SHORT THESIS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY (PHD)

Alternative equations for generalized polynomials

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Debrecen, 2025

This thesis contains the primary results from the doctoral dissertation. Our research has yielded several important lemmas, propositions, theorems and corollaries that are thoroughly elaborated in the papers [**BM23**], [**BM24a**], [**BM24b**] and [**BM25a**], as well as within the doctoral dissertation itself.

Motivation and research plan

Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{N}$ denote the sets of complex numbers, real numbers, rational numbers, and positive integers, respectively.

We introduce the following subsets of $\mathbb{R}^2$ to describe certain geometric structures:

$$\begin{aligned}
S_0 &= \{(x,y)\in\mathbb{R}^2 \,|\, xy=1\},\\
S_1 &= \{(x,y)\in\mathbb{R}^2 \,|\, x^2-y^2=1\},\\
S_2 &= \{(x,y)\in\mathbb{R}^2 \,|\, x^2+y^2=1\}.
\end{aligned}$$

Additionally, given two regular, non-constant real polynomials p and q, and a nonzero real number m, we define:

$$\begin{aligned}
R_{p,q} &= \{(p(t),q(t)) \,|\, t\in\mathbb{R}\},\\
S_{1,m} &= \{(x,y)\in\mathbb{R}^2 \,|\, x^2-my^2=1\}.
\end{aligned}$$

The existence of non-linear additive real functions $f:\mathbb{R}\to\mathbb{R}$ has been established by G. Hamel [**Ham**]. Clearly, these functions are irregular in several senses (for instance, the graph of such a function is dense in the plane). Therefore, the study of additive real functions satisfying additional (possibly conditional) functional equations has been an active area of research. Such investigations were extended to quadratic functions by Z. Boros and Edit Garda-Mátyás (see [**BG**] and the references therein). Z. Kominek, L. Reich, and J. Schwaiger [**KRS**] investigated additive functions $f:\mathbb{R}\to\mathbb{R}$ fulfilling the additional condition:

$$f(x)f(y)=0 \quad \text{for all } (x,y)\in D\,, \tag{1}$$

and proved that the only solution is $f(x)=0$ for all $x\in\mathbb{R}$, where D represents various subsets of $\mathbb{R}^2$:

- $D = R_{p,q}$, with p and q being regular, non-constant polynomials;
- $D = S_2$, the unit circle;
- D is any measurable set in $\mathbb{R}^2$ with a positive planar Lebesgue measure.

Further, Z. Boros and W. Fechner [**BF**] extended these findings for $D = S_2$ to include generalized polynomials. However, P. Kutas [**Kut**] recently established the existence of a nonzero additive function $f : \mathbb{R} \to \mathbb{R}$ that satisfies the same condition (1) for $D = S_0$. The purpose of this dissertation was to extend various results involved in [**KRS**] to generalized polynomials defined on the real line or more general domains.

Let $(G, +)$ denote an Abelian group and m be a positive integer. A function $F : G^m \to \mathbb{R}$ is called *m-additive* if F is additive in each of its variables. If $f : G \to \mathbb{R}$ is defined as a diagonalization (or trace) of an m-additive mapping $F : G^m \to \mathbb{R}$ as

$$f(x) = F(x, \ldots, x) \tag{2}$$

for every $x \in G$, we say that f is a *generalized monomial of degree m*. Replacing F with its symmetric part, we obtain that for every generalized monomial $f : G \to \mathbb{R}$ of degree m there exists a symmetric m-additive function $F : G^m \to \mathbb{R}$ such that (2) holds for every $x \in G$.

Any generalized monomial $f : G \to \mathbb{R}$ of degree m satisfies the m–monomial functional equation

$$\Delta_y^m f(x) = m! f(y) \qquad (x, y \in G), \tag{3}$$

where $\Delta_y f(x) = f(x + y) - f(x)$ $(x, y \in G)$ and Δ_y^m denotes the m-th iterate of the difference operation Δ_y. In fact ([**Kuc**, Chapter 15], [**Sze91**, Chapter 1]), generalized monomials of degree m are characterized as the solutions of the m–monomial functional equation (3) if G is uniquely divisible by $(m + 1)!$.

Moreover, let

$$f(x) = \sum_{k=0}^{p} f_k(x) \qquad (x \in G), \tag{4}$$

where each $f_k : G \to \mathbb{R}$ is a generalized monomial of degree k (in particular f_0 is a constant function). Functions with representation (4) are called *generalized polynomials* (or *polynomial functions*) *of degree at most p*.

As a consequence, generalized polynomials of degree at most p are characterized as solutions of the functional equation

$$\Delta_y^{p+1} f(x) = 0 \qquad (x, y \in G) \tag{5}$$

if G is uniquely divisible by $(p+1)!$.

First we collect some notable properties of these functions that are established in order to provide powerful tools for our subsequent investigations. We establish the following continuity type property of generalized monomials under limits involving linear perturbations.

Lemma. *Let $m \in \mathbb{N}$, and let $f : \mathbb{R} \to \mathbb{R}$ be a generalized monomial of degree m. For any $x, y \in \mathbb{R}$, the following holds:*

$$\lim_{n\to\infty} f\left(x + \frac{y}{n}\right) = f(x).$$

We recall a key result by Halter-Koch, Reich, and Schwaiger [**HKRS**, Theorem 2], claiming that the set of generalized polynomials is an integral domain. Since the cited paper contains only a sketch of the proof of this fundamental property, the details of the argument were discussed in the particular case of real functions.

Equations along pairs of polynomials and conic sections

In the first part, we generalize these results by considering the condition:

$$f(x)g(y) = 0 \quad \text{for all } (x, y) \in D,$$

where f and g are generalized polynomials. Specifically, we examine cases where:

- $D = R_{p,q}$, with p and q as regular, non-constant polynomials;
- $D = S_{1,m}$, where m is an arbitrary nonzero real number.

Concerning the first setting we obtained that when the product of the compositions $f(p(x))$ and $g(q(x))$ vanishes identically, one of the generalized polynomials must be identically zero.

THEOREM. *Let p and q be polynomials of degrees at least one. If the generalized polynomials $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfy the equation*

$$f(p(x))g(q(x)) = 0 \tag{6}$$

for every $x \in \mathbb{R}$, then $f(x) = 0$ for all $x \in \mathbb{R}$ or $g(x) = 0$ for all $x \in \mathbb{R}$.

We may formulate the following corollary, demonstrating that a single generalized polynomial cannot satisfy such a product equation unless it is identically zero.

COROLLARY. *Let p and q be polynomials of degrees at least one. If the generalized polynomial $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation*

$$f(p(x))f(q(x)) = 0 \tag{7}$$

for every $x \in \mathbb{R}$, then $f(x) = 0$ for all $x \in \mathbb{R}$.

Given the conditional equation

$$f(x)g(y) = 0 \quad \text{for every} \quad (x, y) \in S_{1,m} \tag{8}$$

with a non-zero real number m, and two generalized polynomials (of possibly different degrees) f and g, we proved the following statement:

THEOREM. *Let m denote a non-zero real number. Suppose that f, $g\colon \mathbb{R} \to \mathbb{R}$ are generalized polynomials and $f(x)g(y) = 0$ for all $(x, y) \in S_{1,m}$. Then f or g is identically equal to zero.*

COROLLARY. *Let a and b denote positive real numbers and let $\sigma \in \{-1, 1\}$. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a generalized polynomial* *and* $f(x)f(y) = 0$ *for all solutions of the equation* $\frac{x^2}{a^2} - \sigma\frac{y^2}{b^2} = 1$. *Then* f *is identically equal to zero.*

The previous corollary involves hyperbolas and ellipses when $\sigma = 1$ or $\sigma = -1$, respectively.

Investigations involving non-algebraic constraints

In the second part of the dissertation we consider non-algebraic constraints where $f : \mathbb{R}^N \to \mathbb{R}$ and $g : \mathbb{R}^N \to \mathbb{R}$ are generalized polynomials fulfilling the conditional equation

$$f(x)g(y) = 0 \tag{9}$$

for every $(x, y) \in D$, where $D \subseteq \mathbb{R}^{2N}$ has a positive $2N$ dimensional Lebesgue measure or it is a second category Baire set. We prove that $f(x) = 0$ for every $x \in \mathbb{R}^N$ or $g(x) = 0$ for every $x \in \mathbb{R}^N$.

In fact, some statements are established in a considerably more general setting. Namely, we consider Euclidean spaces (or even more general domains, for instance, σ-finite measure spaces) X, Y and we investigate arbitrary functions $f : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$ satisfying the condition (9) for all $(x, y) \in D$, where $D \subseteq X \times Y$ is large in the sense of measure or category. We prove that f or g vanishes on a large subset of X or Y, respectively, in the same sense.

Finally, we make use of a result by László Székelyhidi on the zeros of polynomials [**Sze85**, Theorem 2] (and some of its counterparts) to conclude, in various settings, that whenever the product of generalized polynomials f and g vanishes, in the sense of equation (9), on a large set, then f or g equals zero identically.

Then we investigate generalized monomials $f : \mathbb{R} \to \mathbb{R}$ of even degree fulfilling $f(x)f(y) \geq 0$ for the pairs $(x, y) \in D$, where $D \subseteq \mathbb{R}^2$ has a positive planar Lebesgue measure or it is a second category Baire set. We prove that f cannot change its sign.

All these investigations are extended to almost polynomial (respectively, almost monomial) functions as well.

Equation with measure constraint

The first theorem can be applied for the product of arbitrary functions over σ-finite measure spaces.

Theorem. *For each* $j \in \{ 1 , 2 \}$, *let* $(X_j , \mathcal{A}_j , \mu_j)$ *be a* σ*-finite measure space. Suppose that* $f_j : X_j \to \mathbb{C}$ $(j = 1,2)$ *fulfill*

$$f_1(x)f_2(y) = 0 \tag{10}$$

for all $(x, y) \in D$, *where* $D \subseteq X_1 \times X_2$ *is a* $\mu_1 \times \mu_2$ *measurable subset with positive product measure. Then there exist an index* $j \in \{ 1 , 2 \}$ *and* $A_j \in \mathcal{A}_j$ *such that* $\mu_j(A_j) > 0$ *and* $f_j(x) = 0$ *for every* $x \in A_j$.

The statement is identical if, instead of the product measure $\mu_1 \times \mu_2$ (as defined by Halmos [**Hal**, § 34. Theorem A (p. 141), § 35. Theorems A and B (p. 143–144)]), we refer to its Lebesgue completion $\mu_1 \otimes \mu_2$ as defined, for instance, in a monograph by Bogachev [**Bog**, Theorem 3.3.1].

In the sequel we apply our previous results to products of functions taken from particular families of functions.

Definition. Let $(X , \mathcal{A}, \mu)$ denote a measure space (i.e., let $(X , \mathcal{A})$ denote a measurable space with a non-negative — possibly, but not identically, infinite — and σ-additive set function μ on $\mathcal{A}$). We call a family $\mathcal{F}$ of (possibly non-measurable) functions $f : X \to \mathbb{C}$ *algebraically measure regular* provided that the following implication is valid for every $f \in \mathcal{F}$: if $f(x) = 0$ for every $x \in B$, where $B \in \mathcal{A}$ and $\mu(B) > 0$, then $f(x) = 0$ for every $x \in X$.

So we call a family of functions *algebraically measure regular* if every member of this family that vanishes on a set of positive measure must be identically equal to zero.

Theorem. *For each* $j \in \{ 1 , 2 \}$, *let* $(X_j , \mathcal{A}_j , \mu_j)$ *be a* σ*-finite measure space and let* $\mathcal{F}_j$ *denote an algebraically measure regular family of functions* $f : X_j \to \mathbb{C}$. *Let* $f_j \in \mathcal{F}_j$ $(j = 1,2)$ *such that (10)* *holds for all* $(x, y) \in D$ *, where* $D \subseteq X_1 \times X_2$ *is a* $\mu_1 \otimes \mu_2$ *measurable subset with positive measure. Then* f_1 *or* f_2 *is identically equal to zero.*

Since, for arbitrary positive integers k and m, the $k+m$ dimensional Lebesgue measure can be considered as the Lebesgue completion of the product of the k and m dimensional Lebesgue measures, we can establish the following particular case of the previous theorem.

COROLLARY. *For some* $k, m \in \mathbb{N}$*, let* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *denote algebraically measure regular families of functions* $f_1 : \mathbb{R}^k \to \mathbb{C}$ *and* $f_2 : \mathbb{R}^m \to \mathbb{C}$ *related to the* k *and* m *dimensional Lebesgue measures, respectively. Suppose that* $f_j \in \mathcal{F}_j$ $(j = 1, 2)$ *such that (10) holds for all* $(x, y) \in D$*, where* $D \subseteq \mathbb{R}^{k+m}$ *is a measurable subset with a positive* $k+m$ *dimensional Lebesgue measure. Then* f_1 *or* f_2 *is identically equal to zero.*

Using induction, we can easily extend this corollary to arbitrary finite products.

COROLLARY. *Let* $n \in \mathbb{N}$*,* $k_j \in \mathbb{N}$ $(j = 1, 2 \ldots, n)$ *and* $m = \sum_{j=1}^{n} k_j$*. For each* $j \in \{1, 2 \ldots, n\}$*, let* $\mathcal{F}_j$ *denote an algebraically measure regular family of functions* $f : \mathbb{R}^{k_j} \to \mathbb{C}$ *with respect to the* k_j *dimensional Lebesgue measure. Let* $f_j \in \mathcal{F}_j$ $(j = 1, 2, \ldots, n)$ *such that*

$$f_1(x_1)\, f_2(x_2) \cdots f_n(x_n) = 0 \tag{11}$$

holds for all $(x_1, x_2, \ldots, x_n) \in D$ *where* $D \subseteq \mathbb{R}^m$ *is measurable with positive* m *dimensional Lebesgue measure. Then there exists an index* $j^* \in \{1, 2 \ldots, n\}$ *such that* f_{j^*} *is identically equal to zero.*

Now we consider functions defined on particular locally compact Abelian groups. It is well known [**Hal**, § 58. Theorem B (p. 254)] that there exists a Haar measure on such a group (which is unique on

Borel sets up to a constant factor [**Hal**, § 60. Theorem C (p. 263)]). In our statements and arguments we refer to such a measure.

We shall make use of Székelyhidi's result on the zeros of generalized polynomials ([**Sze85**, Theorem 2], [**Sze91**, Theorem 3.3]):

Theorem. *Let $\mathcal{G}$ be a locally compact Abelian group which is generated by any neighborhood of zero and let Z be a complex linear space. If a generalized polynomial $p : \mathcal{G} \to Z$ vanishes on a Haar measurable set of positive Haar measure, then it vanishes everywhere.*

Clearly, Székelyhidi's theorem states that the family of all generalized polynomials $p : \mathcal{G} \to \mathbb{C}$ constitutes an algebraically measure regular family of functions. Therefore, we obtain the following theorem as a corollary.

Theorem. *For each $j \in \{1, 2\}$, let G_j be a locally compact Abelian group which is generated by any neighborhood of zero, let μ_j denote the Haar measure on G_j, and let us assume that μ_j is σ-finite. Let $f_j : G_j \to \mathbb{C}$ be generalized polynomials $(j = 1, 2)$ fulfilling (10) for all $(x, y) \in D$, where $D \subseteq G_1 \times G_2$ is a $\mu_1 \times \mu_2$ measurable subset with positive measure. Then f_1 or f_2 is identically equal to zero.*

Corollary. *Let $\mathcal{G}$ be a locally compact Abelian group which is generated by any neighborhood of zero. Let μ denote the Haar measure on $\mathcal{G}$, and let us assume that μ is σ-finite. Let $f : \mathcal{G} \to \mathbb{C}$ be a generalized polynomial fulfilling*

$$f(x)f(y) = 0 \tag{12}$$

for all $(x, y) \in D$, where $D \subseteq \mathcal{G}^2$ is a $\mu \times \mu$ measurable subset with positive measure. Then $f(x) = 0$ for every $x \in \mathcal{G}$.

The following theorem is a straightforward corollary of our results as well.

THEOREM. *Let* $n \in \mathbb{N}$, $k_j \in \mathbb{N}$ $(j = 1, 2 \ldots, n)$ *and* $m = \sum_{j=1}^{n} k_j$. *Let* $f_j : \mathbb{R}^{k_j} \to \mathbb{C}$ *be generalized polynomials* $(j = 1, 2, \ldots, n)$ *fulfilling (11) for all* $(x_1, x_2, \ldots, x_n) \in D$ *where* $D \subseteq \mathbb{R}^m$ *is measurable with positive* m *dimensional Lebesgue measure. Then there exists an index* $j^* \in \{1, 2 \ldots, n\}$ *such that* f_{j^*} *is identically equal to zero.*

Equation with category constraint

In this section we elaborate an analogy of the previous results when sets of positive measure are replaced with second category Baire sets in Euclidean spaces. We recall that $B \subseteq \mathbb{R}^N$ has the Baire property (or shortly, B is a Baire set) if there exist an open set $G \subseteq \mathbb{R}^N$ and a first category set $T \subseteq \mathbb{R}^N$ such that $B = G \triangle T$ (where the set operation $\triangle$ denotes the symmetric difference, as usual [**Oxt**], [**Kuc**, Chapter 2]).

In what follows, we wish to establish a category version of Székelyhidi's theorem on the zeros of generalized polynomials. For this purpose we need an analogy of Steinhaus' Theorem.

LEMMA. *Let* $m \in \mathbb{N}$ *and* $A \subseteq \mathbb{R}^N$ *such that* A *is a second category Baire set. Then there exists a neighborhood* U *of zero such that, for every* $y \in U$, *there exists* $x \in \mathbb{R}^N$ *fulfilling*

$$x + ky \in A \quad (k = 0, 1, \ldots, m).$$

In the particular case $m = N = 1$ this result was proved by S. Piccard [**Pic**]. Piccard's theorem has been generalized by several authors (e.g. [**Jar**]).

Now we can establish a category version of Székelyhidi's theorem.

THEOREM. *If a generalized polynomial* $f : \mathbb{R}^N \to \mathbb{C}$ *vanishes on a second category Baire set, then* f *vanishes everywhere.*

In the rest of this section, results for alternative equations with category constraints are presented. In our first theorem we consider arbitrary functions.

Theorem. *Let $k\,, m \in \mathbb{N}$. Suppose that $f_1 : \mathbb{R}^k \to \mathbb{C}$ and $f_2 : \mathbb{R}^m \to \mathbb{C}$ such that (10) holds for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^{k+m}$ is a second category Baire set. Then there exists a second category Baire set $A_1 \subseteq \mathbb{R}^k$ such that $f_1(x) = 0$ for every $x \in A_1$, or there exists a second category Baire set $A_2 \subseteq \mathbb{R}^m$ such that $f_2(y) = 0$ for every $y \in A_2$.*

Definition. Let $k \in \mathbb{N}$. We call a family $\mathcal{F}$ of functions $f : \mathbb{R}^k \to \mathbb{C}$ *algebraically Baire regular* provided that the following implication is valid for every $f \in \mathcal{F}$: if $f(x) = 0$ for every $x \in B$, where $B \subseteq \mathbb{R}^k$ is a second category Baire set, then $f(x) = 0$ for every $x \in \mathbb{R}^k$.

So we call a family of functions *algebraically Baire regular* if every member of this family that vanishes on a second category Baire set must be identically equal to zero.

Theorem. *For some $k\,, m \in \mathbb{N}$, let $\mathcal{F}_1$ and $\mathcal{F}_2$ denote algebraically Baire regular families of functions $f_1 : \mathbb{R}^k \to \mathbb{C}$ and $f_2 : \mathbb{R}^m \to \mathbb{C}$. Suppose that $f_j \in \mathcal{F}_j$ $(j = 1, 2)$ such that (10) holds for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^{k+m}$ is a second category Baire set. Then f_1 or f_2 is identically equal to zero.*

Corollary. *For some $k\,, m \in \mathbb{N}$, let $f : \mathbb{R}^k \to \mathbb{C}$ and $g : \mathbb{R}^m \to \mathbb{C}$ be generalized polynomials fulfilling (9) for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^{k+m}$ is second category Baire set. Then $f(x) = 0$ for every $x \in \mathbb{R}^k$ or $g(y) = 0$ for every $y \in \mathbb{R}^m$.*

Corollary. *Let $N \in \mathbb{N}$ and let $f : \mathbb{R}^N \to \mathbb{C}$ be a generalized polynomial fulfilling (12) for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$ is second category Baire set. Then $f(x) = 0$ for every $x \in \mathbb{R}^N$.*

Products of generalized polynomials

In this section we mention an alternative approach to some of our results for the products of generalized polynomials. This argument was suggested by Eszter Gselmann [**Gse**].

Proposition. *Let G, H be commutative semigroups, n be a nonnegative integer such that the multiplication by $n!$ is bijective in the commutative group K. If $\varphi : G \to H$ is a homomorphism and $p : H \to K$ is a generalized polynomial of degree at most n, then $p \circ \varphi : G \to K$ is a generalized polynomial of degree at most n.*

Applying the previous statement, Eszter Gselmann obtained the following notable result as well.

Proposition. *Let k be a positive integer, $G_1, \ldots, G_k$ be commutative semigroups and X be an algebra over the field $\mathbb{K}$. Let further $p_i : G_i \to X$ be a generalized polynomial for all $i = 1, \ldots, k$. Then the mapping P defined on $\times_{i=1}^k G_i$ by*

$$P(x_1, \ldots, x_k) = p_1(x_1) \cdots p_k(x_k) \quad \big((x_1, \ldots, x_k) \in \times_{i=1}^k G_i\big) \tag{13}$$

is a generalized polynomial on $\times_{i=1}^k G_i$.

Now, if we assume that X denotes the field of (real or) complex numbers and the mapping (13) vanishes on a large set (in the sense of the Haar measure or category) with respect to the product topology on the Cartesian product of locally compact Abelian groups (or, in particular, Euclidean spaces), we may apply Székelyhidi's theorem or its category counterpart to conclude that the mapping (13) vanishes identically on the product space. Then we may apply the formerly cited theorem by Halter-Koch, Reich and Schwaiger [**HKRS**, Theorem 2] claiming that the set of generalized polynomials is an integral domain. Hence, if the product is zero, one of the factors p_l must be identically equal to zero.

Inequalities for monomials with measure constraint

Investigations on the signs of monomial mappings (involving measure constraints) are based on the following preliminary results.

Lemma. *Suppose that* $P \subseteq \mathbb{R}$ *fulfills the following assumptions:*

(i) for all $r \in \mathbb{Q}$ *and* $x \in P$ *we have* $rx \in P$ *;*

(ii) there exists a Lebesgue measurable set A *such that* $A \subseteq P$ *and* $\lambda(A) > 0$ *.*

Then P *has full Lebesgue measure, that is,* $\mathbb{R} \setminus P$ *is Lebesgue measurable and* $\lambda(\mathbb{R} \setminus P) = 0$ *.*

Lemma. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a generalized monomial of degree* $m \geq 1$ *and let* H *denote a closed subset of* $\mathbb{R}$ *such that*

$$P = \{\, x \in \mathbb{R} \,|\, f(x) \in H \,\}$$

has full Lebesgue measure. Then $f(x) \in H$ *for every* $x \in \mathbb{R}$ *.*

Now we can establish a sufficient condition for the non-negativity of monomial functions of even degree [**BM24a**, Theorem 4.3].

Theorem. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a generalized monomial of even degree* $m = 2k$ *(with some positive integer* k*) fulfilling*

$$f(x) \geq 0 \tag{14}$$

for all $x \in A$ *, where* $A \subseteq \mathbb{R}$ *is a Lebesgue measurable subset with positive Lebesgue measure. Then (14) holds for every* $x \in \mathbb{R}$ *.*

This theorem can be combined with a representation theorem due to Gy. Maksa [**Mak**]: If a quadratic function f is non-negative on $\mathbb{R}$, then there exists a Hilbert space H and an additive mapping $\varphi : \mathbb{R} \to H$ such that $f(x) = \|\varphi(x)\|^2$ for every $x \in \mathbb{R}$. In view of our above theorem, the assumption in Maksa's theorem involving the non-negativity of f everywhere could be replaced by the weaker condition that f is non-negative on a set with positive Lebesgue measure (since quadratic functions are generalized monomials of degree 2).

We may obtain an analogous statement for monomial functions with odd degree.

Corollary. *Let $f : \mathbb{R} \to \mathbb{R}$ be a generalized monomial of odd degree $m = 2k - 1$ (with some positive integer k) fulfilling*

$$xf(x) \geq 0 \tag{15}$$

for all $x \in A$, where $A \subseteq \mathbb{R}$ is a Lebesgue measurable subset with positive Lebesgue measure. Then (15) holds for every $x \in \mathbb{R}$.

Now we can establish a sufficient condition for a monomial function of even degree to assure that it preserves its sign.

Theorem. *Let $f : \mathbb{R} \to \mathbb{R}$ be a generalized monomial of even degree fulfilling*

$$f(x)f(y) \geq 0 \tag{16}$$

for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^2$ is a Lebesgue measurable subset with positive planar Lebesgue measure. Then (16) *holds for all $(x, y) \in \mathbb{R}^2$ (i.e., f does not change sign).*

Inequalities for monomials with category constraint

It is natural to investigate the topological analogues of our previous results, when the required properties are satisfied for arguments taken from a second category Baire set.

LEMMA. *Suppose that* $P \subseteq \mathbb{R}$ *fulfills the following assumptions:*
(i) for all $r \in \mathbb{Q}$ *and* $x \in P$ *we have* $rx \in P$ *;*
(ii) there exists a second category Baire set A *such that* $A \subseteq P$ *.*
Then $\mathbb{R} \setminus P$ *is of the first category.*

LEMMA. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a generalized monomial of degree* $m \geq 1$ *and let* H *denote a closed subset of* $\mathbb{R}$ *such that the set*

$$S = \{\, x \in \mathbb{R} \,|\, f(x) \notin H \,\}$$

is of the first category. Then $f(x) \in H$ *for every* $x \in \mathbb{R}$ *.*

THEOREM. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a generalized monomial of even degree* $m = 2k$ *(with some positive integer* k*) fulfilling (14) for all* $x \in A$ *, where* $A \subseteq \mathbb{R}$ *is a second category Baire set. Then (14) holds for every* $x \in \mathbb{R}$ *.*

COROLLARY. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a generalized monomial of odd degree* $m = 2k - 1$ *(with some positive integer* k*) fulfilling (15) for all* $x \in A$ *, where* $A \subseteq \mathbb{R}$ *is a second category Baire set. Then (15) holds for every* $x \in \mathbb{R}$ *.*

THEOREM. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a generalized monomial of even degree fulfilling (16) for all* $(x, y) \in D$ *, where* $D \subseteq \mathbb{R}^2$ *is a second category Baire set. Then (16) holds for all* $(x, y) \in \mathbb{R}^2$ *(i.e.,* f *does not change sign).*

Equations for almost polynomials

Finally it is possible to extend the previous results for almost polynomial functions and almost monomial functions, respectively.

In order to cover the concept and the description of almost polynomial functions both in the sense of measure and in the sense of category, we recall Ger's concepts of conjugate proper linearly independent ideals ([**Ger**, Definitions 2 and 3], [**Kuc**, Definitions in Section 17.5]).

Definition. Let N denote a positive integer. A family $\mathcal{I}$ of subsets of $\mathbb{R}^N$ is called a *proper linearly independent ideal* if

(i) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$;
(ii) $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$;
(iii) $\mathbb{R}^N \notin \mathcal{I}$;
(iv) $\alpha \in \mathbb{R}$, $A \in \mathcal{I}$ and $u \in \mathbb{R}^N$ implies $\alpha A + u \in \mathcal{I}$.

For an arbitrary set $M \subseteq \mathbb{R}^N \times \mathbb{R}^N$ and for every $x \in \mathbb{R}^N$, let

$$M_x = \{ y \in \mathbb{R}^N \mid (x, y) \in M \}.$$

Definition. Let $\mathcal{I}_1$ be a proper linearly independent ideal in $\mathbb{R}^N$ and $\mathcal{I}_2$ be a proper linearly independent ideal in $\mathbb{R}^N \times \mathbb{R}^N$. We say that the ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ are *conjugate* if, for every set $M \in \mathcal{I}_2$ there exists a set $U \in \mathcal{I}_1$ such that $M_x \in \mathcal{I}_1$ for every $x \in \mathbb{R}^N \setminus U$.

Given a proper linearly independent ideal $\mathcal{I}$ in $\mathbb{R}^N$, we say that a condition is satisfied $\mathcal{I}$-*almost everywhere* in $\mathbb{R}^N$ (written $\mathcal{I}$-(a.e.)) if there exists a set $S \in \mathcal{I}$ such that the condition holds for every $x \in \mathbb{R}^N \setminus S$. Using this terminology, we may say that the ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ are conjugate if, for every set $M \in \mathcal{I}_2$, the condition $M_x \in \mathcal{I}_1$ holds $\mathcal{I}_1$-almost everywhere in $\mathbb{R}^N$.

The following examples are corollaries of Fubini's theorem [**Bog**, Theorem 3.4.1 and Corollary 3.4.2] and the Kuratowski–Ulam theorem [**KU**], respectively.

Example. Let $\mathcal{L}_0^N$ denote the family of Lebesgue measurable subsets A of $\mathbb{R}^N$ fulfilling $\lambda^N(A) = 0$, where λ^N denotes the

N-dimensional Lebesgue measure. Then $\mathcal{L}_0^N$ and $\mathcal{L}_0^{2N}$ are conjugate proper linearly independent ideals.

EXAMPLE. Let $\mathcal{F}^N$ denote the family of first category subsets of $\mathbb{R}^N$. Then $\mathcal{F}^N$ and $\mathcal{F}^{2N}$ are conjugate proper linearly independent ideals.

These examples motivate the following concept ([**Ger**, Definition 4], [**Kuc**, Section 17.7]).

DEFINITION. Let $\mathcal{I}$ be a proper linearly independent ideal in $\mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$ and let $p \in \mathbb{N}$. We call a function $f : \mathbb{R}^N \to \mathbb{R}$ an *$\mathcal{I}$-almost polynomial function of degree at most p* if there exists $S \in \mathcal{I}$ such that

$$\Delta_y^{p+1} f(x) = 0 \tag{17}$$

holds for every $(x, y) \in \mathbb{R}^{2N} \setminus S$.

This concept was introduced by Roman Ger in order to establish an abstract description of almost polynomial functions as follows ([**Ger**, Theorem 1], [**Kuc**, Theorem 17.7.2]).

THEOREM. [R. Ger] *Let $\mathcal{I}_1$ be a proper linearly independent ideal in $\mathbb{R}^N$ and $\mathcal{I}_2$ be a proper linearly independent ideal in $\mathbb{R}^{2N}$ such that $\mathcal{I}_1$ and $\mathcal{I}_2$ are conjugate. If $f : \mathbb{R}^N \to \mathbb{R}$ is an $\mathcal{I}_2$-almost polynomial function of degree at most p, then there exists a unique polynomial function $g : \mathbb{R}^N \to \mathbb{R}$ such that $f = g$ $\mathcal{I}_1$-almost everywhere in $\mathbb{R}^N$.*

As immediate consequences of the previous examples and Ger's theorem, as well as the corresponding results for generalized polynomials, we can establish the following corollaries.

Corollary. *Let $f:\mathbb{R}^N\to\mathbb{R}$ be an $\mathcal{L}_0^{2N}$-almost polynomial function of degree at most $p\geq 1$ fulfilling (12) for all $(x,y)\in D$, where $D\subseteq\mathbb{R}^N\times\mathbb{R}^N$ is Lebesgue-measurable and $\lambda^{2N}(D)>0$. Then $f=0$ $\mathcal{L}_0^N$(-a.e.) in $\mathbb{R}^N$ (i.e., $f(x)=0$ for λ^N-almost every $x\in\mathbb{R}^N$).*

Corollary. *Let $f:\mathbb{R}^N\to\mathbb{R}$ be an $\mathcal{F}^{2N}$-almost polynomial function of degree $m\geq 1$ fulfilling (12) for all $(x,y)\in D$, where $D\subseteq\mathbb{R}^N\times\mathbb{R}^N$ is a second category Baire set. Then $f=0$ $\mathcal{F}^N$(-a.e.) in $\mathbb{R}^N$.*

Inequalities for almost monomials

Since we have considered conditional inequalities for monomial functions in a real variable, we may wish to extend our related results to almost monomial functions defined on $\mathbb{R}$. Motivated by the approach in the previous section, we may introduce the required concept as follows.

Definition. Let $\mathcal{I}$ be a proper linearly independent ideal in $\mathbb{R}^2$ and let $m\in\mathbb{N}$. We call a function $f:\mathbb{R}\to\mathbb{R}$ an *$\mathcal{I}$-almost monomial function of degree m* if there exists $S\in\mathcal{I}$ such that

$$\Delta_y^m f(x)=m!f(y) \tag{18}$$

holds for every $(x,y)\in\mathbb{R}^2\setminus S$.

In order to establish an analogy of Ger's theorem to almost monomials, we need a stronger invariance property with respect to the linear transformations of the members of the ideal. So we consider the following concept.

Definition. Let N denote a positive integer. A family $\mathcal{I}$ of subsets of $\mathbb{R}^N$ is called a *proper linearly invariant ideal* if $\mathcal{I}$ is a proper linearly independent ideal such that $S\in\mathcal{I}$ implies $\phi(S)\in\mathcal{I}$ for every linear bijection $\phi : \mathbb{R}^N \to \mathbb{R}^N$.

In other words, we could say that a non-void family $\mathcal{I}$ of subsets of $\mathbb{R}^N$ is a proper linearly invariant ideal if $\mathcal{I}$ fulfills properties (i)–(iii) in the definition of proper linearly independent ideals and (instead of (iv)) it satisfies

(iv*) $S \in \mathcal{I}$ implies $\phi(S) + u \in \mathcal{I}$ for every $u \in \mathbb{R}^N$ and for every linear bijection $\phi : \mathbb{R}^N \to \mathbb{R}^N$.

Using the notation introduced in the former examples, we can claim that sets having Lebesgue measure zero as well as sets of the first category form proper linearly invariant ideals. The first example follows immediately from the identity for the Lebesgue measure of linear transforms of measurable sets [**Bog**, Corollary 3.6.4].

EXAMPLE. $\mathcal{L}_0^N$ is a proper linearly invariant ideal.

The second example can be established as follows.

PROPOSITION. *$\mathcal{F}^N$ is a proper linearly invariant ideal.*

In view of these examples, it is reasonable to formulate the following theorem.

THEOREM. *Let $\mathcal{I}_1$ be a proper linearly independent ideal in $\mathbb{R}$ and $\mathcal{I}_2$ be a proper linearly invariant ideal in $\mathbb{R}^2$ such that $\mathcal{I}_1$ and $\mathcal{I}_2$ are conjugate. Let $f : \mathbb{R} \to \mathbb{R}$ be an $\mathcal{I}_2$-almost monomial function of degree $m \geq 1$. Then there exists a unique monomial function $g : \mathbb{R} \to \mathbb{R}$ of degree m such that $f = g$ $\mathcal{I}_1$-almost everywhere*

Now we can establish the analogies of our former results for generalized monomials for $\mathcal{L}_0^2$-almost monomial functions and $\mathcal{F}^2$-almost monomial functions.

Corollary. *Let $f : \mathbb{R} \to \mathbb{R}$ be an $\mathcal{L}_0^2$-almost monomial function of degree $m = 2k$ fulfilling (16) for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^2$ is a Lebesgue measurable subset with positive planar Lebesgue measure. Then $f(x) \geq 0$ for $\mathcal{L}_0^1$-almost every $x \in \mathbb{R}$ or $f(x) \leq 0$ for $\mathcal{L}_0^1$-almost every $x \in \mathbb{R}$.*

Corollary. *Let $f : \mathbb{R} \to \mathbb{R}$ be an $\mathcal{F}^2$-almost monomial function of degree $m = 2k$ fulfilling (16) for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^2$ is a second category Baire set. Then $f(x) \geq 0$ for $\mathcal{F}^1$-almost every $x \in \mathbb{R}$ or $f(x) \leq 0$ for $\mathcal{F}^1$-almost every $x \in \mathbb{R}$.*

Perspectives of this research project

Concerning the algebraic conditions (along conic sections) for the product of generalized polynomials, the problem is completely solved for parabolas, ellipses and hyperbolas in standard positions. The counterexample by Kutas [**Kut**] demonstrates that our results cannot be extended to arbitrary hyperbolas even in the case when a single additive function is involved. It is a natural (but possibly difficult) research project to give a description of conic sections $S \subseteq \mathbb{R}^2$ that admit the implication that whenever a generalized polynomial $f : \mathbb{R} \to \mathbb{R}$ fulfills the condition $f(x)f(y) = 0$ for all $(x, y) \in S$, then $f = 0$ identrically. The problem is open even in the particular case when f is additive.

Another natural idea is to consider the additional equation

$$f(x_1)f(x_2)\ldots f(x_k) = 0\,,$$

fulfilled under the condition $(x_1\,, x_2\,, \ldots\,, x_k) \in D$, where $D \subseteq \mathbb{R}^k$ is given by an appropriate algebraic equation and $f : \mathbb{R} \to \mathbb{R}$ is additive (or a generalized monomial, respectively, polynomial).

Our results for products vanishing on large subsets of the product space admit applications for algebraically measure/Baire regular families of functions. In the thesis we mention the family of generalized polynomials (on various domains) as an example for such families. It is a natural idea to look for other (different or more general) interesting families of functions fulfilling these properties. Clearly, new examples admit new applications of our main results in this direction. It is another interesting question (suggested by Peter Eliaš at the 38^{th} ISCRFT, September 15–20, 2024, Stará Lesná, Slovakia) whether we can replace the product $f(x)g(y)$ by some other appropriate operation $F(f(x), g(y))$ in our investigations.

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List of talks

(1) *Phd Qualification at the end of the first year*, Institute of Mathematics, University of Debrecen, June 18, 2021.
(2) *Alternative equations for quadratic functions*, Síkfőkút seminar of the department of analysis, August 27–29, 2021.
(3) *Alternative equations for quadratic functions* 19th International Conference on Functional Equations and Inequalities, September 12–18, 2021 (on-line).
(4) *An alternative equation for generalized monomials*, 21st Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities, Brenna, Poland, February 2–5, 2022.
(5) *Complex Exam*, Institute of Mathematics, University of Debrecen, June 29, 2022.
(6) *An alternative equation for generalized monomials involving measure*, 22nd Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities, Hajdúszoboszló, Hungary, February 1–4, 2023.
(7) *Phd Qualification at the End of the Third Year*, Institute of Mathematics, University of Debrecen, June 6, 2023.
(8) *An alternative equation for generalized polynomials of degree two*, The 59th International Symposium on Functional Equations, Hajdúszoboszló, Hungary, June 18–25, 2023.
(9) *An alternative equation for almost polynomials and related inequalities for generalized monomials*, 37th International Summer Conference on Real Functions Theory, Rowy, Poland, September 10–15, 2023.
(10) *An alternative equation for polynomial functions on locally compact abelian groups*, 23rd Katowice–Debrecen Winter

Seminar on Functional Equations and Inequalities, Brenna, Poland, January 31 – February 3, 2024.

(11) *An alternative equation for two polynomial functions on locally compact abelian groups*, 60th International Symposium on Functional Equations, Kościelisko, Poland, June 9–15, 2024.

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Registry number:	DEENK/568/2024.PL
Subject:	PhD Publication List

Candidate: Rayene Menzer
Doctoral School: Doctoral School of Mathematical and Computational Sciences
MTMT ID: 10084607

List of publications related to the dissertation

Foreign language scientific articles in Hungarian journals (2)

1. Boros, Z., **Menzer, R.**: An alternative equation for generalized monomials involving measure.
   Publ. Math. Debr. 104 (1-2), 171-183, 2024. ISSN: 0033-3883.
   DOI: http://dx.doi.org/10.5486/PMD.2024.9664
   IF: 0.4 (2023)

2. Boros, Z., **Menzer, R.**: An alternative equation for generalized polynomials involving measure and category constraints.
   Acta Math. Hung. [Accepted by publisher], 1-18, 2024. ISSN: 0236-5294.
   IF: 0.6 (2023)

Foreign language scientific articles in international journals (2)

3. Boros, Z., **Menzer, R.**: An Alternative Equation for Generalized Polynomials of Degree Two.
   Ann. Math. Silesianae. 38 (2), 214-220, 2024. ISSN: 0860-2107.
   DOI: http://dx.doi.org/10.2478/amsil-2023-0017
   IF: 0.3 (2023)

4. Boros, Z., **Menzer, R.**: An alternative equation for generalized monomials.
   Aequ. Math. 97, 113-120, 2023. ISSN: 0001-9054.
   DOI: http://dx.doi.org/10.1007/s00010-022-00917-y
   IF: 0.9

List of other publications

Foreign language scientific articles in international journals (1)

5. Saeed, H. J., Ali, A. H., **Menzer, R.**, Potcelan, A. D., Arora, H.: New Family of Multi-Step Iterative Methods Based on Homotopy Perturbation Technique for Solving Nonlinear Equations.
   Mathematics. 11 (12), 1-13, 2023. EISSN: 2227-7390.
   DOI: http://dx.doi.org/10.3390/math11122603
   IF: 2.3

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6. Boros, Z., **Menzer, R.**, összeáll. Nagy, G.: A conditional equation for almost polynomials. In: Report of meeting: The 59th International Symposium on Functional Equations Hotel Aurum, Hajdúszoboszló (Hungary), June 18-25, 2023.
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   Ann. Math. Silesianae. 36 (1), 99, 2022. ISSN: 0860-2107.

Total IF of journals (all publications): 4,5
Total IF of journals (publications related to the dissertation): 2,2

The Candidate's publication data submitted to the iDEa Tudóstér have been validated by DEENK on the basis of the Journal Citation Report (Impact Factor) database.

14 November, 2024