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Nullity Distribution in Finsler Geometry

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Salah Gomaa Ahmed Ali
jelölt

Tanúsítom, hogy Salah Gomaa Ahmed Ali doktorjelölt 2015-2016 között a fent megnevezett doktori program keretében irányításommal végezte munkáját. Az értekezésben foglalt eredményekhez a jelölt önálló alkotó tevékenységével meghatározóan hozzájárult. Az értekezés elfogadását javaslom.

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Zoltán Muzsnay
témavezető

I confirm that Salah Gomaa Ahmed Ali, PhD candidate at the University of Debrecen, has worked under my supervision. His contribution in the results of this dissertation is essential. The dissertation completely fits the requirements of the PhD degree.

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Nabil Labib Youssef
témavezető

Nullity distribution in Finsler geometry

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Contents

Introduction

Chern and Kuiper [17] in 1952 defined a distribution on a Riemannian manifold M which assigns to each point $x \in M$ the subspace

$$\mathcal{N}_R(x) = \{X \in T_xM : R(X,Y) = 0, \forall\, Y \in T_xM\},$$

where R is the curvature of the Riemannian connection on M. It is called the nullity space at x. The distribution defined by the subspace $\mathcal{N}_R(x)$ at each point x of M is called the nullity distribution $\mathcal{N}_R$ of the Riemannian manifold M. The dimension $\mu_R(x)$ of $\mathcal{N}_R(x)$ is called the index of nullity at x. Chern and Kuiper showed that, if $\mu_R(x)$ is constant in a neighborhood, then $\mathcal{N}_R$ constitutes a completely integrable distribution there, and that the leaves of the resulting foliation are flat. Later, Maltz and others developed this point in different papers, for example, [19, 21, 23, 32, 33, 41, 49, 50].

In 1972, Akbar Zadeh [3, 4] extended this work to Finsler geometry adopting the *pullback approach* (PB-) approach to Finsler geometry. He studied the nullity distribution of the (classical) curvature of Cartan connection. Recently, Bidabad and Refie-Rad [11] studied a more general case called k-nullity distribution in Finsler geometry.

On the other hand, in 1982, Youssef [51, 53] studied the nullity distributions of the curvature tensors of Barthel connection and Berwald connection, adopting the *Klein-Grifone approach* (KG-) approach to Finsler geometry.

In the PB-approach, the existence and uniqueness theorems for the four fundamental linear connections (Berwald, Cartan, Chern and Hashiguchi connections) on a Finsler manifold have been satisfactorily established [56, 57]. In the KG-approach, Grifone [25] has investigated Cartan and Berwald connections. Szilasi and Vincze [47] have studied Chern and Hasiguchi connections using the technique of lifting vector fields to the tangent bundle.

In this thesis, we study the nullity distribution in Finsler geometry adopting both the KG- and PB-approaches. We investigate the metric (degree of) freedom of a spray. We introduce computational techniques and study many examples. The thesis is organized as follows:

In *Chapter one*, a brief introduction to Klein-Grifone formalism for Finsler geometry is given.

In *Chapter two*, the nullity distributions of the h-curvature R, hv-curvature P and v-curvature Q tensors of Cartan connection adopting the Klein-Grifone approach [24, 25, 30] are introduced and investigated. We show that the nullity distribution $\mathcal{N}_R$ is included in the nullity distribution of the curvature of Barthel connection and we show, by an example, that this inclusion is, in general, proper. Through examples, we show that the nullity distributions $\mathcal{N}_P$ and $\mathcal{N}_Q$ are not completely integrable. Nevertheless, necessary and sufficient conditions for such distributions to be completely integrable are found.

In *Chapter three*, we formulate and prove an intrinsic existence and uniqueness theorem for Chern connection in the KG-approach. We derive its torsion and curvature tensors. The Bianchi identities and the properties of the curvature tensors of Chern connection are investigated. The nullity distributions associated with Chern connection are studied. The integrability and the auto-parallel property of the nullity distribution $\mathcal{N}_{R^*}$ of the Chern h-curvature $\overset{*}{R}$ are proved. We give two interesting examples. The first shows that the nullity distribution $\mathcal{N}_{R^*}$ does not coincide with the kernel distribution of $\overset{*}{R}$ ($\mathcal{N}_{R^*}$ is proper sub-distribution of $\ker_{R^*}$). The second shows that $\mathcal{N}_{P^*}$ is not completely integrable. As a by-product, we provide a class of examples of non regular Landsberg spaces which are not a Berwald spaces.

In 1971, Akbar-Zadeh [2] proved that the kernel $\ker_R$ of the h-curvature operator R of Cartan connection coincides with the nullity distribution $\mathcal{N}_R$ of that operator. This result was reappeared again in [4] and was used to prove that the nullity foliation is auto-parallel. Moreover, Bidabad and Refie-Rad [11] generalized this result to the case of k-nullity distribution following the same pattern of proof as Akbar-Zadeh's. In *Chapter four*, we show by a counterexample that $\ker_R$ and $\mathcal{N}_R$ do not coincide, contrary to Akbar-Zadeh's result. In addition, we find sufficient conditions for $\ker_R$ and $\mathcal{N}_R$ to coincide.

In *Chapter five*, the question of how many essentially different metrics can metricize a spray is discussed. The notion of metric freedom of a spray is introduced and investigated. We show that in the regular case, the holonomy distribution can be used to calculate the metric freedom of a spray. The metric freedom of the isotropic spray is characterized. Different examples are given.

Antonelli et al. have had a good contribution in Finsler geometry computations using MAPLE (cf. [8, 9, 37]). Rutz and Portugal [39] have introduced the remarkable FINSLER package [38] (it is also included in a CD with the "Handbook of Finsler geometry" [7]). They illustrated how to use this package by an example related to general relativity. The calculation of the components P^h_{ijk} of the hv-curvature of Cartan connection leads to wrong expressions. On the other hand, the FINSLER package works only in dimension four. In *Appendix A*, we fix the two problems and solve them. Moreover, we extend this package to compute not only the geometric objects associated with Cartan connection but also those associated with Berwald, Chern and Hashiguchi connections in any dimension. Aso, the problem of simplifying tensor expressions is treated.

In *Appendix B*, we introduce a computational techinque to calculate the nullity and kernel vectors. We give three interesting examples. The first shows that the kernel distribution $\ker_R$ and the nullity distribution $\mathcal{N}_R$ do not coincide, in accordance with [59]. The second proves that the nullity distribution $\mathcal{N}_{P^\circ}$ of the hv-curvature $\overset{\circ}{P}$ of Berwald connection is not completely integrable. The third shows that the nullity distribution $\mathcal{N}_{\mathfrak{R}}$ of the curvature $\mathfrak{R}$ of Barthel connection is not a sub-distribution of the nullity distribution $\mathcal{N}_{R^\circ}$ of the h-curvature $\overset{\circ}{R}$ of Berwald connection.

It should finally be noted that most of the results obtained in this thesis have been published (or submitted for publication) in a series of articles: [35, 59, 60, 61, 63, 62, 64].

Chapter 1

Preliminaries

In this chapter, we give a brief introduction to the Frölicher-Nijenhuis formalism concerning scalar forms, vector forms and derivations. The basic concepts of Klein-Grifone approach to the differential algebraic formalism of connection theory are presented. The definitions of canonically associated geometric objects to a Finsler structure are introduced. Finally, an introduction to the well known two linear connections on the double tangent bundle; namely, Berwald and Cartan connections is given.

1.1 Frölicher-Nijenhuis theory of derivations

In 1956 A. Frolicher and A. Nijenhuis investigated an elegant theory of the derivations of the exterior algebra of the differential forms on a manifold. In this section, we show some results of the Frölicher-Nijenhuis formalism concerning scalar forms, vector forms and derivations. For more details, we refer to [22, 26].

Unless otherwise stated, all geometric objects considered are assumed to be of class C^∞. Throughout, M will denote a real paracompact manifold of finite dimension n and of class C^∞. Moreover, $C^\infty(M)$ is the $\mathbb{R}$-algebra of smooth functions on M and $\mathfrak{X}(M)$ is the $C^\infty(M)$-module of vector fields on M.

Now, a scalar p-form ω on a manifold M is a skew-symmetric $C^\infty(M)$-

multilinear map

$$\omega : \underbrace{\mathfrak{X}(M) \times ... \times \mathfrak{X}(M)}_{p-times} \longrightarrow C^\infty(M).$$

The set of all scalar p-forms on M is a $C^\infty(M)$-module, denoted by $\Lambda^p(M)$.

A vector k-form K on M is a skew-symmetric $C^\infty(M)$-multilinear map

$$K : \underbrace{\mathfrak{X}(M) \times ... \times \mathfrak{X}(M)}_{k-times} \longrightarrow \mathfrak{X}(M).$$

The set of all vector k-forms on M is a $C^\infty(M)$-module, denoted by $\Psi^k(M)$.

Let $\Lambda(M) := \bigoplus_{p=0}^n \Lambda^p(M)$ and $\Psi(M) := \bigoplus_{k=0}^n \Psi^k(M)$ be the graded algebras of scalar forms and vector forms on M respectively.

Definition 1.1. A derivation of degree r on the graded algebra $\Lambda(M)$ over $\mathbb{R}$ is a map

$$D : \Lambda(M) \longrightarrow \Lambda(M)$$

such that

(a) $Dk = 0$ for all $k \in \mathbb{R}$,

(b) $D\Lambda^p(M) \subset \Lambda^{p+r}(M)$,

(c) D is $\mathbb{R}$-linear on $\Lambda(M)$,

(d) $D(\varphi \wedge \psi) = (D\varphi) \wedge \psi + (-1)^{pr} \varphi \wedge D\psi;\ \varphi \in \Lambda^p(M),\ \psi \in \Lambda(M)$.

Theorem 1.2. *A derivation on $\Lambda(M)$ is completely determined by its action on $\Lambda^0(M) = C^\infty(M)$ and $\Lambda^1(M)$.*

Proposition 1.3. *The commutator of two derivations D_1 and D_2 on $\Lambda(M)$ defined by*

$$[D_1, D_2] := D_1\, D_2 - (-1)^{r_1 r_2} D_2\, D_1, \tag{1.1}$$

is a derivation of degree $r_1 + r_2$; r_1 and r_2 being the degrees of D_1 and D_2.

Now, we focus our attention on the most well-known derivations on $\Lambda(M)$, namely, interior product, Lie derivative and exterior derivative:

The interior product of $\omega \in \Lambda^p(M)$ with respect to $X \in \mathfrak{X}(M)$, is defined by

$$(i_X\omega)(Y_1, ..., Y_{p-1}) := \omega(X, Y_1, ..., Y_{p-1}).$$

It is a derivation of degree -1 on $\Lambda(M)$.

The Lie derivative of $\omega \in \Lambda^p(M)$ with respect to $X \in \mathfrak{X}(M)$, is defined by

$$(\mathcal{L}_X\omega)(Y_1, ..., Y_p) := X \cdot \omega(Y_1, ..., Y_p) - \sum_{i=1}^{p} \omega(Y_1, ..., [X, Y_i], ..., Y_p).$$

It is a derivation of degree 0 on $\Lambda(M)$.

The exterior derivative of $\omega \in \Lambda^p(M)$ with respect to $X \in \mathfrak{X}(M)$, is defined by [1]

$$(d\omega)(Y_1, ..., Y_{p+1}) := \sum_{i=1}^{p+1} (-1)^{i+1}\, Y_i \cdot \omega(Y_1, ..., \widehat{Y_i}, ..., Y_{p+1})$$
$$+ \sum_{1 \le i < j \le p+1} (-1)^{i+j}\, \omega([Y_i, Y_j], Y_1 ..., , \widehat{Y_i}, ..., \widehat{Y_j}, ..., Y_{p+1})$$

It is a derivation of degree 1 on $\Lambda(M)$. Note that $d \circ d = d^2 = 0$.

For any vector k-form $K \in \Psi^k(M)$ there are associated two derivations on $\Lambda(M)$ as follows:

- i_K: a derivation of degree $k-1$ defined by

$$i_K|_{\mathfrak{F}}(M) = 0,\ i_K\omega := \omega \circ K, \text{ if } \omega \in \Phi^1(M).$$

- d_K: a derivation of degree k defined by

$$d_K := [i_K, d] \overset{(1.1)}{=} i_K \circ d - (-1)^{k-1} d \circ i_K.$$

In particular, we have the following interesting special cases:

(a) If $X \in \Psi^0(M)$, then i_X is the interior product with respect to X.

(b) If $X \in \Psi^0(M)$, then $d_X = \mathcal{L}_X$ is the Lie derivative with respect to X.

(c) If $K = I \in \Psi^1(M)$, then $d_I = d$ is the exterior derivative.

[1] $\widehat{Y_i}$ means that Y_i is to be omitted

For any two vector forms $K \in \Psi^k(M)$ and $L \in \Psi^l(M)$, there exists a unique vector form $[K, L] \in \Psi^{k+l}(M)$, called the Frölicher-Nijenhuis bracket of K and L, such that

$$d_{[K,L]} := [d_k, d_L] \overset{(1.1)}{=} d_K \circ d_L - (-1)^{kl} d_L \circ d_K.$$

The Frölicher-Nijenhuis bracket of K and L has the properties:

(a) $[K, L] = (-1)^{kl+1}[L, K]$,

(b) $(-1)^{kn}[K, [L, N]] + (-1)^{lk}[L, [N, K]] + (-1)^{nl}[N, [K, L]] = 0$,

(c) $[I, K] = 0$.

In particular, if K and L are vector 0-forms, i.e., vector fields on M, then $[K, L]$ reduces to the Lie bracket of vector fields.

If $L \in \Psi^l(M)$ and $X \in \Psi^0(M)$, then for all $Y_1, ..., Y_l \in \mathfrak{X}(M)$ the Frölicher-Nijenhuis bracket is given by

$$[X, L](Y_1, ..., Y_l) = [X, L(Y_1, ..., Y_l)] - \sum_{i=1}^{l} L(Y_1, ..., [X, Y_i], ..., Y_l).$$

In particular, if $L \in \Psi^1(M)$,

$$[X, L]Y = [X, LY] - L[X, Y].$$

If $K, L \in \Psi^1(M)$, then for all $X, Y \in \mathfrak{X}(M)$ the Frölicher-Nijenhuis bracket is defined by

$$\begin{aligned}[K, L](X, Y) =& [KX, LY] + [LX, KY] + KL[X, Y] + LK[X, Y] \\ & - K[LX, Y] - K[X, LY] - L[KX, Y] - L[X, KY].\end{aligned}$$

In particular, the vector 2-form $N_K := \frac{1}{2}[K, K]$ is said to be the Nijenhuis torsion of the vector 1-form K:

$$N_K(X, Y) = [KX, KY] + K^2[X, Y] - K[KX, Y] - K[X, KY]. \quad (1.2)$$

1.2 Klein-Grifone formalism

In this section, we give a brief account of the basic concepts of the Klein-Grifone approach to global Finsler geometry. For more details, we refer to [24, 25, 30, 52].

We start with important vector bundles that will be needed.

Let M be a differentiable manifold of dimension n. For each $x \in M$, let T_xM be the tangent space to M at x. Consider the set $TM := \bigcup\limits_{x \in M} T_xM$ of all vectors tangent to M together with the canonical projection

$$\pi : TM \longrightarrow M : v \in T_xM \longmapsto x.$$

The triple (TM, π, M) is a vector bundle of rank n and fibre type $\mathbb{R}^n$, called the *tangent bundle* of M.

Similarly, the triple (TTM, π_{TM}, TM) is a vector bundle of rank $2n$ and fibre type $\mathbb{R}^{2n}$, called the *double tangent bundle* of M.

We denote the tangent bundle of non-zero tangent vectors by $(\mathcal{T}M, \pi, M)$ or simply by $\mathcal{T}M(= TM\backslash\{0\})$, where $\{0\}$ is the null section, i.e, $\{0\} := \{0_x \in T_xM : x \in M\}$ is the set of all zero vectors of the tangent spaces $T_xM; x \in M$. The manifold $\mathcal{T}M$ is called the *slit tangent bundle* of M.

Let $\pi^{-1}(TM) := TM \underset{M}{\times} TM = \{(u, v) \in TM \times TM : \pi(u) = \pi(v)\}$ and $pr_1 : TM \underset{M}{\times} TM \longrightarrow TM : (u, v) \longmapsto u$.

The triple $(\pi^{-1}(TM), pr_1, TM)$ is a vector bundle of rank n and fibre type $\mathbb{R}^n$, called the *pullback bundle* of TM.

If (TM, π, M) is the tangent bundle of M. Then, the tangent map $\pi_* : TTM \longrightarrow TM$ is defined by

$$\pi_{*u} : \quad T_uTM \longrightarrow T_{\pi(u)}M : \pi_{*u}(X_u)f := X_u \cdot (f \circ \pi),$$

for all $u \in TM$, $X_u \in T_uTM$, $f \in C^\infty(M)$. The tangent map π_* is a linear map.

A tangent vector $X \in T_u(TM)$ is said to be vertical at $u \in TM$ if $\pi_{*u}(X) = 0$. The set of all vertical vectors at u is a vector space of dimension n, called the vertical space at u and is denoted by $V_u(TM)$:

$V_u(TM) := \ker(\pi_{*u})$. Consider the disjoint union

$$V(TM) := \bigcup_{u \in TM} V_u(TM).$$

The triple $(V(TM), \pi_{TM}|_{V(TM)}, TM)$ is a vector bundle of rank n, called the *vertical bundle* of M.

Remark 1.4. By abuse of language, we often use the notation TM rather (TM, π, M) to denote the tangent bundle of M, and we even say that TM is the tangent bundle of M. The same remark is valid for the pullback bundle $(\pi^{-1}(TM), pr_1, TM)$ and the vertical bundle $(V(TM), \pi_{TM}|_{V(TM)}, TM)$.

Now, we deal with some results from the Klein-Grifone (KG-) formalism necessary for this work.

The following lemma is of extreme importance for the whole work.

Lemma 1.5. *The double tangent bundle (TTM, π_{TM}, TM) and the pull-back bundle $(\pi^{-1}(TM), pr_1, TM)$ are related by the following short exact sequence:*

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} TTM \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0, \tag{1.3}$$

where the bundle morphisms ρ and γ are defined respectively by $\rho := (\pi_{TM}, \pi_*)$ and $\gamma(u, v) := j_u(v)$, where j_u is the natural isomorphism $j_u : T_{\pi(u)}M \longrightarrow T_u(T_{\pi(u)}M)$.

The exact sequence (1.3) is called the fundamental exact sequence.

The π-vector field $\overline{\eta}$ on $\pi^{-1}(TM)$ defined by

$$\overline{\eta}(u) = (u, u) \text{ for all } u \in TM,$$

is called the fundamental π-vector field. The geometry of the tangent bundle is dominated by two canonical geometric objects: the Liouville vector field and the natural almost tangent structure. Here are their definitions.

Definition 1.6.

(a) The vertical vector field C on TM defined by $C := \gamma \circ \overline{\eta}$ is called the Liouville vector field.

(b) The vector 1-form J on TM defined by $J := \gamma \circ \rho$ is called the natural almost tangent structure of TM.

One can show that the natural almost tangent structure J has the properties:

$$J^2 = 0, \;\; [J, J] = 0, \;\; [C, J] = -J, \;\; \mathrm{Im}(J) = \ker(J) = V(TM). \qquad (1.4)$$

Hence, JX is vertical, for all $X \in \mathfrak{X}(TM)$ and any vertical vector field Y can be written as $Y = JX$, for some $X \in \mathfrak{X}(TM)$. Moreover, in view of (1.4) and (1.2), the identity $[J, J] = 0$ may be read as follows: the Lie bracket of two vertical vector fields is a vertical vector field, which means that the vertical distribution is completely integrable.

Moreover, we have

$$[i_J, d_J] = [d, d_J] = 0,$$

$$[i_J, \mathcal{L}_C] = [i_C, d_J] = i_J,$$

$$[d_J, \mathcal{L}_C] = d_J.$$

Definition 1.7.

(a) A scalar form ω on TM is said to be homogenous of degree r (simply, h(r)) if
$$\mathcal{L}_C \omega = r\omega.$$

(b) A vector form L on TM is said to be homogenous of degree r (simply, h(r)) if
$$[C, L] = (r - 1)L.$$

In particular, a function $f \in C^\infty(\mathcal{T}M)$ is k-homogeneous if

$$\mathcal{L}_C f = kf. \qquad (1.5)$$

Definition 1.8.

(a) A scalar p-form ω is semi-basic if $i_{JX}\omega = 0, \;\forall X \in \mathfrak{X}(TM)$.

(b) A vector l-form L is semi-basic if $JL = 0$ and $i_{JX}L = 0, \;\forall X \in \mathfrak{X}(TM)$.

In view of (1.4), the almost tangent structure J is h(0).

Definition 1.9. A semi-spray on M is a vector field X on TM, C^∞ on $\mathcal{T}M$, C^1 on TM, such that $JX = C$.
A semi-spray X which is homogeneous of degree 2 ($[C, X] = X$) is called a spray.

Remark 1.10. If S is a semi-spray on M, then for every vector field $X \in \mathfrak{X}(TM)$, we have

$$J[JX, S] = JX. \tag{1.6}$$

Definition 1.11. A nonlinear connection on M is a vector 1-form Γ on TM, C^∞ on $\mathcal{T}M$, C^0 on TM, such that

$$J\Gamma = J, \qquad \Gamma J = -J.$$

A nonlinear connection Γ is homogeneous if Γ is homogenous of degree 1 as a vector 1-form: $[\mathcal{C}, \Gamma] = 0$.
A homogenous nonlinear connection Γ is linear if Γ is C^1 on TM.

Let Γ be a nonlinear connection on M. The horizontal and vertical projectors h and v associated with Γ are defined by

$$h := \frac{1}{2}(I + \Gamma), \quad v := \frac{1}{2}(I - \Gamma).$$

Thus, the horizontal and vertical projectors has the properties:

$$h^2 = h, \qquad h \circ J = 0, \qquad J \circ h = J,$$

$$v^2 = v, \qquad v \circ J = J, \qquad J \circ v = 0,$$

$$h \circ v = v \circ h = 0, \qquad h + v = I \ (= id_{TTM}).$$

Consequently, Γ gives rise to the direct sum decomposition

$$TTM = H(TM) \oplus V(TM),$$

where $V(TM) := \mathrm{Im}(v) = \ker(h)$ is the vertical bundle and $H(TM) := \mathrm{Im}(h) = \ker(v)$ is a vector bundle called the horizontal bundle. Throughout, the modules of horizontal and vertical vector fields will be denoted by $\mathfrak{X}^h(TM)$ and $\mathfrak{X}^v(TM)$ respectively.

Lemma 1.12. *For a homogeneous connection* Γ*, we have*

$$[C, hX] = h[C, X], \ \forall X \in \mathfrak{X}(TM). \tag{1.7}$$

Definition 1.13. The torsion t of Γ is the vector 2-form on TM, C^∞ on $\mathcal{T}M$, defined by

$$t := \frac{1}{2}[J, \Gamma].$$

Clearly, t is semi-basic (i.e., $\mathrm{Im}(t) \subset V(TM)$ and $i_J t = 0$).

Definition 1.14. The curvature $\mathfrak{R}$ of Γ is the vector 2-form on TM, C^∞ on $\mathcal{T}M$, defined by

$$\mathfrak{R} := -\frac{1}{2}[h, h].$$

Similarly, $\mathfrak{R}$ is also semi-basic. Moreover, $\mathfrak{R}(X, Y) = -v[hX, hY]$, for all $X, Y \in \mathfrak{X}(\mathcal{T}M)$, [51].

We end this section with the following result.

Proposition 1.15. *With any given nonlinear connection Γ, one can associate a semi-spray S which is horizontal with respect to Γ, namely, $S = hS'$, where S' is an arbitrary semi-spray.*
Moreover, if Γ is homogeneous, then its associated semi-spray is a spray.

1.3 Finsler structure and Barthel connection

In this section, we deal with a Finsler structure and the associated canonical nonlinear connection called Barthel connection, following the Klein-Grifone formalism. For more detail, we refer to [24, 25, 30].

We start with the definition.

Definition 1.16. *[30]* A *Finsler manifold* of dimension n is a pair (M, L), where M is a differentiable manifold of dimension n and L is a map

$$L : TM \longrightarrow \mathbb{R},$$

satisfying the axioms:

(a) $L(u) > 0$ for all $u \in \mathcal{T}M$ and $L(0) = 0$,

(b) L is C^∞ on $\mathcal{T}M$, C^0 on TM,

(c) L is homogenous of degree 1 in the directional argument y: $\mathcal{L}_\mathcal{C} L = L$,

(d) The exterior 2-form $\Omega := dd_J E$ has maximal rank (non-degenerate), where $E := L^2/2$.

L is called the *Finsler structure* and E is the *energy function* associated with L. We will use the notation (M, L) for a Finsler manifold.

Remark 1.17. When L is defined on an open subset U of $\mathcal{T}M$ such that U is conic (that is, if $p \in U$ and $\lambda > 0$, then $\lambda p \in U$), then L is called *conic Finsler structure* . In this case (M, L) called a *conic Finsler manifold.*

Theorem 1.18. [30] *Let (M, L) be a Finsler manifold. The vector field S defined by $i_S\Omega = -dE$ is a spray.*
Such a spray is called the canonical spray associated with (M, L).

A nonlinear connection Γ on a Finsler manifold (M, L) is said to be conservative if $d_h E = 0$. Now, we give a fundamental result of Finsler geometry, which ensures the existence and uniqueness of a certain nonlinear connection with remarkable properties. Namely,

Theorem 1.19. [30] *On a Finsler manifold (M, L), there exists a unique conservative homogeneous nonlinear connection with zero torsion. It is given by*:

$$\Gamma = [J, S],$$

where S is the canonical spray.
Such a connection is called the canonical connection, the Barthel connection or the Cartan nonlinear connection associated with (M, L).

It should be noted that the semi-spray associated with the Barthel connection is a spray, which coincides with the canonical spray.

Associated with Γ, an almost complex structure F ($F^2 = -I$) is defined by $FJ = h$ and $Fh = -J$. This F defines an isomorphism of $T_z(TM)$ for all $z \in TM$.

1.4 Berwald and Cartan connections

In this section, we present the necessary material, concerning Berwald and Cartan connections, that will be needed throughout. For more details, we refer to [25, 53].

Theorem 1.20. *[25] For a Finsler space (M, L), there exists a unique linear connection $\overset{\circ}{D}$ on TM satisfying the following properties:*

(a) $\overset{\circ}{D}J = 0$.

(b) $\overset{\circ}{D}C = v$.

(c) $\overset{\circ}{D}\Gamma = 0$ ($\Longleftrightarrow \overset{\circ}{D}h = \overset{\circ}{D}v = 0$).

(d) $\overset{\circ}{D}_{JX}JY = J[JX, Y]$.

(e) $\overset{\circ}{T}(JX, Y) = 0$,

where h and v are the horizontal and vertical projectors of Barthel connection and $\overset{\circ}{T}$ is the (classicl) torsion of $\overset{\circ}{D}$. This connection is called the Berwald connection.

The explicit expression of $\overset{\circ}{D}$ is given by:

$$\left.\begin{array}{rcl} \overset{\circ}{D}_{JX}JY & = & J[JX, Y], \\ \overset{\circ}{D}_{hX}JY & = & v[hX, JY], \\ \overset{\circ}{D}F & = & 0. \end{array}\right\} \tag{1.8}$$

Lemma 1.21. *The h-curvature $\overset{\circ}{R}$, hv-curvature $\overset{\circ}{P}$ and h-torsion $\overset{\circ}{T}$ tensors of Berwald connection are given by:*

(a) $\overset{\circ}{R}(X, Y)Z = (\overset{\circ}{D}_{JZ}\mathfrak{R})(X, Y)$,

(b) $\overset{\circ}{P}(X, Y)Z = v[hX, J[JY, Z]] - J[JY, F[hX, JZ]]$
$\qquad\qquad - v[h[hX, JY], JZ] - J[v[hX, JY], Z]$,

(c) $\overset{\circ}{T}(hX, hY) = \mathfrak{R}(X, Y)$,

where $\mathfrak{R}$ is the curvature of Barthel connection.

Let (M, L) be a Finsler space and $\Omega = dd_J E$. The map $\mathbf{g}$ defined by

$$\mathbf{g}(JX, JY) := \Omega(JX, Y),\ \forall\, X, Y \in T(TM)$$

defines a metric on $V(TM)$. This metric can be extended to a metric g on $T(TM)$ defined by the formula:

$$g(X, Y) = \mathbf{g}(JX, JY) + \mathbf{g}(vX, vY) = \Omega(X, FY). \tag{1.9}$$

Theorem 1.22. *[25] For a Finsler space (M, L), there exists a unique linear connection D on TM satisfying the following properties:*

(a) $DJ = 0$.

(b) $DC = v$.

(c) $D\Gamma = 0$ ($\Longleftrightarrow Dh = Dv = 0$). (d) $Dg = 0$.

(e) $T(JX, JY) = 0$. (f) $JT(hX, hY) = 0$.

This connection is called the Cartan connection.

The explicit expression of D is given by:

$$\left.\begin{array}{rcl} D_{JX}JY &=& \overset{\circ}{D}_{JX}JY + \mathcal{C}(X,Y), \\ D_{hX}JY &=& \overset{\circ}{D}_{hX}JY + \mathcal{C}'(X,Y), \\ DF &=& 0, \end{array}\right\} \tag{1.10}$$

where $\mathcal{C}$ and $\mathcal{C}'$ are the vector 2-forms on TM defined by

$$\Omega(\mathcal{C}(X,Y), Z) = \frac{1}{2}(\mathcal{L}_{JX}(J^*g))(Y,Z),$$

$$\Omega(\mathcal{C}'(X,Y), Z) = \frac{1}{2}(\mathcal{L}_{hX}g)(JY,JZ),$$

with $(J^*g)(Y,Z) = g(JY,JZ)$.

The tensors $\mathcal{C}$ and $\mathcal{C}'$ will be called the first and second Cartan tensors respectively. They are semi-basic, symmetric and

$$\mathcal{C}(X,S) = \mathcal{C}'(X,S) = 0. \tag{1.11}$$

We have the following lemmas.

Lemma 1.23. *The (h)h-torsion $T(hX, hY)$ and (h)v-torsion $T(hX, JY)$ of Cartan connection are given respectively by*

$$T(hX,hY) = \mathfrak{R}(X,Y), \quad T(hX,JY) = (\mathcal{C}' - F\mathcal{C})(X,Y),$$

where $\mathfrak{R}$ is the curvature of Barthel connection.

Lemma 1.24. *The h-curvature R, hv-curvature P and v-curvature Q of Cartan connection are given respectively by:*

(a) $R(X,Y)Z = \overset{\circ}{R}(X,Y)Z + (D_{hX}\mathcal{C}')(Y,Z) - (D_{hY}\mathcal{C}')(X,Z)$
$\quad + \mathcal{C}'(F\mathcal{C}'(X,Z),Y) - \mathcal{C}'(F\mathcal{C}'(Y,Z),X) + \mathcal{C}(F\mathfrak{R}(X,Y),Z),$

(b) $P(X,Y)Z = \overset{\circ}{P}(X,Y)Z + (D_{hX}\mathcal{C})(Y,Z) - (D_{JY}\mathcal{C}')(X,Z)$
$\quad + \mathcal{C}(F\mathcal{C}'(X,Z),Y) + \mathcal{C}(F\mathcal{C}'(X,Y),Z) - \mathcal{C}'(F\mathcal{C}(Y,Z),X)$
$\quad - \mathcal{C}'(F\mathcal{C}(X,Y),Z),$

(c) $Q(X,Y)Z = \mathcal{C}(F\mathcal{C}(X,Z),Y) - \mathcal{C}(F\mathcal{C}(Y,Z),X)$.

Lemma 1.25. *For Cartan connection, the following properties hold:*

(a) $R(X,Y)S = \mathfrak{R}(X,Y)$.

(b) $P(X,Y)S = \mathcal{C}'(X,Y)$.

(c) $P(S,X)Y = P(X,S)Y = 0$, *S is the canonical spray.*

(d) $Q(S,X)Y = Q(X,S)Y = Q(X,Y)S = 0$.

Lemma 1.26. *The Bianchi identities for Cartan connection are given by:*

(a) $\mathfrak{S}_{X,Y,Z}\{R(X,Y)Z\} = \mathfrak{S}_{X,Y,Z}\{\mathcal{C}(F\mathfrak{R}(X,Y),Z)\}$.

(b) $\mathfrak{S}_{X,Y,Z}\{Q(X,Y)Z\} = 0$.

(c) $\mathcal{C}(F\mathfrak{R}(X,Y),Z) = \mathfrak{R}(F\mathcal{C}(X,Z),Y) - \mathfrak{R}(F\mathcal{C}(Y,Z),X)$.

(d) $\mathfrak{S}_{X,Y,Z}\{(D_{hX}\mathfrak{R})(Y,Z)\} = \mathfrak{S}_{X,Y,Z}\{\mathcal{C}'(F\mathfrak{R}(X,Y),Z)\}$.

(e) $\mathfrak{S}_{X,Y,Z}\,\{(D_{hX}R)(Y,Z)\} = \mathfrak{S}_{X,Y,Z}\,\{P(X,F\mathfrak{R}(Y,Z))\}$.

(f) $(D_{hX}P)(Y,Z) - (D_{hY}P)(X,Z) + (D_{JZ}R)(X,Y) =$
$P(X,F\mathcal{C}'(Y,Z)) - P(Y,F\mathcal{C}'(X,Z)) + R(F\mathcal{C}(Y,Z),X)$
$- R(F\mathcal{C}(X,Z),Y) - Q(F\mathfrak{R}(X,Y),Z)$.

(g) $(D_{hX}Q)(Y,Z) - (D_{JY}P)(X,Z) + (D_{JZ}P)(X,Y) =$
$P(F\mathcal{C}(X,Y),Z) - P(F\mathcal{C}(Z,X),Y) - Q(F\mathcal{C}'(X,Y),Z)$
$+ Q(F\mathcal{C}'(Z,X),Y)$.

(h) $\mathfrak{S}_{X,Y,Z}\{(D_{JX}Q)(Y,Z)\} = 0$,

where $\mathfrak{S}_{X,Y,Z}$ is the cyclic sum over the vector fields X, Y and Z.

Lemma 1.27. *For all $X,Y \in \mathfrak{X}(TM)$, we have*

(a) $[JX,JY] = J(D_{JX}Y - D_{JY}X)$.

(b) $[hX,JY] = J(D_{hX}Y) - h(D_{JY}X) - (\mathcal{C}' - F\mathcal{C})(X,Y)$.

(c) $[hX,hY] = h(D_{hX}Y - D_{hY}X) - \mathfrak{R}(X,Y)$.

Proof.
(a) Using (1.8) and (1.10), the symmetry of $\mathcal{C}$, $[J, J] = 0$, $J^2 = 0$ and $DJ = 0$, we get

$$\begin{aligned} J(D_{JX}Y - D_{JY}X) &= D_{JX}JY - D_{JY}JX \\ &= \overset{\circ}{D}_{JX}JY + \mathcal{C}(X,Y) - \overset{\circ}{D}_{JY}JX - \mathcal{C}(Y,X) \\ &= J[JX,Y] - J[JY,X] \\ &= [JX, JY]. \end{aligned}$$

(b) Using (1.8) and (1.10), the symmetry of $\mathcal{C}$, $DJ = Dh = DF = 0$, we obtain

$$\begin{aligned} J(D_{hX}Y) - h(D_{JY}X) &= D_{hX}JY - D_{JY}hX \\ &= \overset{\circ}{D}_{hX}JY + \mathcal{C}'(X,Y) - \overset{\circ}{D}_{JY}hX - F\mathcal{C}(Y,X) \\ &= v[hX, JY] - h[JY, X] + (\mathcal{C}' - F\mathcal{C})(X,Y) \\ &= [hX, JY] + (\mathcal{C}' - F\mathcal{C})(X,Y). \end{aligned}$$

(c) Again using (1.8) and (1.10), by the symmetry property of $\mathcal{C}'$, we have

$$\begin{aligned} h(D_{hX}Y - D_{hY}X) &= D_{hX}hY - D_{hY}hX \\ &= \overset{\circ}{D}_{hX}hY + F\mathcal{C}'(X,Y) - \overset{\circ}{D}_{hY}hX - F\mathcal{C}'(Y,X) \\ &= Fv[hX, JY] + Fv[JX, hY]. \end{aligned}$$

As the torsion of Γ vanishes, then $0 = t(X,Y) = v[JX,hY] + v[hX,JY] - J[hX,hY]$, from which $Fv[JX,hY] + Fv[hX,JY] = FJ[hX,hY] = h[hX,hY]$. Consequently, $h(D_{hX}Y - D_{hY}X) = h[hX,hY] = [hX,hY] - v[hX,hY] = [hX,hY] + \mathfrak{R}(X,Y)$, where we have used the identity $\mathfrak{R}(X,Y) = -v[hX,hY]$ [51]. □

Remark 1.28. It is to be noted that the identity $\mathfrak{R}(X,Y) = -v[hX,hY]$ shows that the Lie bracket $[hX,hY]$ is horizontal if and only if $\mathfrak{R}(X,Y)$ vanishes. This means that a necessary and sufficient condition for the horizontal distribution to be completely integrable is that $\mathfrak{R}$ vanishes. This fact can also be deduced from Lemma 1.27 (c) above.

Chapter 2

Nullity distributions associated with Cartan connection

In this chapter, the nullity distribution of the three curvature tensors of Cartan connection, adopting the Klein-Grifone approach, are investigated. We introduce the nullity distribution $\mathcal{N}_R$ of the h-curvature tensor of Cartan connection, the nullity spaces being subspaces of the horizontal space. We show that $\mathcal{N}_R$ is included in the nullity distribution $\mathcal{N}_{\mathfrak{R}}$ of the curvature of Barthel connection and we show, by an example, that this inclusion can be proper. We study the nullity distributions of the hv-curvature and v-curvature of Cartan connection. We show through examples that these distributions are not completely integrable. Nevertheless, we investigate necessary and sufficient conditions for such distributions to be completely integrable.

The results of this chapter are published [64].

2.1 Nullity distribution of the Cartan h-curvature

It should first be noted that the nullity distributions of Barthel and Berwald connections have been investigated in [51, 53]. In this section, we study the nullity distribution of the h-curvature tensor.

Definition 2.1. Let R be the h-curvature tensor of Cartan connection. The nullity space of R at a point $z \in TM$ is the subspace of $H_z(TM)$ defined by

$$\mathcal{N}_R(z) := \{v \in H_z(TM) |\, R_z(v, w) = 0, \text{ for all } w \in H_z(TM)\}.$$

The dimension of $\mathcal{N}_R(z)$, denoted by $\mu_R(z)$, is the index of nullity of R at z.

If the index of nullity is constant, then the map $\mathcal{N}_R : z \mapsto \mathcal{N}_R(z)$ defines a distribution $\mathcal{N}_R$ of dimension μ_R called nullity distribution of R.

Any smooth section in the nullity distribution $\mathcal{N}_R$ is called a nullity vector field. We shall assume that $\mu_R \neq 0$ and $\mu_R \neq n$.

Throughout the thesis, we denote by $Sec(\mathcal{N}_R)$ the $C^\infty(TM)$-module of the nullity vector fields.

Proposition 2.2. *The nullity distribution $\mathcal{N}_R$ has the following properties:*

(a) $\mathcal{N}_R \neq \phi$.

(b) $\mathcal{N}_R \subseteq \mathcal{N}_{\mathfrak{R}}$, *where $\mathcal{N}_{\mathfrak{R}}$ is the nullity distribution of the curvature $\mathfrak{R}$.*

(c) *If $Z \in Sec(\mathcal{N}_R)$, then $R(X,Y)Z = \mathcal{C}(F\mathfrak{R}(X,Y),Z)$.*

(d) *Let S is arbitrary spray and $S \in Sec(\mathcal{N}_R)$, then $\mathfrak{R} = 0$.*

(e) *If $X \in Sec(\mathcal{N}_R)$, then $[C,X] \in Sec(\mathcal{N}_R)$ and consequently, $[C,X] \in Sec(\mathcal{N}_{\mathfrak{R}})$.*

Proof.
(b) Let X be a nullity vector field. We have

$$\begin{aligned} X \in Sec(\mathcal{N}_R) &\Longrightarrow R(X,Y)Z = 0 \quad \forall\, Y, Z \in \mathfrak{X}(TM) \\ &\Longrightarrow R(X,Y)S = 0 \quad \forall\, Y \in \mathfrak{X}(TM) \\ &\Longrightarrow \mathfrak{R}(X,Y) = 0 \qquad \forall\, Y \in \mathfrak{X}(TM) \\ &\Longrightarrow X \in Sec(\mathcal{N}_{\mathfrak{R}}). \end{aligned}$$

(c) Let $Z \in Sec(\mathcal{N}_R)$, then $Z \in Sec(\mathcal{N}_{\mathfrak{R}})$ and by Lemma 1.26 (a), we have

$$\mathfrak{S}_{X,Y,Z}\{R(X,Y)Z\} = \mathfrak{S}_{X,Y,Z}\{\mathcal{C}(F\mathfrak{R}(X,Y),Z)\}.$$

Since $R(Y,Z)X = R(Z,X)Y = 0$ and $\mathfrak{R}(Y,Z) = \mathfrak{R}(Z,X) = 0$, the result follows.

(d) Let S is arbitrary spray and $S \in Sec(\mathcal{N}_R)$, then by (c), we have $R(X,Y)S = \mathcal{C}(F\mathfrak{R}(X,Y),S)$. Then, the result follows from (1.11) and Lemma 1.25.

(e) Let $X \in Sec(\mathcal{N}_R)$. By the identity $D_C R = 0$ [25], we have

$$(D_C R)(X, Y) = 0,$$

which leads to

$$R(D_C X, Y) = 0.$$

Using (1.8) and (1.10), we have $R([C, X], Y) = 0$. Since h is $h(1)$, then $[C, h] = 0$, from which $[C, hX] = h[C, X]$. That is, $[C, hX]$ is horizontal. Hence, $[C, X] \in Sec(\mathcal{N}_R)$. Consequently, by (b), $[C, X] \in Sec(\mathcal{N}_{\mathfrak{R}})$. $\square$

It is important to note that the converse of property (b) of Proposition 2.2 is not true in general, that is, $\mathcal{N}_{\mathfrak{R}} \not\subset \mathcal{N}_R$. This is shown by the next example (see B.4) in which the calculations are performed using MAPLE program and the NF-package [60] and [61].

From now on, all calculations of the examples throughout are given in Appendix B.

Example 2.3. Let $M = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 : x^4 \neq 0\}$,
$U = \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : x^4 \neq 0;\, y^i \neq 0, i = 1, ..., 4\} \subset TM$. Let L be defined on U by

$$L(x, y) = \sqrt{x^4 y^1((y^2)^3 + (y^3)^3 + (y^4)^3)^{1/3}}.$$

Let $X = X^i \delta_i \in Sec(\mathcal{N}_{\mathfrak{R}})$, where $\delta_i := \frac{\partial}{\partial x^i} - N_i^m \frac{\partial}{\partial y^m}$ form a basis of $\mathfrak{X}^h(\mathcal{T}M)$, then $X^i \mathfrak{R}^h_{ij} = 0$ gives a system of algebraic equations. In this system, we have $X^1 = t_1$, $t_1 \in \mathbb{R}$ and $X^2 = X^3 = 0$. Then, we get $((y^2)^3 + (y^3)^3 + 5(y^4)^3)X^4 = 0$. Now, we have two cases, either $(y^2)^3 + (y^3)^3 + 5(y^4)^3 = 0$ or $(y^2)^3 + (y^3)^3 + 5(y^4)^3 \neq 0$. Firstly, if $(y^2)^3 + (y^3)^3 + 5(y^4)^3 \neq 0$, then $X^4 = 0$ and thus $\mu_{\mathfrak{R}} = 1$. Secondly, if $(y^2)^3 + (y^3)^3 + 5(y^4)^3 = 0$, then $X^4 = t_4$, $t_4 \in \mathbb{R}$ and thus $X = t_1\delta_1 + t_4\delta_4$ and $\mu_{\mathfrak{R}} = 2$. We will be interested in the second case.

Let $Y = Y^i \delta_i \in Sec(\mathcal{N}_R)$, then $Y^j R^h_{ijk} = 0$ gives a system of algebraic equations. This system has the solution $Y^1 = t'_1, t'_1 \in \mathbb{R}$ and $Y^2 = Y^3 = Y^4 = 0$. Thus, $Y = t'_1 \delta_1$ and $\mu_R = 1$. So, the dimension of $\mathcal{N}_R = 1$ and the dimension of $\mathcal{N}_{\mathfrak{R}} = 2$, consequently, $\mathcal{N}_{\mathfrak{R}} \not\subset \mathcal{N}_R$. $\square$

Nevertheless, we have some cases in which $\mathcal{N}_{\mathfrak{R}} \subset \mathcal{N}_R$ as the case of Landsberg spaces satisfying certain conditions.

Definition 2.4. [47] A Finsler space is called Landsberg if the second Cartan tensor vanishes: $\mathcal{C}' = 0$ or, equivalently, if $P = 0$.

Theorem 2.5. *Let (M, L) be a Landsberg space. If, for all $X \in Sec(\mathcal{N}_{\mathfrak{R}}), \overset{\circ}{D}_{JZ}X \in Sec(\mathcal{N}_{\mathfrak{R}})$, then $\mathcal{N}_{\mathfrak{R}} \subset \mathcal{N}_R$ and hence $\mathcal{N}_{\mathfrak{R}} = \mathcal{N}_R$.*

Proof. Let (M, F) be a Landsberg space. Then, using Lemma 1.24, we get

$$R(X, Y)Z = \overset{\circ}{R}(X, Y)Z + \mathcal{C}(F\mathfrak{R}(X, Y), Z).$$

Let X be a vector field in $Sec(\mathcal{N}_{\mathfrak{R}})$. Then, by the above equation and the fact that $\overset{\circ}{R}(X, Y)Z = (\overset{\circ}{D}_{JZ}\mathfrak{R})(X, Y)$ [53], $R(X, Y)Z = -\mathfrak{R}(\overset{\circ}{D}_{JZ}X, Y)$. Since $\overset{\circ}{D}_{JZ}X \in Sec(\mathcal{N}_{\mathfrak{R}}), \forall X \in Sec(\mathcal{N}_{\mathfrak{R}})$, then $R(X, Y)Z = 0$ and we obtain that $X \in Sec(\mathcal{N}_R)$. Consequently, $\mathcal{N}_{\mathfrak{R}} \subset \mathcal{N}_R$ and hence $\mathcal{N}_{\mathfrak{R}} = \mathcal{N}_R$. □

Theorem 2.6. *For a Landsberg space, the nullity distributions $\mathcal{N}_R$ coincides with the nullity distribution $\mathcal{N}_{R^\circ}$ of the h-curvature $\overset{\circ}{R}$ of Berwald connection.*

Proof. Let (M, L) be a Landsberg space. Then, $\mathcal{C}' = 0$. Hence, by Lemma 1.24 (a), we get

$$R(X, Y)Z = \overset{\circ}{R}(X, Y)Z + \mathcal{C}(F\mathfrak{R}(X, Y), Z).$$

Let $X \in Sec(\mathcal{N}_R)$, then $X \in Sec(\mathcal{N}_{\mathfrak{R}})$ and thus $\mathfrak{R}(X, Y) = 0$, hence, $X \in Sec(\mathcal{N}_{R^\circ})$. Consequently, $\mathcal{N}_R \subseteq \mathcal{N}_{R^\circ}$. Conversely, let $X \in Sec(\mathcal{N}_{R^\circ})$, then $X \in Sec(\mathcal{N}_{\mathfrak{R}})$ [53] and thus $\mathfrak{R}(X, Y) = 0$, hence, $X \in SecX(\mathcal{N}_R)$. Consequently, $\mathcal{N}_{R^\circ} \subseteq \mathcal{N}_R$. □

Theorem 2.7. *Let μ_R be constant on an open subset U of TM. Then, the nullity distribution $z \mapsto \mathcal{N}_R(z)$ is completely integrable on U.*

Proof. To prove this theorem we have to show that if $X, Y \in Sec(\mathcal{N}_R)$, then $[X, Y] \in Sec(\mathcal{N}_R)$. So, let $X, Y \in Sec(\mathcal{N}_R)$ and $Z \in \mathfrak{X}^h(TM)$. This implies that X and Y are horizontal and $X, Y \in Sec(\mathcal{N}_{\mathfrak{R}})$. Then, by Lemma 1.26 **(e)**, we have

$$\mathfrak{S}_{X,Y,Z}\{(D_X R)(Y, Z)\} = \mathfrak{S}_{X,Y,Z}\{P(X, F\mathfrak{R}(Y, Z))\}.$$

Since $X, Y \in Sec(\mathcal{N}_{\mathfrak{R}})$, then $\mathfrak{R}(X, Y) = \mathfrak{R}(Y, Z) = \mathfrak{R}(Z, X) = 0$. Making use of Lemma 1.27 and the fact that R is semi-basic and $T(hX, hY) =$

$\mathfrak{R}(X, Y)$, we have

$$\begin{aligned}
0 &= \mathfrak{S}_{X,Y,Z}\{(D_X R)(Y, Z)\} \\
&= \mathfrak{S}_{X,Y,Z}\{D_X R(Y, Z) - R(D_X Y, Z) - R(Y, D_X Z)\} \\
&= -R(D_X Y, Z) - R(Z, D_Y X) \\
&= R(D_X Y - D_Y X, Z) \\
&= R([X, Y] + \mathfrak{R}(X, Y), Z) \\
&= R([X, Y], Z) + R(\mathfrak{R}(X.Y), Z) \\
&= R([X, Y], Z), \quad \forall Z \in \mathfrak{X}^h(TM).
\end{aligned}$$

It remains to show that $[X, Y]$ is horizontal. In fact, as $\mathfrak{R}(X, Y) = -v[hX, hY]$, $0 = \mathfrak{R}(X, Y) = -v[X, Y]$, and hence $[X, Y]$ is horizontal. Hence, we have $[X, Y] \in Sec(\mathcal{N}_R)$. $\square$

Remark 2.8. It should be noted that the nullity distribution $\mathcal{N}_{\mathfrak{R}}$ of the curvature of Barthel connection is completely integrable as it has been proved in [51].

Definition 2.9. [1, 55] A Finsler space (M, E), where $\dim M \geq 3$, is said to be h-isotropic if there exists a scalar function k_o such that the h-curvature tensor R of Cartan connection has the form

$$R(X, Y)Z = k_o\{g(X, Z)Y - g(Y, Z)X\}, \;\; \forall\, X, Y, Z \in \mathfrak{X}(TM).$$

Theorem 2.10. *For an h-isotropic n-Finsler space, the index of nullity μ_R is either 0 or n.*

Proof. Let $\mu_R \neq 0$ and let X be a non zero nullity vector in $Sec(\mathcal{N}_R)$ and $Y, Z, W \in \mathfrak{X}(TM)$. Then, by Definition 2.9, we have

$$\begin{aligned}
0 &= k_o\{g(X, Z)Y - g(Y, Z)X\} \\
&= k_o\{g(g(X, Z)Y, W) - g(g(Y, Z)X, W)\} \\
&= k_o\{g(Y, W)g(X, Z) - g(X, W)g(Y, Z)\}.
\end{aligned}$$

As g is a metric on TM, its trace is thus $2n$. Taking the trace with respect to the pair Y and W, we get

$$k_o\{2ng(X, Z) - g(X, Z)\} = 0,$$

Again, taking the trace of the above equation, we have

$$2n(2n-1)k_o = 0.$$

which gives $k_o = 0$. Consequently, $R = 0$ and hence $\mu_R = n$. □

Definition 2.11. [47, 55] A Finsler space (M, E), is said to be Berwald space if the hv-curvature tensor $\overset{\circ}{P}$ of Berwald connection vanishes or, equivalently, $D_{hX}\mathcal{C} = 0$ for all $X \in \mathfrak{X}(TM)$.

Theorem 2.12. *For a Berwald space, the index of nullity $\mu_{\mathfrak{R}}$ of $\mathcal{N}_{\mathfrak{R}}$ takes its maximal value if and only if the index of nullity μ_R of $\mathcal{N}_R$ takes its maximal value.*

Proof. Let (M, E) be a Berwald space and so $\mathcal{C}' = 0$ [47]. Hence, by Lemma 1.24 **(a)**, the h-curvature of Cartan connection is written in the form

$$R(X,Y)Z = \overset{\circ}{R}(X,Y)Z + \mathcal{C}(F\mathfrak{R}(X,Y), Z). \tag{2.1}$$

Now, let $\mu_{\mathfrak{R}} = n$. Then $\mathfrak{R} = 0$, which is equivalent to $\overset{\circ}{R} = 0$ [53]. Thus, Equation (2.1) yields $R = 0$. Consequently, $\mu_R = n$.

Conversely, let $\mu_R = n$. Hence, by (2.1), $\overset{\circ}{R}(X,Y)Z + \mathcal{C}(Z, F\mathfrak{R}(X,Y)) = 0$. Setting $Z = S$ in this equation, we have $\overset{\circ}{R}(X,Y)S = 0$. But $\overset{\circ}{R}(X,Y)S = \mathfrak{R}(X,Y)$ [53]. Thus, $\mathfrak{R} = 0$, consequently, $\mu_{\mathfrak{R}} = n$. □

2.2 Nullity distribution of the Cartan hv-curvature

In this section, we study the nullity distribution of the hv-curvature of Cartan connection. We show that the nullity distribution $\mathcal{N}_P$ of the hv-curvature P is not completely integrable. We impose a certain condition to make $\mathcal{N}_P$ completely integrable. We present a class of Finsler spaces which fulfills such a condition.

Definition 2.13. Let P be the hv-curvature of Cartan connection. The nullity space of P at a point $z \in TM$ is the subspace of $H_z(TM)$ defined by

$$\mathcal{N}_P(z) := \{u \in H_z(TM) : \ P_z(u,w) = 0, \ \forall w \in H_z(TM)\}.$$

The dimension of $\mathcal{N}_P(z)$, denoted by $\mu_P(z)$, is the index of nullity of P at z.

Proposition 2.14. *The nullity distribution $\mathcal{N}_P$ of P has the following properties:*

(a) $\mathcal{N}_P \neq \phi$.

(b) $S \in Sec(\mathcal{N}_P)$*, S is the canonical spray.*

(c) *If $u \in \mathcal{N}_P(z)$, then $\mathcal{C}'_z(u, v) = 0$, $\forall\, v \in T_z(TM)$.*

(d) *If $X, Y \in Sec(\mathcal{N}_P) \cap Sec(\mathcal{N}_{\mathfrak{R}})$, then $R(X, Y)Z = \mathcal{C}'([X, Y], Z)$.*

Proof.
(b) Follows from the fact that $P(S, X)Y = P(X, S)Y = 0$ (Lemma 1.25).

(c) Let $u \in \mathcal{N}_P(z)$,

$$\begin{aligned} u \in \mathcal{N}_P(z) &\Longrightarrow P_z(u, v)w = 0 \quad \forall\, v, w \in T_z(TM) \\ &\Longrightarrow P_z(u, v)S_z = 0 \quad \forall\, v \in T_z(TM) \\ &\Longrightarrow \mathcal{C}'_z(u, v) = 0 \qquad \forall\, v \in T_z(TM). \end{aligned}$$

(d) Let X, $Y \in Sec(\mathcal{N}_P) \cap Sec(\mathcal{N}_{\mathfrak{R}})$. Then, by (c) above, Lemma 1.24 (a) and the identity $\overset{\circ}{R}(X, Y)Z = (\overset{\circ}{D}_{JZ}\mathfrak{R})(X, Y)$ [53], we have

$$R(X, Y)Z = (D_{hX}\mathcal{C}')(Y, Z) - (D_{hY}\mathcal{C}')(X, Z).$$

By Lemma 1.27 (c) and the fact that $\mathcal{C}'$ is semi-basic, we get

$$R(X, Y)Z = \mathcal{C}'([hX, hY], Z). \qquad \square$$

The nullity distribution $\mathcal{N}_P$ is not completely integrable in general as shown by the following example (see B.5).

Example 2.15. Let $M = \{(x^1, x^2, x^3) \in \mathbb{R}^3 : x^2 \neq 0\}$,
$U = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : x^2 \neq 0;\ y^1, y^2 \neq 0\} \subset TM$. Let L be defined on U by:

$$L(x, y) = \sqrt{e^{-x^1}(e^{-x^1 x^3}(y^1)^2 y^3 + x^2 (y^2)^3)^{2/3}}.$$

Let $X = X^i \delta_i \in Sec(\mathcal{N}_P)$, then $X^j P^h_{ijk} = 0$ leads to a system of algebraic equations. Thus, the solution of this system is $X^1 = t_1, X^2 = \frac{y^2}{y^1} t_1$ and $X^3 = t_3$; $t_1, t_3 \in \mathbb{R}$. Hence, $X = t_1(\delta_1 + \frac{y_2}{y_1}\delta_2) + t_3\delta_3$ and $\mu_P = 2$. Now, we take $X, Y \in Sec(\mathcal{N}_P)$ such that $X = \delta_1 + \frac{y_2}{y_1}\delta_2$ and $Y = h_3$. By simple calculations, we obtain the bracket $[X, Y] = [h_1 + \frac{y^2}{y^1}\delta_2, \delta_3] = -\frac{1}{2}y^1\frac{\partial}{\partial y^1} + y^3\frac{\partial}{\partial y^3}$, which is vertical. Consequently, the nullity distribution $\mathcal{N}_P$ is not completely integrable. $\square$

Nevertheless, we have

Theorem 2.16. *Let μ_P be constant on an open subset U of TM. The nullity distribution $\mathcal{N}_P$ is completely integrable on U if and only if $\mathfrak{R}(X, Y) = 0$ and $(D_{JZ}R)(X, Y) = R(Y, F\mathcal{C}(X, Z)) - R(X, F\mathcal{C}(Y, Z))$, $\forall X, Y \in Sec(\mathcal{N}_P)$.*

Proof. Let $X, Y \in Sec(\mathcal{N}_P)$. Then, $\mathfrak{R}(X, Y) = 0$ and $(D_{JZ}R)(X, Y) = R(Y, F\mathcal{C}(X, Z)) - R(X, F\mathcal{C}(Y, Z))$, $\forall Z \in \mathfrak{X}(TM)$. Firstly, $[X, Y] = [hX, hY]$ is horizontal since $\mathfrak{R}(X, Y) = 0$. Making use of Lemma 1.26 (f) and Lemma 1.27 (c), we get

$$\begin{aligned}(D_{hX}P)(Y, Z) - (D_{hY}P)(X, Z) = 0 &\Longrightarrow P(D_X Y - D_Y X, Z) = 0\\ &\Longrightarrow P([X, Y] + \mathfrak{R}(X, Y), Z) = 0\\ &\Longrightarrow P([X, Y], Z) = 0\\ &\Longrightarrow [X, Y] \in Sec(\mathcal{N}_P).\end{aligned}$$

Hence $\mathcal{N}_P$ be completely integrable.

Conversely, let $\mathcal{N}_P$ be completely integrable. Then, if $X, Y \in Sec(\mathcal{N}_P)$, the bracket $[hX, hY]$ is horizontal, thus, $\mathfrak{R}(X, Y) = 0$. Also, by Lemma 1.26 (f) and the fact $P([hX, hY], Z) = (D_{hX}P)(Y, Z) - (D_{hY}P)(X, Z) = 0$, we have $(D_{JZ}R)(X, Y) = R(Y, F\mathcal{C}(X, Z)) - R(X, F\mathcal{C}(Y, Z))$, $\forall X, Y \in Sec(\mathcal{N}_P)$, $\forall Z \in \mathfrak{X}(TM)$. $\square$

Remark 2.17. *The class of Finsler spaces with vanishing h-curvature satisfy the conditions of Theorem 2.16. Consequently, for such spaces, $\mathcal{N}_P$ is completely integrable.*

Moreover, we have

Proposition 2.18. *A sufficient condition for $\mathcal{N}_P$ to be completely integrable is that*

$$\mathcal{N}_P \subset N_R.$$

Proof. Let $\mathcal{N}_P \subset N_R$ and $X, Y \in Sec(\mathcal{N}_P)$, $Z \in \mathfrak{X}(TM)$. Then, $X, Y \in Sec(\mathcal{N}_R)$ and hence $X, Y \in Sec(\mathcal{N}_{\mathfrak{R}})$, consequently, $\mathfrak{R}(X, Y) = 0$. Also, by Lemma 1.26 (f), we have $(D_{JZ}R)(X, Y) = R(Y, F\mathcal{C}(X, Z)) - R(X, F\mathcal{C}(Y, Z))$, $\forall X, Y \in Sec(\mathcal{N}_P)$, $\forall Z \in \mathfrak{X}(TM)$. Hence, by Theorem 2.16, $\mathcal{N}_P$ is completely integrable. $\square$

2.3 Nullity distribution of the Cartan v-curvature

In this section, we study the nullity distribution of the v-curvature Q of Cartan connection. The nullity distribution of Q is defined in a similar manner as that of R (Definition 4.6)

Proposition 2.19. *The nullity distribution $\mathcal{N}_Q$ of Q has the properties:*

(a) $\mathcal{N}_Q \neq \phi$.

(b) $S \in Sec(\mathcal{N}_Q)$.

(c) *If $Z \in Sec(\mathcal{N}_Q)$, then $Q(X,Y)Z = 0, \forall X, Y \in \mathfrak{X}(TM)$.*

(d) *If $X, Y \in Sec(\mathcal{N}_Q)$, then $F[JX, JY] \in Sec(\mathcal{N}_Q)$.*

Proof.
(b) Follows from the fact that $Q(S,X)Y = 0$. (Lemma 1.25 (d))
(c) Follows from Lemma 1.26 (b).
(d) Let $X, Y \in Sec(\mathcal{N}_Q)$, then Lemmas 1.26 (h) and 1.27 (a) lead to

$$\begin{aligned}
0 &= (D_{JX}Q)(Y,Z) + (D_{JY}Q)(Z,X) + (D_{JZ}Q)(X,Y) \\
&= -Q(D_{JX}Y, Z) - Q(Z, D_{JY}X) \\
&= Q(D_{JY}X - D_{JX}Y, Z) \\
&= -Q(F[JX,JY], Z).
\end{aligned}$$

Since $[JX, JY]$ is vertical and $FJ = h$, then $F[JX, JY]$ is horizontal. Consequently, $F[JX, JY] \in Sec(\mathcal{N}_Q)$. $\square$

The nullity distribution $\mathcal{N}_Q$ is not completely integrable in general as shown by the following example (see B.6).

Example 2.20. $M = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 : x^2 \neq 0\}$,
$U = \{(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4 : x^2 \neq 0;\ y^1, y^3, y^4 \neq 0\} \subset TM$. Let L be defined on U by

$$L(x,y) = \sqrt{x^2 (y^1)^2 e^{-y^3/y^4} + (y^2)^2}.$$

Let $X = X^i \delta_i \in Sec(\mathcal{N}_Q)$, then $X^j Q^h_{ijk} = 0$ gives a system of algebraic equations. In this system, we have $X^2 = t$, $X^4 = t'$, $X^1 = \frac{y^1}{y^4} t'$, $X^3 = \frac{y^3}{y^4} t'$; $t, t' \in \mathbb{R}$. Hence, $X = t\delta_2 + t'(\frac{y^1}{y^4}\delta_1 + \frac{y^3}{y^4}\delta_3 + \delta_4)$ and $\mu_Q = 2$. Now, we take $X, Y \in Sec(\mathcal{N}_Q)$ such that $X = h_2$ and $Y = \frac{y_1}{y_4}h_1 + \frac{y_3}{y_4}h_3 + h_4$. Then, the bracket $[X,Y] = [\delta_2, \frac{y_1}{y_4}\delta_1 + \frac{y^3}{y^4}\delta_3 + \delta_4] = -\frac{y^1y^2}{2(x^2)^2y^4}\frac{\partial}{\partial y^1} + \frac{(y^1)^2(5y^3-2y^4)}{4x^2(y^4)^2}e^{-y^3/y^4}\frac{\partial}{\partial y^2} + \frac{y^4}{2(x^2)^2}\frac{\partial}{\partial y^4}$, which is vertical. Consequently, the nullity distribution $\mathcal{N}_Q$ is not completely integrable. □

Nevertheless, we have

Theorem 2.21. *Let μ_Q be constant on an open subset U of TM. The nullity distribution $\mathcal{N}_Q$ is completely integrable on U if and only if for all $X, Y \in Sec(\mathcal{N}_Q)$, $\mathfrak{R}(X,Y) = 0$ and the tensor*

$$A(X,Y,Z) := P(F\mathcal{C}(Z,X),Y) - (D_{JX}P)(Y,Z) - (D_{JZ}P)(X,Y),$$

is symmetric in X and Y.

Proof. Let $X, Y \in Sec(\mathcal{N}_Q)$. Then, $\mathfrak{R}(X,Y) = 0$ and the tensor $A(X,Y,Z)$ is symmetric in the first two arguments. By Lemma 1.26 (g), we have

$$\begin{aligned}(D_{hX}Q)(Y,Z) = (D_{JY}P)(X,Z) - (D_{JZ}P)(X,Y) + P(F\mathcal{C}(X,Y),Z)\\ - P(F\mathcal{C}(Z,X),Y) - Q(F\mathcal{C}'(X,Y),Z).\end{aligned} \tag{2.2}$$

Interchange X with Y in the above equation, we get

$$\begin{aligned}(D_{hY}Q)(X,Z) = (D_{JX}P)(Y,Z) - (D_{JZ}P)(Y,X) + P(F\mathcal{C}(Y,X),Z)\\ - P(F\mathcal{C}(Z,Y),X) - Q(F\mathcal{C}'(Y,X),Z).\end{aligned} \tag{2.3}$$

Making use of the symmetry of $\mathcal{C}$ and $\mathcal{C}'$, Equations (2.2) and (2.3) give

$$(D_{hX}Q)(Y,Z) - (D_{hY}Q)(X,Z) = A(X,Y,Z) - A(Y,X,Z). \tag{2.4}$$

Then, by the symmetry of $A(X,Y,Z)$ in X and Y, we get

$$Q(D_{hY}X - D_{hX}Y, Z) = 0.$$

Consequently, it follows from Lemma 1.27 that

$$Q([X,Y] + \mathfrak{R}(X,Y), Z) = 0.$$

Since $\mathfrak{R}(X,Y) = 0$, $[hX,hY] = [X,Y]$ is horizontal and so $[X,Y] \in Sec(\mathcal{N}_Q)$. Consequently, $\mathcal{N}_Q(z)$ is completely integrable.

Conversely, let $\mathcal{N}_Q$ be completely integrable. Then, for all $X, Y \in Sec(\mathcal{N}_Q)$, we obtain $[hX,hY] \in Sec(\mathcal{N}_Q)$, i.e., $[hX,hY]$ is horizontal and hence $\mathfrak{R}(X,Y) = 0$. Moreover, by (2.4) and the fact that $Q([hX,hY],Z) = 0$, the tensor $A(X,Y,Z)$ is symmetric in X and Y. □

Remark 2.22. *The class of Minkowski spaces satisfies the conditions of the above theorem. Consequently, a Minkowski space has a completely integrable* $\mathcal{N}_Q$.

Definition 2.23. Let (M, L) be a Finsler manifold. The angular metric $\hbar$ on TM is defined by

$$\hbar(X,Y) = g(X,Y) - \ell(X)\ell(Y),$$

where g is the metric tensor on TM given by (1.9) and $\ell(X) := \frac{1}{L}g(X,C)$.

It should be noted that the trace of $\hbar$ is $(2n-1)$.

Definition 2.24. A Finsler space (M, L), with $\dim M \geq 4$, is said to be S_3-like if

$$Q(X,Y,Z,W) = r\{\hbar(JX,JZ)\hbar(JY,JW) - \hbar(JX,JW)\hbar(JY,JZ)\},$$

where $Q(X,Y,Z,W) = g(Q(X,Y)Z,JW)$ and r is a scalar function.

Theorem 2.25. *Let (M, L) be an S_3-like space. Then, the index of nullity μ_Q takes its maximal value.*

Proof. Let (M, L) be an S_3-like space and $X \in Sec(\mathcal{N}_Q)$, then we have

$$r\{\hbar(JX,JZ)\hbar(JY,JW) - \hbar(JX,JW)\hbar(JY,JZ)\} = 0.$$

Taking the trace with respect to JX and JZ , we get

$$(2n-2)r\hbar(JY,JW) = 0.$$

Again, taking the trace of the above equation, we have

$$(2n-1)(n-1)r = 0.$$

As $n \geq 4$, then $r = 0$ and consequently $Q = 0$. □

Chapter 3

Nullity distributions associated with Chern connection

In this chapter, we formulate and prove an intrinsic existence and uniqueness theorem for Chern connection. Then, the nullity distributions associated with the Chern connection are investigated. We prove the integrability and the autoparallel property of the nullity distribution $\mathcal{N}_{R^*}$ of the Chern h-curvature $\overset{*}{R}$. We give two interesting examples. The first shows that the nullity distribution $\mathcal{N}_{R^*}$ does not coincide with the kernel distribution of $\overset{*}{R}$ ($\mathcal{N}_{R^*}$ is a proper sub-distribution of $\ker_{R^*}$). The second shows that $\mathcal{N}_{P^*}$ is not completely integrable. As a by-product, this allows us to give a simple class of non-Berwaldian Landsberg spaces with singularities.

Some results of this chapter are published [62] and others are submitted for publication [63].

3.1 Existence and uniqueness of Chern connection

In this section, we formulate and prove an intrinsic existence and uniqueness theorem for Chern connection. We begin with some definitions quoted from [25]. These definitions are important for subsequent use throughout.

Definition 3.1. A linear connection ∇ on TM is said to be regular if $\nabla J = 0$ and the map $\varphi : V(TM) \longrightarrow V(TM)$: $X \longmapsto \nabla_X C$ is an isomorphism of $V(TM)$.

For a regular connection ∇ on TM there is associated a nonlinear connection Γ on M defied by $\Gamma = I - 2\varphi^{-1} \circ \nabla C$; Γ is said to be induced by ∇.

Definition 3.2. A regular connection ∇ on TM is said to be reducible if $\nabla\Gamma = 0$, where Γ is the nonlinear connection induced by ∇.

Definition 3.3. A linear connection ∇ on TM is said to be almost-projectable if $\nabla J = 0$ and $\nabla_{JX} C = JX$, for all $X \in T(TM)$. (an almost-projectable connection is necessairly regular).

If we replace the last axiom by the axiom $\nabla_{JX} JY = J[JX, Y]$, the connection ∇ is called normal almost-projectable.

The connection Γ induced on M by an almost-projectable (rep. normal almost-projectable) connection ∇ on TM will be called a projection (rep. normal projection) of ∇. We also say that ∇ projects (resp. projects normally) on Γ.

Definition 3.4. Let Γ be a connection on M. The lift of Γ is a reducible connection ∇ on TM which projects on Γ. The lift of Γ is said to be normal if ∇ is normal.

Lemma 3.5. *For a reducible connection ∇, we have $\nabla F = 0$, where F is the almost-complex structure associated with the connection Γ induced by ∇.*

Now, we prove the following existence and uniqueness theorem.

Theorem 3.6. *For a Finsler manifold (M, L) there exists a unique normal lift $\overset{*}{D}$ of Barthel connection $\Gamma = [J, S]$ such that:*

(a) *$\overset{*}{D}$ is horizontally metric: $\overset{*}{D}_{hX} g = 0, \ \forall X \in \mathfrak{X}(TM)$.*

(b) *The classical torsion $\overset{*}{T}$ has the property that $J\overset{*}{T}(hX, hY) = 0$, for all $X, Y \in \mathfrak{X}(TM)$.*

This connection is called Chern connection.

Proof. Firstly, we prove the *uniqueness*. Since $\overset{*}{D}$ is a normal lift of Barthel connection $\Gamma = [J, S]$, then

$$\overset{*}{D}_{JX} JY = J[JX, Y]. \tag{3.1}$$

Also, by Lemma 3.5, we have

$$\overset{*}{D}F = 0. \tag{3.2}$$

Condition (a) implies that:

$$hX.g(JY, JZ) = g(\overset{*}{D}_{hX}JY, JZ) + g(JY, \overset{*}{D}_{hX}JZ), \tag{3.3}$$
$$hY.g(JZ, JX) = g(\overset{*}{D}_{hY}JZ, JX) + g(JZ, \overset{*}{D}_{hY}JX), \tag{3.4}$$
$$hZ.g(JX, JY) = g(\overset{*}{D}_{hZ}JX, JY) + g(JX, \overset{*}{D}_{hZ}JY). \tag{3.5}$$

By adding (3.3), (3.4) and subtracting (3.5), we get

$$\begin{aligned} &hX.g(JY, JZ) + hY.g(JZ, JX) - hY.g(JZ, JX) = \\ &g(\overset{*}{D}_{hX}JY + \overset{*}{D}_{hY}JX, JZ) + g(JY, \overset{*}{D}_{hX}JZ - \overset{*}{D}_{hZ}JX) \\ &+ g(\overset{*}{D}_{hY}JZ - \overset{*}{D}_{hZ}JY, JX). \end{aligned} \tag{3.6}$$

Condition (b) together with $\overset{*}{D}J = 0$ imply:

$$\overset{*}{D}_{hX}JY - \overset{*}{D}_{hY}JX = J[hX, hY]. \tag{3.7}$$

From (3.6) and (3.7), we get

$$\begin{aligned} g(2\overset{*}{D}_{hX}JY, JZ) = {} & hX.g(JY, JZ) + hY.g(JZ, JX) \\ & - hY.g(JZ, JX) + g(J[hX, hY], JZ) \\ & - g(J[hX, hZ], JY) - g(J[hY, hZ], JX). \end{aligned} \tag{3.8}$$

Since $\Omega(X, Y) = g(X, JY) - g(JX, Y)$ and using $J\mathcal{C}' = 0$, then

$$\frac{1}{2}(\mathcal{L}_{hX}g)(JY, JZ) = \Omega(\mathcal{C}'(X, Y), Z) = g(\mathcal{C}'(X, Y), JZ),$$

which is totally symmetric. Now,

$$g(2\mathcal{C}'(X, Y), JZ) = hX.g(JY, JZ) - g([hX, JY], JZ) - g(JY, [hX, JZ]),$$

$$g(2\mathcal{C}'(Y, Z), JX) = hY.g(JZ, JX) - g([hY, JZ], JX) - g(JZ, [hY, JX]),$$

$$-g(2\mathcal{C}'(Z, X), JY) = -hZ.g(JX, JY) + g([hZ, JX], JY) + g(JX, [hZ, JY]).$$

By adding the above three equations, we get

$$\begin{aligned} g(2\mathcal{C}'(X,Y), JZ) = {} & hX.g(JY, JZ) + hY.g(JZ, JX) - hZ.g(JX, JY) \\ & - g([hX, JY] + [hY, JX], JZ) + g([hZ, JX] - [hX, JZ], JY) \\ & + g([hZ, JY] - [hY, JZ], JX) \qquad (3.9) \end{aligned}$$

From (3.8) and (3.9), we have

$$\begin{aligned} g(2\overset{*}{D}_{hX} JY, JZ) = {} & g(2\mathcal{C}'(X,Y), JZ) \\ & + g([hX, JY] + [hY, JX] + J[hX, hY], JZ) \\ & - g([hZ, JX] - [hX, JZ] + J[hX, hZ], JY) \\ & - g([hZ, JY] - [hY, JZ] + J[hY, hZ], JX). \qquad (3.10) \end{aligned}$$

Since the Barthel connection is torsion-free (Theorem 1.19), then

$$0 = t(X, Y) = v[JX, hY] + v[hX, JY] - J[hX, hY]$$

and so $J[hX, hY] = v[JX, hY] + v[hX, JY]$. Hence, one can write:

$$\begin{aligned} [hX, JY] + [hY, JX] + J[hX, hY] &= 2v[hX, hY] + h[hX, JY] \\ &\quad + h[hY, JX], \\ [hZ, JX] - [hX, JZ] + J[hX, hZ] &= h[JZ, JX] + h[hZ, JX], \\ [hZ, JY] - [hY, JZ] + J[hY, hZ] &= h[hZ, JY] + h[JZ, hY]. \end{aligned}$$

The above relations and the the fact that $g(hX, JY) = 0$ enable us to write (3.10) in the form

$$g(2\overset{*}{D}_{hX} JY, JZ) = g(2\mathcal{C}'(X,Y) + 2v[hX, JY], JZ).$$

Hence,

$$\overset{*}{D}_{hX} JY = v[hX, JY] + \mathcal{C}'(X, Y). \qquad (3.11)$$

Therefore, $\overset{*}{D}_X Y$ is uniquely determined by (3.1), (3.11) and (3.2).

To prove the *existence* of $\overset{*}{D}$, let us define $\overset{*}{D}$ by the requirement that (3.1), (3.11) and (3.2) hold. Now, we have to prove that $\overset{*}{D}$ is a normal lift of $\Gamma = [J, S]$ (i.e., $\overset{*}{D}J = 0$, $\overset{*}{D}C = v$, $\overset{*}{D}\Gamma = 0$, $\overset{*}{D}_{JX} JY = J[JX, Y]$) and conditions(a) and (b) of the theorem are satisfied.

• $\overset{*}{D}J = 0$: From (3.1), (3.2) and (3.11), we have

$$\begin{aligned}
J\overset{*}{D}_{hX}Y &= J\overset{*}{D}_{hX}hY + J\overset{*}{D}_{hX}vY \\
&= JF\overset{*}{D}_{hX}JY + J\overset{*}{D}_{hX}JFY \\
&= (JFv[hX, JY] + v\mathcal{C}'(X,Y)) + (Jv[hX, vY] + J\mathcal{C}'(X, FY)) \\
&= v[hX, JY] + \mathcal{C}'(X,Y), \text{ since } \mathcal{C}' \text{ is semi basic and } Jv = 0 \\
&= \overset{*}{D}_{hX}JY.
\end{aligned}$$

Similarly, one can show that $J\overset{*}{D}_{vX}Y = \overset{*}{D}_{vX}JY$.

• $\overset{*}{D}C = v$: From (3.11), (1.11) and (1.7), we get

$$\overset{*}{D}_{hX}C = \overset{*}{D}_{hX}JS = v[hX, JS] + \mathcal{C}'(X,S) = -v[C, hX] = -vh[C,X] = 0.$$

On the other hand, from (3.1) and (1.10), we obtain $\overset{*}{D}_{JX}C = \overset{*}{D}_{JX}JS = J[JX, S] = JX$.

• $\overset{*}{D}\Gamma = 0$ or, equivalently, $\overset{*}{D}h = 0$:

$$\begin{aligned}
h\overset{*}{D}_{hX}Y &= h\overset{*}{D}_{hX}hY + h\overset{*}{D}_{hX}vY = h\overset{*}{D}_{hX}hY, \ \ \overset{*}{D}_{hX}vY \text{ is vertical by (3.11)} \\
&= h\overset{*}{D}_{hX}FJY = hF\overset{*}{D}_{hX}JY, \text{ by (3.2)} \\
&= hFv[hX, JY] + hF\mathcal{C}'(X,Y) \\
&= Fv^2[hX, JY] + Fv\mathcal{C}'(X,Y) \\
&= Fv[hX, JY] + F\mathcal{C}'(X,Y) = FD_{hX}JY = \overset{*}{D}_{hX}hY.
\end{aligned}$$

Similarly, $\overset{*}{D}_{JX}hY = h\overset{*}{D}_{JX}Y$.

• $\overset{*}{D}$ is h-metrical: As $g(JX, JY) = g(hX, hY)$ and $g(hX, JY) = 0$, it suffices to prove that $(\overset{*}{D}_{hX}g)(JY, JZ) = 0$. By (3.1), (3.2) and (3.11), we have

$$\begin{aligned}
(\overset{*}{D}_{hX}g)(JY, JZ) &= hX.g(JY, JZ) - g(\overset{*}{D}_{hX}JY, JZ) - g(JY, \overset{*}{D}_{hX}JZ) \\
&= hX.g(JY, JZ) - g(v[hX, JY], JZ) - g(\mathcal{C}'(X,Y), JZ) \\
&\quad - g(JY, v[hX, JZ]) - g(JY, \mathcal{C}'(X,Z)) \\
&= hX.g(JY, JZ) - g(v[hX, JY], JZ) \\
&\quad - g(JY, v[hX, JZ]) - 2\mathcal{C}'_b(X,Y,Z) = 0.
\end{aligned}$$

• $J\mathbf{T}(hX, hY) = 0$: By (3.11) and the symmetry of $\mathcal{C}'$, we have

$$\begin{aligned}
J\overset{*}{T}(hX, hY) &= J\overset{*}{D}_{hX}hY - J\overset{*}{D}_{hY}hX - J[hX, hY] \\
&= \overset{*}{D}_{hX}JY - \overset{*}{D}_{hY}JX - J[hX, hY] \\
&= v[hX, JY] + \mathcal{C}'(X, Y) - v[hY, JX] - \mathcal{C}'(Y, X) \\
&\quad - J[hX, hY] = t(X, Y) = 0.
\end{aligned}$$

This completes the proof. □

Corollary 3.7. *The Chern connection $\overset{*}{D}$ is completely determined by:*

(a) $\overset{*}{D}_{JX}JY = J[JX, Y] = \overset{\circ}{D}_{JX}JY.$

(b) $\overset{*}{D}_{hX}JY = v[hX, JY] + \mathcal{C}'(X, Y) = D_{hX}JY.$

(c) $\overset{*}{D}F = 0.$

3.2 Torsion and curvature tensors

In this section, we study the torsion and curvature tensors of Chern connection. We also derive the Bianchi identities and obtain some properties of the curvature tensors. The following lemma will be useful for subsequent use.

Lemma 3.8. *For all $X, Y \in \mathfrak{X}(TM)$, we have*

(a) $[JX, JY] = J(\overset{*}{D}_{JX}Y - \overset{*}{D}_{JY}X).$

(b) $[hX, JY] = J(\overset{*}{D}_{hX})Y - h(\overset{*}{D}_{JY}X) - \mathcal{C}'(X, Y).$

(c) $[hX, hY] = h(\overset{*}{D}_{hX}Y - \overset{*}{D}_{hY}X) - \mathfrak{R}(X, Y).$

Proof. (a) By Corollary 3.7 and the fact that $[J, J] = 0$ (1.4), we get

$$\begin{aligned}
J(\overset{*}{D}_{JX}Y - \overset{*}{D}_{JY}X) &= \overset{*}{D}_{JX}JY - \overset{*}{D}_{JY}JX \\
&= J[JX, Y] - J[JY, X] = [JX, JY].
\end{aligned}$$

(b) By Corollary 3.7 and the identity $h[JY, X] = h[JY, hX]$, we obtain

$$\begin{aligned} J(\overset{*}{D}_{hX}Y) - h(\overset{*}{D}_{JY}X) &= \overset{*}{D}_{hX}JY - \overset{*}{D}_{JY}hX \\ &= v[hX, JY] + \mathcal{C}'(X,Y) - h[JY, X] \\ &= v[hX, JY] - h[JY, X] + \mathcal{C}'(X,Y) \\ &= v[hX, JY] + h[hX, JY] + \mathcal{C}'(X,Y) \\ &= [hX, JY] + \mathcal{C}'(X,Y). \end{aligned}$$

(c) Again by Corollary 3.7 and the symmetry property of $\mathcal{C}'$, we have

$$\begin{aligned} h(\overset{*}{D}_{hX}Y - \overset{*}{D}_{hY}X) &= \overset{*}{D}_{hX}hY - \overset{*}{D}_{hY}hX \\ &= Fv[hX, JY] + F\mathcal{C}'(X,Y) - Fv[hY, JX] \\ &\quad - F\mathcal{C}'(Y,X) \\ &= Fv[hX, JY] + Fv[JX, hY]. \end{aligned}$$

As the torsion of Γ vanishes, then $0 = t(X,Y) = v[JX, hY] + v[hX, JY] - J[hX, hY]$, from which $Fv[JX, hY] + Fv[hX, JY] = FJ[hX, hY] = h[hX, hY]$. Consequently,

$$\begin{aligned} h(\overset{*}{D}_{hX}Y - \overset{*}{D}_{hY}X) &= h[hX, hY] = [hX, hY] - v[hX, hY] \\ &= [hX, hY] + \mathfrak{R}(X,Y). \end{aligned}$$ □

Proposition 3.9. *The h-torsion, hv-torsion and v-torsion of Chern connection $\overset{*}{D}$ are given by:*

(a) $\overset{*}{T}(hX, hY) = \mathfrak{R}(X,Y)$.

(b) $\overset{*}{T}(hX, JY) = \mathcal{C}'(X,Y)$.

(c) $\overset{*}{T}(JX, JY) = 0$.

Proof. (a) Follows directly from the definition of $\overset{*}{T}(hX, hY)$ and Lemma 3.8.
(b) By Corollary 3.7 and using the property that $h[JX, vY] = 0$, we get

$$\begin{aligned} \overset{*}{T}(hX, JY) &= \overset{*}{D}_{hX}JY - \overset{*}{D}_{JY}hX - [hX, JY] \\ &= v[hX, JY] + \mathcal{C}'(X,Y) - h[JY, X] - [hX, JY] \\ &= v[hX, JY] + \mathcal{C}'(X,Y) \\ &\quad - h[JY, hX] - (h[hX, JY] + v[hX, JY]) \\ &= \mathcal{C}'(X,Y). \end{aligned}$$

(c) Is obvious. $\square$

As $\overset{*}{D}F = 0$, the (classical) curvature tensor K of Chern connection is completely determined by the three curvature tensors: h-curvature $\overset{*}{R}$, hv-curvature $\overset{*}{P}$ and v-curvature $\overset{*}{Q}$ defined respectively by:

$$\begin{aligned}
\overset{*}{R}(X,Y)Z &= K(hX,hY)JZ,\\
\overset{*}{P}(X,Y)Z &= K(hX,JY)JZ,\\
\overset{*}{Q}(X,Y)Z &= K(JX,JYJ)Z.
\end{aligned}$$

Proposition 3.10. *The h-curvature* $\overset{*}{R}$*, hv-curvature* $\overset{*}{P}$ *and v-curvature* $\overset{*}{Q}$ *of the Chern connection are given by:*

(a) $\overset{*}{R}(X,Y)Z = R(X,Y)Z - \mathcal{C}(F\mathfrak{R}(X,Y),Z)$.

(b) $\overset{*}{P}(X,Y)Z = \overset{\circ}{P}(X,Y)Z - (\overset{*}{D}_{JY}\mathcal{C}')(X,Z)$.

(c) $\overset{*}{Q}(X,Y)Z = 0$.

Proof. We prove (a) only. The other expressions can be proved similarly. As $\overset{*}{D}_{hX}JY = D_{hX}JY$ (Corollary 3.7(b)), we have

$$\begin{aligned}
\overset{*}{R}(X,Y)Z &= \overset{*}{D}_{hX}\overset{*}{D}_{hY}JZ - \overset{*}{D}_{hY}\overset{*}{D}_{hX}JZ - \overset{*}{D}_{[hX,hY]}JZ\\
&= D_{hX}D_{hY}JZ - D_{hY}D_{hX}JZ - \overset{*}{D}_{[hX,hY]}JZ\\
&= R(X,Y)Z + D_{[hX,hY]}JZ - \overset{*}{D}_{[hX,hY]}JZ\\
&= R(X,Y)Z + D_{JF[hX,hY]}JZ - \overset{*}{D}_{JF[hX,hY]}JZ
\end{aligned}$$

By (1.10) and Corollary 3.7, the last equation takes the form

$$\begin{aligned}
\overset{*}{R}(X,Y)Z &= R(X,Y)Z + \mathcal{C}(F[hX,hY],Z)\\
&= R(X,Y)Z - \mathcal{C}(F\mathfrak{R}(X,Y),Z),
\end{aligned}$$

where we have used the identity $\mathfrak{R}(X,Y) = -v[hX,hY]$ and the fact that C is semi-basic. $\square$

Proposition 3.11. *The h-curvature $\overset{*}{R}$ and hv-curvature $\overset{*}{P}$ of Chern connection have the following properties:*

(a) $\overset{*}{R}(X,Y)S = \mathfrak{R}(X,Y)$.

(b) $\overset{*}{P}(X,Y)S = \overset{*}{P}(S,Y)X = \mathcal{C}'(X,Y)$, *$S$ is the canonical spray.*

(c) $\overset{*}{P}(X,S)Z = 0$.

Proof. (a) Follows from Proposition 3.10, Lemma 1.25 and (1.11).
(b) By Proposition 3.10, (1.10), the properties of $\mathcal{C}'$ and $\overset{\circ}{P}$, we get

$$\begin{aligned}\overset{*}{P}(X,Y)S &= -(\overset{*}{D}_{JY}\mathcal{C}')(X,S) = -(\overset{\circ}{D}_{JY}\mathcal{C}')(X,S) = \mathcal{C}'(X,\overset{\circ}{D}_{JY}S)\\ &= \mathcal{C}'(X,FJ[JY,S]) = \mathcal{C}'(X,FJY) = \mathcal{C}'(X,Y).\end{aligned}$$

(c) Can be proved similarly. □

To study the Bianchi identities for Chern connection, let us first write the Bianchi identities for an arbitrary connection ∇.

Lemma 3.12. *Let ∇ be a linear connection on M with torsion tensor $\boldsymbol{T}$ and curvature tensor $\boldsymbol{K}$. For every $X,Y,Z \in \mathfrak{X}(M)$, we have:*

(I) $\mathfrak{S}_{X,Y,Z}\{\boldsymbol{K}(X,Y)Z\} = \mathfrak{S}_{X,Y,Z}\{\boldsymbol{T}(\boldsymbol{T}(X,Y),Z) + (\nabla_X\boldsymbol{T})(Y,Z)\}$,

(II) $\mathfrak{S}_{X,Y,Z}\{\boldsymbol{K}(\boldsymbol{T}(X,Y),Z) + (\nabla_X\boldsymbol{K})(Y,Z)\} = 0$,

where the symbol $\mathfrak{S}_{X,Y,Z}$ denotes cyclic sum over X,Y and Z.

Applying the identities (I) and (II) on Chern connection, for different triples of vector fields (hX,hY,hZ), (hX,hY,JZ), ..., we obtain many identities. Here, we give only the most important of these identities.

Proposition 3.13. *The first Bianchi identity for Chern connection yields:*

(a) $\mathfrak{S}_{X,Y,Z}\,\{\overset{*}{R}(X,Y)Z\} = 0$.

(b) $\mathfrak{S}_{X,Y,Z}\,\{(\overset{*}{D}_{hX}\,\mathfrak{R})(Y,Z)\} = \mathfrak{S}_{X,Y,Z}\{\,\mathcal{C}'(F\mathfrak{R}(X,Y),Z)\}$.

(c) $\overset{*}{P}(X,Y)Z = \overset{*}{P}(Z,Y)X$.

(d) $\overset{*}{P}(X,Y)Z - \overset{*}{P}(X,Z)Y = (\overset{*}{D}_{JZ}\mathcal{C}')(X,Y) - (\overset{*}{D}_{JY}\mathcal{C}')(X,Z)$.

The second Bianchi identity for Chern connection yields:

(e) $\mathfrak{S}_{X,Y,Z}\{\,(\overset{*}{D}_{hX}\overset{*}{R})(Y,Z)\} = \mathfrak{S}_{X,Y,Z}\{\overset{*}{P}(X, F\mathfrak{R}(Y,Z))\}$.

(f) $(\overset{*}{D}_{hX}\overset{*}{P})(Y,Z) - (\overset{*}{D}_{hY}\overset{*}{P})(X,Z) + (\overset{*}{D}_{JZ}\overset{*}{R})(X,Y) = \overset{*}{P}(X, F\mathcal{C}'(Y,Z))$
$-\overset{*}{P}(Y, F\mathcal{C}'(X,Z))$.

(g) $(\overset{*}{D}_{JY}\overset{*}{P})(X,Z) = (\overset{*}{D}_{JZ}\overset{*}{P})(X,Y)$.

Corollary 3.14. *The h-curvature $\overset{*}{R}$ and the hv-curvature $\overset{*}{P}$ satisfy:*
(a)$\overset{*}{D}_C\overset{*}{R} = 0$, (b) $\overset{*}{D}_C\overset{*}{P} = 0$, (c) $\overset{*}{P}$ *is totally symmetric if* $\overset{*}{D}_{JZ}\,\mathcal{C}' = 0$.

Proposition 3.15. *The h-curvature $\overset{*}{R}$ has the following properties:*

(a) $\overset{*}{R}(X,Y,Z,W) = -\overset{*}{R}(Y,X,Z,W)$,

(b) $\overset{*}{R}(X,Y,Z,W) + \overset{*}{R}(Y,Z,X,W) + \overset{*}{R}(Z,X,Y,W) = 0$,

Moreover, if $\mathfrak{R} = 0$, we have

(c) $\overset{*}{R}(X,Y,Z,W) = -\overset{*}{R}(X,Y,W,Z)$,

(d) $\overset{*}{R}(X,Y,Z,W) = \overset{*}{R}(Z,W,X,Y)$,

where $\overset{*}{R}(X,Y,Z,W) := g(\overset{*}{R}(X,Y)Z, JW)$.

Proof. (a) is clear.
(b) By making use of Proposition 3.13, we have

$$\begin{aligned}
&\overset{*}{R}(X,Y,Z,W) + \overset{*}{R}(Y,Z,X,W) + \overset{*}{R}(Z,X,Y,W)\\
&\quad = g(\overset{*}{R}(X,Y)Z, JW) + g(\overset{*}{R}(Y,Z)X, JW) + g(\overset{*}{R}(Z,X)Y, JW)\\
&\quad = g(\overset{*}{R}(X,Y)Z + \overset{*}{R}(Y,Z)X + \overset{*}{R}(Z,X)Y, JW) = 0.
\end{aligned}$$

(c) By Theorem 3.6, we have

$$hX.g(JY,JZ) = g(\overset{*}{D}_{hX}JY, JZ) + g(JY, \overset{*}{D}_{hX}JZ).$$

Then, we can write

$$\begin{aligned} hW.(hX.g(JY,JZ)) &= g(\overset{*}{D}_{hW}\overset{*}{D}_{hX}JY,JZ) + g(\overset{*}{D}_{hX}JY,\overset{*}{D}_{hW}JZ) \\ &\quad + g(\overset{*}{D}_{hW}JY,\overset{*}{D}_{hX}JZ) + g(JY,\overset{*}{D}_{hW}\overset{*}{D}_{hX}JZ). \end{aligned}$$

Interchanging X and W, we get

$$\begin{aligned} hX.(hW.g(JY,JZ)) &= g(\overset{*}{D}_{hX}\overset{*}{D}_{hW}JY,JZ) + g(\overset{*}{D}_{hW}JY,\overset{*}{D}_{hX}JZ) \\ &\quad + g(\overset{*}{D}_{hX}JY,\overset{*}{D}_{hW}JZ) + g(JY,(\overset{*}{D}_{hX}\overset{*}{D}_{hW}JZ)). \end{aligned}$$

Using the above two equations, we obtain

$$\begin{aligned} [hW,hX].g(JY,JZ) &= g((\overset{*}{D}_{hW}\overset{*}{D}_{hX} - \overset{*}{D}_{hX}\overset{*}{D}_{hW})JY,JZ) \\ &\quad + g(JY,(\overset{*}{D}_{hW}\overset{*}{D}_{hX} - \overset{*}{D}_{hX}\overset{*}{D}_{hW})JZ). \end{aligned}$$

If $\mathfrak{R} = 0$, then the horizontal distribution is completely integrable. Consequently, $[hW,hX]$ is horizontal and so $\overset{*}{D}_{[hW,hX]}\,g = 0$. Hence, we have $[hW,hX].g(JY,JZ) = g(\overset{*}{D}_{[hW,hX]}JY,JZ) + g(JY,\overset{*}{D}_{[hW,hX]}JZ)$. Comparing the above two equations, we get

$$\begin{aligned} &g((\overset{*}{D}_{hW}\overset{*}{D}_{hX} - \overset{*}{D}_{hX}\overset{*}{D}_{hW} - \overset{*}{D}_{[hW,hX]})JY,JZ) \\ &\qquad + g(JY,(\overset{*}{D}_{hW}\overset{*}{D}_{hX} - \overset{*}{D}_{hX}\overset{*}{D}_{hW} - \overset{*}{D}_{[hW,hX]})JZ) = 0. \end{aligned}$$

From which,

$$\overset{*}{R}(W,X,Y,Z) = -\overset{*}{R}(W,X,Z,Y).$$

(d) Using (a), (b) and (c), since $\mathfrak{R} = 0$, we have

$$\overset{*}{R}(X,Y,Z,W) = -\overset{*}{R}(Y,X,Z,W) = \overset{*}{R}(X,Z,Y,W) + \overset{*}{R}(Z,Y,X,W),$$

$$\overset{*}{R}(X,Y,Z,W) = -\overset{*}{R}(X,Y,W,Z) = \overset{*}{R}(Y,W,X,Z) + \overset{*}{R}(W,X,Y,Z).$$

Adding the above two equation, we get

$$\begin{aligned} 2\overset{*}{R}(X,Y,Z,W) &= \overset{*}{R}(X,Z,Y,W) + \overset{*}{R}(Z,Y,X,W) \\ &\quad + \overset{*}{R}(Y,W,X,Z) + \overset{*}{R}(W,X,Y,Z). \end{aligned} \tag{3.12}$$

Similarly,

$$2\overset{*}{R}(Z,W,X,Y) = \overset{*}{R}(Z,X,W,Y) + \overset{*}{R}(X,W,Z,Y) + \overset{*}{R}(W,Y,Z,X) + \overset{*}{R}(Y,Z,W,X). \tag{3.13}$$

Comparing (3.12) and (3.13), we get $\overset{*}{R}(X,Y,Z,W) = \overset{*}{R}(Z,W,X,Y)$. □

Remark 3.16. We noted that if $\mathfrak{R}$ vanishes, we get some interesting results:

- $\overset{*}{R} = R$ (Proposition 3.10 (a)) and also $\mathfrak{S}_{X,Y,Z}\{R(X,Y)Z\} = 0$.
- $\mathfrak{S}_{X,Y,Z}\{(\overset{*}{D}_{hX}\overset{*}{R})(Y,Z)\} = 0$ (Proposition 3.13 (e)).
- The properties (c) and (d) in Proposition 3.15 hold.

The above properties are very similar to the properties of the Riemannian curvature. The reason lies in the condition $\mathfrak{R} = 0$ which is stronger than the condition (b) of Theorem 3.6. More precisely, the condition $\overset{*}{T}(hX,hY) = \Omega(X,Y) = 0$ is stronger than the condition $J\overset{*}{T}(hX,hY) = 0$.

Table 1: Intrinsic Comparison

The following table gives a concise comparison concerning Berwald, Cartan and Chern connections and the fundamental associated geometric objects.

Table 1: Intrinsic Comparison

Connection	Berwald: $\overset{\circ}{D}$	Cartan: D	Chern: $\overset{*}{D}$
Expression	$\overset{\circ}{D}_{JX}JY = J[JX,Y]$ $\overset{\circ}{D}_{hX}JY = v[hX,JY]$ $\overset{\circ}{D}F = 0$	$D_{JX}JY = \overset{\circ}{D}_{JX}JY + \mathcal{C}(X,Y)$ $D_{hX}JY = \overset{\circ}{D}_{hX}JY + \mathcal{C}'(X,Y)$ $DF = 0$	$\overset{*}{D}_{JX}JY = \overset{\circ}{D}_{JX}JY$ $\overset{*}{D}_{hX}JY = D_{hX}JY$ $\overset{*}{D}F = 0$
h-torsion hv-torsion	$\mathfrak{R}$ 0	$\mathfrak{R}$ $\mathcal{C}' - F\mathcal{C}$	$\mathfrak{R}$ $\mathcal{C}'$
h-curvature	$\overset{\circ}{R}(X,Y)Z = (\overset{\circ}{D}_{JZ}\mathfrak{R})(X,Y)$	$R(X,Y)Z = \overset{\circ}{R}(X,Y)Z + (D_{hX}\mathcal{C}')(Y,Z)$ $-(D_{hY}\mathcal{C}')(X,Z) + \mathcal{C}'(F\mathcal{C}'(X,Z),Y)$ $-\mathcal{C}'(F\mathcal{C}'(Y,Z),X) + \mathcal{C}(F\mathfrak{R}(X,Y),Z)$	$\overset{*}{R}(X,Y)Z = R(X,Y)Z$ $-\mathcal{C}(F\mathfrak{R}(X,Y),Z)$
hv-curvature	$\overset{\circ}{P}(X,Y)Z = v[hX,J[JY,Z]]$ $-J[JY,F[hX,JZ]] - v[h[hX,JY],JZ]$ $-J[v[hX,JY],Z]$	$P(X,Y)Z = \overset{\circ}{P}(X,Y)Z + (D_{hX}\mathcal{C})(Y,Z)$ $-(D_{JY}\mathcal{C}')(X,Z) + \mathcal{C}(F\mathcal{C}'(X,Z),Y)$ $+\mathcal{C}(F\mathcal{C}'(X,Y),Z) - \mathcal{C}'(F\mathcal{C}(Y,Z),X)$ $-\mathcal{C}'(F\mathcal{C}(X,Y),Z)$	$\overset{*}{P}(X,Y)Z = \overset{\circ}{P}(X,Y)Z$ $-(\overset{*}{D}_{JY}\mathcal{C}')(X,Z)$
v-curvature	0	$Q(X,Y)Z = \mathcal{C}(F\mathcal{C}(X,Z),Y)$ $-\mathcal{C}(F\mathcal{C}(Y,Z),X)$	0
v-metricity h-metricity	not v-metrical not h-metrical	v-metrical h-metrical	not v-metrical h-metrical

3.3 Nullity distribution of the Chern h-curvature

It is be noted first that the nullity distributions of the Barthel, Berwald and Cartan connections have already been studied in [51, 53, 64], respectively. In this section, we investigate the nullity distribution of the Chern connection. First, we study the nullity distribution of the h-curvature tensor.

Definition 3.17. Let $\overset{*}{R}$ be the h-curvature tensor of the Chern connection. The nullity space of $\overset{*}{R}$ at a point $z \in TM$ is the subspace of $H_z(TM)$ defined by

$$\mathcal{N}_{R^*}(z) := \{v \in H_z(TM) | \overset{*}{R}_z(v, w) = 0, \text{ for all } w \in H_z(TM)\}.$$

The dimension of $\mathcal{N}_{R^*}(z)$, denoted by $\mu_{R^*}(z)$, is the nullity index of $\overset{*}{R}$ at z. If the nullity index μ_{R^*} is constant, then the map $\mathcal{N}_{R^*} : z \mapsto \mathcal{N}_{R^*}(z)$ defines a distribution $\mathcal{N}_{R^*}$ of rank μ_{R^*}, called the nullity distribution of $\overset{*}{R}$. We shall assume that $\mu_{R^*} \neq 0$ and $\mu_{R^*} \neq n$.

Let $\mathcal{N}_{R^*}(x) := \pi_*(\mathcal{N}_{R^*}(z))$ if $\pi(z) = x$. Then $\mathcal{N}_{R^*}(x)$ isomorphic to $\mathcal{N}_{R^*}(z)$ via the isomorphism $\pi_* \upharpoonright_{H_z(TM)}$.

Definition 3.18. The kernel of $\overset{*}{R}$ at the point $z \in TM$ is defined by

$$\ker_{R^*}(z) := \{u \in H_z(TM) | \overset{*}{R}_z(v, w)u = 0, \text{ for all } v, w \in H_z(TM)\}.$$

We have $\ker_{R^*}(x) = \pi_*(\ker_{R^*}(z))$; $x = \pi(z)$.

Proposition 3.19. *The nullity distribution $\mathcal{N}_{R^*}$ has the following properties:*

(a) $\mathcal{N}_{R^*} \neq \phi$ *and* $\ker_{R^*} \neq \phi$.

(b) $\mathcal{N}_{R^*} \subseteq \mathcal{N}_{\mathfrak{R}}$, *where $\mathcal{N}_{\mathfrak{R}}$ is the nullity distribution of the curvature $\mathfrak{R}$ of the Barthel connection.*

(c) $\mathcal{N}_{R^*} \subseteq \ker_{R^*}$.

(d) *If the canonical spray S belongs to $Sec(\mathcal{N}_{R^*})$, then $\mathfrak{R} = 0$.*

(e) *If $X \in Sec(\mathcal{N}_{R^*})$, then $[C, X] \in Sec(\mathcal{N}_{R^*})$ and, consequently, $[C, X] \in Sec(\mathcal{N}_{\mathfrak{R}})$.*

Proof. (b) Let X be a nullity vector field. Using Lemma 3.11 (a), we have

$$\begin{aligned} X \in Sec(\mathcal{N}_{R^*}) &\Longrightarrow \overset{*}{R}(X,Y)Z = 0 \quad \text{for all } Y, Z \in \mathfrak{X}(TM) \\ &\Longrightarrow \overset{*}{R}(X,Y)S = 0 \quad \text{for all } Y \in \mathfrak{X}(TM) \\ &\Longrightarrow \mathfrak{R}(X,Y) = 0 \quad \text{for all } Y \in \mathfrak{X}(TM) \\ &\Longrightarrow X \in Sec(\mathcal{N}_{\mathfrak{R}}). \end{aligned}$$

(c) Let $Z \in Sec(\mathcal{N}_{R^*})$, then, by Proposition 3.15 (b), we have $\mathfrak{S}_{X,Y,Z}\{\overset{*}{R}(X,Y)Z\} = 0$. Since $\overset{*}{R}(Y,Z)X = \overset{*}{R}(Z,X)Y = 0$, then the result follows.

(d) This is an immediate consequence of Lemma 3.11 (a) and (c) above.

(e) Let $X \in Sec(\mathcal{N}_{R^*})$. Since $\overset{*}{D}_C\overset{*}{R} = 0$ [63] , we get $(\overset{*}{D}_C\overset{*}{R})(X,Y) = 0$, which leads to $\overset{*}{R}(\overset{*}{D}_C X, Y) = 0$. Using Corollary 3.7, we have $\overset{*}{R}([C,X],Y) = 0$. By the homogeneity of h, $[C,hX] = h[C,X]$. That is, $[C,hX]$ is horizontal. Hence, $[C,X] \in Sec(\mathcal{N}_{R^*})$. Consequently, by (b), $[C,X] \in Sec(\mathcal{N}_{\mathfrak{R}})$. $\square$

It is important to note that the reverse inclusion in the property (c) of Proposition 3.19 is not true; that is, $\ker_{R^*} \not\subset \mathcal{N}_{R^*}$. This is shown by the next example (see B.1).

Example 3.20. Let $M = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 | x^2 > 0\}$ and
$U = \{(x^1, ..., x^4; y^1, ..., y^4) \in \mathbb{R}^4 \times \mathbb{R}^4 : \ y^1 \neq 0, y^2 \neq 0\} \subset TM$. Define L on U by

$$L(x,y) := \sqrt[4]{(x^2)^2(y^1)^4 + (y^2)^4 + (y^3)^4 + (y^4)^4}.$$

The nullity distribution of the Cartan h-curvature R of (M,F) is

$$\mathcal{N}_R = \{s\delta_3 + t\delta_4 \in \mathfrak{X}^h(\mathcal{T}M) |\, s,t \in \mathbb{R}\} \tag{3.14}$$

and the kernel distribution $\ker_R$ of R is

$$\begin{aligned} \ker_R = \Bigg\{ s' \left(\frac{y^1}{y^2}\delta_1 + \delta_2 + \frac{x^2 (y^1)^4 + (y^2)^4 + 2(y^3)^4 + 2(y^4)^4}{y^2 (y^4)^3}\delta_4 \right) \\ + t' \left(\delta_3 - \frac{(y^3)^3}{(y^4)^3}\delta_4 \right) \in \mathfrak{X}^h(\mathcal{T}M) |\, s', t' \in \mathbb{R} \Bigg\}. \end{aligned} \tag{3.15}$$

The nullity distribution $\mathcal{N}_{R^*}$ is

$$\mathcal{N}_{R^*} = \{s\delta_3 + t\delta_4 | \, s, t \in \mathbb{R}\}. \tag{3.16}$$

The kernel distribution $\ker_{R^*}$ is

$$\ker_{R^*} = \left\{ r\left(\frac{2y1}{y2}\delta_1 + \delta_2\right) + s\delta_3 + t\delta_4 | \, r, s, t \in \mathbb{R} \right\}. \tag{3.17}$$

Equations (3.16) and (3.17) show that $\ker_{R^*}$ can not be a sub-distribution of $\mathcal{N}_{R^*}$. □

Theorem 3.21. *The nullity distribution $\mathcal{N}_{R^*}$ of the Chern h-curvature and the nullity distribution $\mathcal{N}_R$ of the Cartan h-curvature coincide.*

Proof. Let $X \in Sec(\mathcal{N}_{R^*})$. Then, by Proposition 3.10 (a) and Proposition 3.19 (2), $X \in Sec(\mathcal{N}_R)$. Hence $\mathcal{N}_{R^*}$ is a subset of $\mathcal{N}_R$. Conversely, let $X \in Sec(\mathcal{N}_R)$. Then, by Proposition 3.10 (a) and $\mathcal{N}_R \subset \mathcal{N}_{\mathfrak{R}}$ (Proposition 2.2 (b)), we get $X \in Sec(\mathcal{N}_{R^*})$, whence, $\mathcal{N}_R \subset \mathcal{N}_{R^*}$. □

Remark 3.22. The above example shows that $\mathcal{N}_{R^*} \subset \ker_{R^*}$ and the reverse inclusion is false by (3.16) and (3.17). It also shows that, although $\mathcal{N}_{R^*} = \mathcal{N}_R$ (see (3.14) and (3.16)), $\ker_{R^*} \neq \ker_R$ by (3.15), (3.17). In view of the above theorem, the reverse inclusion in (b) of Proposition 3.19 is not true either: $\mathcal{N}_{\mathfrak{R}} \not\subset \mathcal{N}_R = \mathcal{N}_{R^*}$.

Definition 3.23. The conullity space of the h-curvature tensor at z, denoted by ${\mathcal{N}_{R^*}}^{\perp}(z)$, is the orthogonal complement of $\mathcal{N}_{R^*}$ in $H_z(TM)$, where the orthogonality is taken with respect to the metric g defined by (1.9).

Proposition 3.24. *For each point $z \in TM$, either $\mu_{R^*}(z) = n$ or $\mu_{R^*}(z) \leq n-2$. Consequently,* $\dim \ker_{R^*} > n-2$.

Proof. If $\mu_{R^*}(z) \neq n$, then there is a non-zero horizontal vector $v \notin \mathcal{N}_{R^*}(z)$. It follows that there is a vector $w \in H_z(TM)$ such that $\overset{*}{R}_z(v,w) \neq 0$ and so $\overset{*}{R}_z(w,v) \neq 0$. Then $v, w \notin \mathcal{N}_{R^*}(z)$ and hence $v, w \in {\mathcal{N}_{R^*}}^{\perp}(z)$. By the antisymmetry of $\overset{*}{R}$, the vectors v and w are independent. Thus, $\dim {\mathcal{N}_{R^*}}^{\perp}(z) \geq 2$. Consequently, $\mu_{R^*}(z) \leq n-2$. □

Proposition 3.25. *If* $\mathfrak{R} = 0$, *then* $\mathrm{Im}\,(\overset{*}{R}) = (J\mathcal{N}_{R^*})^{\perp}$. Consequently, $\mathrm{rank}\,(\overset{*}{R}) = n - \mu_{R^*}$.

Proof. For all $X \in Sec(\mathcal{N}_{R^*})$ and $Y, Z, W \in \mathfrak{X}^h(\mathcal{T}M)$, we have

$$\begin{aligned} g(\overset{*}{R}(Y,Z)W, JX) =& \overset{*}{R}(Y,Z,W,X) \\ =& \overset{*}{R}(W,X,Y,Z) \ \text{(by Lemma 3.13 (e))} \\ =& -\overset{*}{R}(X,W,Y,Z) \\ =& -g(\overset{*}{R}(X,W)Y, JZ) \\ =& 0 \ \text{(since } X \text{ is a nullity vector field)}, \end{aligned}$$

as wanted. □

As a direct consequence of Theorem 3.21 and the fact that $\mathcal{N}_R$ is completely integrable (Theorem 2.7), we have the following result.

Corollary 3.26. *Let μ_{R^*} be constant on an open subset U of TM. The nullity distribution $z \mapsto \mathcal{N}_{R^*}(z)$ is completely integrable on U.*

Theorem 3.27. *If $\mathfrak{R} = 0$, then the two distributions $\mathcal{N}_{R^*}$ and $\ker_{R^*}$ coincide.*

Proof. By Proposition 3.19 (c), we always have $\mathcal{N}_{R^*} \subset \ker_{R^*}$. Let $X \in \mathfrak{X}(\ker_{R^*})$ and let Y, Z, W be vector fields on TM, then by Proposition 3.15 (d), we have

$$\begin{aligned} \overset{*}{R}(Y,Z)X = 0 \Longrightarrow & \ g(\overset{*}{R}(Y,Z)X, JW) = 0 \\ \Longrightarrow & \overset{*}{R}(Y,Z,X,W) = 0 \\ \Longrightarrow & \overset{*}{R}(X,W,Y,Z) = 0 \\ \Longrightarrow & \ g(\overset{*}{R}(X,W)Y, JZ) = 0 \\ \Longrightarrow & \overset{*}{R}(X,W)Y = 0 \\ \Longrightarrow & \ X \in Sec(\mathcal{N}_{R^*}). \end{aligned}$$

Hence, $\ker_{R^*} \subset \mathcal{N}_{R^*}$. □

We have seen that if the index of nullity μ_{R^*} is constant, then the nullity distribution $\mathcal{N}_{R^*}$ is completely integrable. According to the Frobenius theorem, there exists a foliation of M by μ_{R^*}-dimensional maximal connected submanifolds as leaves, such that the nullity space at a point $x \in M$ is the tangent space to the leaf at x. We call the foliation induced by the nullity distribution $\mathcal{N}_{R^*}$ the nullity foliation and denote it again by $\mathcal{N}_{R^*}$.

Theorem 3.28. *Let (M, L) be a complete Finsler manifold and U the open subset of M on which μ_{R^*} takes its minimum. If $\mathfrak{R}$ vanishes, then every totally geodesic integral manifold of the nullity foliation $\mathcal{N}_{R^*}$ in U is complete.*

Proof. The proof is inspired by [3], taking into account the fact that the two spaces $\mathcal{N}_{R^*}(z)$ and $\mathcal{N}_{R^*}(x)$, $x = \pi(z)$, are isomorphic. Let N be an integral manifold of the nullity foliation $\mathcal{N}_{R^*}$ in U. To prove that N is complete, it suffices to show that every geodesic $\gamma : [0, c) \to N$ on N can be extended to a geodesic $\widetilde{\gamma} : [0, \infty) \to N$ on N. Suppose that such a geodesic extension $\widetilde{\gamma}$ does not exist. As N is totally geodesic, γ is a geodesic on M and thus has a geodesic extension $\widetilde{\gamma} : [0, \infty) \to M$ such that $\gamma = \widetilde{\gamma} \cap N$. It follows that $p := \widetilde{\gamma}(c) \notin U$. Let $p_0 := \gamma(0) = \widetilde{\gamma}(0)$ and set $r_0 := \mu_{R^*}(p_0)$, the dimension of the nullity space $\mathcal{N}_{R^*}(p_0)$. Since μ_{R^*} is positive and minimal on U, then $\mu_{R^*}(p) > r_0 > 0$. Now, consider a basis $B = \{e_1, ..., e_{r_0}, e_{r_0+1}, ..., e_n\}$ for $T_{p_0}M$ such that $\{e_1, ..., e_{r_0}\}$ is a basis for $\mathcal{N}_{R^*}(p_0)$ and e_1 is tangent to γ at $p_0 = \gamma(0)$. Using the system of differential equations

$$\frac{\overset{*}{D}F_i}{dt} = 0, \quad F_i(0) = e_i, \quad i = 1, 2, ..., n,$$

the basis B can be translated into a parallel frame $(F_1, ..., F_{r_0}, F_{r_0+1}, ..., F_n)$ along $\widetilde{\gamma}$. Then $(F_1, ..., F_{r_0})$ is a basis for the nullity space at every point $\widetilde{\gamma}(t)$ in $U \cap V$ for some neighborhood V of $\widetilde{\gamma}(t)$ on M. Since $\mu_{R^*}(p) > r_0$, there is a vector field F_a along $\widetilde{\gamma}$, for a fixed integer a in the range $r_0 + 1, ..., n$, such that for every $t \in [0, c)$, we have

$$F_a(\gamma(t)) \notin \mathcal{N}_{R^*}(\gamma(t)), \quad F_a(p) \in \mathcal{N}_{R^*}(p). \tag{3.18}$$

Now, let $\widehat{\widetilde{\gamma}}$ be the natural lift of $\widetilde{\gamma}$ to $\mathcal{T}M$ and $\{\widehat{F}_1, ..., \widehat{F}_{r_0}, \widehat{F}_{r_0+1}, ..., \widehat{F}_n\}$ the basis of $H_{\widehat{\widetilde{\gamma}}(t)}TM$ such that $\pi_*(\widehat{F}_i) = F_i$. Let ϕ^h_{ijk} be the functions defined by

$$\overset{*}{R}(\widehat{F}_i, \widehat{F}_j)\widehat{F}_k = \phi^h_{ijk} \frac{\partial}{\partial y^h}. \tag{3.19}$$

By Proposition 3.13 (e), taking into account that $\mathfrak{R} = 0$, we have

$$(\overset{*}{D}_{hX}\overset{*}{R})(Y, Z) + (\overset{*}{D}_{hY}\overset{*}{R})(Z, X) + (\overset{*}{D}_{hZ}\overset{*}{R})(X, Y) = 0.$$

Plugging $\widehat{F}_1$, $\widehat{F}_i$ and $\widehat{F}_j$ instead of X, Y and Z, where $i, j = r_0 + 1, ..., n$, we get

$$(\overset{*}{D}_{\widehat{F}_1}\overset{*}{R})(\widehat{F}_i, \widehat{F}_j) + (\overset{*}{D}_{\widehat{F}_i}\overset{*}{R})(\widehat{F}_j, \widehat{F}_1) + (\overset{*}{D}_{\widehat{F}_j}\overset{*}{R})(\widehat{F}_1, \widehat{F}_i) = 0.$$

Since $\widehat{F}_1 \in \mathcal{N}_{R^*}$ and $\overset{*}{T}(hX, hY) = \mathfrak{R}(X, Y) = 0$, the last equality takes the form

$$\overset{*}{D}_{\widehat{F}_1}\overset{*}{R}(\widehat{F}_i, \widehat{F}_j) + \overset{*}{R}(\widehat{F}_j, [\widehat{F}_1, \widehat{F}_i]) + \overset{*}{R}(\widehat{F}_i, [\widehat{F}_j, \widehat{F}_1]) = 0.$$

Applying the above equation on $\widehat{F}_a$, we get

$$\overset{*}{D}_{\widehat{F}_1}\overset{*}{R}(\widehat{F}_i, \widehat{F}_j)\widehat{F}_a + \overset{*}{R}(\widehat{F}_j, [\widehat{F}_1, \widehat{F}_i])\widehat{F}_a + \overset{*}{R}(\widehat{F}_i, [\widehat{F}_j, \widehat{F}_1])\widehat{F}_a = 0. \tag{3.20}$$

Since, $[\widehat{F}_1, \widehat{F}_i]$ is horizontal, it can be written in the form $[\widehat{F}_1, \widehat{F}_i] = \xi^k_{1i}\widehat{F}_k + \xi^\mu_{1i}\widehat{F}_\mu$, where $k = r_0 + 1, ..., n$ and $\mu = 1, ..., r_0$. Consequently, by (3.19) and (3.20), noting that $\widehat{F}_\mu$ are null vector fields, we get

$$(\phi^h_{ija})' + \xi^k_{1i}\,\phi^h_{jka} - \xi^k_{1j}\,\phi^h_{ika} = 0 \tag{3.21}$$

Since F_a is a nullity vector field at p, then for the fixed index a, $\phi^h_{lma}(p) = 0$, where $l, m = r_0 + 1, ..., n$. Hence, the differential equations (3.21) with the initial condition $\phi^h_{lma}(p) = 0$ imply that the functions ϕ^h_{lma} vanish identically. As $\mathfrak{R} = 0$, Theorem 3.27 and (3.19) give rise to

$$F_a(\gamma(t)) \in \mathcal{N}_{R^*}(\gamma(t)), \text{ for all } t \in [0, c] \tag{3.22}$$

Now (3.18) and (3.22) lead to a contradiction. Consequently, γ can be extended to a geodesic $\widetilde{\gamma} : [0, \infty) \longrightarrow N$. $\square$

3.4 Nullity distribution of the Chern hv-curvature

In this section we investigate the nullity distribution of the hv-curvature $\overset{*}{P}$ of the Chern connection. We show, by a counterexample, that the nullity distribution $\mathcal{N}_{P^*}$ is not completely integrable. We find a sufficient condition for $\mathcal{N}_{P^*}$ to be completely integrable.

Definition 3.29. Let $\overset{*}{P}$ be the hv-curvature of the Chern connection. The nullity space of $\overset{*}{P}$ at a point $z \in TM$ is a subspace of $H_z(TM)$ defined by

$$\mathcal{N}_{P^*}(z) := \{v \in H_z(TM) | \overset{*}{P}_z(v, w) = 0, \text{ for all } w \in H_z(TM)\}.$$

The dimension of $\mathcal{N}_{P^*}(z)$, denoted by $\mu_{P^*}(z)$, is the nullity index of $\overset{*}{P}$ at z.

Proposition 3.30. *The nullity distribution of* $\overset{*}{P}$ *has the properties:*

(a) $\mathcal{N}_{P^*} \neq \phi$.

(b) *If* $X \in Sec(\mathcal{N}_{P^*})$, *then* $[C, X] \in Sec(\mathcal{N}_{P^*})$.

(c) *If* $X \in Sec(\mathcal{N}_{P^*})$, *then* $\mathcal{C}'(X, Y) = 0,$ *for all* $Y \in \mathfrak{X}^h(\mathcal{T}M)$.

(d) *If* $\mu_{P^*} = n$, *then* $\mathcal{N}_{R^*} = \mathcal{N}_{R^\circ}$,

where $\mathcal{N}_{R^\circ}$ *is the nullity distribution of the h-curvature of the Berwald connection [53].*

If the nullity index μ_{P^*} takes its maximum, then by Proposition 3.30 (c), $\mathcal{C}' = 0$. Consequently, in view of Definition 2.4, a Finsler manifold (M, L) is Landsbergian if the nullity index μ_{P^*} achieves its maximum.

Theorem 3.31. *A Finsler manifold* (M, L) *is Landsbergian if and only if the canonical spray* S *belongs to the nullity distribution* $\mathcal{N}_{P^*}$.

Proof. We use Lemma 3.11 (b):

$$\begin{aligned}
(M, L) \text{ is Landsbergian} &\Longleftrightarrow \mathcal{C}' = 0 \\
&\Longleftrightarrow \overset{*}{P}(S, Y)X = 0 \ \text{ for all } X, Y \in \mathfrak{X}(TM) \\
&\Longleftrightarrow S \in Sec(\mathcal{N}_{P^*}),
\end{aligned}$$

as was to be shown. □

Remark 3.32. The above theorem shows that the canonical spray S does not belong to the nullity distribution $\mathcal{N}_{P^*}$ except in the Landsbergian case. This is in contrast to the case of Cartan connection, where the canonical spray always belongs to the nullity distribution of the Cartan hv-curvature P.

The nullity distribution $\mathcal{N}_{P^*}$ is not completely integrable in general, as is illustrated by the following example (see B.7).

Example 3.33. Let $U = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : \ y^1, y^2, y^3 \neq 0, y^3 \neq 4y^2\} \subset TM$, where $M := \mathbb{R}^3$. Define L on U by

$$L(x, y) := \sqrt[4]{e^{-x^1 x^2}(y^1)^2 (y^3)^2 e^{-\frac{y^3}{y^2}}}.$$

Let $W \in Sec(\mathcal{N}_{P^*})$, then $W^j P^{*h}_{ijk} = 0$ leads to a system of algebraic equations. This system has a solution if $y^3 = 2y^2$ and $x^1 > 0$: $W^1 = s$, $W^2 = t$, $W^3 = 2t$; $s, t \in \mathbb{R}$. Hence, a $\overset{*}{P}$-nullity vector must have the form $W = s\delta_1 + t(\delta_2 + 2\delta_3)$. Consequently, the nullity index $\mu_{P^*} = 2$. Now, take $X, Y \in Sec(\mathcal{N}_{P^*})$ such that $X = \delta_1$, $Y = \delta_2 + 2\delta_3$. Taking into account that $y^3 = 2y^2$, then $\delta_1 = \frac{\partial}{\partial x^1} + \frac{x^2 y^1}{2}\frac{\partial}{\partial y^1}$, $\delta_2 = \frac{\partial}{\partial x^2} + \frac{x^1 y^2}{2}\frac{\partial}{\partial y^2}$ and $\delta_3 = \frac{\partial}{\partial x^3} + \frac{x^1 y^2}{2}\frac{\partial}{\partial y^3}$. Hence, the bracket $[\delta_1, \delta_2 + 2\delta_3] = -\frac{y^1}{2}\frac{\partial}{\partial y^1} + \frac{y^2}{2}\frac{\partial}{\partial y^2} + \frac{y^2}{2}\frac{\partial}{\partial y^3}$ is vertical and, consequently, $\mathcal{N}_{P^*}$ is not completely integrable. □

Theorem 3.34. *Let μ_{P^*} be constant on an open subset U of TM. The nullity distribution $\mathcal{N}_{P^*}$ is completely integrable on U if and only if $\mathfrak{R}(X, Y) = 0$ and $(\overset{*}{D}_{JZ}\overset{*}{R})(X, Y) = 0$, for all $X, Y \in Sec(\mathcal{N}_{P^*})$.*

Proof. Necessity. Let $\mathcal{N}_{P^*}$ be completely integrable. Then, if $X, Y \in Sec(\mathcal{N}_{P^*})$, the bracket $[hX, hY]$ is horizontal, thus, $\mathfrak{R}(X, Y) = 0$. Also, by Lemma 3.13 (f) and the fact that $\overset{*}{P}([hX, hY], Z) = (\overset{*}{D}_{hX}\overset{*}{P})(Y, Z) - (\overset{*}{D}_{hY}\overset{*}{P})(X, Z) = 0$ (Lemma 3.8), we have $(\overset{*}{D}_{JZ}\overset{*}{R})(X, Y) = 0$, for all $X, Y \in Sec(\mathcal{N}_{P^*})$.

Sufficiency. Let $\mathfrak{R}(X, Y) = 0$ and $(\overset{*}{D}_{JZ}\overset{*}{R})(X, Y) = 0$ for all $X, Y \in Sec(\mathcal{N}_{P^*})$. As $0 = \mathfrak{R}(X, Y) = -v[hX, hY] = -v[X, Y]$, the bracket $[X, Y]$ is horizontal. Making use of Lemma 3.13 (f), Lemma 3.8 and Proposition 3.9, we get

$$\begin{aligned}(\overset{*}{D}_{hX}\overset{*}{P})(Y, Z) - (\overset{*}{D}_{hY}\overset{*}{P})(X, Z) = 0 &\Longrightarrow \overset{*}{P}(\overset{*}{D}_X Y - \overset{*}{D}_Y X, Z) = 0 \\ &\Longrightarrow \overset{*}{P}([X, Y] + \mathfrak{R}(X, Y), Z) = 0 \\ &\Longrightarrow \overset{*}{P}([X, Y], Z) = 0 \\ &\Longrightarrow [X, Y] \in Sec(\mathcal{N}_{P^*}).\end{aligned}$$

Hence $\mathcal{N}_{P^*}$ is completely integrable. □

By the property $\overset{*}{P}(X, Y)Z = \overset{*}{P}(Z, Y)X$ we have the following result.

Theorem 3.35. *The nullity distribution $\mathcal{N}_{P^*}$ and the kernel distribution $\ker_{P^*}$ coincide.*

A Finsler manifold for which the Chern hv-curvature tensor $\overset{*}{P}$ vanishes is called a Berwald space [47]. It is well known that every Berwald space is a Landsberg space, but it is not known whether the converse is true. In [42], Shen introduced a class of non-regular Finsler metrics which is Landsbergian and not Berwaldian. The calculations are not easy, especially, if one wants to study some concrete examples. Here, by using Maple program together with the results of [42] and [60], we give a simple class (see B.8) of proper non-regular non Berwaldian Landsbergian spaces.

Example 3.36. Let $M = \mathbb{R}^3$, $U = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : \, y^2 > 0, y^3 > 0\} \subset TM$. Define L on U by

$$L(x, y) := f(x^1)\sqrt{(y^1)^2 + y^2y^3 + y^1\sqrt{y^2y^3}}\, e^{\frac{1}{\sqrt{3}}\arctan\left(\frac{2y^1}{\sqrt{3y^2y^3}}+\frac{1}{\sqrt{3}}\right)}.$$

By the above example, for any non constant positive smooth function f on $\mathbb{R}$, the Landsberg tensor of (M, L) vanishes (or equivalently, the hv-curvature P of the Cartan connection vanishes) and hence the class is Landsbergian. On the other hand, the hv-curvature $\overset{*}{P}$ of the Chern connection does not vanish and hence the class is not Berwaldian. So we can confirm:

Theorem 3.37. *There are non-regular Landsberg spaces which are not Berwaldian.*

Chapter 4

Nullity distribution in the pull-back approach

In this chapter, following the pullback formalism, we show by a counterexample that $\ker_{\mathcal{R}}$ and $\mathcal{N}_{\mathcal{R}}$ ($\mathcal{R}$ is the h-curvature of Cartan connection on the pullback bundle) do not coincide, contrary to Akbar-Zadeh's result. In addition, we find sufficient conditions for $\ker_{\mathcal{R}}$ and $\mathcal{N}_{\mathcal{R}}$ to coincide.

The results of this chapter are published [59].

4.1 Regular conections on the pullback bundle

In this section, we present a brief review of the geometry of the pullback bundle $\pi^{-1}(TM)$ in relation with regular connections. For more detail concerning the pullback (PB-) formalism, we refer the reader, for example, to [5, 44, 56].

In what follows, we denote by $\pi : \mathcal{T}M \longrightarrow M$ the subbundle of nonzero vectors tangent to M, $\pi_* : T(\mathcal{T}M) \longrightarrow TM$ the linear tangent map of π and $V_z(TM) = (\ker\, \pi_*)_z$ the vertical space at $z \in \mathcal{T}M$. Let $\mathfrak{X}(\pi(M))$ be the $C^\infty(TM)$-module of differentiable sections of the pullback bundle $\pi^{-1}(TM)$. The elements of $\mathfrak{X}(\pi(M))$ will be called π-vector fields and denoted by barred letters $\overline{X}$.

Definition 4.1. Let ∇ be a linear connection on the pullback bundle

$\pi^{-1}(TM)$. We associate with ∇ the map

$$K : TTM \longrightarrow \pi^{-1}(TM) : X \longmapsto \nabla_X \overline{\eta},$$

called the connection map or the deflection map of ∇.
A tangent vector $X \in T_z TM$ is horizontal if $K(X) = 0$.

We have

$H_z(TM) = \{X \in T_z TM : K(X) = 0\}$: the horizontal space of M at z,
$H(TM) = \bigcup_{z \in TM} H_z(TM)$: the horizontal subbundle associated with ∇.

Definition 4.2. *[48]* A linear connection ∇ on $\pi^{-1}(TM)$ is said to be regular if the double tangent bundle TTM has the direct sum decomposition

$$TTM = V(TM) \oplus H(TM). \tag{4.1}$$

For all $z \in \mathcal{T}M$, $H_z(\mathcal{T}M)$ is isomorphic to $T_{\pi z}M$ via the restriction of the tangent map $\pi_* \restriction_{H_z(\mathcal{T}M)}$ on the horizontal space. Moreover, the fibers of the tangent bundle are isomorphic to the fibers of the pullback bundle. Hence, for all $X \in \mathfrak{X}(\mathcal{T}M)$, there exists $\overline{X} \in \mathfrak{X}(\pi(M))$ such that $\pi_* X = \overline{X}$.

The ((h)hv-) torsion tensor of ∇, denoted by T, is defined by $T(\overline{X}, \overline{Y}) = \mathbf{T}(vX, hY)$, for all $\overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M))$, where

$$\mathbf{T}(X,Y) = \nabla_X \pi_* Y - \nabla_Y \pi_* X - \pi_*[X,Y]$$

is the (classical) torsion associated with ∇.

The h-curvature tensor of ∇, denoted by $\mathcal{R}$, is defined by $\mathcal{R}(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(hX, hY)\overline{Z}$, where

$$\mathbf{K}(X,Y)\pi_* Z = \nabla_X \nabla_Y \pi_* Z - \nabla_Y \nabla_X \pi_* Z - \nabla_{[X,Y]} \pi_* Z$$

is the (classical) curvature associated with ∇. The contracted curvature $\widehat{R}$ is defined by $\widehat{R}(\overline{X}, \overline{Y}) = \mathcal{R}(\overline{X}, \overline{Y})\overline{\eta}$.

The following proposition is useful for subsequent use.

Proposition 4.3. *Let (M, L) be a Finsler manifold. The map $\overline{g}$ defined by*

$$\overline{g}(\overline{X}, \overline{Y}) := \Omega(JX, Y), \; \forall\, X, Y \in \mathfrak{X}(TM), \tag{4.2}$$

is a metric on $\pi^{-1}(TM)$.
The map $\overline{g}$ is called the Finsler metric defined on $\pi^{-1}(TM)$ by $E = L^2/2$.

If M is endowed with a metric $\overline{g}$ on $\pi^{-1}(TM)$, we write

$$\mathcal{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) := \overline{g}(\mathcal{R}(\overline{X},\overline{Y})\overline{Z},\overline{W}). \tag{4.3}$$

Theorem 4.4. *Let (M, L) be a Finsler manifold and $\overline{g}$ the Finsler metric defined by L. There exists a unique regular connection ∇ on $\pi^{-1}(TM)$ such that*

(a) ∇ *is metric*: $\nabla\overline{g} = 0$,

(b) *The **(h)**h-torsion of* ∇ *vanishes*: $Q = 0$,

(c) *The (h)hv-torsion T of ∇ satisfies* $\overline{g}(T(\overline{X},\overline{Y}),\overline{Z}) = \overline{g}(T(\overline{X},\overline{Z}),\overline{Y})$.

Such a connection is called the Cartan connection associated with the Finsler manifold (M, L).

Lemma 4.5. *The h-curvature tensor $\mathcal{R}$ of the Cartan connection has the properties:*

(a) $\mathcal{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) = -\mathcal{R}(\overline{Y},\overline{X},\overline{Z},\overline{W})$,

(b) $\mathcal{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) = -\mathcal{R}(\overline{X},\overline{Y},\overline{W},\overline{Z})$,

(c) $\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\mathcal{R}(\overline{X},\overline{Y})\overline{Z} - T(\widehat{R}(\overline{X},\overline{Y}),\overline{Z})\} = 0$.

4.2 Nullity distributions in PB-formalism

Let (M, L) be a Finsler manifold. Let ∇ be the Cartan connection associated with (M, L). It is well known that ∇ is the unique metrical regular connection on $\pi^{-1}(TM)$ such that $\overline{g}(T(\overline{X},\overline{Y}),\overline{Z}) = \overline{g}(T(\overline{X},\overline{Z}),\overline{Y})$ [4, 57]. Note that the bracket $[X, Y]$ is horizontal if and only if $\widehat{R}(\overline{X},\overline{Y}) = 0$, where $\widehat{R}$ is the contracted curvature of the h-curvature tensor of ∇.

Let us now define the concepts of nullity and kernel spaces associated with the curvature **K** of ∇, following Akbar-Zadeh's definitions [2].

Definition 4.6. The subspace $\mathcal{N}_{\mathbf{K}}(z)$ of $H_z(TM)$ at a point $z \in TM$ is defined by

$$\mathcal{N}_{\mathbf{K}}(z) := \{u \in H_z(TM) : \mathbf{K}(u, v) = 0,\ \forall\, v \in H_z(TM)\}.$$

The dimension of $\mathcal{N}_{\mathbf{K}}(z)$ is denoted by $\mu_{\mathbf{K}}(z)$.

The subspace $\mathcal{N}_{\mathbf{K}}(x) = \pi_*(\mathcal{N}_{\mathbf{K}}(z)) \subset T_xM$, $x = \pi z$, is linearly isomorphic to $\mathcal{N}_{\mathbf{K}}(z)$.

Definition 4.7. The kernel of $\mathbf{K}$ at the point $x = \pi z$ is defined by

$$\ker_{\mathbf{K}}(x) := \{\overline{X}_z \in \{z\} \times T_x M \simeq T_x M : \mathbf{K}(u, v)\overline{X}_z = 0, \forall\, u, v \in H_z(TM)\}.$$

Akbar-Zadeh has obtained the following results.

Theorem 4.8. *The nullity spaces $\mathcal{N}_{\mathbf{K}}(x)$ and the kernel space $\ker_{\mathbf{K}}(x)$ coincide.*

4.3 Counterexample

Since $\mathcal{N}_{\mathbf{K}}$ and $\ker_{\mathbf{K}}$ are both subspaces of the horizontal space, we can replace the classical curvature $\mathbf{K}$ by the h-curvature tensor $\mathcal{R}$ of Cartan connection. Akbar-Zadeh [2] proved that the nullity space $\mathcal{N}_{\mathbf{K}}(x)$ and the kernel space $\ker_{\mathbf{K}}(x)$ coincide for each point $x \in M$ at which they are defined. We show by a counterexample that the above mentioned spaces do not coincide.

Theorem 4.9. *The nullity space $\mathcal{N}_{\mathcal{R}}(x)$ and the kernel space $\ker_{\mathcal{R}}(x)$ do not coincide.*

For the detailed calculations see (B.9).

Example 4.10. Let $M = \mathbb{R}^3$ and $U = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : y^i \neq 0; i = 1, 2, 3\} \subset TM$.
Let L be defined on U by:

$$L(x, y) = e^{-x^1 x^2}(y^1 y^2 y^3)^{1/3}.$$

The nullity space $\mathcal{N}_{\mathcal{R}}(x)$, by using $\pi_*(\delta_i) = \frac{\partial}{\partial x^i}$, is given by

$$\mathcal{N}_{\mathcal{R}}(x) = \left\{ t\frac{\partial}{\partial x^3} \,|\, t \in \mathbb{R} \right\}. \tag{4.4}$$

The kernel space $\ker_{\mathcal{R}}(x)$ is given by

$$\ker_{\mathcal{R}}(x) = \left\{ t'\Big(\frac{\partial}{\partial x^1} + \frac{y^2}{y^1}\frac{\partial}{\partial x^2} - \frac{2y^3}{y^1}\frac{\partial}{\partial x^3}\Big) \,|\, t' \in \mathbb{R} \right\}. \tag{4.5}$$

Comparing (4.4) and (4.5), we note that there is no values of t and t' for which $\mathcal{N}_{\mathcal{R}}(x) = \ker_{\mathcal{R}}(x)$. Consequently, $\mathcal{N}_{\mathcal{R}}(x)$ and $\ker_{\mathcal{R}}(x)$ can not coincide. $\square$

According to Akabr-Zadeh's proof, if $X \in Sec(\mathcal{N}_{\mathcal{R}})$, then, by Lemma 4.5, we have $\mathcal{R}(\overline{Y},\overline{Z})\overline{X} = \mathbf{T}(X,[Y,Z])$. But there is no guarantee for the vanishing of the right-hand side. Akabr-Zadeh claimed that by using Theorem 4.4 (c) and Lemma 4.5 (b), one can obtain that $\mathcal{R}(\overline{Y},\overline{Z})\overline{X} = 0$. It seems that the way to reach this goal is as follows:

$$\begin{aligned}\overline{g}(\mathcal{R}(\overline{Y},\overline{Z})\pi_*X,\pi_*W) &= \overline{g}(\mathbf{T}(X,[Y,Z]),\pi_*W)\\ &= \overline{g}(\mathbf{T}(W,[Y,Z]),\pi_*X) \qquad (4.6)\\ &= \overline{g}(\mathcal{R}(\overline{Y},\overline{Z})\pi_*W,\pi_*X) \qquad (4.7)\\ &= -\overline{g}(\mathcal{R}(\overline{Y},\overline{Z})\pi_*X,\pi_*W),\end{aligned}$$

$W \in \mathfrak{X}^h(TM)$. Since $\overline{g}(\mathcal{R}(\overline{Y},\overline{Z})\pi_*X,\pi_*W) = \overline{g}(\mathbf{T}(X,[Y,Z]),\pi_*W)$ is true only for $X \in Sec(\mathcal{N}_{\mathcal{R}})$, we can not move from (4.6) to (4.7). Consequently, we can not use the symmetry or skew-symmetry properties in X and W to conclude that $\overline{g}(\mathcal{R}(\overline{Y},\overline{Z})\overline{X},\overline{W}) = 0$. This can be assured, again, by the previous example: if we take $X = \delta_3 \in Sec(\mathcal{N}_{\mathcal{R}})$ and $Y = \delta_1, Z = \delta_2$, then the bracket $[Y,Z] = -3y^1\frac{\partial}{\partial y^1} + 3y^2\frac{\partial}{\partial y^2}$ is vertical and, moreover, $\mathbf{T}(\delta_3,[\delta_1,\delta_2]) = \frac{1}{y^3}(-y^1\bar{\partial}_1 + y^2\bar{\partial}_2) \neq 0$, where $\bar{\partial}_i$ is the basis of the fibers of the pullback bundle.

As has been shown above, $\mathcal{N}_{\mathcal{R}}$ and $\mathrm{Ker}_{\mathcal{R}}$ do not coincide in general. Nevertheless, we have

Theorem 4.11. *Let (M,L) be a Finsler manifold and $\mathcal{R}$ the h-curvature of Cartan connection. If*

$$\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\mathcal{R}(\overline{X},\overline{Y})\overline{Z} = 0, \qquad (4.8)$$

then the two distributions $\mathcal{N}_{\mathcal{R}}$ and $\ker_{\mathcal{R}}$ coincide.

Proof. If $X \in Sec(\mathcal{N}_{\mathcal{R}})$, then, from (4.8), we have $\mathcal{R}(Y,Z)X = 0$ and consequently $X \in Sec(\ker_{\mathcal{R}})$. On the other hand, it follows also from (4.8) that $\overline{g}(\mathcal{R}(\overline{X},\overline{Y})\overline{Z},\overline{W}) =: \mathcal{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) = \mathcal{R}(\overline{Z},\overline{W},\overline{X},\overline{Y})$. This proves that if $X \in Sec(\ker_{\mathcal{R}})$, then $X \in Sec(\mathcal{N}_{\mathcal{R}})$. □

The following corollary shows that there are nontrivial cases in which (4.8) is verified and consequently the two distributions coincide.

Corollary 4.12. *Let (M,L) be a Finsler manifold and $\overline{g}$ the associated Finsler metric. If one of the following conditions holds:*

(a) $\widehat{R} = 0$ *(the integrability condition for the horizontal distribution),*

(b) $\widehat{R}(\overline{X}, \overline{Y}) = \lambda L(\ell(\overline{X})\overline{Y} - \ell(\overline{Y})\overline{X})$*, where* $\lambda(x, y)$ *is a homogenous function of degree* 0 *in* y *and* $\ell(\overline{X}) := L^{-1}\overline{g}(\overline{X}, \overline{\eta})$ *(the isotropy condition),*

then the two distributions $\mathcal{N}_{\mathcal{R}}$ *and* $\ker_{\mathcal{R}}$ *coincide.*

Proof.
(a) We have $\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\mathcal{R}(\overline{X}, \overline{Y})\overline{Z} - T(\overline{X}, \widehat{R}(\overline{Y}, \overline{Z}))\} = 0$ [58]. Then, if $\widehat{R} = 0$, (4.8) holds.
(b) If $\widehat{R}(\overline{X}, \overline{Y}) = \lambda L(\ell(\overline{X})\overline{Y} - \ell(\overline{Y})\overline{X})$, then, by [44], (4.8) is satisfied. □

Remark 4.13. *It should be noted that the identity (4.8) is a sufficient condition for the validity of the identity (2.1) of [2].*

Chapter 5

Metric freedom of a spray

In this chapter, the question of how many essentially different metrics can metricize a spray is discussed. The notion of metric freedom of a spray is introduced and investigated. We show that in the regular case the holonomy distribution can be used to calculate the metric freedom of a spray. The metric freedom of an isotropic spray is characterized. Different examples are given.

The results of this chapter are submitted for publication [35].

5.1 Notation and preliminaries

We denote by (x^i) local coordinates on the base manifold M and by (x^i, y^i) the induced coordinates on TM. Locally, a spray can be expressed as follows

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}, \tag{5.1}$$

where the *spray coefficients* $G^i = G^i(x, y)$ are 2-homogeneous functions in the $y = (y^1, \ldots, y^n)$ variable. A curve $c : I \to M$ is called is called *geodesic* of a spray S if $S \circ c' = c''$. Locally, $c(t) = (x^i(t))$ is a geodesic of S if and only if it satisfies the equation

$$\frac{d^2x^i}{dt^2} + 2G^i\Big(x, \frac{dx}{dt}\Big) = 0. \tag{5.2}$$

Therefore sprays may be seen as the coordinate-free version of systems of homogeneous second order differential equations.

In this chapter, we mean by a Finsler manifold a pair (M, L) where the *Finsler function* $L\colon TM \to \mathbb{R}$ is smooth and strictly positive on $\mathcal{T}M$, L positively 1-homogeneous in the directional argument $y = (y^i)$ and the matrix

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j} \tag{5.3}$$

is positive definite on $\mathcal{T}M$. The *energy function* associated with L is $E = \frac{1}{2}L^2$ and the *Finsler metric tensor* is defined as $g = g_{ij}dx^i \otimes dx^j$. Because of the positive definiteness of (5.3) the 2-form $\Omega_E := dd_J E$ is non-degenerate, and the Euler-Lagrange equation

$$\omega_E := i_S\Omega_E - d(E - \mathcal{L}_C E) = 0 \tag{5.4}$$

uniquely determines a spray S on TM. This spray is called the *geodesic spray* of the Finsler function. The ω_E is called the Euler-Lagrange form associated with S and E.

Let us consider the *inverse problem*:

Definition 5.1. A given spray S on a manifold M is called *Finsler metrizable* if there exists a Finsler function L whose geodesic spray coincides with S.

Let S be a spray. The *Euler-Lagrange form* associated with S is

$$\omega_E = i_S\Omega_E - d(E - \mathcal{L}_C E). \tag{5.5}$$

It is easy to see that ω_E is semi-basic, and the local expression in the standard coordinate system on TM is $\omega_E = \Big(\mathcal{L}_S\big(\frac{\partial E}{\partial y^i}\big) - \frac{\partial E}{\partial x^i}\Big)\, dx^i$. Therefore along a curve $\gamma = (x(t))$ associated with S we have $\omega_E\big|_\gamma = \big(\frac{d}{dt}\frac{\partial E}{\partial \dot{x}^i} - \frac{\partial E}{\partial x^i}\big)\, dx^i$, where d/dt denotes the derivation along γ.

Definition 5.2. Let S be a given spray. A function $E \in C^\infty(\mathcal{T}M)$ is called *Euler-Lagrange function* of the spray S if it is a solution of the Euler-Lagrange equation (5.4).

If E is a regular Euler-Lagrange function of the spray S, then the stationary curves of the functional

$$I(\gamma) = \int E(\gamma(t), \dot{\gamma}(t))dt$$

are the geodesic curves of S. The set of Euler-Lagrange functions of a spray S will be denoted by $\mathcal{E}_S$ and the subset of k-homogeneous Euler-Lagrange functions will be denoted by $\mathcal{E}_{S,k}$. In particular, taking (1.5) into account, the set of 2-homogeneous Euler-Lagrange functions is denoted by

$$\mathcal{E}_{S,2} = \{E \in C^\infty(\mathcal{T}M) \mid \omega_E = 0,\ \mathcal{L}_C E = 2E\}\,. \tag{5.6}$$

Property 5.3. A spray S is metrizable if and only if there exists a 2-homogeneous Euler-Lagrange function $E \in \mathcal{E}_{S,2}$ such that the matrix field $g_{ij} = \frac{\partial^2 E}{\partial y^i \partial y^j}$ is positive definite at any point of $\mathcal{T}M$.

Several works are devoted to the metrizability problem (see for example [14, 13, 13, 20, 28, 29, 34, 46]). In this chapter, we are considering a different aspect of this problem: *How many essentially different Finsler metrics* exist for a given spray to be metrizable. Moreover, *how to determine* this number in terms of the geometric objects associated to the spray? To formulate, precisely, the problem we introduce the following

Definition 5.4. We say that the *metric freedom* of a metrizable spray S is $m_S \in \mathbb{N}$ if $\mathcal{E}_{S,2}$ can be locally generated by of m_S functionally independent of its elements. If the spray S is non-metrizable, we set $m_S = 0$.

In other words, if the *metric freedom* of a spray S is $m_S > 1$, then for every $E \in \mathcal{E}_{S,2}$ and $v_0 \in \mathcal{T}M$ there exists a neighbourhood $U \subset \mathcal{T}M$ of v_0, a function $\varphi\colon \mathbb{R}^{m_S} \to \mathbb{R}$ and $E_1, ..., E_{m_S} \in \mathcal{E}_{S,2}$ functionally independent on U such that

$$E(v) = \varphi\big(E_1(v), \ldots, E_{m_S}(v)\big), \qquad \forall\, v \in U.$$

To compute the metric freedom we have to determine *how many different energy functions* exist for a given spray? To answer this question we have to investigate the geometric objects associated with the spray. In Section 5.2 we introduce these objects (parallel translation, holonomy distribution, holonomy invariant functions) and in Section 5.3 we determine the metric freedom of sprays in the regular case.

5.2 Geometric objects associated with a spray

In this section, we use the geometric objects associated with a spray S to calculate the metric freedom of S. Every spray S induces an Ehresmann connection (see [45]). The corresponding vertical and horizontal projectors, locally, can be expressed as $v = \frac{\partial}{\partial y^i} \otimes \delta y^i$ and $h = \delta_i \otimes dx^i$ where $\delta y^i = dy^i + N^i_j dx^j$,

$$\delta_i = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}, \tag{5.7}$$

and $N^j_i = \frac{\partial G^j}{\partial y^i}$. The *parallel translation* of a vector along curves is defined through horizontal lifts:

Definition 5.5 (Parallel translation)**.** Let $\gamma\colon [0,1] \to M$ be a curve such that is $\gamma(0) = p$ and $\gamma(1) = q$. Let γ^h be a horizontal lift of the curve γ, that is $\pi \circ \gamma^h = \gamma$ and $\dot{\gamma}^h(t) \in H_{\gamma^h(t)}$. The parallel translation $\tau\colon T_pM \to T_qM$ along γ is defined as follows: $\tau(v) = w$, where $\gamma^h(0) = v$, $\gamma^h(1) = w$.

The curvature tensor $\mathfrak{R}$ of the nonlinear connection is given by Definition 1.14 and characterizes the integrability of the horizontal distribution: the horizontal distribution is integrable if and only if the curvature $\mathfrak{R}$ is identically zero. In a local coordinate system, the curvature is given by $\mathfrak{R} = \mathfrak{R}^i_{jk} dx^j \otimes dx^k \otimes \frac{\partial}{\partial y^i}$ where $\mathfrak{R}^i_{jk} = \delta_k N^i_j - \delta_j N^i_k$.

Definition 5.6 ([34])**.** The *holonomy distribution* $\mathcal{D}_{\mathcal{H}}$ of a spray S is the distribution on TM generated by the horizontal vector fields and their successive Lie-brackets, that is

$$\mathcal{D}_{\mathcal{H}} := \Big\langle \mathfrak{X}^h(TM) \Big\rangle_{Lie} = \Big\{ [X_1, [\dots [X_{m-1}, X_m]\dots]] \;\Big|\; X_i \in \mathfrak{X}^h(TM) \Big\} \tag{5.8}$$

Remark 5.7. From the definition we observe that $\mathcal{D}_{\mathcal{H}}$ contains the horizontal distribution. Using the horizontal and vertical projectors we have

$$\mathcal{D}_{\mathcal{H}} = h(\mathcal{D}_{\mathcal{H}}) \oplus v(\mathcal{D}_{\mathcal{H}}) = H\mathcal{T}M \oplus v(\mathcal{D}_{\mathcal{H}}).$$

By Definition 1.14, the image of the curvature tensor is a subset of the vertical part of the holonomy distribution: $\operatorname{Im}\mathfrak{R} \subsetneq v(\mathcal{D}_{\mathcal{H}})$. Moreover, $\mathcal{D}_{\mathcal{H}} = HTM$ if and only if $\mathfrak{R} \equiv 0$.

Remark 5.8. Because of the construction (5.8), it is clear that $\mathcal{D}_{\mathcal{H}}$ is an involutive distribution. When $\mathcal{D}_{\mathcal{H}}$ is also a regular distribution, then it is integrable. Using the definition of parallel translation via horizontal lifts (Definition 5.5), it is easy to see that the integral manifold through $v \in TM$, $\mathcal{O}_\tau(v)$, is the orbit of v with respect to the parallel translations. By Frobenius integrability theorem one can find a coordinate system (U, z) of $\mathcal{T}M$ in a neighborhood of $v \in \mathcal{T}M$ such that the components of $\mathcal{O}_\tau \cap U$ are the sets

$$\{w \in U \mid z^i(w) = z_0^i,\ \dim \mathcal{O}_\tau + 1 \leq i \leq 2n\}, \qquad |z_0^i| < \epsilon. \tag{5.9}$$

We say that the parallel translation is *regular* if the distribution $\mathcal{D}_\mathcal{H}$ is regular and for any $v \in \mathcal{T}M$ there is a neighbourhood $U \subset \mathcal{T}M$ such that any orbits $\mathcal{O}_\tau$ have at most one connected subset in U. In that case, using the Frobenius integrability theorem, there exists a coordinate system (U, z) of $\mathcal{T}M$ in a neighborhood of any $v \in \mathcal{T}M$ such that in (5.9) different z^i coordinates $(\dim \mathcal{O}_\tau + 1 \leq i \leq 2n)$ correspond to different orbits of the parallel translation.

Definition 5.9. Let S be a spray. A function $E \in C^\infty(TM)$ is called *holonomy invariant*, if it is invariant with respect to parallel translation, that is, for any $v \in TM$ and for any parallel translation τ we have $E(\tau(v)) = E(v)$. The set of holonomy invariant functions will be denoted by $\mathcal{H}_S$.

In the case when the parallel translation is regular, then the tangent spaces of its orbits are given by the holonomy distribution $\mathcal{D}_\mathcal{H}$. Consequently, for any holonomy invariant function $E \in C^\infty(TM)$ we have $\mathcal{L}_X E = 0$, $X \in \mathcal{D}_\mathcal{H}$ that is

$$\mathcal{H}_S = \{E \in C^\infty(\mathcal{T}M) \mid \mathcal{L}_X E = 0,\ X \in \mathcal{D}_\mathcal{H}\}. \tag{5.10}$$

The subset of k-homogeneous holonomy invariant functions will be denoted by $\mathcal{H}_{S,k}$. Using the Euler characterization of homogeneous functions we have

$$\mathcal{H}_{S,k} = \{E \in \mathcal{H}_S \mid \mathcal{L}_C E = kE\}. \tag{5.11}$$

5.3 Metric freedom of sprays

The Property 5.3 shows that, modulo the regularity condition, one may think of $\mathcal{E}_{S,2}$ as the set of possible energy functions of S. Therefore any relevant information on $\mathcal{E}_{S,2}$ can be very interesting from the metrizability point of view. Comparing the holonomy invariance and the Euler-Lagrange property we have the following

Lemma 5.10. *A 2-homogeneous function is an Euler-Lagrange function of a spray S if and only if it is a holonomy invariant function. Using the notation* (5.11) *and* (5.6) *we have*

$$\mathcal{E}_{S,2} = \mathcal{H}_{S,2}. \tag{5.12}$$

Proof. In [34] it was proven that a 2-homogeneous functions $E \in C^\infty(\mathcal{T}M)$ is a solution of the Euler-Lagrange PDE (5.4) if and only if it satisfies the equation

$$d_{\mathfrak{h}} E = 0, \tag{5.13}$$

where $\mathfrak{h}\colon TTM \to \mathcal{D}_{\mathcal{H}}$ is an arbitrary projection on $\mathcal{D}_{\mathcal{H}}$. Moreover, for any $X \in \mathfrak{X}(TM)$ we have $d_{\mathfrak{h}}E(X) = \mathfrak{h}X(E)$, that is, $d_{\mathfrak{h}}E = 0$ if and only if E is an element of $\mathcal{H}_{S,2}$. □

We emphasis the fact that the Euler-Lagrange system (5.4) is a second order PDE system and (5.13) is a first order PDE system. Proposition 5.10 shows that for 2-homogeneous functions they are equivalent. Using this equivalence one can understand more easily the structure of $\mathcal{E}_{S,2}$ and therefore the metrizability property of sprays. We note that there is no such equivalence in general between $\mathcal{E}_S$ and $\mathcal{H}_S$.

We can observe the following

Property 5.11. The $\mathcal{E}_S$, $\mathcal{E}_{S,k}$, ($k \in \mathbb{N}$) are vector spaces over $\mathbb{R}$.

Proof. The Euler-Lagrange equation (5.4), the homogeneity equation (1.5) and the holonomy equation (5.13) are all linear PDE. Therefore linear combination of their solutions with constant coefficients are also solutions. □

In particular, Property 5.11 states that linear combination of 2-homogeneous Euler-Lagrange functions of S is also a 2-homogeneous Euler-Lagrange function of S. We can consider this combination as a "trivial" combination of Euler-Lagrange functions. As the next proposition shows, a much wider combination of homogeneous Euler-Lagrange functions can produce new homogeneous Euler-Lagrange functions:

Proposition 5.12. *A 1-homogeneous functional combination of* 2-*homogeneous Euler-Lagrange functions of a spray S is also a* 2-*homogeneous Euler-Lagrange functions of S.*

Proof. Let $\varphi = \varphi(z_1, ..., z_r)$ be a 1-homogeneous real valued smooth function on $\mathbb{R}^n \setminus \{0\}$ and $E_1, \dots, E_r \in \mathcal{E}_{S,2}$ are 2-homogeneous Euler-Lagrange functions of S. We consider $E\colon \mathcal{T}M \to \mathbb{R}$ defined as

$$E(x,y) := \varphi\big(E_1(x,y), ..., E_r(x,y)\big). \tag{5.14}$$

and we will show that E is a 2-homogeneous Euler-Lagrange functions of the spray S, that is, $E \in \mathcal{E}_{S,2}$. Because of the 2-homogeneity of E_i we have

$$\begin{aligned} E(x,\lambda y) &= \varphi(E_1(x,\lambda y),\dots,E_r(x,\lambda y)) \\ &= \varphi(\lambda^2 E_1(x,y),\dots,\lambda^2 E_r(x,y)) = \lambda^2 E(x,y), \end{aligned}$$

that is E is a 2-homogeneous. Moreover, we can use the characterization of $\mathcal{E}_{S,2}$ via holonomy: according to Proposition 5.10 we have $\mathcal{E}_{S,2} = \mathcal{H}_{S,2}$. Hence $E_i \in \mathcal{H}_{S,2}$ and from (5.10) we get that $\mathcal{L}_X E_i = 0$ for any vector field $X \in \mathcal{D}_{\mathcal{H}}$. Therefore the same is true for E:

$$\mathcal{L}_X E = \frac{\partial \varphi}{\partial z^1} \cdot \mathcal{L}_X E_1 + ... + \frac{\partial \varphi}{\partial z^r} \cdot \mathcal{L}_X E_r = 0,$$

which shows that $E \in \mathcal{H}_{S,2}$ and therefore $E \in \mathcal{E}_{S,2}$. $\square$

Proposition 5.12 shows that the functional combinations of functions of type (5.14), satisfying the regularity condition, are also Finsler energy functions. Measuring the number of functionally independent energy functions can be an important invariance quantity of sprays.

We have the following

Theorem 5.13. *Let S be a metrizable spray on a manifold M. If its parallel translation is regular, then the metric freedom of S is*

$$m_s = \operatorname{codim} \mathcal{D}_{\mathcal{H}}.$$

To prove the above theorem we need the following three lemmas.

Lemma 5.14. *Let S be a Finsler metrizable spray with a Finsler energy function E. Then $\tilde{E}$ is a Finsler energy function associated to S if and only if*

1. *the matrix $\tilde{g}_{ij} = \frac{1}{2}\frac{\partial^2 \tilde{E}}{\partial y^i \partial y^j}$ is positive definite on $\mathcal{T}M$,*
2. *$\theta := \tilde{E}/E$ is 0-homogeneous holonomy invariant function on $\mathcal{T}M$.*

Proof. Suppose that $\tilde{E}$ is a Finsler energy function associated with S. Then it is regular and the condition (1) is satisfied. Moreover, since both E and $\tilde{E}$ are 2-homogeneous Euler-Lagrange functions of S, then, using Proposition 5.10, they are 2-homogeneous holonomy invariant functions. Thus, $\theta := \tilde{E}/E$ is a 0-homogeneous holonomy invariant function, that is, $\theta \in \mathcal{H}_{S,0}$.

On the other hand, assume that $\tilde{E}$ satisfies the conditions (1) and (2). Then, it is a regular, 2-homogeneous, holonomy invariant function. By Proposition 5.10, $\tilde{E}$ is an Euler-Lagrange function of the spray S. $\square$

Similar to (5.8), let us consider the smallest involutive distribution containing $\mathcal{D}_{\mathcal{H}}$ and C:

$$\mathcal{D}_{\mathcal{H}C} := \big\langle \mathcal{D}_{\mathcal{H}}, C \big\rangle_{Lie}.$$

Lemma 5.15. *The distribution $\mathcal{D}_{\mathcal{H}C}$ is linearly generated by $\mathcal{D}_{\mathcal{H}}$ and C, that is, $\mathcal{D}_{\mathcal{H}C} = Span\{\mathcal{D}_{\mathcal{H}}, C\}$.*

Proof. Firstly, let us consider the case when $C \in \mathcal{D}_{\mathcal{H}}$. By using the involutivity property of $\mathcal{D}_{\mathcal{H}}$ we get $\mathcal{D}_{\mathcal{H}C} = \langle \mathcal{D}_{\mathcal{H}}, C\rangle = \langle \mathcal{D}_{\mathcal{H}}\rangle = \mathcal{D}_{\mathcal{H}}$ which shows that the statement in this case is true.

Secondly, suppose that $C \notin \mathcal{D}_{\mathcal{H}}$. Take $X, Y \in \mathcal{D}_{\mathcal{H}C}$. Using the decomposition corresponding to the directions $\mathcal{D}_{\mathcal{H}}$ and C we get

$$[X,Y] = [X_{\mathcal{D}_{\mathcal{H}}}, Y_{\mathcal{D}_{\mathcal{H}}}] + [X_C, Y_C] + [X_C, Y_{\mathcal{D}_{\mathcal{H}}}] + [X_{\mathcal{D}_{\mathcal{H}}}, Y_C] \tag{5.15}$$

We have $[X_C, Y_C] \in Span\{C\}$ and because of the involutivity of $\mathcal{D}_{\mathcal{H}}$, we have $[X_{\mathcal{D}_{\mathcal{H}}}, Y_{\mathcal{D}_{\mathcal{H}}}] \in \mathcal{D}_{\mathcal{H}}$. In order to prove that the third and the fourth terms are in $Span\{\mathcal{D}_{\mathcal{H}}, C\}$, we consider the local basis $\mathcal{B} = \big\{\delta_1, \ldots, \delta_n\big\}$ of the horizontal space $H\mathcal{T}M$. Then the holonomy distribution $\mathcal{D}_{\mathcal{H}}$ can be generated locally by the elements of $\mathcal{B}$ and their successive Lie brackets. Since the spray coefficients $G^i(x,y)$ introduced in (5.1) are 2-homogeneous in the y-variables, we have $[C, \delta_i] = 0$. By the Jacobi identity, this is also true for the successive brackets of the δ_i's. Now, $Y_{\mathcal{D}_{\mathcal{H}}} \in \mathcal{D}_{\mathcal{H}}$ can be written as a linear combination of the elements $Y_{\mathcal{D}_{\mathcal{H}}} = g^\alpha Y_\alpha$, where Y_α can be obtained by successive brackets of the δ_i's, and therefore $[C, Y_\alpha] = 0$. Hence, for the C-directional component X^c of X we have $X_C = X^c C$ with $X^c \in C^\infty(\mathcal{T}M)$ and

$$\begin{aligned}
[X_C, Y_{\mathcal{D}_{\mathcal{H}}}] &= [X^c C, g^\alpha Y_\alpha] = (X_C g^\alpha) Y_\alpha - (Y_{\mathcal{D}_{\mathcal{H}}} X^c) C + X^c g^\alpha [C, Y_\alpha] \\
&= (X_C g^\alpha) Y_\alpha - (Y_{\mathcal{D}_{\mathcal{H}}} X^c) C
\end{aligned}$$

which is clearly an element of $Span\{\mathcal{D}_{\mathcal{H}}, C\}$. The same argument is valid for the fourth term in (5.15). □

Lemma 5.16. *If the spray S is metrizable, then $Span\{\mathcal{D}_{\mathcal{H}}, C\} = \mathcal{D}_{\mathcal{H}} \oplus C$ on $\mathcal{T}M$.*

Proof. If S is metrizable, then there exists a 2-homogeneous, positive definite regular Euler-Lagrarange function $E \in \mathcal{E}_{S,2}$ of S. Because of Proposition 5.10, $E \in \mathcal{H}_{S,2}$. On the other hand, by the homogeneity property of E, we have $\mathcal{L}_C E = 2E > 0$ at any point $v \in \mathcal{T}M$. Since the derivatives of E with respect to the elements of $\mathcal{D}_{\mathcal{H}v}$ is zero we get that $C_v \notin \mathcal{D}_{\mathcal{H}v}$. Consequently we have $Span\{\mathcal{D}_{\mathcal{H}}, C\} = \mathcal{D}_{\mathcal{H}} \oplus C$. □

Proof of Theorem 5.13. We will show that in a neighbourhood of any $v \in \mathcal{T}M$ one has exactly $\operatorname{Codim} \mathcal{D}_{\mathcal{H}} = 2n - \kappa$ locally functionally independent elements in $\mathcal{E}_{S,2}$, where κ is the dimension of $\mathcal{D}_{\mathcal{H}}$.

As S is metrizable, then there exists an energy function $E \in \mathcal{E}_{S,2}$ associated with S. We know that the parallel translation is regular. By Lemma 5.16 we have $\mathcal{D}_{\mathcal{H}C} = \mathcal{D}_{\mathcal{H}} \oplus C$ and $\dim \mathcal{D}_{\mathcal{H}C} = \kappa + 1$. Both $\mathcal{D}_{\mathcal{H}}$ and $\mathcal{D}_{\mathcal{H}C}$ are involutive, C^∞ distributions on $\mathcal{T}M$. By Frobenius integrability theorem one can find a coordinate system (U, z) of $\mathcal{T}M$ in a neighborhood of $v \in \mathcal{T}M$, such that $z^i(v) = 1$, $z(U) =]1-\epsilon, 1+\epsilon[^{2n}$ and for all $z_0^{\kappa+1}, ..., z_0^{2n}$ with $|1 - z_0^i| < \epsilon$, the sets

$$N = \{w \in U \mid z^i(w) = z_0^i,\ \kappa+1 \leq i \leq 2n\},$$
$$N_C = \{w \in U \mid z^i(w) = z_0^i,\ \kappa+2 \leq i \leq 2n\}$$

are integral manifolds of $\mathcal{D}_{\mathcal{H}}$ and $\mathcal{D}_{\mathcal{H}C}$ respectively. Moreover, by the regularity of the parallel translation, one can choose U such that $N \cap U$ has at most one connected subset of N. In this case the z^i coordinates $(\kappa + 1 \leq i \leq 2n)$ parametrise the orbits of the parallel translation. Locally,

$$\mathcal{D}_{\mathcal{H}} = Span\left\{\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^\kappa}\right\}, \ \mathcal{D}_{\mathcal{H}C} = Span\left\{\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^\kappa}, \frac{\partial}{\partial z^{\kappa+1}}\right\} \tag{5.16}$$

where $\frac{\partial}{\partial z^{\kappa+1}}$ and C generate the same direction, that is, $\frac{\partial}{\partial z^{\kappa+1}} = \lambda C$, with $\lambda(v) \neq 0$. Hence, from (1.5) we get

$$\frac{\partial E}{\partial z^{\kappa+1}}(v) = \lambda \mathcal{L}_C E(v) = 2\lambda E(v) \neq 0. \tag{5.17}$$

Considering $\mathcal{H}_{S,0}$, the set of $0-$homogeneous holonomy invariant functions, we have

$$\theta \in \mathcal{H}_{S,0} \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \mathcal{L}_X \theta = 0, \ \forall\, X \in \mathcal{D}_{\mathcal{H}} \\ \mathcal{L}_C \theta = 0, \end{array} \right\} \quad \Leftrightarrow \quad \mathcal{L}_X \theta = 0, \ \forall\, X \in \mathcal{D}_{\mathcal{H}C}. \tag{5.18}$$

From (5.16) and (5.18), it follows that $\theta \in \mathcal{H}_{S,0}$ on U if and only if it is a function of the variables $z^{\kappa+2}, \ldots, z^{2n}$, that is

$$\theta = \theta(z^{\kappa+2}, \ldots, z^{2n}). \tag{5.19}$$

By using a convenient bump function ψ^i in each variable z^i $(\kappa + 2 \leq i \leq 2n)$, we obtain smooth functions $\theta_i(z^i) := \psi^i(z^i) \cdot z^i \in C^\infty(\mathcal{T}M)$ (no summation convention is used here), with $\operatorname{supp}(\theta_i) \subset U$, such that $\theta_i(v) = 1$ and

$\frac{d\theta_i}{dz^i}(v) = 1$. It is locally clear that $z^i = z^i(\theta_i)$ and therefore

$$\theta_{\kappa+2}, \; \dots \; , \theta_{2n} \tag{5.20}$$

are elements of $\mathcal{H}_{S,0}$ and are functionally independent on some neighbourhood $\widetilde{U} \subset U$ of v. Consequently, any element of $\mathcal{H}_{S,0}$ can be expressed on $\widetilde{U}$ as a function of these functions. The functions (5.20) can be used to "*modify*" the original Euler-Lagrange function E to obtain new elements of $\mathcal{E}_{S,2}$, functionally independent on $\widetilde{U}$.

Indeed, let us consider the functions $E_i := (1+\theta_i)E$ for $\kappa+2 \leq i \leq 2n$ and let $E_{\kappa+1} := E$. By Lemma 5.14, the functions

$$E_{\kappa+1}, \; E_{\kappa+2}, \; \dots \; , E_{2n}, \tag{5.21}$$

are elements in $\mathcal{H}_{S,2}$. Moreover, by the construction, we have $dE_i = d\big((1+\theta_i)E\big) = \frac{d\theta_i}{dz^i}E dz^i + (1+\theta_i)dE$ (with no summation on i). Hence, at $v \in \mathcal{T}M$ we get

$$(dE_i)_v = (dz^i)_v + \big(1+\theta_i(v)\big)(dE)_v.$$

Taking (5.17) into account, we get at $v \in \mathcal{T}M$:

$$\begin{aligned}
dE_{\kappa+1} \wedge dE_{\kappa+2} \wedge \cdots \wedge dE_{2n} & \\
&= dE \wedge (dz^{\kappa+2} + \theta_{\kappa+2} dE) \wedge \cdots \wedge (dz^{2n} + \theta_{2n} dE) \\
&= dE \wedge dz^{\kappa+2} \wedge \cdots \wedge dz^{2n} \\
&= 2\lambda E(v)\, dz^{\kappa+1} \wedge dz^{\kappa+2} \wedge \cdots \wedge dz^{2n} \neq 0,
\end{aligned}$$

that is, the functions (5.21) are functionally independent in some neighbourhood $\widehat{U} \subset \widetilde{U}$ of $v \in \mathcal{T}M$.

On the other hand, let us suppose that $\tilde{E}$ is an is an energy function associated with S. Using Lemma 5.14, we know that there exists a $0-$homogeneous holonomy invariant function $\theta \in \mathcal{H}_{S,0}$, such that $\tilde{E} = \theta E$. Then, θ has the form (5.19) on U and it can thus be expressed as a function of the elements (5.20). Consequently $\tilde{E}$ can be expressed locally by the elements (5.21).

□

Remark 5.17. Using the notation of the above proof, we point out that when E is an energy function of S, then, choosing a sufficiently small nonzero constant $c_i \in \mathbb{R}$, the functions $(1+c_i\theta_i)E \in \mathcal{H}_{S,2}$ will be also an energy function of S. We can therefore deform the original energy function (or metric) to define a new one, locally functionally independent from the original one.

Remark 5.18. From the hypothesis of the Theorem 5.13 one cannot omit the metrizability. As Example 5.23 (page 71) shows, it may happen that $\operatorname{codim} \mathcal{D}_{\mathcal{H}} > 0$ but the spray is not metrizable and therefore the metric freedom is zero.

5.4 Examples: Isotropic sprays

Let S be a spray and $\mathfrak{R}$ its curvature tensor. The *Jacobi endomorphism* Φ of S is defined by

$$\Phi = i_S \mathfrak{R}. \tag{5.22}$$

The Ricci curvature, Ric, and the Ricci scalar, ρ are given by $\mathrm{Ric} = (n-1)\rho = \mathfrak{R}^i_i = \mathrm{Tr}(\Phi)$ [43].

Definition 5.19. A spray S is said to be *isotropic* if its Jacobi endomorphism has the form

$$\Phi = \rho J - \alpha \otimes C,$$

where $\rho \in C^\infty(\mathcal{T}M)$ is the Ricci scalar and α is a semi-basic 1-form on $\mathcal{T}M$.

Lemma 5.20. *For an isotropic sprays with non vanishing Ricci scalar one has* $\dim \mathcal{D}_{\mathcal{H}} \geq 2n - 1$.

Proof. Let $X \in H\mathcal{T}M$ be a horizontal vector. We have

$$\Phi(X) = 0 \iff \rho JX - \alpha(X)C = 0 \iff FJX = \frac{i_X\alpha}{\rho} FC \iff X = \frac{i_X\alpha}{\rho} S. \tag{5.23}$$

By (5.23), $\ker \Phi \cap H\mathcal{T}M = Span\{S\}$ and, therefore, using the semi-basic property of Φ, we get $\ker \Phi = V\mathcal{T}M \oplus S$ and $\dim \ker \Phi = n + 1$. Hence, we have $\dim(\mathrm{Im}\Phi) = 2n - (n+1) = n - 1$. On the other hand, by (5.22), $\Phi(X) = (i_S\mathfrak{R})(X) = \mathfrak{R}(S, X)$. Thus $\mathrm{Im}\,\Phi \subset \mathrm{Im}\,R$ and $\dim(\mathrm{Im}\mathfrak{R}) \geq n - 1$. By Remark 5.7, the result follows. □

Proposition 5.21. *Let S be an isotropic spray on an n-dimensional manifold M with regular parallel translation. Then we have $m_s \in \{0, 1, n\}$. More precisely, we have the following possibilities:*

(a) $m_s = 0$ *if and only if S is not metrizable; (in this case $\mathfrak{R} \neq 0$)*

(b) $m_s = 1$ *if and only if $\mathfrak{R} \neq 0$ and S is metrizable;*

(c) $m_S = n$, *that is maximal, if and only if* $\mathfrak{R} = 0$.

Proof. Let us first consider (c). We remark that if $\mathfrak{R} = 0$, then the holonomy is trivial and S is Riemann and Finsler metrizable: an arbitrary Minowski norm extended through parallel translation defines a Finsler norm for S. Moreover, by Remark 5.7 and Theorem 5.13, we have

$$m_S = n \iff \operatorname{codim}\mathcal{D}_{\mathcal{H}} = n \iff \dim\mathcal{D}_{\mathcal{H}} = n \iff \mathfrak{R} = 0.$$

(a) We have $m_S = 0$ if and only if S is not metrizable. In this case we have necessarily $\mathfrak{R} \neq 0$.
(b) Let $m_S = 1$. Then S is metrizable and by Theorem 5.13 we have $\operatorname{codim}\mathcal{D}_{\mathcal{H}} = 1$ and therefore $\dim\mathcal{D}_{\mathcal{H}} = 2n-1$. Using Remark 5.7, we obtain $\mathfrak{R} \neq 0$. Conversely, if S is metrizable and $\mathfrak{R} \neq 0$, then by Lemma 5.20, we have $\dim\mathcal{D}_{\mathcal{H}} \geq 2n-1$. On the other hand, Lemma 5.16 shows that $C \notin \mathcal{D}_{\mathcal{H}}$ and therefore $\dim\mathcal{D}_{\mathcal{H}} \leq 2n-1$. From the two inequalities, we have $\dim\mathcal{D}_{\mathcal{H}} = 2n-1$ and hence $\operatorname{codim}\mathcal{D}_{\mathcal{H}} = m_S = 1$. $\square$

Explicit examples

Example 5.22 ($m_S = 0$, $\operatorname{codim}\mathcal{D}_{\mathcal{H}} = 0$)**.**
Let $M = \{(x^1, x^2) \in \mathbb{R}^2 : x^2 > 0\}$ and S the spray (5.1) given by the coefficents

$$G^1 := y^1\sqrt{x^2(y^1)^2 + (y^2)^2} + \frac{y^1y^2}{2x^2}, \quad G^2 := y^2\sqrt{x^2(y^1)^2 + (y^2)^2} - \frac{(y^1)^2}{4}.$$

Since M is 2-dimensional, then S is isotropic. Moreover, introducing the notation $\varphi := \sqrt{x^2(y^1)^2 + (y^2)^2}$, the coefficients of the nonlinear connection are given by

$$N_1^1 = \frac{y^2}{2x^2} + \varphi + \frac{x^2(y^1)^2}{\varphi}, \quad N_2^1 = -\frac{y^1}{2} + \frac{x^2y^1y^2}{\varphi},$$
$$N_1^2 = \frac{y^1}{2x^2} + \frac{y^1y^2}{\varphi}, \quad N_2^2 = \varphi + \frac{(y^2)^2}{\varphi}.$$

The horizontal basis is $\{\delta_1, \delta_2\}$ where

$$\delta_1 = \frac{\partial}{\partial x^1} - \left(\frac{y^2}{2x^2} + \varphi + \frac{x^2(y^1)^2}{\varphi}\right)\frac{\partial}{\partial y^1} + \left(\frac{y^1}{2} - \frac{x^2y^1y^2}{\varphi}\right)\frac{\partial}{\partial y^2},$$
$$\delta_2 = \frac{\partial}{\partial x^2} - \left(\frac{y^1}{2x^2} + \frac{y^1y^2}{\varphi}\right)\frac{\partial}{\partial y^1} - \left(\varphi + \frac{(y^2)^2}{\varphi}\right)\frac{\partial}{\partial y^2}.$$

We have

$$v_1 := [\delta_1, \delta_2] = \frac{4(x^2)^2+1}{4(x^2)^2}\left(y^1\frac{\partial}{\partial y^2} - y^2\frac{\partial}{\partial y^1}\right)$$
$$v_2 := \big[[\delta_1, \delta_2], \delta_1\big] = \frac{4(x^2)^2+1}{4x^2\varphi}\left(y^1y^2\frac{\partial}{\partial y^1} + \big(\varphi^2 + (y^2)^2\big)\frac{\partial}{\partial y^2}\right).$$

Being v_1 and v_2 linearly independent we have $\mathcal{D}_{\mathcal{H}} = Span\{\delta_1, \delta_2, v_1, v_2\} = T\mathcal{T}M$. Consequently, $C \in \mathcal{D}_{\mathcal{H}}$ and according to Lemma 5.16 the spray is not metrizable; that is $m_s = 0$.

Example 5.23 ($m_s = 0$, $\operatorname{codim}\mathcal{D}_{\mathcal{H}} > 0$)**.**
Let $M = \{(x^1, x^2) \in \mathbb{R}^2 : x^2 > 0\}$ and S the spray (5.1) given by the coefficents $G^1 = \frac{(y^1)^2}{2x^2}$, $G^2 = 0$. The non zero coefficient of the non linear connection is $N_1^1 = \frac{y^1}{x^2}$. The horizontal basis $\{\delta_1, \delta_2\}$ and their commutator are

$$\delta_1 = \frac{\partial}{\partial x^1} - \frac{y^1}{x^2}\frac{\partial}{\partial y^1}, \qquad \delta_2 = \frac{\partial}{\partial x^2}, \qquad v := [\delta_1, \delta_2] = -\frac{y^1}{(x^2)^2}\frac{\partial}{\partial y^1}.$$

One has $\mathcal{D}_{\mathcal{H}} = Span\{\delta_1, \delta_2, v\}$, $\dim\mathcal{D}_{\mathcal{H}} = 3$ and $\operatorname{codim}\mathcal{D}_{\mathcal{H}} = 1$. For any holonomy invariant 2-homogeneous function $E \in \mathcal{H}_{S,2}$, we have $\mathcal{L}_{\delta_1}E = \mathcal{L}_{\delta_2}E = \mathcal{L}_vE = 0$. From the last equation we get $\frac{\partial E}{\partial y^1} = 0$ and threfore E cannot be a regular Lagrange function. From Proposition 5.10, it follows that S has no regular Euler-Lagrange function and, therefore, it can not be metrizable.

Example 5.24 ($m_s = 1$)**.**
Let us consider on the unite disk $\mathbb{D} \subset \mathbb{R}^n$ the spray (5.1) where $G^i = -\dfrac{\mu\langle x, y\rangle}{1+\mu|x|^2}y^i$ with $\mu \in \mathbb{R}\setminus\{0\}$. The spray is metrizable: it is the geodesic spray of the Riemannian norm

$$L_\mu = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1+\mu|x|^2}.$$

Since L_μ is of constant flag curvature $\mu \neq 0$, then the spray is isotropic. Hence, by Proposition 5.21, we get $m_s = 1$.

Example 5.25 (m_s is maximal)**.**
One can consider the trivial example where $M = \mathbb{R}^n$ and the spray (5.1) where $G^i = 0$. In this case the parallel translation is regular and the holonomy group is trivial. Hence we have $m_s = n$.
We prefer to give also another, not so obvious coefficients example: Let $\mathbb{B}^n \subset \mathbb{R}^n$ be the standard unit ball and S the spray with

$$G^i = -\frac{\langle a, y\rangle}{1 + \langle a, x\rangle} y^i, \tag{5.24}$$

where $a \in \mathbb{R}^n$ is a constant vector with $|a| < 1$. Since $\mathfrak{R} = 0$, then $\mathcal{D}_{\mathcal{H}} = H\mathcal{T}M$, the horizontal distribution. Hence, by Theorem 5.13, the metric freedom is maximal.

We remark, that S.S. Chern and Z. Shen investigated in [18] the family of Riemannian metrics associated with the norms

$$L_a = \frac{\sqrt{1 - |a|^2}}{(1 + \langle a, x\rangle)^2} \sqrt{|y|^2 - \frac{2\langle a, y\rangle\langle x, y\rangle}{1 + \langle a, x\rangle} - \frac{(1 - |x|^2)\langle a, y\rangle^2}{1 + \langle a, x\rangle}}. \tag{5.25}$$

The geodesic equation of (5.25) is (5.24), but one can find other generating Finsler metrics too. Indeed, putting $z^i = ((1 + \langle a, x\rangle)y^i - \langle a, y\rangle x^i)/(1 + \langle a, x\rangle)^2$ and considering a 1-homogeneous function $\phi\colon \mathbb{R}^n \to \mathbb{R}$, we get

$$L_\phi(x, y) = \phi\big(z^1(x, y), \dots, z^n(x, y)\big) \tag{5.26}$$

such that $E_\phi = \frac{1}{2}L_\phi^2$ is a (not necessarily regular) element of $\mathcal{E}_{S,2}$. Therefore, if L_ϕ satisfies the regularity condition (5.3), then it is a projectively flat Finsler metric of zero flag curvature with geodesic spray given by (5.24). The family (5.25) can be considered as a special case of (5.26) by choosing $\phi(z) = \big(\langle z, z\rangle - \langle a, z\rangle^2\big)^{1/2}$.

Appendix A

New Finsler Package

Antonelli et. al. have had a good contribution in Finsler geometry computations using MAPLE (cf. [8, 9, 37]). Rutz and Portugal [39] have introduced the remarkable FINSLER package [38] (it is also included in a CD with the "Handbook of Finsler geometry" [7]).

During studying an example in which the coefficients of Berwald connection are functions of positional argument x^i only. Hence, the space under consideration is Berwaldian and is thus Landsbergian. It is well known that for a Landsberg space the hv-curvature P^h_{ijk} of Cartan connection vanishes. But according to the package, the program calculated non-vanishing components of P^h_{ijk}. After a deep reading of the source code (Finsler.mpl), we discovered an error in the definition of P^h_{ijk} (similar error is found in "Handbook of Finsler geometry, II ", page 1154). Another problem with this package is that of dimension. If one considers a Finsler space of dimension three, the package can not compute the components of the h-curvature R^h_{ijk} and hv-curvature P^h_{ijk} of Cartan connection.

In this Appendix we solve the above two mentioned problems. We illustrate our modification and extension of the FINSLER package by treating a concrete example of a three dimensional Finsler space. We also propose a technique for simplifying tensor expressions.

The results of this chapter are published [60].

A.1 Notations and preliminaries

In this section, we give a brief introduction to Finsler connections. For more details, we refer, for example, to [7, 10, 12, 54].

Let (M, L) be a Finsler manifold. Let (x^i) be the coordinates of any point of M and (y^i) a supporting element at this point. Partial differentiation with respect to x^i (resp. y^i) will be denoted by ∂_i (resp. $\dot{\partial}_i$). We use the following notations:

$l_i := \dot{\partial}_i L = g_{ij} l^j = g_{ij} \frac{y^j}{L}$: the normalized supporting element; $l^i := \frac{y^i}{L}$,

$l_{ij} := \dot{\partial}_i l_j$,

$h_{ij} := L l_{ij} = g_{ij} - l_i l_j$: the angular metric tensor,

$C_{ijk} := \frac{1}{2} \dot{\partial}_k g_{ij} = \frac{1}{4} \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L^2$: the Cartan tensor,

$C^i_{jk} := g^{ri} C_{rjk}$: the (h)hv-torsion tensor,

$\gamma^i_{jk} := \frac{g^{ir}}{2}(\partial_j g_{kr} + \partial_k g_{jr} - \partial_r g_{jk})$: the Christoffel symbol with respect to ∂_i,

$G^i := \frac{1}{2} \gamma^i_{jk} y^j y^k$: the components of the canonical spray of (M, L),

$N^i_j := \dot{\partial}_j G^i$: the Barthel (or nonlinear) connection associated with (M, L),

$G^i_{jh} := \dot{\partial}_h N^i_j = \dot{\partial}_h \dot{\partial}_j G^i$: the coefficients of Berwald connection,

$\delta_i := \partial_i - N^r_i \dot{\partial}_r$: the basis vector fields of the horizontal bundle,

$\Gamma^i_{jk} := \frac{1}{2} g^{ir}(\delta_j g_{kr} + \delta_k g_{jr} - \delta_r g_{jk})$: the Christoffel symbol with respect to δ_i.

A *Finsler connection* [6] on M is $F\Gamma = (\mathbf{F}^i_{jk}(x, y), \mathbf{N}^i_j(x, y), \mathbf{C}^i_{jk}(x, y))$ such that, under a change of coordinates $(x^i) \to (\widetilde{x}^i)$, the geometric objects $\mathbf{F}^i_{jk}(x, y)$, $\mathbf{N}^i_j(x, y)$ and $\mathbf{C}^i_{jk}$ transform respectively as follows:

$$\widetilde{\mathbf{F}}^k_{ij} = \frac{\partial \widetilde{x}^k}{\partial x^l} \frac{\partial x^p}{\partial \widetilde{x}^i} \frac{\partial x^q}{\partial \widetilde{x}^j} \mathbf{F}^l_{pq} + \frac{\partial^2 x^p}{\partial \widetilde{x}^i \partial \widetilde{x}^j} \frac{\partial \widetilde{x}^k}{\partial x^p},$$

$$\widetilde{\mathbf{N}}^i_j = \frac{\partial \widetilde{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \widetilde{x}^j} \mathbf{N}^p_q + \frac{\partial x^p}{\partial \widetilde{x}^j} \frac{\partial^2 \widetilde{x}^i}{\partial x^p \partial x^q} y^q, \quad \widetilde{\mathbf{C}}^k_{ij} = \frac{\partial \widetilde{x}^k}{\partial x^l} \frac{\partial x^p}{\partial \widetilde{x}^i} \frac{\partial x^q}{\partial \widetilde{x}^j} \mathbf{C}^l_{pq}.$$

Moreover, $F\Gamma$ defines two types of covariant derivatives:

$$X^i_{j|k} := \delta_k X^i_j + X^r_j \mathbf{F}^i_{rk} - X^i_r \mathbf{F}^r_{jk}.$$
$$X^i_j|_k := \dot{\partial}_k X^i_j + X^r_j \mathbf{C}^i_{rk} - X^i_r \mathbf{C}^r_{jk}.$$

Let $F\Gamma = (\mathbf{F}^i_{jk}, \mathbf{N}^i_j, \mathbf{C}^i_{jk})$ be a Finsler connection. The (h)h-, (h)hv-, (v)h-, (v)hv- and (v)v-torsion tensors of $F\Gamma$ are given respectively by [27]:

$$\mathbf{T}^i_{jk} = \mathbf{F}^i_{jk} - \mathbf{F}^i_{kj}, \qquad \mathbf{C}^i_{jk} = \text{the connection parameters } \mathbf{C}^i_{jk},$$

$$\mathbf{R}^i_{jk} = \delta_k \mathbf{N}^i_j - \delta_j \mathbf{N}^i_k, \qquad \mathbf{P}^i_{jk} = \dot{\partial}_k \mathbf{N}^i_j - \mathbf{F}^i_{jk}, \qquad \mathbf{S}^i_{jk} = \mathbf{C}^i_{jk} - \mathbf{C}^i_{kj}.$$

and the h-, hv- and v-curvature tensors of $F\Gamma$ are given respectively by [27]:

$$\mathbf{R}^i_{hjk} = \mathfrak{A}_{(j,k)}\{\delta_k \mathbf{F}^i_{hj} + \mathbf{F}^m_{hj}\mathbf{F}^i_{mk}\} + \mathbf{C}^i_{hm}\mathbf{R}^m_{jk},$$

$$\mathbf{P}^i_{hjk} = \dot{\partial}_k \mathbf{F}^i_{hj} - \mathbf{C}^i_{hk|j} + \mathbf{C}^i_{hm}\mathbf{P}^m_{jk}, \qquad \mathbf{S}^i_{hjk} = \mathfrak{A}_{(j,k)}\{\dot{\partial}_k \mathbf{C}^i_{hj} + \mathbf{C}^m_{hk}\mathbf{C}^i_{mj}\},$$

where $\mathfrak{A}_{(j,k)}\{A_{jk}\} := A_{jk} - A_{kj}$.

The *Cartan connection* is given by $C\Gamma = (\Gamma^i_{jk}, N^i_j, C^i_{jk})$, where Γ^i_{jk}, N^i_j and C^i_{jk} are as defined above. The torsion tensors of $C\Gamma$ are:

$$C^i_{jk} = \frac{1}{2} g^{ir}\dot{\partial}_k g_{rj}, \qquad R^i_{jk} = \delta_k N^i_j - \delta_j N^i_k, \qquad P^i_{jk} = \dot{\partial}_k N^i_j - \Gamma^i_{jk}.$$

The h-, hv- and v-curvature tensors of $C\Gamma$ are:

$$R^i_{hjk} = \mathfrak{A}_{(j,k)}\{\delta_k \Gamma^i_{hj} + \Gamma^m_{hj}\Gamma^i_{mk}\} + C^i_{hm}R^m_{jk},$$
$$P^i_{hjk} = \dot{\partial}_k \Gamma^i_{hj} - C^i_{hk|j} + C^i_{hm}P^m_{jk}, \quad S^i_{hjk} = \mathfrak{A}_{(j,k)}\{C^m_{hk}C^i_{mj}\}.$$

The *Berwald connection* is given by $B\Gamma = (G^i_{jk}, N^i_j, 0)$. The associated geometric objects will be marked by a circle. The torsion tensor is given by:

$$\overset{\circ}{R}{}^i_{jk} = R^i_{jk} = \delta_k N^i_j - \delta_j N^i_k.$$

The h-, and hv-curvature tensors of $B\Gamma$ are:

$$\overset{\circ}{R}{}^i_{hjk} = \mathfrak{A}_{(j,k)}\{\delta_k G^i_{hj} + G^m_{hj}G^i_{mk}\}, \qquad \overset{\circ}{P}{}^i_{hjk} = \dot{\partial}_k G^i_{hj}.$$

The *Chern (Rund) connection* is given by $R\Gamma = (\Gamma^i_{jk}, N^i_j, 0)$. The associated geometric objects will be marked by a star. The torsion tensors are:

$$\overset{\star}{R}{}^i_{jk} = R^i_{jk} = \delta_k N^i_j - \delta_j N^i_k, \qquad \overset{\star}{P}{}^i_{jk} = P^i_{jk} = \dot{\partial}_k N^i_j - \Gamma^i_{jk}.$$

The h- and hv-curvature tensors of $R\Gamma$ are:

$$\overset{\star}{R}{}^i_{hjk} = \mathfrak{A}_{(j,k)}\{\delta_k \Gamma^i_{hj} + \Gamma^m_{hj}\Gamma^i_{mk}\}, \qquad \overset{\star}{P}{}^i_{hjk} = \dot{\partial}_k \Gamma^i_{hj}.$$

The *Hashiguchi connection* is given by $H\Gamma = (G^i_{jk}, N^i_j, C^i_{jk})$. The associated geometric objects will be marked by an asterisk. The torsion tensors are:

$$\overset{*}{C}{}^i_{jk} = C^i_{jk}, \qquad \overset{*}{R}{}^i_{jk} = R^i_{jk} = \delta_k N^i_j - \delta_j N^i_k.$$

The h-, hv- and v-curvature tensors of $H\Gamma$ are:

$$\overset{*}{R}{}^i_{hjk} = \mathfrak{A}_{(j,k)}\{\delta_k G^i_{hj} + G^m_{hj} G^i_{mk}\} + C^i_{hm} R^m_{jk},$$

$$\overset{*}{P}{}^i_{hjk} = \dot{\partial}_k G^i_{hj} - C^i_{hk\overset{*}{|}j}, \qquad \overset{*}{S}{}^i_{hjk} = \mathfrak{A}_{(j,k)}\{C^m_{hk} C^i_{mj}\}.$$

Table 2: Fundamental linear connections [54]

	Cartan	Berwald	Chern (Rund)	Hashiguchi
$(\mathbf{F}^h_{ij}, \mathbf{N}^h_i, \mathbf{C}^h_{ij})$	$(\Gamma^h_{ij}, N^h_i, C^h_{ij})$	$(G^h_{ij}, N^h_i, 0)$	$(\Gamma^h_{ij}, N^h_i, 0)$	$(G^h_{ij}, N^h_i, C^h_{ij})$
(h)h-tors. $\mathbf{T}^i_{jk}$	0	0	0	0
(h)hv-tors. $\mathbf{C}^i_{jk}$	C^i_{jk}	0	0	C^i_{jk}
(v)h-tors. $\mathbf{R}^i_{jk}$	R^i_{jk}	$\overset{\circ}{R}{}^i_{jk} = R^i_{jk}$	$\overset{\star}{R}{}^i_{jk} = R^i_{jk}$	$\overset{*}{R}{}^i_{jk} = R^i_{jk}$
(v)hv-tors. $\mathbf{P}^i_{jk}$	$P^i_{jk} = C^i_{jk\|h} y^h$	0	$\overset{\star}{P}{}^i_{jk} = P^i_{jk}$	0
(v)v-tors. $\mathbf{S}^i_{jk}$	0	0	0	0
h-curv. $\mathbf{R}^h_{ijk}$	R^h_{ijk}	$\overset{\circ}{R}{}^h_{ijk}$	$\overset{\star}{R}{}^h_{ijk}$	$\overset{*}{R}{}^h_{ijk}$
hv-curv. $\mathbf{P}^h_{ijk}$	P^h_{ijk}	$\overset{\circ}{P}{}^h_{ijk}$	$\overset{\star}{P}{}^h_{ijk}$	$\overset{*}{P}{}^h_{ijk}$
v-curv. $\mathbf{S}^h_{ijk}$	S^h_{ijk}	0	0	$\overset{*}{S}{}^h_{ijk} = S^h_{ijk}$
h-cov. der.	$K^i_{j\|k}$	$K^i_{j\overset{\circ}{\|}k}$	$K^i_{j\overset{\star}{\|}k} = K^i_{j\|k}$	$K^i_{j\overset{*}{\|}k} = K^i_{j\overset{\circ}{\|}k}$
v-cov. der.	$K^i_j\|_k$	$K^i_j\overset{\circ}{\|}_k = \dot{\partial}_k K^i_j$	$K^i_j\overset{\star}{\|}_k = K^i_j\overset{\circ}{\|}_k$	$K^i_j\overset{*}{\|}_k = K^i_j\|_k$

A.2 Notes on the FINSLER package

When performing some applications using the FINSLER package, we have encountered some problems. To show one of these problems, let us consider the following example. Let $M = \mathbb{R}^4$, $U = \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : x^1 \neq 0\}$. Let L be defined on U by:

$$L(x, y) = \sqrt{x^1 y^4 \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}}.$$

Based on this package, the non-vanishing coefficients of Berwald connection are as follows:

$$G^1_{11} = G^2_{12} = G^3_{13} = \frac{1}{x^1}, \quad G^1_{22} = G^1_{33} = -\frac{1}{x^1}.$$

This shows that the coefficients of Berwald connection are functions of the positional argument x^i only. Hence, the space under consideration is Berwaldian and is thus Landsbergian. Consequently, the hv-curvature P^h_{ijk} of Cartan connection should vanish identically. However, the FINSLER package calculated non-vanishing components of P^h_{ijk}.

Summing up, we have two problems with the Rutz and Portugal's package. The first is the wrong calculations of the curvature P^h_{ijk}. The second is the disability of computing R^h_{ijk} and P^h_{ijk} in dimensions different from 4.

A.3 Improvement of the package

In this section, we solve the two above mentioned problems. Moreover, we extend the package in order to compute various geometric objects associated not only with Cartan connection but also with the other fundamental connections in Finsler geometry. And this is for any dimension. Other geometric objects can be similarly added to the package. We illustrate these tasks using a concrete example.

Rutz and Portugal have illustrated how to use the package [39]. However, let us recall some instructions to make the use of this package easier. When we write, for example, N[i,-j] we mean N^i_j, i.e., a positive (resp. negative) index means that it is a contravariant (resp. covariant) index. If one wants to lower or raise an index by the metric or the inverse metric, he just changes its sign from positive to negative or vice versa. The command *tdiff*(N[i,-j], X[k]) means $\partial_k N^i_j$, the command *tddiff*(N[i,-j], Y[k]) means $\dot{\partial}_k N^i_j$ and the command *Hdiff*(N[i,-j], X[k]) means $\delta_k N^i_j$.

In addition to the definitions of geometric objects existing already in the FINSLER package, we add other definitions by using the command *definetensor*. We rewrite the correct expression of P^h_{ijk} and tackle the issue of dimension.

It should be that, in the package, the notations X[i] and Y[i] are used only for the position and direction arguments, respectively.

Now, let us illustrate what have been said before using a concrete example. Let $M = \mathbb{R}^3$, $U = \{(x^1, x^2, x^3; y^1, y^2, y^3) \in \mathbb{R}^3 \times \mathbb{R}^3 : x^3 \neq 0,\, y^i \neq 0\} \subset TM$. Let L be defined on U by

$$L = \sqrt{\frac{x^3(y^1)^3}{y^2} + (y^3)^2}.$$

It should first be noted that, according to Table 2, we have only three independent torsions, namely, C^h_{ij}, R^h_{ij} and P^h_{ij}. So, we will compute these torsions for Cartan connection and we will not do the calculation for the other connections.

Following the instructions of the FINSLER package, the following calculations can be performed.

```
>  Dimension := 3:
>  coordinates(x1,x2,x3):
>  Dcoordinates(y1,y2,y3):
```

Finsler structure F:

```
>  L := sqrt(x3*y1^3/y2+y3^2);
```

$$L := \sqrt{\frac{x3\,y1^3}{y2} + y3^2}$$

Metric tensor g_{ij}:

```
>  F0:= y1^3*x3/y2+y3^2;
```

$$F0 := \frac{x3\,y1^3}{y2} + y3^2$$

```
>  metricfunction(F0):
       `The components of the metric are:`
```

$$g_{x1x1} = \frac{3\,x3\,y1}{y2} \quad g_{x1x2} = -\frac{3}{2}\,\frac{x3\,y1^2}{y2^2} \quad g_{x2x2} = \frac{x3\,y1^3}{y2^3} \quad g_{x3x3} = 1$$

Inverse metric tensor g^{ij}:

```
>  show(g[i,j]);
```

$$g^{x1x1} = \frac{4y2}{3x3\,y1} \quad g^{x1x2} = \frac{2y2^2}{x3\,y1^2} \quad g^{x2x2} = \frac{4y2^3}{x3\,y1^3} \quad g^{x3x3} = 1$$

Supporting element l_i:

```
>  show(l[-i]);
```

$$l_{x1} = \frac{3}{2}\frac{x3\,y1^2}{y2\sqrt{\frac{x3\,y1^3+y2\,y3^2}{y2}}} \qquad l_{x2} = -\frac{1}{2}\frac{x3\,y1^3}{y2^2\sqrt{\frac{x3\,y1^3+y2\,y3^2}{y2}}}$$

$$l_{x3} = \frac{y3}{\sqrt{\frac{x3\,y1^3+y2\,y3^2}{y2}}}$$

Angular metric tensor h_{ij}:

```
> definetensor(h[-i,-j]=g[-i,-j]-l[-i]*l[-j]);
> show(h[-i, -j]);
```

$h_{x1x1} = \frac{3}{4}\frac{x3\,y1\left(x3\,y1^3+4y2\,y3^2\right)}{y2\left(x3\,y1^3+y2\,y3^2\right)}$ $\quad h_{x1x2} = -\frac{3}{4}\frac{x3\,y1^2\left(x3\,y1^3+2y2\,y3^2\right)}{y2\left(x3\,y1^3+y2\,y3^2\right)}$

$h_{x_1x_3} = -\frac{3}{2}\frac{x3\,y1^2\,y3}{x3\,y1^3+y2\,y3^2}$ $\quad h_{x_2x_2} = \frac{1}{4}\frac{x3\,y1^3\left(3x3\,y1^3+4y2\,y3^2\right)}{y2^3\left(x3\,y1^3+y2\,y3^2\right)}$

$h_{x_2x_3} = \frac{1}{2}\frac{x3\,y1^3\,y3}{y2\left(x3\,y1^3+y2\,y3^2\right)}$ $\quad h_{x_3x_3} = \frac{x3\,y1^3}{x3\,y1^3+y2\,y3^2}$

Cartan tensor C_{ijk}:

```
> show(C[-i,-j,-k]);
```

$C_{x1x1x_1} = \frac{3}{2}\frac{x3}{y2}$ $\quad C_{x_1x1x2} = -\frac{3}{2}\frac{x3\,y1}{y2^2}$

$C_{x1x2x_2} = \frac{3}{2}\frac{x3\,y1^2}{y2^3}$ $\quad C_{x_2x_2x_2} = -\frac{3}{2}\frac{x3\,y1^3}{y2^4}$

Spray coefficients G^i:

```
> show(G[i]);
```

$G^{x1} = \frac{1}{2}\frac{y1\;y3}{x3}$ $\quad G^{x2} = \frac{1}{2}\frac{y2\;y3}{x3}$ $\quad G^{x3} = -\frac{1}{4}\frac{y1^3}{y2}$

Nonlinear connection (Barthel connection) N^i_j:

```
> show(N[i,-j]);
```

$N^{x1}_{x1} = \frac{1}{2}\frac{y3}{x3}$ $\quad N^{x1}_{x3} = \frac{1}{2}\frac{y1}{x3}$ $\quad N^{x2}_{x2} = \frac{1}{2}\frac{y3}{x3}$

$N^{x2}_{x3} = \frac{1}{2}\frac{y2}{x3}$ $\quad N^{x3}_{x1} = -\frac{3}{4}\frac{y1^2}{y2}$ $\quad N^{x3}_{x2} = \frac{1}{4}\frac{y1^3}{y2}$

Coefficients of Berwald connection G^i_{jk}:

```
> show(G[i,-j,-k]);
```

$G^{x3}_{x1x1} = -\frac{3}{2}\frac{y1}{y2}$ $\quad G^{x3}_{x1x2} = \frac{3}{4}\frac{y1^2}{y2^2}$ $\quad G^{x1}_{x1x3} = \frac{1}{2x3}$

$G^{x3}_{x2x2} = -\frac{1}{2}\frac{y1^3}{y2^3}$ $\quad G^{x2}_{x2x3} = \frac{1}{2x3}$

Coefficients of Cartan connection Γ^i_{jk}:

```
> show(Gammastar[i,-j,-k]);
```

$Gammastar^{x1}_{x1x1} = \frac{1}{2}\frac{y3}{x3\,y1}$ $\quad Gammastar^{x2}_{x1x1} = \frac{3}{2}\frac{y2\,y3}{x3\,y1^2}$

$Gammastar^{x3}_{x1x1} = -\frac{3}{2}\frac{y1}{y2}$ $\quad Gammastar^{x1}_{x1x2} = -\frac{1}{2}\frac{y3}{x3\,y2}$

$Gammastar^{x2}_{x1x2} = -\frac{3}{2}\frac{y3}{x3\,y1}$ $Gammastar^{x3}_{x1x2} = \frac{3}{4}\frac{y1^2}{y2^2}$

$Gammastar^{x1}_{x1x3} = \frac{1}{2x3}$ $Gammastar^{x1}_{x2x2} = \frac{1}{2}\frac{y1\,y3}{x3\,y2^2}$

$Gammastar^{x2}_{x2x2} = \frac{3}{2}\frac{y3}{x3\,y2}$ $Gammastar^{x3}_{x2x2} = -\frac{1}{2}\frac{y1^3}{y2^3}$

$Gammastar^{x2}_{x2x3} = \frac{1}{2x3}$

Torsion tensors of Cartan connection

- **(h)hv-torsion C^h_{ij}:**

```
>  show(C[i,-j,-k]);
```

$C^{x1}_{x1x1} = -\frac{1}{y1}$ $C^{x1}_{x1x2} = \frac{1}{y2}$ $C^{x1}_{x2x2} = -\frac{y1}{y2^2}$

$C^{x2}_{x1x1} = -\frac{3y2}{y1^2}$ $C^{x2}_{x1x2} = \frac{3}{y1}$ $C^{x2}_{x2x2} = -\frac{3}{y2}$

- **(v)h-torsion R^h_{ij}:**

```
>  definetensor(RG[i,-j,-k]=Hdiff(N[i,-j],X[k])
>  -Hdiff(N[i,-k],X[j]));

>  show(RG[i,-j,-k]);
```

$RG^{x1}_{x1x2} = -\frac{1}{8}\frac{y1^3}{x3\,y2^2}$ $RG^{x2}_{x1x2} = -\frac{3}{8}\frac{y1^2}{x3\,y2}$

$RG^{x1}_{x1x3} = -\frac{1}{4}\frac{y3}{x3^2}$ $RG^{x3}_{x1x3} = \frac{3}{8}\frac{y1^2}{x3\,y2}$

$RG^{x2}_{x2x3} = -\frac{1}{4}\frac{y3}{x3^2}$ $RG^{x3}_{x2x3} = -\frac{1}{8}\frac{y1^3}{x3\,y2^2}$

- **(v)hv-torsion P^h_{ij}:**

```
>  definetensor(PT[i,-j,-k] = G[i,-j,-k]
>  - Gammastar[i,-j,-k]):

>  show(PT[i,-j,-k]);
```

$PT^{x1}_{x1x1} = -\frac{1}{2}\frac{y3}{x3\,y1}$ $PT^{x2}_{x1x1} = -\frac{3}{2}\frac{y2\,y3}{x3\,y1^2}$

$PT^{x1}_{x1x2} = \frac{1}{2}\frac{y3}{x3\,y2}$ $PT^{x2}_{x1x2} = \frac{3}{2}\frac{y3}{x3\,y1}$

$PT^{x1}_{x2x2} = -\frac{1}{2}\frac{y1\,y3}{x3\,y2^2}$ $PT^{x2}_{x2x2} = -\frac{3}{2}\frac{y3}{x3\,y2}$

Curvature tensors of Cartan connection

- **h-curvature tensor R^h_{ijk}:**

```
> definetensor(RC[i,-h,-j,-k] =
Hdiff(Gammastar[i,-h,-j],X[k])
> -Hdiff(Gammastar[i,-h,-k],
X[j])+Gammastar[m,-h,-j]*Gammastar[i,-m, -k]
> -Gammastar[m,-h,-k]*Gammastar[i,-m,-j]
> +C[i,-h,-m]*RG[m,-j,-k], antisymm[3,4]):
```

```
> show(RC[i,-h,-j,-k]);
```

$RC^{x1}_{x1x1x2} = -\frac{3}{8}\frac{y1^2}{x3\,y2^2}$ $RC^{x1}_{x2x1x2} = \frac{1}{4}\frac{y1^3}{x3\,y2^3}$

$RC^{x2}_{x1x1x2} = -\frac{3}{4}\frac{y1}{x3\,y2}$ $RC^{x2}_{x2x1x2} = \frac{3}{8}\frac{y1^2}{x3\,y2^2}$

$RC^{x2}_{x3x1x3} = -\frac{1}{4x3^2}$ $RC^{x3}_{x1x1x3} = \frac{3}{4}\frac{y1}{x3\,y2}$

$RC^{x3}_{x2x1x3} = -\frac{3}{8}\frac{y1^2}{x3\,y2^2}$ $RC^{x2}_{x3x2x3} = -\frac{1}{4x3^2}$

$RC^{x3}_{x1x2x3} = -\frac{3}{8}\frac{y1^2}{x3\,y2^2}$ $RC^{x3}_{x2x2x3} = \frac{1}{4}\frac{y1^3}{x3\,y2^3}$

- **hv-curvature tensor P^h_{ijk}:**

```
> definetensor(FT[i,-j,-k,-h]=
> Hdiff(C[i,-j,-k],X[h])
> +Gammastar[i,-h,-u]*C[u,-k,-j]
> -Gammastar[u,-k,-h]*C[i,-u,-j]
> -Gammastar[u,-h,-j]*C[i,-u,-k]):
```

```
> definetensor(PC[i,-h,-j,-k]=
> tddiff(Gammastar[i,-h,-j],Y[k])-FT[i,-h,-k,-j]
> +C[i,-h,-m]*PT[m,-j,-k]);
```

$PC^{x1}_{x3x1x1} = -\frac{1}{2x3\,y1}$ $PC^{x1}_{x3x1x2} = \frac{1}{2x3\,y2}$

$PC^{x1}_{x3x2x1} = \frac{1}{2x3\,y2}$ $PC^{x1}_{x3x2x2} = -\frac{1}{2}\frac{y1}{x3\,y2^2}$

$PC^{x2}_{x3x1x1} = -\frac{3}{2}\frac{y2}{x3\,y1^2}$ $PC^{x2}_{x3x1x2} = \frac{3}{2x3\,y1}$

$PC^{x2}_{x3x2x1} = \frac{3}{2x3\,y1}$ $PC^{x2}_{x3x2x2} = -\frac{3}{2x3\,y2}$

$PC^{x3}_{x1x1x1} = -\frac{3}{4y2}$ $PC^{x3}_{x1x1x2} = \frac{3}{4}\frac{y1}{y2^2}$

$PC^{x3}_{x1x2x1} = \frac{3}{4}\frac{y1}{y2^2}$ $PC^{x3}_{x1x2x2} = -\frac{3}{4}\frac{y1^2}{y2^3}$

$PC^{x3}_{x2x1x1} = \frac{3}{4}\frac{y1}{y2^2}$ $PC^{x3}_{x2x1x2} = -\frac{3}{4}\frac{y1^2}{y2^3}$

$$PC^{x3}_{x2x2x1} = -\frac{3}{4}\frac{y1^2}{y2^3} \qquad PC^{x3}_{x2x2x2} = \frac{3}{4}\frac{y1^3}{y2^4}$$

- **v-curvature tensor S^h_{ijk}:**

```
>   definetensor(S[i,-h,-j,-k] =
>   C[m,-h,-k]*C[i,-m,-j]-C[m,-h,-j]*C[i,-m,-k]):
```

```
>   show(S[i,-h,-j,-k]);
```

$$S^h_{ijk} = 0$$

Remark A.1. According to the above consideration, if we calculate the hv-curvature P of Cartan connection, in the example mentioned in Section 3, we find that the components P^h_{ijk} vanish identically as expected.

A.4 Tensor simplification

It is well known that the simplification of tensor expressions is not an easy task [36]. However, we have noted that if we have a complicated formula of a geometric object, such as P^h_{ijk}, we can significantly simplify its expression as follows. We let the package compute the tensor $P_{hijk} := g_{rh}P^r_{ijk}$ (instead of P^h_{ijk}) and ask it to show the tensor P^h_{ijk}.

To illustrate this technique let us consider the following example.

Let $M = \mathbb{R}^3$, $U = \{(x^1, x^2, x^3; y^1, y^2, y^3) \in \mathbb{R}^3 \times \mathbb{R}^3 : y^i \neq 0\}$. Let L be defined on U by

$$L = (x^1(y^2)^3 + (y^1)^2y^3)^{1/3}.$$

For example, let us compute the component S^1_{112} of the v-curvature tensor S^h_{ijk} of Cartan connection.

```
>   definetensor(SC[i,-h,-j,-k] =
C[m,-h,-k]*C[i,-m,-j]
>   -C[m,-h,-j]*C[i,-m,-k]):
```

$$SC^i_{hjk} = C^m_{hk}C^i_{mj} - C^m_{hj}C^i_{mk}$$

```
>   show(SC[i,-h,-j,-k]);
```

$$
\begin{aligned}
SC^{x1}_{x1x1x2} = & -\frac{1}{18}\,\frac{y3y1\left(-x1y2^3+y3y1^2\right)x1y2^2\left(y3y1^2-3x1y2^3\right)}{\left(x1y2^3+y3y1^2\right)^4} \\
& -\frac{2}{27}\,\frac{y3^2y1^3x1y2^2\left(y3y1^2+3\,x1y2^3\right)}{\left(x1y2^3+y3y1^2\right)^4} \\
& +\frac{1}{36}\,\frac{y1x1y2^2\left(-x1y2^3+y3y1^2\right)\,y3\left(-3\,x1y2^3+5y3y1^2\right)}{\left(x1y2^3+y3y1^2\right)^4} \\
& +\frac{1}{54}\,\frac{y1x1y2^2\left(4y3^2y1^4+21y3y1^2x1y2^3+9x1^2y2^6\right)y3}{\left(x1y2^3+y3y1^2\right)^4}
\end{aligned}
$$

The above expression is complicated. But, in fact, if we lower the index i in the above definition and use the command *show(SC[i,-h,-j,-k])*, then we have the following simplification.

```
>  definetensor(SC[-i,-h,-j,-k] =
C[m,-h,-k]*C[-i,-m,-j]
>  -C[m,-h,-j]*C[-i,-m,-k]):
```

$$SC_{ihjk} = C^m_{hk}C_{imj} - C^m_{hj}C_{imk}$$

```
>  show(SC[i,-h,-j,-k]);
```

$$SC^{x1}_{x1x1x2} = \frac{1}{12}\,\frac{y3y1x1y2^2}{\left(x1y2^3+y3y1^2\right)^2},$$

which is very simple compared with its expression before simplification.

Remark A.2. Be careful when you lower or raise an index, this index should be lowerable or raisable. For example, in the definition of P^h_{ijk} we encounter the term $\dot\partial_k\Gamma^i_{hj}$ (cf. §1). The index i in this term can not be lowered since $g_{im}(\dot\partial_k\Gamma^m_{hj}) \neq \dot\partial_k(g_{im}\Gamma^m_{hj})$. So we can not use the command tddiff(Gammastar[-i,-h,-j], Y[k]). Such a problem can be treated as illustrated below:

```
>  definetensor(FT[i,-j,-k,-h]=
>  Hdiff(C[i,-j,-k], X[h])
>  +Gammastar[i,-h,-u]*C[u,-k,-j]
>  -Gammastar[u,-k,-h]*C[i,-u,-j]
>  -Gammastar[u,-h,-j]*C[i,-u,-k]);
```

```
>  definetensor(PC[i,-h,-j,-k] =
tddiff(Gammastar[i,-h,-j],Y[k])
>  -FT[i,-h,-k,-j]+C[i,-h,-m]*PT[m,-j,-k]);
```

$$PC^i_{hjk} = tddiff_k\left(Gammastar^i_{hj}\right) - FT^i_{hkj} + C^i_{hm}\,PT^m_{jk}$$

```
>  show(PC[i,-h,-j,-k]);
```

$$PC^{x1}_{x1x1x1} = \frac{1}{72}\frac{1}{\left(x1\,y2^3+y1^2y3\right)^2 y1}\left(\frac{3x1\,y2^6\left(-y1^2y3+3x1\,y2^3\right)y1^2y3}{\left(x1\,y2^3+y1^2y3\right)^2}\right.$$
$$+\frac{10y1^2\left(y1^2y3+3x1\,y2^3\right)y2^6y3\,x1}{\left(x1\,y2^3+y1^2y3\right)^2}+\frac{3}{2}\frac{x1^2y2^9\left(-y1^2y3+3x1\,y2^3\right)}{\left(x1\,y2^3+y1^2y3\right)^2}$$
$$+\frac{5\left(y1^2y3+3x1\,y2^3\right)y2^9x1^2}{\left(x1\,y2^3+y1^2y3\right)^2}+\frac{3}{2}\frac{y2^3y1^4\left(-y1^2y3+3x1\,y2^3\right)y3^2}{\left(x1\,y2^3+y1^2y3\right)^2}$$
$$\left.+\frac{5y2^3y1^4\left(y1^2y3+3x1\,y2^3\right)y3^2}{\left(x1\,y2^3+y1^2y3\right)^2}-15x1\,y2^6+y1^2y3\,y2^3\right)$$

This component can be simplified using the above mentioned technique.

```
>  definetensor(FT[i,-j,-k,-h] =
>  Hdiff(C[i,-j,-k], X[h])
>  +Gammastar[i,-h, u]*C[u,-k,-j]
>  -Gammastar[u,-k,-h]*C[i,-u,-j]
>  -Gammastar[u,-h,-j]*C[i,-u,-k]);
>  definetensor(ST[i,-h,-j,-k] =
tddiff(Gammastar[i,-h,-j], Y[k]));
>  definetensor(PC[-i,-h,-j,-k] =
g[-m,-i]*ST[m,-h,-j,-k]
>  -FT[-i,-h,-k,-j]+C[-i,-h,-m]*PT[m,-j,-k]);
```

$$PC_{ihjk} = g_{mi}\, ST^m_{hjk} - FT_{ihkj} + C_{ihm}\, PT^m_{jk}$$

```
>  show(PC[i,-h,-j,-k]);
```

$$PC^{x1}_{x1x1x1} = \frac{1}{16}\frac{y2^3}{y1\left(x1y2^3+y3\,y1^2\right)},$$

which is simpler compared with its expression before simplification.

Appendix B

Computational technique and explicit examples

Based on the new Finsler package, we introduce a computational technique to calculate the nullity and kernel vectors. Some examples are shown. Some details and calculations of the mentioned examples throughout the thesis are given.

Some results of this appendix are published [61].

B.1 Nullity and kernel vectors

To calculate the nullity vectors using the NF-package, let $Z \in Sec(\mathcal{N}_R)$ be a nullity vector. Then, Z can be written locally in the form $Z = Z^i \delta_i$, where Z^i are the components of the nullity vector Z with respect to the basis $\{\delta_i\}$ of the horizontal space. The equation $R(Z, X)Y = 0, \forall\, X, Y \in \mathfrak{X}^h(TM)$, is written locally in the form

$$Z^j R^i_{hjk} = 0.$$

To derive the resulting system from $Z^j R^i_{hjk} = 0$, we first compute the components R^i_{hjk} using the NF-package. Then, we define a new tensor by the command "*definetensor*"as follows:

```
>  definetensor(RCZ[h,-i,-k] =
RC[h,-i,-j,-k]*Z[j]);
```

```
>  show(RCZ[h,-i,-k]);
```

Putting $RCZ[h, -i, -k] = 0$, we obtain a homogeneous system of algebraic equations. Solving this system, we get the components Z^i.

B.2 Explicit examples

The nullity distributions associated with Cartan connection are studied in Chapter 2. The following example shows that *the nullity space $\mathcal{N}_R$ of the h-curvature R of Cartan connection and the kernel* $\ker_R$ *do not coincide.*

Example B.1. Let $U = \{(x^1, ..., x^4; y^1, ..., y^4) \in \mathbb{R}^4 \times \mathbb{R}^4 : y^2 \neq 0\ , y^4 \neq 0\} \subset TM$, where $M = \{(x^1, ..., x^4) \in \mathbb{R}^4 |\, x^2 > 0\}$. Let L be defined on U by

$$L(x, y) := \sqrt[4]{(x^2)^2(y^1)^4 + (y^2)^4 + (y^3)^4 + (y^4)^4}.$$

From now on, the calculations will be performed by Maple program and the NF-packag.

R**-Nullity vectors**

```
>  definetensor(RCW[h,-i,-k]=RC[h,-i,-j,-k]*W[j]);
>  show(RCW[h,-i,-k]);
```

$RCW^{x1}_{x2x1} =$

$$\frac{\left(3x2^4y1^8+13x2^2y1^4y2^4+2x2^2y1^4y4^4+2y3^4x2^2y1^4+8y2^4y4^4+4y2^8+8y3^4y2^4\right)W^{x2}}{18x2^2\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^4}$$

$RCW^{x1}_{x2x2} =$

$$\frac{\left(3x2^4y1^8+13x2^2y1^4y2^4+2x2^2y1^4y4^4+2y3^4x2^2y1^4+8y2^4y4^4+4y2^8+8y3^4y2^4\right)W^{x1}}{-18x2^2\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^4}$$

Putting $RCW^h_{ij} = 0$, then we have a system of algebraic equations. The system has the following solution: $W^1 = W^2 = 0, W^3 = s, W^4 = t, ;\ s, t \in \mathbb{R}$. Then, any nullity vector W has the form

$$W = s\delta_3 + t\delta_4. \tag{B.1}$$

R**-Kernel vectors**

```
>  definetensor(RCZ[h,-j,-k]=RC[h,-i,-j,-k]*Z[i]);
>  show(RCZ[h, -j, -k]);
```

$$RCZ^{x1}_{x1x2} = \frac{1}{9}\frac{y1^3\left(4y2^4+x2^2y1^4\right)Z^{x1}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^3}$$
$$-\frac{\left(3x2^4y1^8+13x2^2y1^4y2^4+2x2^2y1^4y4^4+2y3^4x2^2y1^4+8y2^4y4^4+4y2^8+8y3^4y2^4\right)Z^{x2}}{18x2^2\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^4}$$
$$+\frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y3^3Z^{x3}}{x2^2y2^3\left(x2^2y1^4+y2^4+y3^4+y4^4\right)}+\frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y4^3Z^{x4}}{x2^2y2^3\left(x2^2y1^4+y2^4+y3^4+y4^4\right)}$$

$$RCZ^{x2}_{x1x2} = -\frac{1}{18}\frac{y1^3y4^3\left(4y2^4+x2^2y1^4\right)Z^{x4}}{y2^6\left(x2^2y1^4+y2^4+y3^4+y4^4\right)}$$
$$+\frac{1}{9}\frac{y1^3\left(4y2^4+x2^2y1^4\right)Z^{x2}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^3}-\frac{1}{18}\frac{y1^3y3^3\left(4y2^4+x2^2y1^4\right)Z^{x3}}{y2^6\left(x2^2y1^4+y2^4+y3^4+y4^4\right)}+$$
$$\frac{\left(x2^4y1^8+7x2^2y1^4y2^4+2y3^4x2^2y1^4+2x2^2y1^4y4^4+12y2^8+8y2^4y4^4+8y3^4y2^4\right)y1^2Z^{x1}}{18y2^6\left(x2^2y1^4+y2^4+y3^4+y4^4\right)}$$

$$RCZ^{x3}_{x1x2} = \frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y3y1^2Z^{x1}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^3}-\frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y3y1^3Z^{x2}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^4}$$

$$RCZ^{x4}_{x1x2} = \frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y1^2y4Z^{x1}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^3}-\frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y4y1^3Z^{x2}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^4}$$

Putting $RCZ^h_{ij} = 0$, we obtain a system of algebraic equations. The NF-package yields the solution:
$Z^1 = \frac{sy^1}{y^2}$, $Z^2 = s$, $Z^3 = t$ and $Z^4 = \frac{s(x^2(y^1)^4+(y^2)^4+2(y^3)^4+2(y^4)^4)-ty^2(y^3)^3}{y^2(y^4)^3}$.

Then, any kernel vector Z should have the form

$$Z = s\left(\frac{y^1}{y^2}\delta_1+\delta_2+\frac{x^2(y^1)^4+(y^2)^4+2(y^3)^4+2(y^4)^4}{y^2(y^4)^3}\delta_4\right)+t\left(\delta_3-\frac{(y^3)^3}{(y^4)^3}\delta_4\right). \tag{B.2}$$

Comparing (B.1) and (B.2), $\mathcal{N}_R$ and $\ker_R$ can not coincide.

Nullity distribution of $\overset{*}{R}$:

```
> definetensor(RchernW[h,-i,-k]=Rchern[h,-i,-j,-k]*W[j]);
> show(RchernW[h,-i,-k]);
```

$$RchernW^{x1}_{x2x2} = -\frac{1}{9}\frac{\left(4y2^4+x2^2y1^4\right)W^{x1}}{x2^2y2^4}$$

$$RchernW^{x2}_{x1x1} = -\frac{1}{9}\frac{\left(4y1^2y2^4+x2^2y1^6\right)W^{x2}}{y2^6}$$

Putting $RchernW^{x1}_{x2x2} = 0$ and $RchernW^{x2}_{x1x1} = 0$, then we have a system of algebraic equations. The system has the following solution: $W^1 = W^2 = 0, W^3 =$

$s, W^4 = t$, where $s, t \in \mathbb{R}$. Then, the nullity distribution is

$$\mathcal{N}_{R^*} = \{s\delta_3 + t\delta_4 : s, t \in \mathbb{R}\}. \tag{B.3}$$

Kernel distribution of $\overset{*}{R}$:

```
>  definetensor(RchernZ[h,-j,-k]=Rchern[h,-i,-j,-k]*Z[i]);
>  show(RchernZ[h,-j,-k]);
```

$$RchernZ^{x1}_{x1x2} = \frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)Z^{x1}}{x2^2y1\,y2^3} - \frac{1}{9}\frac{\left(4y2^4+x2^2y1^4\right)Z^{x2}}{x2^2y2^4}$$

Putting $RchernZ^{x1}_{x1x2} = 0$, we get $Z^1 = \frac{2y^1}{y^2}r, Z^2 = r, Z^3 = s, Z^4 = t;\ r, s, t \in \mathbb{R}$. Then, the kernel distribution $\ker_{R^*}$ is

$$\ker_{R^*} = \left\{ r\left(\frac{2y^1}{y^2}\delta_1 + \delta_2\right) + s\delta_3 + t\delta_4 : r, s, t \in \mathbb{R}\right\}. \tag{B.4}$$

Equations (B.3) and (B.4) show that $\ker_{R^*}$ can not be a sub-distribution of $\mathcal{N}_{R^*}$.

In [53] Youssef proved that the nullity distribution $\mathcal{N}_{R^\circ}$ associated with the h-curvature $\overset{\circ}{R}$ of Berwald connection is completely integrable. He conjectured that the nullity distribution $\mathcal{N}_{P^\circ}$ of the hv-curvature $\overset{\circ}{P}$ of Berwald connection is not completely integrable. In the next example, *we show that his conjecture is true*.

Example B.2. Let $U = \{(x^1, x^2, x^3; y^1, y^2, y^3) \in \mathbb{R}^3 \times \mathbb{R}^3 : y^1 \neq 0\} \subset TM$, where $M = \mathbb{R}^3$. Let L be defined on U by

$$L := \mathrm{e}^{-x^1}\left(y^{2^3} + \mathrm{e}^{-x^1x^3}y^3\,y^{1^2}\right)^{1/3}.$$

By Maple program and the NF-package, we can perform the following calculations.

$\overset{\circ}{P}$-Nullity vectors

```
>  definetensor(PBW[i,-h,-k] =
PB[i,-h,-j,-k]*W[j]);
>  show(PBW[i,-h,-k]);
```

$$PBW^{x3}_{x1x1} = -\frac{9}{2}\frac{y2^3W^{x1}}{y1^4e^{-x1x3}} + \frac{9}{2}\frac{y2^2W^{x2}}{y1^3e^{-x1x3}}$$

$$PBW^{x3}_{x2x2} = -\frac{9}{2}\frac{y2W^{x1}}{y1^2e^{-x1x3}} + \frac{9}{2}\frac{W^{x2}}{y1\,e^{-x1x3}}$$

Putting $PBW^h_{ij} = 0$, we get a system of algebraic equations. We have two cases:
The first case is $y^2 = 0$ and the solution in this case is $W^1 = s$, $W^2 = 0$ and $W^3 = t$. Hence, any $\overset{\circ}{P}$-nullity vector is written in the form $W = s\delta_1 + t\delta_3$. Take two nullity vectors $X, Y \in Sec(\mathcal{N}_{P^\circ})$ such that $X = \delta_1$ and $Y = \delta_3$. Their Lie bracket $[X,Y] = -\frac{y^1}{2}\frac{\partial}{\partial y^1} + y^3\frac{\partial}{\partial y^3}$, which is vertical.
The second case is $y^2 \neq 0$ and the solution in this case is $W^1 = s$, $W^2 = \frac{y^2}{y^1}s$ and $W^3 = t$. Then any $\overset{\circ}{P}$-nullity vector is written in the form $W = s(\delta_1 + \frac{y^2}{y^1}\delta_2) + t\delta_3$. Let X and Y be the two nullity vectors in $\mathcal{N}_{P^\circ}$ given by $X = \delta_1 + \frac{y^2}{y^1}\delta_2$ and $Y = \delta_3$. By computing their Lie bracket, we find that $[X,Y] = -\frac{y^1}{2}\frac{\partial}{\partial y^1} + y^3\frac{\partial}{\partial y^3}$, which is vertical. Consequently, in both cases the Lie bracket $[X,Y]$ is not a section of $\mathcal{N}_{P^\circ}$.

Let $\mathcal{N}_{R^\circ}$ and $\mathcal{N}_{\mathfrak{R}}$ be the nullity distributions associated with the h-curvature $\overset{\circ}{R}$ of Berwald connection and the curvature $\mathfrak{R}$ of the Barthel connection respectively. In [51], Youssef proved that $\mathcal{N}_{R^\circ} \subseteq \mathcal{N}_{\mathfrak{R}}$. The following example shows that *the converse is not true: that is $\mathcal{N}_{R^\circ}$ is a proper sub-distribution of $\mathcal{N}_{\mathfrak{R}}$.*

Example B.3. Let $M = \mathbb{R}^4$ and Let $U = \{(x^1, \cdots, x^4; y^1, \cdots, y^4) \in \mathbb{R}^4 \times \mathbb{R}^4 : y^2 \neq 0,\ y^4 \neq 0\} \subset TM$. Let L be defined on U by

$$L := \sqrt{\mathrm{e}^{-x2}\, y1\, \sqrt[3]{y2^3 + y3^3 + y4^3}}.$$

$\mathfrak{R}$-nullity vectors

```
>  definetensor(RGZ[i,-j]=RG[i,-j,-k]*Z[k]);
>  show(RGZ[i, -j]);
```

$$RGZ^{x2}_{x2} = -\frac{3}{16}\frac{y3^2\left(y2^3+y3^3+y4^3\right)Z^{x3}}{y2^4} - \frac{3}{16}\frac{y4^2\left(y2^3+y3^3+y4^3\right)Z^{x4}}{y2^4}$$

$$RGZ^{x2}_{x3} = \frac{3}{16}\frac{y3^2\left(y2^3+y3^3+y4^3\right)Z^{x2}}{y2^4} \qquad RGZ^{x2}_{x4} = \frac{3}{16}\frac{y4^2\left(y2^3+y3^3+y4^3\right)Z^{x2}}{y2^4}$$

$$RGZ^{x3}_{x4} = -\frac{9}{16}\frac{y4^2 Z^{x3}}{y2} \qquad RGZ^{x3}_{x3} = -\frac{\left(3y2^3+y3^3+y4^3\right)Z^{x2}}{16y2^2} + \frac{9y4^2 Z^{x4}}{16y2}$$

$$RGZ^{x3}_{x2} = \frac{3}{16}\frac{\left(y2^3+y3^3+y4^3\right)Z^{x3}}{y2^2} \qquad RGZ^{x4}_{x2} = \frac{3}{16}\frac{\left(y2^3+y3^3+y4^3\right)Z^{x4}}{y2^2}$$

$$RGZ^{x4}_{x3} = -\frac{9}{16}\frac{y3^2 Z^{x4}}{y2} \qquad RGZ^{x4}_{x4} = \frac{9}{16}\frac{y3^2 Z^{x3}}{y2} - \frac{3}{16}\frac{\left(y2^3+y3^3+y4^3\right)Z^{x2}}{y2^2}$$

Putting $RGZ^h_i = 0$, we get a system of algebraic equations. In the case where $y2^3 + y3^3 + y4^3 = 0$, we get the solution $Z^1 = t_1$, $Z^2 = t_2$ and $Z^3 = Z^4 = 0$, where $t_1, t_2 \in \mathbb{R}$. Then,

$$Z = t_1\delta_1 + t_2\delta_2. \tag{B.5}$$

$\overset{\circ}{R}$-nullity vectors

```
>  definetensor(RBW[i,-h,-k]=RB[i,-h,-j,-k]*W[j]);
>  show(RBW[i,-h,-k]);
```

$RBW^{x2}_{x4x3} = -\frac{9}{16}\frac{W^{x2}y4^2y3^2}{y2^4}$ $\qquad RBW^{x3}_{x4x4} = \frac{9}{8}\frac{W^{x3}y4}{y2}$

$RBW^{x3}_{x4x3} = \frac{9}{16}\frac{W^{x2}y4^2}{y2^2} - \frac{9}{8}\frac{y4W^{x4}}{y2}$ $\qquad RBW^{x3}_{x2x4} = -\frac{9}{16}\frac{W^{x3}y4^2}{y2^2}$

$RBW^{x4}_{x2x2} = -\frac{3}{16}\frac{\left(y2^3-2y3^3-2y4^3\right)W^{x4}}{y2^3}$

$RBW^{x4}_{x2x4} = \frac{3}{16}\frac{\left(y2^3-2y3^3-2y4^3\right)W^{x2}}{y2^3} + \frac{9}{16}\frac{W^{x3}y3^2}{y2^2}$

$RBW^{x4}_{x3x4} = \frac{9}{16}\frac{W^{x2}y3^2}{y2^2} - \frac{9}{8}\frac{y3W^{x3}}{y2}$ $\qquad RBW^{x4}_{x4x2} = -\frac{9}{16}\frac{W^{x4}y4^2}{y2^2}$

Putting $RBW^h_{ij} = 0$, we obtain a system of algebraic equations. This system has the solution $W^1 = t$, $t \in \mathbb{R}$ and $W^2 = W^3 = W^4 = 0$. Then,

$$W = t\delta_1. \tag{B.6}$$

Consequently, (B.5) and (B.6) lead to $\mathcal{N}_{\mathfrak{R}} \not\subset \mathcal{N}_{R^\circ}$.

B.3 Calculations of the examples

Example B.4. Let $M = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 : x^4 \neq 0\}$,
$U = \{(x^1, \cdots, x^4; (y^1, \cdots, y^4) \in \mathbb{R}^4 \times \mathbb{R}^4 : x^4 \neq 0;\ y^i \neq 0\} \subset TM$. Let L be defined on U by:

$$L(x, y) = \sqrt{x^4y^1((y^2)^3 + (y^3)^3 + (y^4)^3)^{1/3}}.$$

Nullity vectors of the curvature of Barthel connection:

```
>  definetensor(RGW[i,-j]=RG[i,-j,-k]*W[k]);
>  show(RGW[i,-j]);
```

$$RGW^{x2}_{x2} = \frac{9}{16}\frac{y3^2 W^{x3}}{y4\, x4^2} - \frac{3}{16}\frac{\left(5y4^3+y2^3+y3^3\right)W^{x4}}{x4^2 y4^2}$$

$$RGW^{x2}_{x3} = -\frac{9}{16}\frac{y3^2 W^{x2}}{y4\, x4^2} \qquad RGW^{x2}_{x4} = \frac{3}{16}\frac{\left(5y4^3+y2^3+y3^3\right)W^{x2}}{x4^2 y4^2}$$

$$RGW^{x3}_{x3} = \frac{9}{16}\frac{y2^2 W^{x2}}{y4\, x4^2} - \frac{3}{16}\frac{\left(5y4^3+y2^3+y3^3\right)W^{x4}}{x4^2 y4^2}$$

$$RGW^{x3}_{x4} = \frac{3}{16}\frac{\left(5y4^3+y2^3+y3^3\right)W^{x3}}{x4^2 y4^2} \qquad RGW^{x3}_{x2} = -\frac{9}{16}\frac{y2^2 W^{x3}}{y4\, x4^2}$$

$$RGW^{x4}_{x2} = \frac{3}{16}\frac{y2^2\left(5y4^3+y2^3+y3^3\right)W^{x4}}{x4^2 y4^4}$$

$$RGW^{x4}_{x3} = \frac{3}{16}\frac{y3^2\left(5y4^3+y2^3+y3^3\right)W^{x4}}{x4^2 y4^4}$$

$$RGW^{x4}_{x4} = -\frac{3}{16}\frac{y2^2\left(5y4^3+y2^3+y3^3\right)W^{x2}}{x4^2 y4^4} - \frac{3}{16}\frac{y3^2\left(5y4^3+y2^3+y3^3\right)W^{x3}}{x4^2 y4^4}$$

Putting $RGW^h_i = 0$, we get a system of algebraic equations. In this system, we have $W^1 = t_1$; $t_1 \in \mathbb{R}$ and $W^2 = W^3 = 0$. Then, we get $(y_2^3 + y_3^3 + 5y_4^3)X^4 = 0$. Now, we have two cases, either $y_2^3 + y_3^3 + 5y_4^3 = 0$ or $y_2^3 + y_3^3 + 5y_4^3 \neq 0$. Firstly, if $y_2^3 + y_3^3 + 5y_4^3 \neq 0$, then $W^4 = 0$ and thus $\mu_{\mathfrak{R}} = 1$. Secondly, if $y_2^3 + y_3^3 + 5y_4^3 = 0$, then $W^4 = t_4$; $t_4 \in \mathbb{R}$ and thus $W = t_1\delta_1 + t_4\delta_4$ and $\mu_{\mathfrak{R}} = 2$. We will be interested in the second case.

Nullity vectors of Cartan connection:

```
>  definetensor(RCW[i,-h,-k]=RC[i,-h,-j,-k]*W[j]);

>  show(RCW[i,-h,-k]);
```

$$RCW^{x1}_{x2x2} = \frac{9}{16}\frac{y2\, y1\, y3^2 W^{x3}}{y4\left(y2^3+y3^3+y4^3\right)x4^2} - \frac{3}{16}\frac{y1\, y2\left(5y4^3+y2^3+y3^3\right)W^{x4}}{x4^2 y4^2\left(y2^3+y3^3+y4^3\right)}$$

$$RCW^{x1}_{x2x3} = -\frac{9}{16}\frac{y2\, y1\, y3^2 W^{x2}}{y4\left(y2^3+y3^3+y4^3\right)x4^2}$$

$$RCW^{x1}_{x2x4} = \frac{3}{16}\frac{y1\, y2\left(5y4^3+y2^3+y3^3\right)W^{x2}}{x4^2 y4^2\left(y2^3+y3^3+y4^3\right)}$$

$$RCW^{x1}_{x3x2} = -\frac{9}{16}\frac{y2^2\, y1\, y3 W^{x3}}{y4\left(y2^3+y3^3+y4^3\right)x4^2}$$

$$RCW^{x2}_{x1x3} = \frac{9}{32}\frac{y3^2 W^{x2}}{x4^2\, y1\, y4} \qquad RCW^{x3}_{x1x2} = \frac{9}{32}\frac{y2^2 W^{x3}}{x4^2\, y1\, y4}$$

$$RCW^{x4}_{x2x3} = \frac{9}{64}\frac{y2\, y3^2\left(3y2^3+3y3^3+5y4^3\right)W^{x2}}{y4^3 x4^2\left(y2^3+y3^3+y4^3\right)}$$

$$-\frac{3}{64}\frac{y3^2y2^2\left(5y2^3+5y3^3-11y4^3\right)W^{x4}}{y4^4\left(y2^3+y3^3+y4^3\right)x4^2}$$

Putting $RCW^h_{ij}=0$, we get a system of algebraic equations. This system has the solution $W^1 = t'_1; t'_1 \in \mathbb{R}$ and $W^2 = W^3 = W^4 = 0$. Thus, $W = t'_1\delta_1$ and $\mu_R = 1$. So, the dimension of $\mathcal{N}_R = 1$ and the dimension of $\mathcal{N}_{\mathfrak{R}} = 2$, consequently, $\mathcal{N}_{\mathfrak{R}} \not\subset \mathcal{N}_R$.

The following example shows that $\mathcal{N}_P$ is not completely integrable.

Example B.5. Let $U = \{(x^1,x^2,x^3;y^1,y^2,y^3) \in \mathbb{R}^3\times\mathbb{R}^3 : x^2 \neq 0;\ y^1,y^2 \neq 0\} \subset TM$ and $M = \{(x^1,x^2,x^3) \in \mathbb{R}^3 : x^2 \neq 0\}$. Let L be defined on U by:

$$L(x,y) = \sqrt{e^{-x^1}(e^{-x^1x^3}(y^1)^2y^3 + x^2(y^2)^3)^{2/3}}.$$

Nullity vector of the hv-curvature of Cartan connection:

```
>  definetensor(PCW[h,-i,-k]=PC[h,-i,-j,-k]*w[j]);
>  show(PCW[h, -i, -k]);
```

$$PCW^{x1}_{x1x1} = -\frac{3}{32}\frac{x2\,y2^3w^{x1}}{y1\left(x2\,y2^3+e^{-x1\,x3}y1^2y3\right)} + \frac{3}{32}\frac{x2\,y2^2w^{x2}}{x2\,y2^3+e^{-x1\,x3}y1^2y3}$$

$$PCW^{x1}_{x1x2} = \frac{3}{32}\frac{x2\,y2^2w^{x1}}{x2\,y2^3+e^{-x1\,x3}y1^2y3} - \frac{3}{32}\frac{y1\,x2\,y2\,w^{x2}}{x2\,y2^3+e^{-x1\,x3}y1^2y3}$$

$$PCW^{x2}_{x1x1} = -\frac{1}{32}\frac{y2\left(7e^{-x1\,x3}y1^2y3+12x2\,y2^3\right)w^{x1}}{y1^2\left(x2\,y2^3+e^{-x1\,x3}y1^2y3\right)}$$

$$+\frac{1}{32}\frac{\left(7e^{-x1\,x3}y1^2y3+12x2\,y2^3\right)w^{x2}}{y1\left(x2\,y2^3+e^{-x1\,x3}y1^2y3\right)}$$

$$PCW^{x3}_{x1x1} = -\frac{9}{32}\frac{y3\,x2\,y2^3w^{x1}}{y1^2\left(x2\,y2^3+e^{-x1\,x3}y1^2y3\right)} + \frac{9}{32}\frac{y3\,x2\,y2^2w^{x2}}{y1\left(x2\,y2^3+e^{-x1\,x3}y1^2y3\right)}$$

Putting $PCW^h_{ij} = 0$, we get a system of algebraic equations. We have two cases: Thus, the solution of this system is $W^1 = t_1, W^2 = \frac{y^2}{y^1}t_1$ and $W^3 = t_3$; $t_1, t_3 \in \mathbb{R}$. Hence, $W = t_1(\delta_1 + \frac{y^2}{y^1}\delta_2) + t_3\delta_3$ and $\mu_P = 2$. Now, we take $X, Y \in Sec(\mathcal{N}_P)$ such that $X = \delta_1 + \frac{y^2}{y^1}\delta_2$ and $Y = h_3$. By simple calculations, we obtain the bracket $[X,Y] = [\delta_1 + \frac{y^2}{y^1}\delta_2, \delta_3] = -\frac{1}{2}y^1\frac{\partial}{\partial y^1} + y^3\frac{\partial}{\partial y^3}$, which is vertical. Consequently, the nullity distribution $\mathcal{N}_P$ is not completely integrable.

The following example shows that $\mathcal{N}_Q$ is not completely integrable.

Example B.6. $U = \{(x^1, \cdots, x^4; y^1, \cdots, y^4) \in \mathbb{R}^4 \times \mathbb{R}^4 : x^2 \neq 0; y^1, y^3, y^4 \neq 0\} \subset TM$ and $M = \{(x^1, \cdots, x^4) \in \mathbb{R}^4 : x^2 \neq 0\}$. Let F be defined on U by

$$F = \sqrt{x^2(y^1)^2 e^{-y^3/y^4} + (y^2)^2}.$$

Nullity vectors:

```
>  show(SCW[i, -h, -k]);
```

$SCW^{x1}_{x3x1} = -\frac{1}{4}\frac{W^{x3}}{y4^2} + \frac{1}{4}\frac{y3W^{x4}}{y4^3}$ $\qquad SCW^{x1}_{x3x3} = \frac{1}{4}\frac{W^{x1}}{y4^2} - \frac{1}{4}\frac{y1W^{x4}}{y4^3}$

$SCW^{x1}_{x3x4} = -\frac{1}{4}\frac{y3W^{x1}}{y4^3} + \frac{1}{4}\frac{y1W^{x3}}{y4^3}$ $\qquad SCW^{x1}_{x4x1} = \frac{1}{4}\frac{y3W^{x3}}{y4^3} - \frac{1}{4}\frac{y3^2W^{x4}}{y4^4}$

$SCW^{x1}_{x4x3} = \frac{y3W^{x1}}{-4y4^3} + \frac{y3y1W^{x4}}{4y4^4}$ $\qquad SCW^{x1}_{x4x4} = \frac{y3^2W^{x1}}{4y4^4} - \frac{y3y1W^{x3}}{4y4^4}$

$SCW^{x3}_{x1x1} = -\frac{1}{2}\frac{y3W^{x3}}{y4\,y1^2} + \frac{1}{2}\frac{y3^2W^{x4}}{y4^2y1^2}$ $\qquad SCW^{x3}_{x1x3} = \frac{1}{2}\frac{y3W^{x1}}{y4\,y1^2} - \frac{1}{2}\frac{y3W^{x4}}{y4^2y1}$

Putting $SCW^h_{ij} = 0$, we get a system of algebraic equations. In this system, we have $W^2 = t$, $W^4 = t'$, $W^1 = \frac{y^1}{y^4}t'$, $W^3 = \frac{y^3}{y^4}t'$; $t, t' \in \mathbb{R}$. Hence, $W = t\delta_2 + t'(\frac{y^1}{y^4}\delta_1 + \frac{y^3}{y^4}\delta_3 + \delta_4)$ and $\mu_Q = 2$. Now, we take $X, Y \in Sec(\mathcal{N}_Q)$ such that $X = \delta_2$ and $Y = \frac{y^1}{y^4}\delta_1 + \frac{y^3}{y^4}\delta_3 + \delta_4$. Then, the bracket $[X, Y] = -\frac{y^1y^2}{2(x^2)^2y^4}\frac{\partial}{\partial y^1} + \frac{(y^1)^2(5y^3-2y^4)}{4x^2(y^4)^2}e^{-y^3/y^4}\frac{\partial}{\partial y^2} + \frac{y^4}{2(x^2)^2}\frac{\partial}{\partial y^4}$, which is vertical. Consequently, the nullity distribution $\mathcal{N}_Q$ is not completely integrable.

The following example shows that $\mathcal{N}_{P^*}$ is not completely integrable.

Example B.7. Let $M := \mathbb{R}^3$ and $U = \{(x^1, x^2, x^3; y^1, y^2, y^3) \in \mathbb{R}^3 \times \mathbb{R}^3 : y^1, y^2, y^3 \neq 0, y^3 \neq 4y^2\} \subset TM$. Define L on U by

$$L(x, y) := \sqrt[4]{e^{-x^1x^2}(y^1)^2(y^3)^2 e^{-\frac{y^3}{y^2}}}.$$

$\overset{*}{P}$-nullity vectors:

```
>  definetensor(PchernW[h,-i,-k]=
>  Pchern[h,-i,-j,-k]*w[j]);
```

```
>  show(PchernW[h,-i,-k]);
```

$$
\begin{aligned}
PchernW^{x2}_{x2x2} &= -\frac{12x1y2\left(8y3^2y2-y3^3-24y2^2y3+24y2^3\right)w^{x2}}{y3(-y3+4y2)^4} \\
&\quad +\frac{6x1y2^2\left(32y2^3+8y3^2y2-28y2^2y3-y3^3\right)w^{x3}}{y3^2(-y3+4y2)^4} \\
PchernW^{x2}_{x3x2} &= \frac{6x1y2^2\left(-28y2^2y3+32y2^3+8y3^2y2-y3^3\right)w^{x2}}{y3^2(-y3+4y2)^4} \\
&\quad -\frac{48x1y2^5(2y2-y3)w^{x3}}{y3^3(-y3+4y2)^4} \\
PchernW^{x3}_{x2x2} &= \frac{6x1y3\left(y3^2-4y2y3+8y2^2\right)w^{x2}}{(-y3+4y2)^4}-\frac{12x1y2^2y3w^{x3}}{(-y3+4y2)^4} \\
PchernW^{x3}_{x3x2} &= -\frac{12x1y2^2y3w^{x2}}{(-y3+4y2)^4}-\frac{6x1y2^2\left(-8y2y3+8y2^2+y3^2\right)w^{x3}}{y3(-y3+4y2)^4}
\end{aligned}
$$

Putting $PchernW^h_{ij} = 0$, we get a system of algebraic equations. This system has a solution if $y^3 = 2y^2$ and $x^1 > 0$: $W^1 = s$, $W^2 = t$, $W^3 = 2t$; $s, t \in \mathbb{R}$. Hence, a $\overset{*}{P}$-nullity vector must have the form $W = s\delta_1 + t(\delta_2 + 2\delta_3)$. Consequently, the nullity index $\mu_{P^*} = 2$. Now, take $X, Y \in Sec(\mathcal{N}_{P^*})$ such that $X = \delta_1$, $Y = \delta_2 + 2\delta_3$. Taking into account that $y^3 = 2y^2$, then $\delta_1 = \frac{\partial}{\partial x^1} + \frac{x^2y^1}{2}\frac{\partial}{\partial y^1}$, $\delta_2 = \frac{\partial}{\partial x^2} + \frac{x^1y^2}{2}\frac{\partial}{\partial y^2}$ and $\delta_3 = \frac{\partial}{\partial x^3} + \frac{x^1y^2}{2}\frac{\partial}{\partial y^3}$. Hence, the bracket $[\delta_1, \delta_2 + 2\delta_3] = -\frac{y^1}{2}\frac{\partial}{\partial y^1} + \frac{y^2}{2}\frac{\partial}{\partial y^2} + \frac{y^2}{2}\frac{\partial}{\partial y^3}$ is vertical and, consequently, $\mathcal{N}_{P^*}$ is not completely integrable.

The following example gives a class of Finsler functions which are Landsbergian and not Berwaldian.

Example B.8. Let $M = \mathbb{R}^3$ and $U = \{(x^1, x^2, x^3; y^1, y^2, y^3) \in \mathbb{R}^3 \times \mathbb{R}^3 : y^2 > 0, y^3 > 0\} \subset TM$. Define L on U by

$$L(x,y) := f(x^1)\sqrt{(y^1)^2 + y^2y^3 + y^1\sqrt{y^2y^3}}\; e^{\frac{1}{\sqrt{3}}\arctan\left(\frac{2y^1}{\sqrt{3y^2y^3}} + \frac{1}{\sqrt{3}}\right)}.$$

The Landsberg tensor L_{ijk} and the Berwald tensor G^h_{ijk} are given by

$$L_{ijk} := \frac{L}{2}\frac{\partial L}{\partial y^h}G^h_{ijk}, \quad G^h_{ijk} := \frac{\partial^3 G^h}{\partial y^i \partial y^j \partial y^k}.$$

We show that the Landsberg tensor vanishes identically while there are some non vanishing components of the Berwald tensor.

$$L := f(x1)\sqrt{y1^2 + y2\,y3 + y1\sqrt{y2\,y3}}\; \mathrm{e}^{\frac{\sqrt{3}}{3}\arctan\left(\frac{2}{3}\frac{y1\sqrt{3}}{\sqrt{y2\,y3}} + \frac{\sqrt{3}}{3}\right)}$$

```
> simplify(G1)
```

$$G1 := \frac{1}{2}\,\frac{\left(y1^2 - y2\;y3\right)\frac{d}{dx1}f(x1)}{f(x1)}$$

```
> simplify(G2)
```

$G2 := \frac{1}{2}\left(\frac{d}{dx1}f(x1)\right)y2^2 y3\,(92\,y2^5 y3^5 y1^3 + 408\,y2^3 y3^3 y1^7 + 230\,y2^2 y3^2 y1^9 + 48\,y2\;y3\;y1^{11} + 8\,y2^6 y3^6 y1 + 306\,y2^4 y3^4 y1^5 + 2\,y1^{13} + (y2^6 y3^6 + 33\,y2^5 y3^5 y1^2 + 190\,y2^4 y3^4 y1^4 + 121\,y2\;y3\;y1^{10} + 342\,y2^2 y3^2 y1^8 + 393\,y2^3 y3^3 y1^6 + 13\,y1^{12})\sqrt{y2\;y3})/(f(x1)\,(50\,y2^5 y3^5 y1^3 + 126\,y2^3 y3^3 y1^7 + 50\,y2^2 y3^2 y1^9 + 6\,y2\;y3\;y1^{11} + 6\,y2^6 y3^6 y1 + 126\,y2^4 y3^4 y1^5 + (y2^6 y3^6 + 21\,y2^5 y3^5 y1^2 + y1^{12} + 90\,y2^4 y3^4 y1^4 + 21\,y2\;y3\;y1^{10} + 90\,y2^2 y3^2 y1^8 + 141\,y2^3 y3^3 y1^6)\sqrt{y2\;y3})\sqrt{y2\;y3})$

```
> simplify(G3)
```

$G3 := \frac{1}{2}\left(\frac{d}{dx1}f(x1)\right)y3^2 y2\,(408\,y2^3 y3^3 y1^7 + 230\,y2^2 y3^2 y1^9 + 8\,y2^6 y3^6 y1 + 2\,y1^{13} + (33\,y1^2 y2^5 y3^5 + 393\,y1^6 y2^3 y3^3 + 342\,y1^8 y3^2 y2^2 + 121\,y1^{10} y3\;y2 + 190\,y1^4 y3^4 y2^4 + 13\,y1^{12} + y2^6 y3^6)\sqrt{y2\;y3} + 92\,y2^5 y3^5 y1^3 + 306\,y2^4 y3^4 y1^5 + 48\,y2\;y3\;y1^{11})/(f(x1)\,(50\,y2^2 y3^2 y1^9 + 6\,y2\;y3\;y1^{11} + 126\,y2^3 y3^3 y1^7 + 6\,y2^6 y3^6 y1 + (90\,y1^4 y3^4 y2^4 + 141\,y1^6 y2^3 y3^3\sqrt{y2\;y3} + 21\,y1^{10} y3\;y2 + 90\,y1^8 y3^2 y2^2 + 21\,y1^2 y2^5 y3^5 + y1^{12} + y2^6 y3^6)\sqrt{y2\;y3} + 126\,y2^4 y3^4 y1^5 + 50\,y2^5 y3^5 y1^3)\sqrt{y2\;y3})$

```
> y1 := y[1]; y2 := y[2]; y3 := y[3]
```

$$y1 := y_1$$
$$y2 := y_2$$
$$y3 := y_3$$

```
> printlevel := 3;
> for i to 3 do
> for j to i do
> for k to j do
> L[i,j,k]:=simplify((diff(L,y1))
> *(diff(G1,y[i],y[j],y[k]))
> +(diff(L,y2))*(diff(G2,y[i],y[j],y[k]))
> +(diff(L,y3))*(diff(G3,y[i],y[j],y[k])));
> end do; end do; end do;
```

$$L_{i,j,k} := 0$$

```
> Berwald[2,2,2]:=diff(G2,y[2],y[2],y[2])
```

```
> y[1] := 0;  y[2] := 1;  y[3] := 1;
```

$$y_1 := 0$$

$$y_2 := 1$$
$$y_3 := 1$$

```
>   simplify(Berwald[2, 2, 2])
```

$$\frac{-3}{16}\,\frac{\frac{d}{dx1}f(x1)}{f(x1)}$$

The following example shows that $\mathcal{N}_{\mathbf{R}}$ and $\ker_{\mathbf{R}}$ do not coincide.

Example B.9. Let $M = \mathbb{R}^3$ and $U = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : y_i \neq 0\} \subset TM$. Define the pseudo-Finsler function L (*of Berwald-Moor type*) on U by:

$$L(x, y) = \sqrt[3]{e^{-x^1x^2}(y^1y^2y^3)}.$$

Nullity vector

```
>   definetensor(RCW[h,-i,-k]=RC[h,-i,-j,-k]*W[j]);
>   show(RCW[h, -i, -k]);
```

$RCW^{x1}_{x1x1} = 2W^{x2}$ $\qquad RCW^{x1}_{x1x2} = -2W^{x1}$

$RCW^{x1}_{x3x1} = \frac{y1\,W^{x2}}{y3}$ $\qquad RCW^{x1}_{x3x2} = -\frac{y1\,W^{x1}}{y3}$

Putting $RCW^h_{ij} = 0$, we have the system of equations $W^1 = 0$, $W^2 = 0$ having the solution $W^3 = t\,(t \in \mathbb{R})$, $W^1 = W^2 = 0$. As $\pi_*(\delta_i) = \frac{\partial}{\partial x^i}$,

$$\mathcal{N}_R(x) = \left\{ t\frac{\partial}{\partial x^3} \,|\, t \in \mathbb{R} \right\}. \tag{B.7}$$

Kernel vector

```
>   definetensor(RCK[h,-j,-k]=RC[h,-i,-j,-k]*Z[i]);
>   show(RCK[h,-j,-k]);
```

$RCK^{x1}_{x1x2} = -2Z^{x1} - \frac{y1\,Z^{x3}}{y3}$ $\qquad RCK^{x2}_{x1x2} = 2Z^{x2} + \frac{y2\,Z^{x3}}{y3}$

$RCK^{x3}_{x1x2} = -\frac{y3\,Z^{x1}}{y1} + \frac{y3\,Z^{x2}}{y2}$ $\qquad RCK^{x3}_{x2x1} = \frac{y3\,Z^{x1}}{y1} - \frac{y3\,Z^{x2}}{y2}$

Putting $RCK^h_{ij} = 0$, we get a system has the solution $Z^1 = t'$, $Z^2 = \frac{y^2}{y^1}t'$ and $Z^3 = -\frac{2y^3}{y^1}t'$, $(t' \in \mathbb{R})$. Thus,

$$\ker_R(x) = \left\{ t'\Big(\frac{\partial}{\partial x^1} + \frac{y^2}{y^1}\frac{\partial}{\partial x^2} - \frac{2y^3}{y^1}\frac{\partial}{\partial x^3}\Big) \,|\, t' \in \mathbb{R} \right\}. \tag{B.8}$$

Comparing (B.7) and (B.8), $\mathcal{N}_R(x)$ and $\ker_R(x)$ can not coincide. $\square$

Chapter 6

Summary

6.1 English summary

Chern and Kuiper [17] in 1952 defined a distribution on a Riemannian manifold M which assigns to each point $x \in M$ the subspace

$$\mathcal{N}_R(x) = \{X \in T_xM : R(X,Y) = 0, \forall\, Y \in T_xM\},$$

where R is the curvature of the Riemannian connection on M. It is called the nullity space at x. The distribution defined by the subspace $\mathcal{N}_R(x)$ at each point x of M is called the nullity distribution $\mathcal{N}_R$ of the Riemannian manifold M. The dimension $\mu_R(x)$ of $\mathcal{N}_R(x)$ is called the index of nullity at x. Chern and Kuiper showed that, if $\mu_R(x)$ is constant in a neighborhood, then $\mathcal{N}_R$ constitutes a completely integrable distribution there, and that the leaves of the resulting foliation are flat. Later, Maltz and others developed this point in different papers, for example, [19, 21, 23, 32, 33, 41, 49, 50].

In 1972, Akbar Zadeh [3, 4] extended this work to Finsler geometry adopting the *pullback approach* (PB-) approach to Finsler geometry. He studied the nullity distribution of the (classical) curvature of Cartan connection. Recently, Bidabad and Refie-Rad [11] studied a more general case called k-nullity distribution in Finsler geometry.

On the other hand, in 1982, Youssef [51, 53] studied the nullity distributions of the curvature tensors of Barthel connection and Berwald connection, adopting the *Klein-Grifone approach*(KG-) approach to Finsler geometry.

In the PB-approach, the existence and uniqueness theorems for the four fundamental linear connections (Berwald, Cartan, Chern and Hashiguchi connections) on a Finsler manifold have been satisfactorily established [56, 57]. In the KG-approach, Grifone [25] has investigated Cartan and Berwald connections. Szilasi and Vincze [47] have studied Chern and Hasiguchi connections using the technique of lifting vector fields to the tangent bundle.

Adopting the Klein-Grifone formalism of Finsler geometry, we investigated the nullity distributions of the h-curvature R, hv-curvature P and v-curvature Q tensors of Cartan connection. We showed that the nullity distribution $\mathcal{N}_R$ is included in $\mathcal{N}_{\mathfrak{R}}$ of the curvature of Barthel connection and we showed, by an example, that this inclusion is proper. We proved that $\mathcal{N}_R$ is completely integrable. Through examples, we show that the distributions $\mathcal{N}_P$ and $\mathcal{N}_Q$ are not completely integrable. Nevertheless, we investigated the necessary and sufficient conditions for these distributions to be completely integrable. A coordinate-free existence and uniqueness theorem for Chern connection is formulated and proved. The torsion and curvature tensors of Chern connection are derived. Some properties and the Bianchi identities for this connection are derived. The nullity distributions of the two curvature tensors $\overset{*}{R}$ and $\overset{*}{P}$ of Chern connection are investigated. The completely integrable property of $\mathcal{N}_{R^*}$ and the completeness of the nullity foliation associated with $\mathcal{N}_{R^*}$ are proved. Two counterexamples are given. The first shows that $\mathcal{N}_{R^*}$ does not coincide with the kernel distribution of $\overset{*}{R}$. The second shows that $\mathcal{N}_{P^*}$ is not completely integrable. An example of non regular Landesbergian non Berwaldian metric is given.

Adopting the pullback formalism of Finsler geometry, we show by a counterexample that the kernel distribution $\ker_{\mathcal{R}}$ of the h-curvature $\mathcal{R}$ of Cartan connection and the associated nullity distribution $\mathcal{N}_{\mathcal{R}}$ do not coincide, contrary to Akbar-Zadeh's result [2]. We give sufficient conditions for $\ker_{\mathcal{R}}$ and $\mathcal{N}_{\mathcal{R}}$ to coincide.

The question of how many essentially different metrics metricize a spray is discussed. The notion of metric freedom of a spray is introduced and investigated. We show that in the regular case, the holonomy distribution can be used to calculate the metric freedom of a spray. The metric freedom of isotropic sprays is characterized. Different examples are given.

Some modifications of the Maple package, FINSLER, (for calculations in Finsler geometry) included in the book "Handbook of Finsler geometry [7]" are performed. A technique for simplifying tensor expressions is proposed. A computational technique for calculating nullity vectors and kernel vectors, using the new Finsler package, is introduced. Three interesting examples are given.

6.2 Hungarian summary

Chern és Kuiper [17] 1952-ben definiálta egy M Riemann-sokaság nullitás-disztribúcióját az alábbi módon. Jelölje minden $x \in M$ esetén

$$\mathcal{N}_R(x) = \{X \in T_xM : R(X,Y) = 0, \forall Y \in T_xM\}$$

az x-beli nullitás teret, ahol R a Riemann-konnexió görbületi tenzora. Ezen alterek összességét hívjuk nullitás-disztribúciónak. Az $\mathcal{N}_R(x)$ dimenzióját az x-beli nullitás indexnek nevezzük, és $\mu_R(x)$-szel jelöljük. Chern és Kuiper megmutatta, hogy ha $\mu_R(x)$ konstans egy x_0 pont egy környezetén, akkor $\mathcal{N}_R$ integrálható disztribúció ezen a környezeten, és az integrálsokaságai laposak. Később Maltz és mások további vizsgálatokat végeztek a témakörben [19, 21, 23, 32, 33, 41, 49, 50].

1972-ben Akbar-Zadeh [3, 4] kiterjesztette ezeket a vizsgálatokat a Finsler-geometriára, a *pull-back* megközelítést alkalmazva. A Cartan-konnexió (klasszikus) görbületének nullitás-disztribúcióját vizsgálta. A közelmúltban Bidabad és Refie-Rad [11] vizsgálta az általánosabb k-nullitás-disztribúciót a Finsler esetben.

Ezek mellett 1982-ben, Youssef [51, 53] tanulmányozta a Barthel- és Berwald-konnexiók görbületi tenzorainak nullitás-disztribúcióját, a Klein–Grifone megközelítést alkalmazva.

A PB-megközelítésben Finsler-sokaságok négy alapvető konnexiójának (Berwald-, Cartan-, Chern- és Hashiguchi-konnexiók) létezése és egyértelműsége már ki van dolgozva [56, 57]. A KG-megközelítésben Grifone [25] vizsgálta a Cartan- és Berwald-konnexiókat. Szilasi és Vincze [47] tanulmányozta a Chern- és Hasiguchi-konnexiókat a vektormezők érintősokaságra való liftelésének technikájával.

A disszertáció 2. fejezetében a Finsler-geometria Klein–Grifone formalizmusát alkalmazva vizsgáltuk a Cartan-konnexió h-görbületének, hv-görbületének és v-görbületének (jelölésük rendre R, P és Q) nullitás-disztribúcióját. Megmutattuk, hogy az $\mathcal{N}_R$ nullitás-disztribúcióját tartalmazza a Barthel-konnexió $\mathcal{N}_{\mathfrak{R}}$ nullitás-disztribúciója, és egy példával megmutattuk, hogy a tartalmazás valódi. Megmutattuk, hogy $\mathcal{N}_R$ integrálható. Példákkal demonstráltuk, hogy az $\mathcal{N}_P$ és $\mathcal{N}_Q$ disztribúciók nem integrálhatók. Mindazonáltal vizsgáltuk ezen disztribúciók integrálhatóságának szükséges és elégséges feltételeit.

A 3. fejezetben koordinátamentes bizonyítást adtunk a Chern-konnexió létezésére és egyértelműségére, és levezettük a torzió és görbületi tenzorait, a konnexió további tulajdonságait és Bianchi-azonosságokat. Vizsgáltuk a Chern-konnexió $\overset{*}{R}$ és $\overset{*}{P}$ görbületi tenzorainak a nullitás-disztribúcióit. Megmutattuk, hogy $\mathcal{N}_{R^*}$ integrálható. Két ellenpéldával rámutattunk, hogy $\mathcal{N}_{R^*}$ nem esik egybe az $\overset{*}{R}$ kernel-disztribúciójával, és hogy $\mathcal{N}_{P^*}$ nem integrálható. Példát adtunk olyan nem reguláris Landsberg-sokaságra, ami nem Berwald-sokaság.

A 4. fejezetben a pull-back formalizmusban ellenpéldával megmutattuk, hogy a Chern-konnexió $\mathcal{R}$ h-görbületének
$ker_{\mathcal{R}}$ magja és a hozzá tartozó $\mathcal{N}_{\mathcal{R}}$ nullitás-disztribúció nem esnek egybe, cáfolva ezzel Akbar-Zadeh eredményét [2]. Elégséges feltételt adtunk $\ker_{\mathcal{R}}$ és $\mathcal{N}_{\mathcal{R}}$ egybeesésére.

Az 5. fejezetben azt vizsgáltuk, hogy hány különböző metrikából származhat egy adott spray. Bevezettük és vizsgáltuk egy spray metrikus szabadságának fogalmát. Megmutattuk, hogy a reguláris esetben a holonómia disztribúció segítségével meghatározható a spray metrikus szabadsága. Megadtuk az izotropikus sprayk lehetséges metrikus szabadságait. Több konkrét példával is szolgáltunk.

Az appendixben továbbfejlesztettük a "Handbook of Finsler geometry" [7] című könyvben található, FINSLER nevű Maple csomagot, többek közt egy tenzor-kifejezések egyszerűsítésére szolgáló technikával Az új csomagot NFP-nek (new Finsler package) neveztük el.

Az NFP csomagban módszert adtunk a nullitás és kernel vektorok kiszámítására. Végül példákon át megmutattuk a következőket: a Ker_R és $\mathcal{N}_R$ disztribúciók nem egyenlőek, a $\mathcal{N}_{P^\circ}$ disztribúció nem integrálható, és a $\mathcal{N}_{\mathfrak{R}}$ disztribúciót nem tartalmazza $\mathcal{N}_{R^\circ}$.

Megjegyezzük, hogy a disszertációban található eredmények többsége publikálásra vagy benyújtásra került ([35, 59, 60, 61, 63, 62, 64]).

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