + 15} \right) = \frac{1}{10} \cdot \mathcal{L}^{-1} \left(\frac{s}{s^2 + 9} \right) + \frac{1}{10} \cdot \mathcal{L}^{-1} \left(\frac{3}{s^2 + 9} \right) + \\ &\quad + \frac{1}{10} \cdot \left(-11 \cdot \mathcal{L}^{-1} \left(\frac{s - 3}{(s - 3)^2 + (\sqrt{6})^2} \right) - \frac{8}{\sqrt{6}} \cdot \mathcal{L}^{-1} \left(\frac{\sqrt{6}}{(s - 3)^2 + (\sqrt{6})^2} \right) \right) = \\ &= \frac{1}{10} \cdot \left(\cos 3t + \frac{1}{3} \cdot \sin 3t - 11 \cdot e^{3t} \cdot \cos(\sqrt{6} \cdot t) - \frac{8}{\sqrt{6}} \cdot e^{3t} \cdot \sin(\sqrt{6} \cdot t) \right) \end{aligned}$$

7 Solution of Systems of Linear Differential Equations with Laplace Transform

Let I be an interval. If $a_{ij}: I \rightarrow \mathbb{R}$ and $f_i: I \rightarrow \mathbb{R}$ are given continuous functions and $x_i: I \rightarrow \mathbb{R}$ are unknown functions ($i, j = 1, \dots, n$), then

$$\begin{aligned}\dot{x}_1(t) &= a_{11}(t) \cdot x_1(t) + \dots + a_{1n}(t) \cdot x_n(t) + f_1(t) \\ \dot{x}_2(t) &= a_{21}(t) \cdot x_1(t) + \dots + a_{2n}(t) \cdot x_n(t) + f_2(t) \\ &\vdots \\ \dot{x}_n(t) &= a_{n1}(t) \cdot x_1(t) + \dots + a_{nn}(t) \cdot x_n(t) + f_n(t)\end{aligned}$$

is called a *system of first order linear differential equations*.

With notations

$$A(t) = \begin{pmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{pmatrix}, \quad \bar{f}(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

$$\bar{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \dot{\bar{x}}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix}$$

we get the matrix form of the linear system

$$\dot{\bar{x}}(t) = A(t) \cdot \bar{x}(t) + \bar{f}(t)$$

Linear dynamic systems that can be modelled with differential equations generally have two equivalent mathematical models: higher order linear differential equations or systems of first order linear differential equations. In system analysis when physical laws are used modelling with a system of first order linear differential equations is generally more natural.

To show the connection between the two models let us consider an n -th order linear differential equation

$$y^{(n)}(t) + g_{n-1}(t) \cdot y^{(n-1)}(t) + \dots + g_1(t) \cdot y'(t) + g_0(t) \cdot y(t) = f(t)$$

Introducing functions $x_1, \dots, x_n$ by the equality

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}$$

our n -th order linear differential equation transforms into the system of first order linear differential equations

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) \\ &\vdots \\ \dot{x}_n(t) &= -g_{n-1}(t) \cdot x_n(t) - \dots - g_1(t) \cdot x_2(t) + g_0(t) \cdot x_1(t) + f(t)\end{aligned}$$

If $a_{ij}: I \rightarrow \mathbb{R}$ are constant functions then we have a *system of first order linear differential equations with constant coefficients*:

$$\begin{aligned}\dot{x}_1(t) &= a_{11} \cdot x_1(t) + \dots + a_{1n} \cdot x_n(t) + f_1(t) \\ \dot{x}_2(t) &= a_{21} \cdot x_1(t) + \dots + a_{2n} \cdot x_n(t) + f_2(t) \\ &\vdots \\ \dot{x}_n(t) &= a_{n1} \cdot x_1(t) + \dots + a_{nn} \cdot x_n(t) + f_n(t)\end{aligned}$$

or in matrix form

$$\dot{\bar{x}}(t) = A \cdot \bar{x}(t) + \bar{f}(t)$$

If we have further information about the solution $\bar{x}$, namely $\bar{x}(t_0) = \bar{x}_0 \in \mathbb{R}^n$ or, in detail,

$\begin{pmatrix} x_1(t_0) \\ \vdots \\ x_n(t_0) \end{pmatrix} = \begin{pmatrix} x_{10} \\ \vdots \\ x_{n0} \end{pmatrix}$ is given for some $t_0 \in I$, we have an *initial value problem of constant coefficients*.

An initial value problem

$$\dot{\bar{x}}(t) = A \cdot \bar{x}(t) + \bar{f}(t), \quad \bar{x}(t_0) = \bar{x}_0$$

can also be solved with the help of the Laplace transform.

>>>Exercise 7.1 – 1

Convert the linear differential equation

$$y''(t) - 5 \cdot y'(t) + 12 \cdot y(t) = \sin 10t$$

into a system of first order linear differential equations.

>>>Exercise 7.1 – 1- Solution

Express y'' from the equation:

$$y''(t) = -12 \cdot y(t) + 5 \cdot y'(t) + \sin 10t$$

Introducing functions x_1 and x_2 by $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}$ our second order linear differential equation transforms into the system of first order linear differential equations

$$\begin{aligned} \dot{x}_1(t) &= y'(t) = x_2(t) \\ \dot{x}_2(t) &= y''(t) = -12 \cdot y(t) + 5 \cdot y'(t) + \sin 10t = -12 \cdot x_1(t) + 5 \cdot x_2(t) + \sin 10t \end{aligned}$$

In matrix form:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -12 & 5 \end{pmatrix} \cdot \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \sin 10t \end{pmatrix}$$

>>>Exercise 7.1 – 2

Convert the linear differential equation

$$y^{(4)}(t) + 2 \cdot y^{(3)}(t) + 7 \cdot y''(t) + 4 \cdot y'(t) - y(t) = t^2$$

into a system of first order linear differential equations.

>>>Exercise 7.1 – 2- Solution

Express $y^{(4)}$ from the equation:

$$y^{(4)}(t) = y(t) - 4 \cdot y'(t) - 7 \cdot y''(t) - 2 \cdot y^{(3)}(t) + t^2.$$

Introducing functions x_1, x_2, x_3 and x_4 by $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' \\ y^{(3)} \end{pmatrix}$ our second order linear differential

equation transforms into the system of first order linear differential equations

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) \\ \dot{x}_3(t) &= x_4(t) \\ \dot{x}_4(t) &= x_1(t) - 4 \cdot x_2(t) - 7 \cdot x_3(t) - 2 \cdot x_4(t) + t^2 \end{aligned}$$

In matrix form:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -4 & -7 & -2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{pmatrix}$$

>>>Exercise 7.1 – 3

Solve the initial value problem

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & 6 \end{pmatrix} \cdot \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} e^t \\ -e^t \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

>>>Exercise 7.1 – 3- Solution

$$I. \quad \dot{x}(t) = 3 \cdot x(t) - 2 \cdot y(t) + e^t$$

$$II. \quad \dot{y}(t) = x(t) + 6 \cdot y(t) - e^t$$

Laplace transform of the equations:

$$I. \quad s \cdot X(s) - 2 = 3 \cdot X(s) - 2 \cdot Y(s) + \frac{1}{s-1}$$

$$II. \quad s \cdot Y(s) + 1 = X(s) + 6 \cdot Y(s) - \frac{1}{s-1}$$

Solving it for $X(s)$ and $Y(s)$ we have

$$I. \quad (s-3) \cdot X(s) + 2 \cdot Y(s) = 2 + \frac{1}{s-1}$$

$$II. \quad -X(s) + (s-6) \cdot Y(s) = -1 - \frac{1}{s-1}$$

$$X(s) = \frac{2 \cdot s^2 - 11 \cdot s + 6}{(s-4) \cdot (s-5) \cdot (s-1)}$$

$$Y(s) = -\frac{s^2 - 5 \cdot s + 1}{(s-4) \cdot (s-5) \cdot (s-1)}$$

The partial fraction decomposition of $X(s)$ gives that

$$X(s) = \frac{2 \cdot s^2 - 11 \cdot s + 6}{(s-4) \cdot (s-5) \cdot (s-1)} = 2 \cdot \frac{1}{s-4} + \frac{1}{4} \cdot \frac{1}{s-5} - \frac{1}{4} \cdot \frac{1}{s-1}$$

so

$$\begin{aligned} x(t) &= \mathcal{L}^{-1} \left(2 \cdot \frac{1}{s-4} + \frac{1}{4} \cdot \frac{1}{s-5} - \frac{1}{4} \cdot \frac{1}{s-1} \right) = \\ &= 2 \cdot \mathcal{L}^{-1} \left(\frac{1}{s-4} \right) + \frac{1}{4} \cdot \mathcal{L}^{-1} \left(\frac{1}{s-5} \right) - \frac{1}{4} \cdot \mathcal{L}^{-1} \left(\frac{1}{s-1} \right) = 2 \cdot e^{4t} + \frac{1}{4} \cdot e^{5t} - \frac{1}{4} \cdot e^t \end{aligned}$$

The partial fraction decomposition of $Y(s)$ gives that

$$Y(s) = -\frac{s^2 - 5 \cdot s + 1}{(s-4) \cdot (s-5) \cdot (s-1)} = -\frac{1}{s-4} - \frac{1}{4} \cdot \frac{1}{s-5} + \frac{1}{4} \cdot \frac{1}{s-1}$$

so

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left(-\frac{1}{s-4} - \frac{1}{4} \cdot \frac{1}{s-5} + \frac{1}{4} \cdot \frac{1}{s-1} \right) = \\ &= -\mathcal{L}^{-1} \left(\frac{1}{s-4} \right) - \frac{1}{4} \cdot \mathcal{L}^{-1} \left(\frac{1}{s-5} \right) + \frac{1}{4} \cdot \mathcal{L}^{-1} \left(\frac{1}{s-1} \right) = -e^{4t} - \frac{1}{4} \cdot e^{5t} + \frac{1}{4} \cdot e^t \end{aligned}$$

Details of the partial fraction decomposition for $X(s)$:

$$\begin{aligned} \frac{2 \cdot s^2 - 11 \cdot s + 6}{(s-4) \cdot (s-5) \cdot (s-1)} &= \frac{A}{s-4} + \frac{B}{s-5} + \frac{C}{s-1} = \\ &= \frac{A \cdot (s-5) \cdot (s-1) + B \cdot (s-4) \cdot (s-1) + C \cdot (s-4) \cdot (s-5)}{(s-4) \cdot (s-5) \cdot (s-1)} = \end{aligned}$$

$$= \frac{(A+B+C) \cdot s^2 + (-6A-5B-9C) \cdot s + (5A+4B+20C)}{(s-4) \cdot (s-5) \cdot (s-1)} = 2 \cdot \frac{1}{s-4} + \frac{1}{4} \cdot \frac{1}{s-5} - \frac{1}{4} \cdot \frac{1}{s-1}$$

$$\begin{bmatrix} \text{I.} & A+B+C & = & 2 \\ \text{II.} & -6A-5B-9C & = & -11 \\ \text{III.} & 5A+4B+20C & = & 6 \end{bmatrix} \Rightarrow \begin{matrix} A=2 \\ B=1/4 \\ C=-1/4 \end{matrix}$$

Details of the partial fraction decomposition for $Y(s)$:

$$-\frac{s^2-5 \cdot s+1}{(s-4) \cdot (s-5) \cdot (s-1)} = \frac{A}{s-4} + \frac{B}{s-5} + \frac{C}{s-1} =$$

$$= \frac{A \cdot (s-5) \cdot (s-1) + B \cdot (s-4) \cdot (s-1) + C \cdot (s-4) \cdot (s-5)}{(s-4) \cdot (s-5) \cdot (s-1)} =$$

$$= \frac{(A+B+C) \cdot s^2 + (-6A-5B-9C) \cdot s + (5A+4B+20C)}{(s-4) \cdot (s-5) \cdot (s-1)} = -\frac{1}{s-4} - \frac{1}{4} \cdot \frac{1}{s-5} + \frac{1}{4} \cdot \frac{1}{s-1}$$

$$\begin{bmatrix} \text{I.} & A+B+C & = & -1 \\ \text{II.} & -6A-5B-9C & = & 5 \\ \text{III.} & 5A+4B+20C & = & -1 \end{bmatrix} \Rightarrow \begin{matrix} A=-1 \\ B=-1/4 \\ C=1/4 \end{matrix}$$

>>>Exercise 7.1 – 4

Solve the initial value problem

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} -7 & 1 \\ -2 & -5 \end{pmatrix} \cdot \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 5 \\ 37 \cdot t \end{pmatrix}, \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

>>>Exercise 7.1 – 4– Solution

$$\begin{aligned} \text{I.} \quad & \dot{x}(t) = -7 \cdot x(t) + y(t) + 5 \\ \text{II.} \quad & \dot{y}(t) = -2 \cdot x(t) - 5 \cdot y(t) + 37 \cdot t \end{aligned}$$

Laplace transform of the equations:

$$\begin{aligned} \text{I.} \quad & s \cdot X(s) = -7 \cdot X(s) + Y(s) + \frac{5}{s} \\ \text{II.} \quad & s \cdot Y(s) = -2 \cdot X(s) - 5 \cdot Y(s) + \frac{37}{s^2} \end{aligned}$$

Solving it for $X(s)$ and $Y(s)$ we have

$$\begin{aligned} \text{I.} \quad & (s+7) \cdot X(s) - Y(s) = \frac{5}{s} \\ \text{II.} \quad & 2 \cdot X(s) + (s+5) \cdot Y(s) = \frac{37}{s^2} \end{aligned}$$

$$X(s) = \frac{5 \cdot s^2 + 25 \cdot s + 37}{s^2 \cdot (s^2 + 12 \cdot s + 37)}$$

$$Y(s) = \frac{-47 \cdot s - 259}{s^2 \cdot (s^2 + 12 \cdot s + 37)}$$

The partial fraction decomposition of $X(s)$ gives that

$$X(s) = \frac{5 \cdot s^2 + 25 \cdot s + 37}{s^2 \cdot (s^2 + 12 \cdot s + 37)} = \frac{1}{s} + \frac{1}{s^2} - \frac{s+6}{(s+6)^2 + 1}$$

so

$$\begin{aligned} x(t) &= \mathcal{L}^{-1} \left(\frac{1}{s} + \frac{1}{s^2} - \frac{s+6}{(s+6)^2 + 1} \right) = \\ &= \mathcal{L}^{-1} \left(\frac{1}{s} \right) + \mathcal{L}^{-1} \left(\frac{1}{s^2} \right) - \mathcal{L}^{-1} \left(\frac{s+6}{(s+6)^2 + 1} \right) = 1 - t - e^{-6t} \cdot \cos t \end{aligned}$$

The partial fraction decomposition of $Y(s)$ gives that

$$Y(s) = \frac{-47 \cdot s - 259}{s^2 \cdot (s^2 + 12 \cdot s + 37)} = \frac{1}{s} - \frac{7}{s^2} - \frac{s+5}{(s+6)^2 + 1}$$

so

$$\begin{aligned}
 y(t) &= \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{7}{s^2} - \frac{s+5}{(s+6)^2+1}\right) = \\
 &= \mathcal{L}^{-1}\left(\frac{1}{s}\right) - 7 \cdot \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{s+6}{(s+6)^2+1}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s+6)^2+1}\right) = \\
 &= 1 - 7 \cdot t + e^{-6t} \cdot \cos t + e^{-6t} \cdot \sin t
 \end{aligned}$$

Details of the partial fraction decomposition for $X(s)$:

$$\begin{aligned}
 \frac{5 \cdot s^2 + 25 \cdot s + 37}{s^2 \cdot (s^2 + 12 \cdot s + 37)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C \cdot s + D}{s^2 + 12 \cdot s + 37} = \\
 &= \frac{A \cdot s \cdot (s^2 + 12 \cdot s + 37) + B \cdot (s^2 + 12 \cdot s + 37) + (C \cdot s + D) \cdot s^2}{s^2 \cdot (s^2 + 12 \cdot s + 37)} = \\
 &= \frac{(A+C) \cdot s^3 + (12A+B+D) \cdot s^2 + (37A+12B) \cdot s + 37B}{s^2 \cdot (s^2 + 12 \cdot s + 37)} = \frac{1}{s} + \frac{1}{s^2} - \frac{s+6}{(s+6)^2+1} \\
 &\quad \left[\begin{array}{lcl} \text{I.} & A+C & = 0 \\ \text{II.} & 12A+B+D & = 5 \\ \text{III.} & 37A+12B & = 25 \\ \text{IV.} & 37B & = 37 \end{array} \Rightarrow \begin{array}{l} A=1 \\ B=1 \\ C=-1 \\ D=-6 \end{array} \right]
 \end{aligned}$$

Details of the partial fraction decomposition for $Y(s)$:

$$\begin{aligned}
 \frac{-47 \cdot s - 259}{s^2 \cdot (s^2 + 12 \cdot s + 37)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C \cdot s + D}{s^2 + 12 \cdot s + 37} = \\
 &= \frac{A \cdot s \cdot (s^2 + 12 \cdot s + 37) + B \cdot (s^2 + 12 \cdot s + 37) + (C \cdot s + D) \cdot s^2}{s^2 \cdot (s^2 + 12 \cdot s + 37)} = \\
 &= \frac{(A+C) \cdot s^3 + (12A+B+D) \cdot s^2 + (37A+12B) \cdot s + 37B}{s^2 \cdot (s^2 + 12 \cdot s + 37)} = \frac{1}{s} - \frac{7}{s^2} - \frac{s+5}{(s+6)^2+1} \\
 &\quad \left[\begin{array}{lcl} \text{I.} & A+C & = 0 \\ \text{II.} & 12A+B+D & = 0 \\ \text{III.} & 37A+12B & = -47 \\ \text{IV.} & 37B & = -259 \end{array} \Rightarrow \begin{array}{l} A=1 \\ B=-7 \\ C=1 \\ D=5 \end{array} \right]
 \end{aligned}$$

>>>Exercise 7.1 – 5

Solve the initial value problem

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ -6 & 0 \end{pmatrix} \cdot \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 2 \\ -t \end{pmatrix}, \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

>>>Exercise 7.1 – 5- Solution

$$\begin{aligned}
 \text{I.} \quad \dot{x}(t) &= 3 \cdot x(t) - 3 \cdot y(t) + 2 \\
 \text{II.} \quad \dot{y}(t) &= -6 \cdot x(t) - t
 \end{aligned}$$

Laplace transform of the equations:

$$\begin{aligned}
 \text{I.} \quad s \cdot X(s) - 1 &= 3 \cdot X(s) - 3 \cdot Y(s) + \frac{2}{s} \\
 \text{II.} \quad s \cdot Y(s) + 1 &= -6 \cdot X(s) - \frac{1}{s^2}
 \end{aligned}$$

Solving it for $X(s)$ and $Y(s)$ we have

$$\begin{aligned}
 \text{I.} \quad (s-3) \cdot X(s) + 3 \cdot Y(s) &= \frac{2}{s} + 1 = \frac{2+s}{s} \\
 \text{II.} \quad 6 \cdot X(s) + s \cdot Y(s) &= -\frac{1}{s^2} - 1 = -\frac{s^2+1}{s^2} \\
 X(s) &= \frac{s^3 + 5 \cdot s^2 + 3}{s^2 \cdot (s^2 - 3 \cdot s - 18)} \\
 Y(s) &= \frac{-s^3 - 3 \cdot s^2 - 13 \cdot s + 3}{s^2 \cdot (s^2 - 3 \cdot s - 18)}
 \end{aligned}$$

The partial fraction decomposition of $X(s)$ gives that

$$X(s) = \frac{s^3 + 5 \cdot s^2 + 3}{s^2 \cdot (s^2 - 3 \cdot s - 18)} = \frac{3}{108} \cdot \frac{1}{s} - \frac{18}{108} \cdot \frac{1}{s^2} + \frac{133}{108} \cdot \frac{1}{s-6} - \frac{28}{108} \cdot \frac{1}{s+3}$$

so

$$\begin{aligned} x(t) &= \mathcal{L}^{-1} \left(\frac{3}{108} \cdot \frac{1}{s} - \frac{18}{108} \cdot \frac{1}{s^2} + \frac{133}{108} \cdot \frac{1}{s-6} - \frac{28}{108} \cdot \frac{1}{s+3} \right) = \\ &= \frac{3}{108} \cdot \mathcal{L}^{-1} \left(\frac{1}{s} \right) - \frac{18}{108} \cdot \mathcal{L}^{-1} \left(\frac{1}{s^2} \right) + \frac{133}{108} \cdot \mathcal{L}^{-1} \left(\frac{1}{s-6} \right) - \frac{28}{108} \cdot \mathcal{L}^{-1} \left(\frac{1}{s+3} \right) = \\ &= \frac{3}{108} - \frac{18}{108} \cdot t + \frac{133}{108} \cdot e^{6t} - \frac{28}{108} \cdot e^{-3t} \end{aligned}$$

The partial fraction decomposition of $Y(s)$ gives that

$$Y(s) = \frac{-s^3 - 3 \cdot s^2 - 13 \cdot s + 3}{s^2 \cdot (s^2 - 3 \cdot s - 18)} = \frac{81}{108} \cdot \frac{1}{s} - \frac{18}{108} \cdot \frac{1}{s^2} - \frac{133}{108} \cdot \frac{1}{s-6} - \frac{56}{108} \cdot \frac{1}{s+3}$$

so

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left(\frac{81}{108} \cdot \frac{1}{s} - \frac{18}{108} \cdot \frac{1}{s^2} - \frac{133}{108} \cdot \frac{1}{s-6} - \frac{56}{108} \cdot \frac{1}{s+3} \right) = \\ &= \frac{81}{108} \cdot \mathcal{L}^{-1} \left(\frac{1}{s} \right) - \frac{18}{108} \cdot \mathcal{L}^{-1} \left(\frac{1}{s^2} \right) - \frac{133}{108} \cdot \mathcal{L}^{-1} \left(\frac{1}{s-6} \right) - \frac{56}{108} \cdot \mathcal{L}^{-1} \left(\frac{1}{s+3} \right) = \\ &= \frac{81}{108} - \frac{18}{108} \cdot t - \frac{133}{108} \cdot e^{6t} - \frac{56}{108} \cdot e^{-3t} \end{aligned}$$

Details of the partial fraction decomposition for $X(s)$:

$$\begin{aligned} \frac{s^3 + 5 \cdot s^2 + 3}{s^2 \cdot (s^2 - 3 \cdot s - 18)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-6} + \frac{D}{s+3} = \\ &= \frac{A \cdot s \cdot (s^2 - 3 \cdot s - 18) + B \cdot (s-6) \cdot (s+3) + C \cdot s^2 \cdot (s+3) + D \cdot s^2 \cdot (s-6)}{s^2 \cdot (s^2 - 3 \cdot s - 18)} = \\ &= \frac{(A+C+D) \cdot s^3 + (-3A+B+3C-6D) \cdot s^2 + (-18A-3B) \cdot s + (-18B)}{s^2 \cdot (s^2 - 3 \cdot s - 18)} = \\ &= \frac{3}{108} \cdot \frac{1}{s} - \frac{18}{108} \cdot \frac{1}{s^2} + \frac{133}{108} \cdot \frac{1}{s-6} - \frac{28}{108} \cdot \frac{1}{s+3} \\ \left[\begin{array}{lll} \text{I.} & A+C+D & = 1 \\ \text{II.} & -3A+B+3C-6D & = 5 \\ \text{III.} & -18A-3B & = 0 \\ \text{IV.} & -18B & = 3 \end{array} \Rightarrow \begin{array}{l} A = 3/108 \\ B = -18/108 \\ C = 133/108 \\ D = -28/108 \end{array} \right] \end{aligned}$$

Details of the partial fraction decomposition for $Y(s)$:

$$\begin{aligned} \frac{-s^3 - 3 \cdot s^2 - 13 \cdot s + 3}{s^2 \cdot (s^2 - 3 \cdot s - 18)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-6} + \frac{D}{s+3} = \\ &= \frac{A \cdot s \cdot (s^2 - 3 \cdot s - 18) + B \cdot (s-6) \cdot (s+3) + C \cdot s^2 \cdot (s+3) + D \cdot s^2 \cdot (s-6)}{s^2 \cdot (s^2 - 3 \cdot s - 18)} = \\ &= \frac{(A+C+D) \cdot s^3 + (-3A+B+3C-6D) \cdot s^2 + (-18A-3B) \cdot s + (-18B)}{s^2 \cdot (s^2 - 3 \cdot s - 18)} = \\ &= \frac{81}{108} \cdot \frac{1}{s} - \frac{18}{108} \cdot \frac{1}{s^2} - \frac{133}{108} \cdot \frac{1}{s-6} - \frac{56}{108} \cdot \frac{1}{s+3} \\ \left[\begin{array}{lll} \text{I.} & A+C+D & = -1 \\ \text{II.} & -3A+B+3C-6D & = -3 \\ \text{III.} & -18A-3B & = -13 \\ \text{IV.} & -18B & = 3 \end{array} \Rightarrow \begin{array}{l} A = 81/108 \\ B = -18/108 \\ C = -133/108 \\ D = -56/108 \end{array} \right] \end{aligned}$$

>>>Second solution

When

$$x(t) = \frac{3}{108} - \frac{18}{108} \cdot t + \frac{133}{108} \cdot e^{6t} - \frac{28}{108} \cdot e^{-3t}$$

is obtained we can continue the calculations as follows.

From the second equation we have

$$\dot{y}(t) = -6 \cdot x(t) - t = \frac{-18}{108} - \frac{798}{108} \cdot e^{6 \cdot t} + \frac{168}{108} \cdot e^{-3 \cdot t}$$

$$y(t) = \int \frac{-18}{108} - \frac{798}{108} \cdot e^{6 \cdot t} + \frac{168}{108} \cdot e^{-3 \cdot t} dt = \frac{-18}{108} \cdot t - \frac{133}{108} \cdot e^{6 \cdot t} - \frac{56}{108} \cdot e^{-3 \cdot t} + c$$

Using the initial condition $y(0) = -1$ we get $-1 = -\frac{133}{108} - \frac{56}{108} + c$, that is, $c = \frac{81}{108}$ and

$$y(t) = \frac{81}{108} - \frac{18}{108} \cdot t - \frac{133}{108} \cdot e^{6 \cdot t} - \frac{56}{108} \cdot e^{-3 \cdot t}$$

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8 Appendix 1 – Complex Numbers, the Complex Exponential Function

8.1 Complex Numbers

Although the physical and geometrical parameters of systems are real numbers, in their mathematical models we often need to calculate with complex numbers.

A complex number is pair of reals (a, b) , where a is the real part and b is the imaginary part. $i = (0,1)$ is the *imaginary unit*, it is generally denoted by j in the technical literature.

The set of complex numbers is denoted by $\mathbb{C}$.

Complex numbers of the form $(a, 0)$ are essentially real numbers. Instead of $(a, 0)$ we simply write a , for example we write 0 when referring $(0,0)$ the zero element of $\mathbb{C}$.

Addition and multiplication in the set of complex numbers are defined by

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d) \\ (a, b) \cdot (c, d) &= (a \cdot c - b \cdot d, a \cdot d + b \cdot c).\end{aligned}$$

These definitions imply that $(a, b) = (a, 0) + (b, 0) \cdot (0, 1) = a + b \cdot i$.

$$z = a + b \cdot i$$

is called the *algebraic form* of the complex number z .

$$|z| = r = \sqrt{a^2 + b^2}$$

is the *magnitude* (or norm) of the complex number $z = a + b \cdot i$.

The basic properties of the magnitude are

$$|z| \geq 0, \text{ for all } z \in \mathbb{C}, |z| = 0 \Leftrightarrow z = 0, \text{ for all } z \in \mathbb{C}$$

$$|\alpha \cdot z| = |\alpha| \cdot |z|, \text{ for all } \alpha \in \mathbb{R}, z \in \mathbb{C}$$

$$|z + w| \leq |z| + |w| \text{ for all } z, w \in \mathbb{C} \text{ (triangle inequality)}$$

The *argument* of the complex number $z = a + b \cdot i$ is

$$\begin{aligned}\varphi & \quad \text{if } a > 0 \text{ and } \operatorname{tg} \varphi = \frac{b}{a} \\ \varphi + \pi & \quad \text{if } a < 0 \text{ and } \operatorname{tg} \varphi = \frac{b}{a} \\ \pi/2 & \quad \text{if } a = 0 \text{ and } b > 0 \\ -\pi/2 & \quad \text{if } a = 0 \text{ and } b < 0\end{aligned}$$

$z^* = a - b \cdot i$ is the *complex conjugate* of $z = a + b \cdot i$.

Remark: z^* is also denoted by $\bar{z}$ in the literature.

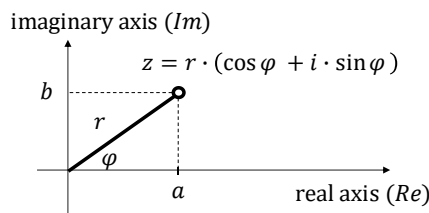
Basic relations between the parameters are

$$a = r \cdot \cos \varphi, \quad b = r \cdot \sin \varphi$$

The formula

$$z = a + b \cdot i = r \cdot (\cos \varphi + i \cdot \sin \varphi)$$

is called the *trigonometric form* of complex number z .



The following formulas show multiplication and division of complex numbers, and the calculation of roots using the trigonometric form.

Let $z = r \cdot (\cos \varphi + i \cdot \sin \varphi)$, $z_1 = r_1 \cdot (\cos \varphi_1 + i \cdot \sin \varphi_1)$, $z_2 = r_2 \cdot (\cos \varphi_2 + i \cdot \sin \varphi_2)$.
Then

$$z_1 \cdot z_2 = r_1 \cdot r_2 \cdot (\cos(\varphi_1 + \varphi_2) + i \cdot \sin(\varphi_1 + \varphi_2))$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot (\cos(\varphi_1 - \varphi_2) + i \cdot \sin(\varphi_1 - \varphi_2))$$

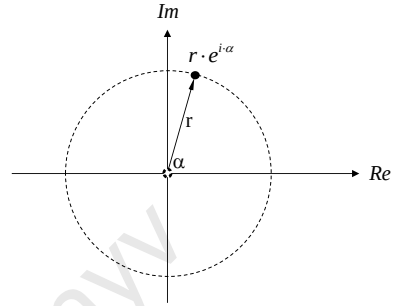
$$\sqrt[n]{z} = \sqrt[n]{r} \cdot \left(\cos\left(\frac{\varphi}{n} + k \cdot \frac{2\pi}{n}\right) + i \cdot \sin\left(\frac{\varphi}{n} + k \cdot \frac{2\pi}{n}\right) \right), \quad n \in \mathbb{N}, k = 0, 1, \dots, n-1$$

Using the so-called Euler's formula

$$e^{i\varphi} = \cos \varphi + i \cdot \sin \varphi$$

we get the *exponential form* of complex numbers

$$z = r \cdot (\cos \varphi + i \cdot \sin \varphi) = r \cdot e^{i\varphi}.$$



The following formulas show multiplication and division of complex numbers, and the calculation of roots using the exponential form.

Let $z = r \cdot e^{i\varphi}$, $z_1 = r_1 \cdot e^{i\varphi_1}$, $z_2 = r_2 \cdot e^{i\varphi_2}$. Then

$$z_1 \cdot z_2 = r_1 \cdot r_2 \cdot e^{i(\varphi_1 + \varphi_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot e^{i(\varphi_1 - \varphi_2)}$$

$$\sqrt[n]{z} = \sqrt[n]{r} \cdot e^{i\left(\frac{\varphi}{n} + k \cdot \frac{2\pi}{n}\right)}, \quad n \in \mathbb{N}, k = 0, 1, \dots, n-1$$

>>>Exercise 8.1 – 1

Represent the following sets of complex numbers on the complex plane.

$$A_1 = \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 2\}$$

$$A_8 = \{z \in \mathbb{C} \mid |z - 1| \leq |z + 4|\}$$

$$A_2 = \{z \in \mathbb{C} \mid -2 \leq \operatorname{Im} z \leq 1\}$$

$$A_9 = \left\{z \in \mathbb{C} \mid \frac{3}{2} < |i - 3z - 1| \leq 4\right\}$$

$$A_3 = \{z \in \mathbb{C} \mid \operatorname{Re} z + \operatorname{Im} z = 1\}$$

$$A_{10} = \left\{z \in \mathbb{C} \mid \frac{1}{2} < \operatorname{Re}\left(\frac{1}{z}\right) \leq 1\right\}$$

$$A_4 = \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$$

$$A_5 = \{z \in \mathbb{C} \mid |z - 1 - 3 \cdot i| \leq 2\}$$

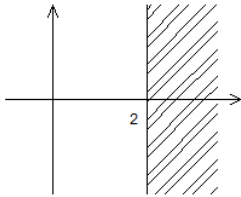
$$A_{10} = \left\{z \in \mathbb{C} \mid \operatorname{Re}\left(\frac{5}{4} - z\right) = \left|z - \frac{3}{4}\right|\right\}$$

$$A_6 = \{z \in \mathbb{C} \mid \operatorname{Im}(z^2) \leq 2\}$$

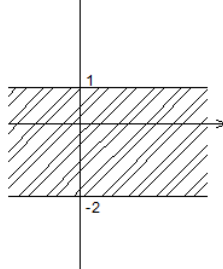
$$A_7 = \{z \in \mathbb{C} \mid \operatorname{Re}(1 + i) \cdot z < 1\}$$

>>>Exercise 8.1 – 1 – Solution

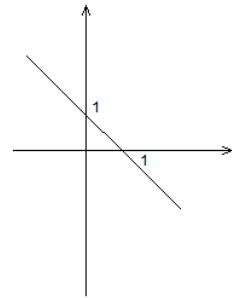
A1



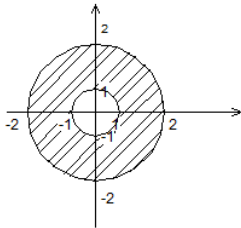
A2



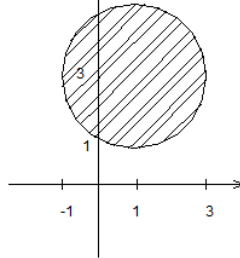
A3



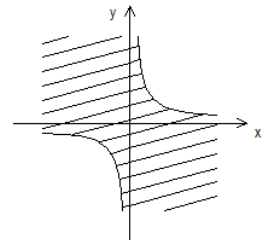
A4



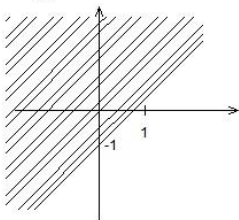
A5



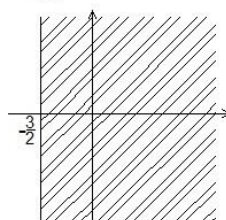
A6



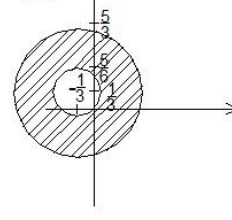
A7



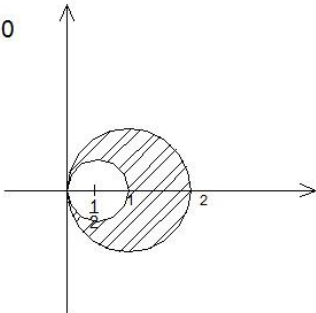
A8



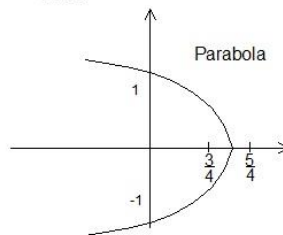
A9



A10



A11



>>>Exercise 8.1 – 2

Give the trigonometric form of the following complex numbers

$$z_1 = 1$$

$$z_5 = 1 + i$$

$$z_9 = 2 + i$$

$$z_2 = -8$$

$$z_6 = -2 + 2i$$

$$z_{10} = -1 - 2i$$

$$z_3 = i$$

$$z_7 = -3 - 3i$$

$$z_{11} = 3 + 4i$$

$$z_4 = -2i$$

$$z_8 = \sqrt{3} - i$$

$$z_{12} = -2 + 5i$$

>>>Exercise 8.1 – 2 – Solution

$$z_1 = \cos 0 + i \cdot \sin 0$$

$$z_3 = \cos \frac{\pi}{2} + i \cdot \sin \frac{\pi}{2}$$

$$z_5 = \sqrt{2} \cdot \left(\cos \frac{\pi}{4} + i \cdot \sin \frac{\pi}{4} \right)$$

$$z_7 = \sqrt{18} \cdot \left(\cos \frac{5\pi}{4} + i \cdot \sin \frac{5\pi}{4} \right)$$

$$z_9 \approx \sqrt{5} \cdot (\cos 0.464 + i \cdot \sin 0.464)$$

$$z_{11} \approx 5 \cdot (\cos 0.927 + i \cdot \sin 0.927)$$

$$z_2 = 8 \cdot (\cos \pi + i \cdot \sin \pi)$$

$$z_4 = 2 \cdot \left(\cos \frac{3\pi}{2} + i \cdot \sin \frac{3\pi}{2} \right)$$

$$z_6 = \sqrt{8} \cdot \left(\cos \frac{3\pi}{4} + i \cdot \sin \frac{3\pi}{4} \right)$$

$$z_8 = 2 \cdot \left(\cos \frac{-\pi}{6} + i \cdot \sin \frac{-\pi}{6} \right)$$

$$z_{10} \approx \sqrt{5} \cdot (\cos 4.249 + i \cdot \sin 4.249)$$

$$z_{12} \approx \sqrt{29} \cdot (\cos 1.951 + i \cdot \sin 1.951)$$

>>>Exercise 8.1 – 3

Give the algebraic form of the following complex numbers

$$z_1 = 5 \cdot \left(\cos \frac{2\pi}{3} + i \cdot \sin \frac{2\pi}{3} \right)$$

$$z_2 = \cos 0 + i \cdot \sin 0$$

$$z_3 = 3 \cdot \left(\cos \frac{7\pi}{6} + i \cdot \sin \frac{7\pi}{6} \right)$$

$$z_4 = 10 \cdot \left(\cos \frac{5\pi}{3} + i \cdot \sin \frac{5\pi}{3} \right)$$

>>>Exercise 8.1 – 3 – Solution

$$z_1 = -\frac{5}{2} + \frac{5\sqrt{3}}{2} \cdot i$$

$$z_2 = 1$$

$$z_3 = -\frac{3\sqrt{3}}{2} - \frac{3}{2} \cdot i$$

$$z_4 = 5 - 5\sqrt{3} \cdot i$$

>>>Exercise 8.1 – 4

Give the exponential form of the following complex numbers

$$z_1 = -4$$

$$z_2 = 2 \cdot i$$

$$z_3 = -\sqrt{2} - \sqrt{2} \cdot i$$

$$z_4 = -i$$

$$z_5 = \sqrt{3} - i$$

>>>Exercise 8.1 – 4 – Solution

$$z_1 = 4 \cdot e^{i\pi}$$

$$z_2 = 2 \cdot e^{i\frac{\pi}{2}}$$

$$z_3 = 2 \cdot e^{i\frac{5\pi}{4}}$$

$$z_4 = e^{i\frac{3\pi}{2}}$$

$$z_5 = 2 \cdot e^{i\left(-\frac{\pi}{6}\right)}$$

>>>Exercise 8.1 – 5

Give the algebraic or trigonometric form of the following expressions

a, i^{18}

b, i^{211}

c, $\frac{(2-i)^2(-1+2 \cdot i)}{3-4i}$

d, $\left(\frac{1+2 \cdot i}{3-i}\right)^4$

e, $\frac{(1+i)^9}{(1-i)^7}$

f, $\left(\frac{4}{\sqrt{3} \cdot i - 1}\right)^{12}$

g, $\sqrt[6]{\frac{3+i}{2-4 \cdot i}}$

h, $\sqrt[5]{i \frac{2-i}{3+4 \cdot i}}$

j, $\sqrt[5]{1+2 \cdot i}$

k, $\sqrt[3]{(1+i)(-2+5 \cdot i)}$

l, $\sqrt[8]{1}$

m, $\sqrt{-64}$

n, $\sqrt[4]{\frac{-4}{(2+i)^3}}$

o, $\frac{(1+2 \cdot i)^2 - (1-i)^2}{(3+2 \cdot i)^3 - (2+i)^2}$

>>>Exercise 8.1 – 5 - Solution

- a, -1
- b, $-i$
- c, $-1 + 2 \cdot i$
- d, $0.2108 - 0.1344 \cdot i$
- e, 2
- f, 4^{12}
- g, $\frac{1}{\sqrt[12]{2}} \cdot \left(\cos \left(0.238 + k \cdot \frac{\pi}{3} \right) + i \cdot \sin \left(0.238 + k \cdot \frac{\pi}{3} \right) \right), k = 0, 1, 2, 3, 4, 5$
- h, $\frac{1}{\sqrt[10]{5}} \cdot \left(\cos \left(0.036 + k \cdot \frac{2\pi}{5} \right) + i \cdot \sin \left(0.036 + k \cdot \frac{2\pi}{5} \right) \right), k = 0, 1, 2, 3, 4$
- j, $\sqrt[10]{5} \cdot \left(\cos \left(0.221 + k \cdot \frac{2\pi}{5} \right) + i \cdot \sin \left(0.221 + k \cdot \frac{2\pi}{5} \right) \right), k = 0, 1, 2, 3, 4$
- k, $\sqrt[6]{58} \cdot \left(\cos \left(0.912 + k \cdot \frac{2\pi}{3} \right) + i \cdot \sin \left(0.912 + k \cdot \frac{2\pi}{3} \right) \right), k = 0, 1, 2$
- l, $\cos \left(k \cdot \frac{\pi}{4} \right) + i \cdot \sin \left(k \cdot \frac{\pi}{4} \right), k = 0, 1, 2, 3, 4, 5, 6, 7$
- m, $16 \cdot \left(\cos \left(-\frac{\pi}{2} + k \cdot \pi \right) + i \cdot \sin \left(-\frac{\pi}{2} + k \cdot \pi \right) \right), k = 0, 1$
- n, $\frac{\sqrt{2}}{\sqrt[8]{125}} \cdot \left(\cos \left(0.438 + k \cdot \frac{\pi}{2} \right) + i \cdot \sin \left(0.438 + k \cdot \frac{\pi}{2} \right) \right), k = 0, 1, 2, 3$
- o, $\frac{16}{106} + \frac{3}{106} \cdot i$

>>>Exercise 8.1 – 6

Find the infimum and supremum of the following sets if they exist.

$$A_1 = \{ |z| : z \in \mathbb{C}, |z - 1 - i| < 1 \}$$

$$A_2 = \{ z \cdot z^* : z \in \mathbb{C}, |\operatorname{Re} z| \leq 2, |\operatorname{Im} z| \leq 2 \}$$

>>>Exercise 8.1 – 6 - Solution

$$\inf A_1 = 0, \quad \sup A_1 = 2$$

$$\inf A_2 = 0, \quad \sup A_2 = 8$$

>>>Exercise 8.1 – 7

Which of the following sets are bounded?

$$A_1 = \{ z \in \mathbb{C} \mid z \cdot z^* = 9 \}$$

$$A_2 = \{ z \in \mathbb{C} \mid |\operatorname{Re} z - \operatorname{Im} z| < 3 \}$$

$$A_3 = \{ z \in \mathbb{C} \mid \operatorname{Re} z = \operatorname{Im} z \}$$

$$A_4 = \{ z \in \mathbb{C} \mid \operatorname{Re} z = z \}$$

$$A_5 = \{ z \in \mathbb{C} \mid z^2 = z \}$$

$$A_6 = \{ z \in \mathbb{C} \mid |\operatorname{Re} z| + |\operatorname{Im} z| = 2 \}$$

>>>Exercise 8.1 – 7 - Solution

	A_1	A_2	A_3	A_4	A_5	A_6
bounded	yes	no	no	no	yes	yes

8.2 Complex Exponential Function

The complex exponential function is defined as

$$\exp(z) = e^z = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot z^k, \quad z \in \mathbb{C}.$$

The definition of complex sine and cosine functions are

$$\sin z = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{(2k+1)!} \cdot z^{2k+1}, \quad z \in \mathbb{C}$$

$$\cos z = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{(2k)!} \cdot z^{2k}, \quad z \in \mathbb{C}$$

Remark: The definitions of the related real functions are

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot x^k, \quad \sin x = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{(2k+1)!} \cdot x^{2k+1}, \quad \cos x = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{(2k)!} \cdot x^{2k}, \quad x \in \mathbb{R}.$$

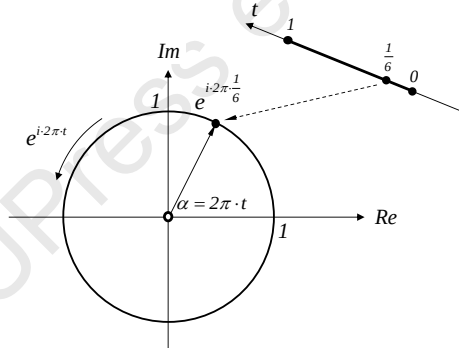
The definitions imply the Euler's formula

$$e^{i \cdot \varphi} = \cos \varphi + i \cdot \sin \varphi, \quad \varphi \in \mathbb{R}.$$

Since

$$|e^{i \cdot \varphi}| = |\cos \varphi + i \cdot \sin \varphi| = \sqrt{\cos^2 \varphi + \sin^2 \varphi} = 1, \quad \varphi \in \mathbb{R}$$

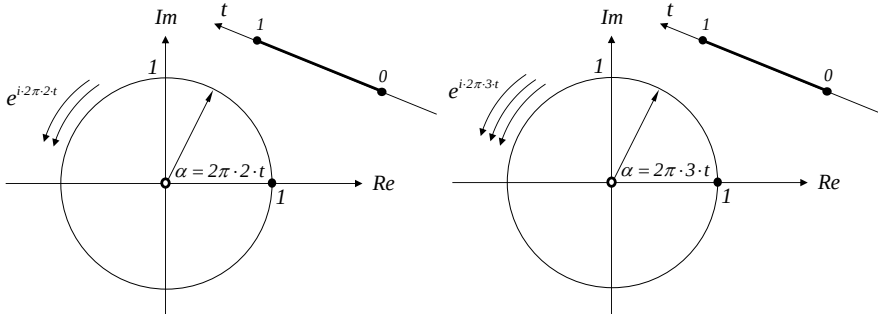
value $e^{i \cdot \varphi}$ is on the complex unit circle for any φ . More precisely, the range of function $t \rightarrow e^{i \cdot 2\pi \cdot t}, t \in [0,1]$ is the complex unit circle.



Since sine and cosine functions are 2π -periodic, function $t \rightarrow e^{i \cdot 2\pi \cdot t}, t \in \mathbb{R}$ is 1-periodic.

Remark: Function $t \rightarrow R \cdot e^{i \cdot (\omega \cdot t + \varphi_0)} = R \cdot e^{i \cdot (2\pi \cdot f \cdot t + \varphi_0)}$ is a model of uniform circular motion, where R is the radius of the circle, ω is the angular frequency, f is the frequency, φ_0 is the initial phase.

The following figures show functions $t \rightarrow e^{i \cdot 2\pi \cdot 2 \cdot t}, t \in [0,1]$ and $t \rightarrow e^{i \cdot 2\pi \cdot 3 \cdot t}, t \in [0,1]$



Remark: The formula of the discrete Fourier transform contains the values of type $e^{i2\pi k \frac{n}{N}}$, $k=0,1,\dots,N-1$

>>>Exercise 8.2 – 1

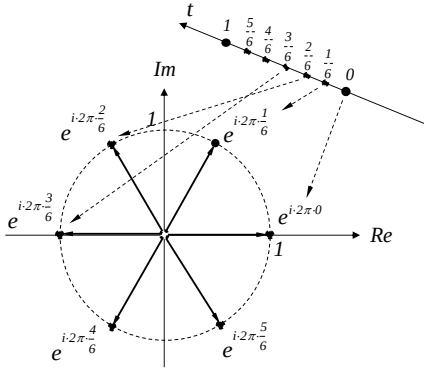
Plot values

$$e^{i2\pi k \frac{n}{6}}, \quad k=1,\dots,3, n=0,\dots,5$$

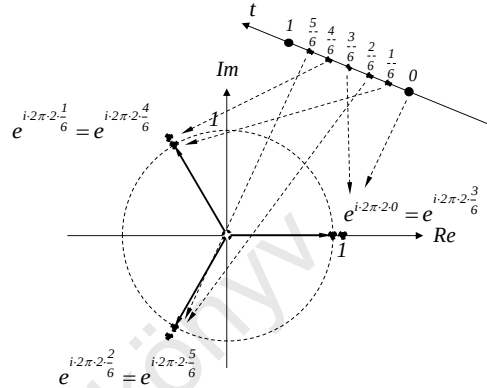
in the complex plane.

>>>Exercise 8.2 – 1 – Solution

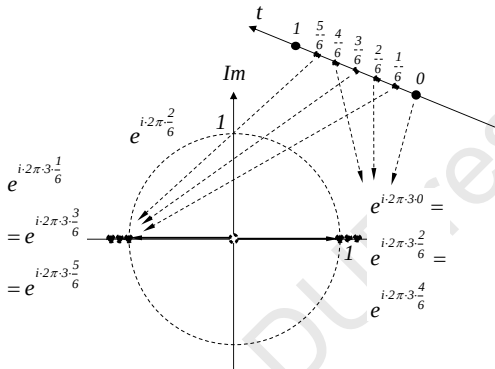
Case $k=1, n=0,\dots,5$



Case $k=2, n=0,\dots,5$



Case $k=3, n=0,\dots,5$



>>>Exercise 8.2 – 2

Show that

$$\sum_{n=0}^5 e^{i2\pi \frac{n}{6}} = 0$$

>>>Exercise 8.2 – 2 – Solution

$$\sum_{n=0}^5 e^{i2\pi \frac{n}{6}} = e^0 + e^{i\frac{2\pi}{6}} + e^{i\frac{4\pi}{6}} + e^{i\frac{6\pi}{6}} + e^{i\frac{8\pi}{6}} + e^{i\frac{10\pi}{6}} =$$

$$\begin{aligned} &= e^0 + e^{i\frac{\pi}{3}} + e^{i\frac{2\pi}{3}} + e^{i\pi} + e^{i\frac{4\pi}{3}} + e^{i\frac{5\pi}{3}} = \\ &= 1 + \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) + \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right) + \\ &+ (\cos \pi + i \sin \pi) + \left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right) + \left(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}\right) = \end{aligned}$$

$$= 1 + \frac{1}{2} + i \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} + i \cdot \frac{\sqrt{3}}{2} - 1 - \frac{1}{2} - i \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} - i \cdot \frac{\sqrt{3}}{2} = 0$$

Remark: It can be proved that for fixed $N \in \mathbb{N}$ and $k \in \mathbb{N}, 0 \leq k < N$ we have

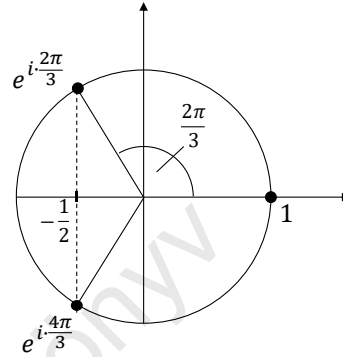
$$\sum_{n=0}^{N-1} e^{i \cdot 2\pi \cdot k \cdot \frac{n}{N}} = 0$$

>>>Exercise 8.2 – 3

Plot values $e^{i \cdot 2\pi \cdot k \cdot \frac{n}{3}}$ for $k = 1$ and $n = 0, 1, 2$ in the complex plane.

>>>Exercise 8.2 – 3 – Solution

$$e^0 = 1, \quad e^{i \cdot \frac{2\pi}{3}}, \quad e^{i \cdot \frac{4\pi}{3}}$$



8.3 Calculations with Phasors

8.3.1 Phasors

In some applications when a real function q versus ω is given

$$q(\omega) = q_0 \cdot \sin \omega,$$

it is sometimes expedient to introduce the complex-valued function, the so-called *phasor*,

$$Q(\omega) = q_0 \cdot e^{i \cdot \omega}$$

for further calculations, and to consider its imaginary part

$$\text{Im } Q(\omega) = q(\omega) = q_0 \cdot \sin \omega.$$

when the (real) value of quantity q is required.

It is clear that for the *magnitude* and the *phase* of q we have

$$q_0 = |Q(\omega)| \quad \text{and} \quad \omega = \arg Q(\omega) = \angle Q(\omega)$$

respectively, and we can write

$$q(\omega) = |Q(\omega)| \cdot \sin(\angle Q(\omega)),$$

A phasor Q can be written in the form

$$Q = |Q| \angle Q.$$

Calculations with phasors are based on the properties of the exponential function:

$$|Q_1 \cdot Q_2| = |Q_1| \cdot |Q_2|$$

$$\angle(Q_1 \cdot Q_2) = \angle Q_1 + \angle Q_2$$

$$\frac{|Q_1|}{|Q_2|} = \frac{|Q_1|}{|Q_2|}$$

$$\angle\left(\frac{Q_1}{Q_2}\right) = \angle Q_1 - \angle Q_2$$

$$|Q^n| = |Q|^n$$

$$\angle(Q^n) = n \cdot \angle Q$$

Calculations with phases are easy, but calculations with magnitudes requires multiplications and divisions. In practice, for instance when drawing an approximate Bode diagram, the magnitude is given in decibel scale as

$$|Q|_{dB} = 20 \cdot \lg|Q|$$

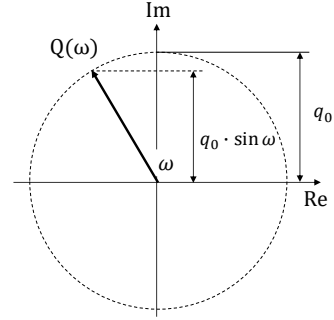
and then

$$|Q_1 \cdot Q_2|_{dB} = |Q_1|_{dB} + |Q_2|_{dB}$$

$$\left| \frac{Q_1}{Q_2} \right|_{dB} = |Q_1|_{dB} - |Q_2|_{dB}$$

$$|Q^n|_{dB} = n \cdot |Q|_{dB}$$

Because of the constant value of the magnitude, all values of $Q(\omega)$ are on a circle of radius q_0 with the centre in the origin in the complex plane. Varying ω the vector of $Q(\omega)$ is 'rotating' at 'angular velocity' ω .



In the special case

$$q(t) = q_0 \cdot \sin(\omega_0 \cdot t + \varphi),$$

where q_0 , ω_0 , and φ are independent from time t , the complex quantity is

$$Q(t) = q_0 \cdot e^{i \cdot (\omega_0 \cdot t + \varphi)}$$

and

$$\text{Im } Q(t) = q(t) = q_0 \cdot \sin(\omega_0 \cdot t + \varphi).$$

Remark: As it was mentioned above, function $t \rightarrow q_0 \cdot e^{i \cdot (\omega_0 \cdot t + \varphi)}$ a model of uniform circular motion of angular velocity ω_0 , so a phasor can be interpreted as a vector of length q_0 rotating with angular velocity ω_0 .

8.3.2 Alternating Current Circuits

Calculations with complex quantities in series AC circuits

Let the voltage-time and the current-time functions of circuit elements be

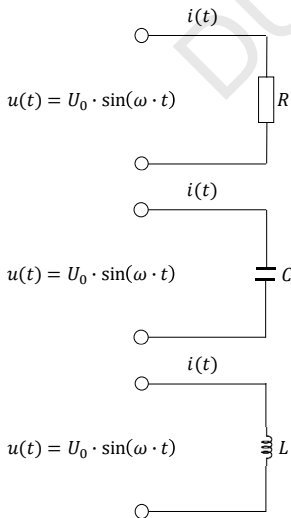
$$u(t) = U_0 \cdot \sin(\omega \cdot t + \varphi_U)$$

and

$$i(t) = I_0 \cdot \sin(\omega \cdot t + \varphi_I)$$

respectively.

Values of I_0 and φ_I in some cases are as follows:



$$i(t) = \frac{U_0}{R} \cdot \sin(\omega \cdot t + \varphi_U)$$

$$I_0 = \frac{U_0}{R}$$

$$\varphi_I = \varphi_U \Rightarrow \varphi = \varphi_U - \varphi_I = 0$$

$$i(t) = U_0 \cdot \omega \cdot C \cdot \sin\left(\omega \cdot t + \varphi_U + \frac{\pi}{2}\right)$$

$$I_0 = U_0 \cdot \omega \cdot C$$

$$\varphi_I = \varphi_U + \frac{\pi}{2} \Rightarrow \varphi = \varphi_U - \varphi_I = -\frac{\pi}{2}$$

$$i(t) = \frac{U_0}{\omega \cdot L} \cdot \sin\left(\omega \cdot t + \varphi_U - \frac{\pi}{2}\right)$$

$$I_0 = \frac{U_0}{\omega \cdot L}$$

$$\varphi_I = \varphi_U - \frac{\pi}{2} \Rightarrow \varphi = \varphi_U - \varphi_I = \frac{\pi}{2}$$

The complex form of the voltage-time and the current-time functions

In this part the complex imaginary unit is denoted by j , and the complex quantities are denoted by $\sim$.

The complex form of voltage-time function $u(t) = U_0 \cdot \sin(\omega \cdot t + \varphi_U)$ is

$$\tilde{u}(t) = U_0 \cdot (\cos(\omega \cdot t + \varphi_U) + j \cdot \sin(\omega \cdot t + \varphi_U)) = U_0 \cdot e^{j \cdot (\omega \cdot t + \varphi_U)}$$

It can be seen that $u = \text{Im } \tilde{u}$.

The following manipulation shows the role of the constant initial phase φ_U and the time dependent phase $\omega \cdot t$

$$\tilde{u}(t) = U_0 \cdot e^{j \cdot (\omega \cdot t + \varphi_U)} = U_0 \cdot e^{j \cdot \varphi_U} \cdot e^{j \cdot \omega \cdot t} = \tilde{U}_0 \cdot e^{j \cdot \omega \cdot t}$$

The complex number $\tilde{U}_0 = U_0 \cdot e^{j \cdot \varphi_U}$ is called the complex amplitude of the voltage-time function. Its magnitude is equal to the (real) maximum voltage $|\tilde{U}_0| = U_0$, while its argument is equal to the initial phase of the voltage $\angle \tilde{U}_0 = \varphi_U$.

The complex form of current-time function $i(t) = I_0 \cdot \sin(\omega \cdot t + \varphi_I)$ is

$$\tilde{i}(t) = I_0 \cdot (\cos(\omega \cdot t + \varphi_I) + j \cdot \sin(\omega \cdot t + \varphi_I)) = I_0 \cdot e^{j \cdot (\omega \cdot t + \varphi_I)}$$

It can be seen that $i = \text{Im } \tilde{i}$.

For the current-time function we have

$$\tilde{i}(t) = I_0 \cdot e^{j \cdot (\omega \cdot t + \varphi_I)} = I_0 \cdot e^{j \cdot \varphi_I} \cdot e^{j \cdot \omega \cdot t} = \tilde{I}_0 \cdot e^{j \cdot \omega \cdot t}$$

The complex number $\tilde{I}_0 = I_0 \cdot e^{j \cdot \varphi_I}$ is called the complex amplitude of the current-time function. Its magnitude is equal to the (real) maximum current $|\tilde{I}_0| = I_0$, while its argument is equal to the initial phase of the current $\angle \tilde{I}_0 = \varphi_I$.

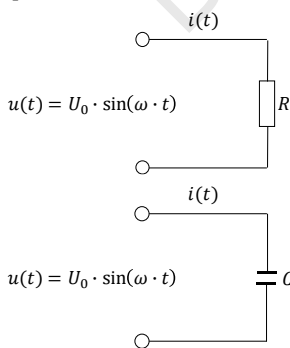
In the case of harmonic excitation, the complex values of voltage and current can be represented as vectors rotating at angular frequency ω . The angle of the two vectors is $\varphi = \varphi_I - \varphi_U$.

Impedance

Impedance of an alternating current circuit is the ratio of the complex amplitude of voltage and current:

$$\tilde{Z} = \frac{\tilde{U}_0}{\tilde{I}_0} = \frac{U_0 \cdot e^{j \cdot \varphi_U}}{I_0 \cdot e^{j \cdot \varphi_I}} = \frac{U_0}{I_0} \cdot e^{j \cdot (\varphi_U - \varphi_I)}$$

Impedance of different circuit elements:



$$\tilde{Z}_R = R \cdot e^{-j \cdot 0} = R$$

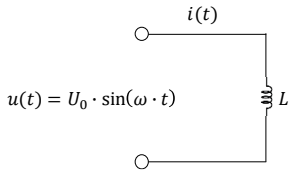
$$\varphi_I - \varphi_U = 0$$

$$|\tilde{Z}| = \frac{U_0}{I_0} = X_R = R$$

$$\tilde{Z}_C = \frac{1}{\omega \cdot C} \cdot e^{-j \cdot \frac{\pi}{2}} = -\frac{1}{\omega \cdot C} \cdot j = -X_C \cdot j$$

$$\varphi_I - \varphi_U = \frac{\pi}{2}$$

$$|\tilde{Z}| = \frac{U_0}{I_0} = X_C = \frac{1}{\omega \cdot C}$$



$$\tilde{Z}_L = \omega \cdot L \cdot e^{-j \cdot \left(-\frac{\pi}{2}\right)} = \omega \cdot L \cdot j = X_L \cdot j$$

$$\varphi_I - \varphi_U = -\frac{\pi}{2}$$

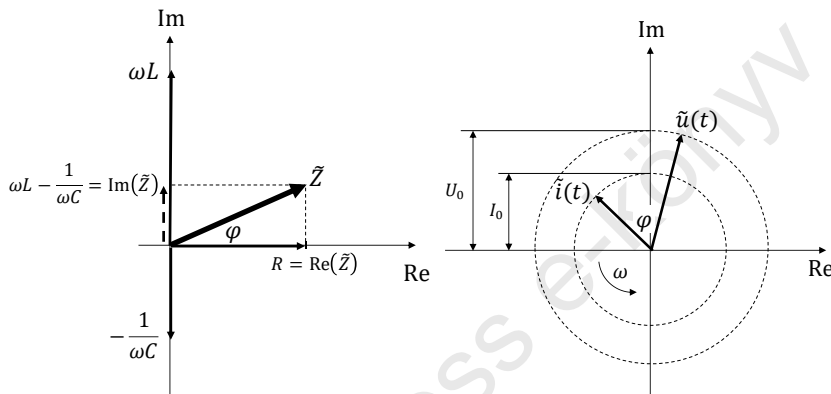
$$|\tilde{Z}| = \frac{U_0}{I_0} = X_L = \omega \cdot L$$

The resultant impedance of series circuit containing elements of impedance $\tilde{Z}_1, \dots, \tilde{Z}_n$ is

$$\tilde{Z}_{\text{res,series}} = \sum_k \tilde{Z}_k.$$

The resultant impedance of parallel circuit containing elements of impedance $\tilde{Z}_1, \dots, \tilde{Z}_n$ is

$$\tilde{Z}_{\text{res,parallel}} = \left(\sum_k \frac{1}{\tilde{Z}_k} \right)^{-1}$$



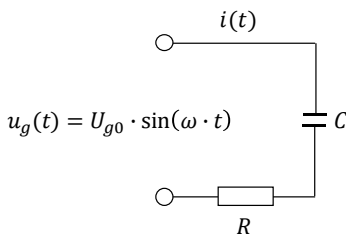
Phase shift of an RLC circuit is

$$\tan \varphi = \frac{X_L - X_C}{X_R} = \frac{\omega \cdot L - \frac{1}{\omega \cdot C}}{R}$$

The generalized form of Ohm's Law and Kirchhoff's Laws are

$$\tilde{U}_0 = \tilde{I}_0 \cdot \tilde{Z}, \quad \sum_k \tilde{I}_{0,k} = 0, \quad \sum_k \tilde{U}_{0,k} = 0$$

Example: Series RC circuit



$$u_g(t) = U_{g0} \cdot \sin(\omega \cdot t)$$

$$\tilde{u}_g(t) = U_{g0} \cdot e^{j \cdot \omega \cdot t}$$

$$(\varphi = 0)$$

$$\tilde{U}_{g0} = U_{g0}$$

The resultant impedance is

$$\tilde{Z}_{RC} = \tilde{Z}_R + \tilde{Z}_C = R - \frac{1}{\omega \cdot C} \cdot j$$

The complex amplitude of current is

$$\tilde{I}_0 = \frac{\tilde{U}_{g0}}{\tilde{Z}_{RC}} = \frac{U_{g0}}{R - \frac{1}{\omega \cdot C} \cdot j}$$

the real amplitude of current is

$$I_0 = |\tilde{I}_0| = \frac{U_{g0}}{\sqrt{R^2 + \left(\frac{1}{\omega \cdot C}\right)^2}}$$

The complex current-time function is

$$\begin{aligned} \tilde{i}(t) &= \tilde{I}_0 \cdot e^{j\omega t} = \frac{U_{g0}}{R - \frac{1}{\omega \cdot C} \cdot j} \cdot e^{j\omega t} = \frac{U_{g0}}{R^2 + \left(\frac{1}{\omega \cdot C}\right)^2} \cdot \left(R + \frac{1}{\omega \cdot C} \cdot j\right) \cdot e^{j\omega t} = \\ &= \frac{U_{g0}}{R^2 + \left(\frac{1}{\omega \cdot C}\right)^2} \cdot \sqrt{R^2 + \left(\frac{1}{\omega \cdot C}\right)^2} \cdot e^{j\varphi} \cdot e^{j\omega t} = \frac{U_{g0}}{\sqrt{R^2 + \left(\frac{1}{\omega \cdot C}\right)^2}} \cdot e^{j(\omega t + \varphi)} = I_0 \cdot e^{j(\omega t + \varphi)}, \end{aligned}$$

where $\tan \varphi = \frac{\frac{1}{\omega \cdot C}}{R} > 0$, so $0 < \varphi < \frac{\pi}{2}$ that is, the current leads the voltage.

The real current-time function is

$$\begin{aligned} i(t) &= \text{Im}(\tilde{i}(t)) = \text{Im}(I_0 \cdot e^{j(\omega t + \varphi)}) = \\ &= \text{Im}(I_0 \cdot (\cos(\omega \cdot t + \varphi) + j \cdot \sin(\omega \cdot t + \varphi))) = I_0 \cdot \sin(\omega \cdot t + \varphi) \end{aligned}$$

The complex voltage amplitudes are

$$\tilde{U}_{R0} = \tilde{I}_0 \cdot \tilde{Z}_R = \frac{U_{g0}}{R - \frac{1}{\omega \cdot C} \cdot j} \cdot R$$

$$\tilde{U}_{C0} = \tilde{I}_0 \cdot \tilde{Z}_C = \frac{U_{g0}}{R - \frac{1}{\omega \cdot C} \cdot j} \cdot \left(-\frac{1}{\omega \cdot C} \cdot j\right)$$

The real voltage amplitudes are

$$U_{R0} = |\tilde{U}_{R0}| = \frac{U_{g0}}{\sqrt{R^2 + \left(\frac{1}{\omega \cdot C}\right)^2}} \cdot R = I_0 \cdot R$$

$$U_{C0} = |\tilde{U}_{C0}| = \frac{U_{g0}}{\sqrt{R^2 + \left(\frac{1}{\omega \cdot C}\right)^2}} \cdot \frac{1}{\omega \cdot C} = I_0 \cdot X_C$$

The complex voltage-time functions of the circuit elements are as follows.

$$\begin{aligned} \tilde{u}_R(t) &= \tilde{U}_{R0} \cdot e^{j\omega t} = \frac{U_{g0}}{R - \frac{1}{\omega \cdot C} \cdot j} \cdot R \cdot e^{j\omega t} = \frac{U_{g0}}{R^2 + \left(\frac{1}{\omega \cdot C}\right)^2} \cdot R \cdot \left(R + \frac{1}{\omega \cdot C} \cdot j\right) \cdot e^{j\omega t} = \\ &= \frac{U_{g0}}{R^2 + \left(\frac{1}{\omega \cdot C}\right)^2} \cdot R \cdot \sqrt{R^2 + \left(\frac{1}{\omega \cdot C}\right)^2} \cdot e^{j\varphi_R} \cdot e^{j\omega t} = \\ &= \frac{U_{g0}}{\sqrt{R^2 + \left(\frac{1}{\omega \cdot C}\right)^2}} \cdot R \cdot e^{j(\omega t + \varphi_R)} = I_0 \cdot R \cdot e^{j(\omega t + \varphi_R)}, \end{aligned}$$

where $\tan \varphi_R = \frac{\frac{1}{\omega \cdot C}}{R}$, $\varphi_R = \arctan \frac{1}{R \cdot \omega \cdot C}$.

$\tilde{u}_R(t) = I_0 \cdot R \cdot e^{j(\omega t + \varphi)}$, where $\varphi = \arctan \frac{1}{R \cdot \omega \cdot C}$.

$$\begin{aligned}
\tilde{u}_C(t) &= \tilde{U}_{C0} \cdot e^{j\omega t} = \frac{U_{g0}}{R - \frac{1}{\omega \cdot C} \cdot j} \cdot \left(-\frac{1}{\omega \cdot C} \cdot j\right) \cdot e^{j\omega t} \\
&= \frac{U_{g0}}{R^2 + \left(\frac{1}{\omega \cdot C}\right)^2} \cdot \left(-\frac{1}{\omega \cdot C} \cdot j\right) \cdot \left(R + \frac{1}{\omega \cdot C} \cdot j\right) \cdot e^{j\omega t} = \\
&= \frac{U_{g0}}{R^2 + \left(\frac{1}{\omega \cdot C}\right)^2} \cdot \frac{1}{\omega \cdot C} \cdot \left(\frac{1}{\omega \cdot C} - j \cdot R\right) \cdot e^{j\omega t} = \\
&= \frac{U_{g0}}{R^2 + \left(\frac{1}{\omega \cdot C}\right)^2} \cdot \frac{1}{\omega \cdot C} \cdot \sqrt{\left(\frac{1}{\omega \cdot C}\right)^2 + R^2} \cdot e^{j\varphi_C} \cdot e^{j\omega t} = \\
&= \frac{U_{g0}}{\sqrt{R^2 + \left(\frac{1}{\omega \cdot C}\right)^2}} \cdot \frac{1}{\omega \cdot C} \cdot e^{j(\omega t + \varphi_C)} = I_0 \cdot \frac{1}{\omega \cdot C} \cdot e^{j(\omega t + \varphi_C)}
\end{aligned}$$

where $\tan \varphi_C = \frac{-R}{\frac{1}{\omega \cdot C}} = \frac{-1}{\tan \varphi} = \tan\left(\varphi - \frac{\pi}{2}\right)$, thus $\varphi_C = \varphi - \frac{\pi}{2} = \varphi_R - \frac{\pi}{2} = \arctan \frac{1}{R \cdot \omega \cdot C} - \frac{\pi}{2}$.

That is the voltage of the capacitor leads to the current by $\frac{\pi}{2}$ (rad):

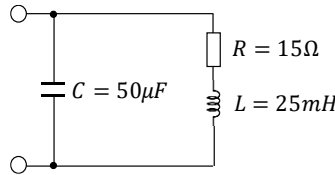
$$\tilde{u}_C(t) = I_0 \cdot R \cdot e^{j(\omega t + \varphi - \frac{\pi}{2})}, \quad \varphi_C = \arctan \frac{1}{R \cdot \omega \cdot C} - \frac{\pi}{2}.$$

The real voltage-time functions of the circuit elements are as follows.

$$\begin{aligned}
u_R(t) &= \text{Im}(\tilde{u}_R(t)) = \text{Im}(I_0 \cdot R \cdot e^{j(\omega t + \varphi)}) = \\
&= \text{Im}(I_0 \cdot R \cdot (\cos(\omega \cdot t + \varphi) + j \cdot \sin(\omega \cdot t + \varphi))) = I_0 \cdot R \cdot \sin(\omega \cdot t + \varphi) \\
u_C(t) &= \text{Im}(\tilde{u}_C(t)) = \text{Im}\left(I_0 \cdot \frac{1}{\omega \cdot C} \cdot e^{j(\omega t + \varphi - \frac{\pi}{2})}\right) = \\
&= \text{Im}\left(I_0 \cdot \frac{1}{\omega \cdot C} \cdot \left(\cos\left(\omega \cdot t + \varphi - \frac{\pi}{2}\right) + j \cdot \sin\left(\omega \cdot t + \varphi - \frac{\pi}{2}\right)\right)\right) = \\
&= I_0 \cdot \frac{1}{\omega \cdot C} \cdot \sin\left(\omega \cdot t + \varphi - \frac{\pi}{2}\right)
\end{aligned}$$

>>>Exercise 8.3 – 1

Calculate the impedance of the circuit in the figure if $\omega = 1000 \left[\frac{\text{rad}}{\text{s}}\right]$.



>>>Exercise 8.3 – 1 – Solution

$$Z_R = 15 [\Omega]$$

$$Z_L = j \cdot \omega \cdot L = j \cdot 1000 \cdot 25 \cdot 10^{-3} = 25 \cdot j [\Omega]$$

$$Z_C = \frac{1}{j \cdot \omega \cdot C} = \frac{1}{j \cdot 1000 \cdot 50 \cdot 10^{-6}} = -20 \cdot j [\Omega]$$

$$Z = \frac{Z_C \cdot (Z_R + Z_L)}{Z_C + (Z_R + Z_L)} = \frac{-20 \cdot j \cdot (15 + 25 \cdot j)}{-20 \cdot j + (15 + 25 \cdot j)} = \frac{500 - 300 \cdot j}{15 + 5 \cdot j} = 24 - 28 \cdot j$$

>>>Exercise 8.3 – 2

Let the complex form of voltage-time function of an element in an AC circuit be

$$\tilde{u}(t) = 42 \cdot e^{j \cdot (150t + 0.4)} [V]$$

Give the complex and the real value of the voltage at $t = 0.02$ [s].

>>>Exercise 8.3 – 2 – Solution

$$\tilde{u}(0.02) = 42 \cdot e^{j \cdot (150 \cdot 0.02 + 0.4)} = 42 \cdot e^{30.4 \cdot j} = (\cos 30.4 + j \cdot \sin 30.4) = (0.53 - 0.85 \cdot j) [V]$$

$$\text{Im } \tilde{u}(0.02) = -0.85 [V]$$

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9 Appendix 2 – Polynomials and Rational Functions

9.1 Polynomials

Polynomials

$$P(z) = a_n \cdot z^n + a_{n-1} \cdot z^{n-1} + \dots + a_2 \cdot z^2 + a_1 \cdot z + a_0$$

where n is a non-negative integer (the *degree of the polynomial*), $a_n \neq 0$ the coefficients a_i and the variable x are complex (or real) numbers plays important role in many fields of mathematics and modelling because of their nice properties.

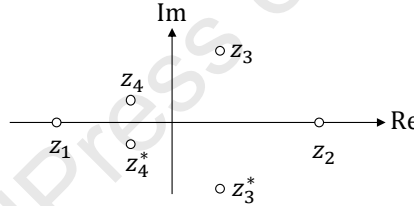
Factorization of polynomials (finding their roots) is a key step in many calculations, for instance in calculation of limits of rational functions or in partial fraction decomposition rational functions when calculating integrals or inverse Laplace transforms.

The *fundamental theorem of algebra* states that every non-zero n -th degree polynomial with complex coefficients has, counted with multiplicity, exactly n complex roots.

If the roots of the polynomial are $z_1, \dots, z_n \in \mathbb{C}$, then its *factorization* is

$$P(z) = a_n \cdot (z - z_1) \cdot (z - z_2) \dots (z - z_n).$$

In the majority of technical applications the coefficients of polynomials are real numbers. In this special case we have that if $z \in \mathbb{C}$ is a root then its complex conjugate z^* is a root as well. Consequently, the roots of a polynomial with real coefficients are real numbers (complex numbers with zero imaginary parts) or pairs complex conjugate numbers (with non-zero imaginary part).



If the coefficients are real numbers and also the variable x is considered to be real, the factorization of the polynomial is

$$P(x) = a_n \cdot Q(x) \cdot R(x)$$

where

$$Q(x) = 1 \text{ or } Q(x) = (x - x_1) \dots (x - x_k),$$

$$R(x) = 1 \text{ or } R(x) = (x^2 + p_1 \cdot x + q_1) \dots (x^2 + p_m \cdot x + q_m),$$

k and m are positive integers, $k + 2m = n$, furthermore the 2nd degree polynomials

$(x^2 + p_i \cdot x + q_i)$, $i = 1, \dots, m$ has no real roots (that is $p_i^2 - 4q_i < 0$, $i = 1, \dots, m$).

Remark: Terms of type $x^2 + p \cdot x + q$ are equal to $(x - z) \cdot (x - z^*)$ for some complex root $z = a + b \cdot i$ of the polynomial:

$$(x - (a + b \cdot i)) \cdot (x - (a - b \cdot i)) = x^2 - 2 \cdot a \cdot x + a^2 + b^2.$$

Here $p = -2 \cdot a$ and $q = a^2 + b^2$ so it is clear that $p^2 - 4 \cdot q = -4 \cdot b^2 < 0$.

It is well-known that the roots of a 2nd degree polynomial $a \cdot z^2 + b \cdot z + c$ are

$$z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4 \cdot a \cdot c}}{2a}$$

In the case of 2nd degree polynomials with real coefficients, depending on the sign of the discriminant $D = b^2 - 4 \cdot a \cdot c$

- there are two different real roots when $D > 0$, or
- there is one real root with multiplicity two when $D = 0$, or
- there is a complex conjugate pair of roots when $D < 0$.

Formulas for the roots of 3rd and 4th degree polynomials are available. These formulas are rather complicated, so we prefer to find some roots using the special form of the polynomial if it is possible.

A general formula for the roots of polynomials of degree higher than four is not available. According to the *Cardano's rule* a root of a 3rd degree polynomial can be given by the following procedure.

It is known that any 3rd degree (cubic) polynomial equation

$$x^3 + a \cdot x^2 + b \cdot x + c = 0$$

with the substitution

$$x = y - \frac{a}{3}$$

can be reduced to the form

$$y^3 + p \cdot y + q = 0,$$

where

$$p = b - \frac{a^2}{3}, \quad q = c + \frac{2}{27} \cdot a^3 - \frac{1}{3} \cdot a \cdot b.$$

Roots of this equation can be given in the form

$$y = A + B,$$

where

$$A = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, \quad B = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

and

$$A \cdot B = -\frac{p}{3}.$$

$\Delta = \frac{q^2}{4} + \frac{p^3}{27}$ is called the *discriminant* of equation $y^3 + p \cdot y + q = 0$.

If $\Delta > 0$, then the cubic equation has one real and two complex conjugate roots.

If $\Delta = 0$, then the cubic equation has three real roots, whereby two roots are equal.

If $\Delta < 0$, then the cubic equation has three distinct real roots, whereby two roots are equal.

>>>Exercise 9.1 – 1

Give the zero sets of the following polynomials.

$$P_1(x) = -3x^2 + 6x + 105$$

$$P_2(x) = 4x^2 + 12x + 9$$

$$P_3(x) = x^3 - x^2 - 6x$$

$$P_4(x) = -6x^5 + 36x^4 + 240x^3$$

$$P_5(x) = x^4 - 13x^2 + 36$$

$$P_6(x) = x^6 - 19x^3 - 216$$

$$P_7(x) = -x^5 + 11x^4 + 22x^3 - 128x^2 + 96x, \text{ known roots: } x_1 = 1, x_2 = 2$$

$$P_8(x) = x^4 + x^3 - 65x^2 - 9x + 504, \text{ known roots: } x_1 = 3, x_2 = -3$$

$$P_9(x) = x^4 + 8x^3 + 18x^2 + 16x + 5, \text{ known root: } x_1 = -5$$

$$P_{10}(x) = 3x^4 - 30x^3 - 32x^2 - 10x - 11$$

$$P_{11}(x) = 2x^5 + 2x^4 - 75x^3 + 69x^2 + 37x - 35$$

>>>Exercise 9.1 – 1 - Solution

$$P_1(x) = -3x^2 + 6x + 105$$

roots	$x_1 = -5, \quad x_2 = 7$
positive	$] -\infty, -5[\cup] 7, +\infty[$
negative	$] -5, 7[$
limits	$\lim_{x \rightarrow -\infty} P_1(x) = -\infty, \quad \lim_{x \rightarrow +\infty} P_1(x) = -\infty$

$$P_2(x) = 4x^2 + 12x + 9$$

roots	$x_1 = -3/2$
positive	$\mathbb{R} \setminus \{3/2\}$
negative	-
limits	$\lim_{x \rightarrow -\infty} P_2(x) = +\infty, \quad \lim_{x \rightarrow +\infty} P_2(x) = +\infty$

$$P_3(x) = x^3 - x^2 - 6x$$

roots	$x_1 = -2, \quad x_2 = 0, \quad x_3 = 3,$
positive	$] -2, 0[\cup] 3, +\infty[$
negative	$] -\infty, -2[\cup] 0, 3[$
limits	$\lim_{x \rightarrow -\infty} P_3(x) = -\infty, \quad \lim_{x \rightarrow +\infty} P_3(x) = +\infty$

$$P_4(x) = -6x^5 + 36x^4 + 240x^3$$

roots	$x_1 = -4, \quad x_2 = 0, \quad x_3 = 10,$
positive	$] -\infty, -4[\cup] 0, 10[$
negative	$] -4, 0[\cup] 10, +\infty[$
limits	$\lim_{x \rightarrow -\infty} P_4(x) = +\infty, \quad \lim_{x \rightarrow +\infty} P_4(x) = -\infty$

$$P_5(x) = x^4 - 13x^2 + 36$$

roots	$x_1 = -3, \quad x_2 = -2, \quad x_3 = 0, \quad x_4 = 2, \quad x_5 = 3$
positive	$] -\infty, -3[\cup] -2, 2[\cup] 3, +\infty[$
negative	$] -3, -2[\cup] 2, 3[$
limits	$\lim_{x \rightarrow -\infty} P_5(x) = +\infty, \quad \lim_{x \rightarrow +\infty} P_5(x) = +\infty$

$$P_6(x) = x^6 - 19x^3 - 216$$

roots	$x_1 = -2, \quad x_2 = 3$
positive	$] -\infty, -2[\cup] 3, +\infty[$
negative	$] -2, 3[$
limits	$\lim_{x \rightarrow -\infty} P_6(x) = +\infty, \quad \lim_{x \rightarrow +\infty} P_6(x) = +\infty$

$$P_7(x) = -x^5 + 11x^4 + 22x^3 - 128x^2 + 96x$$

roots	$x_1 = -4, \quad x_2 = 0, \quad x_3 = 1, \quad x_4 = 2, \quad x_5 = 12$
positive	$] -\infty, -4[\cup] 0, 1[\cup] 2, 12[$
negative	$] -4, 0[\cup] 1, 2[\cup] 12, +\infty[$
limits	$\lim_{x \rightarrow -\infty} P_7(x) = -\infty, \quad \lim_{x \rightarrow +\infty} P_7(x) = -\infty$

$$P_8(x) = x^4 + x^3 - 65x^2 - 9x + 504$$

roots	$x_1 = -8, \quad x_2 = -3, \quad x_3 = 3, \quad x_4 = 7,$
positive	$] -\infty, -8[\cup] -3, 3[\cup] 7, +\infty[$

<i>negative</i>	$] -8, -3[\cup] 3, 7[$
<i>limits</i>	$\lim_{x \rightarrow -\infty} P_8(x) = +\infty, \quad \lim_{x \rightarrow +\infty} P_8(x) = +\infty$
$P_9(x) = x^4 + 8x^3 + 18x^2 + 16x + 5$	
<i>roots</i>	$x_1 = -5, \quad x_2 = -1$
<i>positive</i>	$] -\infty, -5[\cup] -1, +\infty[$
<i>negative</i>	$] -5, -1[$
<i>limits</i>	$\lim_{x \rightarrow -\infty} P_9(x) = +\infty, \quad \lim_{x \rightarrow +\infty} P_9(x) = +\infty$
$P_{10}(x) = 3x^4 - 30x^3 - 32x^2 - 10x - 11$	
<i>roots</i>	$x_1 = -1, \quad x_2 = 11$
<i>positive</i>	$] -\infty, -1[\cup] 11, +\infty[$
<i>negative</i>	$] -1, 11[$
<i>limits</i>	$\lim_{x \rightarrow -\infty} P_{10}(x) = +\infty, \quad \lim_{x \rightarrow +\infty} P_{10}(x) = +\infty$
$P_{11}(x) = 2x^5 + 2x^4 - 75x^3 + 69x^2 + 37x - 35$	
<i>roots</i>	$x_1 = -7, \quad x_2 = -\frac{1}{\sqrt{2}}, \quad x_3 = \frac{1}{\sqrt{2}}, \quad x_4 = 1, \quad x_5 = 5$
<i>positive</i>	$] -\infty, -7[\cup] -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}[\cup] 1, 5[$
<i>negative</i>	$] -7, -\frac{1}{\sqrt{2}}[\cup] \frac{1}{\sqrt{2}}, 1[\cup] 5, +\infty[$
<i>limits</i>	$\lim_{x \rightarrow -\infty} P_{11}(x) = -\infty, \quad \lim_{x \rightarrow +\infty} P_{11}(x) = +\infty$

>>>Exercise 9.1 – 2

Find the complex roots of the following equations

- | | |
|--|---|
| a, $z^3 = 1$ | b, $5z^3 + 5z^2 + z + 1 = 0$ |
| c, $(2+i)^3 \cdot z^4 + 4 = 0$ | d, $z^2 - (3+2i) \cdot z + 1 + 3i = 0$ |
| e, $z^3 \cdot (z-3) - 3 + z = 0$ | f, $z^4 + i^3 - i^2 + i = 0$ |
| g, $(z^3 - \sqrt{3} - i) \cdot (z^2 + 16) = 0$ | h, $\sqrt[3]{z-i+1} + i = \frac{1}{2+i}$ |
| j, $z^3 + z^2 + 2z + 2 = 0$ | k, $z^4 - z^2 - 1 = 0$ |
| l, $z^6 - z^3 + 1 = 0$ | m, $z^5 + 1 + i = 0$ |
| n, $z^3 - i^2 + i - 1 = 0$ | o, $z^3 - 8i = 0$ |
| p, $z^2 + z + 1 = 0$ | q, $\sqrt{\frac{1}{z}} = (1+i) \cdot (2-i)$ |

>>>Exercise 9.1 – 2 – Solution

- a, $z_1 = 1, \quad z_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2} \cdot i, \quad z_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2} \cdot i$
- b, $z_1 = -1, \quad z_2 = \frac{1}{\sqrt{5}} \cdot i, \quad z_3 = -\frac{1}{\sqrt{5}} \cdot i$
- c, $z_k = \frac{\sqrt[3]{2}}{\sqrt[3]{125}} \cdot \left(\cos\left(0.438 + k \cdot \frac{\pi}{2}\right) + i \cdot \sin\left(0.438 + k \cdot \frac{\pi}{2}\right) \right), k = 0, 1, 2, 3$
- d, $z_1 = 1 + i, \quad z_2 = 2 + i$
- e, $z_1 = 3, \quad z_2 = 1, \quad z_3 = -\frac{1}{2} + \frac{\sqrt{3}}{2} \cdot i, \quad z_4 = -\frac{1}{2} - \frac{\sqrt{3}}{2} \cdot i$

- f, $z_1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot i, \quad z_2 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \cdot i$
 $z_3 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot i, \quad z_4 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \cdot i$
- g, $z_1 = \sqrt[3]{2} \cdot \left(\cos \frac{\pi}{18} + i \cdot \sin \frac{\pi}{18} \right), \quad z_2 = \sqrt[3]{2} \cdot \left(\cos \frac{13\pi}{18} + i \cdot \sin \frac{13\pi}{18} \right)$
 $z_3 = \sqrt[3]{2} \cdot \left(\cos \frac{25\pi}{18} + i \cdot \sin \frac{25\pi}{18} \right), \quad z_4 = 4i, \quad z_5 = -4i$
- h, $z = \frac{-333}{125} + \frac{269}{125} \cdot i$
- j, $z_1 = \sqrt{2}i, \quad z_2 = -\sqrt{2}i, \quad z_3 = -1$
- k, $z_1 = \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot i, \quad z_2 = \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot i$
 $z_3 = -\frac{\sqrt{3}}{2} + \frac{1}{2} \cdot i, \quad z_4 = -\frac{\sqrt{3}}{2} - \frac{1}{2} \cdot i$
 $z_1 = \cos \frac{\pi}{9} + i \cdot \sin \frac{\pi}{9}, \quad z_2 = \cos \frac{7\pi}{9} + i \cdot \sin \frac{7\pi}{9}$
- l, $z_3 = \cos \frac{13\pi}{9} + i \cdot \sin \frac{13\pi}{9}, \quad z_4 = \cos \frac{-\pi}{9} + i \cdot \sin \frac{-\pi}{9}$
 $z_5 = \cos \frac{5\pi}{9} + i \cdot \sin \frac{5\pi}{9}, \quad z_6 = \cos \frac{11\pi}{9} + i \cdot \sin \frac{11\pi}{9}$
- m, $z_k = \sqrt[10]{2} \cdot \left(\cos \left(\frac{\pi}{2} + k \cdot \frac{2\pi}{5} \right) + i \cdot \sin \left(\frac{\pi}{2} + k \cdot \frac{2\pi}{5} \right) \right), \quad k = 0, 1, 2, 3, 4$
- n, $z_1 = i, \quad z_2 = -\frac{\sqrt{3}}{2} - \frac{1}{2} \cdot i, \quad z_3 = \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot i$
- o, $z_1 = \sqrt{3} + i, \quad z_2 = \sqrt{3} - i, \quad z_3 = -2 \cdot i$
- p, $z_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} \cdot i, \quad z_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} \cdot i$
- q, $z = 0.8 - 0.6 \cdot i$

>>>Exercise 9.1 – 3

Give the factorization of the following polynomials.

$$P(z) = z^3 + 2i^2 + 3$$

$$T(z) = z^4 - 5z^3 + 12z^2$$

$$Q(z) = z^2 + i$$

$$K(z) = z^4 + 1$$

$$R(z) = z^3 + 3i^3 - 5i$$

$$L(z) = z^6 + 64$$

$$S(z) = z^3 + z^2 + z$$

$$M(z) = z^4 + z^2 + 1$$

>>>Exercise 9.1 – 3 – Solution

$$P(z) = (z + 1) \cdot \left(z - \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot i \right) \cdot \left(z - \frac{1}{2} - \frac{\sqrt{3}}{2} \cdot i \right)$$

$$Q(z) = \left(z + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \cdot i \right) \cdot \left(z - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot i \right)$$

$$R(z) = (z + 2i) \cdot (z - \sqrt{3} + i) \cdot (z + \sqrt{3} - i)$$

$$S(z) = z \cdot \left(z + \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot i \right) \cdot \left(z - \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot i \right)$$

$$T(z) = z^2 \cdot \left(z - \frac{5}{2} - \sqrt{6} \cdot i \right) \cdot \left(z - \frac{5}{2} + \sqrt{6} \cdot i \right)$$

$$K(z) = \frac{1}{4} \cdot (1+i) \cdot (1-i) \cdot (-1+i) \cdot (-1-i)$$

$$L(z) = (z-2) \cdot (z+2) \cdot (z-1-\sqrt{3} \cdot i) \cdot (z-1+\sqrt{3} \cdot i) \cdot (z+1-\sqrt{3} \cdot i) \cdot (z+1+\sqrt{3} \cdot i)$$

$$M(z) = \left(z - \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot i\right) \cdot \left(z - \frac{1}{2} - \frac{\sqrt{3}}{2} \cdot i\right) \cdot \left(z + \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot i\right) \cdot \left(z + \frac{1}{2} - \frac{\sqrt{3}}{2} \cdot i\right)$$

>>>Exercise 9.1 – 4

Give the factorization of the following polynomials.

$$P(x) = x^4 + 1$$

$$S(x) = x^3 + x^2 + x$$

$$Q(x) = x^4 - 5x^3 + 12x^2$$

$$T(x) = x^4 + x^2 + 1$$

$$R(x) = x^6 + 64$$

>>>Exercise 9.1 – 4 – Solution

$$P(x) = (x^2 + \sqrt{2}x + 1) \cdot (x^2 - \sqrt{2}x + 1)$$

$$Q(x) = x^2 \cdot (x^2 - 5x + 12)$$

$$R(x) = (x-2) \cdot (x+2) \cdot \left(x^2 + x + \frac{7}{4}\right) \cdot \left(x^2 - x + \frac{7}{4}\right)$$

$$S(x) = x \cdot (x^2 + x + 1)$$

$$T(x) = (x^2 + x + 1) \cdot (x^2 - x + 1)$$

>>>Exercise 9.1 – 5

Solve the equation

$$x^3 + 2 \cdot x^2 + 10 \cdot x - 20 = 0$$

using Cardano's rule.

>>>Exercise 9.1 – 5 – Solution

$$x^3 + 2 \cdot x^2 + 10 \cdot x - 20 = x^3 + a \cdot x^2 + b \cdot x + c = 0$$

Use substitution $x = y - \frac{2}{3}$ to get the reduced equation.

$$p = b - \frac{a^2}{3} = 10 - \frac{4}{3} = \frac{26}{3}, \quad q = -20 + \frac{16}{27} - \frac{20}{3} = -\frac{704}{27},$$

that is, the new equation is

$$y^3 + \frac{26}{3} \cdot y - \frac{704}{27} = 0,$$

The discriminant is

$$\Delta = \frac{q^2}{4} + \frac{p^3}{27} = \frac{1}{4} \cdot \left(-\frac{704}{27}\right)^2 + \frac{1}{27} \cdot \left(\frac{26}{3}\right)^3 \approx 194.07 > 0,$$

our equation has one real and two complex conjugate roots.

$A = \sqrt[3]{26.97}$	$B = \sqrt[3]{-0.89}$	$y = A + B$	$x = y - \frac{2}{3}$
2.998	-0.962	2.036	1.369
$-1.499 + 2.596 \cdot i$	$0.481 + 0.833 \cdot i$	$-1.018 + 3.429 \cdot i$	$-1.684 + 3.429 \cdot i$
$-1.499 - 2.596 \cdot i$	$0.481 - 0.833 \cdot i$	$-1.018 - 3.429 \cdot i$	$-1.684 - 3.429 \cdot i$

9.2 Rational Functions, Partial Fractions

Rational functions (quotients of polynomials)

$$R(x) = \frac{P(x)}{Q(x)}$$

have important role in many mathematical models, e.g. in calculations with transfer functions of linear systems.

Supposing that P and Q have no common terms in their factorisations, the roots of P are the *zeros* of R , while the roots of Q are *poles* of R .

When there is a common factor in the two factorisations with higher or equal multiplicity in P then there is *removable discontinuity* at the related place.

The limits at $-\infty$, $+\infty$ and at the places of discontinuity are important information about a rational function.

The non-identically zero real rational functions

$$\frac{A}{(x - x_0)^n} \quad \text{and} \quad \frac{B \cdot x + C}{(x^2 + p \cdot x + q)^n}$$

where $A, B, C \in \mathbb{R}$, $p^2 - 4q < 0$ are called *partial fractions*. It is known that each real rational function can be written as a sum of a polynomial and partial fractions.

The denominators in the partial fraction decomposition are determined by the factors in the factorization of Q as follows.

factor in the factorization of Q	the related partial fractions in the decomposition
$(x - x_0)^n$	$\frac{A_0}{x - x_0}, \frac{A_1}{(x - x_0)^2}, \dots, \frac{A_n}{(x - x_0)^n}$
$(x^2 + p \cdot x + q)^n$	$\frac{B_0 \cdot x + C_0}{x^2 + p \cdot x + q}, \frac{B_2 \cdot x + C_2}{(x^2 + p \cdot x + q)^2}, \dots, \frac{B_n \cdot x + C_n}{(x^2 + p \cdot x + q)^n}$

>>>Exercise 9.2 – 1

Give the zero set of the following functions. Give the largest subsets of the reals on which the following expression can be defined, and give the limits.

$$f_1(x) = \frac{x^2 + 3x - 10}{x - 3}$$

$$f_2(x) = \frac{x^2 - 2x - 3}{x^2 + 2x + 1}$$

$$f_3(x) = \frac{x^4 - 3x^3 + 2x^2}{x^4 - 3x^3 + 3x^2 - x}$$

$$f_4(x) = \frac{x^2 + x}{x^3 - x^4}$$

$$f_5(x) = \frac{x^4 - 6x^3 + 8x^2}{x^2 - 2x}$$

$$f_6(x) = x \frac{2x^2 - 4x + 3}{x^2 - 2x + 1}$$

$$f_7(x) = \frac{x}{x^2 - 2x + 1}$$

$$f_8(x) = \frac{x^3 + 4x^2 + 4x}{x^5 + 6x^4 + 12x^3 + 8x^2}$$

$$f_9(x) = \frac{3x - 6}{x^2 - 6x + 9}$$

$$f_{10}(x) = \frac{x^3 - 3x^2 + 3x + 1}{1 - x}$$

$$f_{11}(x) = \frac{4x^3 - 2x^2 - 30x}{x^3 - 9x}$$

>>>Exercise 9.2 – 1 - Solution

$$f_1(x) = \frac{x^2 + 3x - 10}{x - 3} = \frac{(x + 2) \cdot (x - 5)}{x - 3}$$

zeros	$x = -2, \quad x = 5$
poles	$x = 3, \quad \lim_{x \rightarrow 3-0} f_1(x) = -\infty, \quad \lim_{x \rightarrow 3+0} f_1(x) = +\infty$
removable discontinuities	–
$\pm\infty$	$\lim_{x \rightarrow -\infty} f_1(x) = -\infty, \quad \lim_{x \rightarrow +\infty} f_1(x) = +\infty$

$$f_2(x) = \frac{x^2 - 2x - 3}{x^2 + 2x + 1} = \frac{(x - 3) \cdot (x + 1)}{(x + 1)^2}$$

zeros	$x = 3$
poles	$x = -1, \quad \lim_{x \rightarrow -1-0} f_2(x) = +\infty, \quad \lim_{x \rightarrow -1+0} f_2(x) = -\infty$
removable discontinuities	–
$\pm\infty$	$\lim_{x \rightarrow -\infty} f_2(x) = 1, \quad \lim_{x \rightarrow +\infty} f_2(x) = 1$

$$f_3(x) = \frac{x^4 - 3x^3 + 2x^2}{x^4 - 3x^3 + 3x^2 - x} = \frac{x^2 \cdot (x - 2) \cdot (x - 1)}{x \cdot (x - 1)^3}$$

zeros	$x = 2$
poles	$x = 1, \quad \lim_{x \rightarrow 1-0} f_3(x) = -\infty, \quad \lim_{x \rightarrow 1+0} f_3(x) = -\infty$
removable discontinuities	$x = 0, \quad \lim_{x \rightarrow 0-0} f_3(x) = 0, \quad \lim_{x \rightarrow 0+0} f_3(x) = 0$
$\pm\infty$	$\lim_{x \rightarrow -\infty} f_3(x) = 1, \quad \lim_{x \rightarrow +\infty} f_3(x) = 1$

$$f_4(x) = \frac{x^2 + x}{x^3 - x^4} = \frac{x \cdot (x + 1)}{-x^3 \cdot (x - 1)}$$

zeros	$x = -1$
poles	$x = 0, \quad \lim_{x \rightarrow 0-0} f_4(x) = +\infty, \quad \lim_{x \rightarrow 0+0} f_4(x) = +\infty$ $x = 1, \quad \lim_{x \rightarrow 1-0} f_4(x) = +\infty, \quad \lim_{x \rightarrow 1+0} f_4(x) = -\infty$
removable discontinuities	–
$\pm\infty$	$\lim_{x \rightarrow -\infty} f_4(x) = 0, \quad \lim_{x \rightarrow +\infty} f_4(x) = 0$

$$f_5(x) = \frac{x^4 - 6x^3 + 8x^2}{x^2 - 2x} = \frac{x^2 \cdot (x - 2) \cdot (x - 4)}{x \cdot (x - 2)}$$

zeros	$x = 4$
poles	$x = 0, \quad \lim_{x \rightarrow 0-0} f_5(x) = 0, \quad \lim_{x \rightarrow 0+0} f_5(x) = 0$
removable discontinuities	$x = 2, \quad \lim_{x \rightarrow 2-0} f_5(x) = -4, \quad \lim_{x \rightarrow 2+0} f_5(x) = -4$
$\pm\infty$	$\lim_{x \rightarrow -\infty} f_5(x) = +\infty, \quad \lim_{x \rightarrow +\infty} f_5(x) = +\infty$

$$f_6(x) = x \cdot \frac{2x^2 - 4x + 3}{x^2 - 2x + 1} = \frac{x \cdot (2x^2 - 4x + 3)}{(x - 1)^2}$$

zeros	$x = 0$
poles	$x = 1, \quad \lim_{x \rightarrow 1-0} f_6(x) = +\infty, \quad \lim_{x \rightarrow 1+0} f_6(x) = +\infty$

removable discontinuities	–
$\pm\infty$	$\lim_{x \rightarrow -\infty} f_1(x) = -\infty, \quad \lim_{x \rightarrow +\infty} f_1(x) = +\infty$
$f_7(x) = \frac{x}{x^2 - 2x + 1} = \frac{x}{(x-1)^2}$	
zeros	$x = 0$
poles	$x = 1, \quad \lim_{x \rightarrow 1-0} f_7(x) = +\infty, \quad \lim_{x \rightarrow 1+0} f_7(x) = +\infty$
removable discontinuities	–
$\pm\infty$	$\lim_{x \rightarrow -\infty} f_7(x) = 0, \quad \lim_{x \rightarrow +\infty} f_7(x) = 0$
$f_8(x) = \frac{x^3 + 4x^2 + 4x}{x^5 + 6x^4 + 12x^3 + 8x^2} = \frac{x \cdot (x+2)^2}{x^2 \cdot (x+2)^3}$	
zeros	–
poles	$x = -2, \quad \lim_{x \rightarrow -2-0} f_8(x) = -\infty, \quad \lim_{x \rightarrow -2+0} f_8(x) = +\infty$ $x = 0, \quad \lim_{x \rightarrow 0-0} f_8(x) = +\infty, \quad \lim_{x \rightarrow 0+0} f_8(x) = -\infty$
removable discontinuities	–
$\pm\infty$	$\lim_{x \rightarrow -\infty} f_8(x) = 0, \quad \lim_{x \rightarrow +\infty} f_8(x) = 0$
$f_9(x) = \frac{3x - 6}{x^2 - 6x + 9} = \frac{3 \cdot (x-3)}{(x-3)^2}$	
zeros	$x = 2$
poles	$x = 3, \quad \lim_{x \rightarrow 3-0} f_1(x) = +\infty, \quad \lim_{x \rightarrow 3+0} f_1(x) = +\infty$
removable discontinuities	–
$\pm\infty$	$\lim_{x \rightarrow -\infty} f_9(x) = 0, \quad \lim_{x \rightarrow +\infty} f_9(x) = 0$
$f_{10}(x) = \frac{x^3 - 3x^2 + 3x - 1}{1 - x} = -\frac{(x-1)^3}{x-1}$	
zeros	–
poles	–
removable discontinuities	$x = 1, \quad \lim_{x \rightarrow 1-0} f_{10}(x) = 0, \quad \lim_{x \rightarrow 1+0} f_{10}(x) = 0$
$\pm\infty$	$\lim_{x \rightarrow -\infty} f_{10}(x) = -\infty, \quad \lim_{x \rightarrow +\infty} f_{10}(x) = -\infty$
$f_{11}(x) = \frac{4x^3 - 2x^2 - 30x}{x^3 - 9x} = \frac{2x \cdot (x-3) \cdot (x+2.5)}{x \cdot (x-3) \cdot (x+3)}$	
zeros	$x = -2.5$
poles	$x = -3, \quad \lim_{x \rightarrow -3-0} f_{11}(x) = +\infty, \quad \lim_{x \rightarrow -3+0} f_{11}(x) = -\infty$
removable discontinuities	$x = 0, \quad \lim_{x \rightarrow 0-0} f_{11}(x) = 10/3, \quad \lim_{x \rightarrow 0+0} f_{11}(x) = 10/3$ $x = 3, \quad \lim_{x \rightarrow 3-0} f_{11}(x) = 11/3, \quad \lim_{x \rightarrow 3+0} f_{11}(x) = 11/3$
$\pm\infty$	$\lim_{x \rightarrow -\infty} f_{11}(x) = 4, \quad \lim_{x \rightarrow +\infty} f_{11}(x) = 4$

>>> **Exercise 9.2 – 2**

Give the partial fraction decomposition of the following rational functions.

a, $\frac{2x+3}{x^2+5x+6}$

c, $\frac{x+1}{x \cdot (x^2+1)}$

b, $\frac{x}{(x-2)^3}$

>>> **Exercise 9.2 – 2 – Solution**

a,

$$\begin{aligned}\frac{2x+3}{x^2+5x+6} &= \frac{2x+3}{(x+2) \cdot (x+3)} = \frac{A}{x+2} + \frac{B}{x+3} = \frac{Ax+3A+Bx+2B}{(x+2) \cdot (x+3)} = \\ &= \frac{(A+B) \cdot x + (3A+2B)}{(x+2) \cdot (x+3)} = \frac{-1}{x+2} + \frac{3}{x+3} \\ &\left[\begin{array}{lcl} \text{I.} & A+B & = 2 \\ \text{II.} & 3A+2B & = 3 \end{array} \Rightarrow \begin{array}{l} A = -1 \\ B = 3 \end{array} \right]\end{aligned}$$

b,

$$\begin{aligned}\frac{x}{(x-2)^3} &= \frac{A}{(x-2)^3} + \frac{B}{(x-2)^2} + \frac{C}{x-2} = \frac{A+Bx-2B+Cx^2-4Cx+4C}{(x-2)^3} = \\ &= \frac{Cx^2+(B-4C) \cdot x + (A-2B+4C)}{(x-2)^3} = \frac{2}{(x-2)^3} + \frac{1}{(x-2)^2} \\ &\left[\begin{array}{lcl} \text{I.} & C & = 0 & A = 2 \\ \text{II.} & B-4C & = 1 & \Rightarrow B = 1 \\ \text{III.} & A-2B+4C & = 0 & C = 0 \end{array} \right]\end{aligned}$$

c,

$$\begin{aligned}\frac{x+1}{x \cdot (x^2+1)} &= \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{Ax^2+A+Bx^2+Cx}{x \cdot (x^2+1)} = \frac{(A+B) \cdot x^2 + C \cdot x + A}{x \cdot (x^2+1)} = \frac{1}{x} + \frac{-x+1}{x^2+1} \\ &\left[\begin{array}{lcl} \text{I.} & A+B & = 0 & A = 1 \\ \text{II.} & C & = 1 & \Rightarrow B = -1 \\ \text{III.} & A & = 1 & C = 1 \end{array} \right]\end{aligned}$$

10 Appendix 3 – The Real Sine and Cosine Functions

10.1 Basic Transformations of Sine and Cosine Functions

General form of a harmonic signal

$$x(t) = A \cdot \sin(\omega \cdot t + \varphi) + B = A \cdot \sin(2\pi f \cdot t + \varphi) + B = A \cdot \sin\left(\frac{2\pi}{T} \cdot t + \varphi\right) + B$$

where

ω is the angular frequency $\left[\frac{rad}{s}\right]$

f is the frequency $\left[\frac{1}{s}\right] = [Hz]$

T is the period $[s]$

φ is the phase $[rad]$

To understand the calculations with the trigonometric systems it is necessary to see clearly the properties of the sine and the cosine functions and also the way the different transformations alter these properties.

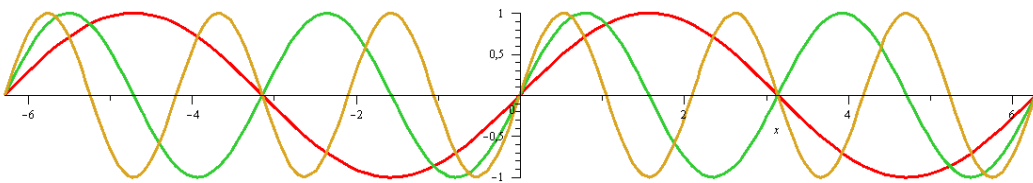
At the end of this chapter some important properties of the trigonometric functions are summarised.

>>>Exercise 10.1 – 1

Sketch the graph of the following functions and characterize them

$$t \rightarrow \sin t, \quad t \rightarrow \sin 2t, \quad t \rightarrow \sin 3t, \quad t \in [-2\pi, 2\pi]$$

>>>Exercise 10.1 – 1 - Solution



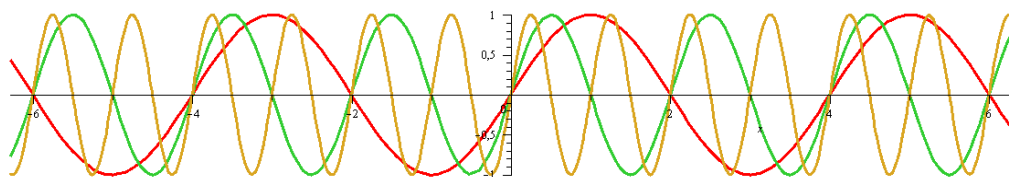
	$t \rightarrow \sin t$	$t \rightarrow \sin 2t$	$t \rightarrow \sin 3t$
amplitude	1	1	1
maximum value	1	1	1
minimum value	-1	-1	-1
period	2π	π	$2\pi/3$
angular frequency	1	2	3
frequency	$1/2\pi$	$1/\pi$	$3/2\pi$
smallest positive zero place	π	$\pi/2$	$\pi/3$

>>>Exercise 10.1 – 2

Sketch the graph of the following functions and characterize them

$$t \rightarrow \sin\left(\frac{\pi}{2}t\right), \quad x \rightarrow \sin(\pi t), \quad t \rightarrow \sin(2\pi t), \quad t \in [-2\pi, 2\pi]$$

>>>Exercise 10.1 – 2 – Solution



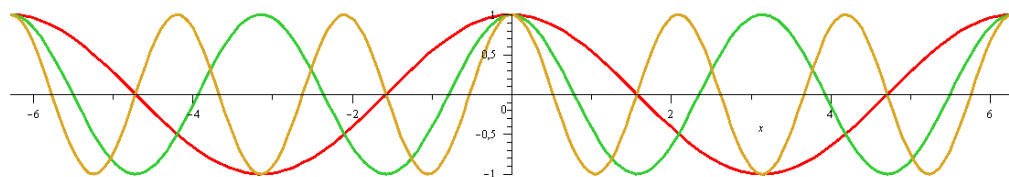
	$t \rightarrow \sin \frac{\pi}{2} t$	$t \rightarrow \sin \pi t$	$t \rightarrow \sin 2\pi t$
amplitude	1	1	1
maximum value	1	1	1
minimum value	-1	-1	-1
period	4	2	1
angular frequency	$\pi/2$	π	2π
frequency	$1/4$	$1/2$	1
smallest positive zero place	2	1	$1/2$

>>>Exercise 10.1 – 3 – Solution

Sketch the graph of the following functions and characterize them

$$t \rightarrow \cos t, \quad t \rightarrow \cos 2t, \quad t \rightarrow \cos 3t, \quad t \in [-2\pi, 2\pi]$$

>>>Exercise 10.1 – 3 – Solution



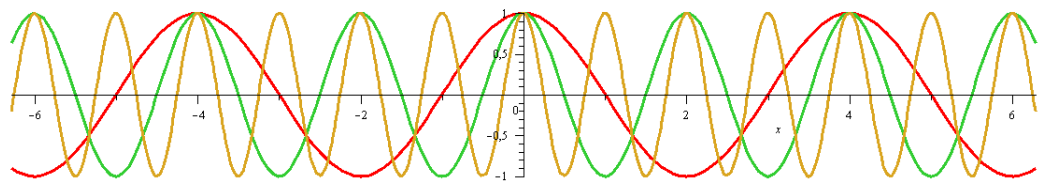
	$t \rightarrow \cos t$	$t \rightarrow \cos 2t$	$t \rightarrow \cos 3t$
amplitude	1	1	1
maximum value	1	1	1
minimum value	-1	-1	-1
period	2π	π	$2\pi/3$
angular frequency	1	2	3
frequency	$1/2\pi$	$1/\pi$	$3/2\pi$
smallest positive zero place	$\pi/2$	$\pi/4$	$\pi/6$

>>>Exercise 10.1 – 4

Sketch the graph of the following functions and characterize them

$$t \rightarrow \cos\left(\frac{\pi}{2}t\right), \quad x \rightarrow \cos(\pi t), \quad t \rightarrow \cos(2\pi t), \quad t \in [-2\pi, 2\pi]$$

>>>Exercise 10.1 – 4 – Solution



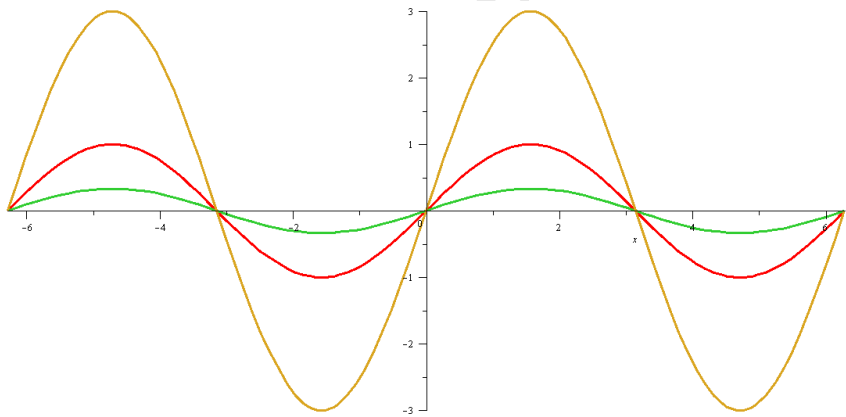
	$t \rightarrow \cos \frac{\pi}{2} t$	$x \rightarrow \cos \pi t$	$t \rightarrow \cos 2\pi t$
amplitude	1	1	1
maximum value	1	1	1
minimum value	-1	-1	-1
period	4	2	1
angular frequency	$\pi/2$	π	2π
frequency	$1/4$	$1/2$	1
smallest positive zero place	1	$1/2$	$1/4$

>>>Exercise 10.1 – 5

Sketch the graph of the following functions and characterize them

$t \rightarrow \sin t, \quad t \rightarrow \frac{1}{3} \sin t, \quad t \rightarrow 3 \sin t, \quad t \in [-2\pi, 2\pi]$

>>>Exercise 10.1 – 5 – Solution



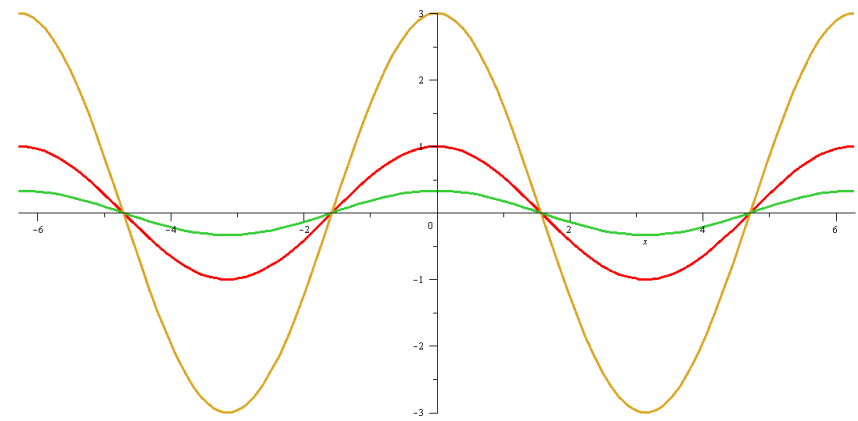
	$t \rightarrow \sin t$	$t \rightarrow \frac{1}{3} \sin t$	$t \rightarrow 3 \sin t$
amplitude	1	$1/3$	3
maximum value	1	$1/3$	3
minimum value	-1	$-1/3$	-3
period	2π	2π	2π
angular frequency	1	1	1
frequency	$1/2\pi$	$1/2\pi$	$1/2\pi$
smallest positive zero place	π	π	π

>>>Exercise 10.1 – 6

Sketch the graph of the following functions and characterize them

$t \rightarrow \cos t, \quad t \rightarrow \frac{1}{3} \sin t, \quad t \rightarrow 3 \sin t, \quad t \in [-2\pi, 2\pi]$

>>>Exercise 10.1 – 6 - Solution



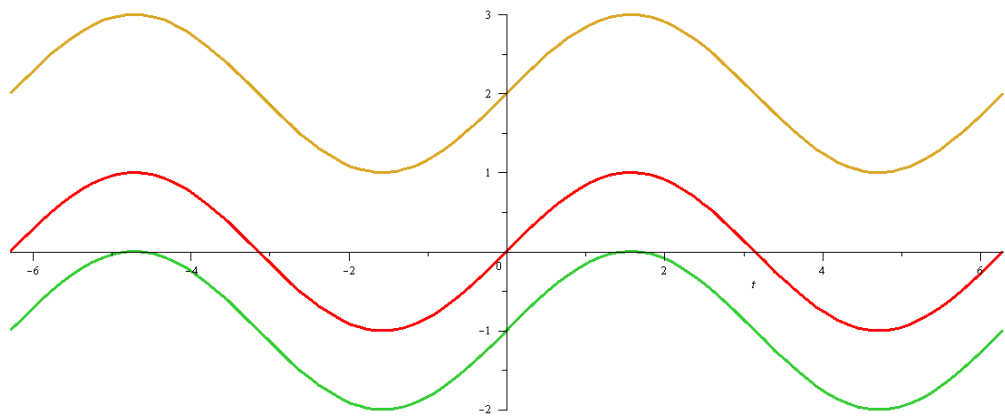
	$t \rightarrow \cos t$	$t \rightarrow \frac{1}{3} \cos t$	$t \rightarrow 3 \cos t$
amplitude	1	1/3	3
maximum value	1	1/3	3
minimum value	-1	-1/3	-3
period	2π	2π	2π
angular frequency	1	1	1
frequency	$1/2\pi$	$1/2\pi$	$1/2\pi$

>>>Exercise 10.1 – 7

Sketch the graph of the following functions and characterize them

$t \rightarrow \sin t, \quad t \rightarrow \sin t - 1, \quad t \rightarrow \sin t + 2, \quad t \in [-2\pi, 2\pi]$

>>>Exercise 10.1 – 7 - Solution



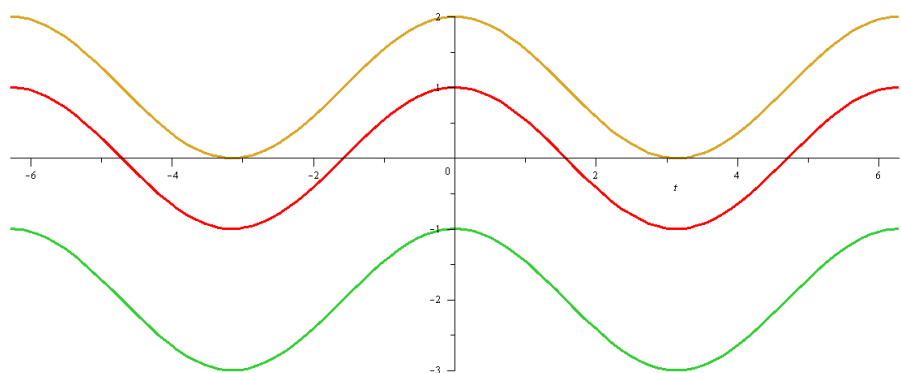
	$t \rightarrow \sin t$	$t \rightarrow \sin t - 1$	$t \rightarrow \sin t + 2$
amplitude	1	1	1
maximum value	1	0	3
minimum value	-1	-2	1
period	2π	2π	2π
angular frequency	1	1	1
frequency	$1/2\pi$	$1/2\pi$	$1/2\pi$

>>>Exercise 10.1 – 8

Sketch the graph of the following functions and characterize them

$$t \rightarrow \cos t, \quad t \rightarrow \cos t - 2, \quad t \rightarrow \cos t + 1, \quad t \in [-2\pi, 2\pi]$$

>>>Exercise 10.1 – 8 – Solution



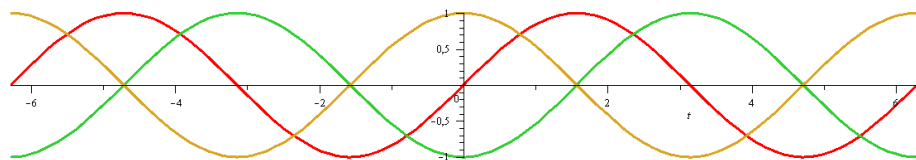
	$t \rightarrow \cos t$	$t \rightarrow \cos t - 2$	$t \rightarrow \cos t + 1$
amplitude	1	1	1
maximum value	1	-1	2
minimum value	-1	-3	0
period	2π	2π	2π
angular frequency	1	1	1
frequency	$1/2\pi$	$1/2\pi$	$1/2\pi$

>>>Exercise 10.1 – 9

Sketch the graph of the following functions and characterize them

$$t \rightarrow \sin t, \quad t \rightarrow \sin\left(t - \frac{\pi}{2}\right), \quad t \rightarrow \sin\left(t + \frac{\pi}{2}\right), \quad t \in [-2\pi, 2\pi]$$

>>>Exercise 10.1 – 9 – Solution



$t \rightarrow \sin t$	$t \rightarrow \sin\left(t - \frac{\pi}{2}\right)$	$t \rightarrow \sin\left(t + \frac{\pi}{2}\right)$
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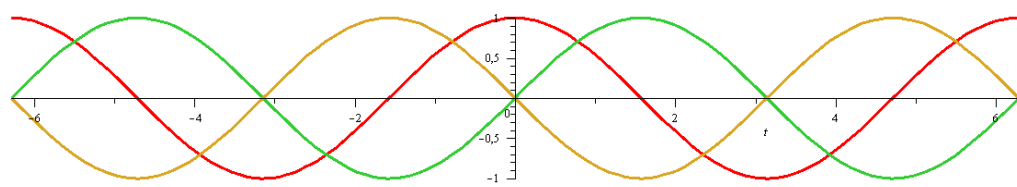
amplitude	1	1	1
maximum value	1	1	1
minimum value	-1	-1	-1
period	2π	2π	2π
angular frequency	1	1	1
frequency	$1/2\pi$	$1/2\pi$	$1/2\pi$
smallest positive zero place	π	$\frac{\pi}{2}$	$\frac{\pi}{2}$

>>> **Exercise 10.1 – 10**

Sketch the graph of the following functions and characterize them

$$t \rightarrow \cos t, \quad t \rightarrow \cos\left(t - \frac{\pi}{2}\right), \quad t \rightarrow \cos\left(t + \frac{\pi}{2}\right), \quad t \in [-2\pi, 2\pi]$$

>>> **Exercise 10.1 – 10 – Solution**



	$t \rightarrow \cos t$	$t \rightarrow \cos\left(t - \frac{\pi}{2}\right)$	$t \rightarrow \cos\left(t + \frac{\pi}{2}\right)$
amplitude	1	1	1
maximum value	1	1	1
minimum value	-1	-1	-1
period	2π	2π	2π
angular frequency	1	1	1
frequency	$1/2\pi$	$1/2\pi$	$1/2\pi$
smallest positive zero place	π	$\pi/2$	$\pi/2$

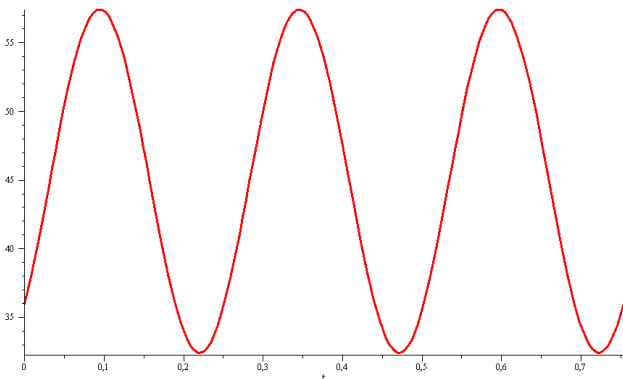
>>> **Exercise 10.1 – 11**

Sketch the graph of the function

$$x(t) = 12.5 \cdot \sin(25 \cdot t - 0.8) + 44.9$$

and characterize it.

>>> **Exercise 10.1 – 11 – Solution**



amplitude	12.5
maximum value	57.4
minimum value	32.4
period	$2\pi/25 \approx 0.251$
frequency	$25/2\pi \approx 3.979$
angular frequency	25
phase	-0.8

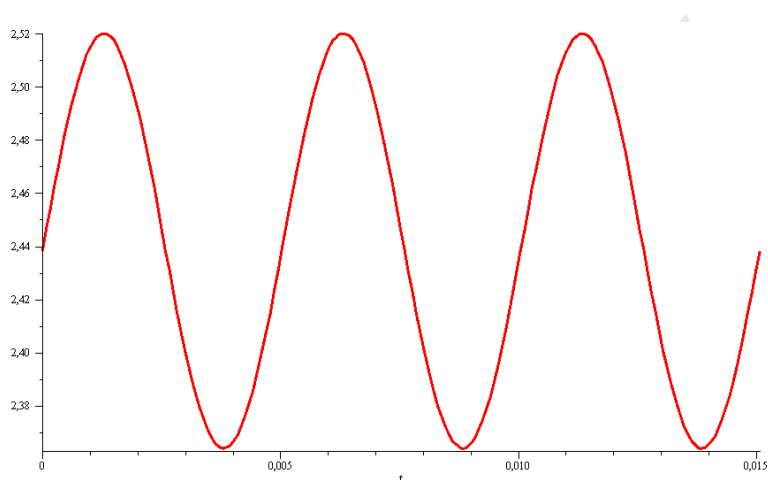
>>>Exercise 10.1 – 12

Sketch the graph of the function

$$x(t) = 0.078 \cdot \sin(1250 \cdot t - 0.05) + 2.442$$

and characterize it.

>>>Exercise 10.1 – 12 – Solution



amplitude	0.078
maximum value	2.520
minimum value	2.364
period	$2\pi/1250 \approx 0.005$
frequency	$1250/2\pi \approx 200$
angular frequency	1250
phase	-0.05

>>>Exercise 10.1 – 12

Give the displacement-time function of an oscillation motion if the amplitude is 0.032 m, the frequency is 560 Hz, and the phase is 0.2 rad.

>>>Exercise 10.1 – 12 – Solution

$$x(t) = 0.032 \cdot \sin(2\pi \cdot 560 \cdot t + 0.2)$$

10.2 Trigonometric Polynomials

Functions

$$t \rightarrow \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right) \quad \text{and} \quad t \rightarrow \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right), k = 2, 3, \dots$$

are called the *harmonics* of the basic functions

$$t \rightarrow \sin\left(\frac{2\pi}{T} \cdot t\right) \quad \text{and} \quad t \rightarrow \cos\left(\frac{2\pi}{T} \cdot t\right),$$

respectively. The periods of the harmonics are half, third, fourth... of the period of the basic function and the frequencies are two, three, four, ... times higher.

Linear combinations of finite many elements of the system

$$\left\{1, \sin\left(\frac{2\pi}{T} \cdot t\right), \cos\left(\frac{2\pi}{T} \cdot t\right), \sin\left(2 \cdot \frac{2\pi}{T} \cdot t\right), \cos\left(2 \cdot \frac{2\pi}{T} \cdot t\right), \dots\right\}$$

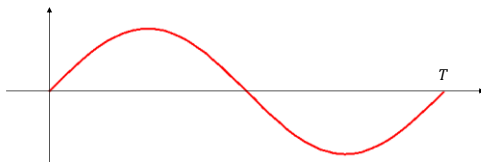
are called trigonometric polynomials:

$$a_0 + \sum_{k=1}^n a_k \cdot \cos\left(k \cdot \frac{2\pi}{T} \cdot t\right) + b_k \cdot \sin\left(k \cdot \frac{2\pi}{T} \cdot t\right)$$

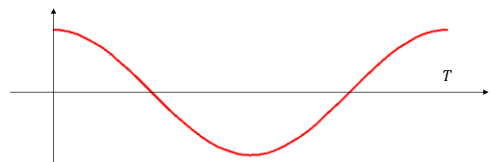
The following table shows the basic functions of period T and some harmonics. Considering the physical meaning of the quantities, period T is related to frequency $f = 1/T$.

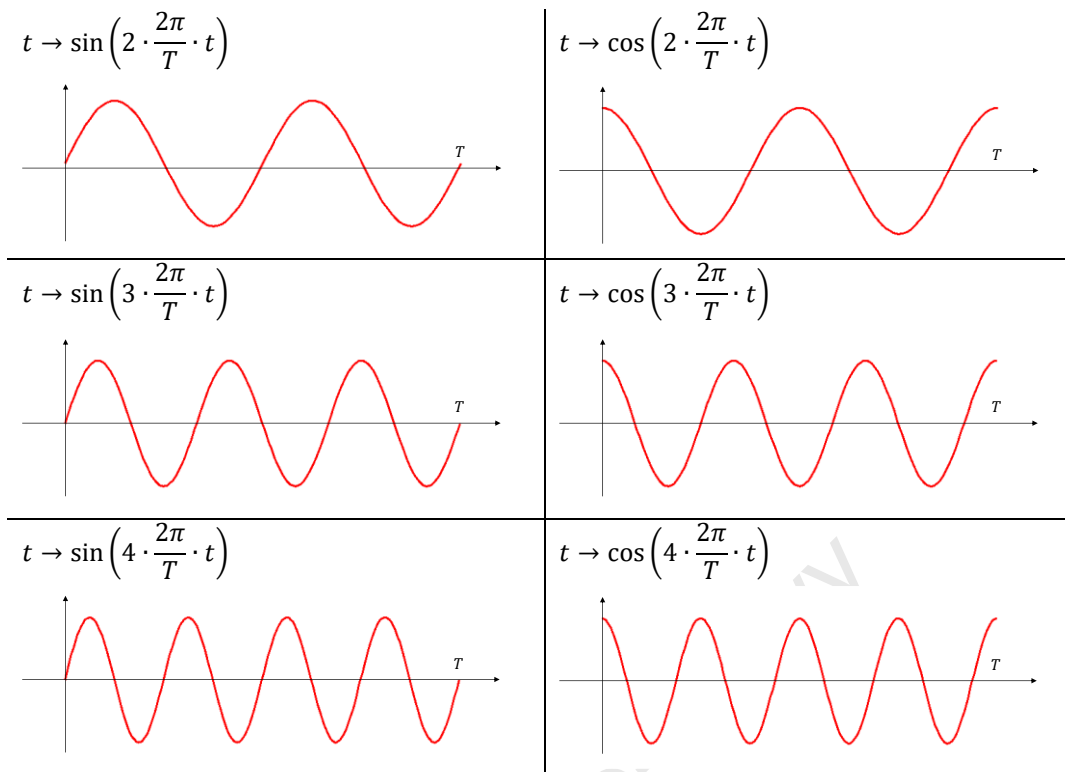
	period	frequency
$\cos\left(\frac{2\pi}{T} \cdot t\right) = \cos(2\pi f \cdot t)$	$T = \frac{1}{f}$	$f = \frac{1}{T}$
$\sin\left(\frac{2\pi}{T} \cdot t\right) = \sin(2\pi f \cdot t)$	$T = \frac{1}{f}$	$f = \frac{1}{T}$
$\cos\left(2 \cdot \frac{2\pi}{T} \cdot t\right) = \cos(2 \cdot 2\pi f \cdot t)$	$\frac{T}{2}$	$2f$
$\sin\left(2 \cdot \frac{2\pi}{T} \cdot t\right) = \sin(2 \cdot 2\pi f \cdot t)$	$\frac{T}{2}$	$2f$
$\cos\left(3 \cdot \frac{2\pi}{T} \cdot t\right) = \cos(3 \cdot 2\pi f \cdot t)$	$\frac{T}{3}$	$3f$
$\sin\left(3 \cdot \frac{2\pi}{T} \cdot t\right) = \sin(3 \cdot 2\pi f \cdot t)$	$\frac{T}{3}$	$3f$
$\cos\left(4 \cdot \frac{2\pi}{T} \cdot t\right) = \cos(4 \cdot 2\pi f \cdot t)$	$\frac{T}{4}$	$4f$
$\sin\left(4 \cdot \frac{2\pi}{T} \cdot t\right) = \sin(4 \cdot 2\pi f \cdot t)$	$\frac{T}{4}$	$4f$

$$t \rightarrow \sin\left(\frac{2\pi}{T} \cdot t\right)$$



$$t \rightarrow \cos\left(\frac{2\pi}{T} \cdot t\right)$$





10.3 Some Properties of Sine and Cosine Functions

$$\sin\left(x + \frac{\pi}{2}\right) = \cos x$$

$$\sin(x + \pi) = -\sin x$$

$$\cos\left(x + \frac{\pi}{2}\right) = -\sin x$$

$$\cos(x + \pi) = -\cos x$$

$$\sin(x + y) = \sin x \cdot \cos y + \cos x \cdot \sin y$$

$$\cos(x + y) = \cos x \cdot \cos y - \sin x \cdot \sin y$$

$$\sin(x - y) = \sin x \cdot \cos y - \cos x \cdot \sin y$$

$$\cos(x - y) = \cos x \cdot \cos y + \sin x \cdot \sin y$$

$$\sin(2x) = 2 \cdot \sin x \cdot \cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x$$

$$\sin(3x) = 3 \cdot \sin x - 4 \cdot \sin^3 x$$

$$\cos(3x) = 4 \cdot \cos^3 x - 3 \cdot \cos x$$

$$\sin(nx) = \sum_{k=0}^n \binom{n}{k} \cdot \cos^k x \cdot \sin^{n-k} x \cdot \sin\left(\frac{(n-k) \cdot \pi}{2}\right)$$

$$\cos(nx) = \sum_{k=0}^n \binom{n}{k} \cdot \cos^k x \cdot \sin^{n-k} x \cdot \cos\left(\frac{(n-k) \cdot \pi}{2}\right)$$

$$\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}}$$

$$\cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}}$$

$$\sin x + \sin y = 2 \cdot \sin \frac{x+y}{2} \cdot \cos \frac{x-y}{2}$$

$$\sin x - \sin y = 2 \cdot \cos \frac{x+y}{2} \cdot \sin \frac{x-y}{2}$$

$$\cos x + \cos y = 2 \cdot \cos \frac{x+y}{2} \cdot \cos \frac{x-y}{2}$$

$$\cos x - \cos y = -2 \cdot \sin \frac{x+y}{2} \cdot \sin \frac{x-y}{2}$$

$$A \cdot \sin x + B \cdot \cos x = \sqrt{A^2 + B^2} \cdot \sin(x + \varphi),$$

where

$$\varphi = \begin{cases} \operatorname{arctg} \frac{B}{A}, & \text{if } A \geq 0 \\ \operatorname{arctg} \frac{B}{A} + \pi, & \text{if } A < 0 \end{cases}$$

$$A \cdot \sin x + B \cdot \sin(x + \alpha) = \sqrt{A^2 + B^2 + 2AB \cdot \cos \alpha} \cdot \sin(x + \beta)$$

where

$$\beta = \operatorname{arctg} \frac{B \cdot \sin \alpha}{A + B \cdot \cos \alpha}$$

With the notation $T_n(x) = \cos(n \cdot \arccos x)$, $|x| \leq 1$, we have

$$\cos(n \cdot x) = T_n(\cos x)$$

The de Moivre formula is

$$\cos(n \cdot x) + i \cdot \sin(n \cdot x) = (\cos x + i \cdot \sin x)^n$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$\sin^3 x = \frac{3 \sin x - \sin(3x)}{4}$$

$$\cos^3 x = \frac{3 \cos x + \cos(3x)}{4}$$

$$\cos x \cdot \cos y = \frac{\cos(x - y) + \cos(x + y)}{2}$$

$$\sin x \cdot \sin y = \frac{\cos(x - y) - \cos(x + y)}{2}$$

$$\sin x \cdot \cos y = \frac{\sin(x - y) + \sin(x + y)}{2}$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^n \cdot \frac{1}{(2k+1)!} \cdot x^{2k+1}$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^n \cdot \frac{1}{(2k)!} \cdot x^{2k}$$

$$\sin x = x \cdot \prod_{k=1}^{\infty} \left(1 - \frac{1}{\pi^2 \cdot k^2} \cdot x^2 \right)$$

$$\cos x = \prod_{k=1}^{\infty} \left(1 - \frac{1}{\pi^2 \cdot \left(k - \frac{1}{2}\right)^2} \cdot x^2 \right)$$

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty} \cos\left(\frac{x}{2^k}\right)$$

11 References

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Lektor:
Dr. Kézi Csaba Gábor
egyetemi docens
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