J. Math. Anal. Appl. 535 (2024) 128214

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

Local and global Hölder- and Minkowski-type inequalities for nonsymmetric generalized Bajraktarević means [☆]

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ARTICLE INFO

Article history:
Received 5 June 2023
Available online 13 February 2024
Submitted by J. Bastero

Keywords:
Generalized Bajraktarević mean
Gini mean
Hölder inequality
Minkowski inequality

ABSTRACT

The aim of this paper is to investigate inequalities that are analogous to the Minkowski and Hölder inequalities by replacing the addition and the multiplication by a more general operation, and instead of using power means, generalized Bajraktarević means are considered, in particular, Gini means. A further aim is to introduce the concept of local and global validity of such inequalities and to characterize them in both senses.

1. Introduction

Throughout this paper, the symbols $\mathbb{N}, \mathbb{R}$, and $\mathbb{R}_+$ will stand for the sets of natural (i.e., positive integer), real, and positive real numbers, respectively, and I will always denote a nonempty open real interval. Let $n, k \in \mathbb{N}$. In the sequel, the ith entry of a real vector

$$x := (x_i)_{i\in\{1,\dots,n\}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

will be denoted by x_i, and analogously, the ith row and jth column of a real matrix

☆ The research of the first author was supported by the ÚNKP-22-3 New National Excellence Program of the Ministry for Innovation and Technology from the source of the National Research, Development and Innovation Fund and by the Scientific Council of the University of Nyíregyháza. The research of the second author was supported by the K-134191 NKFIH Grant.

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https://doi.org/10.1016/j.jmaa.2024.128214

$$x := (x_i^j)_{(i,j)\in\{1,\dots,n\}\times\{1,\dots,k\}} = \begin{pmatrix} x_1^1 & \cdots & x_1^k \\ \vdots & & \vdots \\ x_n^1 & \cdots & x_n^k \end{pmatrix} \in \mathbb{R}^{n\times k}$$

will be denoted by x_i and x^j, respectively. For convenience, we identify $\mathbb{R}^{n\times k}$ by $(\mathbb{R}^n)^k$ in the standard manner. We define the *transpose* $[x]^T \in \mathbb{R}^{k\times n}$ of the matrix $x \in \mathbb{R}^{n\times k}$ by

$$[x]^T := (x_j^i)_{(i,j)\in\{1,\dots,k\}\times\{1,\dots,n\}} = \begin{pmatrix} x_1^1 & \cdots & x_n^1 \\ \vdots & & \vdots \\ x_1^k & \cdots & x_n^k. \end{pmatrix}$$

More generally, for a subset of matrices $X \subseteq \mathbb{R}^{n\times k}$, the transpose X^T of X denotes the set $\{[x]^T \mid x \in X\}$. Finally, the *diagonal* $\operatorname{diag}(I^n)$ of I^n is defined by

$$\operatorname{diag}(I^n) := \{(x,\dots,x) \in \mathbb{R}^n \mid x \in I\}.$$

Let us introduce, for $n \in \mathbb{N}$, the *diagonal map* $\Delta_n\colon \mathbb{R} \to \operatorname{diag}(\mathbb{R}^n)$ by

$$\Delta_n(x) := (x,\dots,x) \in \mathbb{R}^n$$

and, for an n-variable function $G\colon I^n \to \mathbb{R}$, the function $G^\Delta\colon I \to \mathbb{R}$ by

$$G^\Delta(x) := G(\Delta_n(x)) \qquad (x \in I).$$

More generally, for $n, k \in \mathbb{N}$, we can define the map $\Delta_n^k\colon \mathbb{R}^k \to (\operatorname{diag}(\mathbb{R}^n))^k \subseteq \mathbb{R}^{n\times k}$: if $y \in \mathbb{R}^k$, then let $\Delta_n^k(y)$ denote the $n\times k$ matrix whose jth column equals $\Delta_n(y_j)$ for $j \in \{1,\dots,k\}$. Whenever a regularity property is assumed to be valid at each point of the domain of a function, then we do not emphasize the set on which the property in question holds.

The celebrated inequalities discovered by Hölder and Minkowski can be formulated in various contexts, for instance, in the setting of power (or Hölder) means.

To recall the standard Hölder(–Rogers) inequality (which was discovered by Rogers in 1888 and by Hölder in 1889), let $p, q > 1$ with $p^{-1} + q^{-1} = 1$. Then, for all $n \in \mathbb{N}$ and $x, y \in \mathbb{R}_+^n$, the inequality

$$\frac{x_1y_1 + \cdots + x_ny_n}{n} \le \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{\frac{1}{p}} \left(\frac{y_1^q + \cdots + y_n^q}{n}\right)^{\frac{1}{q}}$$

is valid. In the particular case $p = q = 2$, this inequality reduces to the so-called Cauchy–Bunyakovsky–Schwarz inequality, which in the above form was established by Cauchy in 1821. Given a real parameter $p \ge 1$, the standard Minkowski inequality (established in 1910) states that the pth power mean is subadditive, i.e., for all $n \in \mathbb{N}$ and $x, y \in \mathbb{R}_+^n$, the inequality

$$\left(\frac{(x_1+y_1)^p + \cdots + (x_n+y_n)^p}{n}\right)^{\frac{1}{p}} \le \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{\frac{1}{p}} + \left(\frac{y_1^p + \cdots + y_n^p}{n}\right)^{\frac{1}{p}}$$

holds.

Briefly, the aim of this paper is to investigate analogous inequalities by replacing the addition and the multiplication by more general operations, and instead of power means, also using generalized Bajraktarević means and, in particular, Gini means. A further aim is to introduce the concept of local and global validity of such inequalities and to characterize them in both senses.

Let $n \in \mathbb{N}$. Given a strictly monotone continuous function $f: I \to \mathbb{R}$ and an n-tuple of positive valued functions $p = (p_1, \dots, p_n): I \to \mathbb{R}_+^n$, the *$n$-variable nonsymmetric generalized Bajraktarević mean* $A_{f,p}: I^n \to I$ is given by the following formula:

$$A_{f,p}(x) := f^{-1}\left(\frac{p_1(x_1)f(x_1) + \cdots + p_n(x_n)f(x_n)}{p_1(x_1) + \cdots + p_n(x_n)}\right) \qquad (x \in I^n).$$

This is an extension of the notion introduced by Bajraktarević in the symmetric setting in [1] and [2], that is, in the case when $p_1 = \cdots = p_n$, i.e., when all the *weight functions* are the same. In the sequel, the sum of these weight functions will be denoted by p_0, i.e., $p_0 := p_1 + \cdots + p_n$. It is easy to see that $A_{f,p}$ is a strict mean, i.e.,

$$\min\{x_1, \dots, x_n\} \leq A_{f,p}(x) \leq \max\{x_1, \dots, x_n\} \qquad (x \in I^n)$$

holds, and the inequalities are strict if $\min\{x_1, \dots, x_n\} < \max\{x_1, \dots, x_n\}$. The equality and comparison problem of nonsymmetric generalized Bajraktarević means have been investigated by the authors in the recent papers [6] and [7].

The main goal of this article is to investigate Hölder- and Minkowski-type inequality problems for the n-variable nonsymmetric generalized Bajraktarević means. More generally, we are going to derive necessary as well as sufficient conditions for the local as well as for the global validity of the functional inequality

$$M_0(\Phi(x_1), \dots, \Phi(x_n)) \leq \Phi(M_1(x^1), \dots, M_k(x^k)), \tag{1}$$

where $n, k \in \mathbb{N}$, for $\alpha \in \{0, \dots, k\}$, $I_\alpha \subseteq \mathbb{R}$ is a nonempty open interval, $I := I_1 \times \cdots \times I_k$, $M_\alpha: I_\alpha^n \to I_\alpha$ is an n-variable mean and $\Phi: I \to I_0$. If there exists an open set $U \subseteq I^n$ such that $\operatorname{diag}(I^n) \subseteq U$ and (1) holds for all $x \in U^T \subseteq \prod_{\alpha=1}^k I_\alpha^n$, then we say that (1) holds in the local sense. If (1) is valid for all $x \in (I^n)^T = \prod_{\alpha=1}^k I_\alpha^n$, then we say that (1) holds in the global sense. Clearly, the global validity of (1) implies its local validity.

Then we consider the particular case of (1) when all the means are n-variable nonsymmetric generalized Bajraktarević means, i.e., we consider the inequality

$$A_{f_0,p^0}(\Phi(x_1), \dots, \Phi(x_n)) \leq \Phi(A_{f_1,p^1}(x^1), \dots, A_{f_k,p^k}(x^k)), \tag{2}$$

where $n, k \in \mathbb{N}$, for $\alpha \in \{0, \dots, k\}$, $f_\alpha: I_\alpha \to \mathbb{R}$ is a strictly monotone continuous function, $p^\alpha: I_\alpha \to \mathbb{R}_+^n$. We obtain necessary as well as sufficient conditions for its validity in the local and also in the global sense.

We mention some important particular cases of (2).

(1) If $k = 1$, $I_0 = I_1 =: I$ and $\Phi(x) = x$, then (2) reduces to the local and global comparison problem of nonsymmetric generalized Bajraktarević means.
(2) If $k \geq 2$, $I_0 = I_1 = \cdots = I_k =: I$, $\Phi(x_1, \dots, x_k) = \frac{1}{k}(x_1 + \cdots + x_k)$, and $f_0 = f_1 = \cdots = f_k =: f$, $p^0 = p^1 = \cdots = p^k =: p$, then (2) means the Jensen convexity of $A_{f,p}$. In this case, (2) is said to be a Jensen-type inequality.
(3) If $k \geq 2$, $I_0 = I_1 = \cdots = I_k = \mathbb{R}_+$, $\Phi(x_1, \dots, x_k) = x_1 + \cdots + x_k$, and $f_0 = f_1 = \cdots = f_k =: f$, $p^0 = p^1 = \cdots = p^k =: p$, then (2) expresses the subadditivity of $A_{f,p}$, which is often called a Minkowski-type inequality.
(4) If $k \geq 2$, $I_0 = I_1 = \cdots = I_k = \mathbb{R}_+$, $\Phi(x_1, \dots, x_k) = x_1 \cdots x_k$, then (2) reduces to a Hölder-type inequality for the means $A_{f_0,p^0}, A_{f_1,p^1}, \dots, A_{f_k,p^k}$.

There are many results related to the Hölder- and Minkowski-type inequalities. Without completeness, we mention the following standard sources and the references therein: Hardy–Littlewood–Pólya [8], Beckenbach–Bellmann [3], Bullen–Mitrinović–Vasić [4], Mitrinović–Pečarić–Fink [16]. We also quote the papers [9–15] by Losonczi and the papers [5,17–20].

2. Hölder- and Minkowski-type inequalities in the local sense

For the investigation of inequality (1), let us introduce the function $F\colon I_1^n \times \cdots \times I_k^n \subseteq \mathbb{R}^{n\times k} \to \mathbb{R}$ by

$$F(x) = F(x^1, \ldots, x^k) := \Phi(M_1(x^1), \ldots, M_k(x^k)) - M_0(\Phi(x_1), \ldots, \Phi(x_n)), \tag{3}$$

where $n, k \in \mathbb{N}$ and also set $I := I_1 \times \cdots \times I_k$.

Remark 1. Observe that, for all $y \in I$, we have

$$F(\Delta_n^k(y)) = 0.$$

Indeed, by using the mean value property of $M_0, M_1, \ldots, M_k$, it follows that

$$\begin{aligned} F(\Delta_n^k(y)) &= F(\Delta_n(y_1), \ldots, \Delta_n(y_k)) = \Phi(M_1(\Delta_n(y_1)), \ldots, M_k(\Delta_n(y_k))) - M_0(\Delta_n(\Phi(y))) \\ &= \Phi(M_1^\Delta(y_1), \ldots, M_k^\Delta(y_k)) - M_0^\Delta(\Phi(y)) = \Phi(y_1, \ldots, y_k) - \Phi(y) = 0. \end{aligned}$$

For the computation of the partial derivatives of F at points of the form $\Delta_n^k(y)$, we formulate the following Lemma. In what follows, $\delta_{\cdot,\cdot}$ will stand for the standard Kronecker symbol.

Lemma 2. *Let $n, k \in \mathbb{N}$ with $n, k \geq 2$ and, for $\alpha \in \{0, \ldots, k\}$, let $I_\alpha \subseteq \mathbb{R}$ be a nonempty open interval and $M_\alpha\colon I_\alpha^n \to I_\alpha$ be an n-variable mean, define $F\colon I_1^n \times \cdots \times I_k^n \to \mathbb{R}$ by (3) and let $\Phi\colon I \to I_0$.*

(i) Assume, for $\alpha \in \{0, \ldots, k\}$, that M_α is partially differentiable on $\operatorname{diag}(I_\alpha^n)$ and that Φ is differentiable. Then, for all $i \in \{1, \ldots, k\}$, $\ell \in \{1, \ldots, n\}$, and $y \in I$,

$$\partial_{\ell+n(i-1)} F(\Delta_n^k(y)) = \partial_i \Phi(y)\big(\partial_\ell M_i^\Delta(y_i) - \partial_\ell M_0^\Delta(\Phi(y))\big).$$

(ii) Assume, for $\alpha \in \{0, \ldots, k\}$, that M_α is twice partially differentiable on $\operatorname{diag}(I_\alpha^n)$ and that Φ is twice differentiable. Then, for all $i, j \in \{1, \ldots, k\}$, $\ell, m \in \{1, \ldots, n\}$, and $y \in I$,

$$\begin{aligned} \partial_{\ell+n(i-1)}\partial_{m+n(j-1)} F(\Delta_n^k(y)) &= \partial_i\partial_j \Phi(y)\big(\partial_m M_j^\Delta(y_j)\partial_\ell M_i^\Delta(y_i) - \delta_{\ell,m}\partial_m M_0^\Delta(\Phi(y))\big) \\ &\quad - \partial_j \Phi(y)\big(\partial_i \Phi(y)\partial_\ell\partial_m M_0^\Delta(\Phi(y)) - \delta_{i,j}\partial_\ell\partial_m M_j^\Delta(y_j)\big). \end{aligned}$$

Proof. (i) Let $i \in \{1, \ldots, k\}$, $\ell \in \{1, \ldots, n\}$, and $y \in I$ be arbitrary. Then the existence of the partial derivative $\partial_{\ell+n(i-1)} F(\Delta_n^k(y))$ and also the formula for it is a direct consequence of the standard chain rule. More precisely,

$$\begin{aligned} \partial_{\ell+n(i-1)} F(\Delta_n^k(y)) &= \partial_{\ell+n(i-1)} F(\Delta_n(y_1), \ldots, \Delta_n(y_k)) \\ &= \partial_i \Phi\big(M_1(\Delta_n(y_1)), \ldots, M_k(\Delta_n(y_k))\big)\partial_\ell M_i(\Delta_n(y_i)) - \partial_\ell M_0(\Phi(y), \ldots, \Phi(y))\partial_i \Phi(y), \end{aligned}$$

which simplifies to the formula stated in (i).

(ii) For $\alpha \in \{0, \ldots, k\}$, there exists an open set $U_\alpha \subseteq I_\alpha^n$ such that $\operatorname{diag}(I_\alpha^n) \subseteq U_\alpha$, the first-order partial derivatives of M_α exist over U_α, and their first-order partial derivatives, i.e., the second-order partial derivatives of M_α, exist on $\operatorname{diag}(I_\alpha^n)$. Using the continuity of Φ, by shrinking the open sets $U_1, \dots, U_k$, we can also assume

$$(\Phi(x_1), \dots, \Phi(x_n)) \in U_0 \tag{4}$$

provided that $x^1 \in U_1, \dots, x^k \in U_k$.

Let $i, j \in \{1, \dots, k\}$, $\ell, m \in \{1, \dots, n\}$, and $y \in I$ be arbitrary. Computing the partial derivative of F over $U_1 \times \cdots \times U_k$ with respect to its $(m + n(j-1))$th variable, i.e., with respect to the variable x_m^j, which is the jth entry of x_m and the mth entry of x^j, we get

$$\partial_{m+n(j-1)}F(x) = \partial_j\Phi(M_1(x^1), \dots, M_k(x^k))\partial_m M_j(x^j) - \partial_m M_0(\Phi(x_1), \dots, \Phi(x_n))\partial_j\Phi(x_m)$$

for all matrices $x \in \mathbb{R}^{n\times k}$ with $x^1 \in U_1, \dots, x^k \in U_k$. Using this equality, we can compute the partial derivative of $\partial_{m+n(j-1)}F$ at $\Delta_n^k(y)$ with respect to its $(\ell + n(i-1))$th variable, i.e., with respect to the variable x_ℓ^i, which is the ith entry of x_ℓ and the ℓth entry of x^i, as follows

$$\begin{aligned}
&\partial_{\ell+n(i-1)}\partial_{m+n(j-1)}F(\Delta_n^k(y)) \\
&\qquad = \partial_i\partial_j\Phi\big(M_1(\Delta_n(y_1)), \dots, M_k(\Delta_n(y_k))\big)\partial_\ell M_i(\Delta_n(y_i))\partial_m M_j(\Delta_n(y_j)) \\
&\qquad\quad + \partial_j\Phi\big(M_1(\Delta_n(y_1)), \dots, M_k(\Delta_n(y_k))\big)\delta_{i,j}\partial_\ell\partial_m M_j(\Delta_n(y_j)) \\
&\qquad\quad - \partial_\ell\partial_m M_0(\Phi(y), \dots, \Phi(y))\partial_i\Phi(y)\partial_j\Phi(y) - \partial_m M_0(\Phi(y), \dots, \Phi(y))\delta_{\ell,m}\partial_i\partial_j\Phi(y).
\end{aligned}$$

Using the mean value property of $M_0, M_1, \dots, M_k$, this equality simplifies to the formula asserted in statement (ii). □

Our first result describes the first-order necessary condition for the validity of (1) in the local sense.

Theorem 3. *Let $n, k \in \mathbb{N}$ with $n, k \geq 2$ and, for $\alpha \in \{0, \dots, k\}$, let $I_\alpha \subseteq \mathbb{R}$ be a nonempty open interval, $M_\alpha\colon I_\alpha^n \to I_\alpha$ be an n-variable mean which is partially differentiable on $\operatorname{diag}(I_\alpha^n)$ and let $\Phi\colon I \to I_0$ be surjective and differentiable with nonvanishing first-order partial derivatives, where $I := I_1 \times \cdots \times I_k$. Assume that inequality (1) holds in the local sense. Then there exist constants $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$ such that, for all $(y_0, y) \in I_0 \times I$ and $\ell \in \{1, \dots, n\}$,*

$$\lambda_\ell = \partial_\ell M_0^\Delta(y_0) = \partial_\ell M_1^\Delta(y_1) = \cdots = \partial_\ell M_k^\Delta(y_k). \tag{5}$$

If, additionally, for some $\alpha \in \{0, \dots, k\}$ and $y_\alpha \in I_\alpha$, the mean M_α is differentiable at $\Delta_n(y_\alpha)$, then $\lambda_1 + \cdots + \lambda_n = 1$ also holds.

Proof. For $\alpha \in \{1, \dots, k\}$, let $U_\alpha \subseteq I_\alpha^n$ be a nonempty open set containing $\operatorname{diag}(I_\alpha^n)$ such that (1) holds for all matrices $x \in \mathbb{R}^{n\times k}$ with $x^1 \in U_1, \dots, x^k \in U_k$. Then, according to (1), F is nonnegative on $U_1 \times \cdots \times U_k$ and, for all $y \in I$, we have $F(\Delta_n^k(y)) = 0$. Therefore, the first-order partial derivatives of F vanish at the point $\Delta_n^k(y)$. In view of Lemma 2, for all $\ell \in \{1, \dots, n\}$ and $i \in \{1, \dots, k\}$, the equality $\partial_{\ell+n(i-1)}F(\Delta_n^k(y)) = 0$ implies that

$$0 = \partial_i\Phi(y)\big(\partial_\ell M_i^\Delta(y_i) - \partial_\ell M_0^\Delta(\Phi(y))\big).$$

Using that the partial derivatives of Φ do not vanish, for all $\ell \in \{1, \dots, n\}$, $i \in \{1, \dots, k\}$, it follows that

$$\partial_\ell M_i^\Delta(y_i) = \partial_\ell M_0^\Delta(\Phi(y)). \tag{6}$$

We will first prove, for all $\ell \in \{1, \dots, n\}$, that the function $\partial_\ell M_0^\Delta$ is locally constant on I_0. Without loss of generality, we may assume that Φ is strictly increasing in its first variable. To verify the assertion, let $y_0 \in I_0$ be arbitrary. Then, by the surjectivity of Φ, there exists $y \in I$ such that $y_0 = \Phi(y)$. Let $y_1' < y_1 < y_1''$ be arbitrarily fixed elements of I_1. Then, by the assumed monotonicity of Φ, we have

$$y_0' := \Phi(y_1', y_2, \dots, y_k) < y_0 = \Phi(y_1, y_2, \dots, y_k) < y_0'' := \Phi(y_1'', y_2, \dots, y_k).$$

Let $u \in]y_0', y_0''[$ be arbitrary. Then, by the continuity of the function Φ, there exists $v \in]y_1', y_1''[$ such that $u = \Phi(v, y_2, \dots, y_k)$. Applying equality (6) for $i = 2$ and $\ell \in \{1, \dots, n\}$ twice, we get

$$\partial_\ell M_0^\Delta(u) = \partial_\ell M_0^\Delta(\Phi(v, y_2 \dots, y_k)) = \partial_\ell M_2^\Delta(y_2) = \partial_\ell M_0^\Delta(\Phi(y_1, y_2 \dots, y_k)) = \partial_\ell M_0^\Delta(y_0).$$

Therefore $\partial_\ell M_0^\Delta$ is constant on $]y_0', y_0''[$, which is a neighborhood of y_0. It proves that $\partial_\ell M_0^\Delta$ is differentiable at y_0 and $(\partial_\ell M_0^\Delta)'(y_0) = 0$. The choice of y_0 in I_0 was arbitrary, hence $(\partial_\ell M_0^\Delta)'$ is identically zero on I_0, which is an open subinterval of $\mathbb{R}$. This implies that $\partial_\ell M_0^\Delta$ is constant on I_0. We will denote this constant by λ_ℓ. Equality (6) then implies that the partial derivatives $\partial_\ell M_1^\Delta, \dots, \partial_\ell M_k^\Delta$ are also equal to the constant λ_ℓ on their domains.

Finally, assume that, for some $\alpha \in \{0, \dots, k\}$ and $y_\alpha \in I_\alpha$, the mean M_α is differentiable at $(\Delta_n(y_\alpha))$. Then, by the mean value property of M_α, we have that $M_\alpha^\Delta(y) = y$ for all $y \in I_\alpha$. Differentiating this equality with respect to y at $y = y_\alpha$, we get

$$\partial_1 M_i^\Delta(y_\alpha) + \dots + \partial_n M_i^\Delta(y_\alpha) = 1,$$

which implies $\lambda_1 + \dots + \lambda_n = 1$. □

Theorem 4. *Let $n, k \in \mathbb{N}$ with $n, k \geq 2$ and, for $\alpha \in \{0, \dots, k\}$, let $I_\alpha \subseteq \mathbb{R}$ be a nonempty open interval, $M_\alpha \colon I_\alpha^n \to I$ be an n-variable mean which is twice differentiable on $\operatorname{diag}(I_\alpha^n)$ and let $\Phi \colon I \to I_0$ be surjective and twice differentiable with nonvanishing first-order partial derivatives, where $I := I_1 \times \dots \times I_k$. Assume that inequality (1) holds in the local sense. Then there exist constants $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$ with $\lambda_1 + \dots + \lambda_n = 1$ such that, for all $(y_0, y) \in I_0 \times I$ and $\ell \in \{1, \dots, n\}$, the equalities in (5) hold. In addition, for all $y \in I$, the $(nk) \times (nk)$ matrix whose $(\ell + n(i-1), m + n(j-1))$th entry, where $i, j \in \{1, \dots, k\}$ and $\ell, m \in \{1, \dots, n\}$, is given by*

$$\partial_i \partial_j \Phi(y)(\lambda_m \lambda_\ell - \delta_{\ell,m}\lambda_m) - \partial_i \Phi(y) \partial_j \Phi(y) \partial_\ell \partial_m M_0^\Delta(\Phi(y)) + \delta_{i,j} \partial_j \Phi(y) \partial_\ell \partial_m M_j^\Delta(y_j) \tag{7}$$

is positive semidefinite.

Proof. For $\alpha \in \{1, \dots, k\}$, let $U_\alpha \subseteq I_\alpha^n$ be a nonempty open set containing $\operatorname{diag}(I_\alpha^n)$ such that (1) and (4) hold for all matrices $x \in \mathbb{R}^{n \times k}$ with $x^1 \in U_1, \dots, x^k \in U_k$. Then, using Theorem 3, condition (5) is valid with some nonnegative constants $\lambda_1, \dots, \lambda_n$ satisfying also $\lambda_1 + \dots + \lambda_n = 1$.

According to (1), F is nonnegative on $U_1 \times \dots \times U_k$ and, for all $y \in I$, we have $F(\Delta_n^k(y)) = 0$, that is, F has a (local) minimum at $\Delta_n^k(y)$. Therefore, its second derivative, i.e., the $(nk) \times (nk)$ matrix $\big(\partial_\alpha \partial_\beta F(\Delta_n^k(y))\big)$ is positive semidefinite. In view of Lemma 2, for all $y \in I_1 \times \dots \times I_k$, it follows that the $(nk) \times (nk)$ matrix whose $(\ell + n(i-1), m + n(j-1))$th entry, where $i, j \in \{1, \dots, k\}$ and $\ell, m \in \{1, \dots, n\}$, is given by

$$\partial_i \partial_j \Phi(y)\big(\partial_m M_j^\Delta(y_j) \partial_\ell M_i^\Delta(y_i) - \delta_{\ell,m} \partial_m M_0^\Delta(\Phi(y))\big) - \partial_j \Phi(y)\big(\partial_i \Phi(y) \partial_\ell \partial_m M_0^\Delta(\Phi(y)) - \delta_{i,j} \partial_\ell \partial_m M_j^\Delta(y_j)\big)$$

is positive semidefinite. Applying the equalities from (5), the statement follows. □

3. Hölder- and Minkowski-type inequalities for nonsymmetric generalized Bajraktarević means

In order to apply the results from the previous section for nonsymmetric generalized Bajraktarević means, we need to compute their partial derivatives on $\operatorname{diag}(I^n)$. For this aim, we recall the following result, which was obtained by the authors in [6].

Lemma 5. *Let $n, k \in \mathbb{N}, d \in \{1,2\}$, let $f\colon I \to \mathbb{R}$ be a d times differentiable function with a nonvanishing first derivative, $p = (p_1, \dots, p_n)\colon I \to \mathbb{R}_+^n$ and set $p_0 := p_1 + \dots + p_n$. Then we have the following assertions.*

(i) If $d = 1$ and p is continuous, then, for all $\ell \in \{1, \dots, n\}$, the first-order partial derivative $\partial_\ell A_{f,p}$ exists on $\operatorname{diag}(I^n)$ and

$$\partial_\ell A_{f,p}^{\Delta} = \frac{p_\ell}{p_0}.$$

(ii) If $d = 2$ and p is continuously differentiable, then, for all $\ell, m \in \{1, \dots, n\}$, the second-order partial derivatives $\partial_\ell^2 A_{f,p}$ and $\partial_\ell \partial_m A_{f,p}$ exist on $\operatorname{diag}(I^n)$ and

$$\partial_\ell^2 A_{f,p}^{\Delta} = 2\frac{p_\ell'(p_0 - p_\ell)}{p_0^2} + \frac{p_\ell(p_0 - p_\ell)}{p_0^2} \cdot \frac{f''}{f'}, \qquad \partial_\ell \partial_m A_{f,p}^{\Delta} = -\frac{(p_\ell p_m)'}{p_0^2} - \frac{p_\ell p_m}{p_0^2} \cdot \frac{f''}{f'} \qquad (\ell \neq m).$$

Theorem 6. *Let $n, k \in \mathbb{N}$ with $n, k \geq 2$ and, for $\alpha \in \{0, \dots, k\}$, let $I_\alpha \subseteq \mathbb{R}$ be a nonempty open interval, $f_\alpha\colon I_\alpha \to \mathbb{R}$ be a differentiable function with a nonvanishing first derivative and let $p^\alpha = (p_1^\alpha, \dots, p_n^\alpha)\colon I_\alpha \to \mathbb{R}_+^n$ be continuous, set $p_0^\alpha := p_1^\alpha + \dots + p_n^\alpha$ and denote $I := I_1 \times \dots \times I_k$. Let $\Phi\colon I \to I_0$ be surjective and differentiable with nonvanishing first-order partial derivatives. Assume that inequality (2) holds in the local sense. Then there exist constants $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$ with $\lambda_1 + \dots + \lambda_n = 1$ such that, for all $\alpha \in \{0, \dots, k\}$ and $\ell \in \{1, \dots, n\}$,*

$$p_\ell^\alpha = \lambda_\ell p_0^\alpha \tag{8}$$

holds on I_α. If, additionally, for $\alpha \in \{0, \dots, k\}$, f_α is twice differentiable, p^α is continuously differentiable and Φ is twice differentiable, then the $k \times k$ matrix $\Gamma(y)$ given by

$$\Gamma_{i,j}(y) := \left(-\partial_i \partial_j \Phi(y) - \partial_j \Phi(y) \partial_i \Phi(y) \left(2\frac{(p_0^0)'}{p_0^0} + \frac{f_0''}{f_0'} \right)(\Phi(y)) + \delta_{i,j} \partial_j \Phi(y) \left(2\frac{(p_0^j)'}{p_0^j} + \frac{f_j''}{f_j'} \right)(y_j) \right)_{i,j=1}^{k} \tag{9}$$

is positive semidefinite for all $y \in I$.

Proof. For $\alpha \in \{0, \dots, k\}$, let $M_\alpha = A_{f_\alpha, p^\alpha}$ and apply Theorem 4 to this setting. Then M_α is partially differentiable on $\operatorname{diag}(I_\alpha^n)$ and inequality (1) holds in the local sense. According to the first assertion of Theorem 4 and by the first statement of Lemma 5, there exist $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$ such that, for all $(y_0, y) \in I_0 \times I$ and $\ell \in \{1, \dots, n\}$, the equalities in (5) hold, i.e., for $\alpha \in \{0, \dots, k\}$,

$$\frac{p_\ell^\alpha}{p_0^\alpha}(y_\alpha) = \partial_\ell A_{f_\alpha, p^\alpha}^{\Delta}(y_\alpha) = \lambda_\ell.$$

This shows that (8) is valid on I_α for all $\alpha \in \{0, \dots, k\}$ and $\ell \in \{1, \dots, n\}$. In view of the definition of p_0^α, these equalities imply that $\lambda_1 + \dots + \lambda_n = 1$ is also valid.

Assume now that, additionally, for $\alpha \in \{0, \dots, k\}$, f_α is twice differentiable, p^α is continuously differentiable and Φ is twice differentiable. Using (8), according to the second assertion of Lemma 5, we have that

$$\partial_\ell^2 A^{\Delta}_{f_\alpha,p^\alpha} = 2\frac{(p_\ell^\alpha)'(p_0^\alpha - p_\ell^\alpha)}{(p_0^\alpha)^2} + \frac{p_\ell^\alpha(p_0^\alpha - p_\ell^\alpha)}{(p_0^\alpha)^2}\cdot\frac{f_\alpha''}{f_\alpha'} = \lambda_\ell(1-\lambda_\ell)\bigg(2\frac{(p_0^\alpha)'}{p_0^\alpha} + \frac{f_\alpha''}{f_\alpha'}\bigg),$$

$$\partial_\ell\partial_m A^{\Delta}_{f_\alpha,p^\alpha} = -\frac{(p_\ell^\alpha p_m^\alpha)'}{(p_0^\alpha)^2} - \frac{p_\ell^\alpha p_m^\alpha}{(p_0^\alpha)^2}\cdot\frac{f_\alpha''}{f_\alpha'} = -\lambda_\ell\lambda_m\bigg(2\frac{(p_0^\alpha)'}{p_0^\alpha} + \frac{f_\alpha''}{f_\alpha'}\bigg) \qquad (\ell \neq m).$$

Therefore, for all $\alpha \in \{0, \dots, k\}$ and $\ell, m \in \{1, \dots, n\}$,

$$\partial_\ell\partial_m A^{\Delta}_{f_\alpha,p^\alpha} = \lambda_m(\delta_{\ell,m} - \lambda_\ell)\bigg(2\frac{(p_0^\alpha)'}{p_0^\alpha} + \frac{f_\alpha''}{f_\alpha'}\bigg). \tag{10}$$

By the second assertion of Theorem 4, for all $y \in I$, the $(nk)\times(nk)$ matrix whose $(\ell + n(i-1), m + n(j-1))$th entry, where $i, j \in \{1, \dots, k\}$ and $\ell, m \in \{1, \dots, n\}$, is given by (7) is positive semidefinite. Using formula (10), we can conclude that the matrix whose $(\ell + n(i-1), m + n(j-1))$th entry is given by

$$\lambda_m(\delta_{\ell,m} - \lambda_\ell)\Gamma_{i,j}(y)$$

is positive semidefinite.

If a matrix is positive semidefinite, then every minor of the matrix is also positive semidefinite. Therefore, the $k \times k$ submatrix with entries $(1 + n(i-1), 1 + n(j-1))$th, where $i, j \in \{1, \dots, k\}$, is also positive semidefinite, which implies the statement. $\square$

In the next result, we reformulate the positive semidefiniteness condition from the above theorem in terms of a convexity property.

Theorem 7. *Let $n, k \in \mathbb{N}$ with $n, k \geq 2$ and, for $\alpha \in \{0, \dots, k\}$, let $I_\alpha \subseteq \mathbb{R}$ be a nonempty open interval, $f_\alpha\colon I_\alpha \to \mathbb{R}$ be a twice differentiable function with a nonvanishing first derivative and let $p^\alpha = (p_1^\alpha, \dots, p_n^\alpha)\colon I_\alpha \to \mathbb{R}_+^n$ be continuously differentiable, set $p_0^\alpha := p_1^\alpha + \dots + p_n^\alpha$ and denote $I := I_1 \times \dots \times I_k$. Let $\Phi\colon I \to I_0$ be surjective and twice differentiable with nonvanishing first-order partial derivatives. Assume that inequality (2) holds in the local sense. Finally, for $\alpha \in \{0, \dots, k\}$, define the function $\varphi_\alpha\colon I_\alpha \to \mathbb{R}$ and then $\varphi\colon I \to \mathbb{R}^k$ by*

$$\varphi_\alpha := \int (p_0^\alpha)^2 f_\alpha' \qquad \textit{and} \qquad \varphi(y) := (\varphi_1(y_1), \dots, \varphi_k(y_k)).$$

Then $\varphi_1, \dots, \varphi_k$ and φ are twice differentiable and invertible functions and the map $\Psi\colon \varphi(I) \to \mathbb{R}$ defined by

$$\Psi(u) := \varphi_0(\Phi(\varphi^{-1}(u)))$$

is concave if $f_0' > 0$ and convex if $f_0' < 0$.

Proof. According to Theorem 6, our assumptions imply that the matrix-valued map $\Gamma\colon I \to \mathbb{R}^{k\times k}$ defined by (9) has positive semidefinite values.

Without loss of generality, we can assume that $f_0' > 0$. Let $\alpha \in \{0, \dots, k\}$. Then the integrand in the definition of φ_α is either positive everywhere or negative everywhere, therefore φ_α is a twice differentiable function with a nonvanishing first derivative, hence it is strictly monotone and it has a twice differentiable inverse $\varphi_\alpha^{-1}\colon \varphi_\alpha(I_\alpha) \to I_\alpha$. Furthermore, we have that

$$\frac{\varphi_\alpha''}{\varphi_\alpha'} = \frac{((p_0^j)^2 f_\alpha')'}{(p_0^j)^2 f_\alpha'} = \frac{2p_0^j(p_0^j)' f_\alpha' + (p_0^j)^2 f_\alpha''}{(p_0^j)^2 f_\alpha'} = 2\frac{(p_0^j)'}{p_0^j} + \frac{f_\alpha''}{f_\alpha'} \qquad (\alpha \in \{0, \dots, k\}). \tag{11}$$

It follows from the definition of φ that

$$\varphi^{-1}(u) = (\varphi_1^{-1}(u_1), \dots, \varphi_k^{-1}(u_k)) \qquad (u \in \varphi_1(I_1) \times \cdots \times \varphi_k(I_k)).$$

Thus, it is clear that φ and its inverse are also twice differentiable maps.

In order to show that Ψ is concave, we will prove that Ψ'' is negative semidefinite over $\varphi(I)$. First, we compute the first and then the second-order partial derivatives of Ψ. For $i, j \in \{0, \dots, k\}$ and $u \in \varphi(I)$, using standard calculus rules, we obtain

$$\partial_j \Psi(u) = \varphi_0'(\Phi(\varphi^{-1}(u))) \cdot \partial_j \Phi(\varphi^{-1}(u)) \cdot \frac{1}{\varphi_j'(\varphi_j^{-1}(u_j))}$$

and

$$\begin{aligned}
\partial_i \partial_j \Psi(u) &= \varphi_0''(\Phi(\varphi^{-1}(u))) \cdot \partial_i \Phi(\varphi^{-1}(u)) \cdot \partial_j \Phi(\varphi^{-1}(u)) \cdot \frac{1}{\varphi_i'(\varphi_i^{-1}(u_i))} \cdot \frac{1}{\varphi_j'(\varphi_j^{-1}(u_j))} \\
&\quad + \varphi_0'(\Phi(\varphi^{-1}(u))) \cdot \partial_i \partial_j \Phi(\varphi^{-1}(u)) \cdot \frac{1}{\varphi_i'(\varphi_i^{-1}(u_i))} \cdot \frac{1}{\varphi_j'(\varphi_j^{-1}(u_j))} \\
&\quad - \delta_{i,j} \varphi_0'(\Phi(\varphi^{-1}(u))) \cdot \partial_j \Phi(\varphi^{-1}(u)) \cdot \frac{\varphi_j''(\varphi_j^{-1}(u_j))}{\varphi_j'(\varphi_j^{-1}(u_j))^3} \\
&= \frac{\varphi_0'(\Phi(\varphi^{-1}(u)))}{\varphi_i'(\varphi_i^{-1}(u_i)) \cdot \varphi_j'(\varphi_j^{-1}(u_j))} \Big(\partial_i \Phi(\varphi^{-1}(u)) \cdot \partial_j \Phi(\varphi^{-1}(u)) \cdot \frac{\varphi_0''}{\varphi_0'}(\Phi(\varphi^{-1}(u)) \\
&\quad + \partial_i \partial_j \Phi(\varphi^{-1}(u)) - \delta_{i,j} \partial_j \Phi(\varphi^{-1}(u)) \cdot \frac{\varphi_j''}{\varphi_j'}(\varphi_j^{-1}(u_j)) \Big).
\end{aligned}$$

Now using the equalities in (11) and (9), it follows that

$$\partial_i \partial_j \Psi(u) = \frac{\varphi_0'(\Phi(\varphi^{-1}(u)))}{\varphi_i'(\varphi_i^{-1}(u_i)) \cdot \varphi_j'(\varphi_j^{-1}(u_j))} \cdot (-\Gamma_{i,j}(\varphi^{-1}(u))).$$

Therefore, for all $u \in \varphi(I)$, we obtain that $\Psi''(u) = (\partial_i \partial_j \Psi(u))_{i,j=1}^k$ is negative semidefinite. This implies that Ψ is concave on $\varphi(I)$. □

Remark 8. It can be seen from the above argument that the concavity of the auxiliary function Ψ is not merely a consequence of the positive semidefiniteness of the matrix-valued function Γ but, in fact, it is equivalent to it. On the other hand, if all the weight functions are equal to constant 1, then $\varphi_\alpha = f_\alpha$ and, in this case, according to the theory of quasiarithmetic means (cf. [8]), the concavity of the function Ψ is also sufficient for inequality (2) to be valid in the global sense.

The following results establish sufficient conditions for inequality (2) to be valid in the local as well as in the global sense.

Theorem 9. *Let $k \in \mathbb{N}$ and, for $\alpha \in \{0, \dots, k\}$, let $I_\alpha \subseteq \mathbb{R}$ be a nonempty open interval, $f_\alpha \colon I_\alpha \to \mathbb{R}$ be differentiable with a nonvanishing derivative, $p_0^\alpha \colon I_\alpha \to \mathbb{R}_+$ and denote $I := I_1 \times \cdots \times I_k$. Furthermore, let $\Phi \colon I \to I_0$ be partially differentiable. Assume that there exists an open set $V \subseteq I^2$ containing $\operatorname{diag}(I^2)$ such that, for all $(u, y) \in V$, the inequality*

$$\frac{p_0^0(\Phi(y))(f_0(\Phi(y)) - f_0(\Phi(u)))}{p_0^0(\Phi(u)) f_0'(\Phi(u))} \le \sum_{j=1}^k \partial_j \Phi(u) \frac{p_0^j(y_j)(f_j(y_j) - f_j(u_j))}{p_0^j(u_j) f_j'(u_j)} \tag{12}$$

holds. Then, for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}_+^n$ with $\lambda_1 + \cdots + \lambda_n = 1$, the inequality

$$A_{f_0,p_0^0\lambda}(\Phi(x_1),\dots,\Phi(x_n)) \le \Phi(A_{f_1,p_0^1\lambda}(x^1),\dots,A_{f_k,p_0^k\lambda}(x^k)) \tag{13}$$

is valid in the local sense.

Proof. Let $n \in \mathbb{N}$, $\lambda \in \mathbb{R}_+^n$ with $\lambda_1 + \cdots + \lambda_n = 1$ be fixed and construct the set $U \subseteq I^n$ as follows:

$$U := \bigcap_{i=1}^{n} \{x \in \mathbb{R}^{n\times k} \colon [\,x\,]^T \in I^n,\ (A_{f_1,p_0^1\lambda}(x^1),\dots,A_{f_k,p_0^k\lambda}(x^k),x_i^1,\dots,x_i^k) \in V\}. \tag{14}$$

Then, due to the continuity of the mean $A_{f_\alpha,p_0^\alpha\lambda}$, each member of the intersection is open, and hence so is U. On the other hand, if $[\,x\,]^T \in \operatorname{diag}(I^n)$, then, for all $\alpha \in \{1,\dots,k\}$, we have $x_1^\alpha = \cdots = x_n^\alpha = A_{f_\alpha,p_0^\alpha\lambda}(x^\alpha)$, whence, by the properties of V, $(A_{f_1,p_0^1\lambda}(x^1),\dots,A_{f_k,p_0^k\lambda}(x^k),x_i^1,\dots,x_i^k) \in V$ holds for all $i \in \{1,\dots,n\}$. This shows that U contains $\operatorname{diag}(I^n)$.

We now prove that, for all $x \in \mathbb{R}^{n\times k}$ with $[\,x\,]^T \in U$, inequality (13) is valid. Let us define, for $\alpha \in \{1,\dots,k\}$,

$$y_\alpha := A_{f_\alpha,p_0^\alpha\lambda}(x^\alpha)$$

and set $u \in I$. As a consequence of this definition, it follows that

$$\sum_{i=1}^{n} \lambda_i p_0^\alpha(x_i^\alpha)(f_\alpha(x_i^\alpha) - f_\alpha(u_\alpha)) = 0 \qquad (\alpha \in \{1,\dots,k\}). \tag{15}$$

On the other hand, for all $\alpha \in \{1,\dots,k\}$ and $i \in \{1,\dots,n\}$, we have that $(u_1,\dots,u_k,x_i^1,\dots,x_i^k) \in V$. Therefore, we can apply (12) with $(u_1,\dots,u_k,y_1,\dots,y_k) := (u_1,\dots,u_k,x_i^1,\dots,x_i^k)$. Then multiplying each inequality by λ_i, summing up the inequalities so obtained, and using the identities in (15), we obtain

$$\begin{aligned}\sum_{i=1}^{n} \frac{\lambda_i p_0^0(\Phi(x_i))(f_0(\Phi(x_i) - f_0(\Phi(u)))}{p_0^0(\Phi(u))f_0'(\Phi(u))} &\le \sum_{i=1}^{n}\sum_{\alpha=1}^{k} \partial_\alpha\Phi(u)\frac{\lambda_i p_0^\alpha(x_i^\alpha)(f_\alpha(x_i^\alpha) - f_\alpha(u_\alpha))}{p_0^\alpha(u_\alpha)f_\alpha'(u_\alpha)} \\ &= \sum_{\alpha=1}^{k} \frac{\partial_\alpha\Phi(u)}{p_0^\alpha(u_\alpha)f_\alpha'(u_\alpha)} \sum_{i=1}^{n} \lambda_i p_0^\alpha(x_i^\alpha)(f_\alpha(x_i^\alpha) - f_\alpha(u_\alpha)) = 0.\end{aligned}$$

Therefore,

$$\sum_{i=1}^{n} \frac{\lambda_i p_0^0(\Phi(x_i))(f_0(\Phi(x_i) - f_0(\Phi(u)))}{p_0^0(\Phi(u))f_0'(\Phi(u))} \le 0.$$

Assume that f_0' is positive. Then f_0 is strictly increasing and the above inequality is equivalent to

$$\sum_{i=1}^{n} \lambda_i p_0^0(\Phi(x_i))(f_0(\Phi(x_i) - f_0(\Phi(u))) \le 0. \tag{16}$$

Rearranging this inequality, we obtain

$$\frac{\sum_{i=1}^{n} \lambda_i p_0^0(\Phi(x_i))f_0(\Phi(x_i))}{\sum_{i=1}^{n} \lambda_i p_0^0(\Phi(x_i))} \le f_0(\Phi(u)). \tag{17}$$

Applying f_0^{-1} side by side and using that f_0^{-1} is strictly increasing, we can conclude that

$$\begin{aligned}A_{f_0,p_0^0\lambda}(\Phi(x_1),\ldots,\Phi(x_n)) = f_0^{-1}\left(\frac{\sum_{i=1}^n \lambda_i p_0^0(\Phi(x_i))f_0(\Phi(x_i))}{\sum_{i=1}^n \lambda_i p_0^0(\Phi(x_i))}\right)\\ \leq \Phi(u) = \Phi(A_{f_1,p_0^1\lambda}(x^1),\ldots,A_{f_k,p_0^k\lambda}(x^k)),\end{aligned}$$

which completes the proof of inequality (13). In the case, when f_0' is everywhere negative, the inequalities (16) and (17) are reversed, however f_0^{-1} is strictly decreasing, thus we arrive at the same conclusion. $\square$

Remark 10. In view of Theorem 6, the weight functions of the generalized nonsymmetric Bajraktarević means appearing in (2) necessarily are of the form given by (8). Therefore, the local as well as the global validity of (2) immediately follows from the local as well as the global validity of (13), respectively.

Theorem 11. *Let $k \in \mathbb{N}$ and, for $\alpha \in \{0,\ldots,k\}$, let $I_\alpha \subseteq \mathbb{R}$ be a nonempty open interval, $f_\alpha\colon I_\alpha \to \mathbb{R}$ be differentiable with a nonvanishing derivative, $p_0^\alpha\colon I_\alpha \to \mathbb{R}_+$ and let $\Phi\colon I \to I_0$ be partially differentiable, where $I := I_1 \times \cdots \times I_k$. Assume, for all $u, y \in I$, that inequality (12) is satisfied. Then, for all $n \in \mathbb{N}$, $\lambda \in \mathbb{R}_+^n$ with $\lambda_1 + \cdots + \lambda_n = 1$, inequality (13) holds in the global sense.*

Proof. If (12) is satisfied for all $u, y \in I$, then the condition of the previous theorem is validated with the open set $V := I^2$ and the open set U constructed by (14) equals I^n. Hence inequality (13) holds for all $x \in \mathbb{R}^{n\times k}$ with $[\,x\,]^T \in U$, i.e., it holds in the global sense. $\square$

The next result establishes a necessary condition for (12) to be satisfied in the local sense.

Theorem 12. *Let $k \in \mathbb{N}$ and, for $\alpha \in \{0,\ldots,k\}$, let $I_\alpha \subseteq \mathbb{R}$ be a nonempty open interval, $f_\alpha\colon I_\alpha \to \mathbb{R}$ be twice differentiable with a nonvanishing first derivative and $p_0^\alpha\colon I_\alpha \to \mathbb{R}_+$ be twice differentiable. In addition, let $\Phi\colon I \to I_0$ be twice differentiable, where $I := I_1 \times \cdots \times I_k$. Assume that there exists an open set $V \subseteq I^2$ with $\operatorname{diag}(I^2) \subseteq V$ such that (12) is satisfied for all $(u, y) \in V$. Then the matrix-valued function $\Gamma\colon I \to \mathbb{R}^{k\times k}$ defined by (9) takes positive semidefinite values.*

Proof. If (12) is satisfied for all $(u, y) \in V$ then, for all fixed $y \in I$, the map $\Psi_y\colon I \to \mathbb{R}$ defined as

$$\Psi_y(u) := \sum_{\alpha=1}^k \partial_\alpha\Phi(y)\frac{p_0^\alpha(u_\alpha)(f_\alpha(u_\alpha) - f_\alpha(y_\alpha))}{p_0^\alpha(y_\alpha)f_\alpha'(y_\alpha)} - \frac{p_0^0(\Phi(u))(f_0(\Phi(u)) - f_0(\Phi(y)))}{p_0^0(\Phi(y))f_0'(\Phi(y))} \tag{18}$$

has a local minimum at $u = y$. This function is twice differentiable according to our assumptions. Therefore, the second derivative matrix of it at $u = y$ is positive semidefinite. For $i \in \{1,\ldots,k\}$, we have

$$\begin{aligned}\partial_i\Psi_y(u) = \frac{\partial_i\Phi(y)}{p_0^i(y_i)f_i'(y_i)}\big((p_0^i)'(u_i)(f_i(u_i) - f_i(y_i)) + p_0^i(u_i)f_i'(u_i)\big)\\ - \frac{\partial_i\Phi(u)}{p_0^0(\Phi(y))f_0'(\Phi(y))}\big((p_0^0)'(\Phi(u))(f_0(\Phi(u)) - f_0(\Phi(y))) + p_0^0(\Phi(u))f_0'(\Phi(u))\big).\end{aligned} \tag{19}$$

Thus, for $i, j \in \{1,\ldots,k\}$, we obtain

$$\begin{aligned}\partial_i\partial_j\Psi_y(u) = \delta_{ij}\frac{\partial_j\Phi(y)}{p_0^j(y_j)f_j'(y_j)}&\big((p_0^j)''(u_j)(f_j(u_j)-f_j(y_j))+2(p_0^j)'(u_j)f_j'(u_j)+p_0^j(u_j)f_j''(u_j)\big)\\ &-\frac{\partial_i\partial_j\Phi(u)}{p_0^0(\Phi(y))f_0'(\Phi(y))}\big((p_0^0)'(\Phi(u))(f_0(\Phi(u))-f_0(\Phi(y)))+p_0^0(\Phi(u))f_0'(\Phi(u))\big)\\ &-\frac{\partial_i\Phi(u)\partial_j\Phi(u)}{p_0^0(\Phi(y))f_0'(\Phi(y))}\big((p_0^0)''(\Phi(u))(f_0(\Phi(u))-f_0(\Phi(y)))\\ &\qquad+2(p_0^0)'(\Phi(u))f_0'(\Phi(u))+p_0^0(\Phi(u))f_0''(\Phi(u))\big).\end{aligned} \tag{20}$$

Hence, after substituting $u := y$ in the above equality, we get

$$\partial_i\partial_j\Psi_y(y) = \delta_{ij}\partial_j\Phi(y)\left(2\frac{(p_0^j)'}{p_0^j}+\frac{f_j''}{f_j'}\right)(y_j) - \partial_i\partial_j\Phi(y) - \partial_i\Phi(y)\partial_j\Phi(y)\left(2\frac{(p_0^0)'}{p_0^0}+\frac{f_0''}{f_0'}\right)(\Phi(y)) = \Gamma_{i,j}(y). \tag{21}$$

This shows the pointwise positive semidefiniteness of the matrix-valued function $\Gamma\colon I\to\mathbb{R}^{k\times k}$. $\square$

Theorem 13. *Let $k\in\mathbb{N}$ and, for $\alpha\in\{0,\dots,k\}$, let $I_\alpha\subseteq\mathbb{R}$ be a nonempty open interval, $f_\alpha\colon I_\alpha\to\mathbb{R}$ be twice continuously differentiable with a nonvanishing first derivative, $p_0^\alpha\colon I_\alpha\to\mathbb{R}_+$ be twice continuously differentiable and denote $I := I_1\times\cdots\times I_k$. Furthermore, let $\Phi\colon I\to I_0$ be twice continuously differentiable. Assume, for all $y\in I$, that the $k\times k$ matrix $\Gamma(y)$ is positive definite. Then, for all $n\in\mathbb{N}$, $\lambda\in\mathbb{R}_+^n$ with $\lambda_1+\cdots+\lambda_n=1$, inequality (13) holds in the local sense.*

Proof. For all fixed $y\in I$, define the function $\Psi_y\colon I\to\mathbb{R}$ by the formula (18). This function is twice continuously differentiable according to our assumptions and $\Psi_y(y)=0$ holds for all $y\in I$. After simple computations, for $i,j\in\{1,\dots,k\}$ and $u\in I$, we obtain that the equalities (19) and (20) hold.

Putting $u := y$ into equality (19), we can see that

$$\partial_i\Psi_y(y)=0 \qquad (i\in\{1,\dots,k\}),$$

that is, $\Psi_y'(y)=\big(\partial_i\Psi_y(y)\big)_{i=1}^k=0$ holds for all $y\in I$.

Substituting $u := y$ into equality (20), we can conclude that (21) is valid for all $y\in I$. According to the positive definiteness of the matrix-valued function $\Gamma\colon I\to\mathbb{R}^{k\times k}$, it follows that $\Psi_y''(y) := \big(\partial_i\partial_j\Psi_y(y)\big)_{i,j=1}^k$ is positive definite for all $y\in I$.

For $u,y\in I$, denote the smallest eigenvalue of the $k\times k$ symmetric matrix $\Psi_y''(u) := \big(\partial_i\partial_j\Psi_y(u)\big)_{i,j=1}^k$ by $\psi(u,y)$. In view of our twice continuous differentiability assumptions, the map $(u,y)\mapsto\Psi_y''(u)$ is continuous, therefore the map $(u,y)\mapsto\psi(u,y)$ is also continuous. On the other hand, for all $y\in I$, we have that $\Psi_y''(y)=\Gamma(y)$ is positive definite, which implies that $\psi(y,y)>0$. Therefore, there exists an open set $W\subseteq I^2$ containing $\operatorname{diag}(I^2)$ on which ψ is positive.

By the Taylor Mean Value Theorem, for all $u,y\in I$, there exists $t\in[0,1]$ such that

$$\begin{aligned}\Psi_y(u) &= \Psi_y(y)+\Psi_y'(y)(y-u)+\frac{1}{2}(y-u)^T\Psi_y''(tu+(1-t)y)(y-u)\\ &= \frac{1}{2}(y-u)^T\Psi_y''(tu+(1-t)y)(y-u) \ge \frac{1}{2}\psi(tu+(1-t)y,y)\|y-u\|^2.\end{aligned} \tag{22}$$

Define

$$V := \{(u,y)\in I^2 \mid [y,u]\times\{y\} = [(y,y),(u,y)]\subseteq W\}.$$

We will show that V is an open subset of I^2. To see this, let $(u,y)\in V$ be arbitrary. Since the segment $[(y,y),(u,y)]$ is a compact subset of W, it follows that it is disjoint from $W^c := \mathbb{R}^{2k}\setminus W$, which is the complement of W and hence it is a closed set. Therefore, there exists a positive number r so that the distance of every point of the segment $[(y, y), (u, y)]$ from W^c is at least r. Let $v, x \in \mathbb{R}^k$ such that $\|v - u\| < r/2$ and $\|x - y\| < r/2$ hold. Let (w, x) be an arbitrary point of the segment $[(x, x), (v, x)]$. Then there exists $t \in [0, 1]$ such that $w = tv + (1 - t)x$. Therefore,

$$\|w - (tu + (1 - t)y)\| \le t\|v - u\| + (1 - t)\|x - y\| < r/2,$$

which implies that

$$\begin{aligned}\|(w, x) - (tu + (1 - t)y, y)\| &\le \|(w, x) - (tu + (1 - t)y, x)\| + \|(tu + (1 - t)y, x) - (tu + (1 - t)y, y)\| \\ &= \|w - (tu + (1 - t)y)\| + \|x - y\| < r.\end{aligned}$$

In other words, any point of the segment $[(x, x), (v, x)]$ is closer to some point of the segment $[(y, y), (u, y)]$ than r. This yields that the segment $[(x, x), (v, x)] \subseteq W$, i.e., $(v, x) \in V$ whenever $\|v - u\| < r/2$ and $\|x - y\| < r/2$ hold, consequently (u, y) is an interior point of V. This completes the proof of the openness of V. On the other hand, it is obvious that V also contains $\operatorname{diag}(I^2)$.

Using that ψ is positive on W, it follows from (22) that $\Psi_y(u) \ge 0$ for all $(u, y) \in V$. Therefore, by applying Theorem 9, it follows that (13) holds in the local sense. □

4. Inequalities for Gini means

In this section, we apply the above results to some important particular cases of (2). First, we deal with cases obtained by specializing the means M_α in (2) for all $\alpha \in \{0, \dots, k\}$. Then we draw some conclusions by choosing the function Φ in (2) to be the map k-variable addition and multiplication.

Let us recall the definition of the *weighted n-variable Hölder (or power) mean of parameter $r \in \mathbb{R}$ and weight vector $\lambda \in \mathbb{R}^n_+$ with $\lambda_1 + \dots + \lambda_n = 1$*, and the *weighted n-variable Gini mean corresponding to the pair parameters $(r, s) \in \mathbb{R}^2$ and weight vector $\lambda \in \mathbb{R}^n_+$ with $\lambda_1 + \dots + \lambda_n = 1$*:

$$H_{r;\lambda}(x_1, \dots, x_n) := \begin{cases} \left(\lambda_1 x_1^r + \dots + \lambda_n x_n^r\right)^{\frac{1}{r}} & \text{if } r \ne 0, \\ x_1^{\lambda_1} \cdots x_n^{\lambda_n} & \text{if } r = 0; \end{cases}$$

$$G_{r,s;\lambda}(x_1, \dots, x_n) := \begin{cases} \left(\dfrac{\lambda_1 x_1^r + \dots + \lambda_n x_n^r}{\lambda_1 x_1^s + \dots + \lambda_n x_n^s}\right)^{\frac{1}{r-s}} & \text{if } r \ne s, \\ \exp\left(\dfrac{\lambda_1 x_1^r \ln(x_1) + \dots + \lambda_n x_n^r \ln(x_n)}{\lambda_1 x_1^r + \dots + \lambda_n x_n^r}\right) & \text{if } r = s. \end{cases}$$

It is clear that in the particular case $q = 0$, the mean $G_{p,q;\lambda}$ simplifies to $H_{p;\lambda}$.

For $(r, s) \in \mathbb{R}^2$ we also define the function $\chi_{r,s} \colon \mathbb{R}_+ \to \mathbb{R}$ by

$$\chi_{r,s}(t) := \begin{cases} \dfrac{t^r - t^s}{r - s} & \text{if } r \ne s, \\ t^r \ln(t) & \text{if } r = s. \end{cases}$$

Then, for all $\ell \in \{1, \dots, n\}$, with $p_\ell(t) := \lambda_\ell t^s$ and $f(t) := t^{r-s}$ if $r \ne s$ or $f(t) := \ln(t)$ if $r = s$, we can see that $A_{f,p} = G_{r,s;\lambda}$. Using Lemma 5, it follows that

$$\partial_\ell G^\Delta_{r,s;\lambda}(t) = \lambda_\ell \qquad (\ell \in \{1, \dots, n\},\ t \in \mathbb{R}_+),$$

and

$$\partial_\ell \partial_m G^{\Delta}_{r,s;\lambda}(t) = \lambda_m(\delta_{\ell,m} - \lambda_\ell)\frac{r+s-1}{t} \qquad (\ell, m \in \{1,\dots,n\},\, t \in \mathbb{R}_+). \tag{23}$$

Furthermore,

$$\frac{p_0(y)(f(y)-f(u))}{p_0(u)f'(u)} = u\,\chi_{r,s}\Big(\frac{y}{u}\Big) \qquad (u, y \in \mathbb{R}_+). \tag{24}$$

Theorem 14. *Let $n, k \in \mathbb{N}$ with $n, k \geq 2$, $\lambda \in \mathbb{R}^n_+$, $I_1, \dots, I_k$ be nonempty open subintervals of $\mathbb{R}_+$, $I := I_1 \times \cdots \times I_k$, $(r_0, s_0), \dots, (r_k, s_k) \in \mathbb{R}^2$ and $\Phi\colon I \to \mathbb{R}_+$ be twice differentiable with nonvanishing first derivatives. Then, for the inequality*

$$G_{r_0,s_0;\lambda}(\Phi(x_1), \dots, \Phi(x_n)) \leq \Phi\big(G_{r_1,s_1;\lambda}(x^1), \dots, G_{r_k,s_k;\lambda}(x^k)\big) \tag{25}$$

to be valid in the local sense it is necessary that the $k \times k$ matrix $\Gamma(y)$ given by

$$\Gamma(y) := \bigg(-\partial_i\partial_j\Phi(y) - \partial_j\Phi(y)\partial_i\Phi(y)\frac{r_0+s_0-1}{\Phi(y)} + \delta_{i,j}\partial_j\Phi(y)\frac{r_j+s_j-1}{y_j}\bigg)_{i,j=1}^{k}$$

be positive semidefinite for all $y \in I$. Conversely, if this matrix is positive definite for all $y \in I$, then (25) holds in the local sense on I.

Proof. The necessity of the positive semidefiniteness of $\Gamma(y)$ for all $y \in I$ is an immediate consequence of Theorem 9, Theorem 12, and formula (23). The other direction is also obvious due to Theorem 13. □

Theorem 15. *Let $k \in \mathbb{N}$ with $k \geq 2$, $I_1, \dots, I_k$ be nonempty open subintervals of $\mathbb{R}_+$, $I := I_1 \times \cdots \times I_k$, $(r_0, s_0), \dots, (r_k, s_k) \in \mathbb{R}^2$ and let $\Phi\colon I \to \mathbb{R}_+$ be partially differentiable. Assume that, for all $(u, y) \in I^2$, the inequality*

$$\Phi(u)\chi_{r_0,s_0}\bigg(\frac{\Phi(y)}{\Phi(u)}\bigg) \leq \sum_{j=1}^{k} \partial_j\Phi(u)u_j\chi_{r_j,s_j}\Big(\frac{y_j}{u_j}\Big) \tag{26}$$

is valid. Then, for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}^n_+$ with $\lambda_1 + \cdots + \lambda_n = 1$, the inequality (25) holds in the global sense on I.

Proof. The statement directly follows from Theorem 11 because inequality (12) turns out to be equivalent to (26) by applying formula (24). □

For the investigation of the particular cases when Φ is the sum and the product function, we will need the following auxiliary result.

Lemma 16. *Let $k \in \mathbb{N}$ with $k \geq 2$ and $c_0, c_1, \dots, c_k \in \mathbb{R}$. Then the matrix*

$$C := \big(\delta_{i,j}c_i + c_0\big)_{i,j=1}^{k}$$

is positive semidefinite if and only if either $c_0, c_1, \dots, c_k \geq 0$ or there exists $i \in \{0, \dots, k\}$ such that $c_i < 0$ and $c_j > 0$ for all $j \in \{0, \dots, k\} \setminus \{i\}$ and

$$\frac{1}{c_0} + \frac{1}{c_1} + \cdots + \frac{1}{c_k} \leq 0. \tag{27}$$

Furthermore, C is positive definite if and only if either $c_0, c_1, \dots, c_k \geq 0$ and $c_i = 0$ can hold for at most one index $i \in \{0, \dots, k\}$ or there exists $i \in \{0, \dots, k\}$ such that $c_i < 0$ and $c_j > 0$ for all $j \in \{0, \dots, k\} \setminus \{i\}$ and (27) is valid with a strict inequality.

Proof. The quadratic form $Q\colon \mathbb{R}^k \to \mathbb{R}$ generated by C is given by

$$Q(x) := c_0 \left(\sum_{\ell=1}^{k} x_\ell \right)^2 + \sum_{\ell=1}^{k} c_\ell x_\ell^2 \qquad (x \in \mathbb{R}^k).$$

Assume first that Q is positive semidefinite. This means that $Q(x) \geq 0$ for all $x \in \mathbb{R}^k$. With the notation $x_0 := -\sum_{\ell=1}^{k} x_\ell$, this inequality can be rewritten as

$$\sum_{\ell=0}^{k} c_\ell x_\ell^2 \geq 0 \qquad \text{for all } (x_0, x_1, \dots, x_k) \in \mathbb{R}^{k+1} \text{ with } x_0 + x_1 + \dots + x_k = 0. \tag{28}$$

To prove the necessity of the condition, assume that $\min(c_0, c_1, \dots, c_k) < 0$. Choose $i \in \{0, \dots, k\}$ such that $c_i = \min(c_0, c_1, \dots, c_k) < 0$. For a fixed $j \in \{0, \dots, k\} \setminus \{i\}$, define the vector $x = (x_0, x_1, \dots, x_k)$ by $x_\ell := \delta_{i,\ell} - \delta_{j,\ell}$ for $\ell \in \{0, \dots, k\}$, that is, $x_\ell := 0$ if $\ell \in \{0, \dots, k\} \setminus \{i, j\}$ and $x_i := 1$, $x_j := -1$. Thus $x_0 + x_1 + \dots + x_k = 0$ holds, which, by (28), yields that $\sum_{\ell=0}^{k} c_\ell x_\ell^2 \geq 0$, that is, $c_i + c_j \geq 0$. This implies that $c_j > 0$ for all $j \in \{0, \dots, k\} \setminus \{i\}$.

To show that the inequality (27) is also valid, we substitute the vector $x = (x_0, x_1, \dots, x_k)$ given by

$$x_j := \frac{1}{c_j} \qquad (j \in \{0, \dots, k\} \setminus \{i\}) \qquad \text{and} \qquad x_i := -\sum_{j \in \{0, \dots, k\} \setminus \{i\}} \frac{1}{c_j}. \tag{29}$$

Then, obviously, $x_0 + x_1 + \dots + x_k = 0$, which, again by (28), yields $\sum_{\ell=0}^{k} c_\ell x_\ell^2 \geq 0$, that is,

$$\sum_{j \in \{0, \dots, k\} \setminus \{i\}} c_j \left(\frac{1}{c_j} \right)^2 + c_i \left(-\sum_{j \in \{0, \dots, k\} \setminus \{i\}} \frac{1}{c_j} \right)^2 \geq 0. \tag{30}$$

After simplifications, this implies that

$$1 + c_i \left(\sum_{j \in \{0, \dots, k\} \setminus \{i\}} \frac{1}{c_j} \right) \geq 0, \tag{31}$$

which, using that $c_i < 0$, shows that the inequality (27) is valid.

Assume that Q is positive definite. Then $Q(x) > 0$ for all $x \in \mathbb{R}^k \setminus \{0\}$. With the notation $x_0 := -\sum_{\ell=1}^{k} x_\ell$ this inequality can be rewritten as

$$\sum_{\ell=0}^{k} c_\ell x_\ell^2 > 0 \qquad \text{for all } (x_0, x_1, \dots, x_k) \in \mathbb{R}^{k+1} \setminus \{0\} \text{ with } x_0 + x_1 + \dots + x_k = 0. \tag{32}$$

Since the positive definiteness of Q implies its positive semidefiniteness, there are two possible cases:

(a) $c_0, c_1, \dots, c_k \geq 0$;
(b) there exists $i \in \{0, \dots, k\}$ such that $c_i < 0$ and $c_j > 0$ for all $j \in \{0, \dots, k\} \setminus \{i\}$ and (27) holds.

Assume first that case (a) holds. If $c_i = c_j = 0$ were valid for some $i, j \in \{0, \dots, k\}$ with $i \neq j$, then substituting $x_\ell := \delta_{i,\ell} - \delta_{j,\ell}$ for $\ell \in \{0, \dots, k\}$ into (32), we would get that $0 < \sum_{\ell=0}^{k} c_\ell x_\ell^2 = c_i + c_j = 0$. This contradiction shows that $c_i = 0$ can hold only for at most one index $i \in \{0, \dots, k\}$.

Consider now the case (b). Substituting the vector $x = (x_0, x_1, \dots, x_k)$ given by (29) into (32), it follows that $\sum_{\ell=0}^{k} c_\ell x_\ell^2 > 0$. This implies that the inequalities (30) and then (31) hold with strict inequalities. Therefore, (27) is also satisfied with a strict inequality.

Now we show the sufficiency of the conditions. In case (a), it is clear that the inequality (28) holds, whence Q is positive semidefinite. If, in addition, $c_i = 0$ holds for at most one index $i \in \{0, \dots, k\}$, then (32) is also valid, that is, Q is positive definite in this case.

In case (b), it follows from (27) that

$$c_i \geq -\left(\sum_{j \in \{0,\dots,k\} \setminus \{i\}} \frac{1}{c_j} \right)^{-1}. \tag{33}$$

Let $(x_0, x_1, \dots, x_k) \in \mathbb{R}^{k+1}$ with $x_0 + x_1 + \dots + x_k = 0$. Applying the Cauchy–Schwarz inequality to the vectors

$$\left(\frac{1}{\sqrt{c_j}} \right)_{j \in \{0,\dots,k\} \setminus \{i\}} \qquad \text{and} \qquad \left(-\sqrt{c_j} x_j \right)_{j \in \{0,\dots,k\} \setminus \{i\}},$$

we obtain that

$$\left(\sum_{j \in \{0,\dots,k\} \setminus \{i\}} \frac{1}{c_j} \right) \left(\sum_{j \in \{0,\dots,k\} \setminus \{i\}} c_j x_j^2 \right) \geq \left(\sum_{j \in \{0,\dots,k\} \setminus \{i\}} -x_j \right)^2 = x_i^2.$$

Therefore, combining this inequality with (33), we can conclude that

$$\sum_{j=0}^{k} c_j x_j^2 \geq \sum_{j \in \{0,\dots,k\} \setminus \{i\}} c_j x_j^2 - \left(\sum_{j \in \{0,\dots,k\} \setminus \{i\}} \frac{1}{c_j} \right)^{-1} x_i^2 \geq 0. \tag{34}$$

Thus, we have proved that (28) is valid and hence Q is positive semidefinite. If in this case (27) is valid with a strict inequality, then (33) is also strict. Then, for a nonzero vector $(x_0, x_1, \dots, x_k) \in \mathbb{R}^{k+1}$ with $x_0 + x_1 + \dots + x_k = 0$ the first inequality in (34) is strict provided that $x_i \neq 0$ and the second inequality is strict if $x_i = 0$. Thus, (32) is valid, which shows that Q is positive definite. □

4.1. Minkowski-type inequalities

Our next result characterizes the Minkowski-type inequality for Gini means in the local sense.

Theorem 17. *Let $n, k \in \mathbb{N}$ with $n, k \geq 2$, $\lambda \in \mathbb{R}_+^n$, $I_1, \dots, I_k$ be nonempty open subintervals of $\mathbb{R}_+$, $I := I_1 \times \dots \times I_k$, $(r_0, s_0), \dots, (r_k, s_k) \in \mathbb{R}^2$, $\gamma_i := r_i + s_i - 1$ for $i \in \{0, \dots, k\}$. For the inequality*

$$G_{r_0,s_0;\lambda}(x_1^1 + \dots + x_1^k, \dots, x_n^1 + \dots + x_n^k) \leq G_{r_1,s_1;\lambda}(x^1) + \dots + G_{r_k,s_k;\lambda}(x^k) \tag{35}$$

to hold in the local sense on I, it is necessary that exactly one of the following cases be valid:

(i)

$$\gamma_0 \leq 0 \leq \min(\gamma_1, \dots, \gamma_k); \tag{36}$$

(ii) $\gamma_0, \gamma_1, \dots, \gamma_k > 0$ *and*

$$\sum_{i\in J_+}\Big(\frac{1}{\gamma_i}-\frac{1}{\gamma_0}\Big)\sup I_i \le \sum_{i\in J_-}\Big(\frac{1}{\gamma_0}-\frac{1}{\gamma_i}\Big)\inf I_i; \tag{37}$$

(iii) $\gamma_0 < 0$ *and there exists* $i \in \{1, \dots, k\}$ *such that* $\gamma_i < 0$*, for all* $j \in \{1, \dots, k\} \setminus \{i\}$*,* $\gamma_j > 0$*, and inequality* (37) *is also valid,*

where, for the last two cases, we define

$$J_+ := \Big\{i \in \{1,\dots,k\} \colon \frac{1}{\gamma_i} > \frac{1}{\gamma_0}\Big\} \qquad \text{and} \qquad J_- := \Big\{i \in \{1,\dots,k\} \colon \frac{1}{\gamma_0} > \frac{1}{\gamma_i}\Big\}.$$

Conversely, if either (36) *is valid and* $\gamma_i = 0$ *can hold for at most one* $i \in \{0, \dots, k\}$ *or* $\gamma_0 \neq \gamma_\ell$ *for some* $\ell \in \{1, \dots, k\}$ *and one of the conditions (ii) or (iii) hold, then* (35) *is valid in the local sense on* I*.*

Proof. We apply Theorem 14 with the setting $I_0 := \mathbb{R}_+$ and $\Phi\colon I \to I_0$ defined by $\Phi(y) := y_1 + \dots + y_k$. According to Theorem 14, for the validity of (35) in the local sense, it is necessary that the values of the function $\Gamma\colon I \to \mathbb{R}^{k\times k}$ defined by

$$\Gamma(y) := \Big(\delta_{i,j}\frac{\gamma_i}{y_i} - \frac{\gamma_0}{y_1+\dots+y_k}\Big)_{i,j=1}^{k}$$

be positive semidefinite matrices. By Lemma 16, this property is characterized by the following system of conditions: either

$$c_0 := -\frac{\gamma_0}{y_1+\dots+y_k} \ge 0, \qquad c_i := \frac{\gamma_i}{y_i} \ge 0 \qquad (i \in \{1,\dots,k\}); \tag{38}$$

or there exists $i \in \{0, \dots, k\}$ such that $c_i < 0$, for all $j \in \{0, \dots, k\} \setminus \{i\}$, $c_j > 0$, and (27) holds, i.e.,

$$\frac{1}{c_0} + \frac{1}{c_1} + \dots + \frac{1}{c_k} \le 0.$$

Observe that $\operatorname{sign}(\gamma_0) = -\operatorname{sign}(c_0)$ and $\operatorname{sign}(\gamma_i) = \operatorname{sign}(c_i)$ for all $i \in \{1, \dots, k\}$. Therefore, the first alternative can hold if and only if (36) is satisfied. The second alternative can be valid if and only if either $\gamma_0, \gamma_1, \dots, \gamma_k > 0$ or $\gamma_0 < 0$ and there exists $i \in \{1, \dots, k\}$ such that $\gamma_i < 0$, for all $j \in \{1, \dots, k\} \setminus \{i\}$, $\gamma_j > 0$, and

$$0 \le \Big(\frac{1}{\gamma_0}-\frac{1}{\gamma_1}\Big)y_1 + \dots + \Big(\frac{1}{\gamma_0}-\frac{1}{\gamma_k}\Big)y_k \qquad (y \in I).$$

One can easily see that this inequality can be rewritten as (37) and hence the necessity of the other two alternatives has been established.

To prove the reverse implication of the theorem, consider first the case when (36) is valid and $\gamma_i = 0$ for at most one $i \in \{0, \dots, k\}$. Then, for every $y \in I$, the numbers $c_0, c_1, \dots, c_k$ defined in (38) are nonnegative and $c_i = 0$ can hold for at most one $i \in \{0, \dots, k\}$. Thus, in view of the second assertion of Lemma 16, it follows that $\Gamma(y)$ is positive definite.

Now consider the second case when, for some $\ell \in \{1, \dots, k\}$, we have that $\gamma_0 \neq \gamma_\ell$ and either $\gamma_0, \gamma_1, \dots, \gamma_k > 0$ or $\gamma_0 < 0$ and there exists $i \in \{1, \dots, k\}$ such that $\gamma_i < 0$, for all $j \in \{1, \dots, k\} \setminus \{i\}$, $\gamma_j > 0$ and (37) is also valid. Let $y \in I$ be fixed. If $\ell \in J_+$, i.e., $\frac{1}{\gamma_\ell} > \frac{1}{\gamma_0}$, then

$$\Big(\frac{1}{\gamma_\ell}-\frac{1}{\gamma_0}\Big)y_\ell<\Big(\frac{1}{\gamma_\ell}-\frac{1}{\gamma_0}\Big)\sup I_\ell,$$

while if $\ell\in J_-$, i.e., $\frac{1}{\gamma_0}>\frac{1}{\gamma_\ell}$, then

$$\Big(\frac{1}{\gamma_0}-\frac{1}{\gamma_\ell}\Big)\inf I_\ell<\Big(\frac{1}{\gamma_0}-\frac{1}{\gamma_\ell}\Big)y_\ell.$$

Therefore, (37) implies that

$$\sum_{i\in J_+}\Big(\frac{1}{\gamma_i}-\frac{1}{\gamma_0}\Big)y_i<\sum_{i\in J_-}\Big(\frac{1}{\gamma_0}-\frac{1}{\gamma_i}\Big)y_i,$$

which then yields that

$$0<\Big(\frac{1}{\gamma_0}-\frac{1}{\gamma_1}\Big)y_1+\cdots+\Big(\frac{1}{\gamma_0}-\frac{1}{\gamma_k}\Big)y_k.$$

Hence, (27) is valid with a strict inequality sign. On the other hand, with the exception of one index, the numbers $c_0,\dots,c_k$ are positive. Thus, in view of the second assertion of Lemma 16, it follows that $\Gamma(y)$ is positive definite in this case as well. □

Corollary 18. *Let $n,k\in\mathbb{N}$ with $n,k\geq 2$, $\lambda\in\mathbb{R}_+^n$, $I_1,\dots,I_k$ be nonempty open subintervals of $\mathbb{R}_+$ with $\inf I_1=\cdots=\inf I_k=0$, $I:=I_1\times\cdots\times I_k$, $(r_0,s_0),\dots,(r_k,s_k)\in\mathbb{R}^2$. In order that the inequality (35) be valid in the local sense on I, it is necessary that*

$$\max(1,r_0+s_0)\leq\min(r_1+s_1,\dots,r_k+s_k). \tag{39}$$

Conversely, if this inequality is strict, then (35) holds in the local sense on I.

Proof. The proof is based on Theorem 17. Denote $\gamma_i:=r_i+s_i-1$ for $i\in\{0,\dots,k\}$. If condition (36) holds, then (39) is obvious because

$$\max(1,r_0+s_0)=1+\max(0,\gamma_0)=1\leq 1+\min(\gamma_1,\dots,\gamma_k)=\min(r_1+s_1,\dots,r_k+s_k).$$

In the remaining two cases (37) is valid. However, due to our assumptions on the intervals, the right hand side of (37) is equal to 0. Therefore, the left hand side of this equality must be an empty sum, i.e., $J_+=\emptyset$, which means

$$\frac{1}{\gamma_0}\geq\frac{1}{\gamma_i}\qquad(i\in\{1,\dots,k\}) \tag{40}$$

should be valid. In the case $\gamma_0,\gamma_1,\dots,\gamma_k>0$, that is, when $1\leq\min(r_0+s_0,r_1+s_1,\dots,r_k+s_k)$, the inequalities in (40) hold if and only if (39) is satisfied. In the case when $\gamma_0<0$, then for at least one $j\in\{1,\dots,k\}$, we have that $\gamma_j>0$, and hence (40) cannot hold.

Assume now that (39) is satisfied with a strict inequality. Then $\gamma_1,\dots,\gamma_k>0$. If $\gamma_0\leq 0$, then (39) implies that (36) is valid with a strict inequality and thus the first alternative of the sufficiency of Theorem 17 holds. If $\gamma_0>0$, then the strict version of (39) shows that $\gamma_0<\gamma_i$ for all $i\in\{1,\dots,k\}$, therefore, the left and the right hand sides of (37) are equal to zero and the second alternative of the sufficiency of Theorem 17 holds. The third alternative of the sufficiency of Theorem 17 cannot happen if (39) is valid. □

For the global validity of the Minkowski-type inequality, Theorem 15 establishes the following sufficient condition.

Theorem 19. *Let $k \in \mathbb{N}$ with $k \geq 2$, $I_1, \dots, I_k$ be nonempty open subintervals of $\mathbb{R}_+$, $I := I_1 \times \cdots \times I_k$, $(r_0, s_0), \dots, (r_k, s_k) \in \mathbb{R}^2$. Assume that, for all $(u, y) \in I^2$, the inequality*

$$\chi_{r_0,s_0}\Big(\frac{y_1 + \cdots + y_k}{u_1 + \cdots + u_k}\Big) \leq \sum_{j=1}^{k} \frac{u_j}{u_1 + \cdots + u_k} \chi_{r_j,s_j}\Big(\frac{y_j}{u_j}\Big) \tag{41}$$

holds. Then, for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}_+^n$ with $\lambda_1 + \cdots + \lambda_n = 1$, the inequality (35) holds in the global sense on I.

Proof. With $\Phi(y_1, \dots, y_k) := y_1 + \cdots + y_k$, the condition (26) turns out to be equivalent to (41) and hence the result follows from Theorem 15. $\square$

Corollary 20. *Let $k \in \mathbb{N}$ with $k \geq 2$ and $(r_0, s_0), \dots, (r_k, s_k) \in \mathbb{R}^2$. Assume that, for all $z \in \mathbb{R}_+^k$ and $t_1, \dots, t_k \in [0, 1]$ with $t_1 + \cdots + t_k = 1$, the following inequality is valid*

$$\chi_{r_0,s_0}(t_1 z_1 + \cdots + t_k z_k) \leq \sum_{j=1}^{k} t_j \chi_{r_j,s_j}(z_j). \tag{42}$$

Then, for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}_+^n$ with $\lambda_1 + \cdots + \lambda_n = 1$, the inequality (35) holds in the global sense on $\mathbb{R}_+^k$.

Proof. Let $(u, y) \in (\mathbb{R}_+^k)^2$ be arbitrary. Then, with the substitutions

$$z_j := \frac{y_j}{u_j} \qquad \text{and} \qquad t_j := \frac{u_j}{u_1 + \cdots + u_k} \qquad (j \in \{1, \dots, k\})$$

inequality (42) implies (41). Therefore, (41) holds for all $(u, y) \in (\mathbb{R}_+^k)^2$ and, according to Theorem 19, this condition yields that the inequality (35) holds in the global sense on $\mathbb{R}_+^k$. $\square$

In order to compare our results above to existing ones, we recall two theorems related to the global validity of the Minkowski-type inequalities. In the setting of two-variable Gini means the Minkowski inequality was characterized by Czinder and Páles in [5, Theorem 5] (see also [14] for a particular case).

Theorem 21. *Let $k \in \mathbb{N}$ with $k \geq 2$, $(r_0, s_0), \dots, (r_k, s_k) \in \mathbb{R}^2$. Then the inequality*

$$G_{r_0,s_0}(x_1 + \cdots + x_k, y_1 + \cdots + y_k) \leq G_{r_1,s_1}(x_1, y_1) + \cdots + G_{r_k,s_k}(x_k, y_k) \tag{43}$$

is valid for all $x_1, \dots, x_k, y_1, \dots, y_k \in \mathbb{R}_+$ if and only if

(i) $0 \leq \min(r_1, s_1, \dots .r_k, s_k)$,
(ii) $\min(r_0, s_0) \leq \min(1, r_1, s_1, \dots .r_k, s_k)$,
(iii) $\max(1, r_0 + s_0) \leq \min(r_1 + s_1, \dots, r_k + s_k)$.

Remark 22. Observe that the third condition in the above theorem is the necessary condition for the local validity of (43) on $\mathbb{R}_+$. The conditions (i) and (ii) are, however, not necessary for the local validity of (43) on $\mathbb{R}_+$.

Necessary and sufficient conditions for the global validity of the Minkowski-type inequality for Gini means with arbitrary number of variables was established by Páles in [17, Theorem 3.1].

Theorem 23. *Let $k \in \mathbb{N}$ with $k \geq 2$, $(r_0, s_0), \dots, (r_k, s_k) \in \mathbb{R}^2$. Then the inequality*

$$G_{r_0,s_0}(x_1^1 + \cdots + x_1^k, \dots, x_n^1 + \cdots + x_n^k) \leq G_{r_1,s_1}(x^1) + \cdots + G_{r_k,s_k}(x^k) \tag{44}$$

is valid for all $n \in \mathbb{N}$ and $x \in \mathbb{R}_+^{n\times k}$ if and only if

(i) $0 \leq \min(r_1, s_1, \dots, r_k, s_k)$,
(ii) $\min(r_0, s_0) \leq \min(1, r_1, s_1, \dots, r_k, s_k)$,
(iii) $\max(1, r_0, s_0) \leq \min(\max(r_1, s_1), \dots, \max(r_k, s_k))$.

Remark 24. Observe that conditions (i) and (ii) of the above two theorems are identical, therefore they may be necessary for the global validity of (44) for any fixed $n \in \mathbb{N}$. The form of the third condition related to any fixed $n \in \mathbb{N}$ is not known. We also note that conditions (i)-(iii) of Theorem 23 are also necessary and sufficient for the validity of the inequality (42) on the domain indicated in Corollary 20.

4.2. Hölder-type inequalities

Our next results characterize Hölder-type inequalities for Gini means in the local and in the global sense.

Theorem 25. *Let $n, k \in \mathbb{N}$ with $n, k \geq 2$, $\lambda \in \mathbb{R}_+^n$, $I_1, \dots, I_k$ be nonempty open subintervals of $\mathbb{R}_+$, $I := I_1 \times \cdots \times I_k$, $(r_0, s_0), \dots, (r_k, s_k) \in \mathbb{R}^2$, $\gamma_i := r_i + s_i$ for $i \in \{0, \dots, k\}$. Then, in order that the inequality*

$$G_{-r_0,-s_0;\lambda}(x_1^1 \cdots x_1^k, \dots, x_n^1 \cdots x_n^k) \leq G_{r_1,s_1;\lambda}(x^1) \cdots G_{r_k,s_k;\lambda}(x^k) \tag{45}$$

be valid in the local sense on $\mathbb{R}_+^k$ it is necessary that either $\gamma_0, \gamma_1, \dots, \gamma_k \geq 0$ or there exists $i \in \{0, \dots, k\}$ such that $\gamma_i < 0$, for all $j \in \{0, \dots, k\} \setminus \{i\}$, $\gamma_j > 0$ and

$$\frac{1}{\gamma_0} + \frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_k} \leq 0 \tag{46}$$

holds. Conversely, if either $\gamma_0, \gamma_1, \dots, \gamma_k \geq 0$ and $\gamma_i = 0$ for at most one index $i \in \{0, \dots, k\}$ or there exists $i \in \{0, \dots, k\}$ such that $\gamma_i < 0$, for all $j \in \{1, \dots, k\} \setminus \{i\}$, $\gamma_j > 0$ and (46) is valid with a strict inequality, then (45) holds in the local sense on $\mathbb{R}_+^k$.

Proof. We apply Theorem 14 with the function $\Phi\colon I \to \mathbb{R}_+$ defined by $\Phi(y) := y_1 \cdots y_k$. Then for the validity of (45) in the local sense it is necessary (and sufficient) that the values of the function $\Gamma\colon \mathbb{R}_+^k \to \mathbb{R}^{k\times k}$ defined by

$$\begin{aligned}
\Gamma(y) &:= \bigg((\delta_{i,j} - 1)\frac{1}{y_i y_j}\prod_{\ell=1}^k y_\ell - (-\gamma_0 - 1)\frac{1}{y_i y_j}\prod_{\ell=1}^k y_\ell + \delta_{i,j}(\gamma_j - 1)\frac{1}{y_i y_j}\prod_{\ell=1}^k y_\ell\bigg)_{i,j=1}^k \\
&= \bigg(\frac{1}{y_i y_j}\prod_{\ell=1}^k y_\ell\big(\delta_{i,j}\gamma_j + \gamma_0\big)\bigg)_{i,j=1}^k
\end{aligned}$$

be positive semidefinite (positive definite) matrices for all $y \in I$. However, this property holds if and only if the scalar matrix

$$\Gamma^* := \big(\delta_{i,j}\gamma_j + \gamma_0\big)_{i,j=1}^k$$

is positive semidefinite (positive definite). The statement now follows from Lemma 16 with $c_i := \gamma_i$ ($i \in \{0, \dots, k\}$). $\square$

Theorem 26. *Let $k \in \mathbb{N}$, $k \geq 2, I_1, \dots, I_k \subseteq \mathbb{R}_+$ be nonempty open intervals, $(r_0, s_0), \dots, (r_k, s_k) \in \mathbb{R}^2$. Assume that, for all $z_1 \in (I_1/I_1), \dots, z_k \in (I_k/I_k)$, the inequality*

$$\chi_{-r_0,-s_0}(z_1 \cdots z_k) \leq \sum_{j=1}^{k} \chi_{r_j,s_j}(z_j) \tag{47}$$

holds. Then, for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}_+^n$ with $\lambda_1 + \cdots + \lambda_n = 1$, the inequality (45) holds in the global sense on I.

Proof. With the function $\Phi(y_1, \dots, y_k) := y_1 \cdots y_k$, condition (26) turns out to be equivalent to

$$\chi_{-r_0,-s_0}\left(\frac{y_1 \cdots y_k}{u_1 \cdots u_k}\right) \leq \sum_{j=1}^{k} \chi_{r_j,s_j}\left(\frac{y_j}{u_j}\right).$$

Introducing the new variables $z_i := y_i/u_i$ for $i \in \{1, \dots, k\}$, we can conclude that (47) is valid for all $z_1 \in (I_1/I_1), \dots, z_k \in (I_k/I_k)$ if and only if the above inequality holds for all $(y, u) \in (I_1 \times \cdots \times I_k)^2$. Hence the result follows from Theorem 15. □

The global validity of (45) with a non-fixed number of variables was characterized by Páles in [18,19].

Theorem 27. *Let $k \in \mathbb{N}$ with $k \geq 2$, $(r_0, s_0), \dots, (r_k, s_k) \in \mathbb{R}^2$. Then the inequality*

$$G_{-r_0,-s_0}(x_1^1 \cdots x_1^k, \dots, x_n^1 \cdots x_n^k) \leq G_{r_1,s_1}(x^1) \cdots G_{r_k,s_k}(x^k)$$

is valid for all $n \in \mathbb{N}$ and $x \in \mathbb{R}_+^{n \times k}$ if and only if

(i) for all $i \in \{0, \dots, k\}$, $\max(s_i, r_i) \geq 0$ and
(ii) for all $i \in \{0, \dots, k\}$ with $\min(s_i, r_i) < 0$, we have $\max(s_j, r_j) > 0$ for all $j \in \{0, \dots, k\} \setminus \{i\}$ and

$$\frac{1}{\min(s_i, r_i)} + \sum_{\substack{j=0 \\ j \neq i}}^{k} \frac{1}{\max(s_j, r_j)} \leq 0.$$

Remark 28. We note that the conditions (i) and (ii) are necessary and sufficient for the validity of the inequality (47) for all $z_1, \dots, z_k \in \mathbb{R}_+$ (cf. [19]).

Acknowledgment

The authors wish to thank the anonymous referee for her/his detailed comments and suggestions.

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