Geometric vector fields of spray and metric structures

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Contents

Chapter 1

Introduction

1.1 Affine and projective maps

The two kinds of differential geometric structures in the title of the thesis are both generalizations of Finsler structures. The basic idea of Finsler geometry may be found already in Riemann's famous Habilitationsvortrag, where he suggested the study of geometric spaces where 'the metric depends also on the direction'. It was Paul Finsler who started the systematic study of these spaces in 1918 in his dissertation [16], most probably independently of Riemann's talk and motivated by completely different ideas. In our approach, a Finsler manifold will be a special case of a manifold endowed with a generalized metric; the details will be described in section 5.1.

The main domains of differential geometry can be characterized by certain objects specified on a differentiable manifold. A diffeomorphism between two differentiable manifolds completely 'carries' the differentiable structure from the first one to the second. Therefore, given any differential geometric object or structure on the first manifold, we may construct a corresponding object of the same type on the second one with the help of the diffeomorphism. Hence, given a manifold endowed with a structure, it is a meaningful question to ask whether a diffeomorphism of our manifold onto itself preserves the structure or not. If it does, it is said to be a (global) *symmetry*. A certain branch of differential geometry may be regarded as the theory of the invariants of these symmetries. In the case of a diffeomorphism between two open subsets of the manifold we speak of a *local symmetry*. The flow of a vector field on a manifold consists of the integral curves of the vector field as a first order ordinary differential equation. (We shall present a precise definition of flows in section 2.3.) With a fixed value of the parameter, a flow yields a diffeomorphism between two open submanifolds. If it is a local symmetry, the vector field is a *geometric vector field* or an *infinitesimal symmetry*.

In the case of a Finsler manifold there is a natural second-order differential equation on the tangent manifold whose coefficients are positively homogeneous functions of degree 2. It is a natural geometric problem to study differential equations of this type which does not emerge from a Finsler structure, and which are nowadays called *sprays*. A spray is uniquely determined by its parametrized integral curves, therefore the geometry of sprays is sometimes called *the geometry of paths*.

Spray geometry had its first golden age in the 1930's. Several outstanding mathematicians worked in this field, e.g., L. Berwald, É. Cartan, J. Douglas, T. Y. Thomas and J. H. C. Whitehead, among many others. In the 1960's, in relative isolation, A. Rapcsák also obtained important results in spray geometry ([43], see also [55]). Its renaissance began in the 1970's, partly by the geometric foundation of Lagrangian mechanics [8, 9] and partly by the discovery of the fundamental role of sprays in Finsler geometry [14, 18, 19]. Finsler geometry and its generalizations gave a new impulse to these studies in the late 1990's, mainly due to the activity of Zhongmin Shen [50] and the development of Grifone's theory [54, 55], but also through other lines of research [39, 45].

In this renaissance of spray theory, however, an important field has been rather neglected: the modern foundation of the theory of transformations of spray manifolds, mainly the infinitesimal ones, although Youssef's paper [64] and the above mentioned papers [54, 55] represent a definite progress towards the projective theory of spray manifolds. Although several authors have dealt with projective transformations of Finsler manifolds [4, 34, 35, 58], there is hardly any literature available about infinitesimal affine and projective transformations of spray manifolds, save Yano's classical book [63], which is, however, rather laborious to read for the present-day reader. Chapter 3 of this thesis is intended to take the first few steps to cover this important field. We note that two important works on projective connections [11, 12] became available roughly simultaneously with the writing of this thesis.

Apart from the general theoretical motivations outlined above, we find these problems interesting also from the point of view of physical interpretation. Since a spray is a vector field on the tangent manifold, its integral curves run in the tangent manifold. Their projections onto the base manifold are called the *geodesics* of the spray. A diffeomorphism between two spray manifolds is said to be an affine map if it takes (parametrized) geodesic curves to geodesic curves. More generally, if it takes geodesics to pregeodesics, then it is called a projective map. Here, by a pregeodesic we mean a curve which can be reparametrized to be a geodesic by a strictly increasing parameter transformation. An affine (projective) vector field on a spray manifold is a vector field whose flow consists of local affine (projective) transformations. Affine vector fields are geometric vector fields of spray manifolds in the sense described above. Projective vector fields become geometric vector fields in the same sense if we introduce the notion of projective equivalence of sprays. Roughly speaking, two sprays over the same base manifold are projectively equivalent if their geodesics are the same up to a strictly increasing parameter transformations. (The same idea will be formulated in 3.3.1 in a more precise manner.) Then a projective vector field becomes an infinitesimal symmetry of the structure determined by the equivalence class of all sprays projectively equivalent to the given one.

In the geometric interpretation of time-independent classical mechanical systems the base manifold is the configuration space, which is the space of all possible positions of the objects in the mechanical system. The parameter of the paths has a very important physical meaning, namely, the time. In general relativity, the base manifold is the space-time, which is a 4-dimensional Lorentz manifold. Here, time is included in the structure of the base manifold, hence the parameter of the paths of point particles has no real physical relevance. Therefore, the group of transformations leaving the physical interpretation of paths invariant in classical mechanics is the group of affine transformations, whereas in general relativity it is the group of projective transformations.

1.2 Generalized metrics

Working on a Finsler manifold, the Hessian of the Finsler energy is a symmetric, non-degenerate $(0, 2)$ tensor in the pull-back bundle $\tau^*\tau$ of the tangent bundle τ by itself. This tensor is usually known as the Finsler (or Riemann – Finsler) metric of the Finsler manifold. By a generalized metric we shall mean a straightforward generalization of this notion: a symmetric non-degenerate $(0, 2)$ tensor in $\tau^*\tau$ which does not necessarily arise from a Finsler structure. For technical reasons, we shall not assume that this metric tensor is defined on the whole tangent manifold, but only on some open subset of it.

The study of metrics of this type dates back to the 1950's. The Debrecen school contributed to this theory with a pioneering work [40], which inspired further studies [61]. Later, the Romanian differential geometric school achieved important results on generalized metrics [5, 38, 39]. A new classification for them has been published recently [36]. Some of their characteristic properties in which they differ from Finsler structures were already pointed out in [40], e.g., the fact that their autoparallel and extremal curves do not necessarily coincide, even with a natural choice of a covariant derivative. These metrics may be interesting not only from a geometrical, but also from a physical viewpoint, since, as noticed by J. I. Horváth and A. Moór [22], they furnish a natural geometric description of the so-called bilocal field theories introduced by Yukawa in the 1940's. Yukawa's main goal was to explain mass quantization and to eliminate certain types of divergences in quantum field theory. For a review on multi-local field theories, which were developed as generalizations of bilocal field theories, we refer to [44]. In this thesis, however, we restrict ourselves to the geometric aspects of generalized metrics; we wish to consider physical implications later.

In 1987, Makoto Matsumoto wrote the following: 'Through the author's several experiences the author became convinced that there should exist the *best* Finsler connection for every theory of Finsler spaces.' In Chapter 5 of this thesis we tried to extend Matsumoto's principle to the quite strange world of generalized metrics, and to study the following heuristically formulated problem: find the 'best' or, at least, a 'good' metric derivative for the different special classes of generalized metrics.

We say, informally, that a metric derivative is 'good' if there is an Ehresmann connection attached to it determined by the metric alone. In the case of a Moór – Vanstone metric, we have a natural Ehresmann connection: the Barthel connection $\mathcal{H}_E$ determined by the (Finslerian) absolute energy. On the other hand, given an Ehresmann connection on $\overset{\circ}{T}M$, there is a unique metric covariant derivative in $\overset{\circ}{\tau}{}^*\tau$ whose vertical torsion vanishes and whose horizontal torsion coincides with the torsion of the given Ehresmann connection. However, the metric derivative arising from $\mathcal{H}_E$ is not 'good' in the sense above. Thus, to obtain a 'good' metric derivative, we have to look for a more suitable Ehresmann connection than $\mathcal{H}_E$.

The geometric vector fields or infinitesimal symmetries of a manifold endowed with a generalized metric are the so-called Killing vector fields. In general relativity, they express the infinitesimal symmetries of space-time. Therefore, it is an important problem to determine the Killing vector fields of different classes of generalized metrics. In a Euclidean space, translations are distinguished from other types of isometries by the property that their orbits are straight lines. This property is used to generalize the notion of translations to more general classes of metrics: translations are Killing vector fields whose integral curves are at the same time geodesics (in some sense, to be described more precisely in section 6.3). In this work we also study the translations of a certain type of generalized metrics.

I note that I did not aim at completeness in this thesis. Namely, I do not treat infinitesimal conformal transformations at all. The reason for this is that I have no new results in this field, in contrast to the other topics treated. As for conformal transformations of Finsler manifolds, I refer to [3, 20, 21, 56].

1.3 The outline of the thesis

Chapter 2 may be regarded as a preparatory part, since it does not contain any essentially new result; it only makes the thesis more or less self-contained. In Chapter 3 we derive several necessary and sufficient conditions for a vector field to be affine or projective. To the latter we shall use a recently formulated characterization of projectively equivalent sprays in terms of Yano's covariant derivative [53]. Many of these results are already known for manifolds with an affine (i.e., quadratic) spray (cf. [10, 42]). The novelty lies mainly in the fact that we prove these results in a coordinate-free manner and for arbitrary sprays.

Finsler – Minkowski vector spaces are closely related to Finsler manifolds, therefore we devoted Chapter 4 to them. The observation 4.3 is the main original result in this chapter.

Chapters 5 and 6 contain the most of original parts. In section 5.1 we collected some basic facts about generalized metrics. In 5.2 we give a complete description of metric derivatives attached to an Ehresmann connection under certain conditions. These results were motivated by [23], we proceed here, however, much further. In 5.3 we find a covariant derivative having some of the good properties of Cartan's under more general circumstances. Finally, in 5.4 we show how these achievements give back the proof of the uniqueness of Cartan's covariant derivative on a Finsler manifold, with no Ehresmann connection given in advance. To our knowledge, this is the first coordinate-free proof for this result.

Chapter 6 is devoted to our results on Killing vector fields. In section 6.1 we have collected those which are relevant to all generalized metrics. We discuss the Killing vector fields of special types of metrics in section 6.2. In section 6.3 we study the translations of weakly normal and Miron regular metrics, and we also discuss some open problems. Section 6.4 contains applications to Randers manifolds, while in section 6.5 we give a visualizable geometric characterization of the Killing fields of a Funk metric.

In the last two chapters we summarized our main results in English and in Hungarian. Section 7.1 comprises once more our notations. In section 7.2 we enumerated our results in the projective geometry of sprays. In the last two sections we collected our results concerning manifolds endowed with generalized metrics. Chapter 8 is roughly the translation of Chapter 7. The Appendix contains local coordinate expressions of some important objects.

Chapter 2

Preliminary constructions

2.1 The pull-back bundle

In this thesis the term 'manifold' will always mean a finite-dimensional, second countable, Hausdorff-type manifold of class C^∞. We shall work on a connected n-dimensional manifold M ($n \in \mathbb{N}^*$). The ring of smooth functions on the manifold M will be denoted by $C^\infty(M)$, and the $C^\infty(M)$-module of smooth vector fields on M by $\mathfrak{X}(M)$. The tangent manifold of M will be denoted by TM, and $\tau : TM \to M$ will be the tangent bundle of M.

If $\widetilde{TM} \subset TM$ is an open set such that $\tau\left(\widetilde{TM}\right) = M$, then the restriction of τ to $\widetilde{TM}$ will be denoted by $\pi := \tau \restriction \widetilde{TM}$. Then we can construct a vector bundle over $\widetilde{TM}$, called the the *pull-back bundle* $\pi^*\tau$ of τ over π as follows. The total space of $\pi^*\tau$ will be

$$\pi^*TM := \widetilde{TM} \times_M TM := \left\{ (v, w) =: w_v \in \widetilde{TM} \times TM \middle| \pi(v) = \tau(w) \right\}.$$

The projection $\pi^*\tau : \pi^*TM \to \widetilde{TM}$ is defined by $(\pi^*\tau)(w_v) = v$, and the real vector space structure of $(\pi^*TM)_v$ by

$$\alpha(w_1)_v + \beta(w_2)_v := (\alpha w_1 + \beta w_2)_v \quad (\alpha, \beta \in \mathbb{R},\ w_1, w_2 \in T_{\pi(v)}M).$$

Thus, the vector space $(\pi^*TM)_v$ may be canonically identified with $T_{\pi(v)}M$, and a section $\tilde{X} : \widetilde{TM} \to \pi^*TM$ (also called a *vector field along* π) with a smooth map $\underline{X} : \widetilde{TM} \to TM$ such that $\tau \circ \underline{X} = \pi$. We shall omit the distinction between $\tilde{X}$ and $\underline{X}$, and we use the two interpretations of $\tilde{X}$ freely and interchangeably.

These sections form a $C^\infty\left(\widetilde{TM}\right)$-module, which is denoted by $\mathfrak{X}(\pi)$, while $\mathcal{T}(\pi)$ denotes the tensor algebra of this module. The elements of $\mathcal{T}(\pi)$ will be also mentioned as *tensors along* π.

If M and N are two manifolds, and $f : M \to N$ is a smooth map, then, as usual, $f_* : TM \to TN$ will denote its tangent map. However, to avoid confusion, if f is a diffeomorphism, then the resulting isomorphisms between the modules of vector fields will be denoted by $f_\sharp$. Namely, if X is a vector field on M, η on TM and $\tilde{Y}$ a vector field along τ, then

$$f_\sharp X := f_* \circ X \circ f^{-1}, \quad f_\sharp \eta := f_{**} \circ \eta \circ f_*^{-1}, \quad f_\sharp \tilde{Y} := f_* \circ \tilde{Y} \circ f_*^{-1}.$$

If $X \in \mathfrak{X}(M)$, and $\alpha \in \mathcal{T}_s(M)$ is a symmetric or skew-symmetric tensor, then the *substitution operator* i_X acts by

$$(i_X\alpha)(X_1, \dots, X_{s-1}) := \alpha(X, X_1, \dots, X_{s-1}) \quad (X_i \in \mathfrak{X}(M),\ 1 \leqq i \leqq s-1).$$

Now we construct a short exact sequence of vector bundles on $\widetilde{TM}$. If $p \in M$, we use the shorthand $\widetilde{T_pM} := T_pM \cap \widetilde{TM}$. If $v \in \widetilde{T_pM}$, and $w \in T_pM$, then $w_v := (v, w) \in \pi^*TM$, and the map $\mathbf{i} : \pi^*TM \to T\widetilde{TM}$ is defined by the requirement that $\mathbf{i}(w_v)$ is the initial velocity of the parametrized straight line $t \in \mathbb{R} \mapsto v + tw \in TM$. If $v \in \widetilde{TM}$, and $z \in T_v\widetilde{TM}$, then $\mathbf{j}(z) := (\pi_* z)_v \in \pi^*TM$. Then we have the short exact sequence

$$0 \to \pi^*TM \xrightarrow{\mathbf{i}} T\widetilde{TM} \xrightarrow{\mathbf{j}} \pi^*TM \to 0. \tag{2.1}$$

The property that this is an exact sequence means that $\mathbf{i}$ and $\mathbf{j}$ are fibrewise linear maps between the fibres of π^*TM and the tangent bundle of $\widetilde{TM}$, $\mathbf{i}$ is injective, $\mathbf{j}$ is surjective, and $\operatorname{Im}\mathbf{i} = \operatorname{Ker}\mathbf{j}$. The latter is the set of *vertical vectors* over $\widetilde{TM}$, denoted by $V\widetilde{TM} \subset T\widetilde{TM}$, and it is the total space of the *vertical subbundle* of the tangent bundle of $\widetilde{TM}$. The tangent vector fields of $\widetilde{TM}$ whose values are vertical vectors are called *vertical vector fields* on $\widetilde{TM}$. They form a submodule of $\mathfrak{X}\left(\widetilde{TM}\right)$, denoted by $\mathfrak{X}^v\left(\widetilde{TM}\right)$. We did not endow $\mathbf{i}$ and $\mathbf{j}$ with subscripts indicating that they are defined only on $\widetilde{TM}$, since they can be defined on the whole TM, and then in (2.1) we have their natural restrictions to $\widetilde{TM}$.

The maps $\mathbf{i}$ and $\mathbf{j}$ give rise to $C^\infty\left(\widetilde{TM}\right)$-homomorphisms between $\mathfrak{X}(\pi)$ and $\mathfrak{X}\left(\widetilde{TM}\right)$ denoted by the same symbols. Thus we obtain the exact sequence

$$0 \to \mathfrak{X}(\pi) \xrightarrow{\mathbf{i}} \mathfrak{X}\left(\widetilde{TM}\right) \xrightarrow{\mathbf{j}} \mathfrak{X}(\pi) \to 0$$

of $C^\infty\left(\widetilde{TM}\right)$-homomorphisms.

If X is a vector field on M, we define

$$\hat{X}(v) := (v, X(\pi(v))) \quad \left(v \in \widetilde{TM}\right), \quad X^v := \mathbf{i}\hat{X}.$$

Obviously, $\hat{X}$ is a vector field along π, while X^v is a vertical vector field. The vector field $\hat{X}$ is said to be a *basic vector field* along π, and X^v is called the *vertical lift* of X. Further important canonical objects, the *canonical section of* $\pi^*\tau$, the *Liouville vector field* on $\widetilde{TM}$ and the *vertical endomorphism*, are given by

$$\delta(v) := (v, v) \quad \left(v \in \widetilde{TM}\right), \quad C := \mathbf{i}\delta \quad \text{and} \quad J := \mathbf{i} \circ \mathbf{j},$$

respectively. We associate to J the *vertical differential* d_J on $\widetilde{TM}$. By definition,

$$d_J f := df \circ J \quad \left(f \in C^\infty\left(\widetilde{TM}\right)\right).$$

Then $d_J f$ is a (semibasic) one-form on $\widetilde{TM}$. The operator d^v along π is defined by

$$(d^v f)\left(\tilde{X}\right) := \left(\mathbf{i}\tilde{X}\right) f \quad \left(f \in C^\infty\left(\widetilde{TM}\right),\ \tilde{X} \in \mathfrak{X}(\pi)\right).$$

Then $d_J f = d^v f \circ \mathbf{j}$, and d^v as well is usually known as the vertical differential. We note that $\hat{X}$, X^v, δ, C and J may all be defined on the whole TM if needed. In that case, X^v is homogeneous of degree 0, i.e., $[C, X^v] = -X^v$.

If $\alpha \in \mathcal{T}^0_k(M)$ is a symmetric or skew-symmetric k-form on M, then the tensor fields $\hat{\alpha}$ and $\bar{\alpha}$ defined by

$$\hat{\alpha}_v(v_1, \dots, v_k) := \alpha_p(v_1, \dots, v_k), \quad \bar{\alpha}_v(v_1, \dots, v_{k-1}) := \alpha_p(v, v_1, \dots, v_{k-1})$$
$$\left(v \in \widetilde{T_pM}, v_i \in T_pM, 1 \leqq i \leqq k; p \in M\right)$$

are symmetric or skew-symmetric k- and $(k-1)$-forms along τ, respectively. In particular, if $f \in C^\infty(M)$, then $f^v := \hat{f} = f \circ \tau \in C^\infty(TM)$ is the *vertical lift* of f, and $f^c := \overline{df} \in C^\infty(TM)$ is the *complete lift* of f.

If A is a (1,1) tensor over $\widetilde{TM}$, then its *Nijenhuis torsion* may be given by the formula

$$N_A(\eta, \zeta) := [A\eta, A\zeta] + A^2[\eta, \zeta] - A[A\eta, \zeta] - A[\eta, A\zeta] \qquad \left(\eta, \zeta \in \mathfrak{X}\left(\widetilde{TM}\right)\right).$$

We shall use the well-known fact that the Nijenhuis torsion N_J of the vertical endomorphism vanishes [18, 52].

The most important special case will be $\widetilde{TM} = \overset{\circ}{T}M$, the open subset of non-zero tangent vectors to the manifold M. In this case, we shall write $\overset{\circ}{\tau} := \tau \restriction \overset{\circ}{T}M$ rather than π.

2.1.1 Definition. *A map $\mathcal{D} : \mathcal{T}(M) \to \mathcal{T}(M)$ is said to be a* tensor derivation on M *if it satisfies the following conditions:*

(1) *$\mathcal{D}$ is $\mathbb{R}$-linear.*

(2) *$\mathcal{D}$ is type-preserving, i.e. $\mathcal{D}(\mathcal{T}^r_s(M)) \subset \mathcal{T}^r_s(M)$ for each $(r, s) \in \mathbb{N} \times \mathbb{N}$.*

(3) *$\mathcal{D}$ obeys the Leibniz rule: $\mathcal{D}(A \otimes B) = \mathcal{D}A \otimes B + A \otimes \mathcal{D}B$ for any tensor fields A and B on M.*

(4) *$\mathcal{D}$ commutes with all contractions.*

Let $\mathcal{D}$ be a tensor derivation on M. Then it satisfies the *product rule:* if $A \in \mathcal{T}^r_s(M)$, then

$$\begin{aligned}(\mathcal{D}A)(\vartheta^1, \dots, \vartheta^r, X_1, \dots, X_s) = \mathcal{D}[A(\vartheta^1, \dots, \vartheta^r, X_1, \dots, X_s)] \\ - \sum_{i=1}^{r} A(\vartheta^1, \dots, \mathcal{D}\vartheta^i, \dots, \vartheta^r, X_1, \dots, X_s) \\ - \sum_{j=1}^{s} A(\vartheta^1, \dots, \vartheta^r, X_1, \dots, \mathcal{D}X_j, \dots, X_s)\end{aligned}$$

($\vartheta^i \in \mathfrak{X}^*(M)$, $1 \leqq i \leqq r$, $X_j \in \mathfrak{X}(M)$, $1 \leqq j \leqq s$).

2.1.2 Lemma (Willmore's theorem). *Any tensor derivation of the tensor algebra $\mathcal{T}(M)$ is completely determined by its action over the smooth functions and vector fields on M. Conversely, given a vector field $X \in \mathfrak{X}(M)$ and an $\mathbb{R}$-linear map $\mathcal{D}_0 : \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying the condition*

$$\mathcal{D}_0(fY) = (Xf)Y + f\mathcal{D}_0 Y \quad \text{for all } f \in C^\infty(M),\ Y \in \mathfrak{X}(M),$$

there exists a (necessarily unique) tensor derivation $\mathcal{D}$ on M such that

$$\mathcal{D} \restriction C^\infty(M) = X \quad \text{and} \quad \mathcal{D} \restriction \mathfrak{X}(M) = \mathcal{D}_0.$$

Now we wish to formulate Willmore's theorem for tensor derivations in the pull-back bundle.

2.1.3 Definition. *A map* $\mathcal{D} : \mathcal{T}(\pi) \to \mathcal{T}(\pi)$ *is said to be a* tensor derivation along π *if it satisfies the following conditions:*

(1) $\mathcal{D}$ *is* $\mathbb{R}$*-linear.*

(2) $\mathcal{D}$ *is type-preserving, i.e.* $\mathcal{D}(\mathcal{T}^r_s(\pi)) \subset \mathcal{T}^r_s(\pi)$ *for each* $(r, s) \in \mathbb{N} \times \mathbb{N}$.

(3) $\mathcal{D}$ *obeys the Leibniz rule:* $\mathcal{D}\left(\tilde{A} \otimes \tilde{B}\right) = \mathcal{D}\tilde{A} \otimes \tilde{B} + \tilde{A} \otimes \mathcal{D}\tilde{B}$ *for any tensor fields* $\tilde{A}$ *and* $\tilde{B}$ *along* π.

(4) $\mathcal{D}$ *commutes with all contractions.*

If $\mathcal{D}$ is a tensor derivation along π, then the product rule has the following form: if $\tilde{A} \in \mathcal{T}^r_s(\pi)$, then

$$\begin{aligned}\left(\mathcal{D}\tilde{A}\right)\left(\tilde{\vartheta}^1, \dots, \tilde{\vartheta}^r, \tilde{X}_1, \dots, \tilde{X}_s\right) = \mathcal{D}\left[\tilde{A}\left(\tilde{\vartheta}^1, \dots, \tilde{\vartheta}^r, \tilde{X}_1, \dots, \tilde{X}_s\right)\right] \\ - \sum_{i=1}^{r} \tilde{A}\left(\tilde{\vartheta}^1, \dots, \mathcal{D}\tilde{\vartheta}^i, \dots, \tilde{\vartheta}^r, \tilde{X}_1, \dots, \tilde{X}_s\right) \\ - \sum_{j=1}^{s} A\left(\tilde{\vartheta}^1, \dots, \tilde{\vartheta}^r, \tilde{X}_1, \dots, \mathcal{D}\tilde{X}_j, \dots, \tilde{X}_s\right)\end{aligned}$$

$\left(\tilde{\vartheta}^i \in \mathfrak{X}^*(\pi),\ 1 \leqq i \leqq r,\ \tilde{X}_j \in \mathfrak{X}(\pi),\ 1 \leqq j \leqq s\right)$.

2.1.4 Lemma (Willmore's theorem in the pull-back bundle). *Any tensor derivation of the tensor algebra* $\mathcal{T}(\pi)$ *is completely determined by its action over the smooth functions on* $\widetilde{TM}$ *and sections of* $\pi^*\tau$. *Conversely, given a vector field* $\eta \in \mathfrak{X}\left(\widetilde{TM}\right)$ *and an* $\mathbb{R}$*-linear map* $\mathcal{D}_0 : \mathfrak{X}(\pi) \to \mathfrak{X}(\pi)$ *satisfying the condition*

$$\mathcal{D}_0(f\tilde{Y}) = (\eta f)\tilde{Y} + f\mathcal{D}_0\tilde{Y} \quad \textit{for all } f \in C^\infty\left(\widetilde{TM}\right),\ \tilde{Y} \in \mathfrak{X}(\pi),$$

there exists a (necessarily unique) tensor derivation $\mathcal{D}$ *along* π *such that*

$$\mathcal{D} \restriction C^\infty\left(\widetilde{TM}\right) = \eta \quad \textit{and} \quad \mathcal{D} \restriction \mathfrak{X}(\pi) = \mathcal{D}_0.$$

Most of our presentation will be coordinate-free, in a few cases, however, we shall use local coordinates as well. Consider a chart $\left(\mathcal{U}, (u^i)_{i=1}^n\right)$ on M. If

$$x^i := (u^i)^v \qquad \text{and} \qquad y^i := (u^i)^c \qquad (i \in \{1, \dots, n\}),$$

or, equivalently,

$$x^i := u^i \circ \tau \qquad \text{and} \qquad y^i := \overline{du^i},$$

then $\left(\tau^{-1}(\mathcal{U}), (x^i)_{i=1}^n, (y^i)_{i=1}^n\right)$ is a chart on TM, called the *induced chart* by $\left(\mathcal{U}, (u^i)_{i=1}^n\right)$ on TM. The coordinate vector fields of the induced chart satisfy the relations

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i} = \left(\frac{\partial}{\partial u^i}\right)^v.$$

For coordinate expressions of our most important objects see the Appendix.

2.2 Ehresmann connections and covariant derivatives

Following the terminology used e.g. in [31], by an *Ehresmann connection* we shall mean a split short exact sequence:

$$0 \rightleftharpoons \pi^* TM \underset{\mathcal{V}}{\overset{\mathbf{i}}{\rightleftharpoons}} T\widetilde{TM} \underset{\mathcal{H}}{\overset{\mathbf{j}}{\rightleftharpoons}} \pi^* TM \rightleftharpoons 0.$$

The requirement that this is a splitting means that $\mathcal{V} \circ \mathbf{i} = \mathbf{j} \circ \mathcal{H} = 1_{\pi^* TM}$, and $\operatorname{Im} \mathcal{H} = \operatorname{Ker} \mathcal{V}$. Then $\mathcal{H}$ is also said to be a *horizontal map* for π, while $\mathcal{V}$ is called the *vertical map* belonging to $\mathcal{H}$. The horizontal map $\mathcal{H}$ determines $\mathcal{V}$ uniquely. The type $(1,1)$ tensor field $\mathbf{h} := \mathcal{H} \circ \mathbf{j}$ on $\widetilde{TM}$ is said to be the *horizontal projector* belonging to $\mathcal{H}$, and $H_v\widetilde{TM} := \operatorname{Im} \mathbf{h}_v$ is called the *horizontal subspace* of $T_v\widetilde{TM}$ if $v \in \widetilde{TM}$. The module of horizontal vector fields on $\widetilde{TM}$ is $\mathfrak{X}^h\left(\widetilde{TM}\right)$. The Ehresmann connection is completely determined either by $\mathcal{H}$ or $\mathbf{h}$. The map $\mathbf{v} := 1_{T\widetilde{TM}} - \mathbf{h}$ is the *vertical projector* belonging to $\mathbf{h}$. The *almost complex structure* associated to $\mathbf{h}$ is $\mathbf{F} := \mathcal{H} \circ \mathcal{V} - J$. As in the case of $\mathbf{i}$ and $\mathbf{j}$, we denote by the same symbols the arising $C^\infty\left(\widetilde{TM}\right)$-homomorphism between the modules of vector fields as the corresponding bundle maps. Thus we have the exact sequence

$$0 \rightleftharpoons \mathfrak{X}(\pi) \underset{\mathcal{V}}{\overset{\mathbf{i}}{\rightleftharpoons}} \mathfrak{X}\left(\widetilde{TM}\right) \underset{\mathcal{H}}{\overset{\mathbf{j}}{\rightleftharpoons}} \mathfrak{X}(\pi) \rightleftharpoons 0.$$

If $X \in \mathfrak{X}(M)$ is a vector field on M, then $X^h := \mathcal{H}\hat{X} \in \mathfrak{X}^h\left(\widetilde{TM}\right)$ is its *horizontal lift*.

The (weak) *torsion* of an Ehresmann connection is the (1,2) tensor $\mathbf{T}$ along π determined by the formula

$$\mathbf{iT}\left(\hat{X},\hat{Y}\right) := \left[X^h, Y^v\right] - \left[Y^h, X^v\right] - [X,Y]^v \quad (X,Y \in \mathfrak{X}(M)).$$

The *tension* $\mathbf{t}$ is determined by

$$\mathbf{it}\hat{X} := \left[X^h, C\right].$$

An Ehresmann connection is said to be *homogeneous* if its domain satisfies $\overset{\circ}{T}M \subset \widetilde{TM}$, and its tension vanishes. In this case, X^h is homogeneous of degree 1, i.e., $\left[C, X^h\right] = 0$.

A *covariant derivative operator* in $\pi^*\tau$ (or along π) is a map $D : \mathfrak{X}\left(\widetilde{TM}\right) \times \mathfrak{X}(\pi) \to \mathfrak{X}(\pi)$ that satisfies the following conditions:

(1) D is $C^\infty\left(\widetilde{TM}\right)$-linear in its first variable,

(2) $D_\eta\left(\tilde{X}+\tilde{Y}\right) = D_\eta\tilde{X} + D_\eta\tilde{Y}$ $\left(\eta \in \mathfrak{X}\left(\widetilde{TM}\right), \tilde{X},\tilde{Y} \in \mathfrak{X}(\pi)\right)$,

(3) $D_\eta\left(f\tilde{X}\right) = fD_\eta\tilde{X} + \eta f \cdot \tilde{X}$ $\left(\eta \in \mathfrak{X}\left(\widetilde{TM}\right), \tilde{X} \in \mathfrak{X}(\pi), f \in C^\infty\left(\widetilde{TM}\right)\right)$.

Similarly, a *v-covariant derivative operator* is a map $D^v : \mathfrak{X}(\pi) \times \mathfrak{X}(\pi) \to \mathfrak{X}(\pi)$ satisfying

(1) D^v is $C^\infty\left(\widetilde{TM}\right)$-linear in its first variable,

(2) $D^v_{\tilde{X}}\left(\tilde{Y}+\tilde{Z}\right) = D^v_{\tilde{X}}\tilde{Y} + D^v_{\tilde{X}}\tilde{Z}$ $\left(\tilde{X},\tilde{Y},\tilde{Z} \in \mathfrak{X}(\pi)\right)$,

(3) $D^v_{\tilde{X}}\left(f\tilde{Y}\right) = fD^v_{\tilde{X}}\tilde{Y} + \left(\mathbf{i}\tilde{X}\right) f \cdot \tilde{Y}$ $\left(\tilde{X},\tilde{Y} \in \mathfrak{X}(\pi), f \in C^\infty\left(\widetilde{TM}\right)\right)$.

It is easy to see that the operator D_η satisfies the conditions of Willmore's theorem (2.1.4), thus it can be uniquely extended to any tensor along π, to be a tensor derivation. A similar remark holds for $D_{\tilde{X}}$, although it does not fit in the scheme of Willmore's theorem properly, since $\tilde{X}$ is not a tangent vector of $\widetilde{TM}$. The vertical covariant derivative of a one-form $\tilde{\vartheta}$ along π acts by

$$\left(D^v_{\tilde{X}}\tilde{\vartheta}\right)\left(\tilde{Y}\right) = \left(\mathbf{i}\tilde{X}\right)\left(\tilde{\vartheta}\tilde{Y}\right) - \tilde{\vartheta}\left(D_{\tilde{X}}\tilde{Y}\right),$$

and the product rule has the following form if $\tilde{\alpha} \in \mathcal{T}^r_s(\pi)$:

$$\begin{aligned}\left(D^v_{\tilde{X}}\tilde{\alpha}\right)\left(\tilde{\vartheta}^1,\dots,\tilde{\vartheta}^r,\tilde{Y}_1,\dots,\tilde{Y}_s\right) &= \left(\mathbf{i}\tilde{X}\right)\alpha\left(\tilde{\vartheta}^1,\dots,\tilde{\vartheta}^r,\tilde{Y}_1,\dots,\tilde{Y}_s\right)\\ &-\sum_{i=1}^{r}\tilde{\alpha}\left(\tilde{\vartheta}^1,\dots,D^v_{\tilde{X}}\tilde{\vartheta}^i,\dots,\tilde{\vartheta}^r,\tilde{Y}_1,\dots,\tilde{Y}_s\right)\\ &-\sum_{j=1}^{s}\tilde{\alpha}\left(\tilde{\vartheta}^1,\dots,\tilde{\vartheta}^r,\tilde{Y}_1,\dots,D^v_{\tilde{X}}\tilde{Y}_j,\dots,\tilde{Y}_s\right)\end{aligned}$$

$\left(\tilde{\vartheta}^i \in \mathfrak{X}^*(\pi),\ 1 \leqq i \leqq r;\ \tilde{Y}_j \in \mathfrak{X}(\pi),\ 1 \leqq j \leqq s\right)$.

The *deflection* μ of a covariant derivative is defined to be the covariant differential of the canonical section δ. More precisely,

$$\mu : \eta \in \mathfrak{X}\left(\widetilde{TM}\right) \mapsto \mu(\eta) := D_\eta \delta.$$

The *vertical deflection* is

$$\mu^v \tilde{X} := \mu\left(\mathbf{i}\tilde{X}\right) \quad \left(\tilde{X} \in \mathfrak{X}(\pi)\right).$$

The covariant derivative D is said to be *regular* if μ^v has rank n at every point of $\widetilde{TM}$. In the presence of an Ehresmann connection, the *horizontal deflection* may be given by

$$\mu^h \tilde{X} := \mu\left(\mathcal{H}\tilde{X}\right) \quad \left(\tilde{X} \in \mathfrak{X}(\pi)\right).$$

We say that $\mathcal{H}$ *is attached to* D if $\mathfrak{X}^h\left(\widetilde{TM}\right) = \operatorname{Ker}\mu$. This is possible only if D is regular.

Let $\tilde{X}$ and $\tilde{Y}$ be two vector fields along τ. Choose a vector field η on TM such that $\mathbf{j}\eta = \tilde{Y}$. We define the *canonical v-covariant derivative* of $\tilde{Y}$ with respect to $\tilde{X}$ by

$$\nabla^v_{\tilde{X}}\tilde{Y} = \nabla^v_{\tilde{X}}\mathbf{j}\eta := \mathbf{j}\left[\mathbf{i}\tilde{X},\eta\right].$$

It can easily be seen that the definition is independent of the choice of η.

If D is a covariant derivative operator in $\pi^*\tau$, an Ehresmann connection is given on $\widetilde{TM}$, and $\tilde{X}$ is a vector field along π, then the *horizontal covariant differential* of $\tilde{X}$ with respect to D is the tensor $D^h\tilde{X}$ given by

$$\tilde{Y} \in \mathfrak{X}(\pi) \mapsto \left(D^h\tilde{X}\right)\left(\tilde{Y}\right) := D_{\mathcal{H}\tilde{Y}}\tilde{X}.$$

The *horizontal curvature* $\mathbf{R}$ and the *mixed curvature* $\mathbf{P}$ of D are given by

$$\mathbf{R}\left(\tilde{X},\tilde{Y}\right)\tilde{Z} := D_{\mathcal{H}\tilde{X}}D_{\mathcal{H}\tilde{Y}}\tilde{Z} - D_{\mathcal{H}\tilde{Y}}D_{\mathcal{H}\tilde{X}}\tilde{Z} - D_{\left[\mathcal{H}\tilde{X},\mathcal{H}\tilde{Y}\right]}\tilde{Z},$$
$$\mathbf{P}\left(\tilde{X},\tilde{Y}\right)\tilde{Z} := D_{\mathcal{H}\tilde{X}}D_{\mathbf{i}\tilde{Y}}\tilde{Z} - D_{\mathbf{i}\tilde{Y}}D_{\mathcal{H}\tilde{X}}\tilde{Z} - D_{\left[\mathcal{H}\tilde{X},\mathbf{i}\tilde{Y}\right]}\tilde{Z}$$

for $\tilde{X},\tilde{Y},\tilde{Z} \in \mathfrak{X}(\pi)$. We define the *horizontal torsion* $\mathcal{T}$ and the *vertical torsion* $\mathcal{Q}$ by

$$\mathcal{T}\left(\tilde{X},\tilde{Y}\right) := D_{\mathcal{H}\tilde{X}}\tilde{Y} - D_{\mathcal{H}\tilde{Y}}\tilde{X} - \mathbf{j}\left[\mathcal{H}\tilde{X},\mathcal{H}\tilde{Y}\right],$$
$$\mathcal{Q}\left(\tilde{X},\tilde{Y}\right) := D_{\mathbf{i}\tilde{X}}\tilde{Y} - D_{\mathbf{i}\tilde{Y}}\tilde{X} - \mathbf{i}^{-1}\left[\mathbf{i}\tilde{X},\mathbf{i}\tilde{Y}\right].$$

Note that only the horizontal torsion depends on the Ehresmann connection. Similarly, the *torsion* of a v-covariant derivative D^v will be

$$\mathcal{Q}\left(\tilde{X},\tilde{Y}\right) := D^v_{\tilde{X}}\tilde{Y} - D^v_{\tilde{Y}}\tilde{X} - \mathbf{i}^{-1}\left[\mathbf{i}\tilde{X},\mathbf{i}\tilde{Y}\right].$$

Berwald's covariant derivative induced by an Ehresmann connection is the covariant derivative operator $\nabla : \mathfrak{X}\left(\widetilde{TM}\right) \times \mathfrak{X}(\pi) \to \mathfrak{X}(\pi)$ in $\pi^*\tau$ given by

$$\nabla_\eta \tilde{X} := \mathbf{j}\left[\mathbf{v}\eta,\mathcal{H}\tilde{X}\right] + \mathcal{V}\left[\mathbf{h}\eta,\mathbf{i}\tilde{X}\right]$$

for any $\eta \in \mathfrak{X}\left(\widetilde{TM}\right)$, $\tilde{X} \in \mathfrak{X}(\pi)$. We note that the vertical part of Berwald's covariant derivative coincides with the canonical v-covariant derivative, and therefore it is independent of the Ehresmann connection. The horizontal and the mixed curvature of ∇ will be denoted by $\overset{\circ}{\mathbf{R}}$ and $\overset{\circ}{\mathbf{P}}$, respectively. If $X,Y,Z \in \mathfrak{X}(M)$, an easy calculation shows that

$$\mathbf{i}\overset{\circ}{\mathbf{P}}\left(\hat{X},\hat{Y}\right)\hat{Z} = \left[[X^h,Y^v],Z^v\right].$$

2.3 Flows, complete lifts and Lie derivatives

Let $X \in \mathfrak{X}(M)$ be a vector field. Suppose that $W \subset \mathbb{R} \times M$ is a set such that $W \cap (\mathbb{R} \times \{p\})$ is an open interval containing 0 for any $p \in M$, and $\varphi : W \to M$ is a map such that $c_p : t \mapsto \varphi(t,p)$ is the maximal integral curve of X starting from p for each $p \in M$, i.e., $\dot{c}_p = X \circ c_p,$, $c_p(0) = p$, and any other curve with these two properties is a restriction of c_p. Then, as usual, we say that φ is the *flow* of X. The map $p \in M \mapsto \varphi(t,p)$ will also be denoted by φ_t for a fixed $t \in \mathbb{R}$. Then φ_t is a diffeomorphism between two open subsets of M. It can also occur that the domain of φ_t is empty if $|t|$ is large enough. The vector field X is said to be *complete* if $W = \mathbb{R} \times M$, i.e., if its integral curves are defined on the whole real line $\mathbb{R}$.

There is a unique vector field X^c on TM whose flow is $(t,v) \mapsto (\varphi_t)_*(v)$ if $(t,\tau(v)) \in W$. We call this vector field the *complete lift* of X, and it is homogeneous of degree 1, i.e., $[C, X^c] = 0$. If an Ehresmann connection is given, it is easy to show that the horizontal and vertical lifts of X can be expressed as $X^h = \mathbf{h}X^c$ and $X^v = JX^c$. If we have a chart on M and the induced chart on TM as described in section 2.1, then

$$\frac{\partial}{\partial x^i} = \left(\frac{\partial}{\partial u^i}\right)^c.$$

If X and Y are vector fields on M, the Lie derivative of Y with respect to X is

$$\mathcal{L}_X Y := [X,Y].$$

The Lie derivative of an arbitrary tensor field on M with respect to X is defined according to Willmore's theorem.

Suppose that an Ehresmann connection is given on $\widetilde{TM} \subset TM$. Let $\eta \in \mathfrak{X}\left(\widetilde{TM}\right)$ and $\tilde{X} \in \mathfrak{X}(\pi)$. Now we introduce three types of Lie derivatives of $\tilde{X}$ with respect to η:

$$\begin{aligned}
\mathcal{L}^h_\eta \tilde{X} &:= \mathbf{j}\left[\eta, \mathcal{H}\tilde{X}\right] && \text{– horizontal Lie derivative,}\\
\mathcal{L}^t_\eta \tilde{X} &:= \mathbf{j}\left[\mathbf{h}\eta, \mathcal{H}\tilde{X}\right] + \mathcal{V}\left[\mathbf{v}\eta, \mathbf{i}\tilde{X}\right] && \text{– total Lie derivative,}\\
\mathcal{L}^v_\eta \tilde{X} &:= \mathcal{V}\left[\eta, \mathbf{i}\tilde{X}\right] && \text{– vertical Lie derivative.}
\end{aligned}$$

The horizontal Lie derivative was introduced by *Akbar-Zadeh* in the case of Finsler manifolds [2], and these three types were published together in [27] and [52] for the first time. They were motivated by the fact that Cartan's 'magic formula' can be generalized to them in a natural manner. All the three operators $\mathcal{L}^h_\eta$, $\mathcal{L}^t_\eta$ and $\mathcal{L}^v_\eta$ can be extended to the whole tensor algebra along π according to Willmore's theorem. If $X \in \mathfrak{X}(M)$, $\mathcal{L}^h_{X^c}$ coincides with $\mathcal{L}^v_{X^c}$, since

$$\mathbf{i}\mathcal{L}^h_{X^c}\hat{Y} = J\left[X^c, Y^h\right] = [X,Y]^v = [X^c, Y^v] = \mathbf{i}\mathcal{L}^v_{X^c}\hat{Y}.$$

We shall denote the coinciding operators $\mathcal{L}^h_{X^c}$ and $\mathcal{L}^v_{X^c}$ by $\mathcal{L}_X$. Moreover, from the calculation above it can be also seen that

$$\mathcal{L}_X \tilde{Y} = \mathbf{i}^{-1}\left[X^c, \mathbf{i}\tilde{Y}\right]$$

for any $\tilde{Y} \in \mathfrak{X}(\pi)$, and therefore it is independent of the Ehresmann connection. If $Y \in \mathfrak{X}(M)$, we have

$$\mathcal{L}_X \hat{Y} = \widehat{[X, Y]}.$$

In the rest of the thesis we shall use only the operator $\mathcal{L}_X$. This coincides with the Lie derivative formulated in local coordinates in many classical works and first defined by E. T. Davies [13]. It was Gy. Soós who first applied Lie derivatives efficiently in the study of isometries and affine transformations of Finsler manifolds [48].

Let $D : \mathfrak{X}\left(\widetilde{TM}\right) \times \mathfrak{X}(\pi) \to \mathfrak{X}(\pi)$ be a covariant derivative operator. The Lie derivative of D with respect to $X \in \mathfrak{X}(M)$ is given by

$$(\mathcal{L}_X D)\left(\eta, \tilde{Y}\right) := \mathcal{L}_X D_\eta \tilde{Y} - D_{[X^c,\eta]}\tilde{Y} - D_\eta \mathcal{L}_X \tilde{Y}$$

$\left(\eta \in \mathfrak{X}\left(\widetilde{TM}\right), \tilde{Y} \in \mathfrak{X}(\pi)\right)$.

The Lie derivative $\mathcal{L}_X D$ is $C^\infty\left(\widetilde{TM}\right)$-linear in both of its variables, not just in the first one, as D itself, since, e.g., if $f \in C^\infty\left(\widetilde{TM}\right)$, we have

$$\begin{aligned}
&(\mathcal{L}_X D)\left(\eta, f\tilde{Y}\right) = \mathcal{L}_X D_\eta\left(f\tilde{Y}\right) - D_{[X^c,\eta]}\left(f\tilde{Y}\right) - D_\eta \mathcal{L}_X\left(f\tilde{Y}\right) \\
&= \mathcal{L}_X\left(\eta f \cdot \tilde{Y} + f D_\eta \tilde{Y}\right) - [X^c, \eta] f \cdot \tilde{Y} - f D_{[X^c,\eta]}\tilde{Y} - D_\eta\left(X^c f \cdot \tilde{Y} + f\mathcal{L}_X \tilde{Y}\right) \\
&= X^c \eta f \cdot \tilde{Y} + \eta f \cdot \mathcal{L}_X \tilde{Y} + X^c f \cdot D_\eta \tilde{Y} + f\mathcal{L}_X D_\eta \tilde{Y} - X^c \eta f \cdot \tilde{Y} + \eta X^c f \cdot \tilde{Y} \\
&\quad - f D_{[X^c,\eta]}\tilde{Y} - \eta X^c f \cdot \tilde{Y} - X^c f \cdot D_\eta \tilde{Y} - \eta f \cdot \mathcal{L}_X \tilde{Y} - f D_\eta \mathcal{L}_X \tilde{Y} \\
&= f\left(\mathcal{L}_X D_\eta \tilde{Y} - D_{[X^c,\eta]}\tilde{Y} - D_\eta \mathcal{L}_X \tilde{Y}\right) = f(\mathcal{L}_X D)\left(\eta, \tilde{Y}\right).
\end{aligned}$$

Thus $\mathcal{L}_X D$ may be evaluated on individual vectors as well, not only on vector fields.

In this thesis we shall repeatedly use the known fact that the Lie derivative of the vertical endomorphism with respect to a complete lift or a vertical lift vanishes [18, 52], i.e.,

$$\mathcal{L}_{X^c} J = \mathcal{L}_{X^v} J = 0.$$

2.3.1 Lemma. *Let X be a vector field on M, and φ its flow.*

(1) *If $Y \in \mathfrak{X}(M)$,*

$$\mathcal{L}_X Y = \lim_{t\to 0} \frac{1}{t}((\varphi_{-t})_\sharp Y - Y).$$

(2) *If $\tilde{Y} \in \mathfrak{X}(\pi)$,*

$$\mathcal{L}_X \tilde{Y} = \lim_{t\to 0} \frac{1}{t}\left((\varphi_{-t})_\sharp \tilde{Y} - \tilde{Y}\right).$$

(3) *If D is a covariant derivative in $\pi^*\tau$, then*

$$(\mathcal{L}_X D)\left(\eta, \tilde{Z}\right) = \lim_{t\to 0} \frac{1}{t}\left[(\varphi_{-t})_\sharp D_{(\varphi_t)_\sharp \eta}(\varphi_t)_\sharp \tilde{Z} - D_\eta \tilde{Z}\right]$$

for any vector fields $\eta \in \mathfrak{X}\left(\widetilde{TM}\right)$, $\tilde{Z} \in \mathfrak{X}(\pi)$.

Assertion (1) is very well-known, and its proof can be found in most of the books treating smooth manifolds; see e.g. [41, 59]. Assertion (2) is an immediate consequence of (1). The proof of (3) is a quite straightforward generalization and rather lengthy, therefore we omit it. We note that the statements of the lemma may be reformulated in the following 'pointwise' forms:

$$(\mathcal{L}_X Y)(p) = \lim_{t\to 0} \frac{1}{t}[(\varphi_{-t})_* Y(\varphi_t(p)) - Y(p)] \qquad \text{for } p \in M;$$

$$\left(\mathcal{L}_X \tilde{Y}\right)(v) = \lim_{t\to 0} \frac{1}{t}\left[(\varphi_{-t})_* \tilde{Y}((\varphi_t)_* v) - \tilde{Y}(v)\right] \qquad \text{for } v \in \widetilde{TM};$$

$$(\mathcal{L}_X D)\left(w_v, \tilde{Z}(v)\right) = \lim_{t\to 0} \frac{1}{t}\left[(\varphi_{-t})_* D_{(\varphi_t)_* w_v}(\varphi_t)_\sharp \tilde{Z} - D_{w_v} \tilde{Z}\right]$$

$$\text{for } v \in \widetilde{TM},\ w \in T_{\tau(v)}M \text{ and } \tilde{Z} \in \mathfrak{X}(\pi).$$

2.3.2 Lemma. *If the Berwald derivatives induced by two homogeneous Ehresmann connections coincide, then the two Ehresmann connections are identical.*

Proof. Let $\mathcal{H}$ and $\bar{\mathcal{H}}$ be the horizontal maps of the given Ehresmann connections, and then we distinguish the data of $\bar{\mathcal{H}}$ from those of $\mathcal{H}$ by a bar. For any vector fields X, Y on M we have

$$\begin{aligned} 0 &= \mathbf{i}\left(\nabla_{X^c}\hat{Y} - \bar{\nabla}_{X^c}\hat{Y}\right) = \mathbf{i}\left(\nabla_{X^h}\hat{Y} + \nabla_{\mathbf{v}X^c}\hat{Y} - \bar{\nabla}_{X^{\bar{h}}}\hat{Y} - \bar{\nabla}_{\bar{\mathbf{v}}X^c}\hat{Y}\right) \\ &= \left[X^h, Y^v\right] - \left[X^{\bar{h}}, Y^v\right] = \left[X^h - X^{\bar{h}}, Y^v\right]. \end{aligned}$$

Since $X^h - X^{\bar{h}}$ is vertical, and a vertical vector field whose Lie bracket with any vertical vector field vanishes is itself a vertical lift [57], $X^h - X^{\bar{h}}$ is a vertical lift, and hence it is homogeneous of degree 0. On the other hand, since both Ehresmann connections are homogeneous, $X^h - X^{\bar{h}}$ is homogeneous of degree 1 as well. Therefore $X^h - X^{\bar{h}} = 0$. □

Chapter 3

Affine and projective vector fields on spray manifolds

3.1 Sprays and their associated objects

3.1.1 Definition. *A vector field $\xi : TM \to TTM$ on TM of class C^1, smooth on $\overset{\circ}{T}M$ is said to be a* spray *over M if $J\xi = C$ and ξ is positively homogeneous of degree 2, i.e., $[C, \xi] = \xi$. The spray ξ is* quadratic *if it is smooth on the whole TM. A smooth curve $c : I \subset \mathbb{R} \to M$ is called a* geodesic *of ξ if $\ddot{c} = \xi \circ \dot{c}$.*

A constant curve is trivially a geodesic of a spray. On the other hand, from the uniqueness theorem of the solutions of ordinary differential equations, it follows that, provided there is a parameter $t \in I$ such that $\dot{c}(t) \neq 0$, then $\dot{c}$ is nowhere zero along the geodesic c.

From now on in Chapter 3, we shall assume that a spray ξ is specified over the manifold M.

According to Crampin and Grifone's theorem [7, 8, 18, 19], a horizontal projector $\mathbf{h}$ can be associated to a spray by

$$\mathbf{h} := \frac{1}{2}(1_{\mathfrak{X}(TM)} - \mathcal{L}_\xi J).$$

In Chapter 3, $\mathbf{h}$ will always mean the horizontal projector associated to the given spray ξ, and the other notations introduced in 2.2 will mean the data of the Ehresmann connection determined by $\mathbf{h}$. This Ehresmann connection is homogeneous, and its torsion vanishes. Berwald's covariant derivative induced by $\mathbf{h}$ will be ∇. The horizontal projector $\mathbf{h}$ is of class C^0 on TM and C^∞ on $\overset{\circ}{T}M$, therefore some of its associated objects are defined only on $\overset{\circ}{T}M$ (the Berwald derivative ∇ among others). If X is a vector field on M, its horizontal lift may be given by

$$X^h = \frac{1}{2}(X^c + [X^v, \xi]).$$

3.1.2 Lemma. *If the Berwald derivatives induced by two sprays coincide, then the two sprays coincide.*

Proof. Let ξ and $\bar{\xi}$ be the two sprays. From 2.3.2 it follows that the Ehresmann connections given by ξ and $\bar{\xi}$ coincide, and therefore $\mathcal{L}_{\xi-\bar{\xi}}J = 0$, and, since $\xi - \bar{\xi}$ is vertical, this implies that $\xi - \bar{\xi}$ is a vertical lift [57], thus it is homogeneous of degree 0. On the other hand, since ξ and $\bar{\xi}$ are homogeneous of degree 2, $\xi - \bar{\xi}$ is also homogeneous of degree 2. This is possible only if $\xi - \bar{\xi} = 0$. □

It can be shown that the mixed curvature $\overset{\circ}{\mathbf{P}}$ of ∇ is totally symmetric if $\mathbf{h}$ comes from a spray [52]. Therefore, all its three possible contractions coincide, and thus $\operatorname{tr}\overset{\circ}{\mathbf{P}}$ will denote any of them.

3.1.3 Definition. Yano's covariant derivative D *induced by* ξ *in* $\overset{\circ}{\tau}{}^*\tau$ *is defined by the formula*

$$D_\eta \tilde{X} := \nabla_\eta \tilde{X} + \frac{1}{n+1} \operatorname{tr}\overset{\circ}{\mathbf{P}}\left(\mathbf{j}\eta, \tilde{X}\right) \cdot \delta$$

for $\eta \in \mathfrak{X}\left(\overset{\circ}{T}M\right)$, $\tilde{X} \in \mathfrak{X}(\overset{\circ}{\tau})$.

This covariant derivative was first introduced by K. Yano [63], in terms of local coordinates; that is why we call it Yano's derivative (see also [54]).

3.2 Affine vector fields

3.2.1 Definition. *A diffeomorphism* $f : \mathcal{U} \to \mathcal{V}$ *between two open subsets of* M *is said to be a* local affine transformation *if it leaves* ξ *invariant, i.e.,*

$$f_\sharp(\xi \restriction T\mathcal{U}) = \xi \restriction T\mathcal{V}.$$

Let X *be a vector field on* M *and* φ *the flow of* X. *If* φ_t *is a local affine transformation between two open submanifolds of* M *for each* $t \in \mathbb{R}$ *such that the domain of* φ_t *is not empty, then* X *is called an* affine vector field *for* ξ.

It is easy to see that a local affine transformation takes a geodesic of ξ to another geodesic.

3.2.2 Theorem. *Let X be a vector field on M. The following statements are equivalent:*

(1) *X is an affine vector field;*

(2) $[X^c, \xi] = 0$*;*

(3) $\mathcal{L}_X \nabla = 0$*;*

(4) $\mathcal{L}_{X^c} \mathbf{h} = 0$*;*

(5) $\left(\nabla^h \nabla^h \hat{X}\right)\left(\tilde{Y}, \tilde{Z}\right) = -\overset{\circ}{\mathbf{R}}\left(\hat{X}, \tilde{Y}\right)\tilde{Z} + \overset{\circ}{\mathbf{P}}\left(\mathcal{V}X^c, \tilde{Y}\right)\tilde{Z}$ *for any* $\tilde{Y}, \tilde{Z} \in \mathfrak{X}(\overset{\circ}{\tau})$ (generalized Killing equation).

Proof. The scheme of our proof will be the following:

$$\begin{array}{ccccc} (1) & \Longleftrightarrow & (3) & \Longleftrightarrow & (5) \\ & & \Updownarrow & & \\ (2) & \Longleftrightarrow & (4) & & \end{array}$$

(1) $\Longrightarrow$ (3) Suppose that X is affine, and $\varphi : W \subset \mathbb{R} \times M \to M$ is the flow of X. Let $Y, Z \in \mathfrak{X}(M)$ be arbitrary vector fields. It is enough to show that $(\mathcal{L}_X \nabla)\left(Y^h, \hat{Z}\right) = 0$. We begin by showing that $(\varphi_t)_\sharp X^h = ((\varphi_t)_\sharp X)^h$:

$$\begin{aligned} 2(\varphi_t)_\sharp X^h &= (\varphi_t)_\sharp (X^c + [X^v, \xi]) = (\varphi_t)_\sharp X^c + [(\varphi_t)_\sharp X^v, (\varphi_t)_\sharp \xi] \\ &= ((\varphi_t)_\sharp X)^c + [((\varphi_t)_\sharp X)^v, \xi] = 2((\varphi_t)_\sharp X)^h. \end{aligned}$$

Using this, we obtain

$$\begin{aligned} \mathbf{i}(\mathcal{L}_X \nabla)\left(Y^h, \hat{Z}\right) &= \lim_{t \to 0} \frac{1}{t}\left[\mathbf{i}(\varphi_{-t})_\sharp \nabla_{(\varphi_t)_\sharp Y^h} (\varphi_t)_\sharp \hat{Z} - \mathbf{i}\nabla_{Y^h} \hat{Z}\right] \\ &= \lim_{t \to 0} \frac{1}{t}\left[(\varphi_{-t})_\sharp \mathbf{i}\nabla_{((\varphi_t)_\sharp Y)^h} \widehat{(\varphi_t)_\sharp Z} - \mathbf{i}\nabla_{Y^h} \hat{Z}\right] \\ &= \lim_{t \to 0} \frac{1}{t}\left\{(\varphi_{-t})_\sharp [((\varphi_t)_\sharp Y)^h, ((\varphi_t)_\sharp Z)^v] - [Y^h, Z^v]\right\} \\ &= \lim_{t \to 0} \frac{1}{t}\left([Y^h, Z^v] - [Y^h, Z^v]\right) = 0. \end{aligned}$$

(3) $\Longrightarrow$ (1) Now we suppose that the vector field X has the property $\mathcal{L}_X\nabla = 0$. Let $\eta \in \mathfrak{X}\big(\overset{\circ}{T}M\big)$ and $\tilde{Z} \in \mathfrak{X}\big(\overset{\circ}{\tau}\big)$ be arbitrary vector fields. Choose an arbitrary tangent vector $v \in \overset{\circ}{T}M$, and define a function ℓ on a suitable open interval I containing 0 by

$$t \in I \mapsto \ell(t) := \left[(\varphi_{-t})_\sharp \nabla_{(\varphi_t)_\sharp \eta}(\varphi_t)_\sharp \tilde{Z}\right](v).$$

The function ℓ is constant, since for all $t \in I$ we have

$$\begin{aligned}
\ell'(t) &= \lim_{s\to 0}\frac{1}{s}\left[(\varphi_{-t-s})_\sharp \nabla_{(\varphi_{t+s})_\sharp \eta}(\varphi_{t+s})_\sharp \tilde{Z} - (\varphi_{-t})_\sharp \nabla_{(\varphi_t)_\sharp \eta}(\varphi_t)_\sharp \tilde{Z}\right](v)\\
&= \lim_{s\to 0}\frac{1}{s}\left[(\varphi_{-t-s})_* \nabla_{(\varphi_{t+s})_{**}\eta(v)}(\varphi_{t+s})_\sharp \tilde{Z} - (\varphi_{-t})_* \nabla_{(\varphi_t)_{**}\eta(v)}(\varphi_t)_\sharp \tilde{Z}\right]\\
&= (\varphi_{-t})_* \lim_{s\to 0}\frac{1}{s}\left[(\varphi_{-s})_* \nabla_{(\varphi_{t+s})_{**}\eta(v)}(\varphi_{t+s})_\sharp \tilde{Z} - \nabla_{(\varphi_t)_{**}\eta(v)}(\varphi_t)_\sharp \tilde{Z}\right]\\
&= (\varphi_{-t})_*(\mathcal{L}_X\nabla)\Big((\varphi_t)_{**}\eta(v), (\varphi_t)_\sharp \tilde{Z}((\varphi_t)_* v)\Big) = 0.
\end{aligned}$$

Since two sprays coincide provided their associated Berwald's derivatives coincide (3.1.2), this implies that X is indeed an affine vector field.

(3) $\Longrightarrow$ (4) Suppose that $\mathcal{L}_X\nabla = 0$, and let $Y, Z \in \mathfrak{X}(M)$ be arbitrary vector fields. Then we have

$$\begin{aligned}
0 = \mathbf{i}(\mathcal{L}_X\nabla)\left(Y^h, \hat{Z}\right) &= \mathbf{i}\mathcal{L}_X\nabla_{Y^h}\hat{Z} - \mathbf{i}\nabla_{[X^c,Y^h]}\hat{Z} - \mathbf{i}\nabla_{Y^h}\mathcal{L}_X\hat{Z}\\
&= \left[X^c,[Y^h,Z^v]\right] - \mathbf{v}\left[\mathbf{h}[X^c,Y^h],Z^v\right] - \mathbf{v}\left[Y^h,[X^c,Z^v]\right]\\
&= -\left[Y^h,[Z^v,X^c]\right] - \left[Z^v,[X^c,Y^h]\right] - \left[[X,Y]^h,Z^v\right] - \left[Y^h,[X^c,Z^v]\right]\\
&= \left[Z^v,[X,Y]^h - [X^c,Y^h]\right].
\end{aligned}$$

Since $J\left([X,Y]^h - [X^c,Y^h]\right) = 0$, i.e., $[X,Y]^h - [X^c,Y^h]$ is vertical, it follows that it is a vertical lift for each vector field $Y \in \mathfrak{X}(M)$. On the other hand,

$$\left[C,[X,Y]^h - [X^c,Y^h]\right] = -\left[C,[X^c,Y^h]\right] = \left[X^c,[Y^h,C]\right] + \left[Y^h,[C,X^c]\right] = 0,$$

hence $[X,Y]^h - \left[X^c,Y^h\right]$ is homogeneous of degree both 0 and 1. This implies $[X,Y]^h - \left[X^c,Y^h\right] = 0$. Now turning back to the original assertion,

$$(\mathcal{L}_{X^c}\mathbf{h})Y^c = [X^c,\mathbf{h}Y^c] - \mathbf{h}[X^c,Y^c] \overset{\text{Jacobi}}{=} \left[X^c,Y^h\right] - [X,Y]^h = 0.$$

Thus we conclude that $\mathcal{L}_{X^c}\mathbf{h} = 0$.

(4) $\Longrightarrow$ (3) Now we suppose that $\mathcal{L}_{X^c}\mathbf{h} = 0$ holds. Then, using also the previous arguments, we obtain

$$\mathbf{i}(\mathcal{L}_X\nabla)\left(Y^h, \hat{Z}\right) = \left[Z^v, [X,Y]^h - [X^c, Y^h]\right] = -\left[Z^v, (\mathcal{L}_{X^c}\mathbf{h})Y^c\right] = 0,$$

which implies $\mathcal{L}_X\nabla = 0$.

(4) $\Longrightarrow$ (2) If Y is an arbitrary vector field on M,

$$\begin{aligned}0 &= 2(\mathcal{L}_{X^c}\mathbf{h})(Y^c) = 2\left[X^c, Y^h\right] - 2[X,Y]^h \\ &= [X^c, Y^c + [Y^v, \xi]] - [X^c, Y^c] - [[X,Y]^v, \xi] = [X^c, [Y^v, \xi]] - [[X,Y]^v, \xi] \\ &= -[[Y^v, \xi], X^c] - [[X^c, Y^v], \xi] \overset{\text{Jacobi}}{=} [[\xi, X^c], Y^v].\end{aligned}$$

Since $[X^c, \xi]$ is vertical, this implies as above that $[X^c, \xi]$ is a vertical lift and hence positively homogeneous of degree 0. On the other hand,

$$[C, [X^c, \xi]] \overset{\text{Jacobi}}{=} -[X^c, [\xi, C]] - [\xi, [C, X^c]] = -[X^c, [\xi, C]] = [X^c, \xi],$$

thus $[X^c, \xi]$ is homogeneous of degree both 0 and 2, hence, as in the proof of (3) $\Longrightarrow$ (4), it is the zero vector field.

(2) $\Longrightarrow$ (4) Supposing $[X^c, \xi] = 0$ and using the relation obtained in the previous part of the proof, we get

$$(\mathcal{L}_{X^c}\mathbf{h})(Y^c) = \frac{1}{2}[[\xi, X^c], Y^v] = 0$$

for any $Y \in \mathfrak{X}(M)$, and thus $\mathcal{L}_{X^c}\mathbf{h}$ vanishes.

(3) $\Longleftrightarrow$ (5) It is enough to show that

$$(\mathcal{L}_X\nabla)\left(\mathcal{H}\tilde{Y}, \tilde{Z}\right) = \overset{\circ}{\mathbf{R}}\left(\hat{X}, \tilde{Y}\right)\tilde{Z} - \overset{\circ}{\mathbf{P}}\left(\mathcal{V}X^c, \tilde{Y}\right)\tilde{Z} + \left(\nabla_{\mathcal{H}\tilde{Y}}\nabla^h\hat{X}\right)\left(\tilde{Z}\right),$$

for $\tilde{Y}, \tilde{Z} \in \mathfrak{X}(\overset{\circ}{\tau})$, or, equivalently, that

$$(\mathcal{L}_X\nabla)\left(Y^h, \hat{Z}\right) = \overset{\circ}{\mathbf{R}}\left(\hat{X}, \hat{Y}\right)\hat{Z} - \overset{\circ}{\mathbf{P}}\left(\mathcal{V}X^c, \hat{Y}\right)\hat{Z} + \left(\nabla_{Y^h}\nabla^h\hat{X}\right)\left(\hat{Z}\right)$$

if $Y, Z \in \mathfrak{X}(M)$. Starting from the right-hand side of this equation, we find

$$\begin{aligned}
&\overset{\circ}{\mathbf{R}}\left(\hat{X},\hat{Y}\right)\hat{Z} - \overset{\circ}{\mathbf{P}}\left(\hat{Y},\hat{Z}\right)\mathcal{V}X^c + \left(\nabla_{Y^h}\nabla^h\hat{X}\right)\left(\hat{Z}\right)\\
&= \nabla_{X^h}\nabla_{Y^h}\hat{Z} - \nabla_{Y^h}\nabla_{X^h}\hat{Z} - \nabla_{[X^h,Y^h]}\hat{Z}\\
&\quad - \left(\nabla_{Y^h}\nabla_{Z^v}\mathcal{V}X^c - \nabla_{Z^v}\nabla_{Y^h}\mathcal{V}X^c - \nabla_{[Y^h,Z^v]}\mathcal{V}X^c\right)\\
&\quad + \nabla_{Y^h}\nabla_{Z^h}\hat{X} - \nabla_{\mathcal{H}\nabla_{Y^h}\hat{Z}}\hat{X}\\
&= \left(\nabla_{X^h}\nabla_{Y^h}\hat{Z} - \nabla_{\mathcal{H}\nabla_{Y^h}\hat{Z}}\hat{X}\right) - \nabla_{Y^h}\left(\nabla_{X^h}\hat{Z} - \nabla_{Z^h}\hat{X}\right) - \nabla_{[X^h,Y^h]}\hat{Z}\\
&\quad - \mathcal{V}\left[Y^h, J[Z^v, \mathbf{Fv}X^c]\right] + \mathbf{j}\left[Z^v, \mathbf{F}[Y^h, \mathbf{v}X^c]\right] + \mathbf{j}\left[[Y^h, Z^v], \mathbf{Fv}X^c\right].
\end{aligned}$$

where we have used the definition of Berwald's derivative. Since the torsion of $\mathcal{H}$ vanishes, so does the horizontal torsion of ∇. We use this, the disappearence of $\mathcal{L}_{Z^v}J$ and later on the Jacobi identity:

$$\begin{aligned}
&\overset{\circ}{\mathbf{R}}\left(\hat{X},\hat{Y}\right)\hat{Z} - \overset{\circ}{\mathbf{P}}\left(\hat{Y},\hat{Z}\right)\mathcal{V}X^c + \left(\nabla_{Y^h}\nabla^h\hat{X}\right)\left(\hat{Z}\right)\\
&= \mathbf{j}\left[X^h, \mathcal{H}\nabla_{Y^h}\hat{Z}\right] - \nabla_{Y^h}\mathbf{j}\left[X^h, Z^h\right] - \nabla_{[X^h,Y^h]}\hat{Z}\\
&\quad - \mathcal{V}\left[Y^h, [Z^v, \mathbf{v}X^c]\right] + \mathcal{V}\left[Z^v, [Y^h, \mathbf{v}X^c]\right] + \mathbf{j}\left[[Y^h, Z^v], \mathbf{Fv}X^c\right]\\
&= \mathbf{j}\left[X^h, \mathbf{F}[Y^h, Z^v]\right] - \nabla_{Y^h}\widehat{[X,Z]} - \nabla_{[X^c,Y^h]}\hat{Z}\\
&\quad + \mathcal{V}\left[\mathbf{v}X^c, [Y^h, Z^v]\right] - \mathbf{j}\left[\mathbf{Fv}X^c, [Y^h, Z^v]\right]\\
&\overset{(*)}{=} \mathbf{j}\left[X^h, \mathbf{F}[Y^h, Z^v]\right] - \nabla_{Y^h}\mathcal{L}_X\hat{Z} - \nabla_{[X^c,Y^h]}\hat{Z} + \mathbf{j}\left[\mathbf{v}X^c, \mathbf{F}[Y^h, Z^v]\right]\\
&= \mathbf{j}\left[X^c, \mathbf{F}[Y^h, Z^v]\right] - \nabla_{[X^c,Y^h]}\hat{Z} - \nabla_{Y^h}\mathcal{L}_X\hat{Z}\\
&= \mathcal{L}_X\nabla_{Y^h}\hat{Z} - \nabla_{[X^c,Y^h]}\hat{Z} - \nabla_{Y^h}\mathcal{L}_X\hat{Z} = \left(\mathcal{L}_X\nabla\right)\left(Y^h, \hat{Z}\right).
\end{aligned}$$

In the step denoted by $(*)$ we used $N_J = 0$. □

3.2.3 Corollary. *The affine vector fields on M form a sub-Lie-algebra of $\mathfrak{X}(M)$.*

Proof. The closure under addition and multiplication by scalars is obvious. The closure under the Lie bracket follows from

$$[[X,Y]^c, \xi] = [[X^c, Y^c], \xi] = -[[Y^c, \xi], X^c] - [[\xi, X^c], Y^c] = 0$$

where X and Y are affine vector fields, which implies according to the previous theorem that $[X, Y]$ is also affine. □

If ξ is a quadratic spray, $\overset{\circ}{\mathbf{P}}$ vanishes, and Berwald's covariant derivative reduces to a 'lift' of a torsion-free covariant derivative operator $\tilde{\nabla}$ on M in the sense that

$$\nabla_{X^h}\hat{Y} = \widehat{\tilde{\nabla}_X Y}$$

if X and Y are vector fields on M. In this case, ξ is the spray of $\tilde{\nabla}$, and 3.2.2 reduces to the known results for affine sprays [10, 42].

3.3 Projectively equivalent sprays

3.3.1 Definition. *Let ξ and $\bar{\xi}$ be two sprays over M, and suppose that for any geodesic $c : I \to M$ of ξ there is a strictly increasing smooth parameter transformation $\vartheta : J \to I$ such that $\bar{c} := c \circ \vartheta$ is a geodesic of $\bar{\xi}$. Then ξ and $\bar{\xi}$ are said to be* projectively equivalent.

It is easy to see that the projective equivalence of sprays is an equivalence relation and the geodesics of projectively equivalent sprays coincide as point sets.

3.3.2 Proposition. *Let ξ and $\bar{\xi}$ be two sprays over M, and $D, \bar{D}$ the induced Yano derivatives. The following statements are equivalent:*

(1) *ξ and $\bar{\xi}$ are projectively equivalent.*

(2) *There is a continuous function f on TM such that $\bar{\xi} = \xi + fC$.*

(3) *There is a smooth function f on $\overset{\circ}{T}M$ such that*

$$\bar{D}_\eta \tilde{Z} = D_\eta \tilde{Z} - \frac{1}{2}\left[(J\eta)f \cdot \tilde{Z} + \left(\mathbf{i}\tilde{Z}\right) f \cdot \mathbf{j}\eta\right] \quad \left(\eta \in \mathfrak{X}\left(\overset{\circ}{T}M\right), \tilde{Z} \in \mathfrak{X}\left(\overset{\circ}{\tau}\right)\right).$$

If one (and hence all) of these conditions is satisfied, then f in (2) *and* (3) *can be chosen such that they coincide on $\overset{\circ}{T}M$, and, in this case, f is positively homogeneous of degree 1.*

For the proof of the equivalence of (1) and (2), see [25]; for that of (2) and (3), see [53].

3.4 Projective vector fields

In this section D will stand for Yano's derivative.

3.4.1 Definition. *A diffeomorphism $f : \mathcal{U} \to \mathcal{V}$ between two open subsets of M is a* local projective transformation *if it takes $\xi \upharpoonright T\mathcal{U}$ to a spray projectively equivalent to $\xi \upharpoonright T\mathcal{V}$ over $\mathcal{V}$. Let $X \in \mathfrak{X}(M)$, and $\varphi : W \to M$ the flow of X. If φ_t is a local projective transformation between two open submanifolds of M for each $t \in \mathbb{R}$ such that the domain of φ is not empty, then X is a* projective vector field *for ξ.*

The maps φ_t $(t \in \mathbb{R})$ take geodesics to geodesics as point sets.

3.4.2 Theorem. *Let $X \in \mathfrak{X}(M)$ be a vector field. The following statements are equivalent:*

(1) *X is a projective vector field;*

(2) *there is a continuous function F on TM, smooth on $\overset{\circ}{T}M$, such that $[X^c, \xi] = FC$;*

(3) *there is a smooth function F on $\overset{\circ}{T}M$, positively homogeneous of degree 1, such that*

$$(\mathcal{L}_{X^c}\mathbf{h})(\eta) = \frac{1}{2}(FJ\eta + (J\eta)F \cdot C),\ \eta \in \mathfrak{X}\left(\overset{\circ}{T}M\right);$$

(4) *there is a smooth function F on $\overset{\circ}{T}M$ such that*

$$(\mathcal{L}_X D)\left(\eta, \tilde{Z}\right) = -\frac{1}{2}\left((J\eta)F \cdot \tilde{Z} + \left(\mathbf{i}\tilde{Z}\right) F \cdot \mathbf{j}\eta\right),$$
$$\eta \in \mathfrak{X}\left(\overset{\circ}{T}M\right),\ \tilde{Z} \in \mathfrak{X}(\overset{\circ}{\tau}).$$

Proof. The scheme of this proof will be as follows:

$$\begin{array}{ccc} (1) & \Longrightarrow & (2) \\ \Uparrow & & \Downarrow \\ (4) & \Longleftarrow & (3) \end{array}$$

(1) $\Longrightarrow$ (2) For any $(t, \tau(v)) \in W$ (the domain of φ), we define

$$\xi_t(v) := ((\varphi_{-t})_\sharp \xi)(v) = (\varphi_{-t})_{**}\xi((\varphi_t)_* v).$$

Then ξ_t is a spray on its domain, and it is projectively equivalent to ξ. Thus there is a function f_t such that $\xi_t = \xi + f_t C$, and

$$[X^c, \xi] = \lim_{t \to 0} \frac{1}{t}(\xi_t - \xi) = \lim_{t \to 0} \frac{f_t}{t} C,$$

hence $F := \lim_{t \to 0} \frac{f_t}{t}$ is a continuous function on TM, smooth on $\overset{\circ}{T}M$.

(2) $\Longrightarrow$ (3) First we restrict F to $\overset{\circ}{T}M$, and we show that it is 1-homogeneous:

$$(CF)C = [C, FC] = [C, [X^c, \xi]] \overset{\text{Jacobi}}{=} -[X^c, [\xi, C]] - [\xi, [C, X^c]] = [X^c, \xi] = FC,$$

and thus $CF = F$. Now let $Y \in \mathfrak{X}(M)$ be a vector field. We calculate the action of $\mathcal{L}_{X^c}\mathbf{h}$ on its lifts:

$$\begin{aligned}
(\mathcal{L}_{X^c}\mathbf{h})Y^v &= [X^c, \mathbf{h}Y^v] - \mathbf{h}[X^c, Y^v] = 0, \\
2(\mathcal{L}_{X^c}\mathbf{h})Y^h &= 2\left([X^c, Y^h] - \mathbf{h}[X^c, Y^h]\right) = 2\left([X^c, Y^h] - [X, Y]^h\right) \\
&= [X^c, Y^c + [Y^v, \xi]] - [X, Y]^c - [[X, Y]^v, \xi] \\
&= [X^c, [Y^v, \xi]] - [[X^c, Y^v], \xi] = [Y^v, [X^c, \xi]] \\
&= [Y^v, FC] = F[Y^v, C] + (Y^v F)C = FY^v + (Y^v F)C.
\end{aligned}$$

This proves the desired statement (3).

(3) $\Longrightarrow$ (4) Let $Y, Z \in \mathfrak{X}(M)$ be arbitrary vector fields. For $\mathcal{L}_X D$ we obtain

$$(\mathcal{L}_X D)\left(Y^h, \hat{Z}\right) = (\mathcal{L}_X \nabla)\left(Y^h, \hat{Z}\right) + \frac{1}{n+1}\left(\mathcal{L}_X \operatorname{tr} \overset{\circ}{\mathbf{P}}\right)\left(\hat{Y}, \hat{Z}\right)\delta.$$

Now we proceed by treating the two terms separately:

$$\begin{aligned}
\mathbf{i}\,(\mathcal{L}_X \nabla)\left(Y^h, \hat{Z}\right) &= \mathbf{i}\mathcal{L}_X D_{Y^h}\hat{Z} - \mathbf{i}D_{[X^c, Y^h]}\hat{Z} - \mathbf{i}D_{Y^h}\mathcal{L}_X\hat{Z} \\
&= \left[X^c, [Y^h, Z^v]\right] - \left[[X, Y]^h, Z^v\right] - \left[Y^h, [X^c, Z^v]\right] \\
&= \left[[X^c, Y^h] - [X, Y]^h, Z^v\right] = \left[(\mathcal{L}_{X^c}\mathbf{h})Y^h, Z^v\right] \\
= \frac{1}{2}[(FY^v + (Y^v F)C), Z^v] &= -\frac{1}{2}[(Z^v F)Y^v + (Y^v F)Z^v + (Y^v Z^v F)C].
\end{aligned}$$

The operators $\mathcal{L}_X$ and tr commute, since $\mathcal{L}_X$ is a tensor derivation. Thus we calculate $\mathcal{L}_X \overset{\circ}{\mathbf{P}}$:

$$\begin{aligned}
\mathbf{i}\Big(\mathcal{L}_X \overset{\circ}{\mathbf{P}}\Big)\Big(\hat{Y},\hat{Z},\hat{U}\Big) =& \mathbf{i}\mathcal{L}_X\Big(\overset{\circ}{\mathbf{P}}\Big(\hat{Y},\hat{Z}\Big)\hat{U}\Big) - \mathbf{i}\overset{\circ}{\mathbf{P}}\Big(\mathcal{L}_X\hat{Y},\hat{Z}\Big)\hat{U} \\
& - \mathbf{i}\overset{\circ}{\mathbf{P}}\Big(\hat{Y},\mathcal{L}_X\hat{Z}\Big)\hat{U} - \mathbf{i}\overset{\circ}{\mathbf{P}}\Big(\hat{Y},\hat{Z}\Big)\mathcal{L}_X\hat{U} \\
=& \Big[X^c, \mathbf{i}\overset{\circ}{\mathbf{P}}\Big(\hat{Y},\hat{Z}\Big)\hat{U}\Big] - \mathbf{i}\overset{\circ}{\mathbf{P}}\Big(\widehat{[X,Y]},\hat{Z}\Big)\hat{U} \\
& - \mathbf{i}\overset{\circ}{\mathbf{P}}\Big(\hat{Y},\widehat{[X,Z]}\Big)\hat{U} - \mathbf{i}\overset{\circ}{\mathbf{P}}\Big(\hat{Y},\hat{Z}\Big)\widehat{[X,U]} \\
=& \left[X^c, [[Y^h, Z^v], U^v]\right] - \left[\left[[X,Y]^h, Z^v\right], U^v\right] \\
& - \left[\left[Y^h, [X^c, Z^v]\right], U^v\right] - \left[[Y^h, Z^v], [X^c, U^v]\right].
\end{aligned}$$

With the Jacobi identity applied twice, this yields

$$\begin{aligned}
\mathbf{i}\Big(\mathcal{L}_X \overset{\circ}{\mathbf{P}}\Big)\Big(\hat{Y},\hat{Z},\hat{U}\Big) &= \left[\left[[X^c, Y^h] - [X,Y]^h, Z^v\right], U^v\right] \\
&= \left[\left[(\mathcal{L}_{X^c}\mathbf{h})Y^h, Z^v\right], U^v\right] = \frac{1}{2}\left[\left[FY^v + (Y^vF)C, Z^v\right], U^v\right] \\
&= -\frac{1}{2}[(Z^vF)Y^v + (Y^vF)Z^v + (Y^vZ^vF)C, U^v] \\
&= \frac{1}{2}((U^vZ^vF)Y^v + (U^vY^vF)Z^v + (Y^vZ^vF)U^v + (U^vY^vZ^vF)C).
\end{aligned}$$

Thus we need the trace of the map

$$\tilde{U} \in \mathfrak{X}(\overset{\circ}{\tau}) \mapsto \Big(\mathbf{i}\tilde{U}\Big) Z^vF \cdot \hat{Y} + \Big(\mathbf{i}\tilde{U}\Big) Y^vF \cdot \hat{Z} + (Y^vZ^vF)\hat{U} + \Big(\mathbf{i}\tilde{U}\Big) Y^vZ^vF \cdot \delta.$$

We calculate the trace of these terms separately. Around each point $p \in M$ where $Y(p) \neq 0$, Y may be completed to a local basis $(Y, E_1, \dots, E_{n-1})$, thus $\Big(\hat{Y}, \hat{E}_1, \dots, \hat{E}_{n-1}\Big)$ is a local basis of $\mathfrak{X}(\overset{\circ}{\tau})$. If we substitute the vector fields of this local basis into the linear map

$$\tilde{U} \in \mathfrak{X}(\overset{\circ}{\tau}) \mapsto \Big(\mathbf{i}\tilde{U}\Big) Z^vF \cdot \hat{Y},$$

then we see that the component of $E_i^vZ^vF \cdot \hat{Y}$ with respect to $\hat{E}_i$ vanishes for each $i \in \{1, \dots, n-1\}$, and in this basis, the trace consists of only one term:

$$\operatorname{tr}\Big[\tilde{U} \mapsto \Big(\mathbf{i}\tilde{U}\Big) Z^vF \cdot \hat{Y}\Big] = Y^vZ^vF.$$

We obtain in a similar way that the trace of the second term is Y^vZ^vF as well. The third term is the identity transformation multiplied by the funcion Y^vZ^vF, thus its trace is nY^vZ^vF. Finally, to determine the trace of the last term, we complete δ to be a local basis of $\mathfrak{X}(\overset{\circ}{\tau})$ around $v \in \overset{\circ}{T}M$: $\left(\delta, \tilde{E}_1, \dots, \tilde{E}_{n-1}\right)$. We substitute these vector fields into the map

$$\tilde{U} \mapsto \left(\mathbf{i}\tilde{U}\right) Y^vZ^vF \cdot \delta.$$

In the expression of the trace all but the first term vanish:

$$\begin{aligned}
\operatorname{tr}\left[\tilde{U} \mapsto \left(\mathbf{i}\tilde{U}\right) Y^vZ^vF \cdot \delta\right] &= CY^vZ^vF = [C, Y^v]Z^vF + Y^vCZ^vF \\
&= -Y^vZ^vF + Y^v[C, Z^v]F + Y^vZ^vCF \\
&= -Y^vZ^vF - Y^vZ^vF + Y^vZ^vF = -Y^vZ^vF,
\end{aligned}$$

and hence we obtain

$$\left(\mathcal{L}_X \operatorname{tr} \overset{\circ}{\mathbf{P}}\right)(\hat{Y}, \hat{Z}) = \frac{1}{2}(Y^vZ^vF + Y^vZ^vF + nY^vZ^vF - Y^vZ^vF) = \frac{n+1}{2} Y^vZ^vF.$$

Putting these together, we obtain

$$\begin{aligned}
(\mathcal{L}_X D)\left(Y^h, \hat{Z}\right) &= -\frac{1}{2}\left[(Y^vF)\hat{Z} + (Z^vF)\hat{Y} + (Y^vZ^vF)\delta\right] + \frac{1}{2}(Y^vZ^vF)\delta \\
&= -\frac{1}{2}\left[(Y^vF)\hat{Z} + (Z^vF)\hat{Y}\right],
\end{aligned}$$

thereby proving the implication.

(4) $\Longrightarrow$ (1) If $t \in \mathbb{R}, v \in \overset{\circ}{T}M$, and $(t, \tau(v)) \in W$, let

$$f_t(v) := \int_0^t F((\varphi_s)_* v) ds.$$

Then f_t is a smooth function on an open submanifold of TM. The smoothness of f_t follows from the differentiability of parametric integrals. Now we compute:

$$(\varphi_{-t})_\sharp D_{(\varphi_t)_\sharp\eta}(\varphi_t)_\sharp\tilde{Z}(v) - D_\eta\tilde{Z}(v) = \int_0^t \left[s \mapsto (\varphi_{-s})_\sharp D_{(\varphi_s)_\sharp\eta}(\varphi_s)_\sharp\tilde{Z}(v)\right]'$$

$$= \int_0^t \lim_{h\to 0}\frac{1}{h}\left[(\varphi_{-s-h})_\sharp D_{(\varphi_{s+h})_\sharp\eta}(\varphi_{s+h})_\sharp\tilde{Z}(v) - (\varphi_{-s})_\sharp D_{(\varphi_s)_\sharp\eta}(\varphi_s)_\sharp\tilde{Z}(v)\right] ds$$

$$= \int_0^t \lim_{h\to 0}\frac{1}{h}\left[(\varphi_{-s-h})_* D_{(\varphi_{s+h})_{**}\eta(v)}(\varphi_{s+h})_\sharp\tilde{Z} - (\varphi_{-s})_* D_{(\varphi_s)_{**}\eta(v)}(\varphi_s)_\sharp\tilde{Z}\right] ds$$

$$= \int_0^t (\varphi_{-s})_* \lim_{h\to 0}\frac{1}{h}\left[(\varphi_{-h})_* D_{(\varphi_{s+h})_{**}\eta(v)}(\varphi_{s+h})_\sharp\tilde{Z} - D_{(\varphi_s)_{**}\eta(v)}(\varphi_s)_\sharp\tilde{Z}\right] ds$$

$$= \int_0^t (\varphi_{-s})_*(\mathcal{L}_X D)\left((\varphi_s)_{**}\eta(v), (\varphi_s)_\sharp\tilde{Z}((\varphi_s)_* v)\right) ds.$$

Here we used the dynamic interpretation of the Lie derivative (2.3.1 (3)). Now we express D with the help of F as prescribed in condition (4):

$$2(\varphi_{-t})_\sharp D_{(\varphi_t)_\sharp\eta}(\varphi_t)_\sharp\tilde{Z}(v) - 2D_\eta\tilde{Z}(v)$$

$$= -\int_0^t \left\{[J(\varphi_s)_{**}\eta(v)]F\cdot\tilde{Z}(v) + \left[\mathbf{i}(\varphi_s)_*\tilde{Z}(v)\right] F\cdot(\varphi_{-s})_*\mathbf{j}((\varphi_s)_{**}\eta(v))\right\} ds$$

$$= -\int_0^t [(\varphi_s)_{**}J\eta(v)F]ds\cdot\tilde{Z}(v) - \int_0^t [(\varphi_s)_{**}\mathbf{i}\tilde{Z}(v)F]ds\cdot\mathbf{j}\eta(v)$$

$$= -\int_0^t J\eta(v)(F\circ(\varphi_s)_*)ds\cdot\tilde{Z}(v) - \int_0^t \mathbf{i}\tilde{Z}(v)(F\circ(\varphi_s)_*)ds\cdot\mathbf{j}\eta(v)$$

$$= -J\eta(v)\int_0^t (F\circ(\varphi_s)_*)ds\cdot\tilde{Z}(v) - \mathbf{i}\tilde{Z}(v)\int_0^t (F\circ(\varphi_s)_*)ds\cdot\mathbf{j}\eta(v)$$

$$= -(J\eta(v))f_t\cdot\tilde{Z}(v) - \left(\mathbf{i}\tilde{Z}(v)\right) f_t\cdot\mathbf{j}\eta(v).$$

Due to 3.3.2, this result implies that φ_t is a local projective transformation for every $t \in \mathbb{R}$, which means that X is a projective vector field. □

3.4.3 Corollary. *The projective vector fields on M form a sub-Lie-algebra of $\mathfrak{X}(M)$.*

Proof. As in the case of affine vector fields, it is enough to show that $[X, Y]$ is projective if X and Y are projective vector fields on M. If this holds, then, according to the previous theorem, there are smooth functions F and G on $\overset{\circ}{T}M$

such that $[X^c, \xi] = FC$ and $[Y^c, \xi] = GC$, and thus

$$\begin{aligned}[][X,Y]^c,\xi] &= [[X^c,Y^c],\xi] = -[[Y^c,\xi],X^c] - [[\xi,X^c],Y^c] \\ &= -[GC,X^c] + [FC,Y^c] = (X^cG)C - (Y^cF)C = (X^cG - Y^cF)C,\end{aligned}$$

which implies that $[X, Y]$ is also a projective vector field. □

If the spray ξ is affine, a similar remark holds as in Section 3.2, i.e., our conditions are in accordance with the well-known results for covariant derivatives on M.

Chapter 4

Finsler – Minkowski vector spaces

In what follows, by a vector space we shall always mean a *finite dimensional* (non-trivial) real vector space with its canonical linear topology. If $(b_i)_{i=1}^n$ is a basis of V, the map[1]

$$a = \alpha^i b_i \in V \mapsto (\alpha^1, \dots, \alpha^n) \in \mathbb{R}^n,$$

also called a *linear coordinate system*, is a global chart on V, therefore it is contained in a unique maximal smooth atlas of the topological manifold V. Moreover, any two such linear coordinate systems are C^∞-compatible, since the passage map is a linear automorphism of $\mathbb{R}^n$. Hence any two linear coordinate systems determine the same differentiable structure on V. We shall consider a vector space as a manifold endowed with this structure.

If V and W are vector spaces, and $U \subset V$ is an open set, we can speak of the pointwise differentiability of a map $f : U \to W$: f is said to be *differentiable* at a point $p \in U$ if there is a linear map $f'(p) : V \to W$ such that

$$f'(p)(v) = \lim_{t \to 0} \frac{f(p + tv) - f(p)}{t} \quad (v \in V).$$

Then $f'(p)$ is uniquely determined, and it is called *the derivative* of f at p. If f is differentiable at every point of U, we can speak of the differentiability of the

[1] We use Einstein's summation convention, i.e., if an index appears twice in a product, it means summation for all possible values of that index.

map

$$f' : p \in U \mapsto f'(p) \in \mathrm{Hom}(V, W),$$

since the vector space $\mathrm{Hom}(V, W)$ is also finite dimensional. If f' is differentiable at each point of U, its derivative is a map

$$f'' : U \to \mathrm{Hom}(V, \mathrm{Hom}(V, W)).$$

The vector space $\mathrm{Hom}(V, \mathrm{Hom}(V, W))$ is canonically isomorphic to $L^2(V, W)$, the vector space of bilinear maps $V \times V \to W$, therefore they may be considered as identical. Proceeding inductively, we can define the kth derivative $f^{(k)}$ of f. If it exists at every point of U, and it is continuous, then f is called of class C^k on U. A map $f : U \to W$ is of class C^∞ or smooth as a map between two differentiable manifolds if and only if it is of class C^k for all integers $k \geqq 1$ in the sense just defined.

4.1 Definition. *A* Finsler – Minkowski norm *on a vector space V is a function $\varphi : V \to \mathbb{R}$ that satisfies the following axioms:*

(1) $\varphi(v) > 0$ *if* $v \neq 0$ (positivity)*;*

(2) *if* $\lambda > 0$*, then* $\varphi(\lambda v) = \lambda\varphi(v)$ *for all* $v \in V$ (positive homogeneity)*;*

(3) φ *is smooth over* $V \setminus \{0\}$*;*

(4) *if* $E := \frac{1}{2}\varphi^2$*, then for all* $p \in V \setminus \{0\}$ *the symmetric bilinear form*

$$g_p := E''(p) : V \times V \to \mathbb{R}$$

is non-degenerate.

A *Finsler – Minkowski vector space* is a vector space endowed with a Finsler – Minkowski norm. The function E is the *energy function*, the map

$$g : p \in V \setminus \{0\} \mapsto g_p \in L^2(V, \mathbb{R})$$

is the *metric tensor* of the Finsler – Minkowski space (V, F). The (open) *unit ball* in (V, φ) is the set

$$\Omega := \{v \in V | \varphi(v) < 1\} = \varphi^{-1}\,[0, 1[= \left\{ v \in V \,\middle|\, E(v) < \frac{1}{2} \right\};$$

the *unit sphere* in (V, F) is the boundary $\partial\Omega$ of the unit ball. Thus

$$\partial\Omega = \{v \in V | \varphi(v) = 1\} = \varphi^{-1}(1) = \left\{ v \in V \,\middle|\, E(v) = \frac{1}{2} \right\} = E^{-1}\left(\frac{1}{2}\right).$$

A Finsler – Minkowski norm $\varphi : V \to \mathbb{R}$ is called *reversible* if $\varphi(v) = \varphi(-v)$ for all $v \in V$. We note that (2) immediately implies $\varphi(0) = 0$.

We can speak of the curvature of the unit sphere in two different senses. First, since $(V \setminus \{0\}, g)$ is a semi-Riemannian manifold, we may consider the shape operator on $\partial\Omega$ induced by this structure. This way is followed, e.g., in [62]. On the other hand, we may identify V with $\mathbb{R}^n$ by a linear coordinate system, and use the shape operator induced by the Euclidean metric of $\mathbb{R}^n$. Here we shall follow this second way. This identification is no canonical; in the proof of 4.3, however, we shall find an estimate on the sign of the normal curvature which will be independent of the choice of the linear coordinate system.

4.2 Lemma. *The metric tensor g of a Finsler – Minkowski vector space (V, F) has the following properties:*

$$g_p(p, v) = E'(p)(v)$$
$$g_p(p, p) = 2E(p) = \varphi^2(p)$$

for all $p \in V \setminus \{0\},\ v \in V$.

The proof is easy and well-known. Since φ is positively homogeneous of degree 1, the energy function E is positively homogeneous of degree 2, and its derivative is also 1-homogeneous. Taking these into account we get the result.

Also by the homogeneity properties, it is not difficult to check that the Finsler – Minkowski norm φ is *continuous at 0* and the *energy E has a derivative at 0.* However, the existence of the second derivative $E''(0)$ cannot be expected in general, and, what is more, if its existence is supposed, the structure becomes semi-Euclidean, namely

$$E(p) = \frac{1}{2} E''(0)(p, p) \text{ for all } p \in V,$$

thus the function

$$(v, w) \in V \times V \mapsto \langle v, w \rangle := E''(0)(v, w)$$

is a scalar product on V. Our next observation will show that this scalar product is positive definite.

4.3 Proposition. *The metric tensor g of a Finsler – Minkowski norm is positive definite, i.e. $g_p \in L^2(V, \mathbb{R})$ is a positive definite scalar product for all $p \in V \setminus \{0\}$, therefore $(V \setminus \{0\}, g)$ is a Riemannian manifold.*

Proof. For simplicity, we may replace our vector space V with the Euclidean vector space $\mathbb{R}^n$ endowed with the canonical scalar product $\langle\ ,\ \rangle$ and the norm $\|\ \|$ induced by $\langle\ ,\ \rangle$. The tangent space of $\mathbb{R}^n$ at a point p is $T_p\mathbb{R}^n := \{p\} \times \mathbb{R}^n$, sometimes it will be identified with $\mathbb{R}^n$ itself.

The unit sphere $\partial\Omega = E^{-1}(1/2)$ is a smooth hypersurface in $\mathbb{R}^n$. Indeed, the non-degeneracy of the metric tensor g guarantees that at any point $p \in \mathbb{R}^n \setminus \{0\}$ there exists a vector $v \in \mathbb{R}^n$ such that

$$g_p(p, v) = E'(p)(v) \neq 0.$$

Thus E is a submersion at any point of $\partial\Omega$.

If $n \geqq 2$, any two points $p, q \in \mathbb{R}^n \setminus \{0\}$ can be joined by a smooth arc. The parallel translation along this smooth arc with respect to the Levi-Civita connection induced by the semi-Riemannian metric g is an isometry between the semi-Euclidean vector spaces $(T_p\mathbb{R}^n, g_p)$ and $(T_q\mathbb{R}^n, g_q)$, therefore the signature of g is the same at every point of $\mathbb{R}^n \setminus \{0\}$. Hence it is enough to show that g is positive definite at a particular point.

To prove this, we shall need the unit normal vector field

$$\underline{N} : p \in \partial\Omega \mapsto \underline{N}(p) = (p, N(p)) \in T_p\mathbb{R}^n,\ N(p) := -\frac{\operatorname{grad} E(p)}{\|\operatorname{grad} E(p)\|}$$

on $\partial\Omega$. Now we are going to use a quite standard trick (cf. [60, proof of Theorem 4, Chapter 12]). Define the function $f : \mathbb{R}^n \to \mathbb{R}$ by $f(p) := \|p\|^2$. Then f is continuous, and the compactness of $\partial\Omega$ implies the existence of a point $p \in \partial\Omega$ at which the maximum of $f \restriction \partial\Omega$ is attained. Given any unit tangent vector $v_p \in T_p\partial\Omega$ ('unit' means $\|v_p\| := \|v\| = 1$), choose a curve $c : I \subset \mathbb{R} \to \partial\Omega$ such that

$$\dot{c}(0) = v_p,\ \text{i.e. } c(0) = p,\ c'(0) = v.$$

Then the function $f \circ c : I \to \mathbb{R}$ has a maximum at 0. Thus

$$0 = (f \circ c)'(0) = \langle c, c \rangle' (0) = 2 \langle c(0), c'(0) \rangle = 2 \langle p_p, v_p \rangle \,, \tag{4.1}$$

$$0 \geqq (f \circ c)''(0) = 2 \langle c, c' \rangle' (0) = 2(\langle c', c' \rangle + \langle c, c'' \rangle)(0) = 2 + 2 \langle p, c''(0) \rangle \,. \tag{4.2}$$

Since v_p was an arbitrary unit tangent vector to $\partial\Omega$ at p, (4.1) implies that $\left(p, \frac{p}{\|p\|}\right)$ is a unit normal vector to $\partial\Omega$ at p, thus

$$\underline{N}(p) = \left(p, \frac{p}{\|p\|}\right) \quad \text{or} \quad \underline{N}(p) = \left(p, -\frac{p}{\|p\|}\right).$$

To decide between these two possibilities we calculate the scalar product $\langle p, \operatorname{grad} E(p)\rangle$. By Lemma 4.2 we obtain

$$\langle p, \operatorname{grad} E(p)\rangle = E'(p)(p) = g_p(p,p) = 2E(p) = 1 > 0,$$

therefore

$$\underline{N}(p) = \left(p, -\frac{p}{\|p\|}\right).$$

Consider the shape operator

$$L_p : u_p \in T_p\partial\Omega \mapsto L_p(u_p) := (p, -N'(p)(u))$$

of $\partial\Omega$ at p. The normal curvature of $\partial\Omega$ in the direction v_p is

$$k(v_p) := \langle L_p(v_p), v_p\rangle = \langle c''(0), N(p)\rangle = -\frac{1}{\|p\|}\langle p, c''(0)\rangle,$$

therefore (4.2) leads to the relation

$$k(v_p) \geqq \frac{1}{\|p\|}.$$

In particular, all principal curvatures $k_i(p)$ satisfy the inequality

$$k_i(p) \geqq \frac{1}{\|p\|},\ 1 \leqq i \leqq n-1.$$

Let $(\underline{b}_i)_{i=1}^{n-1}$ be an orthonormal basis of the Euclidean vector space $T_p\partial\Omega$ consisting of the eigenvectors of L_p with the eigenvalues $k_1(p), \dots, k_{n-1}(p)$. Then

$$\mathcal{B} := (\underline{b}_1, \dots, \underline{b}_{n-1}, \underline{N}(p))$$

is an orthonormal basis of $T_p\mathbb{R}^n \cong \mathbb{R}^n$. If φ_{pq} is the canonical isomorphism

$$T_p\mathbb{R}^n \to T_q\mathbb{R}^n,\ (p,v) \mapsto (q,v),$$

then the maps

$$B_i : q \in \mathbb{R}^n \mapsto B_i(q) := \varphi_{pq}(\underline{b}_i) \in T_q\mathbb{R}^n \quad (1 \leqq i \leqq n-1)$$

are (constant) vector fields on $\mathbb{R}^n$. Using these, in the next and final step we show that *the basis $\mathcal{B}$ is orthogonal also relative to the scalar product g_p.*

For any indices $i, j \in \{1, \dots, n-1\}$ we have

$$\begin{aligned} g_p(\underline{b}_i, \underline{b}_j) &= \underline{b}_i(B_j E) = \underline{b}_i \langle \operatorname{grad} E, B_j \rangle = \langle (D_{b_i} \operatorname{grad} E)(p), b_j \rangle \\ &= \left\langle D_{b_i} \left(\| \operatorname{grad} E \| \frac{\operatorname{grad} E}{\| \operatorname{grad} E \|} \right)(p), b_j \right\rangle \\ &= \| \operatorname{grad} E(p) \| \left\langle \left(D_{b_i} \frac{\operatorname{grad} E}{\| \operatorname{grad} E \|} \right)(p), b_j \right\rangle = -\| \operatorname{grad} E(p) \| \langle (D_{b_i} N)(p), b_j \rangle \\ &= \| \operatorname{grad} E(p) \| \left\langle L_p(\underline{b}_i), \underline{b}_j \right\rangle = k_i(p) \| \operatorname{grad} E(p) \| \delta_{ij}, \end{aligned}$$

thus $g_p(\underline{b}_i, \underline{b}_i) > 0$ for all $i \in \{1, \dots, n-1\}$. The g_p-length of $\underline{N}(p)$ is also positive, since

$$g_p(\underline{N}(p), \underline{N}(p)) = \frac{1}{\|p\|^2} g_p(p,p) = \frac{2}{\|p\|^2} E(p) = \frac{1}{\|p\|^2} > 0.$$

It remains only to check that $\underline{N}(p)$ is g_p-orthogonal to $T_p \partial \Omega$. An immediate calculation shows that

$$g_p(\underline{b}_i, \underline{N}(p)) = \underline{N}(p)(B_i E) = -\frac{1}{\|p\|} p_p \langle \operatorname{grad} E, B_i \rangle$$

for all $i \in \{1, \dots, n-1\}$. Since $\langle \operatorname{grad} E, B_i \rangle$ vanishes at p, by the homogeneity of E it follows that it also vanishes on the ray with vertex 0 through p. Thus

$$g_p(\underline{b}_i, \underline{N}(p)) = 0 \quad (1 \leqq i \leqq n-1).$$

We have obtained that all the vectors of the basis $\mathcal{B}$ have positive length relative to g_p. This proves that g_p is a positive definite scalar product. □

Chapter 5

Natural metric covariant derivatives of generalized metrics

5.1 Generalized metrics

In this section we introduce generalized metrics and some of their special classes. Our main source is reference [36].

5.1.1 Definition. *Let g be a symmetric and non-degenerate tensor of type (0,2) in the bundle $\pi^*\tau$. Then g is said to be a* generalized metric *or briefly a* metric.

It is crucial that g need not be defined on the zero section, since, if g is homogeneous and is defined in the whole $\tau^*\tau$ (and, of course, is smooth), then it is the lift of a pseudo-Riemannian metric on M.

It is important to note that we mean non-degeneracy pointwise and not on the level of sections. (The latter would be a weaker condition on g.) More precisely, it is required that

$$\forall v \in \widetilde{TM} : g_v : T_{\pi(v)}M \times T_{\pi(v)}M \to \mathbb{R}$$

should be a non-degenerate symmetric bilinear form.

Due to the non-degeneracy of g, if $\tilde{\alpha}$ is a one-form along π, there is a unique vector field $\tilde{\alpha}^\sharp$ along π such that $\tilde{\alpha}\left(\tilde{Y}\right) = g\left(\tilde{\alpha}^\sharp, \tilde{Y}\right)$ for any vector field $\tilde{Y}$

along π (Riesz' lemma). Conversely, if $\tilde{X}$ is a vector field along π, then we have a one-form $\tilde{X}^\flat \in \mathfrak{X}^*(\pi)$ such that $\tilde{X}^\flat\left(\tilde{Y}\right) = g\left(\tilde{X}, \tilde{Y}\right)$ for any vector field $\tilde{Y} \in \mathfrak{X}(\pi)$.

Using again non-degeneracy, the *first Cartan tensor* $\mathcal{C}$ and the *lowered first Cartan tensor* $\mathcal{C}_\flat$ of a generalized metric g are defined by the following formulae:

$$g\left(\mathcal{C}\left(\tilde{X}, \tilde{Y}\right), \tilde{Z}\right) := \mathcal{C}_\flat\left(\tilde{X}, \tilde{Y}, \tilde{Z}\right) := \left(\nabla^v_{\tilde{X}} g\right)\left(\tilde{Y}, \tilde{Z}\right) \quad \left(\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{X}(\pi)\right).$$

The definition formally coincides with the one known from Finsler geometry. Generalized metrics differ from Finsler metrics in the essential point that the lowered Cartan tensor $\mathcal{C}_\flat$ of a generalized metric is not totally symmetric in general, only in its last two variables. The one-form

$$\vartheta_g : \tilde{X} \in \mathfrak{X}(\pi) \mapsto \vartheta_g \tilde{X} := g\left(\tilde{X}, \delta\right)$$

along π is called the *Lagrange one-form* associated to g, and the two-form $\omega_g := d(\vartheta_g \circ \mathbf{j})$ on $\widetilde{TM}$ is the *Lagrange two-form* associated to g. The *absolute energy* of g is $E := \frac{1}{2} g(\delta, \delta)$.

5.1.2 Definition. *A metric g along π is said to be* variational *if the first Cartan tensor $\mathcal{C}$ associated to it is symmetric,* weakly variational *if* $\mathcal{C}_\flat\left(\tilde{X}, \tilde{Y}, \delta\right) = \mathcal{C}_\flat\left(\tilde{Y}, \tilde{X}, \delta\right)$ *for every* $\tilde{X}, \tilde{Y} \in \mathfrak{X}(\pi)$*,* normal *if* $\mathcal{C}\left(\tilde{X}, \delta\right) = 0$ *for every* $\tilde{X} \in \mathfrak{X}(\pi)$*, and* weakly normal *if* $\mathcal{C}_\flat\left(\tilde{X}, \delta, \delta\right) = 0$ *for every* $\tilde{X} \in \mathfrak{X}(\pi)$*. The metric is* regular in Miron's sense *or briefly* Miron regular *[38] if the tensor*

$$A : \tilde{X} \in \mathfrak{X}(\pi) \mapsto A\tilde{X} := \tilde{X} + \mathcal{C}\left(\tilde{X}, \delta\right)$$

has maximal rank at every point of $\widetilde{TM}$*. If* $\gamma := \nabla^v \nabla^v E$ *is also non-degenerate, g is called* energy-regular*. An energy-regular and homogeneous* ($\nabla^v_\delta g = 0$) *metric is also mentioned as a* Moór – Vanstone metric.

We note that the tensors ω_g and A are closely related by the formula

$$\omega_g(J\eta, \zeta) = g\left(A(\mathbf{j}\eta), \mathbf{j}\zeta\right) \quad \left(\eta, \zeta \in \mathfrak{X}\left(\widetilde{TM}\right)\right)$$

Now, for the sake of the reader's convenience, we summarize some results of [36] we shall make use of.

(1) If $\widetilde{T_pM}$ is simply connected for every $p \in M$, then a metric g on $\widetilde{TM}$ is variational if and only if there is a smooth function L on $\widetilde{TM}$ whose Hessian is g, more precisely, $g = \nabla^v\nabla^v L$. In this case, we shall call L a *Lagrangian.*

(2) If $\widetilde{T_pM}$ is simply connected for every $p \in M$, then a metric g is weakly variational if and only if there is a smooth function L on $\widetilde{TM}$ such that $\vartheta_g = d^v L$.

(3) If $\widetilde{TM} = \overset{\circ}{T}M$, and g is weakly normal and Miron regular, then it is energy-regular, and E is positively homogeneous of degree 2 ($CE = 2E$). Furthermore, $\vartheta_g = d^v E$.

(4) If $\widetilde{TM} = \overset{\circ}{T}M$, and g is normal, then $g = \nabla^v\nabla^v E$.

If E is positively homogeneous of degree 2, it can be extended continuously to the zero section. In that case, it will be zero on the zero vectors.

By a Finsler energy function, we shall mean a continuous function E on TM, smooth on $\overset{\circ}{T}M$, which is positively homogeneous of degree 2, and which has the property that $\nabla^v\nabla^v E$ is non-degenerate. In view of this convention, (3) says that E is a Finsler energy function. Moreover, (4) may be interpreted in the following way: there is a natural one-to-one correspondence between the Finsler energy functions and the normal metrics on M. Namely, for a Finsler energy E, $g := \nabla^v\nabla^v E$ is a normal metric; and for a normal metric g, $E := \frac{1}{2}g(\delta, \delta)$ is a Finsler energy. Therefore, by a Finsler manifold we may mean either a pair (M, E) where E is a Finsler energy, or a pair (M, g) where g is a normal metric.

If a Finsler energy E has the property that $E(v) > 0$ for any $v \neq 0$, then $F := \sqrt{2E}$ is a continuous function on TM, smooth on $\overset{\circ}{T}M$. In this case, F is said to be a *Finslerian fundamental function* or simply a *fundamental function.* The restriction of a fundamental function to a tangent space T_pM is a Finsler–Minkowski norm in the sense of 4.1. In this case, by 4.3, g is positive definite. Therefore, we shall call this kind of Finsler energy functions *positive definite.* A positive definite Finsler structure may be given either by the energy function or by the fundamental function.

We shall denote by $\xi : TM \to TTM$ the canonical spray of the Finsler manifold (M, E) determined by the relation $(dd_J E)(\xi, \eta) = -\eta E$ for $\eta \in \mathfrak{X}\left(\overset{\circ}{T}M\right)$. A *geodesic* of the Finsler manifold is a geodesic of ξ. The Ehresmann connection associated to ξ as described in section 3.1 is called the *Barthel connection*. This connection is conservative with respect to E in the sense that $X^h E = 0$ for any vector field X on M. The spray ξ is horizontal with respect to the Barthel connection; moreover, $\xi = \mathcal{H}\delta$.

If a metric g is given on $\widetilde{TM}$, a covariant derivative D in $\pi^*\tau$ is said to be *metric* if $Dg = 0$. If an Ehresmann connection $\mathcal{H}$ is also given on $\widetilde{TM}$, we can construct a (unique) metric covariant derivative D in $\pi^*\tau$ whose vertical torsion vanishes and whose horizontal torsion coincides with the torsion of the given Ehresmann connection, as follows (see [24, 36]). Let ∇ be Berwald's covariant derivative in $\pi^*\tau$ induced by the Ehresmann connection. First, we introduce the *second Cartan tensor* $\mathcal{C}^h$ by means of the relation

$$g\left(\mathcal{C}^h\left(\tilde{X},\tilde{Y}\right),\tilde{Z}\right) := (\nabla_{\mathcal{H}\tilde{X}}g)\left(\tilde{Y},\tilde{Z}\right) \quad \left(\tilde{X},\tilde{Y},\tilde{Z}\in\mathfrak{X}(\pi)\right).$$

Next, using the Christoffel trick, we define two other tensors along π:

$$g\Big(\overset{\circ}{\mathcal{C}}\left(\tilde{X},\tilde{Y}\right),\tilde{Z}\Big) = g\left(\mathcal{C}\left(\tilde{X},\tilde{Y}\right),\tilde{Z}\right) + g\left(\mathcal{C}\left(\tilde{Y},\tilde{Z}\right),\tilde{X}\right) - g\left(\mathcal{C}\left(\tilde{Z},\tilde{X}\right),\tilde{Y}\right),$$

$$\begin{aligned} g\Big(\overset{\circ}{\mathcal{C}}{}^h\left(\tilde{X},\tilde{Y}\right),\tilde{Z}\Big) &= g\left(\mathcal{C}^h\left(\tilde{X},\tilde{Y}\right),\tilde{Z}\right) + g\left(\mathcal{C}^h\left(\tilde{Y},\tilde{Z}\right),\tilde{X}\right) \\ &\quad - g\left(\mathcal{C}^h\left(\tilde{Z},\tilde{X}\right),\tilde{Y}\right). \end{aligned}$$

With the help of $\overset{\circ}{\mathcal{C}}$ and $\overset{\circ}{\mathcal{C}}{}^h$ we define D by the rules

$$D_{\mathbf{i}\tilde{X}}\tilde{Y} := \nabla_{\mathbf{i}\tilde{X}}\tilde{Y} + \frac{1}{2}\overset{\circ}{\mathcal{C}}\left(\tilde{X},\tilde{Y}\right),\ D_{\mathcal{H}\tilde{X}}\tilde{Y} := \nabla_{\mathcal{H}\tilde{X}}\tilde{Y} + \frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h\left(\tilde{X},\tilde{Y}\right)$$
$$\left(\tilde{X},\tilde{Y},\tilde{Z}\in\mathfrak{X}(\pi)\right).$$

Finally, this covariant derivative operator can also be extended to any type of tensors by Willmore's theorem (2.1.4). Then it will be metric, i.e., $Dg = 0$. We note that $\mathcal{H}$ is not necessarily attached to this covariant derivative in the sense described in section 2.2. In the next section we give a necessary and sufficient condition for $\mathcal{H}$ to be attached to D in a special case.

If g arises from a Finsler energy function, and $\mathcal{H}$ is the Barthel connection on the Finsler manifold, then $\mathcal{C}$ is totally symmetric, $\overset{\circ}{\mathcal{C}} = \mathcal{C}$, and D coincides with the well-known *Cartan's covariant derivative* [51, 57]. Therefore, we use here this construction as a definition of Cartan's covariant derivative, and in section 5.4 we shall show that, on a Finsler manifold, this is the unique covariant derivative which possesses certain nice properties.

5.2 A characterization of metric derivatives

In this section, g will be a weakly normal Moór – Vanstone metric on $\overset{\circ}{T}M$, ξ the canonical spray belonging to the Finsler energy E, $\mathcal{H}_E$ the Barthel connection, and $\overset{E}{\nabla}$ Berwald's covariant derivative arising from $\mathcal{H}_E$. Other data of $\mathcal{H}_E$ will be distinguished from those of an arbitrary Ehresmann connection by a subscript E. An Ehresmann connection $\mathcal{H}$ is said to be *conservative* with respect to E if $\mathfrak{X}^h\left(\widetilde{TM}\right) \subset \operatorname{Ker} dE$.

5.2.1 Proposition. *Given a type (1,1) tensor field P along $\overset{\circ}{\tau}$, suppose that $\mathcal{H} = \mathcal{H}_E - \mathbf{i} \circ P$ is a conservative Ehresmann connection with respect to E on $\overset{\circ}{T}M$ such that $\mathcal{H}\delta = \xi$. Then $\mathcal{H}$ is attached to the metric covariant derivative D arising from $\mathcal{H}$ if and only if*

$$g\left((\nabla_C P)\left(\tilde{X}\right), \tilde{Y}\right) + g\left(\tilde{X}, (\nabla_C P)\left(\tilde{Y}\right)\right) = -\left(\overset{E}{\nabla}_\xi g\right)\left(\tilde{X}, \tilde{Y}\right)$$

for any $\tilde{X}, \tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau})$.

Proof. Let ∇ be Berwald's derivative arising from $\mathcal{H}$. Since g is non-degenerate, it is enough to consider the expression $g\left(D_{X^h}\delta, \hat{Y}\right)$. Using our condition $\mathcal{H}\delta = \xi$, we obtain

$$\begin{aligned}
2g\left(D_{X^h}\delta, \hat{Y}\right) &= 2g\left(\nabla_{X^h}\delta, \hat{Y}\right) + g\left(\overset{\circ}{\mathcal{C}}{}^h\left(\hat{X}, \delta\right), \hat{Y}\right) = 2g\left(\mathcal{V}\left[X^h, C\right], \hat{Y}\right) \\
&\quad + g\left(\mathcal{C}^h\left(\hat{X}, \delta\right), \hat{Y}\right) + g\left(\mathcal{C}^h\left(\delta, \hat{Y}\right), \hat{X}\right) - g\left(\mathcal{C}^h\left(\hat{Y}, \hat{X}\right), \delta\right) \\
&= 2g\left(\mathbf{t}\hat{X}, \hat{Y}\right) + (\nabla_{X^h} g)\left(\hat{Y}, \delta\right) + (\nabla_\xi g)\left(\hat{X}, \hat{Y}\right) - (\nabla_{Y^h} g)\left(\hat{X}, \delta\right) \\
&= 2g\left(\mathbf{t}\hat{X}, \hat{Y}\right) + X^h g\left(\hat{Y}, \delta\right) - g\left(\nabla_{X^h}\hat{Y}, \delta\right) - g\left(\hat{Y}, \nabla_{X^h}\delta\right) \\
&\quad + \xi g\left(\hat{X}, \hat{Y}\right) - g\left(\nabla_\xi \hat{X}, \hat{Y}\right) - g\left(\hat{X}, \nabla_\xi \hat{Y}\right) \\
&\quad - Y^h g\left(\hat{X}, \delta\right) + g\left(\nabla_{Y^h}\hat{X}, \delta\right) + g\left(\hat{X}, \nabla_{Y^h}\delta\right).
\end{aligned}$$

Since g is weakly normal, by the property (3) mentioned above we have $\vartheta_g =$

d^vE, and we may use, e.g., $g\left(\hat{Y},\delta\right) = Y^vE$. Thus we get

$$\begin{aligned}
2g\left(D_{X^h}\delta,\hat{Y}\right) &= 2g\left(\mathbf{t}\hat{X},\hat{Y}\right) + X^hY^vE - \mathbf{i}\left(\nabla_{X^h}\hat{Y}\right)E \\
&- g\left(\hat{Y},\mathcal{V}\left[X^h,C\right]\right) + \left(\overset{E}{\nabla}_\xi g\right)\left(\hat{X},\hat{Y}\right) + g\left(\overset{E}{\nabla}_\xi\hat{X},\hat{Y}\right) \\
&+ g\left(\hat{X},\overset{E}{\nabla}_\xi\hat{Y}\right) - g\left(\nabla_\xi\hat{X},\hat{Y}\right) - g\left(\hat{X},\nabla_\xi\hat{Y}\right) \\
&- Y^hX^vE + \mathbf{i}\left(\nabla_{Y^h}\hat{X}\right)E + g\left(\hat{X},\mathcal{V}\left[Y^h,C\right]\right).
\end{aligned}$$

Now we use the definition of the tension as described in section 2.2:

$$\begin{aligned}
2g\left(D_{X^h}\delta,\hat{Y}\right) &= 2g\left(\mathbf{t}\hat{X},\hat{Y}\right) + X^hY^vE - \left[X^h,Y^v\right]E - g\left(\mathbf{t}\hat{X},\hat{Y}\right) \\
&+ \left(\overset{E}{\nabla}_\xi g\right)\left(\hat{X},\hat{Y}\right) + g\left(\overset{E}{\nabla}_\xi\hat{X} - \nabla_\xi\hat{X},\hat{Y}\right) + g\left(\hat{X},\overset{E}{\nabla}_\xi\hat{Y} - \nabla_\xi\hat{Y}\right) \\
&- Y^hX^vE + \left[Y^h,X^v\right]E + g\left(\hat{X},\mathbf{t}\hat{Y}\right) \\
&= g\left(\mathbf{t}\hat{X},\hat{Y}\right) + g\left(\hat{X},\mathbf{t}\hat{Y}\right) + Y^vX^hE + g\left(\mathcal{V}_E[\xi,X^v] - \mathcal{V}[\xi,X^v],\hat{Y}\right) \\
&+ g\left(\hat{X},\mathcal{V}_E[\xi,Y^v] - \mathcal{V}[\xi,Y^v]\right) + \left(\overset{E}{\nabla}_\xi g\right)\left(\hat{X},\hat{Y}\right) - X^vY^hE.
\end{aligned}$$

Now by our condition that $\mathcal{H}$ is conservative with respect to E, the expression obtained in the last step takes the following form:

$$\begin{aligned}
2g\left(D_{X^h}\delta,\hat{Y}\right) &= g\left(\mathbf{t}\hat{X},\hat{Y}\right) + g\left(\hat{X},\mathbf{t}\hat{Y}\right) \\
&- g\left(P\mathbf{j}[\xi,X^v],\hat{Y}\right) - g\left(\hat{X},P\mathbf{j}[\xi,Y^v]\right) + \left(\overset{E}{\nabla}_\xi g\right)\left(\hat{X},\hat{Y}\right) \\
&= g\left(P\hat{X},\hat{Y}\right) + g\left(\mathbf{t}\hat{X},\hat{Y}\right) + g\left(\hat{X},P\hat{Y}\right) + g\left(\hat{X},\mathbf{t}\hat{Y}\right) + \left(\overset{E}{\nabla}_\xi g\right)\left(\hat{X},\hat{Y}\right).
\end{aligned}$$

Hence it remains only to show that $\nabla_C P = P + \mathbf{t}$. Using the relation $\mathcal{L}_C J = -J$, we find:

$$\begin{aligned}
\mathbf{i}P\hat{X} + \mathbf{i}\mathbf{t}\hat{X} &= J\mathcal{H}P\hat{X} - \left[C,X^h\right] = -(\mathcal{L}_C J)\mathcal{H}P\hat{X} - \left[C,X^{h_E} - \mathbf{i}P\hat{X}\right] \\
&= -\left[C,\mathbf{i}P\hat{X}\right] + J\left[C,\mathcal{H}P\hat{X}\right] + \left[C,\mathbf{i}P\hat{X}\right] = \mathbf{i}\nabla_C\left(P\hat{X}\right) = \mathbf{i}\left(\nabla_C P\right)\left(\hat{X}\right),
\end{aligned}$$

thus proving our proposition. □

5.2.2 Theorem. *Let $P \in \mathcal{T}^1_1(\overset{\circ}{\tau})$ be an arbitrary tensor. Consider the Ehresmann connection $\mathcal{H} := \mathcal{H}_E - \mathbf{i} \circ P$ on $\overset{\circ}{T}M$ and the unique metric covariant derivative D along $\overset{\circ}{\tau}$ whose vertical torsion vanishes, and whose horizontal torsion coincides with the torsion of $\mathcal{H}$. The horizontal map $\mathcal{H}$ satisfies the conditions*

(i) $\mathcal{H}\delta = \xi$,

(ii) *$\mathcal{H}$ is conservative with respect to E,*

(iii) *$\mathcal{H}$ is attached to D*

if and only if

$$P\tilde{X} = -\frac{1}{2}\Big(i_{\tilde{X}}\overset{E}{\nabla}_\xi g\Big)^\sharp + P_s\tilde{X} + P_a\tilde{X} \quad \Big(\tilde{X} \in \mathfrak{X}(\overset{\circ}{\tau})\Big),$$

where the tensors $P_s, P_a \in \mathcal{T}^1_1(\overset{\circ}{\tau})$ are such that P_s is self-adjoint (with respect to g) and homogeneous of degree 0, P_a is skew-symmetric (with respect to g), and the image of both P_s and P_a is contained in the orthogonal complement of the canonical section δ.

Proof. By the previous proposition, we have to solve the following mixed system of algebraic equations and a partial differential equation for P:

$$P\delta = 0 \tag{1}$$

$$\Big(\mathbf{i}P\tilde{X}\Big)E = 0 \quad \Big(\tilde{X} \in \mathfrak{X}(\overset{\circ}{\tau})\Big) \tag{2}$$

$$g\Big((\nabla_C P)\Big(\tilde{X}\Big), \tilde{Y}\Big) + g\Big(\tilde{X}, (\nabla_C P)\Big(\tilde{Y}\Big)\Big) = -\Big(\overset{E}{\nabla}_\xi g\Big)\Big(\tilde{X}, \tilde{Y}\Big) \quad \Big(\tilde{X}, \tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau})\Big). \tag{3}$$

As the system is linear, we may search its general solution as the sum of a general solution of the homogeneous part and a particular solution. First we show that the tensor P defined by

$$\tilde{X} \in \mathfrak{X}(\overset{\circ}{\tau}) \mapsto P\tilde{X} := -\frac{1}{2}\Big(i_{\tilde{X}}\overset{E}{\nabla}_\xi g\Big)^\sharp \in \mathfrak{X}(\overset{\circ}{\tau})$$

(roughly speaking, the first term of the main formula in the proposition) is a particular solution, i.e., it satisfies (1) – (3). To show that it satisfies (1), it is enough to check that $g\left(P\delta, \hat{X}\right) = 0$ for all $X \in \mathfrak{X}(M)$:

$$\begin{aligned} g\left(P\delta, \hat{X}\right) &= g\left(\left(i_\delta \overset{E}{\nabla}_\xi g\right)^\sharp, \hat{X}\right) = \left(i_\delta \overset{E}{\nabla}_\xi g\right)\left(\hat{X}\right) = \left(\overset{E}{\nabla}_\xi g\right)\left(\delta, \hat{X}\right) \\ &= \xi g\left(\delta, \hat{X}\right) - g\left(\overset{E}{\nabla}_\xi \delta, \hat{X}\right) - g\left(\delta, \overset{E}{\nabla}_\xi \hat{X}\right) \\ &= \xi\left(\vartheta_g \hat{X}\right) - g\left(\mathcal{V}_E[\xi, C], \hat{X}\right) - \vartheta_g\left(\overset{E}{\nabla}_\xi \hat{X}\right) = \xi X^v E - \left(\mathbf{i}\overset{E}{\nabla}_\xi \hat{X}\right)E \\ &= [\xi, X^v]E - (\mathbf{v}_E[\xi, X^v])E = (\mathbf{h}_E[\xi, X^v])E = 0, \end{aligned}$$

thus (1) is indeed satisfied. On the other hand,

$$\mathbf{i}\left(i_{\hat{X}} \overset{E}{\nabla}_\xi g\right)^\sharp E = g\left(\left(i_{\hat{X}} \overset{E}{\nabla}_\xi g\right)^\sharp, \delta\right) = \left(i_{\hat{X}} \overset{E}{\nabla}_\xi g\right)(\delta) = \left(\overset{E}{\nabla}_\xi g\right)\left(\hat{X}, \delta\right) = 0,$$

as in the proof of (1); this proves (2). To verify that (3) is satisfied as well, first we show that $\nabla_C\left(i_{\hat{X}} \overset{E}{\nabla}_\xi g\right)^\sharp = \left(i_{\hat{X}} \overset{E}{\nabla}_\xi g\right)^\sharp$, i.e., the vector field $\left(i_{\hat{X}} \overset{E}{\nabla}_\xi g\right)^\sharp$ is positively homogeneous of degree 1. Using the homogeneity of g, i.e. the relation $\nabla_C g = 0$, we obtain

$$\begin{aligned} g\left(\nabla_C\left(i_{\hat{X}} \overset{E}{\nabla}_\xi g\right)^\sharp, \hat{Y}\right) &= Cg\left(\left(i_{\hat{X}} \overset{E}{\nabla}_\xi g\right)^\sharp, \hat{Y}\right) - g\left(\left(i_{\hat{X}} \overset{E}{\nabla}_\xi g\right)^\sharp, \nabla_C \hat{Y}\right) \\ &= C\left(i_{\hat{X}} \overset{E}{\nabla}_\xi g\right)\left(\hat{Y}\right) = C\left(\overset{E}{\nabla}_\xi g\right)\left(\hat{X}, \hat{Y}\right) \\ &= C\xi g\left(\hat{X}, \hat{Y}\right) - C\left(\overset{E}{\nabla}_\xi \hat{X}, \hat{Y}\right) - C\left(\hat{X}, \overset{E}{\nabla}_\xi \hat{Y}\right). \end{aligned}$$

Now we treat these terms separately:

$$C\xi g\left(\hat{X}, \hat{Y}\right) = [C, \xi]g\left(\hat{X}, \hat{Y}\right) + \xi Cg\left(\hat{X}, \hat{Y}\right) = [C, \xi]g\left(\hat{X}, \hat{Y}\right) = \xi g\left(\hat{X}, \hat{Y}\right),$$

$$\begin{aligned} Cg\left(\overset{E}{\nabla}_\xi \hat{X}, \hat{Y}\right) &= g\left(\overset{E}{\nabla}_C \overset{E}{\nabla}_\xi \hat{X}, \hat{Y}\right) + g\left(\overset{E}{\nabla}_\xi \hat{X}, \overset{E}{\nabla}_C \hat{Y}\right) = g\left(\mathbf{j}[C, \mathcal{H}_E \mathcal{V}_E[\xi, X^v]], \hat{Y}\right) \\ &= g\left(\mathbf{j}[C, \mathcal{H}_E \mathcal{V}_E X^c]], \hat{Y}\right) = g\left(\mathbf{i}^{-1}([C, \mathbf{v}_E X^c] - (\mathcal{L}_C J)\mathcal{H}_E \mathcal{V}_E X^c), \hat{Y}\right) \\ &= g\left(\mathbf{i}^{-1}\left([C, X^c] - [C, X^{h_E}]\right) + \mathcal{V}_E X^c, \hat{Y}\right) = g\left(\mathcal{V}_E[\xi, X^v], \hat{Y}\right) = g\left(\overset{E}{\nabla}_\xi \hat{X}, \hat{Y}\right) \end{aligned}$$

and similarly,

$$Cg\left(\hat{X}, \overset{E}{\nabla}_\xi \hat{Y}\right) = g\left(\hat{X}, \overset{E}{\nabla}_\xi \hat{Y}\right),$$

thus

$$g\left(\nabla_C\left(i_{\hat{X}}\overset{E}{\nabla}_\xi g\right)^\sharp,\hat{Y}\right)=\xi g\left(\hat{X},\hat{Y}\right)-\left(\overset{E}{\nabla}_\xi\hat{X},\hat{Y}\right)-\left(\hat{X},\overset{E}{\nabla}_\xi\hat{Y}\right)$$
$$=\left(\overset{E}{\nabla}_\xi g\right)\left(\hat{X},\hat{Y}\right)=\left(i_{\hat{X}}\overset{E}{\nabla}_\xi g\right)\left(\hat{Y}\right)=g\left(\left(i_{\hat{X}}\overset{E}{\nabla}_\xi g\right)^\sharp,\hat{Y}\right),$$

which proves that $\left(i_{\hat{X}}\overset{E}{\nabla}_\xi g\right)^\sharp$ is indeed positively homogeneous of degree 1. Finally, substituting it into the third equation, we obtain

$$-\frac{1}{2}g\left(\nabla_C\left(i_{\hat{X}}\overset{E}{\nabla}_\xi g\right)^\sharp,\hat{Y}\right)-\frac{1}{2}g\left(\hat{X},\nabla_C\left(i_{\hat{Y}}\overset{E}{\nabla}_\xi g\right)^\sharp\right)=-\frac{1}{2}g\left(\left(i_{\hat{X}}\overset{E}{\nabla}_\xi g\right)^\sharp,\hat{Y}\right)$$
$$-\frac{1}{2}g\left(\hat{X},\left(i_{\hat{Y}}\overset{E}{\nabla}_\xi g\right)^\sharp\right)=-\frac{1}{2}\left(i_{\hat{X}}\overset{E}{\nabla}_\xi g\right)\left(\hat{Y}\right)-\frac{1}{2}g\left(i_{\hat{Y}}\overset{E}{\nabla}_\xi g\right)\left(\hat{X}\right)$$
$$=-\frac{1}{2}\left(\overset{E}{\nabla}_\xi g\right)\left(\hat{X},\hat{Y}\right)-\frac{1}{2}\left(\overset{E}{\nabla}_\xi g\right)\left(\hat{Y},\hat{X}\right)=-\left(\overset{E}{\nabla}_\xi g\right)\left(\hat{X},\hat{Y}\right),$$

thus (3) is satisfied as well. Now we turn to the solution of the homogeneous part of our system:

$$(1)\qquad P_h\delta=0$$
$$(2)\qquad \left(\mathbf{i}P_h\tilde{X}\right)E=0\quad\left(\tilde{X}\in\mathfrak{X}(\overset{\circ}{\tau})\right)$$
$$(3')\qquad g\left((\nabla_C P_h)\left(\tilde{X}\right),\tilde{Y}\right)+g\left(\tilde{X},(\nabla_C P_h)\left(\tilde{Y}\right)\right)=0\quad\left(\tilde{X},\tilde{Y}\in\mathfrak{X}(\overset{\circ}{\tau})\right).$$

If we decompose P_h into the sum of a self-adjoint part P_s and a skew-symmetric part P_a, then we obtain

$$0=g\left((\nabla_C P_s)\hat{X}+(\nabla_C P_a)\hat{X},\hat{Y}\right)+g\left(\hat{X},(\nabla_C P_s)\hat{Y}+(\nabla_C P_a)\hat{Y}\right)$$
$$=g\left(\nabla_C\left(P_s\hat{X}\right)+\nabla_C\left(P_a\hat{X}\right),\hat{Y}\right)+g\left(\hat{X},\nabla_C\left(P_s\hat{Y}\right)+\nabla_C\left(P_a\hat{Y}\right)\right)$$
$$=Cg\left(P_s\hat{X},\hat{Y}\right)+Cg\left(\hat{X},P_s\hat{Y}\right)+Cg\left(P_a\hat{X},\hat{Y}\right)+Cg\left(\hat{X},P_a\hat{Y}\right)$$
$$=2Cg\left(P_s\hat{X},\hat{Y}\right)=2g\left((\nabla_C P_s)\hat{X},\hat{Y}\right),$$

thus (3') holds if and only if P_s is homogeneous of degree 0. Now it remains only to find the conditions imposed by (1) and (2) on P_s and P_a. Equation (1) implies

$$g\left(P_h\delta,\tilde{X}\right) = g\left(P_s\delta + P_h\delta,\tilde{X}\right) = g\left(\delta, P_s\tilde{X}\right) - \left(\delta, P_h\tilde{X}\right)$$
$$= g\left(\delta, \left(P_s - P_h\right)\left(\tilde{X}\right)\right) = 0,$$

whereas equation (2) is equivalent to

$$\left(\mathbf{i}P_s\tilde{X}\right)E + \left(\mathbf{i}P_a\tilde{X}\right)E = \vartheta_g\left(P_s\tilde{X}\right) + \vartheta_g\left(P_a\tilde{X}\right)$$
$$= g\left(P_s\tilde{X},\delta\right) + g\left(P_a\tilde{X},\delta\right) = g\left(\left(P_s + P_a\right)\left(\tilde{X}\right),\delta\right) = 0,$$

thus (1) and (2) holds if and only if both the sum and the difference of P_s and P_a are contained in the orthogonal complement of δ, which implies the desired statement. □

5.3 A generalization of Cartan's covariant derivative

In this section g will be a weakly normal metric along $\overset{\circ}{\tau}$. Then it is weakly variational as well, and the tensors A_v introduced in 5.1.2 are self-adjoint for any $v \in \overset{\circ}{T}M$, since

$$\begin{aligned} g_v\left(A_v w_1, w_2\right) &= g_v\left(w_1, w_2\right) + g_v\left(\mathcal{C}_v\left(w_1, v\right), w_2\right) \\ &= g_v\left(w_1, w_2\right) + g_v\left(\mathcal{C}_v\left(w_1, w_2\right), v\right) = g_v\left(w_1, w_2\right) + g_v\left(\mathcal{C}_v\left(w_2, w_1\right), v\right) \\ &= g_v\left(w_1, w_2\right) + g_v\left(w_1, \mathcal{C}_v\left(w_2, v\right)\right) = g_v\left(w_1, A_v w_2\right). \end{aligned}$$

We shall need the following lemma whose proof may be found in [17].

5.3.1 Lemma. *Let K be a commutative ring with a unit element 1 such that the element $2 := 1 + 1$ has a multiplicative inverse in K. Consider a K-module V. Let the map $f : V \to V^*$ be an isomorphism, and suppose that the function*

$$\langle\ ,\ \rangle : V \times V \to K,\ (v, w) \mapsto \langle v, w\rangle := f(v)(w)$$

is symmetric. Then every skew-symmetric (K-)bilinear map $\omega : V \times V \to V$ determines a unique (K-)bilinear map

$$\psi : V \times V \to V$$

such that

(i) $\forall u, v, w \in V : \langle \psi(u, v), w \rangle + \langle v, \psi(u, w) \rangle = 0$,

(ii) $\forall v, w \in V : \omega(v, w) = \psi(v, w) - \psi(w, v)$.

5.3.2 Lemma. *There is a unique metric v-covariant derivative operator* $D^v : \mathfrak{X}(\overset{\circ}{\tau}) \times \mathfrak{X}(\overset{\circ}{\tau}) \to \mathfrak{X}(\overset{\circ}{\tau})$ *whose torsion vanishes.*

Proof. From the construction sketched in section 5.1, it can be seen that there exists such a v-covariant derivative. To show that it is unique, suppose that $\tilde{D}^v$ is another one. Let ψ^v be the difference tensor of D^v and $\tilde{D}^v$, more precisely:

$$\psi^v : \tilde{X}, \tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau}) \mapsto \psi^v\left(\tilde{X}, \tilde{Y}\right) := D^v_{\tilde{X}}\tilde{Y} - \tilde{D}^v_{\tilde{X}}\tilde{Y}.$$

Since we supposed that both D^v and $\tilde{D}^v$ are metric, we have

$$\begin{aligned} 0 = (D^v_{\tilde{X}} g)\left(\tilde{Y}, \tilde{Z}\right) &= \left(\mathbf{i}\tilde{X}\right) g\left(\tilde{Y}, \tilde{Z}\right) - g\left(D^v_{\tilde{X}}\tilde{Y}, \tilde{Z}\right) - g\left(\tilde{Y}, D^v_{\tilde{X}}\tilde{Z}\right) \\ &= g\left(\tilde{D}^v_{\tilde{X}}\tilde{Y}, \tilde{Z}\right) + g\left(\tilde{Y}, \tilde{D}^v_{\tilde{X}}\tilde{Z}\right) - g\left(D^v_{\tilde{X}}\tilde{Y}, \tilde{Z}\right) - g\left(\tilde{Y}, D^v_{\tilde{X}}\tilde{Z}\right) \\ &= -g\left(\psi^v\left(\tilde{X}, \tilde{Y}\right), \tilde{Z}\right) - g\left(\tilde{Y}, \psi^v\left(\tilde{X}, \tilde{Z}\right)\right) \end{aligned}$$

for any $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{X}(\overset{\circ}{\tau})$. On the other hand, if $X, Y \in \mathfrak{X}(M)$, by the vanishing of $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$ we obtain

$$\begin{aligned} 0 &= D^v_{\hat{X}}\hat{Y} - D^v_{\hat{Y}}\hat{X} = \tilde{D}^v_{\hat{X}}\hat{Y} + \psi^v\left(\hat{X}, \hat{Y}\right) - \tilde{D}^v_{\hat{Y}}\hat{X} - \psi^v\left(\hat{Y}, \hat{X}\right) \\ &= \psi^v\left(\hat{X}, \hat{Y}\right) - \psi^v\left(\hat{Y}, \hat{X}\right). \end{aligned}$$

Thus ψ^v satisfies the conditions of the previous lemma with $\omega = 0$, and it follows that $\psi^v = 0$, and $D^v = \tilde{D}^v$. □

Recall that the first Cartan tensor $\mathcal{C}$ is self-adjoint if we fix its first argument. In the rest of this section $\mathcal{C}^*$ will denote its adjoint linear transformation with its *second* variable fixed, or in other words, we take the adjoint of $\mathcal{C}$ 'with respect to the first variable', i.e., $\mathcal{C}^*$ is defined by

$$g\left(\mathcal{C}\left(\tilde{X}, \tilde{Y}\right), \tilde{Z}\right) = g\left(\tilde{X}, \mathcal{C}^*\left(\tilde{Z}, \tilde{Y}\right)\right) \quad \left(\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{X}(\overset{\circ}{\tau})\right).$$

5.3.3 Lemma. *Let* $\mathcal{H}$, $\tilde{\mathcal{H}}$ *be Ehresmann connections on* $\overset{\circ}{T}M$ *and* D, $\tilde{D}$ *metric covariant derivatives along* $\overset{\circ}{\tau}$ *with vanishing vertical torsions. Suppose that the* *torsion of* $\tilde{\mathcal{H}}$*, the horizontal torsion of* D *with respect to* $\mathcal{H}$ *and the horizontal torsion of* $\tilde{D}$ *with respect to* $\tilde{\mathcal{H}}$ *vanish. Then for all* $\tilde{X}, \tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau})$

$$\begin{aligned} D_{\mathcal{H}\tilde{X}}\tilde{Y} &= \tilde{D}_{\mathcal{H}\tilde{X}}\tilde{Y} \\ &\quad + \frac{1}{2}\left[\mathcal{C}\left(P\tilde{Y}, \tilde{X}\right) - \mathcal{C}\left(\tilde{Y}, P\tilde{X}\right) + \mathcal{C}^*\left(\tilde{Y}, P\tilde{X}\right) - P^*\mathcal{C}^*\left(\tilde{Y}, \tilde{X}\right)\right], \end{aligned}$$

where P *is the difference tensor field of* $\mathcal{H}$ *and* $\tilde{\mathcal{H}}$ *(more precisely,* $\mathbf{i} \circ P = \mathcal{H} - \tilde{\mathcal{H}}$*), and* P^* *is the adjoint of* P *with respect to the metric tensor* g*.*

Proof. First we note that, since the torsion of $\tilde{\mathcal{H}}$ vanishes, $\tilde{D}$ has to be the metric covariant derivative arising from $\tilde{\mathcal{H}}$. Let ψ^h be the tensor along $\overset{\circ}{\tau}$ determined by

$$D_{\mathcal{H}\tilde{X}}\tilde{Y} = \tilde{D}_{\mathcal{H}\tilde{X}}\tilde{Y} + \psi^h\left(\tilde{Y}, \tilde{Z}\right) \quad \left(\tilde{X}, \tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau})\right).$$

Since 5.3.2 implies that the vertical parts of D and $\tilde{D}$ coincide, we can write this last equation also in the form

$$D_{\tilde{\mathcal{H}}\tilde{X}}\tilde{Y} = \tilde{D}_{\tilde{\mathcal{H}}\tilde{X}}\tilde{Y} + \psi^h\left(\tilde{Y}, \tilde{Z}\right) \quad \left(\tilde{X}, \tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau})\right).$$

Now we show that ψ^h satisfies the conditions of 5.3.1 with

$$\omega\left(\tilde{X}, \tilde{Y}\right) = -\frac{1}{2}\left[\overset{\circ}{\mathcal{C}}\left(P\tilde{X}, \tilde{Y}\right) - \overset{\circ}{\mathcal{C}}\left(P\tilde{Y}, \tilde{X}\right)\right].$$

First, using the condition that both D and $\tilde{D}$ are metric, we get

$$\begin{aligned} 0 = \left(D_{\tilde{\mathcal{H}}\tilde{X}}g\right)\left(\tilde{Y}, \tilde{Z}\right) &= \left(\tilde{\mathcal{H}}\tilde{X}\right)g\left(\tilde{Y}, \tilde{Z}\right) - g\left(D_{\tilde{\mathcal{H}}\tilde{X}}\tilde{Y}, \tilde{Z}\right) - g\left(\tilde{Y}, D_{\tilde{\mathcal{H}}\tilde{X}}\tilde{Z}\right) \\ &= g\left(\tilde{D}_{\tilde{\mathcal{H}}\tilde{X}}\tilde{Y} - D_{\tilde{\mathcal{H}}\tilde{X}}\tilde{Y}, \tilde{Z}\right) + g\left(\tilde{Y}, \tilde{D}_{\tilde{\mathcal{H}}\tilde{X}}\tilde{Z} - D_{\tilde{\mathcal{H}}\tilde{X}}\tilde{Z}\right) \\ &= -g\left(\psi^h\left(\tilde{X}, \tilde{Y}\right), \tilde{Z}\right) - g\left(\tilde{Y}, \psi^h\left(\tilde{X}, \tilde{Z}\right)\right). \end{aligned}$$

On the other hand, using the disappearence of the horizontal torsion $\mathcal{T}$ of D

and the horizontal torsion $\tilde{\mathcal{T}}$ of $\tilde{D}$, if $X, Y \in \mathfrak{X}(M)$,

$$\begin{aligned}
0 &= \mathcal{T}\left(\hat{X}, \hat{Y}\right) = D_{X^h}\hat{Y} - D_{Y^h}\hat{X} - \widehat{[X,Y]} \\
&= D_{X^{\tilde{h}}}\hat{Y} + D_{\mathbf{i}P\hat{X}}\hat{Y} - D_{Y^{\tilde{h}}}\hat{X} - D_{\mathbf{i}P\hat{Y}}\hat{X} - \widehat{[X,Y]} = \tilde{D}_{X^{\tilde{h}}}\hat{Y} + \psi^h\left(\hat{X}, \hat{Y}\right) \\
&\quad + \frac{1}{2}\overset{\circ}{\mathcal{C}}\left(P\hat{X}, \hat{Y}\right) - \tilde{D}_{Y^{\tilde{h}}}\hat{X} - \psi^h\left(\hat{Y}, \hat{X}\right) - \frac{1}{2}\overset{\circ}{\mathcal{C}}\left(P\hat{Y}, \hat{X}\right) - \widehat{[X,Y]} \\
&= \tilde{\mathcal{T}}\left(\hat{X}, \hat{Y}\right) + \psi^h\left(\hat{X}, \hat{Y}\right) - \psi^h\left(\hat{Y}, \hat{X}\right) + \frac{1}{2}\overset{\circ}{\mathcal{C}}\left(P\hat{X}, \hat{Y}\right) - \frac{1}{2}\overset{\circ}{\mathcal{C}}\left(P\hat{Y}, \hat{X}\right) \\
&= \psi^h\left(\hat{X}, \hat{Y}\right) - \psi^h\left(\hat{Y}, \hat{X}\right) + \frac{1}{2}\overset{\circ}{\mathcal{C}}\left(P\hat{X}, \hat{Y}\right) - \frac{1}{2}\overset{\circ}{\mathcal{C}}\left(P\hat{Y}, \hat{X}\right),
\end{aligned}$$

thus ψ^h satisfies (i) and (ii) in the lemma, and therefore these two equations determine it uniquely. We show that the only possible choice for ψ^h is given by the formula

(5.1)
$$\psi^h\left(\tilde{X}, \tilde{Y}\right) = \frac{1}{2}\left[\mathcal{C}\left(P\tilde{Y}, \tilde{X}\right) - \mathcal{C}\left(\tilde{Y}, P\tilde{X}\right) + \mathcal{C}^*\left(\tilde{Y}, P\tilde{X}\right) - P^*\mathcal{C}^*\left(\tilde{Y}, \tilde{X}\right)\right].$$

To see this it is enough to check that the tensor ψ^h so defined also satisfies conditions (i), (ii) of 5.3.1. The calculation is quite immediate:

$$\begin{aligned}
&2g\left(\psi^h\left(\tilde{X}, \tilde{Y}\right), \tilde{Z}\right) + 2g\left(\tilde{Y}, \psi^h\left(\tilde{X}, \tilde{Z}\right)\right) \\
&\quad = g\left(\mathcal{C}\left(P\tilde{Y}, \tilde{X}\right), \tilde{Z}\right) - g\left(\mathcal{C}\left(\tilde{Y}, P\tilde{X}\right), \tilde{Z}\right) + g\left(\mathcal{C}^*\left(\tilde{Y}, P\tilde{X}\right), \tilde{Z}\right) \\
&\qquad - g\left(P^*\mathcal{C}^*\left(\tilde{Y}, \tilde{X}\right), \tilde{Z}\right) + g\left(\tilde{Y}, \mathcal{C}\left(P\tilde{Z}, \tilde{X}\right)\right) - g\left(\tilde{Y}, \mathcal{C}\left(\tilde{Z}, P\tilde{X}\right)\right) \\
&\qquad + g\left(\tilde{Y}, \mathcal{C}^*\left(\tilde{Z}, P\tilde{X}\right)\right) - g\left(\tilde{Y}, P^*\mathcal{C}^*\left(\tilde{Z}, \tilde{X}\right)\right) \\
&\quad = g\left(\mathcal{C}\left(P\tilde{Y}, \tilde{X}\right), \tilde{Z}\right) - g\left(\mathcal{C}\left(\tilde{Y}, P\tilde{X}\right), \tilde{Z}\right) + g\left(\tilde{Y}, \mathcal{C}\left(\tilde{Z}, P\tilde{X}\right)\right) \\
&\qquad - g\left(\tilde{Y}, \mathcal{C}\left(P\tilde{Z}, \tilde{X}\right)\right) + g\left(\tilde{Y}, \mathcal{C}\left(P\tilde{Z}, \tilde{X}\right)\right) - g\left(\tilde{Y}, \mathcal{C}\left(\tilde{Z}, P\tilde{X}\right)\right) \\
&\qquad + g\left(\mathcal{C}\left(\tilde{Y}, P\tilde{X}\right), \tilde{Z}\right) - g\left(\mathcal{C}\left(P\tilde{Y}, \tilde{X}\right), \tilde{Z}\right) = 0,
\end{aligned}$$

thus (i) is satisfied. Using the non-degeneracy of g, one gets that

$$\begin{aligned} &2g\left(\psi^h\left(\tilde{X},\tilde{Y}\right)-\psi^h\left(\tilde{Y},\tilde{X}\right),\tilde{Z}\right)\\ &\quad= g\left(\mathcal{C}\left(P\tilde{Y},\tilde{X}\right),\tilde{Z}\right)-g\left(\mathcal{C}\left(\tilde{Y},P\tilde{X}\right),\tilde{Z}\right)+g\left(\mathcal{C}^*\left(\tilde{Y},P\tilde{X}\right),\tilde{Z}\right)\\ &\quad\quad -g\left(P^*\mathcal{C}^*\left(\tilde{Y},\tilde{X}\right),\tilde{Z}\right)-g\left(\mathcal{C}\left(P\tilde{X},\tilde{Y}\right),\tilde{Z}\right)+g\left(\mathcal{C}\left(\tilde{X},P\tilde{Y}\right),\tilde{Z}\right)\\ &\quad\quad -g\left(\mathcal{C}^*\left(\tilde{X},P\tilde{Y}\right),\tilde{Z}\right)+g\left(P^*\mathcal{C}^*\left(\tilde{X},\tilde{Y}\right),\tilde{Z}\right). \end{aligned}$$

Now, remembering that $\mathcal{C}^*$ is the adjoint of $\mathcal{C}$ with the second variable fixed, we obtain

$$\begin{aligned} &2g\left(\psi^h\left(\tilde{X},\tilde{Y}\right)-\psi^h\left(\tilde{Y},\tilde{X}\right),\tilde{Z}\right)\\ &\quad= g\left(\mathcal{C}\left(P\tilde{Y},\tilde{X}\right),\tilde{Z}\right)-g\left(\mathcal{C}\left(\tilde{Y},\tilde{Z}\right),P\tilde{X}\right)+g\left(\mathcal{C}\left(\tilde{Z},P\tilde{X}\right),\tilde{Y}\right)\\ &\quad\quad -g\left(\mathcal{C}\left(P\tilde{Z},\tilde{X}\right),\tilde{Y}\right)-g\left(\mathcal{C}\left(P\tilde{X},\tilde{Y}\right),\tilde{Z}\right)+g\left(\mathcal{C}\left(\tilde{X},\tilde{Z}\right),P\tilde{Y}\right)\\ &\quad\quad -g\left(\mathcal{C}\left(\tilde{Z},P\tilde{Y}\right),\tilde{X}\right)+g\left(\mathcal{C}\left(P\tilde{Z},\tilde{Y}\right),\tilde{X}\right)\\ &\quad= -g\left(\overset{\circ}{\mathcal{C}}\left(P\tilde{X},\tilde{Y}\right),\tilde{Z}\right)+g\left(\overset{\circ}{\mathcal{C}}\left(P\tilde{Y},\tilde{X}\right),\tilde{Z}\right)=2g\left(\omega\left(\tilde{X},\tilde{Y}\right),\tilde{Z}\right), \end{aligned}$$

hence condition (ii) of 5.3.1 is also satisfied. Thus we have

$$\begin{aligned} D_{\mathcal{H}\tilde{X}}\tilde{Y} &= \tilde{D}_{\mathcal{H}\tilde{X}}\tilde{Y}\\ &\quad+\frac{1}{2}\left[\mathcal{C}\left(P\tilde{Y},\tilde{X}\right)-\mathcal{C}\left(\tilde{Y},P\tilde{X}\right)+\mathcal{C}^*\left(\tilde{Y},P\tilde{X}\right)-P^*\mathcal{C}^*\left(\tilde{Y},\tilde{X}\right)\right]. \end{aligned}$$

□

5.3.4 Theorem. *Suppose that g is positive definite and the tensor $A:\tilde{X}\mapsto\tilde{X}+\mathcal{C}\left(\tilde{X},\delta\right)$ has the following property: for a fixed $v\in\overset{\circ}{T}M$, the self-adjoint linear transformation A_v has no (not necessarily different) eigenvalues $\lambda_i,\lambda_j\in\mathbb{R}$ such that $\lambda_i+\lambda_j=0$. Then there is a unique covariant derivative operator $D:\mathfrak{X}\left(\overset{\circ}{T}M\right)\times\mathfrak{X}(\overset{\circ}{\tau})\to\mathfrak{X}(\overset{\circ}{\tau})$ and a unique Ehresmann connection $\mathcal{H}$ on $\overset{\circ}{T}M$ such that*

(1) *D is metric,*

(2) *its vertical torsion vanishes,*

(3) *its horizontal torsion with respect to $\mathcal{H}$ vanishes,*

(4) *the horizontal subbundle is contained in the kernel of the deflection of D.*

In particular, if D is regular, then $\mathcal{H}$ is attached to it.

Proof. Let $\tilde{\mathcal{H}}$ be an arbitrary Ehresmann connection on $\overset{\circ}{T}M$ with vanishing torsion (such an object exists: consider, e.g., the Barthel connection of the Finsler energy E). Let $\tilde{D}$ be the covariant derivative arising from $\tilde{\mathcal{H}}$. The symbol $\tilde{\mu}^h$ will denote the horizontal deflection of $\tilde{D}$ with respect to $\tilde{\mathcal{H}}$. We shall look for $\tilde{\mathcal{H}}$ in the form

$$\mathcal{H} = \tilde{\mathcal{H}} + \mathbf{i} \circ P.$$

By the previous lemma, D has necessarily the form

$$D_{\mathcal{H}\tilde{X}}\tilde{Y} = \tilde{D}_{\mathcal{H}\tilde{X}}\tilde{Y} + \frac{1}{2}\left[\mathcal{C}\left(P\tilde{Y},\tilde{X}\right) - \mathcal{C}\left(\tilde{Y},P\tilde{X}\right) + \mathcal{C}^*\left(\tilde{Y},P\tilde{X}\right) - P^*\mathcal{C}^*\left(\tilde{Y},\tilde{X}\right)\right].$$

From the proof of the lemma it can also be seen that D satisfies conditions (1)−(3). It remains to show that the additional condition (4) determines P uniquely:

$$\begin{aligned}
0 &= 2g\left(D_{\mathcal{H}\tilde{X}}\delta,\tilde{Y}\right) \\
&= g\left(2\tilde{D}_{\mathcal{H}\tilde{X}}\delta + \mathcal{C}\left(P\delta,\tilde{X}\right) - \mathcal{C}\left(\delta,P\tilde{X}\right) + \mathcal{C}^*\left(\delta,P\tilde{X}\right) - P^*\mathcal{C}^*\left(\delta,\tilde{X}\right),\tilde{Y}\right) \\
&= 2g\left(\tilde{D}_{\tilde{\mathcal{H}}\tilde{X}}\delta,\tilde{Y}\right) + 2g\left(D_{\mathbf{i}P\tilde{X}}\delta,\tilde{Y}\right) + g\left(\mathcal{C}\left(P\delta,\tilde{X}\right),\tilde{Y}\right) \\
&\quad - g\left(\mathcal{C}\left(\delta,P\tilde{X}\right),\tilde{Y}\right) + g\left(\mathcal{C}^*\left(\delta,P\tilde{X}\right),\tilde{Y}\right) - g\left(P^*\mathcal{C}^*\left(\delta,\tilde{X}\right),\tilde{Y}\right).
\end{aligned}$$

Using again the definition of $\mathcal{C}^*$, we have

$$\begin{aligned}
0 &= 2g\left(\tilde{\mu}^h\tilde{X},\tilde{Y}\right) + g\Big(2\nabla_{\mathbf{i}P\tilde{X}}\delta + \overset{\circ}{\mathcal{C}}\left(P\tilde{X},\delta\right),\tilde{Y}\Big) + g\left(\mathcal{C}\left(P\delta,\tilde{X}\right),\tilde{Y}\right) \\
&\quad - g\left(\mathcal{C}\left(\delta,P\tilde{X}\right),\tilde{Y}\right) + g\left(\mathcal{C}\left(\tilde{Y},P\tilde{X}\right),\delta\right) - g\left(\mathcal{C}\left(P\tilde{Y},\tilde{X}\right),\delta\right) \\
&= 2g\left(\tilde{\mu}^h\tilde{X},\tilde{Y}\right) + 2g\left(P\tilde{X},\tilde{Y}\right) + g\left(\mathcal{C}\left(P\tilde{X},\delta\right),\tilde{Y}\right) \\
&\quad + g\left(\mathcal{C}\left(\delta,\tilde{Y}\right),P\tilde{X}\right) - g\left(\mathcal{C}\left(\tilde{Y},P\tilde{X}\right),\delta\right) + g\left(\mathcal{C}\left(P\delta,\tilde{X}\right),\tilde{Y}\right) \\
&\quad - g\left(\mathcal{C}\left(\delta,\tilde{Y}\right),P\tilde{X}\right) + g\left(\mathcal{C}\left(\tilde{Y},P\tilde{X}\right),\delta\right) - g\left(\mathcal{C}\left(P\tilde{Y},\tilde{X}\right),\delta\right),
\end{aligned}$$

i.e., we have to solve the equation

$$\begin{aligned} 2g\left(\tilde{\mu}^h\tilde{X},\tilde{Y}\right)+2g\left(P\tilde{X},\tilde{Y}\right)+g\left(\mathcal{C}\left(P\tilde{X},\delta\right),\tilde{Y}\right) \\ +g\left(\mathcal{C}\left(P\delta,\tilde{X}\right),\tilde{Y}\right)-g\left(\mathcal{C}\left(P\tilde{Y},\tilde{X}\right),\delta\right)=0 \end{aligned} \tag{5.2}$$

for P. If we substitute $\tilde{X}=\delta$, the last term vanishes, and, due to the non-degeneracy of g, it follows that

$$2\tilde{\mu}^h\delta+2P\delta+2\mathcal{C}\left(P\delta,\delta\right)=0,$$

or, equivalently, $A(P\delta)=-\tilde{\mu}^h\delta$. Since our condition on the spectrum of A is a special case of Miron regularity, A has an inverse, and $P\delta=-A^{-1}\tilde{\mu}^h\delta$. Substituting this into (5.2) we find

$$\begin{aligned} 0 &= 2g\left(\tilde{\mu}^h\tilde{X},\tilde{Y}\right)+2g\left(P\tilde{X},\tilde{Y}\right)+g\left(\mathcal{C}\left(P\tilde{X},\delta\right),\tilde{Y}\right) \\ &\quad+g\left(\mathcal{C}\left(-A^{-1}\tilde{\mu}^h\delta,\tilde{X}\right),\tilde{Y}\right)-g\left(\mathcal{C}\left(P\tilde{Y},\tilde{X}\right),\delta\right) \\ &= 2g\left(\tilde{\mu}^h\tilde{X},\tilde{Y}\right)+2g\left(P\tilde{X},\tilde{Y}\right)+g\left(\mathcal{C}\left(P\tilde{X},\delta\right),\tilde{Y}\right) \\ &\quad-g\left(\mathcal{C}\left(A^{-1}\tilde{\mu}^h\delta,\tilde{X}\right),\tilde{Y}\right)-g\left(P^*\mathcal{C}^*\left(\delta,\tilde{X}\right),\tilde{Y}\right), \end{aligned}$$

which, in turn, by the non-degeneracy of g, is equivalent to

$$2P\tilde{X}+\mathcal{C}\left(P\tilde{X},\delta\right)-P^*\mathcal{C}^*\left(\delta,\tilde{X}\right)=\mathcal{C}\left(A^{-1}\tilde{\mu}^h\delta,\tilde{X}\right)-2\tilde{\mu}^h\tilde{X}. \tag{5.3}$$

If we introduce the tensor Q by

$$Q\tilde{X}:=\mathcal{C}\left(A^{-1}\tilde{\mu}^h\delta,\tilde{X}\right)-2\tilde{\mu}^h\tilde{X},$$

and the transformation φ by

$$(\varphi P)\left(\tilde{X}\right):=2P\tilde{X}+\mathcal{C}\left(P\tilde{X},\delta\right)-P^*\mathcal{C}^*\left(\delta,\tilde{X}\right),$$

then (5.3) takes the form

$$\varphi P=Q.$$

To show that this equation has a unique solution for P, it suffices to show that the linear transformation φ of the endomorphism algebra of each fibre is of rank

n^2, or that $(\varphi P)_v = 0$ implies $P_v = 0$ for any $v \in \overset{\circ}{T}M$. Since φ is tensorial, we may well suppress the index v and simply suppose $(\varphi P)\left(\tilde{X}\right) = 0$. Forming the scalar product of both sides of this equation with $\tilde{X}$ we get

$$\begin{aligned} 0 &= 2g\left(P\tilde{X}, \tilde{X}\right) + g\left(\mathcal{C}\left(P\tilde{X}, \delta\right), \tilde{X}\right) - g\left(P^*\mathcal{C}^*\left(\delta, \tilde{X}\right), \tilde{X}\right) \\ &= 2g\left(P\tilde{X}, \tilde{X}\right) + g\left(\mathcal{C}\left(P\tilde{X}, \tilde{X}\right), \delta\right) - g\left(\mathcal{C}\left(P\tilde{X}, \tilde{X}\right), \delta\right) = 2g\left(P\tilde{X}, \tilde{X}\right), \end{aligned}$$

thus P is skew-symmetric with respect to g, i.e., $P^* = -P$, and the equation $\varphi P = 0$ has the form

$$2P\tilde{X} + \mathcal{C}\left(P\tilde{X}, \delta\right) + P\mathcal{C}^*\left(\delta, \tilde{X}\right) = 0. \tag{5.4}$$

We transform further the last term as follows:

$$g\left(\mathcal{C}^*\left(\delta, \tilde{X}\right), \tilde{Y}\right) = g\left(\mathcal{C}\left(\tilde{Y}, \tilde{X}\right), \delta\right) \overset{(*)}{=} g\left(\mathcal{C}\left(\tilde{X}, \tilde{Y}\right), \delta\right) = g\left(\mathcal{C}\left(\tilde{X}, \delta\right), \tilde{Y}\right).$$

At the step denoted by $(*)$ we used the weak normality of g, which implies weak variationality as well. Thus it follows that

$$\mathcal{C}^*\left(\delta, \tilde{X}\right) = \mathcal{C}\left(\tilde{X}, \delta\right);$$

substituting this into (5.4), we obtain

$$2P\tilde{X} + \mathcal{C}\left(P\tilde{X}, \delta\right) + P\mathcal{C}\left(\tilde{X}, \delta\right) = (A \circ P + P \circ A)\left(\tilde{X}\right) = 0.$$

We have seen that A_v is self-adjoint, thus, by the positive definiteness of g, there is an orthonormal base $(e_i)_{i=1}^n$ of $T_{\tau(v)}M$ such that $A_v(e_i) = \lambda_i e_i$ for any $i \in \{1, \dots, n\}$. Therefore

$$\begin{aligned} 0 &= g_v\left((A_v \circ P_v + P_v \circ A_v)\, e_i, e_j\right) = g_v\left(P_v e_i, A_v e_j\right) + \lambda_i g_v\left(P_v e_i, e_j\right) \\ &= (\lambda_i + \lambda_j) g_v\left(P_v e_i, e_j\right), \end{aligned}$$

which implies $P = 0$, taking into account our condition on the eigenvalues of A_v. Thus the linear map φ is indeed regular, and (5.3) has a unique solution for P. Finally, since (5.2) and (5.3) are equivalent only if $P\delta = -A^{-1}\tilde{\mu}^h\delta$, thus, to show that the unique solution of (5.3) is a solution of (5.2) as well, we have to check that it satisfies the condition $P\delta = -A^{-1}\tilde{\mu}^h\delta$. To this end, we substitute

$\tilde{X} = \delta$ into (5.3). The third term vanishes due to the weak normality of g, and we have

$$2P\delta + \mathcal{C}(P\delta, \delta) = \mathcal{C}\left(A^{-1}\tilde{\mu}^h\delta, \delta\right) - 2\tilde{\mu}^h\delta = \mathcal{C}\left(A^{-1}\tilde{\mu}^h\delta, \delta\right) - 2AA^{-1}\tilde{\mu}^h\delta$$
$$= \mathcal{C}\left(A^{-1}\tilde{\mu}^h\delta, \delta\right) - 2A^{-1}\tilde{\mu}^h\delta - 2\mathcal{C}\left(A^{-1}\tilde{\mu}^h\delta, \delta\right) = -A^{-1}\tilde{\mu}^h\delta - AA^{-1}\tilde{\mu}^h\delta,$$

or, equivalently,

$$(1 + A)(P\delta) = (1 + A)\left(-A^{-1}\tilde{\mu}^h\delta\right).$$

By the weak normality of g again, δ is an eigenvector field of A corresponding to the eigenvalue 1, and from our conditions it follows that A has no eigenvalue -1, and the linear transformation $1+A$ is invertible at each point. We conclude that (5.3) implies $P\delta = -A^{-1}\tilde{\mu}^h\delta$, thus (5.2) and (5.3) are indeed equivalent. □

5.4 Cartan's covariant derivative

Scrutinizing carefully the proof of 5.3.4, it turns out that the positive definiteness of g is not needed if we start from a Finsler manifold. In this case, the normality of g and the total symmetry of the lowered first Cartan tensor $\mathcal{C}_\flat$ simplify the circumstances considerably. We include here our proof, since this is a new and coordinate-free deduction of the existence and uniqueness of Cartan's covariant derivative under the condition that no Ehresmann connection is specified in advance. For a proof that uses local coordinates, see [1].

In this section we shall work on a Finsler manifold. The Barthel connection will be denoted by $\mathcal{H}_B$, and Cartan's covariant derivative, as defined in section 5.1, by $\overset{\circ}{D}$.

5.4.1 Lemma. *If $\mathcal{H}$ is an Ehresmann connection on $\overset{\circ}{T}M$, and D is a metric covariant derivative with vanishing vertical and horizontal torsions (the latter with respect to $\mathcal{H}$), then for all $\tilde{X}, \tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau})$*

$$D_{\mathcal{H}\tilde{X}}\tilde{Y} = \overset{\circ}{D}_{\mathcal{H}\tilde{X}}\tilde{Y} + \frac{1}{2}\left(\mathcal{C}\left(\tilde{X}, P\tilde{Y}\right) - P^*\mathcal{C}\left(\tilde{X}, \tilde{Y}\right)\right),$$

where P is the difference tensor field of $\mathcal{H}$ and $\mathcal{H}_B$ (more precisely, $\mathbf{i} \circ P = \mathcal{H} - \mathcal{H}_B$), and P^* is the adjoint of P with respect to the Finsler metric tensor.

Proof. Let us define the mapping ψ^h by

$$\psi^h\left(\tilde{X}, \tilde{Y}\right) := D_{\mathcal{H}\tilde{X}}\tilde{Y} - \overset{\circ}{D}_{\mathcal{H}\tilde{X}}\tilde{Y}.$$

We show that ψ^h satisfies (i) and (ii) of 5.3.1 with ω given by

$$\omega\left(\tilde{X},\tilde{Y}\right) := \frac{1}{2}\left(\mathcal{C}\left(\tilde{X},P\tilde{Y}\right) - \mathcal{C}\left(\tilde{Y},P\tilde{X}\right)\right).$$

Since D and $\overset{\circ}{D}$ are metric,

$$\begin{aligned}
0 &= \left(D_{\mathcal{H}_B\tilde{X}}g\right)\left(\tilde{Y},\tilde{Z}\right) = \left(\mathcal{H}_B\tilde{X}\right)g\left(\tilde{Y},\tilde{Z}\right) - g\left(D_{\mathcal{H}_B\tilde{X}}\tilde{Y},\tilde{Z}\right) - g\left(\tilde{Y},D_{\mathcal{H}_B\tilde{X}}\tilde{Z}\right)\\
&= g\Big(\overset{\circ}{D}_{\mathcal{H}_B\tilde{X}}\tilde{Y} - D_{\mathcal{H}_B\tilde{X}}\tilde{Y},\tilde{Z}\Big) + g\Big(\tilde{Y},\overset{\circ}{D}_{\mathcal{H}_B\tilde{X}}\tilde{Z} - D_{\mathcal{H}_B\tilde{X}}\tilde{Z}\Big)\\
&\overset{(*)}{=} g\Big(\overset{\circ}{D}_{\mathcal{H}\tilde{X}}\tilde{Y} - D_{\mathcal{H}\tilde{X}}\tilde{Y},\tilde{Z}\Big) + g\Big(\tilde{Y},\overset{\circ}{D}_{\mathcal{H}\tilde{X}}\tilde{Z} - D_{\mathcal{H}\tilde{X}}\tilde{Z}\Big)\\
&= -g\left(\psi^h\left(\tilde{X},\tilde{Y}\right),\tilde{Z}\right) - g\left(\tilde{Y},\psi^h\left(\tilde{X},\tilde{Z}\right)\right),
\end{aligned}$$

thus condition 5.3.1(i) is satisfied. At the step denoted by $(*)$ we used the coincidence of the vertical parts of D and $\overset{\circ}{D}$ assured by 5.3.2. On the other hand, by the vanishing of the horizontal torsion $\mathcal{T}$ of D and the horizontal torsion $\overset{\circ}{\mathcal{T}}$ of $\overset{\circ}{D}$, for any $X,Y \in \mathfrak{X}(M)$ we have

$$\begin{aligned}
0 &= \mathcal{T}\left(\hat{X},\hat{Y}\right) = D_{X^h}\hat{Y} - D_{Y^h}\hat{X} - \mathbf{j}\left[X^h,Y^h\right]\\
&= D_{X^{h_B}}\hat{Y} - D_{Y^{h_B}}\hat{X} - \widehat{[X,Y]} + D_{\mathbf{i}P\hat{X}}\hat{Y} - D_{\mathbf{i}P\hat{Y}}\hat{X}\\
&= \overset{\circ}{D}_{X^{h_B}}\hat{Y} - \overset{\circ}{D}_{Y^{h_B}}\hat{X} - \widehat{[X,Y]} + \psi^h\left(\hat{X},\hat{Y}\right) - \psi^h\left(\hat{Y},\hat{X}\right) + \overset{\circ}{D}_{\mathbf{i}P\hat{X}}\hat{Y} - \overset{\circ}{D}_{\mathbf{i}P\hat{Y}}\hat{X}\\
&= \overset{\circ}{\mathcal{T}}\left(\hat{X},\hat{Y}\right) + \psi^h\left(\hat{X},\hat{Y}\right) - \psi^h\left(\hat{Y},\hat{X}\right)\\
&\quad + \nabla_{\mathbf{i}P\hat{X}}\hat{Y} + \frac{1}{2}\mathcal{C}\left(P\hat{X},\hat{Y}\right) - \nabla_{\mathbf{i}P\hat{Y}}\hat{X} - \frac{1}{2}\mathcal{C}\left(P\hat{Y},\hat{X}\right)\\
&= \psi^h\left(\hat{X},\hat{Y}\right) - \psi^h\left(\hat{Y},\hat{X}\right) + \frac{1}{2}\mathcal{C}\left(P\hat{X},\hat{Y}\right) - \frac{1}{2}\mathcal{C}\left(P\hat{Y},\hat{X}\right),
\end{aligned}$$

taking into account that the v-covariant derivatives of basic vector fields vanish. This implies

$$\omega\left(\tilde{X},\tilde{Y}\right) = \psi^h\left(\tilde{X},\tilde{Y}\right) - \psi^h\left(\tilde{Y},\tilde{X}\right)$$

as we claimed. Thus, by lemma 5.3.1, ψ^h is uniquely determined. Finally we show that the only possible choice for ψ^h is the map given by

$$\psi^h\left(\tilde{X},\tilde{Y}\right) := \frac{1}{2}\left(\mathcal{C}\left(\tilde{X},P\tilde{Y}\right) - P^*\mathcal{C}\left(\tilde{X},\tilde{Y}\right)\right). \tag{5.5}$$

Indeed,

$$
\begin{aligned}
&2g\left(\psi^h\left(\tilde{X},\tilde{Y}\right),\tilde{Z}\right)+2g\left(\tilde{Y},\psi^h\left(\tilde{X},\tilde{Z}\right)\right)\\
&\quad= g\left(\mathcal{C}\left(\tilde{X},P\tilde{Y}\right),\tilde{Z}\right)-g\left(P^*\mathcal{C}\left(\tilde{X},\tilde{Y}\right),\tilde{Z}\right)+g\left(\tilde{Y},\mathcal{C}\left(\tilde{X},P\tilde{Z}\right)\right)\\
&\qquad- g\left(\tilde{Y},P^*\mathcal{C}\left(\tilde{X},\tilde{Z}\right)\right)= g\left(\mathcal{C}\left(\tilde{X},P\tilde{Y}\right),\tilde{Z}\right)-g\left(\mathcal{C}\left(\tilde{X},\tilde{Z}\right),P\tilde{Y}\right)\\
&\qquad+ g\left(\mathcal{C}\left(\tilde{X},P\tilde{Z}\right),\tilde{Y}\right)-g\left(\mathcal{C}\left(\tilde{X},\tilde{Y}\right),P\tilde{Z}\right)\\
&\quad= \mathcal{C}_\flat\left(\tilde{X},P\tilde{Y},\tilde{Z}\right)-\mathcal{C}_\flat\left(\tilde{X},\tilde{Z},P\tilde{Y}\right)+\mathcal{C}_\flat\left(\tilde{X},P\tilde{Z},\tilde{Y}\right)-\mathcal{C}_\flat\left(\tilde{X},\tilde{Y},P\tilde{Z}\right)=0,
\end{aligned}
$$

due to the symmetry of $\mathcal{C}_\flat$. Thus the map (5.5) satisfies 5.3.1(i). It also satisfies 5.3.1(ii), since by the same reason,

$$
\begin{aligned}
&2\psi^h\left(\tilde{X},\tilde{Y}\right)-2\psi^h\left(\tilde{Y},\tilde{X}\right)\\
&\quad= \mathcal{C}\left(\tilde{X},P\tilde{Y}\right)-P^*\mathcal{C}\left(\tilde{X},\tilde{Y}\right)-\mathcal{C}\left(\tilde{Y},P\tilde{X}\right)-P^*\mathcal{C}\left(\tilde{Y},\tilde{X}\right)\\
&\quad= \mathcal{C}\left(\tilde{X},P\tilde{Y}\right)-\mathcal{C}\left(\tilde{Y},P\tilde{X}\right)=2\omega\left(\tilde{X},\tilde{Y}\right).
\end{aligned}
$$

□

5.4.2 Proposition. *If $\mathcal{H}$ is an Ehresmann connection and $D : \mathfrak{X}\left(\overset{\circ}{T}M\right) \times \mathfrak{X}(\overset{\circ}{\tau}) \to \mathfrak{X}(\overset{\circ}{\tau})$ is a covariant derivative such that*

(1) *D is metric,*

(2) *its vertical torsion vanishes,*

(3) *its horizontal torsion with respect to $\mathcal{H}$ vanishes,*

(4) *$\mathcal{H}$ is attached to it,*

then D is Cartan's covariant derivative, and $\mathcal{H}$ is the Barthel connection of the Finsler manifold.

Proof. We have already known that Cartan's derivative and the Barthel connection satisfy these conditions. We suppose that another D and another $\mathcal{H}$ also satisfy them. As before, we look for $\mathcal{H}$ in the form

$$\mathcal{H} = \mathcal{H}_B + \mathbf{i} \circ P.$$

By the previous lemma, D has necessarily the form

$$D_{\mathcal{H}\tilde{X}}\tilde{Y} = \overset{\circ}{D}_{\mathcal{H}\tilde{X}}\tilde{Y} + \frac{1}{2}\left(\mathcal{C}\left(\tilde{X}, P\tilde{Y}\right) - P^*\mathcal{C}\left(\tilde{X}, \tilde{Y}\right)\right).$$

It remains to show that the additional condition (4) implies $P = 0$. Since

$$\begin{aligned} 2D_{\mathcal{H}\tilde{X}}\delta &= 2\overset{\circ}{D}_{\mathcal{H}\tilde{X}}\delta + \mathcal{C}\left(\tilde{X}, P\delta\right) - P^*\mathcal{C}\left(\tilde{X}, \delta\right) \\ &= 2\overset{\circ}{D}_{\mathcal{H}_B\tilde{X}}\delta + 2\overset{\circ}{D}_{\mathbf{i}P\tilde{X}}\delta + \mathcal{C}\left(\tilde{X}, P\delta\right) \\ &= 2\nabla_{\mathbf{i}P\tilde{X}}\delta + \mathcal{C}\left(P\tilde{X}, \delta\right) + \mathcal{C}\left(\tilde{X}, P\delta\right) = 2P\tilde{X} + \mathcal{C}\left(\tilde{X}, P\delta\right), \end{aligned}$$

we have to solve the equation

$$2P\tilde{X} + \mathcal{C}\left(\tilde{X}, P\delta\right) = 0.$$

If we substitute $\tilde{X} = \delta$, the second term vanishes, hence $P\delta = 0$. Substituting this into the original equation, we conclude $P\tilde{X} = 0$ for any $\tilde{X}$. ☐

Chapter 6

Killing vector fields of generalized metrics

6.1 Killing vector fields in general

In this section g will be a generalized metric on an open set $\widetilde{TM} \subset TM$ such that $\tau\left(\widetilde{TM}\right) = M$, and π will be the restriction of τ to $\widetilde{TM}$, as in section 2.1.

6.1.1 Definition. *A diffeomorphism $f : \mathcal{U} \to \mathcal{V}$ between two open subsets of M is a* local isometry *of g if its tangent map leaves g invariant, i.e.,*

$$g_{f_*(v)}(f_*(w_1), f_*(w_2)) = g_v(w_1, w_2)$$

for any $v \in T\mathcal{U} \cap \widetilde{TM}$ and $w_1, w_2 \in T_{\pi(v)}M$. A vector field $X \in \mathfrak{X}(M)$ with flow $\varphi : W \subset \mathbb{R} \times M \to M$ is said to be an infinitesimal isometry *of g if φ_t is a local isometry between two open subsets of M for all $t \in \mathbb{R}$ such that the domain of φ_t is not empty. A vector field $X \in \mathfrak{X}(M)$ is called a* Killing vector field *if $\mathcal{L}_X g = 0$.*

6.1.2 Proposition. *Let $X \in \mathfrak{X}(M)$ be a vector field. Then X is an infinitesimal isometry of g if and only if it is a Killing vector field.*

Proof. In this proof we shall repeatedly use the dynamic interpretation of the Lie derivatives of vector fields (2.3.1 (1), (2)). Now let us begin with proving the necessity, and assume that X is an infinitesimal isometry. For arbitrarily chosen vector fields Y and Z on M, define a function $f \in C^\infty\left(\widetilde{TM}\right)$ by $f := g\left(\hat{Y}, \hat{Z}\right)$. If $v \in \widetilde{T_pM}$, $t \in \mathbb{R}$ and $(t,p) \in W$, we have

$$\begin{aligned} f((\varphi_t)_* v) &= g_{(\varphi_t)_*(v)}(Y(\varphi_t(p)), Z(\varphi_t(p))) \\ &= g_{(\varphi_t)_*(v)}((\varphi_t)_*[(\varphi_{-t})_\sharp Y](p), (\varphi_t)_*[(\varphi_{-t})_\sharp Z](p)) \\ &= g_v([(\varphi_{-t})_\sharp Y](p), [(\varphi_{-t})_\sharp Z](p)), \end{aligned}$$

using in the last step that φ_t is a local isometry for every sufficiently small $t \in \mathbb{R}$. Now we use the fact that the curve $c_v : t \mapsto (\varphi_t)_*(v)$ is an integral curve of X^c to obtain

$$\begin{aligned} X^c(v)f &= \lim_{t\to 0} \frac{1}{t}[f((\varphi_t)_*(v)) - f(v)] \\ &= \lim_{t\to 0} \frac{1}{t}[g_v((\varphi_{-t})_\sharp Y(p), (\varphi_{-t})_\sharp Z(p)) - g_v(Y(p), Z(p))] \\ = \lim_{t\to 0} &\left[\frac{g_v((\varphi_{-t})_\sharp Y(p) - Y(p), (\varphi_{-t})_\sharp Z(p))}{t} + \frac{g_v(Y(p), (\varphi_{-t})_\sharp Z(p) - Z(p))}{t}\right] \\ = g_v &\left(\lim_{t\to 0} \frac{1}{t}((\varphi_{-t})_\sharp Y(p) - Y(p)), \lim_{t\to 0}(\varphi_{-t})_\sharp Z(p)\right) \\ + g_v &\left(Y(p), \lim_{t\to 0} \frac{1}{t}((\varphi_{-t})_\sharp Z(p) - Z(p))\right) = g_v([X,Y](p), Z(p)) \\ + g_v&(Y(p), [X,Z](p)) = \left(g\left(\widehat{[X,Y]}, \hat{Z}\right) + g\left(\hat{Y}, \widehat{[X,Z]}\right)\right)(v), \end{aligned}$$

and therefore

$$\begin{aligned} X^c g\left(\hat{Y}, \hat{Z}\right) &= X^c f = g\left(\widehat{[X,Y]}, \hat{Z}\right) + g\left(\hat{Y}, \widehat{[X,Z]}\right) \\ &= g\left(\mathcal{L}_X \hat{Y}, \hat{Z}\right) + g\left(\hat{Y}, \mathcal{L}_X \hat{Z}\right). \end{aligned}$$

Thus we conclude

$$(\mathcal{L}_X g)\left(\hat{Y}, \hat{Z}\right) = X^c g\left(\hat{Y}, \hat{Z}\right) - g\left(\mathcal{L}_X \hat{Y}, \hat{Z}\right) - g\left(\hat{Y}, \mathcal{L}_X \hat{Z}\right) = 0,$$

i.e., X is a Killing vector field.

To prove the converse, assume that X is a Killing vector field, consider the flow $\varphi : W \subset \mathbb{R} \times M \to M$ of X, and let $p \in M$, $v \in \widetilde{T_pM}$, $w_1, w_2 \in T_pM$ be arbitrary. We shall again denote the maximal integral curve of X^c starting from v by $c_v : I_p \to TM$. (The domain of this curve depends only on p.) We define the function $\ell : I_v \subset I_p \to \mathbb{R}$ in the following way:

$$\ell(t) := g_{(\varphi_t)_*(v)}((\varphi_t)_*(w_1), (\varphi_t)_*(w_2)) = g_{c_v(t)}((\varphi_t)_*(w_1), (\varphi_t)_*(w_2)).$$

(If $\widetilde{TM} \neq TM$, ℓ need not necessarily be defined on the whole I_p, since the curve c_v may leave $\widetilde{TM}$.) It is enough to show that ℓ is constant. To this end, we define two vector fields Y and Z along c_v as follows:

$$Y(t) := (\varphi_t)_*(w_1),\ Z(t) := (\varphi_t)_*(w_2) \quad (t \in I_v).$$

If $X^c(v) = 0$, c_v is constant, and the assertion is trivial. If $X^c(v) \neq 0$, Y and Z can be extended, at least locally, to vector fields $\tilde{Y}$ and $\tilde{Z}$ on an open subset U of $\widetilde{TM}$ such that

$$Y(t) = \tilde{Y}(c_v(t)),\ Z(t) = \tilde{Z}(c_v(t)) \quad (t \in I)$$

($I \subset I_v$ is another open interval). Now with the help of the function h given by

$$h(q) := g_q\left(\tilde{Y}(q), \tilde{Z}(q)\right) \quad (q \in U),$$

we have $\ell \restriction I = h \circ c_v$. Thus,

$$\begin{aligned} \ell'(t) &= (h \circ c_v)'(t) = \dot{c}_v(t)h = X^c(c_v(t))h = (X^c h)(c_v(t)) \\ &= \left[g\left(\mathcal{L}_X\tilde{Y}, \tilde{Z}\right) + g\left(\tilde{Y}, \mathcal{L}_X\tilde{Z}\right)\right](c_v(t)), \\ \left(\mathcal{L}_X\tilde{Y}\right)(q) &= \lim_{t\to 0}\frac{1}{t}\left[(\varphi_{-t})_*\tilde{Y}((\varphi_t)_* q) - \mathbf{i}\tilde{Y}(q)\right] = 0 \quad (q \in c_v(I)) \end{aligned}$$

due to the construction of $\tilde{Y}$. We obtain in a similar way that $\mathcal{L}_X\tilde{Z} = 0$. Hence ℓ is indeed constant. □

If the metric g is defined at least on $\overset{\circ}{T}M$, and it is positive definite and homogeneous ($\nabla^v_\delta g = 0$, or, equivalently, the function $g\left(\hat{X}, \hat{Y}\right)$ is positively homogeneous of degree 0 for any $X, Y \in \mathfrak{X}(M)$), then we may define the length of an arc $c : [\alpha, \beta] \to M$ by

$$\ell(c) := \int_\alpha^\beta \sqrt{2E \circ \dot{c}} = \int_\alpha^\beta \sqrt{g_{\dot{c}(t)}(\dot{c}(t), \dot{c}(t))}dt.$$

The distance of two points $p, q \in M$ is then given by

$$d(p,q) := \inf\{\ell(c) | c : [0,1] \to M, c(0) = p, c(1) = q\}.$$

We say that g is *reversible* if $g_{-v}(w_1, w_2) = g_v(w_1, w_2)$ for any $v, w_1, w_2 \in T_pM$ and $p \in M$. In this case, d is symmetric, and (M, d) becomes a metric space.

It is known that every Killing field is complete on a complete Riemannian manifold [47]. This result can easily be generalized as follows.

6.1.3 Proposition. *Let g be a homogeneous, reversible and positive definite metric, and suppose that X is a Killing vector field of g. If M is complete as a metric space, the vector field X is complete as well.*

Proof. Let $c_p : [0, \alpha[\to M$ be an integral curve of X starting from p. We show that c_p can be extended to $[0, \alpha]$. Since $\ddot{c}_p = X^c \circ \dot{c}_p$, and

$$X^c E = \frac{1}{2} X^c g(\delta, \delta) = \frac{1}{2} (\mathcal{L}_X g)(\delta, \delta) = 0,$$

the function $E \circ \dot{c}_p$ is constant. Let $\lambda := \sqrt{2E(\dot{c}_p(t))}$ ($t \in [0, \alpha[$ is arbitrary). Then, if $t, t' \in]0, \alpha[$,

$$d(c_p(t), c_p(t')) \leqq \left| \int_t^{t'} \sqrt{2E \circ \dot{c}_p} \right| = \lambda |t - t'|.$$

This implies, by the completeness of M, that the limit $\lim_{t \to \alpha} c_p(t)$ exists. □

Now we suppose that an Ehresmann connection is specified on $\widetilde{TM}$ whose torsion vanishes. Let D be the metric covariant derivative operator on $\widetilde{TM}$ constructed in section 5.1.

The following proposition was formulated in [49] for the special case of Finsler manifolds. It generalizes the skew-symmetry of the covariant differential of a Killing field in Riemannian geometry.

6.1.4 Proposition. *Let X be a Killing vector field on M. Then*

$$g\left(D_{\mathcal{H}\tilde{Y}} \hat{X}, \tilde{Z}\right) + g\left(\tilde{Y}, D_{\mathcal{H}\tilde{Z}} \hat{X}\right) + g\left(\mathcal{C}\left(\mathcal{V}X^c, \tilde{Y}\right), \tilde{Z}\right) = 0$$

for any $\tilde{Y}, \tilde{Z} \in \mathfrak{X}(\pi)$.

Proof. Since the left-hand side is tensorial in $\tilde{Y}$, $\tilde{Z}$, it is enough to verify the formula for basic vector fields $\hat{Y}$, $\hat{Z}$. Using the condition that X is a Killing field, we obtain

$$\begin{aligned}
0 = (\mathcal{L}_X g)\left(\hat{Y},\hat{Z}\right) &= X^c g\left(\hat{Y},\hat{Z}\right) - g\left(\widehat{[X,Y]},\hat{Z}\right) - g\left(\hat{Y},\widehat{[X,Z]}\right)\\
&= g\left(D_{X^c}\hat{Y},\hat{Z}\right) - g\left(\widehat{[X,Y]},\hat{Z}\right) + g\left(\hat{Y},D_{X^c}\hat{Z}\right) - g\left(\hat{Y},\widehat{[X,Z]}\right)\\
&= g\left(\nabla_{X^h}\hat{Y} + \frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h\left(\hat{X},\hat{Y}\right) + \frac{1}{2}\overset{\circ}{\mathcal{C}}\left(\mathcal{V}X^c,\hat{Y}\right) - \widehat{[X,Y]},\hat{Z}\right) + (Y \leftrightarrow Z)\\
&= g\left(\mathcal{V}\left([X^h,Y^v] - [X,Y]^v\right) + \frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h\left(\hat{X},\hat{Y}\right) + \frac{1}{2}\overset{\circ}{\mathcal{C}}\left(\mathcal{V}X^c,\hat{Y}\right),\hat{Z}\right)\\
&\quad + (Y \leftrightarrow Z)
\end{aligned}$$

where the symbol $(Y \leftrightarrow Z)$ means an expression consisting of all preceding terms, with Y and Z interchanged. By the vanishing of the torsion of our Ehresmann connection,

$$\begin{aligned}
0 &= g\left(\mathcal{V}\left[Y^h,X^v\right] + \frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h\left(\hat{X},\hat{Y}\right) + \frac{1}{2}\overset{\circ}{\mathcal{C}}\left(\mathcal{V}X^c,\hat{Y}\right),\hat{Z}\right) + (Y \leftrightarrow Z)\\
&= g\left(\nabla_{Y^h}\hat{X} + \frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h\left(\hat{Y},\hat{X}\right),\hat{Z}\right)\\
&\quad + \frac{1}{2}\left[\mathcal{C}_\flat\left(\mathcal{V}X^c,\hat{Y},\hat{Z}\right) + \mathcal{C}_\flat\left(\hat{Y},\hat{Z},\mathcal{V}X^c\right) - \mathcal{C}_\flat\left(\hat{Z},\mathcal{V}X^c,\hat{Y}\right)\right] + (Y \leftrightarrow Z)\\
&= g\left(D_{Y^h}\hat{X},\hat{Z}\right) + g\left(\hat{Y},D_{Z^h}\hat{X}\right) + \mathcal{C}_\flat\left(\mathcal{V}X^c,\hat{Y},\hat{Z}\right)\\
&= g\left(D_{Y^h}\hat{X},\hat{Z}\right) + g\left(\hat{Y},D_{Z^h}\hat{X}\right) + g\left(\mathcal{C}\left(\mathcal{V}X^c,\hat{Y}\right),\hat{Z}\right),
\end{aligned}$$

thus concluding the proof. □

6.2 Results for some special classes of generalized metrics

For any metric g, we introduce the (1,1) tensor $\overset{*}{\mathcal{C}}$ along π by the prescription

$$\overset{*}{\mathcal{C}} : \tilde{X} \in \mathfrak{X}(\pi) \mapsto \mathcal{C}\left(\tilde{X},\delta\right),$$

where $\mathcal{C}$ is the first Cartan tensor of the metric.

6.2.1 Theorem. *Suppose that $\widetilde{T_pM}$ is connected for all $p \in M$, and let g be a weakly variatonal and Miron regular metric on $\widetilde{TM}$ with $\vartheta_g = d^v L$. A vector field X on M is a Killing vector field for g if and only if the function $X^c L$ is a vertical lift and $\mathcal{L}_X \overset{*}{\mathcal{C}} = 0$.*

Proof.

(1) *Necessity*

Suppose that X is a Killing field. If $Y \in \mathfrak{X}(M)$, we have

$$\begin{aligned} Y^v X^c L &= X^c Y^v L - [X^c, Y^v] L = X^c Y^v L - [X, Y]^v L \\ &= X^c (d^v L)\hat{Y} - (d^v L)\widehat{[X,Y]} = X^c \left(\vartheta_g \hat{Y}\right) - \vartheta_g \widehat{[X,Y]} \\ &= X^c g\left(\hat{Y}, \delta\right) - g\left(\widehat{[X,Y]}, \delta\right) = (\mathcal{L}_X g)\left(\hat{Y}, \delta\right) = 0. \end{aligned}$$

If we substitute coordinate vector fields of a chart for Y, then we see that all partial derivatives of the function $X^c L$ vanish on the intersection of a fibre with $\widetilde{TM}$, and, taking into account that this set is connected, it follows that $X^c L$ is constant on it. Thus $X^c L$ is a vertical lift. To verify the necessity of the second condition, let Z be another vector field on M. Using our assumption $\mathcal{L}_X g = 0$ repeatedly, we get

$$\begin{aligned} & g\left(\left(\mathcal{L}_X \overset{*}{\mathcal{C}}\right)\hat{Y}, \hat{Z}\right) = g\left(\mathcal{L}_X\left(\overset{*}{\mathcal{C}}\hat{Y}\right) - \overset{*}{\mathcal{C}}\left(\mathcal{L}_X \hat{Y}\right), \hat{Z}\right) \\ &= X^c g\left(\overset{*}{\mathcal{C}}\hat{Y}, \hat{Z}\right) - g\left(\overset{*}{\mathcal{C}}\hat{Y}, \mathcal{L}_X \hat{Z}\right) - g\left(\overset{*}{\mathcal{C}}\widehat{[X,Y]}, \hat{Z}\right) \\ &= X^c g\left(\overset{*}{\mathcal{C}}\hat{Y}, \hat{Z}\right) - g\left(\overset{*}{\mathcal{C}}\widehat{[X,Y]}, \hat{Z}\right) - g\left(\overset{*}{\mathcal{C}}\hat{Y}, \widehat{[X,Z]}\right) \\ &= X^c g\left(\mathcal{C}\left(\hat{Y}, \delta\right), \hat{Z}\right) - g\left(\mathcal{C}\left(\widehat{[X,Y]}, \delta\right), \hat{Z}\right) - g\left(\mathcal{C}\left(\hat{Y}, \delta\right), \widehat{[X,Z]}\right), \end{aligned}$$

where we have used the definition of $\overset{*}{\mathcal{C}}$. By the definition of $\mathcal{C}$,

$$\begin{aligned}
&g\left(\left(\mathcal{L}_X\overset{*}{\mathcal{C}}\right)\hat{Y},\hat{Z}\right)\\
&\quad= X^c\left(\nabla^v_{\hat{Y}}g\right)\left(\delta,\hat{Z}\right)-\left(\nabla^v_{\widehat{[X,Y]}}g\right)\left(\delta,\hat{Z}\right)-\left(\nabla^v_{\hat{Y}}g\right)\left(\delta,\widehat{[X,Z]}\right)\\
&\quad= X^cY^vg\left(\delta,\hat{Z}\right)-X^cg\left(\hat{Y},\hat{Z}\right)-[X^c,Y^v]g\left(\delta,\hat{Z}\right)+g\left(\widehat{[X,Y]},\hat{Z}\right)\\
&\qquad-Y^vg\left(\delta,\widehat{[X,Z]}\right)+g\left(\hat{Y},\widehat{[X,Z]}\right)\\
&\quad= Y^vX^cg\left(\delta,\hat{Z}\right)-Y^vg\left(\delta,\widehat{[X,Z]}\right)=Y^v(\mathcal{L}_Xg)\left(\delta,\hat{Z}\right)=0,
\end{aligned}$$

which implies, by the non-degeneracy of g, that $\mathcal{L}_X\overset{*}{\mathcal{C}}=0$.

(2) *Sufficiency*

If X^cL is a vertical lift, we obtain

$$\begin{aligned}
(\mathcal{L}_X\vartheta_g)\hat{Y}&=X^c\left(\vartheta_g\hat{Y}\right)-\vartheta_g\left(\mathcal{L}_X\hat{Y}\right)=X^c(d^vL)\hat{Y}-d^vL\left(\mathbf{i}^{-1}[X^c,Y^v]\right)\\
&=X^cY^vL-[X^c,Y^v]L=Y^vX^cL=0
\end{aligned}$$

for any vector field Y on M, and therefore

$$\begin{aligned}
\mathcal{L}_{X^c}(\vartheta_g\circ\mathbf{j})(\eta)&=X^c(\vartheta_g\mathbf{j}\eta)-\vartheta_g(\mathbf{j}[X^c,\eta])=X^c(\vartheta_g\mathbf{j}\eta)-\vartheta_g(\mathcal{L}_X\mathbf{j}\eta)\\
&=(\mathcal{L}_X\vartheta_g)(\mathbf{j}\eta)=0
\end{aligned}$$

for $\eta\in\mathfrak{X}\left(\widetilde{TM}\right)$, which implies $\mathcal{L}_{X^c}(\vartheta_g\circ\mathbf{j})=0$. Since the Lie derivative and the exterior derivative commute, we also have

$$\mathcal{L}_{X^c}\omega_g=\mathcal{L}_{X^c}d(\vartheta_g\circ\mathbf{j})=d\mathcal{L}_{X^c}(\vartheta_g\circ\mathbf{j})=0.$$

The second condition implies

$$\mathcal{L}_XA=\mathcal{L}_X\left(1_{\mathfrak{X}(\tau)}+\overset{*}{\mathcal{C}}\right)=0.$$

As $\mathcal{L}_Xg$ is tensorial, and g is Miron regular, it is sufficient to show that $(\mathcal{L}_Xg)\left(A\hat{Y},\hat{Z}\right)=0$ for any vector fields Y and Z on M. Using

$$\omega_g(J\eta,\zeta)=g\left(A(\mathbf{j}\eta),\mathbf{j}\zeta\right)\qquad\left(\eta,\zeta\in\mathfrak{X}\left(\widetilde{TM}\right)\right),$$

we get

$$\begin{aligned}
(\mathcal{L}_X g)\left(A\hat{Y},\hat{Z}\right) &= X^c g\left(A\hat{Y},\hat{Z}\right) - g\left(\mathcal{L}_X\left(A\hat{Y}\right),\hat{Z}\right) - g\left(A\hat{Y},\widehat{[X,Z]}\right)\\
&= X^c g\left(A\hat{Y},\hat{Z}\right) - g\left(A\widehat{[X,Y]},\hat{Z}\right) - g\left(A\hat{Y},\widehat{[X,Z]}\right)\\
&= X^c\omega_g(Y^v,Z^c) - \omega_g([X^c,Y^v],Z^c) - \omega_g(Y^v,[X^c,Z^c])\\
&= (\mathcal{L}_{X^c}\omega_g)(Y^v,Z^c) = 0,
\end{aligned}$$

thus concluding the proof.

□

Now we introduce two canonical inclusions. The first one will be

$$i_1 : M \to TM,\ p \in M \mapsto i_1(p) := 0_p.$$

In other words, i_1 is an embedding of M into TM that assigns to each point p the zero vector at p. The second inclusion is given by the prescription

$$\begin{gathered}
i_2 : TM \to TTM,\ v \in TM \mapsto i_2(v) := \dot{c}_v(0),\\
\text{where } c_v : t \in \mathbb{R} \mapsto 0_{\tau(v)} + tv.
\end{gathered}$$

We shall also use the shorthand $\bar{\tau} := i_1 \circ \tau$. The image $\operatorname{Im} i_1 =: \widetilde{M} \subset TM$ is an embedded submanifold, and it consists of the zero tangent vectors. From the coordinate expression of a complete lift X^c (see Appendix) it can be seen that X^c is tangent to $\widetilde{M}$: at the points of $\widetilde{M}$ all the functions y^i vanish, hence the second term vanishes, and since the coordinate vector fields $\frac{\partial}{\partial x^i}$ are tangent to $\widetilde{M}$, it follows that X^c is tangent to $\widetilde{M}$ as well.

The metric g does not determine L uniquely, since a vertical lift can be added to L without changing $d^v L$. Moreover, we have

6.2.2 Corollary. *With conditions similar to those in 6.2.1, if g is defined on the whole TM, and X is a Killing vector field on M, L can be chosen such that $X^c L = 0$.*

Proof. By 6.2.1, there is a Lagrangian $\widetilde{L}$ on TM for g such that $X^c\widetilde{L}$ is a vertical lift. Let us define L by

$$L := \widetilde{L} - \widetilde{L} \circ \bar{\tau},$$

or, more explicitely,

$$L(v) = \widetilde{L}(v) - \widetilde{L}\left(0_{\tau(v)}\right) \quad (v \in TM).$$

The second term is a vertical lift, and therefore $d^v L = d^v \widetilde{L} = \vartheta_g$. We show that $X^c L = 0$. In the expression

$$X^c L = X^c \widetilde{L} - X^c \left(\widetilde{L} \circ \bar{\tau} \right)$$

the first term is a vertical lift by the choice of $\widetilde{L}$, and the second is a vertical lift as well, since the action of a complete lift on a vertical lift yields a vertical lift. Therefore it suffices to show that $X^c L$ vanishes on $\widetilde{M}$, i.e., on the zero section. This follows immediately from our previous observation that X^c is tangent to the submanifold $\widetilde{M}$. □

6.2.3 Theorem. *Let g be a variational metric defined on the whole TM. A vector field X on M is a Killing vector field if and only if there is a Lagrangian L for g such that $X^c L = 0$.*

Proof.

(1) *Necessity*

Suppose that X is a Killing vector field, and $\widetilde{L}$ is an arbitrary Lagrangian for g. Then we obtain

$$0 = (\mathcal{L}_X g) \left(\hat{Y}, \hat{Z} \right) = X^c g \left(\hat{Y}, \hat{Z} \right) - g \left(\widehat{[X,Y]}, \hat{Z} \right) - g \left(\hat{Y}, \widehat{[X,Z]} \right)$$

for any vector fields $Y, Z \in \mathfrak{X}(M)$. Using $g = \nabla^v \nabla^v \widetilde{L}$, this further implies

$$\begin{aligned} 0 &= X^c Y^v Z^v \widetilde{L} - [X,Y]^v Z^v \widetilde{L} - Y^v [X,Z]^v \widetilde{L} \\ &= X^c Y^v Z^v \widetilde{L} - [X^c, Y^v] Z^v \widetilde{L} - Y^v [X^c, Z^v] \widetilde{L} = Y^v Z^v X^c \widetilde{L}. \end{aligned}$$

It follows that $X^c \widetilde{L}$ is an affine function on each fibre of TM, i.e., on each tangent space. Now we define a new Lagrangian L by

$$L := \widetilde{L} - \widetilde{L} \circ i_1 \circ \tau - d\widetilde{L} \circ i_2.$$

It is easy to see that the difference of $\widetilde{L}$ and L is also a fibrewise affine function, thus their Hessians are the same, i.e.,

$$\nabla^v \nabla^v \widetilde{L} = \nabla^v \nabla^v L = g.$$

We compute the action of X^c on the difference $\widetilde{L} - L$ over an induced chart $\left(\tau^{-1}(\mathcal{U}), (x^i)_{i=1}^n, (y^i)_{i=1}^n\right)$ on TM by a chart $\left(\mathcal{U}, (u^i)_{i=1}^n\right)$ on M:

$$X^c\left(\widetilde{L}\circ i_1\circ\tau + d\widetilde{L}\circ i_2\right) = \left[X\left(\widetilde{L}\circ i_1\right)\right]^v + X^c\left(d\widetilde{L}\circ i_2\right)$$

$$= \left[X^i\frac{\partial\left(\widetilde{L}\circ i_1\right)}{\partial u^i}\right]^v + \left(X^i\right)^v\frac{\partial}{\partial x^i}\left(d\widetilde{L}\circ i_2\right) + y^j\left(\frac{\partial X^i}{\partial u^j}\right)^v\frac{\partial}{\partial y^i}\left(d\widetilde{L}\circ i_2\right).$$

We shall further transform the expression $d\widetilde{L}\circ i_2$ by evaluating it at a particular tangent vector $v \in TM$. Thus c_v is the curve $t \mapsto 0_{\tau(v)} + tv$, whose component functions are the following:

$$x^i(c_v(t)) = u^i(v),\ y^i(c_v(t)) = ty^i(v) \quad (t \in \mathbb{R}, i \in \{1,\dots,n\}).$$

Substituting this into the definition of i_2, we get

$$i_2(v) = \dot{c}_v(0) = y^i(v)\left(\frac{\partial}{\partial y^i}\right)_{\bar{\tau}(v)},$$

and thus

$$\left(d\widetilde{L}\circ i_2\right)(v) = y^i(v)\cdot\frac{\partial\widetilde{L}}{\partial y^i}(\bar{\tau}(v)),$$

which is equivalent to

$$d\widetilde{L}\circ i_2 = y^i\left(\frac{\partial\widetilde{L}}{\partial y^i}\circ\bar{\tau}\right).$$

Finally, turning back to the interrupted computation,

$$X^c\left(\widetilde{L}\circ i_1\circ\tau + d\widetilde{L}\circ i_2\right)$$

$$= \left(X^i\right)^v\left(\frac{\partial\widetilde{L}}{\partial x^i}\circ\bar{\tau}\right) + \left(X^i\right)^v y^j\left(\frac{\partial^2\widetilde{L}}{\partial x^i\partial y^j}\circ\bar{\tau}\right) + y^j\left(\frac{\partial X^i}{\partial u^j}\right)^v\left(\frac{\partial\widetilde{L}}{\partial y^i}\circ\bar{\tau}\right).$$

This is a fibrewise affine function, just like $X^c\widetilde{L}$. To show that they are equal, it is enough to check that they coincide on the zero section and so do their linear parts on each fibre. The expression of $X^c\widetilde{L}$ over our induced chart is

$$X^c\widetilde{L} = \left(X^i\right)^v\frac{\partial\widetilde{L}}{\partial x^i} + y^j\left(\frac{\partial X^i}{\partial u^j}\right)^v\frac{\partial\widetilde{L}}{\partial y^i}.$$

Thus, $X^cL = X^c\widetilde{L} - X^c\left(\widetilde{L} - L\right)$ vanishes indeed on the zero section:

$$X^c\widetilde{L} \circ i_1 - X^c\left(\widetilde{L} - L\right) \circ i_1 = X^i\left(\frac{\partial \widetilde{L}}{\partial x^i} \circ i_1\right) - X^i\left(\frac{\partial \widetilde{L}}{\partial x^i} \circ i_1\right) = 0,$$

whereas the linear part of X^cL is

$$y^i\left(\frac{\partial}{\partial y^i}X^c\widetilde{L}\right) \circ \overline{\tau} - \left(X^i\right)^v y^j\left(\frac{\partial^2\widetilde{L}}{\partial x^i \partial y^j} \circ \overline{\tau}\right) - y^j\left(\frac{\partial X^i}{\partial u^j}\right)^v\left(\frac{\partial \widetilde{L}}{\partial y^i} \circ \overline{\tau}\right) = 0.$$

(2) *Sufficiency*

If L is a Lagrangian for g such that $X^cL = 0$, then we have

$$\begin{aligned}(\mathcal{L}_X g)\left(\hat{Y}, \hat{Z}\right) &= X^c g\left(\hat{Y}, \hat{Z}\right) - g\left(\widehat{[X,Y]}, \hat{Z}\right) - g\left(\hat{Y}, \widehat{[X,Z]}\right) \\ &= X^cY^vZ^vL - [X^c, Y^v]Z^vL - Y^v[X^c, Z^v]L = Y^vZ^vX^cL = 0,\end{aligned}$$

thus X is indeed a Killing vector field.

□

6.2.4 Corollary. *If (M, E) is a Finsler manifold with Finslerian metric $g = \nabla^v\nabla^v E$, then a vector field X on M is a Killing vector field of g if and only if $X^cE = 0$.*

Proof. In this case, g is variational, and the energy function E is a Lagrangian for g. From the proof of the previous theorem it can be seen that the function X^cE is affine on each tangent space. On the other hand, since E is positively homogeneous of degree 2, we also have

$$CX^cE = [C, X^c]E + X^cCE = X^cCE = 2X^cE,$$

hence X^cE is also positively homogeneous of degree 2. This is possible only if $X^cE = 0$. □

6.3 Translations

In this section we shall work on a manifold M endowed with a weakly normal and Miron regular metric on $\overset{\circ}{T}M$. We recall that in this case the absolute energy E is a Finsler energy function (see (3) in section 5.1). We shall apply the Barthel connection, Cartan's covariant derivative and the canonical spray corresponding to the Finsler energy E.

The well-known straightening-out theorem will be used repeatedly in our considerations. For the sake of completeness we include it here as a lemma. For a proof, we can refer to e.g. [6].

6.3.1 Lemma (straightening-out theorem). *If X is a vector field on the manifold M such that $X(p) \neq 0$ for some $p \in M$, then there exists a chart $\left(\mathcal{U}, (u^i)_{i=1}^n\right)$ around p such that $X \restriction \mathcal{U} = \frac{\partial}{\partial u^1}$.*

6.3.2 Definition. *A Killing vector field X of g is called a* translation *if every (non-constant) integral curve of X is a geodesic of the Finsler manifold (M, E).*

For classical results on translations of Riemannian manifolds, see [15, 46, 63]. Now we generalize the important *conservation lemma* from Riemannian geometry [41, p. 252] to our setting as follows.

6.3.3 Proposition. *If $X \in \mathfrak{X}(M)$ is a Killing vector field, and $c : I \to M$ is a geodesic of E, then the function*

$$t \in I \mapsto g_{\dot{c}(t)}(X(c(t)), \dot{c}(t))$$

is constant.

Proof. Let us denote the function in question by f. Obviously, it may also be written in the form

$$f = g\left(\hat{X}, \delta\right) \circ \dot{c}.$$

The curve $\dot{c}$ is an integral curve of the spray ξ arising from E, thus we have

$$f' = \xi g\left(\hat{X}, \delta\right) \circ \dot{c}.$$

Using (3) in section 5.1 and the relation $X^c E = 0$ (which follows from the condition that X is a Killing field), we obtain

$$\begin{aligned}\xi g\left(\hat{X}, \delta\right) &= \xi\left(\vartheta_g \hat{X}\right) = \xi(d^v E)\left(\hat{X}\right) = \xi X^v E = -X^c E - X^v \xi E + \xi X^v E \\ &= -(X^c + [X^v, \xi])E = -2X^h E = 0,\end{aligned}$$

and therefore $f' = 0$, which implies that f is constant. □

We recall from section 5.1 that E can be extended continuously to the zero section. In the next proposition E will be this extended function.

6.3.4 Proposition. *Let X be a Killing vector field of g. Then X is a translation if and only if the function*

$$p \in M \mapsto E(X_p)$$

is constant.

Proof.

(1) *Necessity*
Suppose that X is a translation. If $E(X_q) = 0$ for any $q \in M$, the statement is obvious. Hence we assume that there is a point $q \in M$ such that $E(X_q) \neq 0$. We define the following subset of M:

$$V := \{p \in M | E(X_p) = E(X_q)\}.$$

We shall show that $V = M$. First, $V \neq \emptyset$, since $q \in V$. Furthermore, V is closed, since it is the inverse image of the closed set $\{E(X_q)\} \subset \mathbb{R}$ under the continuous function

$$f : p \in M \mapsto f(p) := E(X_p).$$

As agreed in section 2.1, M is connected, thus it remains only to show that V is open.

To see this, take a point $p \in V$. By the straightening-out theorem (6.3.1), there is a chart $\left(\mathcal{U}, (u^i)_{i=1}^n\right)$ around p such that $X \restriction \mathcal{U} = \frac{\partial}{\partial u^1}$. Consider an integral curve $c : I \to M$ of X, which is, by the definition of translations, a geodesic of the Finsler manifold (M, E) as well. Its components $c^i := u^i \circ c$ have the following form:

$$c^1(t) = c^1(0) + t, \quad c^i(t) = c^i(0) \quad (i \in \{2, \ldots, n\}),$$

hence $c^{i\prime\prime} = 0$ for any $i \in \{1, \ldots, n\}$. On the other hand, c satisfies the differential equations of geodesics:

$$c^{i\prime\prime} + 2G^i \circ \dot{c} = 0,$$

where

$$G^i = \frac{1}{2}\gamma^{ij}\left(y^k \frac{\partial^2 E}{\partial x^k \partial y^j} - \frac{\partial E}{\partial x^j}\right),$$

and (γ^{ij}) is the inverse matrix of

$$(\gamma_{ij}) := \left(\frac{\partial^2 E}{\partial y^i \partial y^j}\right)$$

(see the Appendix). Putting these together, we infer that $G^i \circ \frac{\partial}{\partial u^1} = 0$ on U. Since the matrix (γ^{ij}) is regular, using $y^i \circ \frac{\partial}{\partial u^1} = \delta_1^i$, this implies that

$$\begin{aligned} 0 &= \left(y^k \frac{\partial^2 E}{\partial x^k \partial y^j} - \frac{\partial E}{\partial x^j}\right) \circ \frac{\partial}{\partial u^1} = \left(\frac{\partial^2 E}{\partial x^1 \partial y^j} - \frac{\partial E}{\partial x^j}\right) \circ \frac{\partial}{\partial u^1} \\ &= -\frac{\partial E}{\partial x^j} \circ \frac{\partial}{\partial u^1} = -\frac{\partial}{\partial u^j}\left(E \circ \frac{\partial}{\partial u^1}\right), \end{aligned}$$

which, in turn, implies that the function $E \circ \frac{\partial}{\partial u^1}$ is constant on $\mathcal{U}$. Hence $p \in V$ is contained together with an open neighbourhood in V. We conclude that $V = M$.

(2) *Sufficiency*
Suppose that the function $f : p \in M \mapsto f(p) := E(X_p)$ is constant, and choose a chart similar to that in the previous part. Then the components of the integral curves of X satisfy $c^{i\prime\prime} = 0$. By the same reasoning as in the previous part, $G^i \circ \frac{\partial}{\partial u^1} = 0$, thus $G^i \circ \dot{c} = 0$ along the integral curves of X, and c satisfies the differential equation of the geodesics. Therefore the integral curves of X are geodesics as well.

□

It is known that a geodesic on a Riemannian manifold meets a translation at constant angles [15, 46, 63]. In the general case, if g is positive definite, the angle φ of a translation X and a geodesic c may be given by

$$\cos\varphi(t) := \frac{g_{\dot{c}(t)}(X(c(t)), \dot{c}(t))}{\sqrt{g_{\dot{c}(t)}(X(c(t)), X(c(t)))g_{\dot{c}(t)}(\dot{c}(t), \dot{c}(t))}},$$

or, equivalently,

$$\cos\varphi := \frac{g\left(\hat{X}, \delta\right)}{\left(g\left(\hat{X}, \hat{X}\right) g(\delta, \delta)\right)^{1/2}} \circ \dot{c}.$$

The numerator is constant by 6.3.3, and the second factor in the denominator is constant as well even in the most general case. It follows from 6.3.4 that in the Riemannian case the first factor is also constant, since then the function $g\left(\hat{X}, \hat{X}\right)$ is constant on each fibre. From our results, however, it does not follow that the first factor is constant in general, even for Finsler manifolds. Therefore, it does not follow that φ is constant. It remains an open question whether there exists any class of metrics in which this angle is constant and which is more general than the Riemannian case.

Moreover, there is a broad class of (even Riemannian) metrics that have no non-trivial translations at all. For example, the hyperbolic plane does not have any. Consider Poincaré's upper half-plane model:

$$\mathbb{H}^2 := \left\{ p = (p^1, p^2) \in \mathbb{R}^2 \middle| \, p^2 > 0 \right\}$$

with canonical coordinate functions $\left(u^1, u^2\right)$ and metric tensor given by

$$g_p(v_p, w_p) := \frac{\langle v, w \rangle}{(u^2(p))^2} \qquad \left(p \in \mathbb{H}^2, v, w \in \mathbb{R}^2\right).$$

The Killing fields of the Riemannian manifold $\left(\mathbb{H}^2, g\right)$ have the form

$$X = \left[\alpha\left((u^1)^2 - (u^2)^2\right) + \beta u^1 + \gamma\right] \frac{\partial}{\partial u^1} + \left(2\alpha u^1 + \beta\right) u^2 \frac{\partial}{\partial u^2}$$

with some $\alpha, \beta, \gamma \in \mathbb{R}$. The upper half-plane $\mathbb{H}^2$ coincides with the set of complex numbers with positive imaginary part. Suppose that $\alpha \neq 0$, and introduce the notation $k := \sqrt{|\beta^2/4 - \alpha\gamma|}$. Then the integral curves of X are given by

$$\begin{aligned}
z(t) &= -\frac{k}{\alpha} \frac{c \cosh kt - \sinh kt}{c \sinh kt - \cosh kt} - \frac{\beta}{2\alpha} && \text{if } \frac{\beta^2}{4} - \alpha\gamma > 0, \\
z(t) &= -\frac{k}{\alpha} \frac{c \cos kt + \sin kt}{c \sin kt - \cos kt} - \frac{\beta}{2\alpha} && \text{if } \frac{\beta^2}{4} - \alpha\gamma < 0, \\
z(t) &= -\frac{1}{\alpha t + c} - \frac{\beta}{2\alpha} && \text{if } \frac{\beta^2}{4} - \alpha\gamma = 0
\end{aligned}$$

with $c \in \mathbb{C}$ such that $\operatorname{Im} c > 0$, which are no geodesics. That is, however, not surprising, since, if the hyperbolic plane had a non-trivial translation, its flow could be used to construct a geodesic quadrangle with angle sum 2π, in contradiction with the Gauss–Bonnet theorem (cf. Figure 6.1).

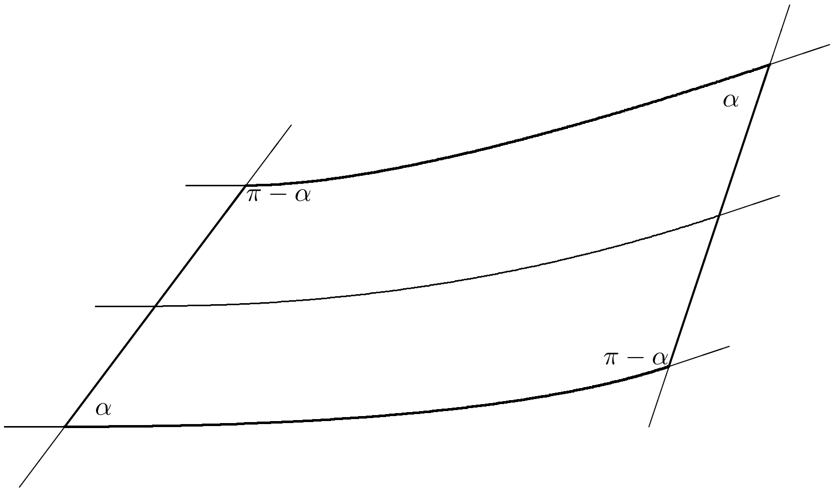

Figure 6.1: There are no translations on the hyperbolic plane

6.4 An application to Randers manifolds

Let (M, α) be a Riemannian manifold and β a one-form on M. We recall from section 2.1 that the tensor $\hat{\alpha}$ along τ and the function $\bar{\beta}$ on TM are given by

$$\hat{\alpha}_v(w_1, w_2) = \alpha_p(w_1, w_2),\ \bar{\beta}(v) = \beta_p(v) \quad (v, w_1, w_2 \in T_pM, p \in M).$$

We define the following functions on TM:

$$F_\alpha(v) := \sqrt{\alpha_{\tau(v)}(v, v)} \quad (v \in TM), \quad F := F_\alpha + \bar{\beta},\ E := \frac{1}{2}F^2.$$

Then F and E are smooth on $\overset{\circ}{T}M$.

On the Riemannian manifold (M, α) we have the same type of canonical one-to-one correspondence between vector fields and one-forms as in the case of metrics along the projection π as described in section 5.1, namely:

$$X^\flat(Y) := \alpha(X, Y),\ g\left(\gamma^\sharp, Y\right) := \gamma(Y) \quad (X, Y \in \mathfrak{X}(M), \gamma \in \mathfrak{X}^*(M)).$$

If $\left\|\beta^\sharp\right\| < 1$, (M, E) is a Finsler manifold, called the *Randers manifold* obtained from the Riemannian manifold (M, α) by the perturbation with the one-form

β. For the expression of the metric tensor of this Finsler structure we use the following formula from [37]:

6.4.1 Lemma. *Let (M, E) be the Randers manifold arising from the Riemannian manifold (M, α) by perturbation with β such that $\|\beta^\sharp\| < 1$. Then the metric tensor g of (M, E) takes the form*

$$g = \frac{F}{F_\alpha}\hat{\alpha} - \frac{\bar{\beta}}{F_\alpha^3}\bar{\alpha} \otimes \bar{\alpha} + \frac{1}{F_\alpha}\bar{\alpha} \odot \hat{\beta} + \hat{\beta} \otimes \hat{\beta},$$

where $\odot$ stands for the symmetric product.

In his paper [33], M. Matsumoto proved that $\beta^\sharp$ is a Killing vector field of the Randers manifold if and only if it is a Killing vector field of the original Riemannian manifold (M, α) as well. Now we give a new proof of the sufficiency of this condition:

6.4.2 Proposition. *Suppose that (M, α) is a Riemannian manifold, and $X \in \mathfrak{X}(M)$ is a Killing vector field of (M, α) such that $\|X\| < 1$. Let $\beta := X^\flat$, $F := F_\alpha + \bar{\beta}$ and $E = \frac{1}{2}F^2$. Then X is a Killing vector field of the Randers manifold (M, E).*

Proof. First, suppose that $X(p) \neq 0$ at $p \in M$. Consider a chart $\big(\mathcal{U}, (u^i)_{i=1}^n\big)$ around p and the induced chart $\big(\tau^{-1}(\mathcal{U}), (x^i)_{i=1}^n, (y^i)_{i=1}^n\big)$ on TM. Let $i, j \in \{1, \dots, n\}$ be arbitrary, then

$$\begin{aligned}(\mathcal{L}_X g)&\left(\widehat{\frac{\partial}{\partial u^i}}, \widehat{\frac{\partial}{\partial u^j}}\right) \\ &= X^c g\left(\widehat{\frac{\partial}{\partial u^i}}, \widehat{\frac{\partial}{\partial u^j}}\right) + g\left(\mathcal{L}_X \widehat{\frac{\partial}{\partial u^i}}, \widehat{\frac{\partial}{\partial u^j}}\right) + g\left(\widehat{\frac{\partial}{\partial u^i}}, \mathcal{L}_X \widehat{\frac{\partial}{\partial u^j}}\right).\end{aligned}$$

By the straightening-out theorem (6.3.1), we can choose a chart such that $X = \frac{\partial}{\partial u^1}$ over $\mathcal{U}$. Then the last two terms vanish since, e.g.,

$$\mathcal{L}_X \widehat{\frac{\partial}{\partial u^i}} = \widehat{\left[X, \frac{\partial}{\partial u^i}\right]} = \widehat{\left[\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^i}\right]} = 0.$$

It remains to show that the first term also vanishes. We have the following coordinate expressions:

$$\hat{\alpha}\left(\widehat{\frac{\partial}{\partial u^i}},\widehat{\frac{\partial}{\partial u^j}}\right)=\alpha^v_{ij},\quad \hat{\beta}\left(\widehat{\frac{\partial}{\partial u^i}}\right)=\beta^v_i,\quad \bar{\alpha}\left(\widehat{\frac{\partial}{\partial u^i}}\right)=\alpha^v_{ij}y^j,$$
$$\bar{\beta}=\beta^v_i y^i,\quad F_\alpha=\sqrt{\alpha^v_{ij}y^iy^j},\quad F=\sqrt{\alpha^v_{ij}y^iy^j}+\beta^v_i y^i.$$

We substitute the expression in the preceding lemma for g:

$$(6.1)\qquad (\mathcal{L}_X g)\left(\widehat{\frac{\partial}{\partial u^i}},\widehat{\frac{\partial}{\partial u^j}}\right)=\frac{\partial}{\partial x^1}g\left(\widehat{\frac{\partial}{\partial u^i}},\widehat{\frac{\partial}{\partial u^j}}\right)$$
$$=\frac{\partial}{\partial x^1}\left[\left(1+\frac{\beta^v_k y^k}{(\alpha^v_{lm}y^ly^m)^{1/2}}\right)\alpha^v_{ij}-\frac{\beta^v_k y^k}{(\alpha^v_{lm}y^ly^m)^{3/2}}\alpha^v_{ir}\alpha^v_{js}y^ry^s\right.$$
$$\left.+\frac{1}{(\alpha^v_{lm}y^ly^m)^{1/2}}(\alpha^v_{ir}y^r\beta^v_j+\beta^v_i\alpha^v_{jr}y^r)+\beta^v_i\beta^v_j\right],$$

and

$$\beta_i=\alpha_{ij}X^j=\alpha_{ij}\delta^j_1=\alpha_{i1}.$$

On the other hand, since X is a Killing vector field of (M,α), it follows that

$$0=\left(\mathcal{L}_{\frac{\partial}{\partial u^1}}\alpha\right)\left(\frac{\partial}{\partial u^i},\frac{\partial}{\partial u^j}\right)=\frac{\partial\alpha_{ij}}{\partial u^1}.$$

Thus we have shown that all functions in the square bracket of (6.1) have vanishing partial derivatives with respect to x^1, and hence $\mathcal{L}_X g=0$ on T_pM if $X(p)\neq 0$. On the other hand, if $X(p)=0$, and there is a series $(p_n)_{n=0}^\infty$ such that $p_n\to p$ and $X(p_n)\neq 0$ $(n\in\mathbb{N})$, then $\mathcal{L}_X g$ vanishes on T_pM by continuity. Finally, if there is a neighbourhood of p on which X vanishes, then $\mathcal{L}_X g=0$ on T_pM automatically. □

6.5 Killing vector fields of a Funk metric

In this section we shall work on an open subset of $\mathbb{R}^n$; D_v will denote the directional derivative with respect to a vector $v\in\mathbb{R}^n$ and D_i the ith partial derivative $(i=1,\dots,n)$.

Let $\varphi:\mathbb{R}^n\to\mathbb{R}$ be a Finsler – Minkowski norm as defined in Chapter 4, and $\Omega=\varphi^{-1}\,[0,1[$ the open unit ball. We shall use the canonical identification $T\Omega\cong\Omega\times\mathbb{R}^n$ and the natural projections $\mathrm{pr}_1:T\Omega\to\Omega$ and $\mathrm{pr}_2:T\Omega\to\mathbb{R}^n$. A

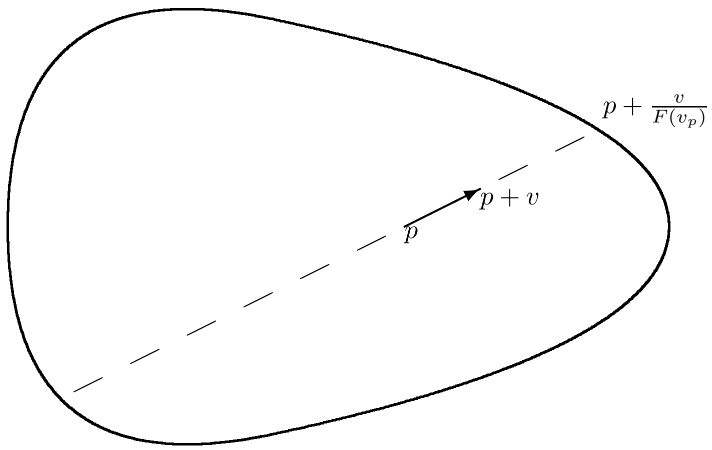

Figure 6.2: Funk metric on Ω

Finslerian fundamental function $F : T\Omega \to \mathbb{R}$ on Ω is determined by the relation

$$\varphi \circ \left(\mathrm{pr}_1 + \frac{\mathrm{pr}_2}{F}\right) = 1 \quad \text{on } \overset{\circ}{T}\Omega. \tag{6.2}$$

The Finsler structure given by F is traditionally called the *Funk metric* on Ω (cf. Figure 6.2). The Finsler energy is then $E = \frac{1}{2}F^2$. If we substitute $v \in \overset{\circ}{T}\Omega$ into (6.2), we see that it is equivalent to

$$p + \frac{v}{F(v_p)} \in \partial\Omega,$$

as shown in the figure. For more about Funk metrics, see [50].

6.5.1 Proposition. *With notations and hypotheses as above, for a vector field X on Ω the following conditions are equivalent:*

(1) *X is a Killing vector field of (Ω, F);*

(2) *for every point $p \in \Omega$ and vector $v \in \mathbb{R}^n$ such that $p + v \in \partial\Omega$, the vector $X(p) + D_vX(p)$ is parallel to the tangent hyperplane of $\partial\Omega$ in $p + v$.*

Proof. Let $(u^i)_{i=1}^n$ be the restriction of the canonical coordinates of $\mathbb{R}^n$ to Ω, thus we have a global chart $\left(\Omega, (u^i)_{i=1}^n\right)$. Let $\left(T\Omega, (x^i)_{i=1}^n, (y^i)_{i=1}^n\right)$ be the induced chart on $T\Omega$. If the coordinate expression of X is $X^i \frac{\partial}{\partial u^i}$, its complete lift is

$$X^c = \left(X^i\right)^v \frac{\partial}{\partial x^i} + y^j \left(\frac{\partial X^i}{\partial u^j}\right)^v \frac{\partial}{\partial y^i}$$

(see the Appendix). Now we act by X^c on both sides of (6.2). Using the chain rule for the compound map on the left-hand side, we obtain

$$\begin{aligned} 0 &= \left[D_k\varphi \circ \left(\mathrm{pr}_1 + \frac{\mathrm{pr}_2}{F}\right)\right] \left[\left(X^i\right)^v \frac{\partial}{\partial x^i}\left(x^k + \frac{y^k}{F}\right)\right. \\ &\qquad \left. + y^j \left(\frac{\partial X^i}{\partial u^j}\right)^v \frac{\partial}{\partial y^i}\left(x^k + \frac{y^k}{F}\right)\right] \\ &= \left[D_k\varphi \circ \left(\mathrm{pr}_1 + \frac{\mathrm{pr}_2}{F}\right)\right] \left[\left(X^i\right)^v \left(\delta_i^k - \frac{y^k}{F^2}\frac{\partial F}{\partial x^i}\right)\right. \\ &\qquad \left. + y^j \left(\frac{\partial X^i}{\partial u^j}\right)^v \left(\frac{\delta_i^k}{F} - \frac{y^k}{F^2}\frac{\partial F}{\partial y^i}\right)\right]. \end{aligned}$$

Regrouping these terms yields

$$\begin{aligned} 0 &= \left[D_k\varphi \circ \left(\mathrm{pr}_1 + \frac{\mathrm{pr}_2}{F}\right)\right] \left[\left(X^k\right)^v + \frac{y^j}{F}\left(\frac{\partial X^k}{\partial u^j}\right)^v\right. \\ &\qquad \left. - \frac{y^k}{F^2}\left(\left(X^i\right)^v \frac{\partial F}{\partial x^i} + y^j \left(\frac{\partial X^i}{\partial u^j}\right)^v \frac{\partial F}{\partial y^i}\right)\right] \\ &= \left[D_k\varphi \circ \left(\mathrm{pr}_1 + \frac{\mathrm{pr}_2}{F}\right)\right] \left[\left(X^k\right)^v + \frac{y^j}{F}\left(\frac{\partial X^k}{\partial u^j}\right)^v - \frac{y^k}{F^2} X^c F\right]. \end{aligned}$$

By 6.2.4, X is a Killing field if and only if $X^c F = 0$. Furthermore, if $v_p (\neq 0) \in T\Omega$ is arbitrary, and $z := p + \frac{v}{F(v_p)} (\in \partial\Omega)$, then

$$\begin{aligned} v^k D_k\varphi(z) &= D_v\varphi(z) = D_v\left(\sqrt{2E}\right)(z) = \frac{1}{\sqrt{2E(z)}} D_v E(z) \\ &= \frac{1}{\varphi(z)} E'(z)(v) = E'(z)(v) = 0. \end{aligned}$$

Therefore, it follows that X is a Killing field if and only if

$$\left[D_k\varphi \circ \left(\mathrm{pr}_1 + \frac{\mathrm{pr}_2}{F}\right)\right] \left[\left(X^k\right)^v + \frac{y^j}{F}\left(\frac{\partial X^k}{\partial u^j}\right)^v\right] = 0. \tag{6.3}$$

From now on, we suppose that v_p is of the form as in the proposition, i.e., $p + v \in \partial\Omega$. By the homogeneity of F, if (6.3) is satisfied for such v_p's, it is satisfied for all. In that case $F(v_p) = 1$, and evaluating (6.3) at v_p we obtain

$$\begin{aligned}(D_k\varphi)(p+v)(X^k(p) + v^j D_j X^k(p)) &= (D_k\varphi)(p+v)(X^k(p) + D_v X^k(p)) \\ &= \langle \operatorname{grad} \varphi(p+v), X(p) + D_v X(p)\rangle = 0,\end{aligned}$$

or, equivalently, the vector $X(p) + D_v X(p)$ is parallel to the tangent hyperplane of the indicatrix at $p + v$. $\square$

Chapter 7

Summary

7.1 Notations

In this thesis we work on an n-dimensional connected manifold M. If $\widetilde{TM} \subset TM$ is an open set such that $\tau\left(\widetilde{TM}\right) = M$, then the restriction of the tangent bundle projection τ to $\widetilde{TM}$ is denoted by π. The pull-back bundle of τ by π is $\pi^*\tau$. The $C^\infty\left(\widetilde{TM}\right)$-module of its sections (also called *vector fields along* π) is denoted by $\mathfrak{X}(\pi)$. The most important special case is $\widetilde{TM} = \overset{\circ}{T}M$, the open subset of non-zero tangent vectors to the manifold M. The restriction of τ to $\overset{\circ}{T}M$ is $\overset{\circ}{\tau}$.

We have the following short exact sequences on $\widetilde{TM}$:

$$0 \to \pi^*TM \xrightarrow{\mathbf{i}} T\widetilde{TM} \xrightarrow{\mathbf{j}} \pi^*TM \to 0 \text{ and } 0 \to \mathfrak{X}(\pi) \xrightarrow{\mathbf{i}} \mathfrak{X}\left(\widetilde{TM}\right) \xrightarrow{\mathbf{j}} \mathfrak{X}(\pi) \to 0.$$

The vertical and the complete lift of a smooth function f on M are denoted by f^v and f^c, respectively. The vertical lift of a vector field X on M is $X^v = \mathbf{i}\hat{X}$, and its complete lift is X^c. The *canonical section of* $\pi^*\tau$, the *Liouville vector field* on $\widetilde{TM}$ and the *vertical endomorphism* are given by

$$\delta(v) := (v, v) \quad \left(v \in \widetilde{TM}\right), \quad C := \mathbf{i}\delta \quad \text{and} \quad J := \mathbf{i} \circ \mathbf{j},$$

respectively. If f is a smooth function on $\widetilde{TM}$, $d^v f$ is defined by the formula

$$(d^v f)\left(\tilde{X}\right) := \left(\mathbf{i}\tilde{X}\right) f \quad \left(\tilde{X} \in \mathfrak{X}(\pi)\right).$$

By an *Ehresmann connection* we mean a split short exact sequence:

$$0 \rightleftharpoons \pi^*TM \underset{\mathcal{V}}{\overset{\mathbf{i}}{\rightleftharpoons}} T\widetilde{TM} \underset{\mathcal{H}}{\overset{\mathbf{j}}{\rightleftharpoons}} \pi^*TM \rightleftharpoons 0 \quad \text{or} \quad 0 \rightleftharpoons \mathfrak{X}(\pi) \underset{\mathcal{V}}{\overset{\mathbf{i}}{\rightleftharpoons}} \mathfrak{X}\left(\widetilde{TM}\right) \underset{\mathcal{H}}{\overset{\mathbf{j}}{\rightleftharpoons}} \mathfrak{X}(\pi) \rightleftharpoons 0.$$

The *horizontal* and *vertical projectors* are $\mathbf{h} := \mathcal{H} \circ \mathbf{j}$ and $\mathbf{v} := \mathbf{i} \circ \mathcal{V}$.

A *covariant derivative operator* in $\pi^*\tau$ is a map $D : \mathfrak{X}\left(\widetilde{TM}\right) \times \mathfrak{X}(\pi) \to \mathfrak{X}(\pi)$ satisfying the familiar conditions. Its *deflection* μ is the covariant differential of δ. The vertical deflection μ^v is, roughly speaking, the restriction of μ to the vertical subbundle. The covariant derivative D is said to be *regular* if μ^v has rank n at every point of $\widetilde{TM}$. We say that an Ehresmann connection is *attached to* D if $\mathfrak{X}^h\left(\widetilde{TM}\right) = \operatorname{Ker}\mu$.

We denote by ∇ Berwald's covariant derivative arising from an Ehresmann connection. The horizontal and mixed curvatures of an arbitrary covariant derivative are $\mathbf{R}$ and $\mathbf{P}$, respectively, whereas those of Berwald's derivative are $\overset{\circ}{\mathbf{R}}$ and $\overset{\circ}{\mathbf{P}}$.

The Lie derivative of a vector field $\tilde{Y}$ along π with respect to $X \in \mathfrak{X}(M)$ may be given by

$$\mathcal{L}_X\tilde{Y} = \mathbf{i}^{-1}\left[X^c, \mathbf{i}\tilde{Y}\right].$$

A *Finsler – Minkowski norm* on a finite-dimensional real vector space V is a function $\varphi : V \to \mathbb{R}$ that satisfies the following axioms:

(1) $\varphi(v) > 0$ if $v \neq 0$ *(positivity)*;

(2) if $\lambda > 0$, then $\varphi(\lambda v) = \lambda\varphi(v)$ for all $v \in V$ *(positive homogeneity)*;

(3) φ is smooth over $V \setminus \{0\}$;

(4) if $E := \frac{1}{2}\varphi^2$, then for all $p \in V \setminus \{0\}$ the symmetric bilinear form $g_p := E''(p)$ is non-degenerate.

Proposition. *The metric tensor g of a Finsler – Minkowski norm is positive definite, therefore $(V \setminus \{0\}, g)$ is a Riemannian manifold.*

7.2 Affine and projective vector fields

The Barthel connection of a spray manifold (M, ξ) is used in the results summarized in this section.

A diffeomorphism between two open subsets of M is said to be a *local affine transformation* if its tangent map leaves ξ invariant. If X is a vector field on M whose flow consists of local affine transformations, then X is called an *affine vector field* for ξ. The affine vector fields are characterized by the following

Theorem. *Let X be a vector field on M. The following statements are equivalent:*

(1) *X is an affine vector field;*

(2) $[X^c, \xi] = 0$*;*

(3) $\mathcal{L}_X \nabla = 0$*;*

(4) $\mathcal{L}_{X^c} \mathbf{h} = 0$*;*

(5) $\left(\nabla^h \nabla^h \hat{X}\right)\left(\tilde{Y}, \tilde{Z}\right) = -\overset{\circ}{\mathbf{R}}\left(\hat{X}, \hat{Y}\right)\tilde{Z} + \overset{\circ}{\mathbf{P}}\left(\mathcal{V}X^c, \tilde{Y}\right)\tilde{Z}$ *for any* $\tilde{Y}, \tilde{Z} \in \mathfrak{X}(\overset{\circ}{\tau})$ (generalized Killing equation).

Yano's covariant derivative D induced by ξ in $\overset{\circ}{\tau}{}^*\tau$ is defined by the formula

$$D_\eta \tilde{X} := \nabla_\eta \tilde{X} + \frac{1}{n+1} \operatorname{tr} \overset{\circ}{\mathbf{P}}\left(\mathbf{j}\eta, \tilde{X}\right) \cdot \delta.$$

To sprays are *projectively equivalent* if their geodesics differ only in an increasing parameter transformation. A diffeomorphism between two open subsets of M is a *local projective transformation* if its tangent map takes ξ to another spray projectively equivalent to it. If the flow of $X \in \mathfrak{X}(M)$ consists of local projective transformations, then X is a *projective vector field*. We have characterized projective vector fields as follows:

Theorem. *If $X \in \mathfrak{X}(M)$, the following statements are equivalent:*

(1) *X is a projective vector field;*

(2) *there is a continuous function F on TM, smooth on $\overset{\circ}{T}M$, such that* $[X^c, \xi] = FC$*;*

(3) *there is a smooth function F on $\overset{\circ}{T}M$, positively homogeneous of degree 1, such that*

$$(\mathcal{L}_{X^c}\mathbf{h})(\eta) = \frac{1}{2}(FJ\eta + (J\eta)F \cdot C),\ \eta \in \mathfrak{X}\left(\overset{\circ}{T}M\right);$$

(4) *there is a smooth function F on $\overset{\circ}{T}M$ such that*

$$(\mathcal{L}_X D)\left(\eta, \tilde{Z}\right) = -\frac{1}{2}\left((J\eta)F \cdot \tilde{Z} + \left(\mathbf{i}\tilde{Z}\right) F \cdot \mathbf{j}\eta\right),$$
$$\eta \in \mathfrak{X}\left(\overset{\circ}{T}M\right),\ \tilde{Z} \in \mathfrak{X}(\overset{\circ}{\tau}).$$

7.3 Natural metric covariant derivatives

Let g be a symmetric and non-degenerate tensor of type (0,2) in the bundle $\pi^*\tau$. Then g is said to be a *(generalized) metric.* Using non-degeneracy, the *first Cartan tensor* $\mathcal{C}$ and the *lowered first Cartan tensor* $\mathcal{C}_\flat$ of a generalized metric g are defined by the following formulae:

$$g\left(\mathcal{C}\left(\tilde{X}, \tilde{Y}\right), \tilde{Z}\right) := \mathcal{C}_\flat\left(\tilde{X}, \tilde{Y}, \tilde{Z}\right) := (\nabla^v_{\tilde{X}} g)\left(\tilde{Y}, \tilde{Z}\right) \quad \left(\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{X}(\pi)\right).$$

The one-form $\vartheta_g : \tilde{X} \mapsto g\left(\tilde{X}, \delta\right)$ along π is called the *Lagrange one-form* associated to g. The *absolute energy* of g is $E := \frac{1}{2}g(\delta, \delta)$.

A metric g along π is said to be *variational* if its first Cartan tensor $\mathcal{C}$ is symmetric, *weakly variational* if $\mathcal{C}_\flat\left(\tilde{X}, \tilde{Y}, \delta\right) = \mathcal{C}_\flat\left(\tilde{Y}, \tilde{X}, \delta\right)$ for every $\tilde{X}, \tilde{Y} \in \mathfrak{X}(\pi)$, *normal* if $\mathcal{C}\left(\tilde{X}, \delta\right) = 0$ for every $\tilde{X} \in \mathfrak{X}(\tau)$, and *weakly normal* if $\mathcal{C}_\flat\left(\tilde{X}, \delta, \delta\right) = 0$ for every $\tilde{X} \in \mathfrak{X}(\pi)$. The metric is *Miron regular* [38] if the tensor $A : \tilde{X} \mapsto \tilde{X} + \mathcal{C}\left(\tilde{X}, \delta\right)$ has maximal rank at every point of $\widetilde{TM}$. If $\gamma := \nabla^v\nabla^v E$ is also non-degenerate, g is called *energy-regular.* An energy-regular and homogeneous ($\nabla^v_\delta g = 0$) metric is also mentioned as a *Moór – Vanstone metric.*

Given an Ehresmann connection on $\widetilde{TM}$, there is a unique metric covariant derivative in $\pi^*\tau$ whose vertical torsion vanishes and whose horizontal torsion coincides with the torsion of the given Ehresmann connection.

If $\widetilde{T_pM}$ is simply connected for every $p \in M$, then g is variational if and only if there is a smooth function L on $\widetilde{TM}$ whose Hessian is g, more precisely, $g = \nabla^v\nabla^v L$. In this case, we call L a *Lagrangian.* If $\widetilde{T_pM}$ is simply connected for every $p \in M$, then g is weakly variational if and only if there is a smooth function L on $\widetilde{TM}$ such that $\vartheta_g = d^v L$. If $\widetilde{TM} = \overset{\circ}{T}M$, and g is weakly normal and Miron regular, then E is a possibly indefinite Finsler energy (in the usual sense). Furthermore, $\vartheta_g = d^v E$. Finally, if $\widetilde{TM} = \overset{\circ}{T}M$, and g is normal, then g is itself a Finsler metric.

7.3.1 Theorem. *Let g be a weakly normal Moór – Vanstone metric on $\overset{\circ}{T}M$, ξ the canonical spray belonging to the Finsler energy E, $\mathcal{H}_E$ the Barthel connection, and $\overset{E}{\nabla}$ Berwald's covariant derivative arising from $\mathcal{H}_E$. Let $P \in \mathcal{T}^1_1(\overset{\circ}{\tau})$ be an arbitrary tensor. Consider the Ehresmann connection $\mathcal{H} := \mathcal{H}_E - \mathbf{i} \circ P$ on $\overset{\circ}{T}M$ and the unique metric covariant derivative D along $\overset{\circ}{\tau}$ whose vertical torsion vanishes, and whose horizontal torsion coincides with the torsion of $\mathcal{H}$. The horizontal map $\mathcal{H}$ satisfies the conditions*

(i) $\mathcal{H}\delta = \xi$,

(ii) *$\mathcal{H}$ is conservative with respect to E* $\left(\mathfrak{X}^h\left(\overset{\circ}{T}M\right) \subset \operatorname{Ker} dE\right)$,

(iii) *$\mathcal{H}$ is attached to D*

if and only if

$$P\tilde{X} = -\frac{1}{2}\left(i_{\tilde{X}} \overset{E}{\nabla}_\xi g\right)^\sharp + P_s\tilde{X} + P_a\tilde{X} \quad \left(\tilde{X} \in \mathfrak{X}(\overset{\circ}{\tau})\right),$$

where $P_s, P_a \in \mathcal{T}^1_1(\overset{\circ}{\tau})$ are tensors such that P_s is self-adjoint (with respect to g) and homogeneous of degree 0, P_a is skew-symmetric (with respect to g), and the image of both P_s and P_a is contained in the orthogonal complement of the canonical section δ.

Theorem. *Let g be a positive definite weakly normal metric along $\overset{\circ}{\tau}$. Suppose that the tensor $A : \tilde{X} \mapsto \tilde{X} + \mathcal{C}\left(\tilde{X}, \delta\right)$ has the following property: for a fixed $v \in \overset{\circ}{T}M$, the self-adjoint linear transformation A_v has no (not necessarily different) eigenvalues $\lambda_i, \lambda_j \in \mathbb{R}$ such that $\lambda_i + \lambda_j = 0$. Then there is a unique covariant derivative operator $D : \mathfrak{X}\left(\overset{\circ}{T}M\right) \times \mathfrak{X}(\overset{\circ}{\tau}) \to \mathfrak{X}(\overset{\circ}{\tau})$ and a unique Ehresmann connection $\mathcal{H}$ on $\overset{\circ}{T}M$ such that*

(1) *D is metric,*

(2) *its vertical torsion vanishes,*

(3) *its horizontal torsion with respect to $\mathcal{H}$ vanishes,*

(4) *the horizontal subbundle is contained in the kernel of the deflection of D.*

In particular, if D is regular, then $\mathcal{H}$ is attached to it.

In the special case of a Finsler manifold, this theorem gives an axiomatic characterization of Cartan's covariant derivative:

Proposition. *If $\mathcal{H}$ is an Ehresmann connection on a Finsler manifold and $D : \mathfrak{X}\left(\overset{\circ}{T}M\right) \times \mathfrak{X}(\overset{\circ}{\tau}) \to \mathfrak{X}(\overset{\circ}{\tau})$ is a covariant derivative such that*

(1) *D is metric,*

(2) *its vertical torsion vanishes,*

(3) *its horizontal torsion with respect to $\mathcal{H}$ vanishes,*

(4) *$\mathcal{H}$ is attached to it,*

then D is Cartan's covariant derivative, and $\mathcal{H}$ is the Barthel connection of the Finsler manifold.

7.4 Killing vector fields

Let g be a generalized metric along π. A diffeomorphism between two open subsets of M is a *local isometry* of g if its tangent map leaves g invariant. A vector field $X \in \mathfrak{X}(M)$ is said to be an *infinitesimal isometry* of g if its flow consists of local isometries. A vector field $X \in \mathfrak{X}(M)$ is called a *Killing vector field* if $\mathcal{L}_X g = 0$.

Proposition. *Let $X \in \mathfrak{X}(M)$ be a vector field. Then X is an infinitesimal isometry of g if and only if it is a Killing vector field.*

In Riemannian geometry, the covariant differential of a Killing field is skew-symmetric. This result is generalized as follows:

Proposition. *Suppose that an Ehresmann connection is specified on $\widetilde{TM}$ whose torsion vanishes, and D is the unique metric covariant derivative operator whose vertical and horizontal torsions vanish. Let X be a Killing vector field on M. Then*

$$g\left(D_{\mathcal{H}\tilde{Y}}\hat{X}, \tilde{Z}\right) + g\left(\tilde{Y}, D_{\mathcal{H}\tilde{Z}}\hat{X}\right) + g\left(\mathcal{C}\left(\mathcal{V}X^c, \tilde{Y}\right), \tilde{Z}\right) = 0$$

for any $\tilde{Y}, \tilde{Z} \in \mathfrak{X}(\pi)$.

For any metric g, we introduce the (1,1) tensor $\overset{*}{\mathcal{C}}$ along π by the prescription $\overset{*}{\mathcal{C}} : \tilde{X} \mapsto \mathcal{C}\left(\tilde{X}, \delta\right)$, where $\mathcal{C}$ is the first Cartan tensor of the metric.

Theorem. *Suppose that $\widetilde{T_pM}$ is connected for all $p \in M$, and let g be a weakly variatonal and Miron regular metric on $\widetilde{TM}$ with $\vartheta_g = d^v L$. A vector field X on M is a Killing vector field for g if and only if the function $X^c L$ is a vertical lift and $\mathcal{L}_X \overset{*}{\mathcal{C}} = 0$.*

The metric g does not determine L uniquely, since a vertical lift can be added to L without changing $d^v L$. Moreover, we have

Corollary. *With conditions similar to those in the theorem, if g is defined on the whole TM, and X is a Killing vector field on M, L can be chosen such that $X^c L = 0$.*

Theorem. *Let g be a variational metric defined on the whole TM. A vector field X on M is a Killing vector field if and only if there is a Lagrangian L for g such that $X^c L = 0$.*

Corollary. *If (M, g) is a Finsler manifold, then a vector field X on M is a Killing vector field of g if and only if $X^c E = 0$.*

If g is weakly normal and Miron regular, then a Killing vector field X of g is called a *translation* if every integral curve of X is a geodesic of the Finsler manifold (M, E).

Proposition. *If $X \in \mathfrak{X}(M)$ is a Killing vector field, and $c : I \to M$ is a geodesic of E, then the function*

$$t \in I \mapsto g_{\dot{c}(t)}(X(c(t)), \dot{c}(t))$$

is constant.

The next proposition gives a characterization of translations.

Proposition. *Let X be a Killing vector field of g. Then X is a translation if and only if the function $p \in M \mapsto E(X_p)$ is constant.*

There is a broad class of (even Riemannian) metrics that have no non-trivial translations at all. For example, the hyperbolic plane does not have any. The Killing fields of Poincaré's upper half plane model have the form

$$X = \left[\alpha\left((u^1)^2 - (u^2)^2\right) + \beta u^1 + \gamma\right]\frac{\partial}{\partial u^1} + \left(2\alpha u^1 + \beta\right)u^2\frac{\partial}{\partial u^2}$$

with some $\alpha, \beta, \gamma \in \mathbb{R}$. The upper half-plane coincides with the set of complex numbers with positive imaginary part. Suppose that $\alpha \neq 0$, and introduce the notation $k := \sqrt{|\beta^2/4 - \alpha\gamma|}$. Then the integral curves of X are given by

$$z(t) = -\frac{k}{\alpha}\frac{c\cosh kt - \sinh kt}{c\sinh kt - \cosh kt} - \frac{\beta}{2\alpha} \qquad \text{if } \frac{\beta^2}{4} - \alpha\gamma > 0,$$
$$z(t) = -\frac{k}{\alpha}\frac{c\cos kt + \sin kt}{c\sin kt - \cos kt} - \frac{\beta}{2\alpha} \qquad \text{if } \frac{\beta^2}{4} - \alpha\gamma < 0,$$
$$z(t) = -\frac{1}{\alpha t + c} - \frac{\beta}{2\alpha} \qquad \text{if } \frac{\beta^2}{4} - \alpha\gamma = 0$$

with $c \in \mathbb{C}$ such that $\operatorname{Im} c > 0$, which are no geodesics. That is, however, not surprising, since, if the hyperbolic plane had a non-trivial translation, a geodesic quadrangle with angle sum 2π could be constructed, in contradiction with the Gauss – Bonnet theorem (cf. Figure 6.1).

We have given a new proof for the following classical result:

Proposition. *Suppose that (M, α) is a Riemannian manifold, $X \in \mathfrak{X}(M)$ is a Killing vector field of (M, α) such that $\|X\| < 1$, and $\beta := X^\flat$. Then X is a Killing vector field of the Randers manifold arising from (M, α) by the perturbation with β.*

Suppose that $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a Finsler – Minkowski norm, and $\Omega := \varphi^{-1}\,[0, 1[$ is the unit ball. Consider the Funk metric on Ω (cf. Figure 6.2). The following proposition gives a description of its Killing vector fields:

Proposition. *For a vector field X on Ω the following conditions are equivalent:*

(1) *X is a Killing vector field;*

(2) *for every point $p \in \Omega$ and vector $v \in \mathbb{R}^n$ such that $p + v \in \partial\Omega$, the vector $X(p) + D_v X(p)$ is parallel to the tangent hyperplane of $\partial\Omega$ in $p + v$.*

Chapter 8

Magyar nyelvű összefoglaló (Summary in Hungarian)

8.1 Jelölések

Ebben a disszertációban egy n-dimenziós M összefüggő sokaságon dolgozunk. Ha $\widetilde{TM} \subset TM$ olyan nyílt részhalmaz, amelyre $\tau\left(\widetilde{TM}\right) = M$ teljesül, akkor a τ érintőnyaláb-projekció $\widetilde{TM}$-ra való leszűkítését π-vel jelöljük. A τ-nak a π általi visszahúzottja $\pi^*\tau$. Ezen nyaláb metszeteinek a $C^\infty\left(\widetilde{TM}\right)$-modulusát (amelyeket *$\pi$-menti vektormezőknek* is nevezünk) $\mathfrak{X}(\pi)$ jelöli. A legfontosabb speciális eset: $\widetilde{TM} = \overset{\circ}{T}M$, vagyis az M sokaság nemnulla érintővektorainak nyílt halmaza. A τ projekció $\overset{\circ}{T}M$-re való leszűkítését $\overset{\circ}{\tau}$ jelöli.

Rendelkezésünkre állnak a következő rövid egzakt sorok $\widetilde{TM}$ fölött:

$$0 \to \pi^*TM \xrightarrow{\mathbf{i}} T\widetilde{TM} \xrightarrow{\mathbf{j}} \pi^*TM \to 0 \quad \text{és} \quad 0 \to \mathfrak{X}(\pi) \xrightarrow{\mathbf{i}} \mathfrak{X}\left(\widetilde{TM}\right) \xrightarrow{\mathbf{j}} \mathfrak{X}(\pi) \to 0.$$

Egy M-en adott f sima függvény vertikális és teljes liftjét f^v, ill. f^c jelöli. Egy M-en adott X vektormező vertikális liftje $X^v = \mathbf{i}\hat{X}$, teljes liftje pedig X^c. A $\pi^*\tau$ *kanonikus metszetét*, a $\widetilde{TM}$ fölötti *Liouville-vektormezőt* és a *vertikális endomorfitmust* rendre a

$$\delta(v) := (v, v) \quad \left(v \in \widetilde{TM}\right), \quad C := \mathbf{i}\delta \quad \text{és} \quad J := \mathbf{i} \circ \mathbf{j}$$

összefüggések definiálják. Ha f sima függvény $\widetilde{TM}$-on, akkor $d^v f$-et a következő képlettel definiáljuk:

$$(d^v f)\left(\tilde{X}\right) := \left(\mathbf{i}\tilde{X}\right) f \quad \left(\tilde{X} \in \mathfrak{X}(\pi)\right).$$

Ehresmann-konnexió alatt egy hasított rövid egzakt sort értünk:

$$0 \rightleftharpoons \pi^* TM \underset{\mathcal{V}}{\overset{\mathbf{i}}{\rightleftharpoons}} T\widetilde{TM} \underset{\mathcal{H}}{\overset{\mathbf{j}}{\rightleftharpoons}} \pi^* TM \rightleftharpoons 0, \text{ vagy } 0 \rightleftharpoons \mathfrak{X}(\pi) \underset{\mathcal{V}}{\overset{\mathbf{i}}{\rightleftharpoons}} \mathfrak{X}\left(\widetilde{TM}\right) \underset{\mathcal{H}}{\overset{\mathbf{j}}{\rightleftharpoons}} \mathfrak{X}(\pi) \rightleftharpoons 0.$$

A *horizontális* és *vertikális projektor*: $\mathbf{h} := \mathcal{H} \circ \mathbf{j}$ és $\mathbf{v} := \mathbf{i} \circ \mathcal{V}$.

A $\pi^*\tau$ nyaláb menti *kovariáns deriválás* olyan $D : \mathfrak{X}\left(\widetilde{TM}\right) \times \mathfrak{X}(\pi) \to \mathfrak{X}(\pi)$ leképezés, amely kielégíti a szokásos feltételeket. *Deflexió*ja a δ kovariáns differenciálja, amelyet μ-vel jelölünk. A μ^v vertikális deflexió, durván szólva, a μ-nek a vertikális résznyalábra való leszűkítése. A D kovariáns deriválást *reguláris*nak mondjuk, ha μ^v a $\widetilde{TM}$ minden pontjában n rangú. Egy Ehresmann-konnexiót *D-hez csatolt*nak nevezünk, ha $\mathfrak{X}^h\left(\widetilde{TM}\right) = \operatorname{Ker} \mu$.

Egy Ehresmann-konnexióból származó Berwald-féle kovariáns deriválást ∇-val jelöljük. Egy tetszőleges kovariáns deriválás horizontális és vegyes görbületének a jele $\mathbf{R}$, ill. $\mathbf{P}$, míg egy Berwald-féle deriválás görbületeié $\overset{\circ}{\mathbf{R}}$ és $\overset{\circ}{\mathbf{P}}$.

Egy $\tilde{Y}$ π-menti vektormező $X \in \mathfrak{X}(M)$ szerinti Lie-deriváltját az

$$\mathcal{L}_X \tilde{Y} = \mathbf{i}^{-1}\left[X^c, \mathbf{i}\tilde{Y}\right]$$

előírással adhatjuk meg.

Egy V végesdimenziós valós vektortéren adott *Finsler – Minkowski-norma* alatt olyan $\varphi : V \to \mathbb{R}$ függvényt értünk, amely eleget tesz az alábbi axiómáknak:

(1) $\varphi(v) > 0$, ha $v \neq 0$ *(φ pozitív definit)*;

(2) ha $\lambda > 0$, akkor $\varphi(\lambda v) = \lambda\varphi(v)$ minden $v \in V$ esetén *(φ pozitív homogén)*;

(3) φ sima $V \setminus \{0\}$ fölött;

(4) ha $E := \frac{1}{2}\varphi^2$, akkor a $g_p := E''(p)$ szimmetrikus bilineáris forma minden $p \in V \setminus \{0\}$ esetén nemelfajuló.

Állítás. *Minden Finsler – Minkowski-norma metrikus tenzora pozitív definit, tehát $(V \setminus \{0\}, g)$ Riemann-sokaság.*

8.2 Affin és projektív vektormezők

Az ezen szakaszban összefoglalt eredmények tárgyalásában egy (M, ξ) permetsokaság (spraysokaság) Barthel-konnexióját használjuk.

Az M sokaság két nyílt részhalmaza közti diffeomorfizmust *lokális affin transzformáció*nak nevezzük, ha érintőleképezése ξ-t invariánsan hagyja. Ha X olyan vektormező M-en, amelynek a folyama lokális affin transzformációkból áll, akkor X-et ξ *affin vektormezőjé*nek nevezzük. Az affin vektormezőket jellemzi az alábbi

Tétel. *Legyen X vektormező M-en. A következő kijelentések ekvivalensek:*

(1) *X affin vektormező;*

(2) $[X^c, \xi] = 0$*;*

(3) $\mathcal{L}_X \nabla = 0$*;*

(4) $\mathcal{L}_{X^c} \mathbf{h} = 0$*;*

(5) $\left(\nabla^h \nabla^h \hat{X}\right)\left(\tilde{Y}, \tilde{Z}\right) = -\overset{\circ}{\mathbf{R}}\left(\hat{X}, \tilde{Y}\right)\tilde{Z} + \overset{\circ}{\mathbf{P}}\left(\mathcal{V}X^c, \tilde{Y}\right)\tilde{Z}$ *bármely* $\tilde{Y}, \tilde{Z} \in \mathfrak{X}(\overset{\circ}{\tau})$ *esetén* (általánosított Killing-egyenlet).

A ξ által indukált *Yano-féle kovariáns deriválást* a következő összefüggés értelmezi:

$$D_\eta \tilde{X} := \nabla_\eta \tilde{X} + \frac{1}{n+1} \operatorname{tr} \overset{\circ}{\mathbf{P}}\left(\mathbf{j}\eta, \tilde{X}\right) \cdot \delta.$$

Két permet *projektíven ekvivalens*, ha geodetikusaik legfeljebb csak növekvő (irányítástartó) paramétertranszformáció erejéig különböznek egymástól. Az M két nyílt részhalmaza közti diffeomorfizmus *lokális projektív transzformáció*, ha érintőleképezése ξ-t vele projektíven ekvivalens másik permetbe viszi. Ha $X \in \mathfrak{X}(M)$ folyama lokális projektív transzformációkból áll, akkor X *projektív vektormező*. A projektív vektormezőknek az alább következő jellemzését adtuk:

Tétel. *Ha $X \in \mathfrak{X}(M)$, akkor a következő kijelentések ekvivalensek:*

(1) *X projektív vektormező;*

(2) *létezik olyan folytonos F függvény TM-en, amely sima $\overset{\circ}{T}M$-en, és amelyre* $[X^c, \xi] = FC$ *teljesül;*

(3) *létezik olyan elsőfokú pozitív homogén sima F függvény $\overset{\circ}{T}M$-en, hogy*

$$(\mathcal{L}_{X^c}\mathbf{h})(\eta) = \frac{1}{2}(FJ\eta + (J\eta)F \cdot C), \ \eta \in \mathfrak{X}\left(\overset{\circ}{T}M\right);$$

(4) *létezik olyan sima F függvény $\overset{\circ}{T}M$-en, hogy*

$$(\mathcal{L}_X D)\left(\eta, \tilde{Z}\right) = -\frac{1}{2}\left((J\eta)F \cdot \tilde{Z} + \left(\mathbf{i}\tilde{Z}\right)F \cdot \mathbf{j}\eta\right),$$
$$\eta \in \mathfrak{X}\left(\overset{\circ}{T}M\right), \ \tilde{Z} \in \mathfrak{X}(\overset{\circ}{\tau}).$$

8.3 Természetes metrikus kovariáns deriválások

Legyen g szimmetrikus nemelfajuló (0,2)-típusú tenzor a $\pi^*\tau$ nyalábon. Ekkor g-t *(általánosított) metriká*nak nevezzük. A nemelfajultságot felhasználva, egy g általánosított metrika $\mathcal{C}$ *első Cartan-tenzorát* és $\mathcal{C}_\flat$ *leszállított első Cartan-tenzorát* az alábbi összefüggésekkel értelmezzük:

$$g\left(\mathcal{C}\left(\tilde{X}, \tilde{Y}\right), \tilde{Z}\right) := \mathcal{C}_\flat\left(\tilde{X}, \tilde{Y}, \tilde{Z}\right) := (\nabla^v_{\tilde{X}} g)\left(\tilde{Y}, \tilde{Z}\right) \quad \left(\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{X}(\pi)\right).$$

A π-menti $\vartheta_g : \tilde{X} \mapsto g\left(\tilde{X}, \delta\right)$ 1-formát a g metrika *Lagrange-1-formájának* nevezzük. A metrikához tartozó *abszolút energia* az $E := \frac{1}{2}g(\delta, \delta)$ függvény.

A π-menti g metrikát *variációs*nak mondjuk, ha $\mathcal{C}$ első Cartan-tenzora szimmetrikus; *gyengén variációsnak*, ha $\mathcal{C}_\flat\left(\tilde{X}, \tilde{Y}, \delta\right) = \mathcal{C}_\flat\left(\tilde{Y}, \tilde{X}, \delta\right)$ minden $\tilde{X}, \tilde{Y} \in \mathfrak{X}(\pi)$ esetén; *normális*nak, ha $\mathcal{C}\left(\tilde{X}, \delta\right) = 0$ minden $\tilde{X} \in \mathfrak{X}(\tau)$ esetén; és *gyengén normális*nak, ha $\mathcal{C}_\flat\left(\tilde{X}, \delta, \delta\right) = 0$ minden $\tilde{X} \in \mathfrak{X}(\pi)$ esetén. A metrika *Miron-reguláris* [38], ha az $A : \tilde{X} \mapsto \tilde{X} + \mathcal{C}\left(\tilde{X}, \delta\right)$ tenzor $\widetilde{TM}$ minden pontjában maximális rangú. Ha $\gamma := \nabla^v\nabla^v E$ szintén nemelfajuló, akkor g-t *energiareguláris*nak nevezzük. Egy energiareguláris és homogén ($\nabla^v_\delta g = 0$) metrikát *Moór – Vanstone-metriká*ként is említünk.

Ha adva van $\widetilde{TM}$-on egy Ehresmann-konnexió, akkor egyértelműen létezik egy olyan metrikus kovariáns deriválás a $\pi^*\tau$ nyalábon, amelynek a vertikális torziója eltűnik, horizontális torziója pedig megegyezik az adott Ehresmann-konnexió torziójával.

Ha $\widetilde{T_pM}$ minden $p \in M$ esetén egyszeresen összefüggő, akkor g variációssága ekvivalens egy olyan L sima függvény létezésével $\widetilde{TM}$-on, amelynek a Hesse-formája g, pontosabban: $g = \nabla^v \nabla^v L$. Ebben az esetben L-et *Lagrange-függvény*nek nevezzük. Ha $\widetilde{T_pM}$ egyszeresen összefüggő minden $p \in M$ esetén, akkor g gyengén variációs volta ekvivalens egy olyan L sima függvény létezésével $\widetilde{TM}$-on, amelyre $\vartheta_g = d^v L$ teljesül. Ha $\widetilde{TM} = \overset{\circ}{T}M$, és g gyengén normális és Miron-reguláris, akkor E (esetleg indefinit) Finsler-energia (a szokásos értelemben), továbbá $\vartheta_g = d^v E$. Végül, ha $\widetilde{TM} = \overset{\circ}{T}M$, és g normális, akkor maga a g Finsler-metrika.

Tétel. *Legyen g gyengén normális Moór – Vanstone-metrika, ξ az E Finsler-energiához tartozó permet, $\mathcal{H}_E$ a Barthel-konnexió és $\overset{E}{\nabla}$ a $\mathcal{H}_E$-ből származó Berwald-féle kovariáns deriválás. Legyen $P \in \mathcal{T}^1_1(\overset{\circ}{\tau})$ tetszőleges tenzor. Tekintsük a $\mathcal{H} := \mathcal{H}_E - \mathbf{i} \circ P$ Ehresmann-konnexiót $\overset{\circ}{T}M$-en és azt az egyértelműen létező D kovariáns deriválást $\overset{\circ}{\tau}$ mentén, amelynek a vertikális torziója eltűnik, a horizontális torziója pedig megegyezik $\mathcal{H}$ torziójával. A $\mathcal{H}$ horizonzális leképezés akkor és csak akkor elégíti ki az*

(I) $\mathcal{H}\delta = \xi$,

(II) *$\mathcal{H}$ konzervatív E-re nézve* $\left(\mathfrak{X}^h\left(\overset{\circ}{T}M\right) \subset \operatorname{Ker} dE\right)$,

(III) *$\mathcal{H}$ csatolva van D-hez*

feltételeket, ha léteznek olyan $P_s, P_a \in \mathcal{T}^1_1(\overset{\circ}{\tau})$ tenzorok, hogy

$$P\tilde{X} = -\frac{1}{2}\left(i_{\tilde{X}} \overset{E}{\nabla}_\xi g\right)^\sharp + P_s\tilde{X} + P_a\tilde{X} \quad \left(\tilde{X} \in \mathfrak{X}(\overset{\circ}{\tau})\right),$$

P_s önadjungált (g-re nézve) és nulladfokú homogén, P_a ferdeszimmetrikus (g-re nézve), és mind P_s, mind P_a képhalmaza benne van a δ kanonikus metszet ortogonális komplementerében.

Tétel. *Legyen g pozitív definit és gyengén normális metrika $\overset{\circ}{\tau}$ mentén. Tegyük fel, hogy az $A : \tilde{X} \mapsto \tilde{X} + \mathcal{C}\left(\tilde{X}, \delta\right)$ tenzor rendelkezik a következő tulajdonsággal: rögzített $v \in \overset{\circ}{T}M$ esetén az A_v önadjungált lineáris transzformációnak nincsenek* *olyan $\lambda_i, \lambda_j \in \mathbb{R}$ (nem feltétlenül különböző) sajátértékei, hogy $\lambda_i + \lambda_j = 0$. Akkor egyértelműen létezik olyan $D : \mathfrak{X}\big(\mathring{T}M\big) \times \mathfrak{X}(\mathring{\tau}) \to \mathfrak{X}(\mathring{\tau})$ kovariáns deriválás és olyan $\mathcal{H}$ Ehresmann-konnexió $\mathring{T}M$-en, hogy*

(1) *D metrikus,*

(2) *vertikális torziója eltűnik,*

(3) *$\mathcal{H}$-ra vonatkozó horizontális torziója eltűnik,*

(4) *a D deflexiójának a magtere tartalmazza a horizontális résznyalábot.*

Speciálisan, ha D reguláris, akkor $\mathcal{H}$ csatolva van hozzá.

Abban a speciális esetben, ha g Finsler-metrika, ez a tétel a Cartan-féle kovariáns deriválásnak egy axiomatikus jellemzését adja:

Állítás. *Ha $\mathcal{H}$ Ehresmann-konnexió egy Finsler-sokaságon, és $D : \mathfrak{X}\big(\mathring{T}M\big) \times \mathfrak{X}\big(\mathring{\tau}\big) \to \mathfrak{X}\big(\mathring{\tau}\big)$ olyan kovariáns deriválás, amelyre az alábbiak teljesülnek:*

(1) *D metrikus,*

(2) *a vertikális torziója eltűnik,*

(3) *a $\mathcal{H}$-ra vonatkozó horizontális torziója eltűnik,*

(4) *$\mathcal{H}$ csatolva van hozzá,*

akkor D megegyezik a Cartan-féle kovariáns deriválással, $\mathcal{H}$ pedig a Finsler-sokaság Barthel-konnexiója.

8.4 Killing-vektormezők

Legyen g általánosított metrika π mentén. Az M két nyílt részhalmaza közti diffeomorfizmust *lokális izometriá*nak nevezzük, ha érintőleképezése g-t invariánsan hagyja. Egy $X \in \mathfrak{X}(M)$ vektormezőt *infinitezimális izometriá*nak mondunk, ha folyama lokális izometriákból áll. Egy $X \in \mathfrak{X}(M)$ vektormezőt *Killing-vektormező*nek nevezünk, ha $\mathcal{L}_X g = 0$.

Állítás. *Egy $X \in \mathfrak{X}(M)$ vektormező akkor és csak akkor infinitezimális izometriája g-nek, ha Killing-vektormező.*

A Riemann-geometriában egy Killing-mező kovariáns differenciálja ferdeszimmetrikus. Ez az eredmény a következőképpen általánosítható:

Állítás. *Tegyük fel, hogy adva van $\widetilde{TM}$-on egy torziómentes Ehresmann-konnexió, D pedig az az egyértelműen létező metrikus kovariáns deriválás, amelynek a vertikális és a horizontális torziója eltűnik. Legyen X Killing-vektormező M-en. Akkor*

$$g\left(D_{\mathcal{H}\tilde{Y}}\hat{X},\tilde{Z}\right)+g\left(\tilde{Y},D_{\mathcal{H}\tilde{Z}}\hat{X}\right)+g\left(\mathcal{C}\left(\mathcal{V}X^c,\tilde{Y}\right),\tilde{Z}\right)=0$$

bármely $\tilde{Y},\tilde{Z}\in\mathfrak{X}(\pi)$ esetén.

Tetszőleges g metrika esetén bevezetjük a $\overset{*}{\mathcal{C}}$ (1,1)-típusú tenzort π mentén a $\overset{*}{\mathcal{C}}:\tilde{X}\mapsto\mathcal{C}\left(\tilde{X},\delta\right)$ előírással, ahol $\mathcal{C}$ a metrika első Cartan-tenzora.

Tétel. *Tegyük fel, hogy $\widetilde{T_pM}$ minden $p\in M$ esetén összefüggő, legyen g gyengén variációs Miron-reguláris metrika $\widetilde{TM}$-on, és legyen $\vartheta_g=d^vL$. Az X vektormező M-en akkor és csak akkor Killing-vektormezője g-nek, ha az X^cL függvény vertikális lift, és $\mathcal{L}_X\overset{*}{\mathcal{C}}=0$.*

A g metrika nem határozza meg egyértelműen L-et, hiszen L-hez hozzáadhatunk egy vertikális liftet, anélkül hogy d^vL megváltozna. Fennáll továbbá az alábbi

Következmény. *A tételbelivel megegyező feltételek mellett, ha g ráadásul az egész TM-en értelmezve van, és X Killing-vektormező M-en, akkor L megválasztható úgy, hogy $X^cL=0$ teljesüljön.*

Tétel. *Legyen g az egész TM-en definiált variációs metrika. Az X vektormező M-en akkor és csak akkor Killing-vektormező, ha létezik olyan L Lagrange-függvény g számára, amelyre $X^cL=0$ teljesül.*

Következmény. *Amennyiben (M,g) Finsler-sokaság, úgy az M-en adott X vektormező akkor és csak akkor Killing-vektormezője g-nek, ha $X^cE=0$.*

Ha g gyengén normális és Miron-reguláris, akkor g-nek az X Killing-vektormezőjét *transzlációnak* nevezzük, ha minden integrálgörbéje egyúttal geodetikusa az (M,E) Finsler-sokaságnak.

Állítás. *Ha $X\in\mathfrak{X}(M)$ Killing-vektormező, és $c:I\to M$ geodetikusa E-nek, akkor a*

$$t\in I\mapsto g_{\dot{c}(t)}(X(c(t)),\dot{c}(t))$$

függvény konstans.

A következő állításban egy lehetséges jellemzését adjuk a transzlációknak.

Állítás. *Amennyiben X a g metrika Killing-vektormezője, úgy X akkor és csak akkor transzláció, ha a $p \in M \mapsto E(X_p)$ függvény konstans.*

A metrikák egy széles osztályának egyáltalán nincs triviálistól különböző transzlációja. A Riemann-struktúrák közül ilyen például a hiperbolikus sík. A Poincaré-féle felsőfélsík-modellben a Killing-mezők a következő alakúak:

$$X = \left[\alpha\left((u^1)^2 - (u^2)^2\right) + \beta u^1 + \gamma\right]\frac{\partial}{\partial u^1} + \left(2\alpha u^1 + \beta\right) u^2 \frac{\partial}{\partial u^2}$$

valamely $\alpha, \beta, \gamma \in \mathbb{R}$ számokkal. A felső félsík megegyezik a pozitív képzetes részű komplex számok halmazával. Tegyük fel, hogy $\alpha \neq 0$, és vezessük be a $k := \sqrt{|\beta^2/4 - \alpha\gamma|}$ jelölést. Akkor X integrálgörbéit a következő alakú görbék szolgáltatják:

$$z(t) = -\frac{k}{\alpha}\frac{c\,\mathrm{ch}\,kt - \mathrm{sh}\,kt}{c\,\mathrm{sh}\,kt - \mathrm{ch}\,kt} - \frac{\beta}{2\alpha}, \qquad \text{ha } \frac{\beta^2}{4} - \alpha\gamma > 0;$$
$$z(t) = -\frac{k}{\alpha}\frac{c\cos kt + \sin kt}{c\sin kt - \cos kt} - \frac{\beta}{2\alpha}, \qquad \text{ha } \frac{\beta^2}{4} - \alpha\gamma < 0;$$
$$z(t) = -\frac{1}{\alpha t + c} - \frac{\beta}{2\alpha}, \qquad \text{ha } \frac{\beta^2}{4} - \alpha\gamma = 0;$$

ahol $c \in \mathbb{C}$, és $\mathrm{Im}\, c > 0$. Ezek a görbék nem geodetikusok. Ez nem is meglepő, hiszen ha a hiperbolikus síknak volna triviálistól különböző transzlációja, akkor konstruálni tudnánk rajta egy olyan geodetikus négyszöget, amelynek a belső szögösszege 2π, ami ellentmondana a Gauss–Bonnet-tételnek (lásd 6.1. ábra).

Új bizonyítását adtuk a következő klasszikus eredménynek:

Állítás. *Tegyük fel, hogy (M, α) Riemann-sokaság, és X az (M, α)-nak olyan Killing-vektormezője, amelyre $\|X\| < 1$ teljesül. Legyen továbbá $\beta := X^\flat$. Akkor X az (M, α) Riemann-sokaság β-val való perturbálásával keletkező Randers-sokaságnak is Killing-vektormezője.*

Tegyük fel, hogy $\varphi : \mathbb{R}^n \to \mathbb{R}$ Finsler–Minkowski-norma, amelynek $\Omega := \varphi^{-1}\,[0, 1[$ az egységgömbje. A következő állítás az Ω-n megadott Funk-metrika (6.2. ábra) Killing-vektormezőit jellemzi.

Állítás. *Egy Ω-n adott X vektormezőre a következő feltételek ekvivalensek:*

(1) *X Killing-vektormező;*

(2) *Minden olyan* $p \in \Omega$ *pont és* $v \in \mathbb{R}^n$ *vektor esetén, amelyekre* $p + v \in \partial\Omega$ *teljesül, az* $X(p) + D_v X(p)$ *vektor párhuzamos* $\partial\Omega$*-nak a* $p + v$ *pontbeli érintő-hipersíkjával.*

Appendix: formulae in local coordinates

In this appendix, $\big(\mathcal{U}, (u^i)_{i=1}^n\big)$ will be a chart on M and $\big(\tau^{-1}(\mathcal{U}), (x^i)_{i=1}^n, (y^i)_{i=1}^n\big)$ the induced chart on TM, as described in section 2.1. Every formula should be meant where it has sense, since we shall suppress the deluge of signs of restriction.

The members of the short exact sequence (2.1) act in the following way:

$$\mathbf{i}\left(\widehat{\frac{\partial}{\partial u^i}}\right) = \frac{\partial}{\partial y^i}, \quad \mathbf{j}\left(\frac{\partial}{\partial x^i}\right) = \widehat{\frac{\partial}{\partial u^i}} \quad \text{and} \quad \mathbf{j}\left(\frac{\partial}{\partial y^i}\right) = 0.$$

If $X = X^i \frac{\partial}{\partial u^i}$ is a vector field on M, then we have

$$\hat{X} = \left(X^i\right)^v \widehat{\frac{\partial}{\partial u^i}}, \quad X^v = \left(X^i\right)^v \frac{\partial}{\partial y^i} \quad \text{and} \quad X^c = \left(X^i\right)^v \frac{\partial}{\partial x^i} + y^j \left(\frac{\partial X^i}{\partial u^j}\right)^v \frac{\partial}{\partial y^i}.$$

The canonical section and the Liouville vector field may be described as

$$\delta = y^i \widehat{\frac{\partial}{\partial u^i}}, \quad \text{and} \quad C = y^i \frac{\partial}{\partial y^i}.$$

In section 2.2 we have seen that an Ehresmann connection is completely determined by its horizontal map $\mathcal{H}$, and therefore it is also determined by the functions G_i^j defined by

$$\mathcal{H}\left(\widehat{\frac{\partial}{\partial u^i}}\right) = \frac{\partial}{\partial x^i} - G_i^j \frac{\partial}{\partial y^j}.$$

A covariant derivative operator is characterized by the so-called *connection parameters* or *Christoffel symbols*:

$$D_{\frac{\partial}{\partial x^i}} \widehat{\frac{\partial}{\partial u^j}} = \Gamma^k_{ij} \widehat{\frac{\partial}{\partial u^k}}, \quad D_{\frac{\partial}{\partial y^i}} \widehat{\frac{\partial}{\partial u^j}} = C^k_{ij} \widehat{\frac{\partial}{\partial u^k}}.$$

For Berwald's covariant derivative induced by an Ehresmann connection, these reduce to

$$\nabla_{\frac{\partial}{\partial x^i}} \widehat{\frac{\partial}{\partial u^j}} = G^k_{ij} \widehat{\frac{\partial}{\partial u^k}}, \quad \nabla_{\frac{\partial}{\partial y^i}} \widehat{\frac{\partial}{\partial u^j}} = 0, \quad \text{where} \quad G^k_{ij} := \frac{\partial G^k_i}{\partial y^j}.$$

Now suppose that $X = X^i \frac{\partial}{\partial u^i}$ is a vector field on M and $\tilde{Y} = Y^i \widehat{\frac{\partial}{\partial u^i}}$ is a vector field along π. Then

$$\mathcal{L}_X \tilde{Y} = \left[(X^j)^v \frac{\partial Y^i}{\partial x^j} + y^k \left(\frac{\partial X^j}{\partial u^k} \right)^v \frac{\partial Y^i}{\partial y^j} - Y^j \left(\frac{\partial X^i}{\partial u^j} \right)^v \right] \widehat{\frac{\partial}{\partial u^i}}.$$

If D is a covariant derivative whose vertical part coincides with the canonical v-covariant derivative (or, equivalently, $C^k_{ij} = 0$), then

$$\begin{aligned}(\mathcal{L}_X D) \left(\frac{\partial}{\partial x^i}, \widehat{\frac{\partial}{\partial u^i}} \right) = \Bigg[& (X^l)^v \frac{\partial \Gamma^k_{ij}}{\partial x^l} + y^m \left(\frac{\partial X^l}{\partial u^m} \right)^v \frac{\partial \Gamma^k_{ij}}{\partial y^l} - \Gamma^l_{ij} \left(\frac{\partial X^k}{\partial u^l} \right)^v \\ & + \Gamma^k_{lj} \left(\frac{\partial X^l}{\partial u^i} \right)^v + \left(\frac{\partial^2 X^k}{\partial u^i \partial u^j} \right)^v + \Gamma^k_{il} \left(\frac{\partial X^l}{\partial u^j} \right)^v \Bigg] \widehat{\frac{\partial}{\partial u^k}}.\end{aligned}$$

Among the covariant derivatives we work with, Berwald's and Yano's satisfy this condition.

If ξ is a spray over M, there are functions G^i of class C^1 on $T\mathcal{U}$, smooth on $\overset{\circ}{T}\mathcal{U}$, such that

$$\xi = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}.$$

Then the parameters of the Ehresmann connection associated to ξ are

$$G^j_i = \frac{\partial G^j}{\partial y^i}.$$

These functions are smooth on $\overset{\circ}{T}\mathcal{U}$ and only continuous on $T\mathcal{U}$ in general. With the notations

$$G^k_{ij} := \frac{\partial G^k_i}{\partial y^j}, \quad G^l_{ijk} := \frac{\partial G^l_{ij}}{\partial y^k} \quad \text{and} \quad G_{ij} := G^k_{ijk},$$

the coordinate expression of Yano's covariant derivative is

$$D_{\frac{\partial}{\partial x^i}} \widehat{\frac{\partial}{\partial u^j}} = \left(G^k_{ij} + \frac{1}{n+1} G_{ij} y^k \right) \widehat{\frac{\partial}{\partial u^k}}, \quad D_{\frac{\partial}{\partial y^i}} \widehat{\frac{\partial}{\partial u^j}} = 0.$$

A vector field $X = X^i \frac{\partial}{\partial u^i}$ on M is affine if and only if

$$2 \left(X^j \right)^v \frac{\partial G^i}{\partial x^j} + 2 y^j G^i_k \left(\frac{\partial X^k}{\partial u^j} \right)^v + y^j y^k \left(\frac{\partial^2 X^i}{\partial u^j \partial u^k} \right)^v + 2 G^j \left(\frac{\partial X^i}{\partial u^j} \right) = 0$$

for any $i = 1, \dots, n$. The same vector field is projective if and only if there is a continuous function F on $T\mathcal{U}$, smooth on $\overset{\circ}{T}\mathcal{U}$, such that

$$2 \left(X^j \right)^v \frac{\partial G^i}{\partial x^j} + 2 y^j G^i_k \left(\frac{\partial X^k}{\partial u^j} \right)^v + y^j y^k \left(\frac{\partial^2 X^i}{\partial u^j \partial u^k} \right)^v + 2 G^j \left(\frac{\partial X^i}{\partial u^j} \right) = F y^i$$

for any $i = 1, \dots, n$.

If g is a generalized metric on $\widetilde{TM} \subset TM$, then its components are the following smooth functions on $T\mathcal{U} \cap \widetilde{TM}$:

$$g_{ij} := g \left(\widehat{\frac{\partial}{\partial u^i}}, \widehat{\frac{\partial}{\partial u^j}} \right).$$

Due to the nondegeneracy of g, the matrix (g_{ij}) is invertible in every point of $T\mathcal{U} \cap \widetilde{TM}$, and its iverse matrix is denoted by (g^{ij}).

The components of the first Cartan tensor $\mathcal{C}$ and the lowered first Cartan tensor $\mathcal{C}_\flat$ of g are

$$\mathcal{C}^k_{ij} := g^{kl} \frac{\partial g_{jl}}{\partial y^i} \quad \text{and} \quad \mathcal{C}_{ijk} := \frac{\partial g_{jk}}{\partial y^i},$$

respectively; thus we have

$$\mathcal{C} \left(\widehat{\frac{\partial}{\partial u^i}}, \widehat{\frac{\partial}{\partial u^j}} \right) = \mathcal{C}^k_{ij} \widehat{\frac{\partial}{\partial u^k}} \quad \text{and} \quad \mathcal{C}_\flat \left(\widehat{\frac{\partial}{\partial u^i}}, \widehat{\frac{\partial}{\partial u^j}}, \widehat{\frac{\partial}{\partial u^k}} \right) = \mathcal{C}_{ijk}.$$

The Lagrange one-form acts as

$$\vartheta_g \left(\widehat{\frac{\partial}{\partial u^i}} \right) = g_{ij} y^j,$$

and the absolute energy is $E = \frac{1}{2} g_{ij} y^i y^j$.

The Miron regularity of a metric is equivalent to the (pointwise) regularity of the matrix function

$$A^i_j := \delta^i_j + \mathcal{C}^i_{jk} y^k,$$

and energy-regularity is equivalent to the regularity of

$$\gamma_{ij} := \frac{\partial^2 E}{\partial y^i \partial y^j}.$$

If (M, g) is a Finsler manifold, then its canonical spray has the coefficients

$$G^i = \frac{1}{2} g^{ij} \left(y^k \frac{\partial^2 E}{\partial x^k \partial y^j} - \frac{\partial E}{\partial x^j} \right).$$

If an Ehresmann connection is given on $\widetilde{TM}$ with parameters G^j_i, then the parameters of the unique metric covariant derivative whose vertical torsion vanishes and whose horizontal torsion coincides with the torsion of the Ehresmann connection are

$$C^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial y^i} + \frac{\partial g_{il}}{\partial y^j} - \frac{\partial g_{ij}}{\partial y^l} \right) = \frac{1}{2} g^{kl} \left(\mathcal{C}_{ijl} + \mathcal{C}_{jil} - \mathcal{C}_{lij} \right),$$
$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} - G^m_i \frac{\partial g_{jl}}{\partial y^m} - G^m_j \frac{\partial g_{il}}{\partial y^m} + G^m_l \frac{\partial g_{ij}}{\partial y^m} \right).$$

We can reformulate 5.2.2 with the help of local coordinates in the following way:

Let g be a weakly normal Moór−Vanstone metric, G^i the coefficients of the canonical spray belonging to the Finsler energy E, and G^j_i the parameters of the Barthel connection. Consider an Ehresmann connection $\mathcal{H}$ on $\overset{\circ}{T\mathcal{U}}$ with parameters N^j_i and the unique covariant derivative D along $\overset{\circ}{\tau}$ whose vertical torsion vanishes, and whose horizontal torsion coincides with the torsion of $\mathcal{H}$. Then $\mathcal{H}$ satisfies the conditions

(i) $y^j N^i_j = G^i$,

(ii) $\dfrac{\partial E}{\partial x^i} = N^j_i \dfrac{\partial E}{\partial y^j} \quad (i = 1, \ldots, n)$,

(iii) $\mathcal{H}$ is attached to D

if and only if there are smooth functions P^s_{ij}, P^a_{ij} on $\overset{\circ}{T}\mathcal{U}$ $(i = 1, \ldots, n)$ such that

$$P^s_{ij} = P^s_{ji},\ P^a_{ij} = -P^a_{ji},\ y^j P^s_{ij} = y^j P^a_{ij} = 0,\ y^k \frac{\partial P^s_{ij}}{\partial y^k} = 0, \text{ and}$$

$$N^j_i = \frac{1}{2}\left(G^j_i - g_{ik} g^{jl} G^k_l\right) + \frac{1}{2} g^{jl}\left(y^k \frac{\partial g_{il}}{\partial x^k} - 2G^k \frac{\partial g_{il}}{\partial y^k} + P^s_{il} + P^a_{il}\right).$$

We note that the coordinate expressions of the covariant derivative and the Ehresmann connection described by 5.3.4 are horrid, and therefore we do not include them here.

Finally, a vector field $X = X^i \frac{\partial}{\partial u^i}$ is a Killing vector field of the metric g if and only if

$$\left(X^k\right)^v \frac{\partial g_{ij}}{\partial x^k} + y^l \left(\frac{\partial X^k}{\partial u^l}\right)^v \frac{\partial g_{ij}}{\partial y^k} + g_{ik}\left(\frac{\partial X^k}{\partial u^j}\right)^v + g_{jk}\left(\frac{\partial X^k}{\partial u^i}\right)^v = 0$$

for any $i, j = 1, \ldots, n$.

List of symbols

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GEOMETRIC VECTOR FIELDS OF SPRAY AND METRIC STRUCTURES

Értekezés a doktori (Ph.D.) fokozat megszerzése érdekében
a matematika tudományágban

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