



# Moment Functions on Groups

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**Abstract.** In this paper generalized moment functions are considered. They are closely related to the well-known functions of binomial type which have been investigated on various abstract structures. The main purpose of this work is to prove characterization theorems for generalized moment functions on commutative groups. At the beginning a multivariate characterization of moment functions defined on a commutative group is given. Next the notion of generalized moment functions of higher rank is introduced and some basic properties on groups are listed. The characterization of exponential polynomials by means of complete (exponential) Bell polynomials is given. The main result is the description of generalized moment functions of higher rank defined on a commutative group as the product of an exponential and composition of multivariate Bell polynomial and an additive function. Furthermore, corollaries for generalized moment function of rank one are also stated. At the end of the paper some possible directions of further research are discussed.

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## 1. Introduction

Let  $(G, +)$  be an Abelian group. Recall that a nonzero function  $m: G \rightarrow \mathbb{C}$  is called *exponential*, if

$$m(x + y) = m(x)m(y)$$

holds for all  $x, y$  in  $G$ . Let  $N$  be a nonnegative integer. A function  $\varphi: G \rightarrow \mathbb{C}$  is termed to be a *moment function of order  $N$* , if there exist functions  $\varphi_k: G \rightarrow \mathbb{C}$  such that  $\varphi_0 = 1$ ,  $\varphi_N = \varphi$  and

$$\varphi_k(x + y) = \sum_{j=0}^k \binom{k}{j} \varphi_j(x) \varphi_{k-j}(y) \tag{1}$$

for all  $x$  and  $y$  in  $G$  and  $k = 0, 1, \dots, N$ . We generalize this concept by relaxing the assumption  $\varphi_0 = 1$  to  $\varphi_0(0) = 1$ . In this case  $\varphi_0$  is an arbitrary exponential function and we say that  $\varphi_0$  *generates the generalized moment sequence of order  $N$*  and the function  $\varphi_k$  is a *generalized moment function of order  $k$* , or, if we want to specify the exponential  $\varphi_0$ , then we say that  $\varphi_k$  is a *generalized moment function of order  $k$  associated with the exponential  $\varphi_0$* . Problems about moment functions have been extensively studied on different type of abstract structures, in particular on hypergroups. They are measure algebras where convolution has specific probabilistic interpretation. In the hypergroup-setting the justification of the name ‘moment problems’ is more visible than on abstract groups without additional structure as we can interpret moment functions by means of properly defined moments of random elements. For a more detailed discussion see e.g. [3] and [8]. More on moment functions can be found e.g. in [2, 4–6, 10–12] and references therein. As already mentioned groups are special hypergroups and in this paper we are going to focus on a generalized moment problem on Abelian groups.

Equation (1) is closely related to the well-known functions of binomial type. We are particular interested in (1) defined on abstract structures. A detailed discussion about binomial type equations in abstract setting, which has been the motivation for the present research, can be found in [1]. It was shown in [1] there that if  $(G, +)$  is a grupoid and  $R$  is a commutative ring, then functions  $\varphi_n : G \rightarrow R$  satisfying (1) for each  $n$  in  $\mathbb{N}$  are of the form

$$\varphi_n(t) = n! \sum_{j_1+2j_2+\dots+nj_n=n} \prod_{k=1}^n \frac{1}{j_k!} \left( \frac{a_k(t)}{k!} \right)^{j_k} \tag{2}$$

for all  $t$  in  $G$  and  $k$  in  $\mathbb{N}$  and arbitrary homomorphisms  $a_k$  from  $(G, +)$  into  $(R, +)$ .

The present paper is organized as follows: first we give the definition and a multi-variable characterization of generalized moment functions. Next we introduce the notion of generalized moment functions of rank  $r$ , a generalization of the solutions of (1), which represent generalized moment functions of rank 1. Our main result is the description of generalized moment functions of higher rank by means of Bell polynomials. As a corollary we get a new characterization of generalized moment functions on Abelian groups, as well.

We recall the definition of the multinomial coefficient which will be used below: let  $n$  be a nonnegative integer and  $l$  a positive integer. Then we write

$$\binom{n}{k_1, k_2, \dots, k_l} = \frac{n!}{k_1! k_2! \cdots k_l!}$$

for all  $k_1, k_2, \dots, k_l$  in  $\mathbb{N}$  satisfying  $k_1 + k_2 + \dots + k_l = n$ . Observe that for  $l = 2$  we obtain the binomial coefficient.

For the sake of completeness we clarify that in this paper the set of complex numbers is denoted by  $\mathbb{C}$ , the set of nonzero complex numbers by  $\mathbb{C}^\times$  and the set of nonnegative integers by  $\mathbb{N}$ .

## 2. Generalized Moment Functions on Groups

We begin with a multi-variable characterization of moment functions.

**Proposition 1.** *Let  $G$  be an Abelian group,  $l, N$  positive integers with  $l \geq 2$ , and let  $\varphi_0, \varphi_1, \dots, \varphi_N : G \rightarrow \mathbb{C}$  be functions for which*

$$\varphi_n(x_1 + x_2 + \dots + x_l) = \sum_{\substack{k_1, k_2, \dots, k_l \geq 0 \\ k_1 + k_2 + \dots + k_l = n}} \binom{n}{k_1, k_2, \dots, k_l} \prod_{t=1}^l \varphi_{k_t}(x_t) \quad (3)$$

holds for each  $x_1, x_2, \dots, x_l$  in  $G$  and  $n = 0, 1, \dots, N$ . If  $\varphi_0(0) = 0$ , then all the functions  $\varphi_0, \varphi_1, \dots, \varphi_N$  are identically zero. If  $\varphi_0(0) = 1$ , then the functions  $\varphi_0, \varphi_1, \dots, \varphi_N : G \rightarrow \mathbb{C}$  form a generalized moment sequence of order  $N$ .

*Proof.* Assume that the conditions of the proposition are satisfied. If  $l = 2$ , then equation (3) reduces to (1).

Suppose that  $l > 2$  and let  $x, y$  be in  $G$ . With the substitution  $x_1 = x, x_2 = y$  and  $x_i = 0$  for  $i = 3, 4, \dots, l$  we get from equation (3) that

$$\varphi_n(x + y) = \sum_{\substack{k_1, k_2, \dots, k_l \geq 0 \\ k_1 + k_2 + \dots + k_l = n}} \binom{n}{k_1, k_2, \dots, k_l} \varphi_{k_1}(x) \varphi_{k_2}(y) \cdot \prod_{t=3}^l \varphi_{k_t}(0)$$

for each  $n = 0, 1, \dots, N$ . Thus the only thing we should prove is that  $\varphi_n(0) = 0$  if  $n \geq 1$ . Indeed, in this case the right hand side of the above identity is nothing but  $\sum_{k=0}^n \binom{n}{k} \varphi_k(x) \varphi_{n-k}(y) \varphi_0(0)^{l-2}$ .

With  $n = 0$  and  $x_i = 0$  for  $i = 1, 2, \dots, l$  equation (3) leads to

$$\varphi_0(0) = \varphi_0(0)^l,$$

that is,

$$\varphi_0(0) \cdot (\varphi_0(0)^{l-1} - 1) = 0,$$

from which we derive that either  $\varphi_0(0) = 0$ , or  $\varphi_0(0)$  is an  $(l - 1)^{\text{th}}$  root of unity.

Furthermore, Eq. (3) for  $n = 0$  and  $x_1, x_2, \dots, x_l$  in  $G$  yields that

$$\varphi_0(x_1 + x_2 + \dots + x_l) = \prod_{i=1}^l \varphi_0(x_i),$$

If  $\varphi_0(0) = 0$ , then  $\varphi_0(x) = 0$  for all  $x$  in  $G$ . If  $\varphi_0^{l-1} = 1$ , then taking  $x_3 = \dots = x_l = 0$  and multiplying both sides of the above equation by  $\varphi_0^{l-2}$  we get

$$\varphi_0(0)^{l-2} \varphi_0(x_1 + x_2) = \varphi_0(0)^{l-2} \varphi_0(x_1) \varphi_0(x_2) \varphi_0(0)^{l-2}.$$

Thus  $\varphi_0(0)^{l-2} \varphi_0$  is an exponential, which we denote by  $m$ . Thus  $\varphi_0(0)^{l-2} \varphi_0(x) = m(x)$  for all  $x$  in  $G$ . Using the fact that  $\varphi_0(0)$  is an  $(l - 1)^{\text{th}}$  root of unity we obtain that

$$\varphi_0(x) = \varphi_0(0) \cdot m(x)$$

holds for each  $x$  in  $G$ .

Similarly, Eq. (3) for  $n = 1$  with  $x_1 = x$  and  $x_i = 0$  whenever  $x$  is in  $G$  and  $i = 2, 3, \dots, n$  yields that

$$\varphi_1(x) = \varphi_1(x) \varphi_0(0)^{l-1} + (l - 1) \varphi_0(x) \varphi_1(0) \varphi_0(0)^{l-2}.$$

It follows that

- (A) either  $\varphi_0(0) = 0$ , implying that both  $\varphi_0$  and  $\varphi_1$  are identically zero;
- (B) or  $\varphi_0(0)$  is an  $(l - 1)^{\text{th}}$  root of unity and

$$(l - 1) \varphi_0(x) \varphi_1(0) \varphi_0(0)^{l-2} = 0,$$

which happens only when  $\varphi_1(0) = 0$ , otherwise  $\varphi_0 \equiv 0$ , contrary to the fact that  $\varphi_0(0)$  is an  $(l - 1)^{\text{th}}$  root of unity.

Assume now that there exists  $k$  in  $\{1, 2, \dots, N - 1\}$  such that  $\varphi_0(0) \neq 0$  and

$$\varphi_1(0) = \varphi_2(0) = \dots = \varphi_k(0) = 0.$$

In this case Eq. (3), with  $k + 1$  instead of  $n$  and with the substitution  $x_1 = x$  in  $G$  and  $x_i = 0$  for  $i = 2, 3, \dots, l$ , yields that

$$\begin{aligned} \varphi_{k+1}(x) &= \sum_{\substack{t_1, t_2, \dots, t_l \geq 0 \\ t_1 + t_2 + \dots + t_l = k+1}} \binom{k+1}{t_1, t_2, \dots, t_l} \varphi_{t_1}(x) \varphi_{t_2}(0) \dots \varphi_{t_l}(0) \\ &= \varphi_{k+1}(x) \varphi_0^{l-1}(0) + \varphi_k(x) \varphi_1(0) \varphi_0(0)^{l-2} + \dots + \varphi_0(x) \varphi_0^{l-2}(0) \dots \varphi_{k+1}(0) \\ &= \varphi_{k+1}(x) \varphi_0^{l-1}(0) + \varphi_0(x) \varphi_0^{l-2}(0) \dots \varphi_{k+1}(0). \end{aligned}$$

Again, we have the following two alternatives:

- (A) either  $\varphi_0(0) = 0$ , implying that  $\varphi_{k+1}$  is identically zero;
- (B) or  $\varphi_0$  is an  $(l - 1)^{\text{th}}$  root of unity and

$$\varphi_k(x) \varphi_1(0) \varphi_0(0)^{l-2} + \dots + \varphi_0(x) \varphi_0(0)^{l-2} \dots \varphi_{k+1}(0) = 0,$$

or equivalently

$$(l - 1) \varphi_0(x) \varphi_0^{l-2}(0) \dots \varphi_{k+1}(0) = 0$$

for each  $x$  in  $G$ .

Due to the induction hypothesis  $\varphi_1(0) = \varphi_2(0) = \dots = \varphi_k(0) = 0$ , thus

$$\varphi_0(x)\varphi_0^{l-2}(0) \cdots \varphi_{k+1}(0) = 0.$$

Since  $\varphi_0(0) \neq 0$ ,  $\varphi_0$  is not identically zero, which shows that the only possibility is that  $\varphi_{k+1}(0) = 0$ .

Finally we conclude that  $\varphi_1(0) = \varphi_2(0) = \dots = \varphi_N(0) = 0$ . However, this implies for  $x, y$  in  $G$ :

$$\begin{aligned} & \sum_{\substack{k_1, k_2, \dots, k_l \geq 0 \\ k_1 + k_2 + \dots + k_l = n}} \binom{n}{k_1, k_2, \dots, k_l} \varphi_{k_1}(x) \varphi_{k_2}(x) \cdots \prod_{t=3}^l \varphi_{k_t}(0) \\ &= (\varphi_0(0))^{l-2} \sum_{k=0}^n \binom{n}{k} \varphi_k(x) \varphi_{n-k}(y). \end{aligned}$$

Therefore, if  $\varphi_0(0) = 1$ , then the functions  $\varphi_0, \varphi_1, \dots, \varphi_N$  form a generalized moment sequence of order  $N$ . □

**Proposition 2.** *Let  $G$  be an Abelian group, and let  $l, N$  be positive integers with  $l \geq 2$ . If the functions  $\varphi_0, \varphi_1, \dots, \varphi_N: G \rightarrow \mathbb{C}$  form a generalized moment sequence of order  $N$ , then the system of equations*

$$\varphi_n(x_1 + x_2 + \dots + x_l) = \sum_{\substack{k_1, k_2, \dots, k_l \geq 0 \\ k_1 + k_2 + \dots + k_l = n}} \binom{n}{k_1, k_2, \dots, k_l} \prod_{t=1}^l \varphi_{k_t}(x_t) \quad (4)$$

is satisfied for each  $x_1, x_2, \dots, x_l$  in  $G$  and  $n = 0, 1, \dots, N$ .

*Proof.* We prove the statement by induction on  $l$ .

Assume that functions  $\varphi_0, \varphi_1, \dots, \varphi_N$  constitute a moment sequence of order  $N$ , i.e.

$$\varphi_n(x + y) = \sum_{k=0}^n \binom{n}{k} \varphi_k(x) \varphi_{n-k}(y) \quad (5)$$

is satisfied for each  $x, y$  in  $G$ . First observe that for  $n = 0$  the statement holds trivially, because equation (4) applied for  $n = 0$  and the  $\varphi_0$  function in the moment sequence functions are the same, namely

$$\varphi_0(x + y) = \varphi_0(x) \varphi_0(y)$$

for  $x, y$  in  $G$ . Thus it is enough to consider the case  $n \geq 1$ . Secondly, note that from (5) we have  $\varphi_0(0) = 0$  or  $\varphi_0(0) = 1$  and for all  $n \geq 1$  we have  $\varphi_n(0) = 0$ .

Let now  $n$  in  $\{1, 2, \dots, N\}$  be arbitrarily fixed and take  $x_1, x_2$  in  $G$ . Observe that

$$\sum_{k=0}^n \binom{n}{k} \varphi_k(x_1) \varphi_{n-k}(x_2) = \sum_{k_1, k_2 \geq 0, k_1 + k_2 = n} \binom{n}{k_1, k_2} \varphi_{k_1}(x_1) \varphi_{k_2}(x_2),$$

for  $x_1, x_2$  in  $G$ , which yields that the statement holds for  $l = 2$ . Assume that there exists an integer  $l \geq 2$  for which the statement holds. Then

$$\begin{aligned} & \varphi_n(x_1 + \dots + (x_l + x_{l+1})) \\ &= \sum_{\substack{k_1, \dots, k_{l-1}, k_l + k_{l+1} \geq 0 \\ k_1 + \dots + k_{l-1} + k_l + k_{l+1} = n}} \binom{n}{k_1, \dots, k_{l-1}, k_l + k_{l+1}} \prod_{t=1}^{l-1} \varphi_{k_t}(x_t) \varphi_{k_l + k_{l+1}}(x_l + x_{l+1}) \end{aligned}$$

for any  $x_1, \dots, x_l, x_{l+1}$  in  $G$ . If we use that the statement holds for  $l = 2$  and the induction hypothesis, then we get that

$$\begin{aligned} & \sum_{\substack{k_1, \dots, k_{l-1}, k_l + k_{l+1} \geq 0 \\ k_1 + \dots + k_{l-1} + k_l + k_{l+1} = n}} \binom{n}{k_1, \dots, k_{l-1}, k_l + k_{l+1}} \prod_{t=1}^{l-1} \varphi_{k_t}(x_t) \varphi_{k_l + k_{l+1}}(x_l + x_{l+1}) \\ &= \sum_{\substack{k_1, \dots, k_{l-1}, k_l + k_{l+1} \geq 0 \\ k_1 + \dots + k_l + k_{l+1} = n}} \binom{n}{k_1, \dots, k_{l-1}, k_l + k_{l+1}} \\ & \prod_{t=1}^{l-1} \varphi_{k_t}(x_t) \binom{k_l + k_{l+1}}{k_l} \varphi_l(x_l) \varphi_{l+1}(x_{l+1}) = \\ & \sum_{\substack{k_1, k_2, \dots, k_{l+1} \geq 0 \\ k_1 + \dots + k_l + k_{l+1} = n}} \binom{n}{k_1, k_2, \dots, k_{l+1}} \prod_{t=1}^{l+1} \varphi_{k_t}(x_t) \end{aligned}$$

for  $x_1, \dots, x_l, x_{l+1}$  in  $G$ . Thus the statement holds for  $l + 1$ , too. □

### 3. Generalized Moment Functions of Rank $r$

In this section we extend the notion of generalized moment functions. At the same time, while stating and proving our results about generalized moment functions of higher rank, notions and results from spectral analysis and synthesis are required.

Let  $G$  be a discrete Abelian group. Subsequently, if it is not otherwise stated, the group  $G$  is always endowed with this topology.

Given a function  $f$  on this group and an element  $y$  in  $G$ , we define

$$\Delta_{f;y} = \delta_{-y} - f(y)\delta_0,$$

where  $\delta_{-y}$  and  $\delta_0$  denote the point masses concentrated at  $-y$  and  $0$ , respectively. For the product of modified differences we use the notation

$$\Delta_{f;y_1, y_2, \dots, y_{n+1}} = \prod_{i=1}^{n+1} \Delta_{f;y_i},$$

for any positive integer  $n$  and for each  $y_1, y_2, \dots, y_{n+1}$  in  $G$ . Here the product  $\prod$  on the right hand side is meant as a convolution product of the measures  $\Delta_{f;y_i}$  for all  $i = 1, 2, \dots, n + 1$ .

In the sequel, the symbol  $\mathbb{C}G$  denotes the group algebra of  $G$  and  $\mathcal{C}(G)$  the space of all (continuous) complex-valued functions on  $G$ . For each function  $f: G \rightarrow \mathbb{C}$  the ideal in  $\mathbb{C}G$  generated by all modified differences of the form  $\Delta_{f;y}$  with  $y$  in  $G$ , is denoted by  $M_f$ . Due to Theorem 12.5 of [9], this ideal  $M_f$  is proper if and only if  $f$  is an exponential and in this case we call  $M_f = \text{Ann } \tau(f)$  an *exponential maximal ideal*. Recall that  $\tau(f)$  denotes the variety of the function  $f$  and it is nothing but the smallest closed translation invariant subspace of  $\mathcal{C}(G)$  that contains the function  $f$ .

Let  $m$  be an exponential on the group  $G$  and  $n$  a positive integer. The function  $f: G \rightarrow \mathbb{C}$  is a *generalized exponential monomial* of degree at most  $n$  corresponding to the exponential  $m$  if

$$\Delta_{m;y_1,y_2,\dots,y_{n+1}} * f = 0$$

holds for each  $y_1, y_2, \dots, y_{n+1}$  in  $G$ . Equivalently,  $f$  is a generalized exponential monomial if there exists an exponential  $m$  and a positive integer  $n$  such that its annihilator includes a positive power of an exponential maximal ideal, that is  $M_m^{n+1} \subset \text{Ann } \tau(f)$  holds.

A special class of exponential monomials is formed by those corresponding to the exponential identically 1: these are called *generalized polynomials*.

The function  $f: G \rightarrow \mathbb{C}$  is called an *exponential monomial* if it is a generalized exponential monomial and  $\tau(f)$  is finite dimensional. Finite sums of exponential monomials are called *exponential polynomials*. Exponential monomials corresponding to the exponential identically 1 are called *polynomials*.

In connection with exponential polynomials here we also recall Theorem 12.31 from [9].

**Theorem 1.** *Let  $G$  be an Abelian group, and let  $f: G \rightarrow \mathbb{C}$  be an exponential polynomial. Then there exist positive integers  $n, k$  and for each  $i = 1, 2, \dots, n$  and  $j = 1, 2, \dots, k$  there exists a polynomial  $P_i: \mathbb{C}^k \rightarrow \mathbb{C}$ , an exponential  $m_i$  and a homomorphism  $a_j$  of  $G$  into the additive group of complex numbers such that*

$$f(x) = \sum_{i=1}^n P_i(a_1(x), a_2(x), \dots, a_k(x)) m_i(x)$$

for each  $x$  in  $G$ . Conversely, every function of this form is an exponential polynomial.

A *composition* of a nonnegative integer  $n$  is a sequence of nonnegative integers  $\alpha = (\alpha_k)_{k \in \mathbb{N}}$  such that

$$n = \sum_{k=1}^{\infty} \alpha_k.$$

For a positive integer  $r$ , an  $r$ -composition of a nonnegative integer  $n$  is a composition  $\alpha = (\alpha_k)_{k \in \mathbb{N}}$  with  $\alpha_k = 0$  for  $k > r$ .

Given a sequence of variables  $x = (x_k)_{k \in \mathbb{N}}$  and compositions  $\alpha = (\alpha_k)_{k \in \mathbb{N}}$  and  $\beta = (\beta_k)_{k \in \mathbb{N}}$  we define

$$\alpha! = \prod_{k=1}^{\infty} \alpha_k, \quad |\alpha| = \sum_{k=1}^{\infty} \alpha_k, \quad x^\alpha = \prod_{k=1}^{\infty} x_k^{\alpha_k}, \quad \binom{\alpha}{\beta} = \prod_{k=1}^{\infty} \binom{\alpha_k}{\beta_k}.$$

Furthermore,  $\beta \leq \alpha$  means that  $\beta_k \leq \alpha_k$  for all  $k \in \mathbb{N}$  and  $\beta < \alpha$  stands for  $\beta \leq \alpha$  and  $\beta \neq \alpha$ .

**Definition 1.** Let  $G$  be an Abelian group,  $r$  a positive integer, and for each multi-index  $\alpha$  in  $\mathbb{N}^r$  let  $f_\alpha: G \rightarrow \mathbb{C}$  be a continuous function. We say that  $(f_\alpha)_{\alpha \in \mathbb{N}^r}$  is a *generalized moment sequence of rank  $r$* , if

$$f_\alpha(x + y) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} f_\beta(x) f_{\alpha-\beta}(y) \tag{6}$$

holds whenever  $x, y$  are in  $G$ . The function  $f_0$ , where  $0$  is the zero element in  $\mathbb{N}^r$ , is called the *generating function* of the sequence.

*Remark 1.* (i) For  $r = 1$ , instead of multi-indices, we have ‘ordinary’ indices and (6) is nothing but

$$f_\alpha(x + y) = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} f_\beta(x) f_{\alpha-\beta}(y)$$

for each  $x, y$  in  $G$  and nonnegative integer  $\alpha$ , yielding that generalized moment functions of rank 1 are moment sequences.

(ii) For  $\alpha = (0, \dots, 0)$  we have

$$f_{0, \dots, 0}(x + y) = f_{0, \dots, 0}(x) \cdot f_{0, \dots, 0}(y)$$

for each  $x, y$  in  $G$  hence  $f_{0, \dots, 0} = m$  is an exponential, or identically zero. In what follows, when considering generalized moment function sequences of any rank, we always assume that the generating function is not identically zero, hence it is always an exponential.

(iii) Now let  $\alpha$  be in  $\mathbb{N}^r$  with  $|\alpha| = 1$ . In this case we have for each positive integer  $n$  that

$$f_{n \cdot \alpha}(x + y) = \sum_{k=0}^n \binom{n}{k} f_{k \cdot \alpha}(x) f_{(n-k) \cdot \alpha}(y)$$

for each  $x, y$  in  $G$ . Hence  $(f_{n \cdot \alpha})_{n \in \mathbb{N}}$  is a generalized moment function sequence associated with the exponential  $m = f_{0 \cdot \alpha}$ .

(iv) Using the above definition for  $r = 2$ , for example for  $|\alpha| \leq 2$  we have

$$\begin{aligned} f_{0,0}(x + y) &= f_{0,0}(x)f_{0,0}(y) \\ f_{1,0}(x + y) &= f_{1,0}(x)f_{0,0}(y) + f_{1,0}(y)f_{0,0}(x) \\ f_{0,1}(x + y) &= f_{0,1}(x)f_{0,0}(y) + f_{0,1}(y)f_{0,0}(x) \\ f_{2,0}(x + y) &= f_{2,0}(x)f_{0,0}(y) + 2f_{1,0}(x)f_{1,0}(y) + f_{2,0}(y)f_{0,0}(x) \\ f_{0,2}(x + y) &= f_{0,2}(x)f_{0,0}(y) + 2f_{0,1}(x)f_{0,1}(y) + f_{0,2}(y)f_{0,0}(x) \\ f_{1,1}(x + y) &= f_{1,1}(x)f_{0,0}(y) + f_{1,0}(x)f_{0,1}(y) \\ &\quad + f_{1,0}(y)f_{0,1}(x) + f_{1,1}(y)f_{0,0}(x) \end{aligned}$$

for each  $x, y$  in  $G$ .

In view of the first equation (or using remark (ii)), we immediately get that the function  $f_{0,0} = m$  is an exponential. Furthermore, due to the second and the third equation, the functions  $f_{0,1}$  and  $f_{1,0}$  are  $m$ -sine functions. In other words,  $\{f_{0,0}, f_{0,1}\}$  and  $\{f_{0,0}, f_{1,0}\}$  form a moment sequence of order 1. More generally, due to remark (iii),  $(f_{0,n})_{n \in \mathbb{N}}$  and also  $(f_{n,0})_{n \in \mathbb{N}}$  is a moment function sequence associated with the exponential  $f_{0,0} = m$ .

(v) Assume that  $(f_\alpha)_{\alpha \in \mathbb{N}^2}$  is a generalized moment sequence of rank two. For all nonnegative integer  $n$ , define the function  $\varphi_n : G \rightarrow \mathbb{C}$  by

$$\varphi_n(x) = \sum_{k=0}^n \binom{n}{k} f_{k,n-k}(x)$$

for  $x$  in  $G$ . Then  $(\varphi_n)_{n \in \mathbb{N}}$  is a moment function sequence associated with the exponential  $\varphi_0 = f_{0,0} = m$ .

Indeed, for  $n = 0$ ,

$$\varphi_0(x) = f_{0,0}(x)$$

for  $x$  in  $G$ , which is an exponential, as we wrote above.

For each positive  $n$ , we have

$$\begin{aligned} &\sum_{k=0}^n \binom{n}{k} \varphi_k(x)\varphi_{n-k}(y) \\ &= \sum_{k=0}^n \binom{n}{k} \cdot \left[ \sum_{i=0}^k \binom{k}{i} f_{i,k-i}(x) \right] \cdot \left[ \sum_{j=0}^{n-k} \binom{n-k}{j} f_{j,n-k-j}(y) \right] \\ &= \sum_{k=0}^n \binom{n}{k} \left[ \sum_{i=0}^k \sum_{j=0}^{n-k} \binom{k}{i} \binom{n-k}{j} f_{i,k-i}(x)f_{j,n-k-j}(y) \right] \\ &= \sum_{k=0}^n \binom{n}{k} f_{k,n-k}(x + y) = \varphi_n(x + y) \end{aligned}$$

for each  $x, y$  in  $G$ .

(vi) The sequence of functions  $(f_\alpha)_{\alpha \in \mathbb{N}^r}$  is a generalized moment sequence of rank  $r$  associated with the nonzero exponential  $f_{0,\dots,0} = m$  if and only if

$(f_\alpha/m)_{\alpha \in \mathbb{N}^r}$  is a generalized moment sequence of rank  $r$  associated with the exponential which is identically one.

- (vii) Let  $(f_\alpha)_{\alpha \in \mathbb{N}^r}$  be a generalized moment sequence of rank  $r$  and denote  $m = f_{0, \dots, 0}$ . Then for all multi-index  $\alpha$  in  $\mathbb{N}^r$ , the function  $f_\alpha$  is a generalized exponential monomial of degree at most  $|\alpha|$  corresponding to the exponential  $m$ . Thus for any multi-index  $\alpha$  in  $\mathbb{N}^r$ , there exists an a generalized polynomial  $p_\alpha : G \rightarrow \mathbb{C}$  of degree at most  $|\alpha|$  such that  $f_\alpha = p_\alpha \cdot m$ . If  $|\alpha| = 0$ , that is, if  $\alpha = (0, 0, \dots, 0)$ , then  $f_{0, \dots, 0} = m$  is an exponential. Thus the above statement holds trivially.

In case  $|\alpha| = 1$ , then there exists  $i \in \{1, 2, \dots, r\}$  such that

$$\alpha_i = 1 \quad \text{and} \quad \alpha_j = 0 \quad \text{for any } j \in \{1, \dots, r\}, j \neq i.$$

This, in view of remark (iii) implies that  $f_\alpha$  is an  $m$ -sine function, i.e.

$$f_\alpha(x + y) = f_\alpha(x)m(y) + m(x)f_\alpha(y)$$

for  $x, y$  in  $G$ . This implies however that

$$\Delta_{m; y_1, y_2} f_\alpha(x) = 0,$$

$x, y_1, y_2$  in  $G$ , in other words  $f_\alpha$  is a generalized exponential monomial of degree at most one corresponding to the exponential  $m$ .

Assume now that there exists a multi-index  $\alpha$  in  $\mathbb{N}^r$  such that the statement holds for any multi-index  $\beta$  for which  $\beta < \alpha$ . Since

$$f_\alpha(x + y) = \sum_{\beta < \alpha} \binom{\alpha}{\beta} f_\beta(x) f_{\alpha - \beta}(y)$$

for each  $x, y \in G$ , we have

$$\Delta_{m; y} f_\alpha(x) = \sum_{\beta < \alpha} \binom{\alpha}{\beta} f_\beta(x) f_{\alpha - \beta}(y)$$

for every  $x, y$  in  $G$ . Observe that the right hand side of this equation (as a functions of the variable  $x$ ) is, due to the induction hypothesis, a generalized exponential polynomial of degree at most  $|\alpha| - 1$ , showing that  $f_\alpha$  is a generalized exponential monomial of degree at most  $|\alpha|$ .

- (viii) The previous remark can be strengthened. Namely, generalized moment sequences of rank  $r$  not only generalized exponential monomials, but they are exponential monomials. This means that the variety of any such function is finite dimensional. Indeed, let  $(f_\alpha)_{\alpha \in \mathbb{N}^r}$  be a generalized moment sequence of rank  $r$ . Then for any multi-index  $\alpha$  in  $\mathbb{N}^r$  we have

$$f_\alpha(x + y) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} f_\beta(x) f_{\alpha - \beta}(y)$$

for every  $x, y$  in  $G$  showing that

$$\tau(f_\alpha) \subseteq \text{span} \{f_\beta \mid \beta \leq \alpha\}.$$

The linear space on the right hand side is obviously finite dimensional, implying the same for  $\tau(f)$ .

We conclude that the polynomial  $p_\alpha$ , appearing in remark (viii), is a complex polynomial of some complex-valued additive functions defined on  $G$ .

We can summarize that, for any multi-index  $\alpha \in \mathbb{N}^r$ , there exists a positive integer  $n(\alpha)$  and a  $P_\alpha: \mathbb{C}^{n(\alpha)} \rightarrow \mathbb{C}$  and additive functions  $a_i: G \rightarrow \mathbb{C}$ ,  $i = 1, \dots, n(\alpha)$  such that

$$f_\alpha(x) = P_\alpha(a(x))m(x)$$

for each  $x \in G$ . Here  $a_i$  denotes the  $i^{\text{th}}$  coordinate functions of the additive function  $a$ .

- (ix) If  $k$  and  $1 \leq n_1 < n_2 < \dots < n_k \leq r$  are positive integers, then let  $\pi_{n_1, \dots, n_k} \alpha$  denote that element in  $\mathbb{N}^r$  which differs from  $\alpha$  in only those coordinates that do not belong to the set  $\{n_1, \dots, n_k\}$ , and those coordinates are zero.

With this notation, let  $k$  and  $r$  be positive integers with  $k \leq r$  and  $1 \leq n_1 < n_2 < \dots < n_k \leq r$  be also positive integers. If  $(f_\alpha)_{\alpha \in \mathbb{N}^r}$  is a generalized moment sequence of rank  $r$  on the group  $G$ , then the sequence of functions  $(f_{\pi_{n_1, \dots, n_k} \alpha})_{\alpha \in \mathbb{N}^r}$  form a generalized moment sequence of rank  $k$ .

As we will see, while describing the polynomial  $P$  appearing in remark (ix), a well-known sequence of polynomials and an appropriate addition formula will play a distinguished role. Let us consider the sequence of complex polynomials  $(B_n)_{n \in \mathbb{N}}$  defined through the following recurrence: for each  $t, t_1, t_2, \dots, t_{n+1}$  in  $\mathbb{C}$  we let

$$B_0(t) = 1,$$

$$B_{n+1}(t_1, \dots, t_{n+1}) = \sum_{i=0}^n \binom{n}{i} B_{n-i}(t_1, \dots, t_{n-i}) t_{i+1}$$

for each  $n$  in  $\mathbb{N}$ . Alternatively, we can also use the double series expansion of the generating function

$$\exp \left( \sum_{j=1}^{\infty} x_j \frac{t^j}{j!} \right) = \sum_{n=0}^{\infty} B_n(x_1, \dots, x_n) \frac{t^n}{n!}.$$

We call  $B_n$  the  $n^{\text{th}}$  complete (exponential) Bell polynomial.

For these functions the following addition formula holds:

$$B_n(t_1 + u_1, \dots, t_n + u_n)$$

$$= B_n(t_1, \dots, t_n) + B_n(u_1, \dots, u_n) + \sum_{k=1}^{n-1} \binom{n}{k} B_{n-k}(t_1, \dots, t_{n-k}) B_k(u_1, \dots, u_k)$$

for any positive integer  $n$ , and for each complex numbers  $t_1, \dots, t_n$  and  $u_1, \dots, u_n$ .

The first few complete (exponential) Bell polynomials are

$$\begin{aligned} B_0 &= 1 \\ B_1(x_1) &= x_1 \\ B_2(x_1, x_2) &= x_1^2 + x_2 \\ B_3(x_1, x_2, x_3) &= x_1^3 + 3x_1x_2 + x_3 \\ B_4(x_1, x_2, x_3, x_4) &= x_1^4 + 6x_1^2x_2 + 4x_1x_3 + 3x_2^2 + x_4 \end{aligned}$$

It turns out, however, that these functions are useful in the rank one case, only. To cover the general case as well, a multivariate extension of the Bell polynomials is necessary. Here we follow the notation and the terminology of [7].

The *multivariate Bell polynomials* can be introduced using the double series expansion of the generating function

$$\sum_{0 \leq |\alpha|} B_\alpha(x) \frac{t^\alpha}{\alpha!} = \exp \left( \sum_{1 \leq |\mu|} x_\mu \frac{t^\mu}{\mu!} \right).$$

Let now  $\alpha = (\alpha_k)_{k \in \mathbb{N}}$  be a composition and  $x = (x_k)_{k \in \mathbb{N}}$  and  $y = (y_k)_{k \in \mathbb{N}}$  be sequences of variables. Then

$$B_\alpha(x + y) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} B_\beta(x) B_{\alpha - \beta}(y).$$

For instance, if  $r = 2$ , then the first few multivariate Bell polynomials are the following:

$$\begin{aligned} B_{0,0}(x) &= 1 \\ B_{0,1}(x_{0,1}) &= x_{0,1} \\ B_{1,0}(x_{1,0}) &= x_{1,0} \\ B_{1,1}(x_{0,1}, x_{1,0}, x_{1,1}) &= x_{0,1}x_{1,0} + x_{1,1} \\ B_{2,0}(x_{0,1}, x_{1,0}, x_{1,1}, x_{2,0}) &= x_{1,0}^2 + x_{2,0} \\ B_{0,2}(x_{0,1}, x_{1,0}, x_{1,1}, x_{2,0}) &= x_{0,1}^2 + x_{0,2} \\ B_{2,1}(x_{0,1}, x_{1,0}, x_{1,1}, x_{2,0}, x_{2,1}) &= x_{0,1}x_{1,0}^2 + 2x_{1,0}x_{1,1} + x_{0,1}x_{2,0} + x_{2,1} \\ B_{2,2}(x_{0,1}, x_{1,0}, x_{1,1}, x_{2,0}, x_{1,2}, x_{2,1}, x_{2,2}) &= x_{0,1}^2x_{1,0}^2 + x_{0,2}x_{1,0}^2 + 4x_{0,1}x_{1,0}x_{1,1} \\ &\quad + 2x_{1,1}^2 + 2x_{1,0}x_{1,2} + x_{0,1}^2x_{2,0} \\ &\quad + x_{0,2}x_{2,0} + 2x_{0,1}x_{2,1} + x_{2,2} \end{aligned}$$

It is important to point out that if  $r = 1$  then the multivariate Bell polynomials reduce to to the complete (exponential) Bell polynomials. For any multi-index  $\alpha \in \mathbb{N}^r$ , the polynomial  $B_\alpha$  has at most  $\prod_{i=1}^r (\alpha_i + 1)$  variables, where  $\alpha = (\alpha_1, \dots, \alpha_r)$  and its degree is exactly  $|\alpha|$ .

**Proposition 3.** *Let  $G$  be a commutative group,  $r$  a positive integer,  $m: G \rightarrow \mathbb{C}$  an exponential and let  $a = (a_\alpha)_{\alpha \in \mathbb{N}^r}$  be a sequence of complex-valued additive functions defined on  $G$ . We define the functions  $f_\alpha: G \rightarrow \mathbb{C}$  by*

$$f_\alpha(x) = B_\alpha(a(x))m(x)$$

for every  $x$  in  $G$ . Then  $(f_\alpha)_{\alpha \in \mathbb{N}^r}$  forms a generalized moment sequence of rank  $r$  associated with the exponential  $m$ .

*Proof.* If  $|\alpha| = 0$ , then the above formula is

$$f_{0,\dots,0}(x) = m(x)$$

for each  $x$  in  $G$ , that is,  $f_{0,\dots,0}$  is an exponential.

Let now  $\alpha$  be in  $\mathbb{N}^r$  with  $|\alpha| > 0$ , then

$$\begin{aligned} f_\alpha(x + y) &= B_\alpha(a(x + y))m(x + y) = B_\alpha(a(x) + a(y))m(x)m(y) \\ &= \sum_{\beta} \binom{\alpha}{\beta} B_\beta(a(x))B_{\alpha-\beta}(a(y))m(x)m(y) \\ &= \sum_{\beta} \binom{\alpha}{\beta} B_\beta(a(x))m(x) \cdot B_{\alpha-\beta}(a(y))m(y) = \sum_{\beta} \binom{\alpha}{\beta} f_\beta(x)f_{\alpha-\beta}(y) \end{aligned}$$

for each  $x, y$  in  $G$ . □

The following result is about the converse of the previous statement.

**Theorem 2.** *Let  $G$  be a commutative group,  $r$  a positive integer, and for each  $\alpha$  in  $\mathbb{N}^r$ , let  $f_\alpha: G \rightarrow \mathbb{C}$  be a function. If the sequence of functions  $(f_\alpha)_{\alpha \in \mathbb{N}^r}$  forms a generalized moment sequence of rank  $r$ , then there exists an exponential  $m: G \rightarrow \mathbb{C}$  and a sequence of complex-valued additive functions  $a = (a_\alpha)_{\alpha \in \mathbb{N}^r}$  such that for every multi-index  $\alpha$  in  $\mathbb{N}^r$  and  $x$  in  $G$  we have*

$$f_\alpha(x) = B_\alpha(a(x))m(x).$$

*Proof.* Let  $r$  be a fixed positive integer,  $G$  be a commutative group, let furthermore for each  $\alpha \in \mathbb{N}^r$ ,  $f_\alpha: G \rightarrow \mathbb{C}$  be a function. Assume that the sequence of functions  $(f_\alpha)_{\alpha \in \mathbb{N}^r}$  forms a generalized moment sequence of rank  $r$ . We define a sequence  $a = (a_\alpha)$  and prove the statement by induction on the height of the multi-index  $\alpha$ .

If  $|\alpha| = 0$ , then we necessarily have  $\alpha = (0, 0, \dots, 0)$  and

$$f_{0,0,\dots,0}(x + y) = f_{0,0,\dots,0}(x)f_{0,0,\dots,0}(y)$$

for every  $x, y$  in  $G$ , which gives immediately that there exists an exponential  $m: G \rightarrow \mathbb{C}$  such that  $f_{0,0,\dots,0} = m$ . Furthermore, if  $|\alpha| = 1$ , then there exists  $i \in \{1, 2, \dots, r\}$  such that  $\alpha_i = 1$  and  $\alpha_j = 0$  for each  $j$  in  $\{1, 2, \dots, r\}$ ,  $j \neq i$ . In view of remark (iii) this implies that  $f_\alpha$  is an  $m$ -sine function, that is

$$f_\alpha(x) = B_\alpha(a(x))m(x)$$

holds for  $x$  in  $G$ .

Now we assume that there exists a multi-index  $\alpha$  in  $\mathbb{N}^r$  such that the statement holds true for every multi-index  $\beta$  with  $\beta < \alpha$ . This means that for any multi-index  $\beta < \alpha$  there exists an additive function  $a_\beta: G \rightarrow \mathbb{C}$  such that the representation in the statement of the theorem holds. We have to show

how the additive function  $a_\alpha: G \rightarrow \mathbb{C}$  should be constructed. Since  $f_\alpha$  is a generalized moment function of order  $\alpha$  and of rank  $r$ , we have

$$f_\alpha(x + y) - f_\alpha(x)m(y) - m(x)f_\alpha(y) = \sum_{0 < \beta < \alpha} \binom{\alpha}{\beta} f_\beta(x)f_{\alpha-\beta}(y) = \sum_{0 < \beta < \alpha} \binom{\alpha}{\beta} B_\beta(a(x))m(x)B_{\alpha-\beta}(a(y))m(y)$$

for every  $x, y$  in  $G$ . Then, due to the addition formula of the function  $B_\alpha$ , we have that

$$B_\alpha(a(x + y), 0)m(x + y) - B_\alpha(a(x), 0)m(x) \cdot m(y) - B_\alpha(a(y), 0)m(y) \cdot m(x) = \sum_{0 < \beta < \alpha} \binom{\alpha}{\beta} B_\beta(a(x))m(x)B_{\alpha-\beta}(a(y))m(y)$$

holds for each  $x, y$  in  $G$ . Observe, that this means that the function  $S_\alpha: G \rightarrow \mathbb{C}$  defined by

$$S_\alpha(x) = f_\alpha(x) - B_\alpha(a(x), 0)$$

for  $x$  in  $G$  is an  $m$ -sine function. Thus, there exists an additive function  $\eta: G \rightarrow \mathbb{C}$  such that  $S(x) = \eta(x)m(x)$  for all  $x$  in  $G$ , yielding that

$$f_\alpha(x) = B_\alpha(a(x), 0)m(x) + \eta(x)m(x) = B_\alpha(a(x), \eta(x))m(x)$$

for each  $x$  in  $G$ , since  $B_\alpha$  is additive in its last variable.

This means that there exists an additive function  $\xi: G \rightarrow \mathbb{C}$  such that

$$f_\alpha(x) = B_\alpha(a(x), \xi(x)) \cdot m(x)$$

for each  $x$  in  $G$ . This shows that how the  $\alpha^{\text{th}}$  element of the sequence  $a$  should be constructed. □

As a consequence of this result, the characterization of generalized moment functions (of rank one) follows. Namely, functions of the form  $(B_n \circ a) \cdot m$  are generalized moment functions. More precisely, we have the following.

**Corollary 1.** *Let  $G$  be a commutative group,  $n$  an arbitrary positive integer,  $m: G \rightarrow \mathbb{C}$  an exponential, and let  $a_1, a_2, \dots, a_n: G \rightarrow \mathbb{C}$  be additive functions. We define the sequence of functions  $(f_n)_{n \in \mathbb{N}}$  for each  $x$  in  $G$  by*

$$f_n(x) = B_n(a_1(x), \dots, a_n(x))m(x).$$

*Then  $(f_n)_{n \in \mathbb{N}}$  forms a generalized moment sequence associated with the exponential  $m$ .*

On the other hand, we can deduce the following result:

**Corollary 2.** *Let  $G$  be a commutative group, and for each natural number  $n$  let  $f_n: G \rightarrow \mathbb{C}$  be a function. If the sequence  $(f_n)_{n \in \mathbb{N}}$  forms a generalized moment sequence, then there exists an exponential  $m: G \rightarrow \mathbb{C}$  and there are additive functions  $a_1, a_2, \dots, a_n: G \rightarrow \mathbb{C}$  such that*

$$f_n(x) = B_n(a_1(x), \dots, a_n(x))m(x)$$

holds for every  $x$  in  $G$  and  $n = 1, 2, \dots$ .

It should be noted that the results from this section coincide with the formula (2) on the page 2 proved by J. Aczél [1] in a much more general settings.

## 4. Summary and Further Research

In this paper we have introduced the notion of generalized moment functions of higher rank defined on a commutative group. We characterize them as the product of an exponential and composition of multivariate Bell polynomial and an additive function. Our next aim is to find a characterization of generalized moment functions of higher rank in the hypergroup settings.

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### Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.

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