

AN EXTENSION OF THE ABEL–LIOUVILLE IDENTITY

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Abstract. In this note, we present an extension of the celebrated Abel–Liouville identity in terms of noncommutative complete Bell polynomials for generalized Wronskians. We also characterize the range equivalence of n -dimensional vector-valued functions in the subclass of n -times differentiable functions with a nonvanishing Wronskian.

1. Introduction

Throughout this paper let \mathbb{R} , \mathbb{N} and \mathbb{N}_0 denote the set of real and the sets of positive and nonnegative integers, respectively, and let I stand for a nonempty open real interval.

For an n -dimensional vector-valued $(n - 1)$ -times continuously differentiable function $f: I \rightarrow \mathbb{R}^n$, its *Wronskian* $W_f: I \rightarrow \mathbb{R}$ is defined by

$$W_f := \begin{vmatrix} f^{(n-1)} & \dots & f' & f \end{vmatrix}.$$

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Here we usually interpret the elements of \mathbb{R}^n as column vectors. In the sequel, the standard inner product on \mathbb{R}^n will be denoted by $\langle \cdot, \cdot \rangle$.

Consider now the n th-order homogeneous linear differential equation

$$(1) \quad y^{(n)} = a_1 y^{(n-1)} + \cdots + a_n y,$$

where $a_1, \dots, a_n: I \rightarrow \mathbb{R}$ are continuous functions. By the classical *Abel–Liouville identity* (cf. [4]), if $f: I \rightarrow \mathbb{R}^n$ is a fundamental system of solutions of (1), then W_f does not vanish on I and

$$W'_f = a_1 W_f.$$

For a sufficiently smooth function $f: I \rightarrow \mathbb{R}^n$ and $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$, we introduce now the *generalized Wronskian* $W_f^k: I \rightarrow \mathbb{R}$ by

$$W_f^k := \begin{vmatrix} f^{(k_1)} & \cdots & f^{(k_n)} \end{vmatrix}.$$

One can easily see that, with this notation, we have

$$W_f = W_f^{(n-1, n-2, \dots, 0)} \quad \text{and} \quad W'_f = W_f^{(n, n-2, \dots, 0)}.$$

Therefore, the Abel–Liouville identity can be rewritten as

$$(2) \quad W_f^{(n, n-2, \dots, 0)} = a_1 W_f^{(n-1, n-2, \dots, 0)}.$$

One of the main goals of this short paper is to establish a formula for W_f^k in terms of the coefficients of differential equation (1). Another goal is to introduce the range equivalence for n -dimensional vector-valued functions and to characterize this equivalence relation in the subclass of n -times differentiable functions with a nonvanishing Wronskian.

2. Main results

For the description of our main result, we recall the notion of *noncommutative complete Bell polynomials*, which was introduced by Schimming and Rida ([3]). Let $\mathbb{R}^{n \times n}$ denote the ring of $n \times n$ matrices with real entries and

let \mathbb{I}_n denote the $n \times n$ unit matrix. Now define $B_m : (\mathbb{R}^{n \times n})^m \rightarrow \mathbb{R}^{n \times n}$ by the following recursive formula

$$B_0 := \mathbb{I}_n, \quad B_{m+1}(X_1, \dots, X_{m+1}) := \sum_{j=0}^m \binom{m}{j} B_j(X_1, \dots, X_j) X_{m+1-j}.$$

The notion of *complete Bell polynomials* in the commutative setting (i.e., when $n = 1$) was introduced by Bell ([1], [2]). One can easily compute the first few Bell polynomials as follows:

$$\begin{aligned} B_1(X_1) &= X_1, \\ B_2(X_1, X_2) &= X_1^2 + X_2, \\ B_3(X_1, X_2, X_3) &= X_1^3 + 2X_1X_2 + X_2X_1 + X_3, \\ B_4(X_1, X_2, X_3, X_4) &= X_1^4 + 3X_1^2X_2 + 2X_1X_2X_1 + 3X_1X_3 + 3X_2^2 \\ &\quad + X_2X_1^2 + X_3X_1 + X_4, \\ B_5(X_1, X_2, X_3, X_4, X_5) &= X_1^5 + 4X_1^3X_2 + 3X_1^2X_2X_1 + 6X_1^2X_3 + 8X_1X_2^2 \\ &\quad + 2X_1X_2X_1^2 + 3X_1X_3X_1 + 4X_1X_4 + 3X_2^2X_1 \\ &\quad + X_2X_1^3 + 4X_2X_1X_2 + 6X_2X_3 + X_3X_1^2 \\ &\quad + 4X_3X_2 + X_4X_1 + X_5. \end{aligned}$$

The statement of the next basic lemma was proved in the paper [3].

LEMMA 1. *For every $j \in \mathbb{N}_0$, and j -times differentiable matrix-valued function $X : I \rightarrow \mathbb{R}^{n \times n}$,*

$$B_{j+1}(X, \dots, X^{(j)}) = XB_j(X, \dots, X^{(j-1)}) + \left(B_j(X, \dots, X^{(j-1)}) \right)'$$

LEMMA 2. *Let $n, m \in \mathbb{N}$, let $X : I \rightarrow \mathbb{R}^{n \times n}$ be an $(m - 1)$ -times continuously differentiable function and $Y : I \rightarrow \mathbb{R}^{n \times n}$ be a differentiable function such that*

$$(3) \quad Y' = YX$$

holds on I . Then Y is m -times continuously differentiable and

$$(4) \quad Y^{(j)} = YB_j(X, \dots, X^{(j-1)}) \quad (j \in \{0, \dots, m\}).$$

PROOF. If $m = 1$, then X is continuous, hence the continuity of Y and (3) imply that Y is continuously differentiable. If $m > 1$, then using (3), a simple inductive argument shows that Y is m -times continuously differentiable.

The equality (4) is trivial if $j = 0$, because $B_0 = \mathbb{I}_n$. For $j = 1$, the equality (4) is equivalent to (3). Now assume that (4) has been verified for some j with $1 \leq j < m$. Then, using (3) and Lemma 1, we get

$$\begin{aligned} Y^{(j+1)} &= (Y^{(j)})' = \left(Y B_j(X, \dots, X^{(j-1)}) \right)' \\ &= Y' B_j(X, \dots, X^{(j-1)}) + Y \left(B_j(X, \dots, X^{(j-1)}) \right)' \\ &= Y \left[X B_j(X, \dots, X^{(j-1)}) + \left(B_j(X, \dots, X^{(j-1)}) \right)' \right] \\ &= Y B_{j+1}(X, \dots, X^{(j)}). \end{aligned}$$

This proves the assertion for $j + 1$. □

In what follows, let e_1, \dots, e_n denote the elements of the standard basis in \mathbb{R}^n .

COROLLARY 3. *Let $n, m \in \mathbb{N}$, let $a = (a_1, \dots, a_n): I \rightarrow \mathbb{R}^n$ be an $(m-1)$ -times continuously differentiable function and let $f: I \rightarrow \mathbb{R}^n$ be a fundamental system of solutions of the differential equation (1). Let the matrix-valued functions $X_a: I \rightarrow \mathbb{R}^{n \times n}$ and $Y_f: I \rightarrow \mathbb{R}^{n \times n}$ be defined by*

$$(5) \quad X_a := \begin{pmatrix} a & e_1 & \dots & e_{n-1} \end{pmatrix} \quad \text{and} \quad Y_f := \begin{pmatrix} f^{(n-1)} & \dots & f' & f \end{pmatrix}.$$

Then Y_f is m -times continuously differentiable and

$$Y_f^{(j)} = Y_f B_j(X_a, \dots, X_a^{(j-1)}) \quad (j \in \{0, \dots, m\}).$$

PROOF. The function f satisfies the differential equation (1), therefore $f^{(n)} = Y_f \cdot a$. On the other hand, $f^{(n-i)} = Y_f \cdot e_i$ holds for $i \in \{1, \dots, n-1\}$. These equalities imply that

$$\begin{aligned} Y_f' &= \begin{pmatrix} f^{(n)} & f^{(n-1)} & \dots & f' \end{pmatrix} \\ &= \begin{pmatrix} Y_f \cdot a & Y_f \cdot e_1 & \dots & Y_f \cdot e_{n-1} \end{pmatrix} = Y_f X_a. \end{aligned}$$

Therefore, equation (3) holds with $Y := Y_f$ and $X := X_a$, consequently, the statement is a consequence of Lemma 2. □

Using the above corollary, we can easily establish a formula for the computation of the generalized Wronskian W_f^k .

THEOREM 4. *Let $n, m \in \mathbb{N}$, let $a = (a_1, \dots, a_n): I \rightarrow \mathbb{R}^n$ be an $(m - 1)$ -times continuously differentiable function and let $f: I \rightarrow \mathbb{R}^n$ be a fundamental system of solutions of the differential equation (1). Let the matrix-valued functions $X_a: I \rightarrow \mathbb{R}^{n \times n}$ be defined by (5). Then, for $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$ with $\max(k_1, \dots, k_n) \leq m + n - 1$,*

$$(6) \quad W_f^k = W_f \left| B_{\ell_1}(X_a, \dots, X_a^{(\ell_1-1)})e_{n+\ell_1-k_1} \right. \\ \left. B_{\ell_n}(X_a, \dots, X_a^{(\ell_n-1)})e_{n+\ell_n-k_n} \right|,$$

where, for $i \in \{1, \dots, n\}$, $\ell_i := (k_i - n + 1)^+ := \max(k_i - n + 1, 0)$.

PROOF. Define the matrix valued function $Y_f: I \rightarrow \mathbb{R}^{n \times n}$ by (5) and observe that, in view of Corollary 3, for all $\ell \in \{0, \dots, m + n - 1\}$ and $i \in \{(\ell - n + 1)^+, \dots, \min(\ell, m)\}$, we have that

$$f^{(\ell)} = Y_f^{(i)} e_{n+i-\ell} = Y_f B_i(X_a, \dots, X_a^{(i-1)}) e_{n+i-\ell}.$$

By taking the smallest possible value for i in the above formula, we get

$$f^{(\ell)} = Y_f B_{(\ell-n+1)^+}(X_a, \dots, X_a^{((\ell-n+1)^+-1)}) e_{n+(\ell-n+1)^+-\ell}.$$

Applying this equality for $\ell \in \{k_1, \dots, k_n\}$, we obtain

$$\begin{pmatrix} f^{(k_1)} & \dots & f^{(k_n)} \end{pmatrix} = Y_f \begin{pmatrix} B_{\ell_1}(X_a, \dots, X_a^{(\ell_1-1)})e_{n+\ell_1-k_1} \\ \dots \\ B_{\ell_n}(X_a, \dots, X_a^{(\ell_n-1)})e_{n+\ell_n-k_n} \end{pmatrix}.$$

Now taking the determinant side by side and using the product rule for determinants, the equality (6) follows. \square

In the subsequent corollary, we consider the case when $\ell_i = 0$ for $i \in \{2, \dots, n\}$. In this particular setting, the determinant on the left hand side of (6) can easily be computed.

COROLLARY 5. *Let $n, m \in \mathbb{N}$, let $a = (a_1, \dots, a_n): I \rightarrow \mathbb{R}^n$ be an $(m - 1)$ -times continuously differentiable function and let $f: I \rightarrow \mathbb{R}^n$ be a fundamental system of solutions of the differential equation (1). Let the matrix-valued*

functions $X_a: I \rightarrow \mathbb{R}^{n \times n}$ be defined by (5) and let $d \in \{0, \dots, m-1\}$ and $j \in \{0, \dots, n-1\}$. Then

$$(7) \quad W_f^{(n+d, n-1, \dots, j+1, j-1, \dots, 0)} = (-1)^{n-j-1} W_f \langle B_{d+1}(X_a, \dots, X_a^{(d)})e_1, e_{n-j} \rangle.$$

If $d = 0$ and $j = n-1$, then this equality reduces to the Abel–Liouville identity (2). More generally, for $d = 0, 1, 2$, we get the following formulas:

$$\begin{aligned} W_f^{(n, n-1, \dots, j+1, j-1, \dots, 0)} &= (-1)^{n-j-1} W_f a_{n-j}, \\ W_f^{(n+1, n-1, \dots, j+1, j-1, \dots, 0)} &= (-1)^{n-j-1} W_f (a_1 a_{n-j} + a_{n-j+1} + a'_{n-j}), \\ W_f^{(n+2, n-1, \dots, j+1, j-1, \dots, 0)} &= (-1)^{n-j-1} W_f (a_1^2 a_{n-j} + a_1 a_{n-j+1} + a_2 a_{n-j} \\ (8) \quad &+ a_{n-j+2} + a_1 a'_{n-j} + 2a'_1 a_{n-j} + 2a'_{n-j+1} + a''_{n-j}). \end{aligned}$$

(Here we define $a_{n+1} := a_{n+2} := 0$.)

PROOF. We apply the previous theorem for $k := (n+d, n-1, \dots, j+1, j-1, \dots, 0)$, where $d \in \{0, \dots, m-1\}$ and $j \in \{0, \dots, n-1\}$. Then we get that $\ell_1 = d+1$, and $\ell_i = 0$ for $i \in \{2, \dots, n\}$. Therefore,

$$\begin{aligned} W_f^{(n+d, n-1, \dots, j+1, j-1, \dots, 0)} &= W_f \left| B_{d+1}(X_a, \dots, X_a^{(d)})e_1 \quad \mathbb{I}_n e_1 \right. \\ &\quad \left. \dots \quad \mathbb{I}_n e_{n-j-1} \quad \mathbb{I}_n e_{n-j+1} \quad \dots \quad \mathbb{I}_n e_n \right| \\ &= (-1)^{n-j-1} W_f \langle B_{d+1}(X_a, \dots, X_a^{(d)})e_1, e_{n-j} \rangle. \end{aligned}$$

Thus, equality (7) has been shown. In the case $d = 0$, we have that

$$\langle B_1(X_a)e_1, e_{n-j} \rangle = \langle X_a e_1, e_{n-j} \rangle = a_{n-j}$$

because the $(n-j)$ th entry of X_a equals a_{n-j} . This implies the first equality in (8) for $j \in \{0, \dots, n-1\}$. In particular, for $j = n-1$, this equality is equivalent to the Abel–Liouville identity (2).

In the case $d = 1$, a simple computation gives that

$$\langle B_2(X_a, X'_a)e_1, e_{n-j} \rangle = \langle (X_a^2 + X'_a)e_1, e_{n-j} \rangle = a_1 a_{n-j} + a_{n-j+1} + a'_{n-j},$$

which yields the second equality in (8) for $j \in \{0, \dots, n-1\}$.

In the case $d = 2$, a somewhat more difficult computation gives that

$$\begin{aligned} \langle B_3(X_a, X'_a, X''_a)e_1, e_{n-j} \rangle &= \langle (X_a^3 + 2X_aX'_a + X'_aX_a + X''_a)e_1, e_{n-j} \rangle \\ &= a_1^2a_{n-j} + a_1a_{n-j+1} + a_2a_{n-j} + a_{n-j+2} \\ &\quad + a_1a'_{n-j} + 2a'_1a_{n-j} + 2a'_{n-j+1} + a''_{n-j}, \end{aligned}$$

which then yields the third equality in (8). \square

For the sake of convenience and brevity, we introduce the following notation: for an n -times continuously differentiable function $f: I \rightarrow \mathbb{R}^n$ such that W_f is nonvanishing and $j \in \{0, \dots, n-1\}$, the function $\Phi_f^{[j]}: I \rightarrow \mathbb{R}$ is defined by

$$\Phi_f^{[j]} := (-1)^{n-j-1} \frac{W_f^{(n, \dots, j+1, j-1, \dots, 0)}}{W_f}.$$

For instance, if f is n -times continuously differentiable function whose components form a fundamental system of solutions for (1), then the Abel–Liouville identity (2) can be rewritten as

$$\Phi_f^{[n-1]} = a_1.$$

More generally, the first equality in (8) gives that

$$\Phi_f^{[j]} = a_{n-j} \quad (j \in \{0, \dots, n-1\})$$

or, equivalently,

$$(9) \quad a_j = \Phi_f^{[n-j]} \quad (j \in \{1, \dots, n\}).$$

LEMMA 6. *Let $f: I \rightarrow \mathbb{R}^n$ be an n -times continuously differentiable function such that W_f is nonvanishing. Then the components of f form a fundamental system of solutions of the n th-order homogeneous linear differential equation*

$$(10) \quad y^{(n)} = \sum_{j=0}^{n-1} \Phi_f^{[j]} y^{(j)}.$$

PROOF. This equation is equivalent to the following identity

$$\begin{aligned} & |f^{(n-1)} \quad \dots \quad f^{(0)}| y^{(n)} \\ &= \sum_{j=0}^{n-1} (-1)^{n-j-1} |f^{(n)} \quad \dots \quad f^{(j+1)} \quad f^{(j-1)} \quad \dots \quad f^{(0)}| y^{(j)}. \end{aligned}$$

We can now rearrange this equation to obtain

$$\begin{vmatrix} y^{(n)} & y^{(n-1)} & \dots & y \\ f_1^{(n)} & f_1^{(n-1)} & \dots & f_1 \\ \vdots & \vdots & \ddots & \vdots \\ f_n^{(n)} & f_n^{(n-1)} & \dots & f_n \end{vmatrix} = 0.$$

It is easily seen that if $y \in \{f_1, \dots, f_n\}$, then the determinant vanishes. Therefore, f_1, \dots, f_n are solutions of (10). Due to the condition that W_f is nonvanishing, the components of f are linearly independent, therefore they form a fundamental solution system for (10). \square

COROLLARY 7. *Let $n, m \in \mathbb{N}$ with $m \geq n$ and let $f: I \rightarrow \mathbb{R}^n$ be an m -times continuously differentiable function such that W_f is nonvanishing. Define $a = (a_1, \dots, a_n): I \rightarrow \mathbb{R}^n$ by (9) and $X_a: I \rightarrow \mathbb{R}^{n \times n}$ by (5). Then the equality (6) holds for $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$, if $k_i \leq m$ and $\ell_i := (k_i - n + 1)^+$ for $i \in \{1, \dots, n\}$.*

PROOF. It follows from the definition of a , that it is $(m - n)$ -times continuously differentiable. On the other hand, by Lemma 6, we have that f satisfies the n -th order homogeneous linear differential equation (1). Thus, the statement is a consequence of Theorem 4. \square

We say that two continuous functions $f, g: I \rightarrow \mathbb{R}^n$ are *range equivalent*, denoted by $f \sim g$, if there exists a nonsingular $n \times n$ -matrix A such that

$$(11) \quad f = Ag.$$

THEOREM 8. *Let $f, g: I \rightarrow \mathbb{R}^n$ be n -times continuously differentiable functions such that W_f and W_g are nonvanishing. Then $f \sim g$ holds if and only if*

$$(12) \quad \Phi_f^{[j]} = \Phi_g^{[j]} \quad (j \in \{0, \dots, n-1\}).$$

PROOF. If $f \sim g$, then there exists a nonsingular $n \times n$ -matrix A such that $f = Ag$. The product rule for determinants shows that $W_f^k = |A|W_g^k$ for every $k \in \mathbb{N}_0^n$. Using this identity and the definition of $\Phi_f^{[j]}$ and $\Phi_g^{[j]}$, we obtain the equalities in (12).

On the other hand, if the identities (12) are valid on I , then the n th-order homogeneous linear differential equation (10) is equivalent to the following one

$$y^{(n)} = \sum_{j=0}^{n-1} \Phi_g^{[j]} y^{(j)}.$$

Therefore, the (n -dimensional) solution spaces of these differential equations are identical, which in view of Lemma 6 yields that the components of f are linear combinations of the components of g . Thus identity (11) holds for some nonsingular $n \times n$ -matrix A . \square

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