Exponential polynomials and polynomial equations

Thesis for the degree
of Doctor of Philosophy (PhD)

written by Mehak Iqbal

and supervised by Eszter Novák–Gselmann, PhD, dr. habil

University of Debrecen
Doctoral Council for Natural Sciences and Engineering
Doctoral School of Mathematical and Computational Sciences

Debrecen, 2025

Hereby I declare that I prepared this thesis within the framework of Doctoral Council for Natural Sciences and Engineering, Doctoral School of Mathematical and Computational Sciences of the University of Debrecen in order to obtain a PhD Degree in Natural Sciences from the University of Debrecen.
I declare that the results published in this thesis are not reported in any other PhD thesis.

Debrecen, June 4, 2025

Mehak Iqbal
signature of the candidate

Hereby I confirm that Mehak iqbal, PhD candidate conducted her studies with my supervision within the Mathematical Analysis Program of the Doctoral School of Mathematical and Computational Sciences of the University of Debrecen between 2021 and 2025. The independent studies and research work of the candidate significantly contributed to the results published in this thesis.
I also declare that the results published in the thesis are not reported in any other PhD thesis.
I support the acceptance of the dissertation.
Debrecen, June 4, 2025

Eszter Novák–Gselmann
signature of the supervisor

Exponential polynomials and polynomial equations

Dissertation submitted in partial fulfilment of the requirements for the doctoral (PhD) degree in Mathematics and Computing.

Written by: Mehak Iqbal

The dissertation was written in the framework of the Mathematical Analysis, Functional Equations and Inequalities Program of the Doctoral School of Mathematical and Computational Sciences of the University of Debrecen.

Dissertation advisor: Eszter Novák–Gselmann

The official opponents of the dissertation:

	Dr.	....	....
	Dr.	....	....

The evaluation committee:

Chairperson:	Dr.	....	....
Members:	Dr.	....	....
	Dr.	....	....
	Dr.	....	....
	Dr.	....	....

The date of the dissertation defense: . . . . 2025.

Contents

Acknowledgment

With deep reverence and heartfelt gratitude, I offer my thanks to my supervisor, Eszter Novák Gselmann, whose presence has been a guiding light and a steady hand through every step of this journey. In her wisdom, kindness, and grace, I have found not only a mentor but an enduring source of inspiration. Her influence will echo quietly in all that I do, both academically and beyond. Her guidance has shaped not only this thesis but the person I have become.

I extend my heartfelt thanks to the reviewers and the members of my committee for their insightful guidance and direction. I am also profoundly grateful to my colleagues, whose kindness and generosity made this journey even more rewarding.

To my dearest parents, who carried my dreams long before I could stand on my own. Your boundless love, quiet sacrifices, and unwavering belief in me gave me the strength to dream and the courage to endure. In moments of doubt, when even I faltered, your faith became my light. Without you, this journey would not have begun, nor could I have endured its trials. I may carry this milestone in my hand, but it was built, piece by piece, from your hopes, prayers, and dreams for me. With all my heart and soul I dedicate this work to you.

To my brother, who believes in tough love (mostly the tough part), but has always been there when it mattered the most. To my sister, who argues with me like it is a full-time job, but still calls because she knows I am stuck adulting far from home. To my friends, who caught my tears, shared my anxieties, carried my doubts and never let me fall alone. My sincere gratitude flows from the core of my being, and I thank you all with deep appreciation.

A sincere thank you to each of my teachers, from my early years to today, who have believed in me and supported me at every step of the way.

Last but not least, to my grandparents, whose pride in my accomplishments knew no bounds. Though they are gone, but their faith in me lives on, and I remain forever grateful.

Notations

$\mathbb{C}$	the set of complex numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Z}$	the set of integers
$\mathbb{N}$	the set of positive integers
$\mathbb{K}$	a field of characteristic zero
$\mathbb{F}$	a subfield of $\mathbb{K}$
$\mathbb{F}^{\times}$	the set of units of the field $\mathbb{F}$
A^*	the diagonalization of an n-additive function A
Δ	the difference operator
$\mathscr{D}_n(\mathbb{F}, \mathbb{K})$	the set of derivations of order at most n defined on $\mathbb{F}$ with values in $\mathbb{K}$
$\mathscr{O}_n(\mathbb{F}, \mathbb{K})$	the set of differential operators of order atmost n defined on $\mathbb{F}$ with values in $\mathbb{K}$
$\mathscr{M}_n(\mathbb{F}, \mathbb{K})$	the set of monomials of degree n from $\mathbb{F}$ to $\mathbb{K}$

Introduction

G. Ancochea was likely the first who investigated additive mappings from a ring into another ring which also fulfill a 'polynomial equation'. More concretely, in [**4**] he described those additive functions that preserve squares. Later, these results were strengthened by (among others) Kaplansky [**25**] and Jacobson–Rickart [**24**].

Recall that if R, R' are rings, then the mapping $\varphi\colon R \to R'$ is called a *homomorphism* if

$$\varphi(a+b) = \varphi(a) + \varphi(b) \qquad (a, b \in R)$$

and

$$\varphi(ab) = \varphi(a)\varphi(b) \qquad (a, b \in R)\,.$$

Furthermore, the function $\varphi : R \to R'$ is an *anti-homomorphism* if

$$\varphi(a+b) = \varphi(a) + \varphi(b) \qquad (a, b \in R)$$

and

$$\varphi(ab) = \varphi(b)\varphi(a) \qquad (a, b \in R)\,.$$

From now on, the set of the positive integers will be denoted by $\mathbb{N}$. Let $n \in \mathbb{N}, n \geq 2$ be fixed. The function $\varphi\colon R \to R'$ is called an *n-Jordan homomorphism* if

$$\varphi(a+b) = \varphi(a) + \varphi(b) \qquad (a, b \in R)$$

and

$$\varphi(a^n) = \varphi(a)^n \qquad (a \in R)\,.$$

In case $n = 2$, the mapping φ is called a *Jordan homomorphism*. It was G. Ancochea who first explored the connection between Jordan homomorphisms and homomorphisms, see [**4**]. These results have since been generalized and expanded in various ways, as seen in works [**24**], [**25**], [**36**]. In [**23**] I.N. Herstein showed that if φ is a Jordan homomorphism of a ring R *onto* a prime ring R' of characteristic different from 2 and 3, then either φ is a homomorphism or an anti-homomorphism.

In addition to homomorphisms, derivations also play an essential role in the theory of rings and fields. Concerning this notion, we will follow [**27**, Chapter 14].

Let Q be a ring and let P be a subring of Q. A function $d\colon P \to Q$ is called a *derivation* if it is additive, i.e.

$$d(x+y) = d(x) + d(y) \quad (x, y \in P)$$

and also satisfies the so-called *Leibniz rule*, i.e. equation

$$d(xy) = d(x)y + xd(y) \quad (x, y \in P)\,.$$

It is well-known that in case of additive functions, Hamel bases play an important role. As [**27**, Theorem 14.2.1] shows in case of derivations, algebraic bases are fundamental.

THEOREM 1. *Let $(\mathbb{K}, +, \cdot)$ be a field of characteristic zero, let $(\mathbb{F}, +, \cdot)$ be a subfield of $(\mathbb{K}, +, \cdot)$, let S be an algebraic base of $\mathbb{K}$ over $\mathbb{F}$, if it exists, and let $S = \emptyset$ otherwise. Let $f\colon \mathbb{F} \to \mathbb{K}$ be a derivation. Then, for every function $u\colon S \to \mathbb{K}$, there exists a unique derivation $g\colon \mathbb{K} \to \mathbb{K}$ such that $g|_{\mathbb{F}} = f$ and $g|_S = u$.*

Let $\mathbb{F} \subset \mathbb{C}$ be a field. In Chapter 2, we examine generalized monomial functions $f, g\colon \mathbb{F} \to \mathbb{C}$ (of possibly different degree) that also satisfy

$$f(P(x)) = Q(g(x)) \qquad (x \in \mathbb{F})\,,$$

where $P \in \mathbb{F}[x]$ and $Q \in \mathbb{C}[x]$ are given (classical) polynomials. The results of this chapter appeared in the article Gselmann–Iqbal [**17**].

In particular, we will focus on generalized monomials $f\colon \mathbb{F} \to \mathbb{C}$ of degree n under the condition that the mapping

$$\mathbb{F} \ni x \longmapsto f(P(x))$$

is a (normal) polynomial, where n is a positive integer and $P \in \mathbb{F}[x]$ is a polynomial.

Similar to homomorphisms, characterization theorems related to derivations also have extensive literature, see e.g. the monographs [**27, 35**].

According to a classical result in connection to derivations, if $\mathbb{F}$ is a subfield of the field $\mathbb{K}$ with characteristic zero, $P \in \mathbb{F}[x]$ is a polynomial and the additive function $a\colon \mathbb{F} \to \mathbb{K}$ fulfills

$$a(P(x)) = P'(x)a(x) \qquad (x \in \mathbb{F})\,,$$

then a is a derivation.

It is important to note that all the aforementioned problems can be viewed as special cases of a more general problem. Indeed, let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F} \subset \mathbb{K}$ be a subfield, let further $P \in \mathbb{F}[x]$ and $Q \in \mathbb{K}[x_1, x_2]$ be given polynomials and $a\colon \mathbb{F} \to \mathbb{K}$ be an additive function such that

$$a(P(x)) = Q(x, a(x)) \qquad (x \in \mathbb{F})\,.$$

The key question then arises: Does this identity imply that the additive function a takes a "special form"? For certain polynomials P and Q, in the case of classical results, the unknown additive function a is a homomorphism, a derivation, or a linear combination of these. Naturally, the question arises as to whether this is the case for all polynomials P and Q. Similar questions can be raised about generalized monomial functions instead of additive functions: assume that n and k are positive integers, $P_{i,j} \in \mathbb{F}[x]$ and $P \in \mathbb{K}[z, x_{1,1}, \dots, x_{n,k}]$ are given polynomials for $i = 1, \dots, n; j = 1, \dots, k$. Suppose further that $f_1, \dots, f_n \colon \mathbb{F} \to \mathbb{K}$ are generalized monomials (of possibly different degree) such that

$$(\bullet)\quad P(x, f_1(P_{1,1}(x)), \dots, f_1(P_{1,k}(x)), \dots, f_n(P_{n,1}(x)), \dots, f_n(P_{n,k}(x))) = 0$$

holds for all $x \in \mathbb{F}$? Is it true that the monomial functions here necessarily have a 'special' form? If so, is there a method to determine these special forms?

In the case where the unknown generalized monomial functions are additive, some results can be found e.g. in [**10, 11, 12, 13, 20, 21**]. The papers [**16, 17**] contain related results, but there the unknown functions were assumed to be quadratic. The Chapter 3 and 4 is a continuation of these.

We also mention papers [**3, 7, 6, 8, 9**] where the authors also dealt with quadratic functions that satisfy additional identities, typically an alternative equation, or a conditional equation. As the papers [**1**] and [**2**] (and also their references) show, such and similar so-called polynomial equations as $(\bullet)$ can appear in many areas of mathematics. Of course, most often in algebra, the theory of functional equations, but also in the field of probability theory or statistics.

One of the main challenges in solving equation $(\bullet)$ is that it contains a single independent variable but n unknown functions. Thus, the primary goal is usually to be able to increase the 'degree of freedom' provided by this independent variable. In other words, we would like to formulate an equation for the unknown functions that is equivalent to equation $(\bullet)$ but contains many more independent variables. This is achieved by the assumption that the functions $f_1, \dots, f_n$ in the equation are *monomial* functions. More precisely, the so-called Polarization formula gives the possibility that (if certain additional conditions are met) the unknown functions in the equation satisfy a Levi-Civita equation

$$f(xy) = \sum_{i=1}^{k} g_i(x)h_i(y)$$

on $\mathbb{F}^{\times}$, i.e. on the *multiplicative* structure of the domain. Therefore, our main objective in Chapter 3 is to determine all those quadratic functions q that satisfy a Levi-Civita equation on the multiplicative structure, i.e., that can be written as

$$q(xy) = \sum_{i=1}^{k} g_i(x)h_i(y)$$

with some positive integer k and with some appropriate functions g_i, h_i, $i = 1, \ldots, k$. For this, those quadratic functions q that satisfy the equations

$$q(xy) = q(x)q(y) \qquad \left(x, y \in \mathbb{F}^{\times}\right)$$

and

$$q(xy) = x^2 q(y) + q(x)y^2 \qquad \left(x, y \in \mathbb{F}^{\times}\right),$$

respectively, must be determined first.
In Chapter 4, we focus on addressing this problem by examining the equation

$$f(x^2) = \kappa \cdot x^n f(x) \qquad (x \in \mathbb{F})$$

for the unknown monomial function $f\colon \mathbb{F} \to \mathbb{K}$. Our objective is to present a method for determining the solutions to this functional equation. Our study, inspired by the work in [**7**], seeks to build upon the work of the authors who investigated real quadratic functions f that satisfy the equation

$$f(x^2) = K \cdot x^2 f(x) \qquad (x \in \mathbb{R}).$$

Their research laid the groundwork for our further exploration of functional equations involving monomial functions. Assuming that the functions involved are generalized monomials, we started the systematic examination of polynomial equations in the papers Gselmann–Iqbal [**17, 18**]. Our work delves into a natural extension of the aforementioned equation, as we set out to uncover the properties of monomial functions $f\colon \mathbb{F} \to \mathbb{K}$ of degree n that satisfy the equation

$$f(x^2) = \kappa \cdot x^n f(x) \qquad (x \in \mathbb{F}).$$

As we will demonstrate, similar to the additive and quadratic functions, the monomial functions in the equation in some cases can be represented with the aid of homomorphisms and higher-order derivations.

CHAPTER 1

Preliminaries

1. Generalized polynomial functions

Polarization formula for multi-additive functions and the symmetrization method will play an important role while proving our results. However, first we need to familiarize ourselves with some basic concepts, such as multi-additive functions, generalized monomial functions, and generalized polynomial functions. In this section the most important notations and statements are summarized, based on the monograph Székelyhidi [**32**].

Definition 1. *Let G, S be commutative semigroups, $n \in \mathbb{N}$ and let $A\colon G^n \to S$ be a function. We say that A is* n-additive *if it is a homomorphism of G into S in each variable. If $n = 1$ or $n = 2$ then the function A is simply termed to be* additive *or* bi-additive*, respectively.*

The *diagonalization* or *trace* of an n-additive function $A\colon G^n \to S$ is defined as

$$A^*(x) = A(x, \ldots, x) \qquad (x \in G).$$

As a direct consequence of the definition each n-additive function $A\colon G^n \to S$ satisfies

$$\begin{aligned} &A(x_1, \ldots, x_{i-1}, kx_i, x_{i+1}, \ldots, x_n) \\ &\qquad = kA(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \qquad (x_1, \ldots, x_n \in G) \end{aligned}$$

for all $i = 1, \ldots, n$, where $k \in \mathbb{N}$ is arbitrary. The same identity holds for any $k \in \mathbb{Z}$ provided that G and S are groups, and for $k \in \mathbb{Q}$, provided that G and S are linear spaces over the rationals. For the diagonalization of A we have

$$A^*(kx) = k^n A^*(x) \qquad (x \in G).$$

The above notion can also be extended for the case $n = 0$ by letting $G^0 = G$ and by calling 0-additive any constant function from G to S.

One of the most important theoretical results concerning multiadditive functions is the so-called *Polarization formula*, that briefly expresses that every n-additive symmetric function is *uniquely* determined by its diagonalization under some conditions on the domain as well as on the range. Suppose that G is a commutative semigroup and S is a commutative group. The action of the *difference operator* Δ on a function $f\colon G \to S$ is defined by the formula

$$\Delta_y f(x) = f(x+y) - f(x) \qquad (x, y \in G)\,.$$

Note that the addition in the argument of the function is the operation of the semigroup G and the subtraction means the inverse of the operation of the group S.

THEOREM 2 (Polarization formula). *Suppose that G is a commutative semigroup, S is a commutative group, $n \in \mathbb{N}$. If $A\colon G^n \to S$ is a symmetric, n-additive function, then for all $x, y_1, \ldots, y_m \in G$ we have*

$$\Delta_{y_1,\ldots,y_m} A^*(x) = \begin{cases} 0 & \text{if} \quad m > n \\ n!A(y_1, \ldots, y_m) & \text{if} \quad m = n. \end{cases}$$

COROLLARY 1. *Suppose that G is a commutative semigroup, S is a commutative group, $n \in \mathbb{N}$. If $A\colon G^n \to S$ is a symmetric, n-additive function, then for all $x, y \in G$*

$$\Delta_y^n A^*(x) = n!A^*(y).$$

LEMMA 1. *Let $n \in \mathbb{N}$ and suppose that the multiplication by $n!$ is surjective in the commutative semigroup G or injective in the commutative group S. Then for any symmetric, n-additive function $A\colon G^n \to S$, $A^* \equiv 0$ implies that A is identically zero, as well.*

DEFINITION 2. *Let G and S be commutative semigroups, a function $p\colon G \to S$ is called a* generalized polynomial *from G to S if it has a representation as the sum of diagonalizations of symmetric multi-additive functions from G to S. In other words, a function $p\colon G \to S$ is a generalized polynomial if and only if, it has a representation*

$$p = \sum_{k=0}^{n} A_k^*,$$

where n is a nonnegative integer and $A_k\colon G^k \to S$ is a symmetric, k-additive function for each $k = 0, 1, \ldots, n$. In this case we also say that p is a generalized polynomial of degree at most n.

Let n be a nonnegative integer, functions $p_n\colon G \to S$ of the form

$$p_n = A_n^*,$$

where $A_n\colon G^n \to S$ is symmetric and n-additive function are the so-called generalized monomials of degree n.

Remark 1. *Obviously, generalized monomials of degree* 0 *are constant functions and generalized monomials of degree* 1 *are additive functions.*

Furthermore, generalized monomials of degree 2 *will be termed* quadratic functions*.*

Definition 3. *Let* G *be a commutative semigroup. We say that the nonzero function* $m\colon G \to \mathbb{C}$ *is an* exponential *if*

$$m(x+y) = m(x)m(y)$$

holds for all x, y *in* G*.*

Remark 2. *Recall that on a commutative semigroup, the identically* 1 *function is always an exponential.*

Definition 4. *Let* G *be a commutative semigroup,* n *be a positive integer and* $m\colon G \to \mathbb{C}$ *be an exponential. The function* $f\colon G \to \mathbb{C}$ *is called a* generalized exponential monomial of degree at most n *corresponding to the exponential* m*, if there exists a generalized polynomial* $p\colon G \to \mathbb{C}$ *such that*

$$f(x) = p(x)m(x) \qquad (x \in G)\,.$$

Finite sums of generalized exponential monomials are called *generalized exponential polynomials.*

2. Polynomial functions

There are several notions of polynomials defined on groups, as provided by Laczkovich [**28**]. One of them is the notion of generalised polynomials, that we introduced in Section 1. As we will see in the upcomming sections, not only this notion, but also that of *(normal) polynomials* will be important. The definitions and results presented here can be found in [**32**].

Throughout this section G is assumed to be a commutative group (written additively).

Definition 5. *Polynomials are elements of the algebra generated by additive functions over* G*. Namely, if* n *is a positive integer,* $P\colon \mathbb{C}^n \to \mathbb{C}$ *is a (classical) complex polynomial in* n *variables and* $a_k\colon G \to \mathbb{C}$ $(k = 1, \ldots, n)$ *are additive functions, then the function*

$$x \longmapsto P(a_1(x), \ldots, a_n(x))$$

is a polynomial and, also conversely, every polynomial can be represented in such a form.

Remark 3. *For the sake of easier distinction, at some places polynomials will be called normal polynomials.*

Remark 4. *We recall that the elements of* $\mathbb{N}^n$ *for any positive integer* n *are called (n-dimensional)* multi-indices*. Addition, multiplication and inequalities between multi-indices of the same dimension are defined component-wise. Further, we define* x^α *for any* n*-dimensional multi-index* α *and for any* $x=(x_1,\dots,x_n)$ *in* $\mathbb{C}^n$ *by*

$$x^\alpha=\prod_{i=1}^n x_i^{\alpha_i},$$

where we always adopt the convention $0^0=1$*. We also use the notation* $|\alpha|=\alpha_1+\cdots+\alpha_n$*. With these notations any polynomial of degree at most* N *on the commutative semigroup* G *has the form*

$$p(x)=\sum_{|\alpha|\leq N} c_\alpha a(x)^\alpha \qquad (x\in G)\,,$$

where $c_\alpha\in\mathbb{C}$ *and* $a=(a_1,\dots,a_n)\colon G\to\mathbb{C}^n$ *is an additive function. Furthermore, the* homogeneous term of degree k *of* p *is*

$$\sum_{|\alpha|=k} c_\alpha a(x)^\alpha.$$

It is easy to see that each polynomial, that is, any function of the form

$$x\longmapsto P(a_1(x),\dots,a_n(x)),$$

where n is a positive integer, $P\colon\mathbb{C}^n\to\mathbb{C}$ is a (classical) complex polynomial in n variables and $a_k\colon G\to\mathbb{C}$ $(k=1,\dots,n)$ are additive functions, is a generalized polynomial. The converse however is in general not true. A complex-valued generalized polynomial p defined on a commutative group G is a polynomial *if and only if* its variety (the linear space spanned by its translates) is of *finite* dimension. To make the situation more clear, here we also recall Theorem 13.4 from Székelyhidi [**33**].

Theorem 3. *The torsion free rank of a commutative group is finite* if and only if *every generalized polynomial on the group is a polynomial.*

Definition 6. *Let* G *be commutative semigroup,* n *be a positive integer and* $m\colon G\to\mathbb{C}$ *be an exponential. The function* $f\colon G\to\mathbb{C}$ *is called a* exponential monomial of degree at most n *corresponding to the exponential* m*, if there exists a polynomial* $p\colon G\to\mathbb{C}$ *such that*

$$f(x)=p(x)m(x) \qquad (x\in G)\,.$$

Finite sums of exponential monomials are called *exponential polynomials.*

In the next section the lemma below will be used, see [**16**, Lemma 6], too.

LEMMA 2. *Let k and n be positive integers and $f\colon \mathbb{F} \to \mathbb{C}$ be a generalized monomial of degree n, where $\mathbb{F}$ is assumed to be a field with* $\mathrm{char}(\mathbb{F}) = 0$. *Then the mapping*

$$\mathbb{F} \ni x \longmapsto f(x^k)$$

is a generalized monomial of degree $n \cdot k$.

Henceforth, not only the notion of (exponential) polynomials, but also that of *decomposable functions* will be used. The basics of this concept are due to Shulman [**31**], besides this we heavily rely on the work of Laczkovich [**29**].

DEFINITION 7. *Let G be a group and $n \in \mathbb{N}, n \geq 2$. A function $F\colon G^n \to \mathbb{C}$ is said to be* decomposable *if it can be written as a finite sum of products $F_1 \cdots F_k$, where all F_i depend on disjoint sets of variables.*

REMARK 5. *Without loss of generality we can suppose that $k = 2$ in the above definition, that is, decomposable functions are those mappings that can be written in the form*

$$F(x_1, \ldots, x_n) = \sum_E \sum_j A_j^E B_j^E,$$

where E runs through all non-void proper subsets of $\{1, \ldots, n\}$ and for each E and j the function A_j^E depends only on variables x_i with $i \in E$, while B_j^E depends only on the variables x_i with $i \notin E$.

The connection between decomposable functions and generalized exponential polynomials was described in Laczkovich [**29**].

THEOREM 4. *Let G be a commutative topological semigroup with unit. A continuous function $f\colon G \to \mathbb{C}$ is a generalized exponential polynomial* if and only if *there is a positive integer $n \geq 2$ such that the mapping*

$$G^n \ni (x_1, \ldots, x_n) \longmapsto f(x_1 + \cdots + x_n)$$

is decomposable.

The notion of derivations can be extended in several ways. We will employ the concept of higher order derivations according to Reich [**30**] and Unger–Reich [**34**]. For further results on characterization theorems on higher order derivations consult e.g. [**10, 11, 13**] and [**20**].

DEFINITION 8. *Let $\mathbb{F} \subset \mathbb{C}$ be a field. The identically zero map is the only* derivation of order zero. *For each $n \in \mathbb{N}$, an additive mapping $f\colon \mathbb{F} \to \mathbb{C}$ is termed to be a* derivation of order n, *if there exists $B\colon \mathbb{F} \times$*

$\mathbb{F} \to \mathbb{C}$ such that B is a bi-derivation of order $n-1$ (that is, B is a derivation of order $n-1$ in each variable) and

$$f(xy) - xf(y) - f(x)y = B(x, y) \qquad (x, y \in \mathbb{F})\,.$$

The set of derivations of order n of the field $\mathbb{F}$ will be denoted by $\mathscr{D}_n(\mathbb{F})$.

Remark 6. *Since $\mathscr{D}_0(\mathbb{F}) = \{0\}$, the only bi-derivation of order zero is the identically zero function, thus $f \in \mathscr{D}_1(\mathbb{F})$ if and only if*

$$f(xy) = xf(y) + f(x)y \qquad (x, y \in \mathbb{F})\,,$$

that is, the notions of first order derivations and derivations coincide. On the other hand, for any $n \in \mathbb{N}$ the set $\mathscr{D}_n(\mathbb{F}) \backslash \mathscr{D}_{n-1}(\mathbb{F})$ is nonempty because $d_1 \circ \cdots \circ d_n \in \mathscr{D}_n(\mathbb{F})$, but $d_1 \circ \cdots \circ d_n \notin \mathscr{D}_{n-1}(\mathbb{F})$, where $d_1, \ldots, d_n \in \mathscr{D}_1(\mathbb{F})$ are non-identically zero derivations.

The main result of [**26**] is Theorem 1.1 that reads in our settings as follows.

Theorem 5. *Let $\mathbb{F} \subset \mathbb{C}$ be a field and let n be a positive integer. Then, for every function $D\colon \mathbb{F} \to \mathbb{C}$, $D \in \mathscr{D}_n(\mathbb{F})$ if and only if D is additive on $\mathbb{F}$, $D(1) = 0$, and $\dfrac{D}{\mathrm{id}}$, as a map from the group $\mathbb{F}^{\times}$ to $\mathbb{C}$, is a generalized polynomial of degree at most n. Here* id *stands for the identity map defined on $\mathbb{F}$.*

Remark 7. *Recall the notions of generalized polynomials, polynomials, generalized exponential polynomials and exponential polynomials, respectively were introduced on commutative groups (i.e; on an algebraic structure where we have one binary operation, the addition). Note that however in Definition 8 and in Theorem 5 we considered functions that are defined on $\mathbb{F} \subset \mathbb{C}$, where we have two binary operations. On fields two commutative groups arise naturally. Namely, the additive group $(\mathbb{F}, +)$, but also the multiplicative group $(\mathbb{F}^{\times}, \cdot)$ where $\mathbb{F}^{\times} = \{x \in \mathbb{F} \mid x \neq 0\}$. The previous theorem says in an illustrative way that these functions are well-connected to both the additive and multiplicative structures. The same holds for field homomorphisms, since the assumption that the function $\varphi\colon \mathbb{F} \to \mathbb{C}$ is a field homomorphism can be expressed (using previous notations) as φ is a additive function on the additive group $(\mathbb{F}, +)$, while it is an exponential on the multiplicative group $(\mathbb{F}^{\times}, \cdot)$. This observation will play a very important role in our research. Our goal will be to show that if a monomial function satisfies the conditions presented there, then this function is necessarily a generalized polynomial on the* multiplicative *structure. This is why homomorphisms and higher order derivations appear in these theorems.*

Definition 9. *Let* $\mathbb{F} \subset \mathbb{K}$ *be fields of characteristic zero. We say that the map* $D\colon \mathbb{F} \to \mathbb{K}$ *is a* differential operator of order at most n *if* D *is the linear combination, with coefficients from* $\mathbb{F}$*, of finitely many maps of the form* $d_1 \circ \cdots \circ d_k$*, where* $d_1, \ldots, d_k$ *are* $\mathbb{K}$*-valued derivations on* $\mathbb{F}$ *and* $k \leq n$*. If* $k = 0$ *then we interpret* $d_1 \circ \cdots \circ d_k$ *as the identity function. We denote by* $\mathscr{O}_n(\mathbb{F}, \mathbb{K})$ *the set of differential operators of order at most* n *defined on* $\mathbb{F}$*. We say that the order of a differential operator* D *is* n *if* $D \in \mathscr{O}_n(\mathbb{F}, \mathbb{K}) \setminus \mathscr{O}_{n-1}(\mathbb{F}, \mathbb{K})$ *(where* $\mathscr{O}_{-1}(\mathbb{F}, \mathbb{K}) = \emptyset$*, by definition).*

The main result of [**26**] is Theorem 1.1 which reads in our settings as follows.

Theorem 6. *Let* $\mathbb{F} \subset \mathbb{K}$ *be fields of characteristic zero and let* n *be a positive integer. Then, for every function* $D\colon \mathbb{F} \to \mathbb{C}$*, the following are equivalent.*

(i) $D \in \mathscr{D}_n(\mathbb{F}, \mathbb{K})$
(ii) $D \in \mathrm{cl}\,(\mathscr{O}_n(\mathbb{F}, \mathbb{K}))$
(iii) D *is additive on* $\mathbb{F}$*,* $D(1) = 0$*, and* D/id*, as a map from the group* $\mathbb{F}^\times$ *to* $\mathbb{K}$*, is a generalized polynomial of degree at most* n*. Here* id *stands for the identity map defined on* $\mathbb{F}$*.*

3. Moment sequences

A *composition* of a nonnegative integer n is a sequence of nonnegative integers $\alpha = (\alpha_k)_{k\in\mathbb{N}}$ such that

$$n = \sum_{k=1}^{\infty} \alpha_k.$$

For a positive integer r, an r*-composition* of a nonnegative integer n is a composition $\alpha = (\alpha_k)_{k\in\mathbb{N}}$ with $\alpha_k = 0$ for $k > r$.

Given a sequence of variables $x = (x_k)_{k\in\mathbb{N}}$ and compositions $\alpha = (\alpha_k)_{k\in\mathbb{N}}$ and $\beta = (\beta_k)_{k\in\mathbb{N}}$ we define

$$\alpha! = \prod_{k=1}^{\infty} \alpha_k, \quad |\alpha| = \sum_{k=1}^{\infty} \alpha_k, \quad x^\alpha = \prod_{k=1}^{\infty} x_k^{\alpha_k}, \quad \binom{\alpha}{\beta} = \prod_{k=1}^{\infty} \binom{\alpha_k}{\beta_k}.$$

Furthermore, $\beta \leq \alpha$ means that $\beta_k \leq \alpha_k$ for all $k \in \mathbb{N}$ and $\beta < \alpha$ stands for $\beta \leq \alpha$ and $\beta \neq \alpha$.

Definition 10. *Let* G *be an Abelian group,* r *a positive integer, and for each multi-index* α *in* $\mathbb{N}^r$ *let* $f_\alpha\colon G \to \mathbb{C}$ *be a continuous function. We say that* $(f_\alpha)_{\alpha\in\mathbb{N}^r}$ *is a* generalized moment sequence of rank r*, if*

$$f_\alpha(x + y) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} f_\beta(x) f_{\alpha-\beta}(y) \tag{1}$$

holds whenever x, y are in G. The function f_0, where 0 is the zero element in $\mathbb{N}^r$, is called the generating function of the sequence.

Remark 8. *(a) For $r = 1$, instead of multi-indices, we have 'ordinary' indices and (1) is nothing but*

$$f_\alpha(x+y) = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} f_\beta(x) f_{\alpha-\beta}(y)$$

for each x, y in G and nonnegative integer α, yielding that generalized moment functions of rank 1 are moment sequences.

(b) For $\alpha = (0, \dots, 0)$ we have

$$f_{0,\dots,0}(x+y) = f_{0,\dots,0}(x) \cdot f_{0,\dots,0}(y)$$

for each x, y in G hence $f_{0,\dots,0} = m$ is an exponential, or identically zero. An easy computation shows that if $f_{0,\dots,0}$ is identically zero, then for any multi-index $\alpha \in \mathbb{N}^r$ we have $f_\alpha \equiv 0$. Thus we always assume that the generating function is not identically zero, hence it is always an exponential.

(c) The sequence of functions $(f_\alpha)_{\alpha\in\mathbb{N}^r}$ is a generalized moment sequence of rank r associated with the nonzero exponential $f_{0,\dots,0} = m$ if and only if *$(f_\alpha/m)_{\alpha\in\mathbb{N}^r}$ is a generalized moment sequence of rank r associated with the exponential which is identically one.*

(d) Generalized moment sequences of rank r are exponential monomials.

As a supplement to part (d), we recall Theorem 2 from [**14**]. In the statement below, B_α stands for the Bell polynomial corresponding to the multi-index $\alpha \in \mathbb{N}^r$.

Theorem 7. *Let G be a commutative group, r a positive integer, and for each α in $\mathbb{N}^r$, let $f_\alpha \colon G \to \mathbb{K}$ be a function. If the sequence of functions $(f_\alpha)_{\alpha\in\mathbb{N}^r}$ forms a generalized moment sequence of rank r, then there exists an exponential $m \colon G \to \mathbb{K}$ and a sequence of $\mathbb{K}$-valued additive functions $a = (a_\alpha)_{\alpha\in\mathbb{N}^r}$ such that for every multi-index α in $\mathbb{N}^r$ and x in G we have*

$$f_\alpha(x) = B_\alpha(a(x)) m(x).$$

And also conversely, if r is a positive integer, $m \colon G \to \mathbb{C}$ is an exponential and $a = (a_\alpha)_{\alpha\in\mathbb{N}^r}$ is a sequence of complex-valued additive functions and for all $\alpha \in \mathbb{N}^r$, we define the function by the above formula, then $(f_\alpha)_{\alpha\in\mathbb{N}^r}$ forms a generalized moment sequence of rank r associated with the exponential m.

CHAPTER 2

Monomial functions, normal polynomials and polynomial equations

Based on the paper Gselmann–Iqbal [**17**], in this chapter, we consider generalized monomial functions $f, g\colon \mathbb{F} \to \mathbb{C}$ (of possibly different degree) that also fulfill

$$f(P(x)) = Q(g(x)) \qquad (x \in \mathbb{F}),$$

where $P \in \mathbb{F}[x]$ and $Q \in \mathbb{C}[x]$ are given (classical) polynomials.

1. Preliminary results

Henceforth let $\mathbb{F} \subset \mathbb{C}$ be a field, n be a positive integer and $P \in \mathbb{F}[x]$ be a polynomial. In what follows we will study generalized monomials $f\colon \mathbb{F} \to \mathbb{C}$ of degree n under the condition that the mapping

$$\mathbb{F} \ni x \longmapsto f(P(x))$$

is a (normal) polynomial.

At first we show that instead of polynomials P, we always may restrict ourselves to (classical) monomials. For this we need the following statement which is in some sense an extension of Lemma 2.

Lemma 3. *Let $\mathbb{F} \subset \mathbb{C}$ be a field, $n \in \mathbb{N}$, $\alpha_1, \dots, \alpha_n$ be non-negative integers and $F\colon \mathbb{F} \to \mathbb{C}$ be a symmetric and n-additive function. Then the function $g\colon \mathbb{F} \to \mathbb{C}$ defined by*

$$g(x) = F(x^{\alpha_1}, \dots, x^{\alpha_n}) \qquad (x \in \mathbb{F})$$

is a generalized monomial of degree $(\alpha_1 + \cdots + \alpha_n)$.

Proof. Let $\mathbb{F} \subset \mathbb{C}$ be a field, $n \in \mathbb{N}$, $\alpha_1, \dots, \alpha_n$ be non-negative integers and $F\colon \mathbb{F} \to \mathbb{C}$ be a symmetric and n-additive function and consider the function $g\colon \mathbb{F} \to \mathbb{C}$ defined by

$$g(x) = F(x^{\alpha_1}, \dots, x^{\alpha_n}) \qquad (x \in \mathbb{F}).$$

Let $N = \alpha_1 + \cdots + \alpha_n$ and define the mapping $\mathscr{F} : \mathbb{F}^N \to \mathbb{C}$ through

$$\begin{aligned}\mathscr{F}&(x_1, \ldots, x_N)\\ &= \frac{1}{N!} \sum_{\sigma \in \mathscr{S}_n} F(x_{\sigma(1)} \cdots x_{\sigma(\alpha_1)}, \ldots, x_{\sigma(N-\alpha_n+1)} \cdots x_{\sigma(N)})\\ &\qquad\qquad\qquad\qquad\qquad\qquad (x_1, \ldots, x_N \in \mathbb{F}) .\end{aligned}$$

Since F is a symmetric and n-additive function, $\mathscr{F}$ is also symmetric and N-additive, further we have

$$\mathscr{F}(x, \ldots, x) = F(x^{\alpha_1}, \ldots, x^{\alpha_n}) = g(x) \qquad (x \in \mathbb{F}) .$$

Thus g can be represented as the trace of a symmetric and $(\alpha_1 + \cdots + \alpha_n)$-additive mapping, showing that the function g is indeed a generalized monomial of degree $(\alpha_1 + \cdots + \alpha_n)$. □

LEMMA 4. *Let $k, n \in \mathbb{N}$, $k \geq 2$, $\mathbb{F} \subset \mathbb{C}$ be a field, $P \in \mathbb{F}[x]$ be a (classical) polynomial of degree k with leading coefficient 1 and $f : \mathbb{F} \to \mathbb{C}$ be a generalized monomial of degree n. If the mapping*

$$\mathbb{F} \ni x \longmapsto f(P(x))$$

is a normal polynomial, then the mapping

$$\mathbb{F} \ni x \longmapsto f(x^k)$$

is a normal polynomial as well.

PROOF. Let $k, n \in \mathbb{N}$, $\mathbb{F} \subset \mathbb{C}$ be a field, $P \in \mathbb{F}[x]$ be a (classical) polynomial of degree k and $f : \mathbb{F} \to \mathbb{C}$ be a generalized monomial of degree n. Suppose further that the mapping

$$\mathbb{F} \ni x \longmapsto f(P(x))$$

is a normal polynomial.

Since $P \in \mathbb{F}[x]$ is a (classical) polynomial of degree k, we have

$$P(x) = \sum_{l=0}^{k} \alpha_l x^l \qquad (x \in \mathbb{F}) ,$$

with some constants $\alpha_l \in \mathbb{F}$, $l = 0, 1, \ldots, k$ such that $\alpha_k = 1$. Further, as $f : \mathbb{F} \to \mathbb{C}$ is a monomial of degree n, there exists a uniquely determined symmetric and n-additive function $F : \mathbb{F}^n \to \mathbb{C}$ such that

$$f(x) = F(x, \ldots, x) \qquad (x \in \mathbb{F}) .$$

This together yield that there exists a positive integer m, linearly independent additive functions $a_1, \ldots, a_m$ over $\mathbb{C}$ and a complex polynomial

$Q \in \mathbb{C}[x]$ such that

$$
\begin{aligned}
F\left(\sum_{l=0}^{k}\alpha_l x^l, \ldots, \sum_{l=0}^{k}\alpha_l x^l\right)
&= Q(a_1(x), \ldots, a_m(x)) \\
&= \sum_{\beta\in\mathbb{N}^m |\beta|\leq kn} c_\beta a(x)^\beta \qquad (x \in \mathbb{F})\,.
\end{aligned}
$$

Using that F is symmetric and additive in each of its variables, we get that

$$
\begin{aligned}
F(x^k, \ldots, x^k) + \lambda_{k-1,k,\ldots,k}F(\alpha_{k-1}x^{k-1}, \alpha_k x^k, \ldots, \alpha_k x^k) + \cdots \\
\cdots + \lambda_{0,\ldots,0}F(\alpha_0, \ldots, \alpha_0) = \sum_{\substack{\beta\in\mathbb{N}^m\\ |\beta|\leq kn}} c_\beta a(x)^\beta
\end{aligned}
$$

for all $x \in \mathbb{F}$ with some complex constants λ_γ, $\gamma \in \mathbb{N}^n$, $|\gamma| \leq kn$. Due to Lemma 3, the left and also the right hand side of this identity is a generalized polynomial (being linear combinations of generalized monomials) and these agree for all possible values. This can however happen only if the monomial terms are the same on each side. From this we get especially that

$$
F(x^k, \ldots, x^k) = \sum_{\substack{\beta\in\mathbb{N}^m\\ |\beta|= kn}} c_\beta a(x)^\beta,
$$

showing that the mapping

$$
\mathbb{F} \ni x \longmapsto f(x^k)
$$

is a normal polynomial as well. □

Remark 9. *According to the previous lemma once $f\colon \mathbb{F} \to \mathbb{C}$ is a generalized monomial of degree n such that the mapping*

$$
\mathbb{F} \ni x \longmapsto f(P(x))
$$

is a normal polynomial, then the mapping

$$
\mathbb{F} \ni x \longmapsto f(x^k)
$$

is a normal polynomial as well. This enables us to restrict ourselves to considering generalized monomials for which the mapping

$$
\mathbb{F} \ni x \longmapsto f(x^k)
$$

is a normal polynomial for a fixed $k \geq 2$. At the same time, we have to emphasize that the assumption that the mapping

$$
\mathbb{F} \ni x \longmapsto f(P(x))
$$

is a normal polynomial for a fixed polynomial $P \in \mathbb{F}[x]$ *is more restrictive than the previous one. To illustrate this let us consider the following example. Let* $f\colon \mathbb{F} \to \mathbb{C}$ *be a quadratic function and let*

$$P(x) = x^2 + \alpha_1 x + \alpha_0 \qquad (x \in \mathbb{F}),$$

where $\alpha_1, \alpha_0 \in \mathbb{F}$ *are fixed. The assumption that the mapping*

$$\mathbb{F} \ni x \longmapsto f(P(x))$$

is a normal polynomial means that we have

$$F(x^2 + \alpha_1 x + \alpha_0, x^2 + \alpha_1 x + \alpha_0) = Q(a_1(x), \dots, a_m(x)) \qquad (x \in \mathbb{F})$$

with appropriate linearly independent additive functions $a_1, \dots, a_m\colon \mathbb{F} \to \mathbb{C}$ *and a complex polynomial* $Q \in \mathbb{C}[x_1, \dots, x_m]$*, where* $F\colon \mathbb{F}^2 \to \mathbb{C}$ *denotes the uniquely determined bi-additive mapping whose trace is the quadratic function* f*. Using the symmetry and also the bi-additivity of* F*, we get that*

$$\begin{aligned} F(x^2, x^2) + 2F(x^2, \alpha_1 x) + 2F(x^2, \alpha_0) \\ + F(\alpha_1 x, \alpha_1 x) + 2F(\alpha_1 x, \alpha_0) + F(\alpha_0, \alpha_0) \\ = Q(a_1(x), \dots, a_m(x)) \quad (x \in \mathbb{F}). \end{aligned}$$

Due to Lemma 3, the mappings

$$\mathbb{F} \ni x \longmapsto F(x^i, x^j)$$

are generalized monomials of degree $(i+j)$*. Further, generalized monomials of different degree are linearly independent. Therefore, not only the mapping*

$$\mathbb{F} \ni x \longmapsto F(x^2, x^2)$$

is a normal polynomial, but also the mappings

$$\begin{array}{lcl} \mathbb{F} \ni & x & \longmapsto F(x^2, \alpha_1 x) \\ \mathbb{F} \ni & x & \longmapsto 2F(x^2, \alpha_0) + F(\alpha_1 x, \alpha_1 x) \\ \mathbb{F} \ni & x & \longmapsto F(\alpha_1 x, \alpha_0) \\ \mathbb{F} \ni & x & \longmapsto F(\alpha_0, \alpha_0), \end{array}$$

too. Obviously, this is always true for the last two functions. Nevertheless, the fact that the first three functions have this property carries more information than that only the first of them is a normal polynomial.

Our second lemma says that while considering this problem, we may restrict ourselves to homogeneous (normal) polynomials.

Lemma 5. *Let* $k, n \in \mathbb{N}$*,* $k \geq 2$*,* $\mathbb{F} \subset \mathbb{C}$ *be a field and* $f\colon \mathbb{F} \to \mathbb{C}$ *be a generalized monomial of degree* n*. If the mapping*

$$\mathbb{F} \ni x \longmapsto f(x^k)$$

is a normal polynomial, then there exists a homogeneous *complex polynomial* $\tilde{P}$ *and there are linearly independent, complex valued additive functions* $a_1, \ldots, a_m$ *on* $\mathbb{F}$ *such that*

$$f(x^k) = \tilde{P}(a_1(x), \ldots, a_m(x)) \qquad (x \in \mathbb{F}),$$

in other words, we have

$$f(x^k) = \sum_{\substack{\alpha \in \mathbb{N}^m \\ |\alpha| = kn}} \lambda_\alpha a^\alpha(x) = \sum_{\substack{\alpha_1, \ldots, \alpha_m \geq 0 \\ \alpha_1 + \cdots + \alpha_m = kn}} \lambda_{\alpha_1, \ldots, \alpha_m} a_1^{\alpha_1}(x) \cdots a_m^{\alpha_m}(x)$$

for each $x \in \mathbb{F}$.

Proof. Let $k, n \in \mathbb{N}$, $k \geq 2$, $\mathbb{F}$ be a field and $f\colon \mathbb{F} \to \mathbb{C}$ be a generalized monomial of degree n. Assume further that the mapping

$$\mathbb{F} \ni x \longmapsto f(x^k)$$

is a normal polynomial. Then due to Lemma 2 this mapping is a generalized monomial of degree kn and hence it is $\mathbb{Q}$-homogeneous of degree kn.

If additionally, this mapping is a normal polynomial, there exists a complex polynomial P and there are linearly independent, complex valued additive functions $a_1, \ldots, a_m$ on $\mathbb{F}$ such that

$$f(x^k) = P(a_1(x), \ldots, a_m(x)) = P(a(x)) = \sum_{l=0}^{N} \sum_{\substack{\alpha \in \mathbb{N}^m \\ |\alpha| = l}} \lambda_\alpha a^\alpha(x) \qquad (x \in \mathbb{F}).$$

Let $r \in \mathbb{Q}$ be arbitrary and let us substitute rx in place of x in the above identity. Using the $\mathbb{Q}$-homogeneity of the involved functions, we deduce

$$r^{kn} f(x^k) - \sum_{l=0}^{N} r^l \sum_{\substack{\alpha \in \mathbb{N}^m \\ |\alpha| = l}} \lambda_\alpha a^\alpha(x) = 0 \qquad (r \in \mathbb{Q}, x \in \mathbb{F}).$$

Observe that the left hand side of this identity is a polynomial in r that is identically zero. So all of its coefficients should vanish, yielding that

$$f(x^k) = \sum_{\substack{\alpha \in \mathbb{N}^m \\ |\alpha| = kn}} \lambda_\alpha a^\alpha(x)$$

holds for all $x \in \mathbb{F}$. $\square$

2. Illustrative examples

At first glance the assumption of the lemma above, i.e., the mapping

$$\mathbb{F} \ni x \longmapsto f(x^k)$$

is a normal polyomial, seems a bit artificial. Nevertheless, the following examples show that this is not the case.

EXAMPLE 1. *Let k be a positive integer, $\varphi_1, \ldots, \varphi_k \colon \mathbb{F} \to \mathbb{C}$ be linearly independent homomorphisms and $\lambda_{i,j} \in \mathbb{C}$ for all $i, j = 1, \ldots, k$. Then the mapping $f \colon \mathbb{F} \to \mathbb{C}$ defined by*

$$f(x) = \sum_{i,j=1}^{k} \lambda_{i,j} \varphi_i(x) \varphi_j(x) \qquad (x \in \mathbb{F})$$

is a quadratic function. Further if $n \in \mathbb{N}$, then we also have

$$f(x^n) = \sum_{i,j=1}^{k} \lambda_{i,j} \varphi_i(x)^n \varphi_j(x)^n \qquad (x \in \mathbb{F}) .$$

In other words, we have

$$f(x^n) = P(\varphi_1(x), \ldots, \varphi_k(x)) \qquad (x \in \mathbb{F}) ,$$

where the k-variable complex, homogeneous polynomial P is defined by

$$P(x_1, \ldots, x_k) = \sum_{i,j=1}^{k} \lambda_{i,j} x_i^n x_j^n \qquad (x_1, \ldots, x_k \in \mathbb{C}) .$$

EXAMPLE 2. *Suppose now that $\mathbb{F}$ is a subfield of $\mathbb{C}$. Let k be a positive integer, $d_1, \ldots, d_k \colon \mathbb{F} \to \mathbb{C}$ be linearly independent derivations and $\lambda_{i,j} \in \mathbb{C}$ for all $i, j = 1, \ldots, k$. Then the mapping $f \colon \mathbb{F} \to \mathbb{C}$ defined by*

$$f(x) = \sum_{i,j=1}^{k} \lambda_{i,j} d_i(x) d_j(x) \qquad (x \in \mathbb{F})$$

is a quadratic function. Further if $n \in \mathbb{N}$, then we also have

$$f(x^n) = \sum_{i,j=1}^{k} \lambda_{i,j} n^2 x^{2n-2} d_i(x) d_j(x) \qquad (x \in \mathbb{F}) .$$

In other words,

$$f(x^n) = P(x, d_1(x), \ldots, d_k(x)) \qquad (x \in \mathbb{F}) ,$$

where the $(k+1)$-variable complex polynomial P is defined by

$$P(x_1, \ldots, x_k, z) = \sum_{i,j=1}^{n} \lambda_{i,j} n^2 z^{2n-2} x_i x_j \qquad (x_1, \ldots, x_k, z \in \mathbb{C}) .$$

EXAMPLE 3. *Assume in this example that* $\mathbb{F}$ *is a subfield of* $\mathbb{C}$*. Let* k *be a positive integer and* $d\colon \mathbb{F} \to \mathbb{C}$ *be a derivation and define the quadratic function* $f\colon \mathbb{F} \to \mathbb{C}$ *by*

$$f(x) = d^k(x^2) = \underbrace{d \circ \cdots \circ d}_{k \text{ times}}(x^2) \qquad (x \in \mathbb{C}) .$$

Further, if n *is a positive integer, then we have*

$$f(x^n) = d^k(x^{2n}) = \sum_{\substack{l_1,\ldots,l_{2n} \geq 0 \\ l_1+\cdots+l_{2n}=k}} \binom{k}{l_1, \ldots, l_{2n}} d^{l_1}(x) \cdots d^{l_{2n}}(x)$$

for all $x \in \mathbb{F}$*. In other words, we have*

$$f(x^n) = P(x, d(x), d \circ d(x), \ldots, d^k(x)) \qquad (x \in \mathbb{F}) .$$

REMARK 10. *All of the above examples can easily be generalized from quadratic functions to monomial functions. To get similar examples instead of quadratic functions, for monomials of degree* n*, where* n *is fixed it is enough to consider the mappings*

$$f(x) = \sum_{i=1}^{k} \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha|=n}} \lambda_i \Phi_i(x)^{\alpha} \qquad (x \in \mathbb{F}) ,$$

$$f(x) = \sum_{i=1}^{k} \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha|=n}} \lambda_i D_i(x)^{\alpha} \qquad (x \in \mathbb{F}) ,$$

and

$$f(x) = d^k(x^n) = \underbrace{d \circ \cdots \circ d}_{k \text{ times}}(x^n) \qquad (x \in \mathbb{F}) ,$$

here

$$\Phi_i(x) = (\varphi_{1,i}(x), \ldots, \varphi_{n,i}(x)) \qquad (x \in \mathbb{F}) ,$$

and

$$D_i(x) = (d_{1,i}(x), \ldots, d_{n,i}(x)) \qquad (x \in \mathbb{F}) ,$$

where the functions $\varphi_{l,i}$ *are homomorphisms, while* d *and* $d_{l,i}$ *are derivations for all possible indices* l *and* i*.*

3. Main results

Regarding 'polynomial equations' for generalized monomials, we note that in the literature, there are several results for additive functions $a\colon \mathbb{F} \to \mathbb{C}$ that also satisfy a polynomial equation. Based on the results presented above, the following statement can be deduced.

PROPOSITION 1. *Let $\mathbb{F} \subset \mathbb{C}$ be a field, $k \in \mathbb{N}$, $k \geq 2$ and $P \in \mathbb{Q}[x]$ be a (classical) polynomial of degree k.*

(i) If the additive function $a\colon \mathbb{F} \to \mathbb{C}$ fulfills

$$a(P(x)) = P(a(x)) \qquad (x \in \mathbb{F}),$$

then there exists a homomorphism $\varphi\colon \mathbb{F} \to \mathbb{C}$ such that $a(x) = a(1)\varphi(x)$ for all $x \in \mathbb{F}$. Further, we also have $a(1) \in \{0, 1\}$.

(ii) If the additive function $a\colon \mathbb{F} \to \mathbb{C}$ fulfills

$$a(P(x)) = P'(x)a(x) \qquad (x \in \mathbb{F}),$$

then a is a derivation.

PROOF. (i) Let $\mathbb{F} \subset \mathbb{C}$ be a field, $k \in \mathbb{N}$, $k \geq 2$ and $P \in \mathbb{Q}[x]$ be a (classical) polynomial of degree k. Suppose further that the additive function $a\colon \mathbb{F} \to \mathbb{C}$ fulfills

$$a(P(x)) = P(a(x)) \qquad (x \in \mathbb{F}).$$

In other words, we have

$$a\left(\sum_{i=0}^{k} \alpha_i x^i\right) = \sum_{i=0}^{k} \alpha_i a(x)^i$$

for all $x \in \mathbb{F}$ with some rational numbers $\alpha_k, \ldots, \alpha_0$. Observe that this especially yields that the mapping

$$\mathbb{F} \ni x \longmapsto a\left(\sum_{i=0}^{k} \alpha_i x^i\right)$$

is a normal polynomial. Due to Lemma 4 we infer that then the mapping

$$\mathbb{F} \ni x \longmapsto a(x^k)$$

is also a normal polynomial, where the $\mathbb{Q}$-homogeneity of a was used, too. However, then necessarily

$$a(x^k) = a(x)^k \qquad (x \in \mathbb{F})$$

holds. From this, we deduce (e.g. using the results of [**23**]) that there exists a homomorphism $\varphi\colon \mathbb{F} \to \mathbb{C}$ such that $a(x) = a(1)\varphi(x)$ for all $x \in \mathbb{F}$. Substituting this back to our equation, we finally get that the only possibility is that $a(1) \in \{0, 1\}$.

(ii) Using a similar reasoning that we did in case (i), here we deduce that the assumptions imply that necessarily

$$a(x^k) = kx^{k-1}a(x) \qquad (x \in \mathbb{F})$$

holds for the additive function $a\colon \mathbb{F} \to \mathbb{C}$. Using some classical characterization theorems concerning derivations (for instance the results of [**27**, Chapter 14]), we finally get that a is indeed a derivation. □

REMARK 11. *We emphasize that the results of the previous statement are classical ones. Nevertheless, we would like to indicate that from one hand the problem we would like to consider in this paper has some prior results both in algebra and the theory of functional equations. Further, with the help of the statements presented here, the proofs can be significantly simplified (at least for mappings $a\colon \mathbb{F} \to \mathbb{C}$).*

In the papers [**10, 11, 13, 20, 21**] *further results can be found concerning additive functions that also fulfill certain polynomial equations.*

REMARK 12. *We also note that related problems have already been considered by Z. Boros and E. Garda–Mátyás in* [**7, 5**] *by Z. Boros and R. Menzer in* [**8**] *and also by M. Amou in* [**3**]. *In these papers the authors consider real monomial functions, which satisfy certain conditional equations on a specified planar curve. In* [**16**], *the polynomial equation $f(P(x)) = Q(f(x))$ for the monomial function $f\colon \mathbb{F} \to \mathbb{C}$ was considered.*

The simplest special case of the problem we are interested in is when the generalized monomial $f\colon \mathbb{F} \to \mathbb{C}$ is of degree 1, i.e., when f is an additive function. In this regard, we have the following statement.

THEOREM 8. *Let k be a positive integer, $\mathbb{F} \subset \mathbb{C}$ be a field and $a\colon \mathbb{F} \to \mathbb{C}$ be an additive function. If the mapping*

$$\mathbb{F} \ni x \longmapsto a(x^k)$$

is a (normal) polynomial, then a is a generalized higher order derivation.

PROOF. Let k be a positive integer, $\mathbb{F} \subset \mathbb{C}$ be a field and $a\colon \mathbb{F} \to \mathbb{C}$ be an additive function. Suppose further that the mapping

$$\mathbb{F} \ni x \longmapsto a(x^k)$$

is a (normal) polynomial. Due to Lemma 5, we can assume that this mapping is a *homogeneous* (normal) polynomial, that is, we have

$$a(x^k) = \sum_{\substack{\alpha_1,\dots,\alpha_m \geq 0 \\ \alpha_1+\cdots+\alpha_m=k}} \lambda_{\alpha_1,\dots,\alpha_m} a_1^{\alpha_1}(x)\cdots a_m^{\alpha_m}(x)$$

for all $x \in \mathbb{F}$. Observe that both sides of this identity is the trace of a symmetric and k-additive mapping. Indeed, the left hand side is the trace of the symmetric k-additive function

$$A(x_1,\dots,x_k) = a(x_1\cdots x_k) \qquad (x_1,\dots,x_k \in \mathbb{F})$$

while the right hand side is the trace of the symmetric k-additive mapping

$$\begin{aligned}\widetilde{A}(x_1, \dots, x_k) \\ = \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} \sum_{\substack{\alpha_1, \dots, \alpha_m \geq 0 \\ \alpha_1 + \dots + \alpha_m = k}} \lambda_{\alpha_1, \dots, \alpha_m} a_1(x_{\sigma(1)}) \cdots a_1(x_{\sigma(\alpha_1)}) \cdots \times \\ \times \cdots a_m(x_{\sigma(k-\alpha_m+1)}) \cdots a_m(x_{\sigma(k)}) \\ (x_1, \dots, x_k \in \mathbb{F}) .\end{aligned}$$

Therefore we have

$$a(x_1 \cdots x_k) = \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} \sum_{\substack{\alpha_1, \dots, \alpha_m \geq 0 \\ \alpha_1 + \dots + \alpha_m = k}} \lambda_{\alpha_1, \dots, \alpha_m} a_1(x_{\sigma(1)}) \cdots a_1(x_{\sigma(\alpha_1)}) \times \\ \times a_m(x_{\sigma(k-\alpha_m+1)}) \cdots a_m(x_{\sigma(k)})$$

for all $x_1, \dots, x_k \in \mathbb{F}$.

In other words, the mapping

$$\mathbb{F}^{\times k} \ni (x_1, \dots, x_k) \longmapsto a(x_1 \cdots x_k)$$

is decomposable. Thus from Theorem 4 we deduce that $a\colon \mathbb{F}^{\times} \to \mathbb{C}$ is a generalized exponential polynomial on the multiplicative group $\mathbb{F}^{\times}$ corresponding to the identity function, as exponential. Using Theorem 5, we finally obtain that a is a generalized higher order derivation. □

Now we turn to quadratic functions. During the proof of Theorem 9 we will use [**21**, Theorem 4.5] in a rather special case that is the following lemma.

Lemma 6. *Let $n \in \mathbb{N}$ and $\mathbb{F}$ be a field and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$. If the additive function $f\colon \mathbb{F} \to \mathbb{C}$ satisfies*

$$\alpha_1 f(x^{2n}) + \alpha_2 f(x^n)^2 + \alpha_3 f(x)^{2n} = 0$$

for all $x \in \mathbb{F}$. Then there exist a complex constant α and a homomorphism $\varphi\colon \mathbb{F} \to \mathbb{C}$ such that

$$f(x) = \alpha\varphi(x) \qquad (x \in \mathbb{F}) .$$

Theorem 9. *Let $n \in \mathbb{N}$, $n \geq 2$ and $\mathbb{F} \subset \mathbb{C}$ be a field. Assume that $f\colon \mathbb{F} \to \mathbb{C}$ is a quadratic function, while $a\colon \mathbb{F} \to \mathbb{C}$ is additive and we have*

$$f(x^n) = a(x)^{2n} \qquad (x \in \mathbb{F}) . \tag{2}$$

Then there exists a complex constant $\alpha \in \mathbb{C}$ and a homomorphism $\varphi\colon \mathbb{F} \to \mathbb{C}$ such that

$$a(x) = \alpha\varphi(x) \qquad \text{and} \qquad f(x) = \alpha^{2n}\varphi(x)^2 \qquad (x \in \mathbb{F}) .$$

And also conversely, if we define the functions a and f through the above formula then they satisfy equation (15) *for all $x \in \mathbb{F}$.*

PROOF. Since f is quadratic, there exists a uniquely determined symmetric and bi-additive mapping $F\colon \mathbb{F}^2 \to \mathbb{C}$ such that

$$F(x,x) = f(x) \qquad (x \in \mathbb{F}).$$

Define the mapping $E\colon \mathbb{F}^{2n} \to \mathbb{C}$ by

$$\begin{aligned} &E(x_1, \dots, x_{2n}) \\ &\quad = \frac{1}{(2n)!} \sum_{\sigma \in \mathscr{S}_{2n}} F(x_{\sigma(1)} \cdots x_{\sigma(n)}, x_{\sigma(n+1)} \cdots x_{\sigma(2n)}) - a(x_1) \cdots a(x_{2n}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x_1, \dots, x_{2n} \in \mathbb{F}), \end{aligned}$$

where $\mathscr{S}_{2n}$ denotes the symmetric group of order $2n$. Since F is bi-additive and a is additive, the mapping E is a symmetric and $2n$-additive mapping. Further its trace is

$$E(x, \dots, x) = F(x^n, x^n) - a(x)^{2n} = f(x^n) - a(x)^{2n} = 0 \qquad (x \in \mathbb{F}),$$

due to the above equation. At the same time, its trace uniquely determines E. Thus the function E vanishes identically, that is,

$$\frac{1}{(2n)!} \sum_{\sigma \in \mathscr{S}_{2n}} F(x_{\sigma(1)} \cdots x_{\sigma(n)}, x_{\sigma(n+1)} \cdots x_{\sigma(2n)}) - a(x_1) \cdots a(x_{2n}) = 0 \tag{3}$$

holds for all $x_1, \dots, x_{2n} \in \mathbb{F}$. With the substitution $x_i = 1$ for $i = 1, \dots, 2n$ we get that

$$F(1,1) - a(1)^{2n} = f(1) - a(1)^{2n} = 0.$$

Let now $x \in \mathbb{F}$ be arbitrary and let

$$x_1 = x \qquad \text{and} \qquad x_i = 1 \qquad \text{for} \qquad i = 2, \dots, 2n$$

to deduce that

$$\mu_1 F(x,1) - \mu_2 a(x) = 0 \qquad (x \in \mathbb{F})$$

holds with some nonzero complex numbers μ_1, μ_2. Thus we have

$$F(x,1) = \lambda a(x) \qquad (x \in \mathbb{F}) \tag{4}$$

with an appropriate complex constant λ.

Let again $x \in \mathbb{F}$ be arbitrary and let now

$$x_1 = x,\ x_2 = x \qquad \text{and} \qquad x_i = 0 \qquad \text{for} \qquad i = 3, \dots, 2n$$

in (3) to obtain that

$$\nu_1 F(x,x) + \nu_2 F(x^2, 1) + \nu_3 a(x)^2 = 0$$

holds for all $x \in \mathbb{F}$ with certain complex constants ν_1, ν_2 and ν_3. The latter identity together with (4) yields that there exists complex constants α_1, α_2 such that

$$F(x,x) = \alpha_1 a(x^2) + \alpha_2 a(x)^2 \qquad (x \in \mathbb{F}),$$

in other words, we have

$$f(x) = \alpha_1 a(x^2) + \alpha_2 a(x)^2 \qquad (x \in \mathbb{F}).$$

If we substitute this representation back to (15), we obtain that

$$\alpha_1 a(x^{2n}) + \alpha_2 a(x^n)^2 = a(x)^{2n}$$

for all $x \in \mathbb{F}$. According to Lemma 6, there exists a complex constant α and a homomorphism $\varphi\colon \mathbb{F} \to \mathbb{C}$ such that

$$a(x) = \alpha \cdot \varphi(x) \qquad (x \in \mathbb{F}),$$

and hence

$$f(x) = \alpha^{2n} \varphi(x)^2$$

is fulfilled for all $x \in \mathbb{F}$.

The converse is an easy computation. □

THEOREM 10. *Let $\mathbb{F} \subset \mathbb{C}$ be a field, $f\colon \mathbb{F} \to \mathbb{C}$ be a quadratic function, while $a_1, a_2\colon \mathbb{F} \to \mathbb{C}$ be additive functions. Then equation*

$$f(x^2) = a_1(x)^2 a_2(x)^2$$

holds for all $x \in \mathbb{F}$ if and only if there exist homomorphisms $\varphi_1, \varphi_2\colon \mathbb{F} \to \mathbb{C}$ such that

$$f(x) = f(1) \cdot \varphi_1(x)\varphi_2(x) \qquad \text{and} \qquad a_i(x) = a_i(1)\varphi_i(x)\, (x \in \mathbb{F}, i = 1,2).$$

PROOF. Let $\mathbb{F} \subset \mathbb{C}$ be a field, $f\colon \mathbb{F} \to \mathbb{C}$ be a quadratic function, while $a_1, a_2\colon \mathbb{F} \to \mathbb{C}$ be additive function. Suppose further that for all $x \in \mathbb{F}$, equation

$$f(x^2) = a_1(x)^2 a_2(x)^2$$

is fulfilled. Since f is quadratic, there exists a uniquely determined symmetric and bi-additive mapping F such that

$$F(x,x) = f(x) \qquad (x \in \mathbb{F}).$$

Define the mapping Φ on $\mathbb{F}^4$ by

$$\begin{aligned}\Phi(x_1,x_2,x_3,x_4) = {} & \frac{F(x_1 x_4, x_2 x_3) + F(x_1 x_3, x_2 x_4) + F(x_1 x_2, x_3 x_4)}{3} \\ & -\frac{1}{6}\left[a_1(x_1)\, a_1(x_2)\, a_2(x_3)\, a_2(x_4) + a_1(x_1)\, a_2(x_2)\, a_1(x_3)\, a_2(x_4)\right. \\ & + a_2(x_1)\, a_1(x_2)\, a_1(x_3)\, a_2(x_4) + a_1(x_1)\, a_2(x_2)\, a_2(x_3)\, a_1(x_4) \\ & \left. + a_2(x_1)\, a_1(x_2)\, a_2(x_3)\, a_1(x_4) + a_2(x_1)\, a_2(x_2)\, a_1(x_3)\, a_1(x_4)\right]\end{aligned}$$

whenever $x_1, x_2, x_3, x_4 \in \mathbb{F}$. Since F is symmetric and bi-additive and a_1 and a_2 are additive, the function Φ is a symmetric 4-additive mapping. Moreover its trace is

$$F\left(x^2, x^2\right) - a_1(x)^2\, a_2(x)^2 = 0 \qquad (x \in \mathbb{F})\,.$$

Thus Φ is identically zero on $\mathbb{F}^4$. From this we especially get that

$$2F\left(x^2, 1\right) - \, a_1(1)^2\, a_2(1)\, a_2(x^2) - \, a_1(1)\, a_2(1)^2\, a_1(x^2) = 0 \qquad (x \in \mathbb{F})$$

and also

$$\begin{aligned} &2\,F\left(x^2, 1\right) + 4\,F\,(x,x) - a_1(1)^2\, a_2(x)^2 \\ &\qquad - 4\,a_1(1)\,a_2(1)\,a_1(x)\,a_2(x) - a_2(1)^2\,a_1(x)^2 = 0 \qquad (x \in \mathbb{F})\,. \end{aligned}$$

These identities together imply that

$$\begin{aligned} &4\,F\,(x,x) + a_1(1)^2\, a_2(1)\, a_2(x^2) + a_1(1)\, a_2(1)^2\, a_1(x^2) - a_1(1)^2\, a_2(x)^2 \\ &\qquad - 4\,a_1(1)\,a_2(1)\,a_1(x)\,a_2(x) - a_2(1)^2\,a_1(x)^2 = 0 \\ &\qquad\qquad (x \in \mathbb{F})\,. \end{aligned}$$

In other words, we have

$$\begin{aligned} 4F(x,x) = &-a_1(1)^2\, a_2(1)\, a_2(x^2) - a_1(1)\, a_2(1)^2\, a_1(x^2) + a_1(1)^2\, a_2(x)^2 \\ &+ 4\,a_1(1)\,a_2(1)\,a_1(x)\,a_2(x) + a_2(1)^2\,a_1(x)^2 \end{aligned}$$

for all $x \in \mathbb{F}$. Since F is a symmetric and bi-additive mapping, we have

$$\begin{aligned} 4F(x,y) = &-a_1(1)^2\, a_2(1)\, a_2(x\,y) - a_1(1)\, a_2(1)^2\, a_1(x\,y) \\ &+ 2\,\left(a_1(1)\,a_2(1)\,a_1(x)\,a_2(y) + a_1(1)\,a_2(1)\,a_2(x)\,a_1(y)\right) \\ &+ a_1(1)^2\,a_2(x)\,a_2(y) + a_2(1)^2\,a_1(x)\,a_1(y) \end{aligned}$$

for all $x, y \in \mathbb{F}$. Combining this with our functional equation, we deduce

$$\begin{aligned} &-a_1(1)^2\, a_2(1)\, a_2(x^4) - a_1(1)\, a_2(1)^2\, a_1(x^4) + a_1(1)^2\, a_2(x^2)^2 \\ &\quad + 4\,a_1(1)\,a_2(1)\,a_1(x^2)\,a_2(x^2) + a_2(1)^2\,a_1(x^2)^2 - 4\,a_1(x)^2\,a_2(x)^2 = 0 \\ &\qquad (x \in \mathbb{F}). \end{aligned}$$

Observe that if $a_1(1) \cdot a_2(1) = 0$, then F and thus f is identically zero. So we may assume $a_1(1) \cdot a_2(1) \neq 0$. Without the loss of generality we can then suppose that $a_1(1) = a_1(1) = 1$, otherwise we consider the mappings

$$\tilde{f}(x) = \frac{f(x)}{a_1(1) \cdot a_2(1)} \qquad \text{and} \qquad \tilde{a}_i(x) = \frac{a_i(x)}{a_i(1)} \qquad (x \in \mathbb{F}, i = 1, 2)\,.$$

Then this equation reduces to

$$\begin{aligned}&a_2(x^4) + a_1(x^4)\\&\qquad = a_2(x^2)^2 + a_1(x^2)^2 + 4\,a_1(x^2)\,a_2(x^2) - 4\,a_1(x)^2\,a_2(x)^2 \quad (x \in \mathbb{F})\,.\end{aligned}$$

After symmetrization, we get that this is equivalent to

$$\begin{aligned}&- a_2(x_1\,x_2\,x_3\,x_4) - a_1(x_1\,x_2\,x_3\,x_4)\\&+ \frac{1}{3}\,[a_2(x_1\,x_2)\,a_2(x_3\,x_4) + a_2(x_1\,x_3)\,a_2(x_2\,x_4) + a_2(x_2\,x_3)\,a_2(x_1\,x_4)]\\&+ \frac{2}{3}\,[a_1(x_1\,x_2)\,a_2(x_3\,x_4) + a_2(x_1\,x_2)\,a_1(x_3\,x_4) + a_1(x_1\,x_3)\,a_2(x_2\,x_4)\\&+a_2(x_1\,x_3)\,a_1(x_2\,x_4) + a_1(x_2\,x_3)\,a_2(x_1\,x_4) + a_2(x_2\,x_3)\,a_1(x_1\,x_4)]\\&+ \frac{1}{3}\,[a_1(x_1\,x_2)\,a_1(x_3\,x_4) + a_1(x_1\,x_3)\,a_1(x_2\,x_4) + a_1(x_2\,x_3)\,a_1(x_1\,x_4)]\\&- \frac{2}{3}\;[a_1(x_1)\,a_1(x_2)\,a_2(x_3)\,a_2(x_4) + a_1(x_1)\,a_2(x_2)\,a_1(x_3)\,a_2(x_4)\\&+a_2(x_1)\,a_1(x_2)\,a_1(x_3)\,a_2(x_4) + a_1(x_1)\,a_2(x_2)\,a_2(x_3)\,a_1(x_4)\\&+a_2(x_1)\,a_1(x_2)\,a_2(x_3)\,a_1(x_4) + a_2(x_1)\,a_2(x_2)\,a_1(x_3)\,a_1(x_4)] = 0\\&(x_1, x_2, x_3, x_4 \in \mathbb{F})\,.\end{aligned}$$

From this we deduce that for all $x, y, z \in \mathbb{F}$ we have

$$\begin{aligned}&3\,a_2(x\,y\,z) + 3\,a_1(x\,y\,z)\\&\qquad = a_2(x)\,a_2(y\,z) + 2\,a_1(x)\,a_2(y\,z) + 2\,a_2(x)\,a_1(y\,z)\\&\qquad + a_1(x)\,a_1(y\,z) + a_2(y)\,a_2(x\,z) + 2\,a_1(y)\,a_2(x\,z)\\&\qquad + 2\,a_2(y)\,a_1(x\,z) + a_1(y)\,a_1(x\,z) + a_2(x\,y)\,a_2(z)\\&\qquad + 2\,a_1(x\,y)\,a_2(z) - 2\,a_1(x)\,a_2(y)\,a_2(z) - 2\,a_2(x)\,a_1(y)\,a_2(z)\\&\qquad - 2\,a_1(x)\,a_1(y)\,a_2(z) + 2\,a_2(x\,y)\,a_1(z) + a_1(x\,y)\,a_1(z)\\&\qquad - 2\,a_2(x)\,a_2(y)\,a_1(z) - 2\,a_1(x)\,a_2(y)\,a_1(z) - 2\,a_2(x)\,a_1(y)\,a_1(z).\end{aligned}$$

For all $z \in \mathbb{F}$, define the additive function by

$$\begin{aligned}A_z(x) &= 3a_1(xz) - a_1(x)\,[a_1(z) + 2a_2(x)]\\&\qquad + 3a_2(xz) - a_2(x)\,[a_2(z) + 2a_1(x)] \qquad (x \in \mathbb{F})\,.\end{aligned}$$

With this notation the above identity turns to

$$\begin{aligned}A_z(xy) &= a_1(x)g_z(y) + a_2(x)h_z(y)\\&\qquad + k_z(x)a_1(y) + l_z(x)a_2(y) \qquad (x, y, z \in \mathbb{F})\,,\end{aligned}$$

with appropriately defined functions g_z, h_z, k_z, l_z, yielding that the mapping A_z is a (normal) exponential polynomial of degree at most 4 on the multiplicative group $\mathbb{F}^{\times}$. Further,

(A) either the system a_1, a_2, k_z, l_z is linearly dependent
(B) or the system a_1, a_2, k_z, l_z is linearly independent.

Alternative (A) holds only if the functions a_1 and a_2 are the same (note that we assumed that $a_1(1) = a_2(1) = 1$), but then Theorem 9 applies and we deduce that there exists a homomorphism $\varphi\colon \mathbb{F} \to \mathbb{C}$ such that

$$f(x) = f(1)\varphi(x)^2 \qquad a_i(x) = a_i(1)\varphi(x) \qquad (x \in \mathbb{F}).$$

If alternative (B) holds, then we get that not only the mapping A_z, but also the functions a_1, a_2, k_z, l_z are (normal) exponential polynomials on the multiplicative group $\mathbb{F}^{\times}$ of degree at most four. Especially, a_1 and a_2 are linearly independent (normal) exponential polynomials of degree at most four. Checking all the possibilities, we finally get that this can happen only if a_1 and a_2 are exponentials on the multiplicative group $\mathbb{F}^{\times}$. Since these functions were assumed also to be additive on $\mathbb{F}$, we get that they are homomorphisms on $\mathbb{F}$.

Suming up, there exists homomorphisms $\varphi_1, \varphi_2\colon \mathbb{F} \to \mathbb{C}$ such that

$$f(x) = f(1)\varphi_1(x)\varphi_2(x) \qquad a_i(x) = a_i(1)\varphi_i(x) \qquad (x \in \mathbb{F}),$$

where the above constants $f(1), a_1(1)$ and $a_2(1)$ also fulfill

$$f(1) = a_1(1)a_2(1).$$

$\square$

The roots of this topic can be found in the papers [**16**] and [**17**]. The results presented there in connection with this problem can be considered as initial steps. Thus we close this chapter with several open questions and we would like to give some perspectives, too.

In the special cases we considered the above general problem, it turned out that the solutions of the functional equations are always (normal) monomials. Additionally, we also showed that the proof is much easier if we know this, see [**16**, Proposition 2]. We conjecture that this is true in general, too. Thus, prove or disprove that if the generalized monomial $f\colon \mathbb{F} \to \mathbb{K}$ solves equation

$$f(P(x)) = Q(f(x)) \qquad (x \in \mathbb{F}),$$

then f is a normal monomial, i.e., a (classical) polynomial of additive functions.

Remark 13. *The above problem is clearly meaningful for polynomials P and Q with degree one. At the same time, in this case we cannot expect* *representations similar to that ones that appeared in the statements proved in this chapter. As an example, here we consider only the case quadratic functions. Accordingly, assume that for the quadratic function* $f\colon \mathbb{F} \to \mathbb{C}$ *equation*

$$f(ax+b) = Af(x) + B \qquad (x \in \mathbb{F}),$$

with some $a, b \in \mathbb{F}$ *and* $A, B \in \mathbb{K}$. *Let us denote* $F\colon \mathbb{F}^2 \to \mathbb{K}$ *the uniquely determined symmetric, bi-additive function whose trace is* f. *With* $x = 0$ *we immediately get that* $f(b) = F(b, b) = B$. *Furthermore,*

$$F(ax+b, ax+b) = AF(x, x) + F(b, b) \qquad (x \in \mathbb{F}),$$

that is,

$$F(ax, ax) + 2F(ax, b) + F(b, b) = AF(x, x) + F(b, b) \qquad (x \in \mathbb{F}),$$

or after some simplification

$$(F(ax, ax) - AF(x, x)) + 2F(ax, b) = 0$$

for all $x \in \mathbb{F}$. *Since the left hand side of this equation is a generalized polynomial of degree two which has to be identically zero, all of its monomial terms should vanish. This means from one hand that*

$$F(ax, b) = 0 \qquad (x \in \mathbb{F}),$$

especially, $f(b, b) = B = 0$. *On the other hand, we also have*

$$F(ax, ax) = AF(x, x) \qquad (x \in \mathbb{F}),$$

from this however

$$F(ax, ay) = AF(x, y) \qquad (x, y \in \mathbb{F}),$$

follows, that is, the symmetric, bi-additive function is semi-homogeneous. *From* [**22**, Theorem 3] *it is known that a non-identically zero, symmetric and bi-additive function* F *fulfilling the above semi-homogeneity exists if and only if there are injective homomorphisms* $\delta_1, \delta_2\colon \mathbb{F} \to \mathbb{K}$ *such that*

$$\delta_1(a)\delta_2(a) = A.$$

Summing up, if $f\colon \mathbb{F} \to \mathbb{K}$ *is a non-identically zero quadratic function such that*

$$f(ax+b) = Af(x) + B \qquad (x \in \mathbb{F}),$$

with some $a, b \in \mathbb{F}$ *and* $A, B \in \mathbb{K}$, *then*

(i) $B = 0$.

(ii) for the uniquely determined symmetric and bi-additive function $F\colon \mathbb{F}^2 \to \mathbb{K}$, *identity*

$$F(ax, b) = 0$$

is satisfied for all $x \in \mathbb{F}$.

(iii) there are injective homomorphisms $\delta_1, \delta_2 \colon \mathbb{F} \to \mathbb{K}$ *such that*

$$\delta_1(a)\delta_2(a) = A.$$

Observe that for instance with $b = 0$*, with arbitrary fixed* $a \in \mathbb{Q}$ *and with* $A = a^2$ *the above identity is fulfilled by* any *quadratic function* $f \colon \mathbb{F} \to \mathbb{K}$ *(this is obviously consistent with the fact that quadratic functions are* $\mathbb{Q}$*-homogeneous of degree two). This shows that in this case we do not get in general any information for the form of the involved quadratic function* f*. With an analogous method, we obtain the same for higher-order generalized monomials.*

In this chapter only equations with one unknown function were considered. At the same time, the investigated problem can clearly be extended: let $P \in \mathbb{F}[x]$ and $Q \in \mathbb{K}[x]$ be polynomials, $f, g \colon \mathbb{F} \to \mathbb{K}$ be generalized polynomials (of possibly *different order*) such that $\deg(f)\deg(P) = \deg(g)\deg(Q)$. Prove or disprove that if

$$f(P(x)) = Q(g(x))$$

holds for all $x \in \mathbb{F}$, then f and g can be represented as products of homomorphisms.

A particularly interesting and presumably the simplest case of the following problem is when $\deg(g) = 1$, i.e., when $g \colon \mathbb{F} \to \mathbb{C}$ is an additive function.

In connection to this problem we recall that above we contributed (among others) to the special case

$$\deg(f) = 2 \qquad P(x) = x^n \qquad \text{and} \qquad Q(x) = x^{2n}.$$

The assumption that the field $\mathbb{K}$ (and thus the field $\mathbb{F}$) has to have zero characteristic is caused by the limitations of the Polarization formula, since n-additive functions are uniquely determined by their diagonalizations only if the characteristic of the domain is large enough or zero. Therefore, we formulate the following open problem.

Let $\mathbb{F}$ and $\mathbb{K}$ be fields, n be a positive integer, $P \in \mathbb{F}[x]$ and $Q \in \mathbb{K}[x]$ be polynomials. Determine those generalized monomials $f, g \colon \mathbb{F} \to \mathbb{K}$ of degree at most n that also fulfill equation

$$f(P(x)) = Q(f(x))$$

for each $x \in \mathbb{F}$.

A particularly interesting case of this problem is when at least one of the fields $\mathbb{F}$ and $\mathbb{K}$ has finite characteristic.

CHAPTER 3

Quadratic functions as solutions of polynomial equations

Polynomial equations are crucial in both algebra and the theory of functional equations. When the unknown functions in these equations are additive, there are already many known results. Still, there remain numerous open questions. In certain cases, classical results suggest that the unknown additive functions are either homomorphisms, derivations, or linear combinations of these. This raises the question of whether the solutions can be characterized when the unknown functions are not additive but instead generalized monomials. This chapter begins by focusing on quadratic functions, aiming to demonstrate that quadratic functions solving specific polynomial equations must take on a "special" form. We will also introduce a method for identifying these special forms. In particular, the primary goal of Chapter 3 is to identify all quadratic functions q that satisfy the Levi-Civita equation on the multiplicative structure, meaning those that can be expressed as

$$q(xy) = \sum_{i=1}^{k} g_i(x)h_i(y)$$

with some positive integer k and with some appropriate functions g_i, h_i, $i = 1, \ldots, k$. For this, those quadratic functions q that satisfy the equations

$$q(xy) = q(x)q(y) \qquad \big(x, y \in \mathbb{F}^{\times}\big)$$

and

$$q(xy) = x^2 q(y) + q(x)y^2 \qquad \big(x, y \in \mathbb{F}^{\times}\big),$$

respectively, must be determined first.

1. Preliminary results

In this section let $\mathbb{F}$ and $\mathbb{K}$ be fields with zero characteristic such that $\mathbb{F} \subset \mathbb{K}$. The following lemma is immediate.

LEMMA 7. *Let us endow* $\mathbb{F}^\times$ *and also* $\mathbb{K}$ *with the discrete topology and let* $\mathscr{C}(\mathbb{F}^\times, \mathbb{K})$ *denote the linear space of all those functions* $f\colon \mathbb{F}^\times \to \mathbb{K}$ *that are continuous. Then the set*

$$\mathscr{V} = \{f|_{\mathbb{F}^\times} \mid f\colon \mathbb{F} \to \mathbb{K} \text{ is quadratic}\}$$

is a closed, translation-invariant linear subspace of $\mathscr{C}(\mathbb{F}^\times, \mathbb{K})$*. So* $\mathscr{V} \subset \mathscr{C}(\mathbb{F}^\times, \mathbb{K})$ *is a* variety*. Note that the translate of a function* $f \in \mathscr{C}(\mathbb{F}^\times, \mathbb{K})$ *by an element* $y \in \mathbb{F}^\times$ *is defined as*

$$(\tau_y f) = f(x \cdot y) \qquad \big(x \in \mathbb{F}^\times\big)\,.$$

2. Illustrative examples

EXAMPLE 4. *Let* $\varphi_1, \varphi_2\colon \mathbb{F} \to \mathbb{K}$ *be homomorphisms and let us consider the function* $q\colon \mathbb{F} \to \mathbb{K}$ *defined by*

$$q(x) = \varphi_1(x)\varphi_2(x) \qquad (x \in \mathbb{F})\,.$$

Then q *is the trace of the symmetric and bi-additive mapping*

$$B(x,y) = \frac{1}{2}\left[\varphi_1(x)\varphi_2(y) + \varphi_1(y)\varphi_2(x)\right] \qquad (x, y \in \mathbb{F})\,.$$

So q *is a quadratic function. Further, we have*

$$\begin{aligned} q(xy) &= \varphi_1(xy)\varphi_2(xy) = (\varphi_1(x)\varphi_1(y)) \cdot (\varphi_2(x)\varphi_2(y)) \\ &= (\varphi_1(x)\varphi_2(x)) \cdot (\varphi_1(y)\varphi_2(y)) = q(x) \cdot q(y) \qquad (x, y \in \mathbb{F})\,. \end{aligned}$$

EXAMPLE 5. *Let* $d\colon \mathbb{F} \to \mathbb{K}$ *be a derivation and define the quadratic mapping* $q\colon \mathbb{F} \to \mathbb{K}$ *by*

$$q(x) = d(x^2) \qquad (x \in \mathbb{F})\,.$$

An easy computation shows that in this case, q *determines uniquely a symmetric and bi-additive mapping, namely*

$$B(x,y) = d(xy) \qquad (x, y \in \mathbb{F})\,.$$

Moreover, we have

$$\begin{aligned} \frac{q(xy)}{x^2y^2} &= \frac{1}{x^2y^2} d((xy)^2) = \frac{1}{x^2y^2} \cdot 2xyd(xy) = \frac{2}{xy}\left[xd(y) + d(x)y\right] \\ &= 2\frac{d(x)}{x} + 2\frac{d(y)}{y} = \frac{d(x^2)}{x^2} + \frac{d(y^2)}{y^2} \\ &= \frac{q(x)}{x^2} + \frac{q(y)}{y^2} \end{aligned}$$

for all $x, y \in \mathbb{F}^\times$*.*

EXAMPLE 6. *Analogously, if $n \geq 2$ is a positive integer then the mapping $f\colon \mathbb{F} \to \mathbb{K}$ defined by*

$$f(x) = d(x^n) \qquad (x \in \mathbb{F})$$

is a (generalized) monomial of degree n. The symmetric, n-additive mapping $B_n\colon \mathbb{F}^n \to \mathbb{K}$ determined by f is

$$B_n(x_1, \dots, x_n) = d(x_1 \cdots x_n) \qquad (x_1, \dots, x_n \in \mathbb{F})\,.$$

Further

$$\begin{aligned}\frac{f(xy)}{(xy)^n} &= \frac{1}{(xy)^n} d((xy)^n) = \frac{1}{(xy)^n} \cdot n(xy)^{n-1} d(xy) \\ &= \frac{n}{xy}\left[d(x)y + xd(y)\right] = n\frac{d(x)}{x} + n\frac{d(y)}{y} = \frac{d(x^n)}{x^n} + \frac{d(y^n)}{y^n} \\ &= \frac{f(x)}{x^n} + \frac{f(y)}{y^n}\end{aligned}$$

for all $x, y \in \mathbb{F}^{\times}$.

For any positive integer $n \geq 2$, define the mapping $\pi_n\colon \mathbb{F} \to \mathbb{K}$ by

$$\pi_n(x) = x^n \qquad (x \in \mathbb{F})\,.$$

Observe that in this case, the above examples show that if we consider the (generalized) monomial of degree n

$$f(x) = d(x^n) \qquad (x \in \mathbb{F})\,,$$

then the mapping $\dfrac{f}{\pi_n}\colon \mathbb{F}^{\times} \to \mathbb{K}$ will be additive on the Abelian group $(\mathbb{F}^{\times}, \cdot)$, that is, we have

$$\frac{f}{\pi_n}(xy) = \frac{f}{\pi_n}(x) + \frac{f}{\pi_n}(y)$$

for all $x, y \in \mathbb{F}^{\times}$.

With the notations and assumptions of Lemma 7, Example 1 shows that the mapping q defined with the aid of the homomorphisms φ_1 and φ_2 by the formula

$$q(x) = \varphi_1(x)\varphi_2(x) \qquad (x \in \mathbb{F})\,,$$

is an exponential in the variety $\mathscr{V}$.

Similarly, Example 2 shows that the mapping q defined with the help of the derivation $d\colon \mathbb{F} \to \mathbb{K}$ by

$$q(x) = d(x^2) \qquad (x \in \mathbb{F})$$

is a moment function of degree 1 corresponding to the exponential $q_0(x) = x^2$ $(x \in \mathbb{F})$.

In what follows, we study the converse direction.

3. Main results

Quadratic functions that are multiplicative on the multiplicative group $\mathbb{F}^{\times}$.

THEOREM 11. *Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F} \subset \mathbb{K}$ be a subfield. Let further $q\colon \mathbb{F} \to \mathbb{K}$ be a quadratic function. If the mapping q is multiplicative on $\mathbb{F}^{\times}$, that is,*

$$q(xy) = q(x)q(y)$$

holds for all $x, y \in \mathbb{F}^{\times}$, then there exist homomorphisms $\varphi_1, \varphi_2\colon \mathbb{F} \to \mathbb{K}$ such that

$$q(x) = c \cdot \varphi_1(x)\varphi_2(x) \qquad (x \in \mathbb{F}),$$

where $c \in \{0, 1\}$.

PROOF. Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F} \subset \mathbb{K}$ be a subfield. Suppose further that the quadratic mapping $q\colon \mathbb{F} \to \mathbb{K}$ is multiplicative on $\mathbb{F}^{\times}$, that is, we have

$$q(xy) = q(x)q(y)$$

holds for all $x, y \in \mathbb{F}^{\times}$. Let $B\colon \mathbb{F} \times \mathbb{F} \to \mathbb{C}$ be the uniquely determined symmetric bi-additive mapping for which we have

$$B(x, x) = q(x) \qquad (x \in \mathbb{F}).$$

The assumption that the quadratic mapping q is multiplicative on $\mathbb{F}^{\times}$ yields for the mapping B that

$$B(xy, xy) = B(x, x)B(y, y) \qquad \left(x, y \in \mathbb{F}^{\times}\right),$$

especially, we have

$$B(x^2, x^2) = B(x, x)^2 \qquad \left(x \in \mathbb{F}^{\times}\right).$$

Define the mapping B_4 on $\mathbb{F}^4$ by

$$\begin{aligned} B_4(x_1, x_2, x_3, x_4) &= B\left(x_1 x_4, x_2 x_3\right) + B\left(x_1 x_3, x_2 x_4\right) + B\left(x_1 x_2, x_3 x_4\right) \\ &- B\left(x_1, x_2\right) B\left(x_3, x_4\right) - B\left(x_1, x_3\right) B\left(x_2, x_4\right) - B\left(x_1, x_4\right) B\left(x_2, x_3\right) \\ &\qquad (x_1, x_2, x_3, x_4 \in \mathbb{F}). \end{aligned}$$

Since B is a symmetric and bi-additive mapping, B_4 is a symmetric and 4-additive mapping. Further the trace of B_4 is

$$B_4(x, x, x, x) = 3\left(B(x^2, x^2) - B(x, x)^2\right) = 0$$

for all $x \in \mathbb{F}^{\times}$. Thus B_4 vanishes identically on $\mathbb{F}^4$. Especially,

$$0 = B_4(1, 1, 1, 1) = -3\left(B(1, 1) - 1\right) B(1, 1).$$

Thus $q(1) = B(1,1) \in \{0,1\}$. Further, for all $x \in \mathbb{F}^{\times}$ we have

$$0 = B_4(x,1,1,1) = -3\,(B\,(1,1) - 1)\,B\,(x,1)\,.$$

Finally,

$$0 = B_4(x,x,1,1) = B\left(x^2,1\right) + (2 - B\,(1,1))\,B\,(x,x) - 2\,B\,(x,1)^2$$

for all $x \in \mathbb{F}^{\times}$. Firstly assume that $q(1) = B(1,1) = 0$. Then

$$B(x,1) = 0 \qquad (x \in \mathbb{F})\,.$$

Therefore

$$2B(x,x) = 2B(x,1)^2 - B(x^2,1) = 0$$

for all $x \in \mathbb{F}^{\times}$. This means that q is identically zero.

Assume now that $q(1) = B(1,1) = 1$. Consider the additive function $a\colon \mathbb{F} \to \mathbb{K}$ defined by

$$a(x) = B(x,1) \qquad (x \in \mathbb{F})\,.$$

Then identity $B_4(x,x,1,1) = 0$ can be re-formulated as

$$B(x,x) = 2a(x)^2 - a(x^2) \qquad \left(x \in \mathbb{F}^{\times}\right),$$

from which

$$B(x,y) = 2a(x)a(y) - a(xy) \qquad (x,y \in \mathbb{F})$$

and

$$q(x) = 2a(x)^2 - a(x^2) \qquad (x \in \mathbb{F})$$

follow. Observe that by the definition of the function a,

$$a(x) = B(x,1) = 2a(x)a(1) - a(x) = (2a(1) - 1)a(x) \qquad (x \in \mathbb{F})\,.$$

Thus $a(1) = 1$, otherwise the function a and thus the q function are identically zero. Since we have

$$B(x^2,x^2) = B(x,x)^2,$$

the additive function a has to fulfill

$$-a\left(x^4\right) + a\left(x^2\right)^2 + 4\,a\,(x)^2\;a\left(x^2\right) - 4\,a\,(x)^4 = 0$$

for all $x \in \mathbb{F}$. Again, the left-hand side of this equation is the trace of the symmetric and 4-additive mapping A_4 defined on $\mathbb{F}^4$ by

$$\begin{aligned}
&A_4(x_1, x_2, x_3, x_4) = -a\,(x_1\,x_2\,x_3\,x_4) \\
&+\frac{1}{3}\,[a\,(x_1\,x_2)\;a\,(x_3\,x_4) + a\,(x_1\,x_3)\;a\,(x_2\,x_4) + a\,(x_2\,x_3)\;a\,(x_1\,x_4)] \\
&+\frac{2}{3}\;[a\,(x_1)\;a\,(x_2)\;a\,(x_3\,x_4) + a\,(x_1)\;a\,(x_3)\;a\,(x_2\,x_4) \\
&+a\,(x_2)\;a\,(x_3)\;a\,(x_1\,x_4) + a\,(x_1)\;a\,(x_2\,x_3)\;a\,(x_4) + a\,(x_2)\;a\,(x_1\,x_3)\;a\,(x_4)] \\
&+a\,(x_1\,x_2)\;a\,(x_3)\;a\,(x_4) - 4\,a\,(x_1)\;a\,(x_2)\;a\,(x_3)\;a\,(x_4)\,.
\end{aligned}$$

Therefore we have

$$-\,a\,(x\,y\,z) + a\,(x)\;a\,(y\,z) + a\,(y)\;a\,(x\,z) + (\,a\,(x\,y) - 2\,a\,(x)\;a\,(y))\;a\,(z) = 0$$

for all $x, y, z \in \mathbb{F}$. For any fixed $z \in \mathbb{F}$ consider the additive function $A_z \colon \mathbb{F} \to \mathbb{K}$ defined by

$$A_z(x) = a(xz) - a(x)a(z) \qquad (x \in \mathbb{F})\,.$$

The above equation, in terms of this function A_z is

$$A_z(xy) = a(x)A_z(y) + A_z(x)a(y) \qquad (x, y \in \mathbb{F})\,.$$

Thus A_z is a normal exponential polynomial of degree at most two on the multiplicative group $\mathbb{F}^{\times}$. Further,

(i) either for all $z \in \mathbb{F}$ the system $\{a, A_z\}$ is linearly dependent
(ii) or there exists an element $z \in \mathbb{F}$ such that the system $\{a, A_z\}$ is linearly independent.

If we have (i), then

$$A_z(x) = c(z)a(x)$$

holds for all $x, z \in \mathbb{F}^{\times}$. Due to the definition of the function A_z this means however that

$$A(xz) = (c(z) + a(z))a(x) \qquad (x, z \in \mathbb{F})\,.$$

Thus a is a constant multiple of an exponential of the Abelian group $\mathbb{F}^{\times}$. Thus because of the additivity of a, yields that a is a constant multiple of a homomorphism $\varphi \colon \mathbb{F} \to \mathbb{K}$. However, $a(1) = 1$, therefore $a = \varphi$.

If we have (ii), then exists an element $z \in \mathbb{F}$ such that the system $\{a, A_z\}$ is linearly independent. In this case, however, not only A_z, but also a is a normal exponential polynomial of degree at most two on the multiplicative group $\mathbb{F}^{\times}$. Thus

(a) either

$$a(x) = (\alpha_1 l(x) + \alpha_2)m(x) \qquad (x \in \mathbb{F})$$

(b) or

$$a(x) = \alpha_1 m_1(x) + \alpha_2 m_2(x) \qquad (x \in \mathbb{F})$$

with some constants $\alpha_1, \alpha_2 \in \mathbb{K}$ and with an additive function $l\colon \mathbb{F}^\times \to \mathbb{K}$ and exponentials $m, m_1, m_2\colon \mathbb{F}^\times \to \mathbb{K}$. We emphasize that now we work on the multiplicative group $\mathbb{F}^\times$. Therefore, e.g. the fact that l is an additive function means that

$$l(xy) = l(x) + l(y) \qquad \left(x, y \in \mathbb{F}^\times\right)$$

and similarly, the fact that m is exponential means that

$$m(xy) = m(x)m(y) \qquad \left(x, y \in \mathbb{F}^\times\right).$$

Using these representations in the equation for the function q, it turns out that in case (a) the function l is identically zero, and $\alpha_2 = 1$, in case (b) $\alpha_1, \alpha_2 = \frac{1}{2}$ and m_1, m_2 are homomorphisms, due to the additivity of a. Thus there exist homomorphisms $\varphi_1, \varphi_2\colon \mathbb{F} \to \mathbb{K}$ such that

$$a(x) = \frac{\varphi_1(x) + \varphi_2(x)}{2} \qquad (x \in \mathbb{F}).$$

Note, however, that in this case we have

$$\begin{aligned} q(x) = 2a(x)^2 - a(x^2) &= 2\left(\frac{\varphi_1(x) + \varphi_2(x)}{2}\right)^2 - \frac{\varphi_1(x^2) + \varphi_2(x^2)}{2} \\ &= \frac{\varphi_1(x)^2 + 2\varphi_1(x)\varphi_2(x) + \varphi_2(x)^2}{2} \\ &\quad - \frac{\varphi_1(x)^2 + \varphi_2(x)^2}{2} \\ &= \varphi_1(x)\varphi_2(x) \end{aligned}$$

for all $x \in \mathbb{F}$. □

Remark 14. *One of the reviewers of this PhD dissertation, Professor Włodzimierz Fechner, drew our attention to the fact that Theorem 11 aligns closely with the main result presented in* [**15**]. *Although our proof employs a different and more elementary argument, the essential conclusion is the same. We were unaware of this prior work at the time of submission and wish to acknowledge the author's earlier contribution.*

Quadratic functions that are additive on the multiplicative group $\mathbb{F}^\times$.

Theorem 12. *Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F} \subset \mathbb{K}$ be a subfield. Let further $q\colon \mathbb{F} \to \mathbb{K}$ be a quadratic function. If the mapping $\dfrac{q}{\pi_2}$ is additive on $\mathbb{F}^\times$, that is,*

$$q(xy) = q(x)y^2 + x^2 q(y)$$

holds for all $x, y \in \mathbb{F}^{\times}$*, then there exists a second order derivation* $d\colon \mathbb{F} \to \mathbb{K}$ *such that*

$$q(x) = 4xd(x) - d(x^2) \qquad (x \in \mathbb{F})\,.$$

Proof. Let $\mathbb{F} \subset \mathbb{K}$ be fields with characteristic zero and assume that $q\colon \mathbb{F} \to \mathbb{K}$ is a quadratic mapping which is additive on $\mathbb{F}^{\times}$. In other words, suppose that the quadratic mapping q also fulfills

$$q(xy) = q(x)y^2 + x^2 q(y)$$

for all $x, y \in \mathbb{F}^{\times}$.

Since the mapping q is quadratic, there exists a uniquely determined symmetric and bi-additive mapping $B\colon \mathbb{F} \times \mathbb{F} \to \mathbb{K}$ such that

$$B(x,x) = q(x) \qquad (x \in \mathbb{F})\,.$$

Then for this symmetric and bi-additive mapping B, we have

$$B(xy, xy) = y^2 B(x,x) + x^2 B(y,y) \qquad (x, y \in \mathbb{F})\,.$$

Let us consider the mapping $B_4\colon \mathbb{F}^4 \to \mathbb{K}$ defined by

$$\begin{aligned} B_4&(x_1,x_2,x_3,x_4) \\ &= \frac{1}{3}\left\{B\left(x_1\,x_4, x_2\,x_3\right) + B\left(x_1\,x_3, x_2\,x_4\right) + B\left(x_1\,x_2, x_3\,x_4\right)\right\} \\ &- \frac{1}{3}\left\{x_1\,B\left(x_3,x_2\right)\,x_4 + x_2\,B\left(x_3,x_1\right)\,x_4 + B\left(x_1,x_2\right)\,x_3\,x_4\right. \\ &\quad \left.+x_1\,x_2\,B\left(x_3,x_4\right) + x_1\,B\left(x_2,x_4\right)\,x_3 + B\left(x_1,x_4\right)\,x_2\,x_3\right\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad (x_1,x_2,x_3,x_4 \in \mathbb{F})\,. \end{aligned}$$

Since B is symmetric and bi-additive, B_4 is symmetric and 4-additive. Further, its trace is

$$B_4(x,x,x,x) = B\left(x^2,x^2\right) - 2\,x^2\,B\left(x,x\right) = 0$$

for all $x \in \mathbb{F}$. Thus B_4 is identically zero on $\mathbb{F}^4$. Especially, we have

$$B_4(1,1,1,1) = -B(1,1) = 0,$$

so $B(1,1) = q(1) = 0$. Further, we have for all $x, y \in \mathbb{F}$ that

$$\begin{aligned} 0 = B_4(x,y,1,1) = & \\ &= B\left(x\,y,1\right) - x\,B\left(y,1\right) - B\left(x,1\right)\,y \\ &- B\left(1,1\right)\,x\,y - B\left(1,x\right)\,y + B\left(x,y\right) - B\left(1,y\right)\,x \\ &= B(xy,1) - 2xB(y,1) - 2yB(x,1) + B(x,y). \end{aligned}$$

Define the additive mapping $a\colon \mathbb{F} \to \mathbb{K}$ by

$$a(x) = B(x,1) \qquad (x \in \mathbb{F})$$

to deduce that the latter identity is

$$B(x,y) = 2xa(y) + 2ya(y) - a(xy) \qquad (x, y \in \mathbb{F})\,.$$

Observe that this already shows that

$$q(x) = B(x,x) = 4xa(x) - a(x^2) \qquad (x \in \mathbb{F})\,.$$

Now we show that a is a second-order derivation. Since the quadratic function q fulfills

$$q(x^2) = 2x^2 q(x) \qquad (x \in \mathbb{F})\,,$$

for the additive function a we have

$$a\left(x^4\right) - 6\,x^2\,a\left(x^2\right) + 8\,x^3\,a\,(x) = 0$$

for all $x \in \mathbb{F}$. Using [**20**, Corollary 4], the additive function a has to be a derivation of order two. Therefore, there exists a mapping $d \in \mathscr{D}_2(\mathbb{F}, \mathbb{K})$ such that

$$q(x) = B(x,x) = 4xd(x) - d(x^2)$$

holds for all $x \in \mathbb{F}$. □

Remark 15. *Note that the quadratic mapping that appears in Example 2 is covered in Theorem 12. Indeed, if $d\colon \mathbb{F} \to \mathbb{K}$ is a derivation, then $d \in \mathscr{D}_1(\mathbb{F}, \mathbb{K}) \subset \mathscr{D}_2(\mathbb{F}, \mathbb{K})$. So d is a derivation of order two, too. Thus*

$$q(x) = 4xd(x) - d(x^2) = 2d(x^2) - d(x^2) = d(x^2) \qquad (x \in \mathbb{F})\,,$$

showing that mappings appearing in Example 2 can indeed be written as mappings appearing in Theorem 12.

However, Theorem 12 shows that the equation

$$q(xy) = x^2 q(y) + y^2 q(x) = 0 \qquad (x, y \in \mathbb{F})$$

has other quadratic solutions than those found in Example 2. Indeed, if $d_1, d_2\colon \mathbb{F} \to \mathbb{K}$ are non-identically zero derivations, then $d_1 \circ d_2 \in \mathscr{D}_2(\mathbb{F}, \mathbb{K}) \setminus \mathscr{D}_1(\mathbb{F}, \mathbb{K})$. Then the mapping $q\colon \mathbb{F} \to \mathbb{K}$ defined by

$$q(x) = 4xd_1 \circ d_2(x) - d_1 \circ d_2(x^2) \qquad (x \in \mathbb{F})$$

is quadratic. Further, we have

$$\begin{aligned} q(x^2) &= 4x^2 d_1 \circ d_2(x^2) - d_1 \circ d_2(x^4) \\ &= 4x^2\left(2xd_1 \circ d_2(x) + 2d_1(x)d_2(x)\right) \\ &- \left(4x^3 d_1 \circ d_2(x) + 12x^2 d_1(x)d_2(x)\right) \\ &= 4x^3 d_1 \circ d_2(x) - 4x^2 d_1(x)d_2(x) \end{aligned}$$

and

$$\begin{aligned}2x^2q(x) &= 2x^2\left(4xd_1\circ d_2(x) - d_1\circ d_2(x^2)\right)\\ &= 8x^3d_1\circ d_2(x) - 2x^2\left(2xd_1\circ d_2(x) + 2d_1(x)d_2(x)\right)\\ &= 4x^3d_1\circ d_2(x) - 4x^2d_1(x)d_2(x)\end{aligned}$$

for all $x\in\mathbb{F}$. Thus

$$q(x^2) = 2x^2q(x) \qquad (x\in\mathbb{F})\,.$$

Note, however, that due to the methods described in the proof of Theorem 12, this equation is equivalent to

$$q(xy) = x^2q(y) + y^2q(x) \qquad (x,y\in\mathbb{F})\,.$$

In view of Theorem 11, if $\varphi\colon\mathbb{F}\to\mathbb{K}$ is a homomorphism then the mapping q defined on $\mathbb{F}$ by

$$q(x) = \varphi(x)^2 \qquad (x\in\mathbb{F})$$

is quadratic and is also an exponential on the multiplicative group $\mathbb{F}^\times$ Thus, using the previous theorem, we obtain the following statement.

Corollary 2. *Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F}\subset\mathbb{K}$ be a subfield. Let further $q\colon\mathbb{F}\to\mathbb{K}$ be a quadratic function, while $\varphi\colon\mathbb{F}\to\mathbb{K}$ be a homomorphism such that*

$$q(xy) = \varphi(x)^2q(y) + q(x)\varphi(y)^2 \qquad (x,y\in\mathbb{F})\,.$$

Then there exists a second-order derivation $d\in\mathscr{D}_2(\mathbb{F},\mathbb{K})$ such that

$$q(x) = \varphi\left(4xd(x) - d(x^2)\right) \qquad (x\in\mathbb{F})\,.$$

And vice versa, that is, if d is a second-order derivation and φ is a homomorphism, then the function q given by the above formula is a quadratic function that also satisfies the above equation.

Proof. In addition to the conditions of the statement, let us first consider the case when φ is identically zero. In this case, the above equation reduces to the identity

$$q(xy) = 0 \qquad (x,y\in\mathbb{F})\,.$$

Therefore, q is identically zero, from which it immediately follows that it has the desired representation.

Assume now that φ is not the identically zero homomorphism. Then φ is one-to-one and φ^{-1} is also a homomorphism. Define the mapping $\tilde{q}$ by

$$\tilde{q}(x) = \varphi^{-1}(q(x)) \qquad (x\in\mathbb{F})\,.$$

Then $\tilde{q}$ will be a quadratic mapping for which we have

$$\tilde{q}(xy) = x^2 q(y) + y^2 q(x) \qquad (x, y \in \mathbb{F}).$$

Theorem 12 yields that there exists a second-order derivation $d \in \mathscr{D}_2(\mathbb{F}, \mathbb{K})$ such that

$$\tilde{q}(x) = 4xd(x) - d(x^2) \qquad (x \in \mathbb{F}),$$

from which the statement of this corollary already follows. □

Finally, we note that the method described in the proof of Theorem 12 is also suitable for determining the solutions to the equation

$$q(xy) = \varphi_1(x)\varphi_2(x)q(y) + \varphi_1(y)\varphi_2(y)q(x) \qquad (x, y \in \mathbb{F})$$

for the unknown quadratic function $q\colon \mathbb{F} \to \mathbb{K}$, where $\varphi_1, \varphi_2\colon \mathbb{F} \to \mathbb{K}$ are (non-identically zero) homomorphisms. In contrast to the previous two statements, however, in this case, the description is unfortunately not complete.

Notice that the above equation says that the quadratic function q is a moment function of degree 1 corresponding to the exponential $q_0 = \varphi_1 \cdot \varphi_2$.

Proposition 2. *Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F} \subset \mathbb{K}$ be a subfield. Let further $q\colon \mathbb{F} \to \mathbb{K}$ be a quadratic function, while $\varphi_1, \varphi_2\colon \mathbb{F} \to \mathbb{K}$ be homomorphisms such that*

$$q(xy) = \varphi_1(x)\varphi_2(x)q(y) + \varphi_1(y)\varphi_2(y)q(x) \qquad (x, y \in \mathbb{F}).$$

Then there exists an additive function $a\colon \mathbb{F} \to \mathbb{K}$ such that

$$q(x) = 2(\varphi_1(x) + \varphi_2(x))a(x) - a(x^2) \qquad (x \in \mathbb{F}),$$

where the additive function also fulfills

$$(\spadesuit) \qquad \begin{aligned} &2\,a\,(x\,y\,z) - (\varphi_1(x) + \varphi_2(x))\;a\,(y\,z) - (\varphi_1(y) + \varphi_2(y))\;a\,(x\,z) \\ &\quad - (\varphi_1(z) + \varphi_2(z))\;a\,(x\,y) + (\varphi_1(x)\,\varphi_2(y) + \varphi_2(x)\,\varphi_1(y))\;a\,(z) \\ &\quad + (\varphi_1(x)\,\varphi_2(z) + \varphi_2(x)\,\varphi_1(z))\;a\,(y) \\ &\quad + (\varphi_1(y)\,\varphi_2(z) + \varphi_2(y)\,\varphi_1(z))\;a\,(x) = 0 \end{aligned}$$

for all $x, y, z \in \mathbb{F}$. And also conversely, if a is an additive function fulfilling ($\spadesuit$) with the homomorphisms φ_1, φ_2, then the function q given by the above formula is a quadratic function that also satisfies the above equation.

Proof. Let us assume that the quadratic function $q\colon \mathbb{F} \to \mathbb{K}$ and the homomorphisms $\varphi_1, \varphi_2\colon \mathbb{F} \to \mathbb{K}$ satisfy

$$q(xy) = \varphi_1(x)\varphi_2(x)q(y) + \varphi_1(y)\varphi_2(y)q(x) \qquad (x, y \in \mathbb{F}).$$

If any of φ_1 and φ_2 were the identically zero homomorphisms, it would immediately follow that q is identically zero. Thus, without loss of generality, we can assume that neither φ_1 nor φ_2 is the identically zero homomorphism. And then $\varphi_1(1) = \varphi_2(1) = 1$.

With $y = x$, we immediately obtain

$$q(x^2) = 2\varphi_1(x)\varphi_2(x)q(x) \qquad (x \in \mathbb{F})\,.$$

Observe that the mapping

$$\mathbb{F} \ni x \longmapsto q(x^2) - 2\varphi_1(x)\varphi_2(x)q(x)$$

is a generalized monomial of degree 4. Thus there exists a symmetric and 4-additive mapping $B_4 \colon \mathbb{F}^4 \to \mathbb{K}$ such that

$$B_4(x, x, x, x) = q(x^2) - 2\varphi_1(x)\varphi_2(x)q(x) \qquad (x \in \mathbb{F})\,.$$

Indeed, this mapping is given by

$$\begin{aligned} &B_4(x_1, x_2, x_3, x_4) \\ &= \frac{1}{4!} \sum_{\sigma \in \mathscr{S}_4} \left[B(x_{\sigma(1)}x_{\sigma(2)}, x_{\sigma(3)}x_{\sigma(1)}) - 2\varphi_1(x_{\sigma(1)})\varphi_2(x_{\sigma(2)})B(x_{\sigma(3)}, x_{\sigma(4)}) \right] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x_1, x_2, x_3, x_4 \in \mathbb{F})\,. \end{aligned}$$

Here $B \colon \mathbb{F}^2 \to \mathbb{K}$ is the uniquely determined symmetric and bi-additive mapping for which we have $B(x, x) = q(x)$ for all $x \in \mathbb{F}$.

With these notations, the equation in our statement means just that

$$B_4(x, x, x, x) = 0 \qquad (x \in \mathbb{F})\,.$$

Thus B_4 is identically zero on $\mathbb{F}^4$. Especially,

$$0 = B_4(1, 1, 1, 1) = (2\,\varphi_1(1)\,\varphi_2(1) - 1)\; B\,(1, 1)\,.$$

So $B(1, 1) = q(1) = 0$.

Identity

$$B_4(x, 1, 1, 1) = 0 \qquad (x \in \mathbb{F})\,,$$

i.e.,

$$\begin{aligned} &(\varphi_1(1)\,\varphi_2(1) - 1)\; B\,(x, 1) + \varphi_1(1)\,B\,(1, 1)\;\varphi_2(x) \\ &\qquad + \varphi_2(1)\,B\,(1, 1)\;\varphi_1(x) + (\varphi_1(1)\,\varphi_2(1) - 1)\; B\,(1, x) = 0 \quad (x \in \mathbb{F}) \end{aligned}$$

contains no information, since $\varphi_1(1) = \varphi_2(1) = 1$ and $B(1, 1) = 0$. Define the additive function $a \colon \mathbb{F} \to \mathbb{K}$ by

$$a(x) = B(x, 1) \qquad (x \in \mathbb{F})$$

to deduce that the equation

$$B_4(x, x, 1, 1) = 0 \qquad (x \in \mathbb{F})$$

takes the form

$$q(x) = B(x,x) = 2(\varphi_1(x) + \varphi_2(x))a(x) - a(x^2) \qquad (x \in \mathbb{F}).$$

Therefore

$$\begin{aligned} q(x^2) =& 2(\varphi_1(x^2) + \varphi_2(x^2))a(x^2) - a(x^4) \\ &= 2(\varphi_1(x)^2 + \varphi_2(x)^2)a(x^2) - a(x^4) \end{aligned}$$

and also

$$2\varphi_1(x)\varphi_2(x)q(x) = 2\varphi_1(x)\varphi_2(x)\left(2(\varphi_1(x) + \varphi_2(x))a(x) - a(x^2)\right)$$

holds for all $x \in \mathbb{F}$. This means that the additive function a has to fulfill

$$\begin{aligned} &2(\varphi_1(x)^2 + \varphi_2(x)^2)a(x^2) - a(x^4) \\ &\qquad = 2\varphi_1(x)\varphi_2(x)\left(2(\varphi_1(x) + \varphi_2(x))a(x) - a(x^2)\right) \qquad (x \in \mathbb{F}). \end{aligned}$$

In other words, we have

$$\begin{aligned} &a\left(x^4\right) + \left(-2\,\varphi_2(x)^2 - 2\,\varphi_1(x)\,\varphi_2(x) - 2\,\varphi_1(x)^2\right)\,a\left(x^2\right) \\ &\qquad + \left(4\,\varphi_1(x)\,\varphi_2(x)^2 + 4\,\varphi_1(x)^2\,\varphi_2(x)\right)\,a\,(x) = 0 \qquad (x \in \mathbb{F}). \end{aligned}$$

Again, the left-hand side of this equation, as a function of the variable x is a generalized monomial of degree 4. With the application of the symmetrization method, we obtain that $a(1) = 0$, and the above equation is equivalent to

$$\begin{aligned} &2\,a\,(x\,y\,z) - (\varphi_1(x) + \varphi_2(x))\;a\,(y\,z) - (\varphi_1(y) + \varphi_2(y))\;a\,(x\,z) \\ &\quad - (\varphi_1(z) + \varphi_2(z))\;a\,(x\,y) + (\varphi_1(x)\,\varphi_2(y) + \varphi_2(x)\,\varphi_1(y))\;a\,(z) \\ &\quad + (\varphi_1(x)\,\varphi_2(z) + \varphi_2(x)\,\varphi_1(z))\;a\,(y) \\ &\quad + (\varphi_1(y)\,\varphi_2(z) + \varphi_2(y)\,\varphi_1(z))\;a\,(x) = 0 \end{aligned}$$

for all $x, y, z \in \mathbb{F}$ which proves the statement. $\square$

Remark 16. *If* $\varphi_1 = \varphi_2 = \mathrm{id}$ *in equation* (♠)*, then we have*

$$a(x^3) - 6xa(x^2) + 3x^2a(x) = 0 \qquad (x \in \mathbb{F})$$

and $a(1) = 0$ *for the additive function* $a\colon \mathbb{F} \to \mathbb{K}$*. In view of* [**20**, Corollary 2] *we deduce that* $a \in \mathscr{D}_2(\mathbb{F}, \mathbb{K})$*.*

Similarly, if $\varphi\colon \mathbb{F} \to \mathbb{K}$ *is a non-identically zero homomorphism and we take* $\varphi_1 = \varphi_2 = \varphi$*, then equation* (♠) *reduces to*

$$a(x^3) - 6\varphi(x)a(x^2) + 3\varphi(x)^2a(x) = 0 \qquad (x \in \mathbb{F}).$$

This means that the mapping $d\colon \mathbb{F} \to \mathbb{K}$ *defined by*

$$d(x) = \varphi^{-1}(a(x)) \qquad (x \in \mathbb{F})$$

satisfies

$$d(x^3) - 6xd(x^2) + 3x^2d(x) = 0 \qquad (x \in \mathbb{F})\,.$$

So $a = \varphi \circ d$ with an appropriate $d \in \mathscr{D}_2(\mathbb{F}, \mathbb{K})$.

These two special cases allow us to conclude that in the general case (see identity (♠)), the function a can be represented with the help of a second-order derivation and homomorphisms. Therefore, we formulate the following open problem. After answering this question, Proposition 2 will take on a more compact form.

Open problem 1. *Let $\mathbb{K}$ be a field of characteristic zero, $\mathbb{F} \subset \mathbb{K}$ be a subfield and $\varphi_1, \varphi_2 \colon \mathbb{F} \to \mathbb{K}$ be non-identically zero homomorphisms. Determine all those additive functions $a \colon \mathbb{F} \to \mathbb{K}$ such that $a(1) = 0$ and*

$$\begin{aligned}
&2\,a\,(x\,y\,z) - (\varphi_1(x) + \varphi_2(x))\;a\,(y\,z) - (\varphi_1(y) + \varphi_2(y))\;a\,(x\,z)\\
&\quad - (\varphi_1(z) + \varphi_2(z))\;a\,(x\,y) + (\varphi_1(x)\,\varphi_2(y) + \varphi_2(x)\,\varphi_1(y))\;a\,(z)\\
&\quad + (\varphi_1(x)\,\varphi_2(z) + \varphi_2(x)\,\varphi_1(z))\;a\,(y)\\
&\quad + (\varphi_1(y)\,\varphi_2(z) + (\varphi_2(y)\,\varphi_1(z))\;a\,(x) = 0
\end{aligned}$$

for all $x, y, z \in \mathbb{F}$.

Quadratic functions as solutions of polynomial equations.

Theorem 13. *Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F} \subset \mathbb{K}$ be a subfield. Let r be a positive integer and suppose that for all multi-index $\alpha \in \mathbb{N}^r$ the mapping $q_\alpha \colon \mathbb{F} \to \mathbb{K}$ is quadratic such that*

$$q_0(x) = x^2 \qquad (x \in \mathbb{F})\,.$$

Assume further that we have

$$q_\alpha(xy) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} q_\beta(x) q_{\alpha-\beta}(y) \qquad \left(x, y \in \mathbb{F}^\times\right)$$

for all multi-index $\alpha \in \mathbb{N}^r$. Then there exists a sequence of $\mathbb{K}$-valued 'additive functions' $a = (a_\alpha)_{\alpha \in \mathbb{N}^r}$ on $\mathbb{F}^\times$ such that

$$q_\alpha(x) = B_\alpha(a(x))x^2$$

holds for all $x \in \mathbb{F}$ and for each multi-index $\alpha \in \mathbb{N}^r$. Here the fact that $a_\alpha \colon \mathbb{F} \to \mathbb{K}$ is an 'additive function' means that there exists $d_\alpha \in \mathscr{D}_2(\mathbb{F}, \mathbb{K})$ such that

$$a_\alpha(x) = 4\frac{d_\alpha(x)}{x} - \frac{d_\alpha(x^2)}{x^2} \qquad \left(x \in \mathbb{F}^\times\right).$$

Proof. Let r be a positive integer and suppose that for all multi-index $\alpha \in \mathbb{N}^r$ the mapping $q_\alpha \colon \mathbb{F} \to \mathbb{K}$ is quadratic such that

$$q_0(x) = x^2 \qquad (x \in \mathbb{F})$$

and we have also

$$q_\alpha(xy) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} q_\beta(x) q_{\alpha-\beta}(y) \qquad \left(x, y \in \mathbb{F}^\times\right)$$

for all multi-index $\alpha \in \mathbb{N}^r$.

In view of Definition 10, this means that the sequence of functions $(q_\alpha)_{\alpha\in\mathbb{N}^r}$ forms a generalized moment sequence of rank r on the group $\mathbb{F}^\times$, corresponding to the 'exponential' $q_0 = x^2$ $(x \in \mathbb{F})$. Equivalently, this means that the sequence of functions $(q_\alpha/q_0)_{\alpha\in\mathbb{N}^r}$ forms a moment sequence of rank r. Thus, we obtain from Theorem 7, that there exists a sequence of $\mathbb{K}$-valued additive functions $a = (a_\alpha)_{\alpha\in\mathbb{N}^r}$ such that for every multi-index α in $\mathbb{N}^r$ and x in $\mathbb{F}^\times$ we have

$$\frac{q_\alpha(x)}{q_0(x)} = B_\alpha(a(x)).$$

Note that for all multi-index $\alpha \in \mathbb{N}^r$ 'additivity' on the group $\mathbb{F}^\times$ means that we have

$$a_\alpha(xy) = a_\alpha(x) + a_\alpha(y) \qquad \left(x \in F^\times\right).$$

Thus, by Theorem 12 we have

$$a_\alpha(x) = 4\frac{d_\alpha(x)}{x} - \frac{d_\alpha(x^2)}{x^2} \qquad \left(x \in \mathbb{F}^\times\right)$$

with an appropriate $d_\alpha \in \mathscr{D}_2(\mathbb{F}, \mathbb{K})$. □

CHAPTER 4

A functional equation for monomial functions

In this chapter, we aim to determine those monomials $f\colon \mathbb{F} \to \mathbb{K}$ of degree n for which

$$f(x^2) = \kappa \cdot x^n f(x)$$

holds for all $x \in \mathbb{F}$. We show that similar to the classical results, where additive functions were considered, the monomial functions in the equation can be represented with the aid of homomorphisms and higher-order derivations.

Let n be a positive integer. Define

$$\mathscr{M}_n(\mathbb{F}, \mathbb{K}) = \{f\colon \mathbb{F} \to \mathbb{K} \,|\, f \text{ is a monomial of degree } n\}\,.$$

1. Preliminary results

At first glance, the above problem may seem quite specific, and naturally the question arises whether we should consider the more general equation

$$f(x^2) = \kappa \cdot x^l f(x) \qquad (x \in \mathbb{F})$$

with a fixed positive integer l. However, as we will see, this equation can have a non-identically zero solutions in only the case when $n = l$.

LEMMA 8. *Let n, l be positive integers, κ be a nonzero element of the field $\mathbb{K}$ and $f \in \mathscr{M}_n(\mathbb{F}, \mathbb{K})$ such that*

$$f(x^2) = \kappa \cdot x^l f(x) \qquad (x \in \mathbb{F})$$

holds. If $n \neq l$, then the function f is identically zero.

PROOF. Let n, l be positive integers, $0 \neq \kappa \in \mathbb{K}$ and $f\colon \mathbb{F} \to \mathbb{K}$. Assume that f is not identically zero. Then by Lemma 2 the mapping

$$\mathbb{F} \ni x \longmapsto f(x^2)$$

is a generalized polynomial of degree $2n$. Similarly, the mapping

$$\mathbb{F} \ni x \longmapsto \kappa \cdot x^l f(x)$$

is a generalized polynomial of degree $n + l$. Due to the above equation, these generalized polynomials are identically equal on $\mathbb{F}$. However, this is possible only if their degree is the same. So $2n = n+l$, that is $n = l$. From this we obtain that if $n \neq l$, then necessarily f is identically zero. $\square$

In the case of additive functions, the proof is a simple calculation. For the sake of completeness, we will present this case along with the proof. Additionally, one of our goals with this is to show that this line of reasoning cannot be easily generalized to higher-degree monomial functions. Therefore, in the general case, it was necessary to develop another method.

PROPOSITION 3. *Let $f \in \mathscr{M}_1(\mathbb{F}, \mathbb{K})$, i.e. assume f to be an additive function and let $\kappa \in \mathbb{K}$. Suppose that*

$$f(x^2) = \kappa \cdot x f(x) \tag{5}$$

holds for all $x \in \mathbb{F}$. Then the following cases are possible

(i) $\kappa = 1$, and we have

$$f(x) = f(1) \cdot x \qquad (x \in \mathbb{F}),$$

(ii) $\kappa = 2$, and f is a derivation,

(iii) $\kappa \notin \{1, 2\}$ and f is identically zero.

PROOF. Let $\kappa \in \mathbb{K}$ and suppose that the additive function $f \colon \mathbb{F} \to \mathbb{K}$ fulfills (5) for all $x \in \mathbb{F}$. Let $x, y \in \mathbb{F}$ be arbitrary and substitute $x + y$ in place of x in equation (5) to deduce that

$$f((x+y)^2) = \kappa \cdot (x+y) f(x+y)$$

for all $x, y \in \mathbb{F}$. As the function f is additive,

$$f((x+y)^2) = f(x^2 + 2xy + y^2) = f(x^2) + 2f(xy) + f(y^2)$$

and

$$\kappa(x+y) \cdot (f(x) + f(y)) = \kappa x f(x) + \kappa y f(y) + \kappa(x f(y) + y f(x))$$

for all $x, y \in \mathbb{F}$. Therefore

$$f(xy) = \frac{\kappa}{2}(x f(y) + y f(x)) \qquad (x, y \in \mathbb{F}).$$

From this, with $x = y = 1$, we get that $\kappa = 1$ or $f(1) = 0$. Further, the above identity with $y = 1$ leads to

$$\left(1 - \frac{\kappa}{2}\right) f(x) = \frac{\kappa}{2} f(1) \cdot x \qquad (x \in \mathbb{F}).$$

If $\kappa = 1$, then this means that

$$f(x) = f(1) \cdot x \qquad (x \in \mathbb{F}).$$

Further, if $\kappa \notin \{1, 2\}$, then we have

$$f(x) = \frac{\kappa}{2 - \kappa} f(1) \cdot x = 0 \qquad (x \in \mathbb{F}),$$

as $f(1) = 0$ in this case. Finally, if $\kappa = 2$, then the above identity is

$$f(xy) = xf(y) + yf(x) \qquad (x, y \in \mathbb{F}).$$

Due to our assumptions, f is an additive function, so this identity precisely states that f is a derivation. $\square$

In [**7**, Proposition 1 and Theorem 3] the authors focused on real additive and quadratic functions satisfying (5) and (6), respectively. They proved the following results.

PROPOSITION 4. *Let $f \in \mathscr{M}_2(\mathbb{F}, \mathbb{K})$, i.e. assume f to be a quadratic function and let $\kappa \in \mathbb{K}$. Suppose that*

$$f(x^2) = \kappa \cdot x^2 f(x) \tag{6}$$

for all $x \in \mathbb{F}$. Then the following cases are possible

(i) $\kappa = 1$, and then

$$f(x) = f(1) \cdot x^2 \qquad (x \in \mathbb{F}).$$

(ii) $\kappa = 2$, and then there exists a $\varphi \in \mathscr{D}_2(\mathbb{F}, \mathbb{K})$ such that

$$f(x) = 4x\varphi(x) - \varphi(x^2) \qquad (x \in \mathbb{F}).$$

(iii) if f is the trace of a symmetric bi-derivation, then f solves (6) with $\kappa = 4$.

(iv) $\kappa \notin \{1, 2, 4\}$ and f is identically zero.

Our results extend these findings. On one hand, instead of the field of the real numbers, the domain and the range of the involved functions can also be more general. On the other hand, this problem can be posed not only for first- and second-degree, but for arbitrary monomial functions as well. Additionally, as we will see below, the statement found in (iii) of Proposition 4 can be strengthened.

2. Main results

THEOREM 14. *Let n be a given positive integer, $\kappa \in \mathbb{K}$ and $f \in \mathscr{M}_n(\mathbb{F}, \mathbb{K})$ such that*

$$f(x^2) = \kappa \cdot x^n f(x) \tag{7}$$

holds for all $x \in \mathbb{F}$. Then

(i) if $\kappa \notin \left\{2^k \,|\, k = 0, 1, \ldots, n\right\}$, then f is identically zero,

(ii) if $\kappa = 1$, then

$$f(x) = f(1) \cdot x^n \qquad (x \in \mathbb{F}),$$

(iii) if $\kappa = 2$, then there exists an $a \in \mathscr{D}_{2n-1}(\mathbb{F}, \mathbb{K})$ such that

$$f(x) = \sum_{j=1}^{n} \lambda_{n,j} x^{n-j} a(x^j) \qquad (x \in \mathbb{F})$$

holds with appropriate constants $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$,

(iv) if $\kappa = 2^n$, then there exists a symmetric and n-additive mapping $A_n \colon \mathbb{F}^n \to \mathbb{K}$ such that

$$\sum_{\sigma \in \mathscr{S}_{n+1}} \Big\{ A_n(x_{\sigma(1)} \cdot x_{\sigma(2)}, x_{\sigma(3)}, \ldots, x_{\sigma(n+1)}) \\ - x_{\sigma(1)} \cdot A_n(x_{\sigma(2)}, \ldots, x_{\sigma(n+1)}) - x_{\sigma(2)} \cdot A_n(x_{\sigma(1)}, \ldots, x_{\sigma(n+1)}) \Big\} = 0 \\ (x_1, \ldots, x_{n+1} \in \mathbb{F})$$

and

$$f(x) = A_n(x, \ldots, x) \qquad (x \in \mathbb{F})$$

holds.

Proof. Since $f \in \mathscr{M}_n(\mathbb{F}, \mathbb{K})$, there exists a uniquely determined symmetric n-additive function $A_n \colon \mathbb{F}^n \to \mathbb{K}$ such that

$$f(x) = A_n(x, \ldots, x) = A_n([x]_n) \qquad (x \in \mathbb{F}).$$

In terms of the mapping A_n, equation (16) reads as

$$A_n(x^2, \ldots, x^2) - \kappa x^n A_n(x, \ldots, x) = 0 \qquad (x \in \mathbb{F}).$$

If we write $x+1$ in place of x in the above identity and expand the terms,

$$\sum_{\substack{\alpha_1+\alpha_2+\alpha_3=n \\ \alpha_1,\alpha_2,\alpha_3 \geq 0}} \binom{n}{\alpha_1, \alpha_2, \alpha_3} A_n\left([x^2]_{\alpha_1}, [2x]_{\alpha_2}, [1]_{\alpha_3}\right) \\ - \kappa \left(\sum_{\substack{\beta_1+\beta_2=n \\ \beta_1,\beta_2 \geq 0}} \binom{n}{\beta_1, \beta_2} x^{\beta_1} \right) \cdot \left(\sum_{\substack{\gamma_1+\gamma_2=n \\ \gamma_1,\gamma_2 \geq 0}} \binom{n}{\gamma_1, \gamma_2} A_n([x]_{\gamma_1}, [1]_{\gamma_2}) \right) = 0 \tag{8}$$

follows for all $x \in \mathbb{F}$. Observe that the left-hand side of this equation is a generalized polynomial of degree $2n$, which is identically zero. Thus all of its monomial terms should vanish. If $\alpha_1, \alpha_2, \alpha_3$ are nonnegative and $\alpha_1 + \alpha_2 + \alpha_3 = n$, then the degree of the mapping

$$x \longmapsto \binom{n}{\alpha_1, \alpha_2, \alpha_3} A_n\left([x^2]_{\alpha_1}, [2x]_{\alpha_2}, [1]_{\alpha_3}\right)$$

is $2\alpha_1+\alpha_2$. Similarly, if $\beta_1,\beta_2,\gamma_1,\gamma_2$ are nonnegative integers with $\beta_1+\beta_2=n$ and $\gamma_1+\gamma_2=n$, then the degree of the generalized monomial

$$x\longmapsto \kappa\binom{n}{\beta_1,\beta_2}x^{\beta_1}\cdot\binom{n}{\gamma_1,\gamma_2}A_n([x]_{\gamma_1},[1]_{\gamma_2})$$

is $\beta_1+\gamma_1$.

Computing the zero-degree terms in (8),

$$(1-\kappa)A_n([1]_n)=0$$

follows. So $\kappa=1$ or $A_n([1]_n)=f(1)=0$. The first-degree terms of the generalized polynomial in equation (8) must vanish, thus we have

$$\begin{aligned}\binom{n}{0,1,n-1}&A_n([2x_1],[1]_{n-1})\\ &-\kappa\left\{\binom{n}{0,n}\binom{n}{1,n-1}A_n([x]_1,[1]_{n-1})\right.\\ &\left.+\binom{n}{1,n-1}\binom{n}{0,n}xA_n([1]_n)\right\}=0\end{aligned}$$

that is,

$$(2-\kappa)A_n([x]_1,[1]_{n-1})-\kappa xA_n([1]_n)=0$$

for all $x\in\mathbb{F}$. If $\kappa=1$, then this leads to

$$A_n([x]_1,[1]_{n-1})=A_n([1]_n)\cdot x\qquad(x\in\mathbb{F})\,.$$

If $\kappa\neq1$, then due to the previous step $A_n([1]_n)=0$, the above identity reduces to

$$(2-\kappa)A_n([x]_1,[1]_{n-1})=0\qquad(x\in\mathbb{F})\,.$$

So $\kappa=2$ (and in this case, we do not have any information for the values $A_n([x]_1,[1]_{n-1})$), or $\kappa\neq1,2$ and then $A_n([x]_1,[1]_{n-1})=0$ for all $x\in\mathbb{F}$. In general, if $k=0,1,\ldots,2n$, we have to distinguish two cases, depending on whether k is even or odd.

If k *is even*, then the k^{th}-degree term in (8) is

$$\begin{aligned}\sum_{l=0}^{\frac{k}{2}}&\binom{n}{l,k-2l,n-k}A_n\left([x^2]_l,[2x]_{k-2l},[1]_{n-k}\right)\\ &-\kappa\sum_{m=0}^{k}\binom{n}{m,n-m}\binom{n}{k-m,n-k+m}x^mA_n([x]_{k-m},[1]_{n-k-m}).\end{aligned}$$

If k *is odd*, then the only difference is that in the first expression, we must not sum up to $\frac{k}{2}$, but to $\frac{k-1}{2}$, i.e.,

$$\sum_{l=0}^{\frac{k-1}{2}} \binom{n}{l, k-2l, n-k} A_n\left([x^2]_l, [2x]_{k-2l}, [1]_{n-k}\right) \\ - \kappa \sum_{m=0}^{k} \binom{n}{m, n-m} \binom{n}{k-m, n-k+m} x^m A_n([x]_{k-m}, [1]_{n-k-m}).$$

This means that for all $k = 0, 1, \ldots, 2n$, we have

$$\sum_{l=0}^{\lfloor\frac{k}{2}\rfloor} \binom{n}{l, k-2l, n-k} A_n\left([x^2]_l, [2x]_{k-2l}, [1]_{n-k}\right) \\ - \kappa \sum_{m=0}^{k} \binom{n}{m, n-m} \binom{n}{k-m, n-k+m} x^m A_n([x]_{k-m}, [1]_{n-k-m}).$$

By selecting the members with indices $l = 0$ and $m = 0$ from the sums, we get that

$$\text{(9)} \quad \binom{n}{k, n-k} (2^k - \kappa) A_n([x]_k, [1]_{n-k}) \\ = -\sum_{l=1}^{\lfloor\frac{k}{2}\rfloor} \binom{n}{l, k-2l, n-k+l} A_n\left([x^2]_l, [2x]_{k-2l}, [1]_{n-k+l}\right) \\ + \kappa \sum_{m=1}^{k} \binom{n}{m, n-m} \binom{n}{k-m, n-k+m} x^m A_n([x]_{k-m}, [1]_{n-k-m})$$

for all $x \in \mathbb{F}$. This identity indicates that the values of the left-hand k-variable function are determined by the right-hand side, with the aid of the at most $(k-1)$-variable functions, for all $k = 0, 1, \ldots, n$.

First assume that for all $k = 0, 1, \ldots, n$, we have $\kappa \neq 2^k$. Then for all $k = 0, 1, \ldots, n$,

$$A_n([x]_k, [1]_{n-k}) = 0 \qquad (x \in \mathbb{F}),$$

thus especially,

$$f(x) = A_n([x]_n) = 0 \qquad (x \in \mathbb{F}).$$

We will show this by induction on k. Above, in the comparison of zero-degree terms, we have already seen that

$$(1 - \kappa) A_n([1]_n) = 0.$$

Since $\kappa \neq 1$ in this case, $A_n([1]_n) = 0$. Therefore the statement holds for $k = 0$. Assume now that there exists a positive integer, such that the statement holds for all indices less than k. In other words, suppose that for all $l = 0, 1, \ldots, k-1$

$$A_n([x]_l, [1]_{n-l}) = 0 \qquad (x \in \mathbb{F}) .$$

Since the left-hand side is a generalized monomial of degree l, which is identically zero, the symmetric, l-additive mapping, defined uniquely by it, must also be identically zero. This means however that

$$A_n(x_1, \ldots, x_l, [1]_{n-l}) = 0$$

holds for all $x_1, \ldots, x_l \in \mathbb{F}$. This means that the right-hand side of equation (9) is identically zero, from which

$$\binom{n}{k, n-k} (2^k - \kappa) A_n([x]_k, [1]_{n-k}) = 0 \qquad (x \in \mathbb{F})$$

can be deduced. So the statement holds also for k. Summing up, if for all $k = 0, 1, \ldots, n$, we have $\kappa \neq 2^k$, then

$$A_n([x]_k, [1]_{n-k}) = 0 \qquad (x \in \mathbb{F}) ,$$

thus especially,

$$f(x) = A_n([x]_n) = 0 \qquad (x \in \mathbb{F}) .$$

We now turn to discussing the case $\kappa = 1$. In this case, we will also use equation (9) by choosing different indices k. We show that for all $k = 0, 1, \ldots, n$, we have

$$A_n([x]_k, [1]_{n-k}) = f(1) \cdot x^k \qquad (x \in \mathbb{F}) .$$

For $k = 0$, this holds true trivially since $A_n([1]_n) = f(1)$. Assume now that there exists a positive k less or equal to n such that the statement holds for $l = 0, \ldots, k-1$, i.e.,

$$A_n([x]_l, [1]_{n-l}) = f(1) \cdot x^l \qquad (x \in \mathbb{F}) .$$

Since A_n is a symmetric and n-additive mapping, for all $l = 0, \ldots, n$, the mappings

$$\mathbb{F}^l \ni (x_1, \ldots, x_l) \longmapsto A_n(x_1, \ldots, x_l, [1]_{n-l})$$

are symmetric and l-additive. Due to the induction hypothesis, for all fixed $l = 0, \ldots, k-1$, the mapping

$$\mathbb{F} \ni x \longmapsto A_n([x]_l, [1]_{n-l}) - f(1) \cdot x^l$$

is identically zero. At the same time, for all fixed $l = 0, \ldots, k-1$, this mapping is the trace of the symmetric and l-additive mapping

$$\mathbb{F}^l \ni (x_1, \ldots, x_l) \longmapsto A_n(x_1, \ldots, x_l, [1]_{n-l}) = f(1) \cdot x_1 \cdots x_l .$$

As the trace is identically zero, this mapping should be identically zero, too. Therefore,

$$A_n(x_1, \ldots, x_l, [1]_{n-l}) = f(1) \cdot x_1 \cdots x_l \qquad (x_1, \ldots, x_l \in \mathbb{F}).$$

holds for all $l = 0, \ldots, k-1$. In this case, however, equation (9) takes the form

$$\begin{aligned}
&\binom{n}{k, n-k}(2^k - 1)A_n([x]_k, [1]_{n-k}) \\
&\quad = -\sum_{l=1}^{\lfloor \frac{k}{2} \rfloor} \binom{n}{l, k-2l, n-k+l} A_n\left([x^2]_l, [2x]_{k-2l}, [1]_{n-k}\right) \\
&\quad + \sum_{m=1}^{k} \binom{n}{m, n-m} \binom{n}{k-m, n-k+m} x^m A_n([x]_{k-m}, [1]_{n-k-m}) \\
&\quad = -\sum_{l=1}^{\lfloor \frac{k}{2} \rfloor} \binom{n}{l, k-2l, n-k+l} f(1) \cdot x^k \\
&\quad + \sum_{m=1}^{k} \binom{n}{m, n-m} \binom{n}{k-m, n-k+m} x^m \cdot f(1) \cdot x^{k-m} \\
&\quad = \binom{n}{k, n-k}(2^k - 1) \cdot f(1) x^k \qquad (x \in \mathbb{F}),
\end{aligned}$$

showing that the statement also holds for k. Note, however, that this means (with $k = n$) that

$$A_n([x]_n) = f(1) \cdot x^n \qquad (x \in \mathbb{F}),$$

that is,

$$f(x) = f(1) \cdot x^n \qquad (x \in \mathbb{F}).$$

We continue with the case $\kappa = 2$. Recall that in this case, we have already seen that $A_n([1]_n) = 0$ and equation (9) for $k = 1$ does not contain any information for the values of the mapping $x \longmapsto A_n([x]_1, [1]_{n-1})$. Consider the additive function a defined on $\mathbb{F}$ by

$$a(x) = A_n([x]_1, [1]_{n-1}) \qquad (x \in \mathbb{F}).$$

We show that for all $k = 1, \ldots, n$,

$$A_n([x]_k, [1]_{n-k}) = \sum_{l=1}^{k} \lambda_{k,l} x^{k-l} a(x^l)$$

holds with some appropriate constants $\lambda_{k,1}, \ldots, \lambda_{k,l} \in \mathbb{K}$. Due to the definition of the function a, this statement is true for $k = 1$. Assume an index $k \geq 2$ exists now, such that the above statement holds for all index $l = 1, \ldots, k-1$. Note from the induction hypothesis it follows that we have

$$\begin{aligned} A_n(x_1, \ldots, x_l, [1]_{n-l}) &= \frac{1}{l!} \sum_{\sigma \in \mathscr{S} l} \sum_{m=1}^{l} \lambda_{l,m} x_{\sigma(m+1)} \cdots x_{\sigma(l)} \cdot a\big(x_{\sigma(1)} \cdots x_{\sigma(m)}\big) \\ &= \frac{1}{l!} \sum_{m=1}^{l} \Delta_{x_1, \ldots, x_l} x^{l-m} a(x^m) \end{aligned}$$

for all $x_1, \ldots, x_l$ and for all $l = 1, \ldots, k-1$. Therefore, we have

$$\begin{aligned} A_n\left([x^2]_l, [2x]_{k-2l}, [1]_{n-k+l}\right) &= \frac{1}{(k-l)!} \sum_{m=1}^{l} \Delta_{[x^2]_l, [2x]_{k-2l}} y^{l-m} a(y^m) \\ &= \sum_{j=1}^{k} \widetilde{\lambda_{k,j}} x^{k-j} a(x^j) \end{aligned}$$

for all $x_1, \ldots, x_l$ and for all $l = 1, \ldots, k-1$. Due to identity (9) and the induction hypothesis, we have

$$\binom{n}{k, n-k} (2^k - \kappa) A_n([x]_k, [1]_{n-k})$$

$$= -\sum_{l=1}^{\lfloor \frac{k}{2} \rfloor} \binom{n}{l, k-2l, n-k+l} A_n\left([x^2]_l, [2x]_{k-2l}, [1]_{n-k+l}\right)$$
$$+ \kappa \sum_{m=1}^{k} \binom{n}{m, n-m} \binom{n}{k-m, n-k+m} x^m A_n([x]_{k-m}, [1]_{n-k-m})$$
$$= \sum_{j=1}^{k} \widetilde{\lambda_{k,j}} x^{k-j} a(x^j)$$
$$+ \kappa \sum_{m=1}^{k} \binom{n}{m, n-m} \binom{n}{k-m, n-k+m} x^m \sum_{j=1}^{k-m} \lambda_{k-m,j} x^{k-m-j} a(x^j)$$
$$= \sum_{j=1}^{k} \widetilde{\lambda_{k,j}} x^{k-j} a(x^j)$$
$$+ \kappa \sum_{m=1}^{k} \sum_{j=1}^{k-m} \binom{n}{m, n-m} \binom{n}{k-m, n-k+m} \lambda_{k-m,j} x^{k-j} a(x^j)$$
$$= \sum_{j=1}^{k} \lambda^*_{k,j} x^{k-j} a(x^j)$$

for all $x \in \mathbb{F}$. From this, we obtain that the statement also holds for k. Observe that this means that

$$f(x) = A_n([x_n]) = \sum_{j=1}^{n} \lambda_{n,j} x^{n-j} a(x^j) \qquad (x \in \mathbb{F})$$

holds with appropriate constants $\lambda_1, \ldots, \lambda_n$. Using this representation and equation (16), identity

$$\sum_{j=1}^{n} \lambda_{n,j} x^{2n-2j} a(x^{2j}) - \sum_{j=1}^{n} 2\lambda_{n,j} x^{2n-j} a(x^j) = 0 \qquad (x \in \mathbb{F})$$

follows for the additive function a. In view of Corollary 4 of [**20**], this means that $a \in \mathscr{D}_{2n-1}(\mathbb{F}, \mathbb{K})$.

Let us consider the case $\kappa = 2^n$. We show that in this case

$$A_n([x]_k, [1]_{n-k}) = 0 \qquad (x \in \mathbb{F})$$

holds for all $k = 0, \ldots, n-1$. Since in case $\kappa = 2^n$, we have $A_n([1]_n) = 0$, the statement holds for $k = 0$. Assume now that there exists a $k \in \{1, \ldots, n\}$ such that the statement holds for all $l = 0, \ldots, k-1$, that is, we have

$$A_n([x]_l, [1]_{n-l}) = 0 \qquad (x \in \mathbb{F})$$

for all $l = 0, \ldots, k-1$. Since the left-hand side of this equation is the trace of a symmetric and l-additive function, we have

$$A_n\left(x_1, \ldots, x_l, [1]_{n-l}\right) = 0 \qquad (x_1, \ldots, x_l \in \mathbb{F})$$

for all $l = 0, \ldots, k-1$. Using this, and equation (9), we obtain that

$$\begin{aligned}
&\binom{n}{k, n-k}(2^k - \kappa)A_n([x]_k, [1]_{n-k}) \\
&\qquad = -\sum_{l=1}^{\lfloor \frac{k}{2} \rfloor} \binom{n}{l, k-2l, n-k+l} A_n\left([x^2]_l, [2x]_{k-2l}, [1]_{n-k+l}\right) \\
&+ \kappa \sum_{m=1}^{k} \binom{n}{m, n-m} \binom{n}{k-m, n-k+m} x^m A_n([x]_{k-m}, [1]_{n-k-m}) = 0
\end{aligned}$$

for all $x \in \mathbb{F}$. Indeed, since $l + (k-2l) \leq k-1$ and $k - m \leq k-1$ holds for all $l = 1, \ldots, \lfloor \frac{k}{2} \rfloor$ and for all $m = 1, \ldots, k$, the mappings

$$\mathbb{F} \ni x \longmapsto A_n\left([x^2]_l, [2x]_{k-2l}, [1]_{n-k+l}\right)$$

and

$$\mathbb{F} \ni x \longmapsto A_n([x]_{k-m}, [1]_{n-k-m})$$

are identically zero.

Since the left-hand side of equation (9) vanishes for $\kappa = 2^n$, we have no information for the mapping $x \longmapsto A_n([x]_n)$.

Using this, and computing the $(n+1)^{\text{st}}$-degree terms in (8), we deduce that

$$n \cdot A_n(x^2, [2x]_{n-1}) - n \cdot x \cdot 2^n \cdot A_n([x]_n) = 0 \qquad (x \in \mathbb{F}),$$

that is,

$$A_n([x^2]_1, [x]_{n-1}) - 2x \cdot A_n([x]_n) = 0 \qquad (x \in \mathbb{F}).$$

The left-hand side of this equation (as a mapping of the variable x) is the trace of a symmetric and $(n+1)$-additive function. However, the $(n+1)$-additive mapping is necessarily identically zero if the trace vanishes. Thus we have

$$\begin{aligned}
\sum_{\sigma \in \mathscr{S}_{n+1}} \{\, & A_n(x_{\sigma(1)} \cdot x_{\sigma(2)}, x_{\sigma(3)}, \ldots, x_{\sigma(n+1)}) \\
& - x_{\sigma(1)} A_n(x_{\sigma(2)}, \ldots, x_{\sigma(n+1)}) \\
& - x_{\sigma(2)} A_n(x_{\sigma(1)}, \ldots, x_{\sigma(n+1)}) \big\} = 0
\end{aligned}$$

for all $x_1, \ldots, x_{n+1} \in \mathbb{F}$. $\square$

REMARK 17. *Notice that the above theorem does not state anything about the cases when* $\kappa \in \left\{2^k \,|\, k = 2, 3, 4, \quad \ldots, n-1\right\}$. *In addition, we make a new conjecture that in the case of* $k = 2$, *the order of the higher-order derivative included in the representation of the function* f *can be reduced. This conjecture is supported by our results for the case* $n = 3$, *which can be found in the next section.*

Results on the special case $n = 3$**.** As a supplement to the result in the previous section, we will deal with the special case $n = 3$ below. Compared to Theorem 14, we can prove two more things. On one hand, we show that in the case $\kappa = 2$, the order of the higher-order derivation appearing in the representation of the function f is at most 3, not at most 5. On the other hand, we show that in the case $\kappa = 4$ the function f is identically zero.

First, we prove the following lemma, which will help in part (ii) of Theorem 15.

LEMMA 9. *Let* $a\colon \mathbb{F} \to \mathbb{K}$ *be an additive mapping such that*

$$2\,a\left(x^6\right) - 9\,x^2\,a\left(x^4\right) - 4\,x^3\,a\left(x^3\right) \\ + 36\,x^4\,a\left(x^2\right) - 36\,x^5\,a\left(x\right) = 0 \qquad (x \in \mathbb{F})\,.$$

Then $a \in \mathscr{D}_3(\mathbb{F}, \mathbb{K})$.

PROOF. Let $a\colon \mathbb{F} \to \mathbb{K}$ be an additive function such that

$$2\,a\left(x^6\right) - 9\,x^2\,a\left(x^4\right) - 4\,x^3\,a\left(x^3\right) \\ + 36\,x^4\,a\left(x^2\right) - 36\,x^5\,a\left(x\right) = 0 \qquad (x \in \mathbb{F})\,.$$

Applying the operator Δ_1 to both sides of this equation, we get that

$$4\,a\left(x^5\right) - 6\,x\,a\left(x^4\right) + 7\,a\left(x^4\right) - 16\,x^2\,a\left(x^3\right) - 28\,x\,a\left(x^3\right) \\ + 44\,x^3\,a\left(x^2\right) + 42\,x^2\,a\left(x^2\right) - 36\,x^4\,a\left(x\right) - 28\,x^3\,a\left(x\right) = 0$$

for all $x \in \mathbb{F}$. Observe that the left-hand side of this equation is a generalized polynomial of degree 5, that has fifth-degree and also fourth-degree monomial terms that should vanish simultaneously. Thus collecting the fourth-degree terms, we arrive to

$$7\,a\left(x^4\right) - 28\,x\,a\left(x^3\right) + 42\,x^2\,a\left(x^2\right) - 28\,x^3\,a\left(x\right) = 0 \qquad (x \in \mathbb{F})$$

that is to

$$a\left(x^4\right) - 4\,x\,a\left(x^3\right) + 6\,x^2\,a\left(x^2\right) - 4\,x^3\,a\left(x\right) = 0 \qquad (x \in \mathbb{F})\,.$$

In view of [**20**, Corollary 2] this means that $a \in \mathscr{D}_3(\mathbb{F}, \mathbb{K})$. $\square$

THEOREM 15. *Let $\kappa \in \mathbb{K}$ be arbitrarily fixed and $f \in \mathscr{M}_3(\mathbb{F}, \mathbb{K})$ be a monomial for which*

$$f(x^2) = \kappa \cdot x^3 f(x) \tag{10}$$

holds for all $x \in \mathbb{F}$. Then the following cases are possible

(i) $\kappa = 1$ *and then*

$$f(x) = f(1) \cdot x^3 \qquad (x \in \mathbb{F}) .$$

(ii) $\kappa = 2$ *and then there exists a $d \in \mathscr{D}_3(\mathbb{F}, \mathbb{K})$ such that*

$$f(x) = d\left(x^3\right) - \frac{9\, x\, d\left(x^2\right)}{2} + 9\, x^2\, d\left(x\right) \qquad (x \in \mathbb{F}) .$$

(iii) $\kappa = 8$ *and then there exists a symmetric and 3-additive mapping D_3 for which*

$$\begin{aligned}\sum_{\sigma \in \mathscr{S}_4} \{ D_3\left(x_{\sigma(1)} x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}\right) & \\ - x_{\sigma(1)} D_3(x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}) & \\ - x_{\sigma(2)} D_3(x_{\sigma(1)}, x_{\sigma(3)}, x_{\sigma(4)}) & \} = 0\end{aligned}$$

holds for all $x_1, x_2, x_3, x_4 \in \mathbb{F}$ such that

$$f(x) = D_3(x, x, x) \qquad (x \in \mathbb{F}) .$$

(iv) $\kappa \in \mathbb{K} \setminus \{1, 2, 8\}$ *and f is identically zero.*

PROOF. The cases $\kappa \in \mathbb{K} \setminus \{2, 4\}$ immediately follow from Theorem 15. Therefore, it is sufficient to deal only with the cases $\kappa \in \{2, 4\}$. In case $n = 3$ equation (8) reads as

$$\begin{aligned}\sum_{\alpha_1+\alpha_2+\alpha_3=3} \binom{3}{\alpha_1, \alpha_3, \alpha_3} A_3([x^2]_{\alpha_1}, [2x]_{\alpha_2}, [1]_{\alpha_3}) & \\ - \kappa(x^3 + 3x^2 + 3x + 1) \sum_{\alpha_1+\alpha_2=3} \binom{3}{\alpha_1, \alpha_2} A_3([x]_{\alpha_1}, [1]_{\alpha_2}) = 0 \quad (x \in \mathbb{F}) .\end{aligned} \tag{11}$$

At first, we consider the case $\kappa = 2$. Collecting the constant terms we get that $f(1) = A_3(1, 1, 1) = 0$. Further, computing the first-degree monomial terms, we do not get any information for the values of $A_3(x, 1, 1)$.

The second-degree monomial terms must also vanish, thus

$$\begin{aligned}\binom{3}{1,0,2}&A_3([x^2]_1,[1]_2)\\&+\binom{3}{0,2,1}A_3([2x]_2,[1]_1)-3\kappa x^2\binom{3}{0,3}A_3([1]_3)\\&-3\kappa x\binom{3}{1,2}A_3([x]_1,[1]_2)-\kappa\binom{3}{2,1}A_3([x]_2,[1]_1)=0,\end{aligned}$$

or after some rearrangement

$$(4-\kappa)A_3(x,x,1)+A_3(x^2,1,1)-\kappa x^2A_3(1,1,1)-3\kappa xA_3(x,1,1)=0$$

for all $x\in\mathbb{F}$. If $\kappa=2$, then we have

$$2A_3(x,x,1)+A_3(x^2,1,1)-6xA_3(x,1,1)=0 \qquad (x\in\mathbb{F}),$$

where the identity $A_3(1,1,1)=0$ was also used. Thus,

$$2A_3(x,x,1)+a(x^2)-6xa(x)=0$$

holds with the additive function $a\colon\mathbb{F}\to\mathbb{K}$, where

$$a(x)=A_3(x,1,1) \qquad (x\in\mathbb{F}).$$

Finally, the fact that the third-degree terms should vanish, yields

$$\begin{aligned}12A_3(x^2,x,1)&+8A_3(x,x,x)\\&-\kappa x^3A_3(1,1,1)-9\kappa x^2A_3(x,1,1)\\&-9\kappa xA_3(x,x,1)-\kappa A_3(x,x,x)=0 \qquad (x\in\mathbb{F}),\end{aligned}$$

that is,

$$\begin{aligned}(8-\kappa)&A_3(x,x,x)\\&+12A_3(x^2,x,1)-9\kappa xA_3(x,x,1)\\&-9\kappa x^2A_3(x,1,1)-\kappa x^3A_3(1,1,1)=0 \qquad (x\in\mathbb{F}).\end{aligned} \tag{12}$$

If $\kappa=2$, then equation (12) furnishes

$$A_3(x,x,x)=a\left(x^3\right)-\frac{9\,x\,a\left(x^2\right)}{2}+9\,x^2\,a\left(x\right)$$

for all $x\in\mathbb{F}$. This means that the function f can be represented as

$$f(x)=a\left(x^3\right)-\frac{9\,x\,a\left(x^2\right)}{2}+9\,x^2\,a\left(x\right)$$

for all $x \in \mathbb{F}$. Since f satisfies equation (17), the additive function a necessarily fulfills

$$2\,a\left(x^{6}\right) - 9\,x^{2}\,a\left(x^{4}\right) - 4\,x^{3}\,a\left(x^{3}\right) \\ + 36\,x^{4}\,a\left(x^{2}\right) - 36\,x^{5}\,a\left(x\right) = 0 \qquad (x \in \mathbb{F}).$$

This however, in view of Lemma 9 implies that $a \in \mathscr{D}_3(\mathbb{F})$.

Finally, in case $\kappa = 4$, we show that f is identically zero. Similarly, as in case $\kappa = 2$, we determine the first, second, and third-degree terms in (11). Computing the first-degree terms,

$$A_3(x,1,1) = 0 \qquad (x \in \mathbb{F})$$

follows. The second-degree terms should also vanish. Thus we have

$$(4-\kappa)A_3(x,x,1) + A_3(x^2,1,1) - \kappa x^2 A_3(1,1,1) - 3\kappa x A_3(x,1,1) = 0$$

for all $x \in \mathbb{F}$, yielding no information for the values $A_3(x,x,1)$ in the case $\kappa = 4$. Further, if we compute the third-degree terms, we obtain that

$$\begin{aligned} A_3(x,x,x) =& 9xA_3(x,x,1) - 3A_3(x^2,x,1) \\ =& 9xB(x,x) - 3B(x^2,x) \qquad (x \in \mathbb{F}), \end{aligned}$$

where the symmetric and bi-additive function $B\colon \mathbb{F}^2 \to \mathbb{K}$ is defined through

$$B(x,y) = A_3(x,y,1) \qquad (x,y \in \mathbb{F}).$$

This means that the values of function B completely determine function A_3. Indeed, we have

$$\begin{aligned} A_3(x,y,z) = 3\left[xB(y,z) + yB(x,z) + zB(x,y)\right] \\ - \left[B(xy,z) + B(xz,y) + B(yz,x)\right] \qquad (x,y,z \in \mathbb{F}). \end{aligned}$$

Reformulating equation (17) with the aid of the mapping B,

$$\begin{aligned} 0 =& f(x^2) - 4x^3 f(x) \\ &= -3B(x^4,x^2) + 9x^2B(x^2,x^2) + 12x^3B(x^2,x) - 36x^4B(x,x), \end{aligned}$$

that is,

$$B(x^4,x^2) - 3x^2B(x^2,x^2) - 4x^3B(x^2,x) + 12x^4B(x,x) = 0$$

follows for all $x \in \mathbb{F}$. The left side, as a function of the variable x, i.e.,

$$g(x) = B(x^4,x^2) - 3x^2B(x^2,x^2) - 4x^3B(x^2,x) + 12x^4B(x,x)$$

is a monomial of degree 6, which is identically zero. Thus $\tau_1 g$ is a generalized polynomial of degree at most 6. Computing the fourth degree monomial terms of $\tau_1 g$, we get that

$$3B(x^2,x^2) - 36xB(x^2,x) + 8B(x^3,x) + 36x^2B(x,x) = 0 \qquad (x \in \mathbb{F}). \tag{13}$$

On the other hand, if we collect the fourth-degree terms in (11),

$$A_3(x^2, x, x) - 3x^2 A_3(x, x, 1) + x A_3(x, x, x) = 0,$$

expressing this, with the aid of the function B,

$$-2B(x^3, x) - B(x^2, x^2) + 9xB(x^2, x) - 9x^2 B(x, x) = 0 \tag{14}$$

follows for all $x \in \mathbb{F}$. Combining equations (13) and (14),

$$B(x^2, x^2) = 0$$

can be concluded for all $x \in \mathbb{F}$. Since the left-hand side of this identity is the trace of the symmetric and 4-additive mapping

$$\begin{aligned} &B_4(x_1, x_2, x_3, x_4) \\ &\qquad = \frac{1}{3}\left[B(x_1x_2, x_3x_4) + B(x_1x_3, x_2x_4) + B(x_1x_4, x_2x_3)\right] \\ &\qquad\qquad\qquad\qquad\qquad\qquad (x_1, x_2, x_3, x_4 \in \mathbb{F}), \end{aligned}$$

we have

$$0 = B_4(x, y, 1, 1) = \frac{B(xy, 1)}{3} + \frac{2B(x, y)}{3} = \frac{2}{3}B(x, y)$$

for all $x, y \in \mathbb{F}$. So B is identically zero. Therefore

$$f(x) = A_3(x, x, x) = 9xB(x, x) - 3B(x^2, x) = 0 \qquad (x \in \mathbb{F}),$$

as stated. $\square$

Summary

The following section provides a summary of the key accomplishments achieved in this dissertation. Our research has resulted in several lemmas, propositions, theorems, examples and corollaries which are derived from the papers Gselmann–Iqbal [**17, 18, 19**].

Additive functions are significant in the field of functional equations and commutative algebra. It is an important and challenging question how special morphisms can be characterized among additive mappings in general. The investigation of additive mappings that satisfy a polynomial equation from one ring to another was started by G. Ancochea in [**4**], who specifically studied additive functions that preserve squares. Subsequently, these findings were further developed among others by Kaplansky [**25**] and Jacobson-Rickart [**24**]. The results of [**4**] were generalized and extended in several ways, see for instance [**24**], [**25**], [**36**]. In [**23**] I.N. Herstein showed that if φ is a Jordan homomorphism of a ring R *onto* a prime ring R' of characteristic different from 2 and 3, then either φ is a homomorphism or an anti-homomorphism.

Besides homomorphisms, derivations also play a key role in the theory of rings and fields. Concerning this notion, we will follow [**27**, Chapter 14]. Hamel basis play a very crucial role in case of additive functions. As [**27**, Theorem 14.2.1] shows in case of derivations, algebraic bases are fundamental. Similar to homomorphisms, characterization theorems related to derivations also have extensive literature, see e.g. the monographs [**27, 35**].

According to a classical result in connection to derivations, if $\mathbb{F}$ is a subfield of the field $\mathbb{K}$ with characteristic zero, $P \in \mathbb{F}[x]$ is a polynomial and the additive function $a\colon \mathbb{F} \to \mathbb{K}$ fulfills

$$a(P(x)) = P'(x) \cdot a(x) \qquad (x \in \mathbb{F}),$$

then a is a derivation.

Note that this problem can be viewed as a special case of a more general problem. Indeed, let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F} \subset \mathbb{K}$ be a subfield, let further $P \in \mathbb{F}[x]$ and $Q \in \mathbb{K}[x_1, x_2]$ be given polynomials and $a\colon \mathbb{F} \to \mathbb{K}$ be an additive function such that

$$a(P(x)) = Q(x, a(x)) \qquad (x \in \mathbb{F}).$$

The question arises: Does the above identity imply that this additive function a has some 'special form'? For certain polynomials P and Q, in the case of classical results, the unknown additive function a is a homomorphism, a derivation, or a linear combination of these. Naturally, the question arises as to whether this is the case for all polynomials P and Q. Similar questions can be raised about generalized monomial functions instead of additive functions: assume that n and k are positive integers, $P_{i,j} \in \mathbb{F}[x]$ and $P \in \mathbb{K}[z, x_{1,1}, \ldots, x_{n,k}]$ are given polynomials for $i = 1, \ldots, n; j = 1, \ldots, k$. Suppose further that $f_1, \ldots, f_n \colon \mathbb{F} \to \mathbb{K}$ are generalized monomials (of possibly different degree) such that

($\bullet$)

$$P(x, f_1(P_{1,1}(x)), \ldots, f_1(P_{1,k}(x)), \ldots, f_n(P_{n,1}(x)), \ldots, f_n(P_{n,k}(x))) = 0$$

holds for all $x \in \mathbb{F}$? Is it true that the monomial functions here necessarily have a 'special' form? If so, is there a method to determine these special forms?

In the case where the unknown generalized monomial functions are additive, some results can be found e.g. in [**10, 11, 12, 13, 20, 21**]. The papers [**16, 17**] contain related results, but there the unknown functions were assumed to be quadratic. Our papers [**17**] and [**18**] are continuations of these.

Monomial functions, normal polynomials and polynomial equations. Let $\mathbb{F} \subset \mathbb{C}$ be a field, n be a positive integer and $P \in \mathbb{F}[x]$ be a polynomial. In Chapter 2, firstly we study generalized monomials $f\colon \mathbb{F} \to \mathbb{C}$ of degree n under the condition that the mapping

$$\mathbb{F} \ni x \longmapsto f(P(x))$$

is a (normal) polynomial.

This problem is not an end in itself. We undertake this investigation because it allows the original problem to be reduced, in many cases, to a much simpler one.

Accordingly, at first we show that instead of polynomials P in the original problem, we always may restrict ourselves to (classical) monomials. In fact, we can always restrict ourselves to homogeneous normal polynomials. More precisely, to carry out the aforementioned reduction, we prove the following two things.

Let $k, n \in \mathbb{N}$, $k \geq 2$, $\mathbb{F} \subset \mathbb{C}$ be a field, $P \in \mathbb{F}[x]$ be a (classical) polynomial of degree k with leading coefficient 1 and $f\colon \mathbb{F} \to \mathbb{C}$ be a generalized monomial of degree n. If the mapping

$$\mathbb{F} \ni x \longmapsto f(P(x))$$

is a normal polynomial, then the mapping

$$\mathbb{F} \ni x \longmapsto f(x^k)$$

is a normal polynomial as well.

Secondly we show that if the mapping

$$\mathbb{F} \ni x \longmapsto f(x^k)$$

is a normal polynomial, then there exists a *homogeneous* complex polynomial $\tilde{P}$ and there are linearly independent, complex valued additive functions $a_1, \dots, a_m$ on $\mathbb{F}$ such that

$$f(x^k) = \tilde{P}(a_1(x), \dots, a_m(x)) \qquad (x \in \mathbb{F}),$$

in other words, we have

$$f(x^k) = \sum_{\substack{\alpha \in \mathbb{N}^m \\ |\alpha| = kn}} \lambda_\alpha a^\alpha(x) = \sum_{\substack{\alpha_1, \dots, \alpha_m \geq 0 \\ \alpha_1 + \cdots + \alpha_m = kn}} \lambda_{\alpha_1, \dots, \alpha_m} a_1^{\alpha_1}(x) \cdots a_m^{\alpha_m}(x)$$

for each $x \in \mathbb{F}$.

At first glance, the assumption that the mapping

$$\mathbb{F} \ni x \longmapsto f(x^k)$$

is a normal polyomial, seems a bit artificial. Nevertheless, the following examples show that this is indeed the case in connection to characterization theorems of homomorphisms and derivations.

(1) Let k be a positive integer, $\varphi_1, \dots, \varphi_k \colon \mathbb{F} \to \mathbb{C}$ be linearly independent homomorphisms and $\lambda_{i,j} \in \mathbb{C}$ for all $i, j = 1, \dots, k$. Then the mapping $f \colon \mathbb{F} \to \mathbb{C}$ defined by

$$f(x) = \sum_{i,j=1}^{k} \lambda_{i,j} \varphi_i(x) \varphi_j(x) \qquad (x \in \mathbb{F})$$

is a quadratic function. Further if $n \in \mathbb{N}$, then we also have

$$f(x^n) = \sum_{i,j=1}^{k} \lambda_{i,j} \varphi_i(x)^n \varphi_j(x)^n \qquad (x \in \mathbb{F}).$$

In other words, we have

$$f(x^n) = P(\varphi_1(x), \dots, \varphi_k(x)) \qquad (x \in \mathbb{F}),$$

where the k-variable complex, homogeneous polynomial P is defined by

$$P(x_1, \ldots, x_k) = \sum_{i,j=1}^{k} \lambda_{i,j} x_i^n x_j^n \qquad (x_1, \ldots, x_k \in \mathbb{C}).$$

(2) Suppose now that $\mathbb{F}$ is a subfield of $\mathbb{C}$. Let k be a positive integer, $d_1, \ldots, d_k \colon \mathbb{F} \to \mathbb{C}$ be linearly independent derivations and $\lambda_{i,j} \in \mathbb{C}$ for all $i, j = 1, \ldots, k$. Then the mapping $f \colon \mathbb{F} \to \mathbb{C}$ defined by

$$f(x) = \sum_{i,j=1}^{k} \lambda_{i,j} d_i(x) d_j(x) \qquad (x \in \mathbb{F})$$

is a quadratic function. Further if $n \in \mathbb{N}$, then we also have

$$f(x^n) = \sum_{i,j=1}^{k} \lambda_{i,j} n^2 x^{2n-2} d_i(x) d_j(x) \qquad (x \in \mathbb{F}).$$

In other words,

$$f(x^n) = P(x, d_1(x), \ldots, d_k(x)) \qquad (x \in \mathbb{F}),$$

where the $(k+1)$-variable complex polynomial P is defined by

$$P(x_1, \ldots, x_k, z) = \sum_{i,j=1}^{n} \lambda_{i,j} n^2 z^{2n-2} x_i x_j \qquad (x_1, \ldots, x_k, z \in \mathbb{C}).$$

Regarding 'polynomial equations' for generalized monomials, we note that in the literature, there are several results for additive functions $a \colon \mathbb{F} \to \mathbb{C}$ that also satisfy a polynomial equation. Based on the results presented above, the following statement can be deduced.

Let $\mathbb{F} \subset \mathbb{C}$ be a field, $k \in \mathbb{N}$, $k \geq 2$ and $P \in \mathbb{Q}[x]$ be a (classical) polynomial of degree k.

(1) If the additive function $a \colon \mathbb{F} \to \mathbb{C}$ fulfills

$$a(P(x)) = P(a(x)) \qquad (x \in \mathbb{F}),$$

then there exists a homomorphism $\varphi \colon \mathbb{F} \to \mathbb{C}$ such that $a(x) = a(1)\varphi(x)$ for all $x \in \mathbb{F}$. Further, we also have $a(1) \in \{0, 1\}$.

(2) If the additive function $a \colon \mathbb{F} \to \mathbb{C}$ fulfills

$$a(P(x)) = P'(x) a(x) \qquad (x \in \mathbb{F}),$$

then a is a derivation.

We emphasize that the results of the previous statement are classical ones. Nevertheless, we would like to indicate that from one hand the problem we consider in this dissertation has some prior results both in algebra and the theory of functional equations. Further, with the help of the statements presented here, the proofs can be significantly simplified (at least for mappings $a\colon \mathbb{F} \to \mathbb{C}$).

In the papers [**10, 11, 13, 20, 21**] further results can be found concerning additive functions that also fulfill certain polynomial equations.

We also note that related problems have already been considered by Z. Boros and E. Garda–Mátyás in [**7, 5**] by Z. Boros and R. Menzer in [**8**] and also by M. Amou in [**3**]. In these papers the authors consider real monomial functions, which satisfy certain conditional equations on a specified planar curve. In [**16**], the polynomial equation $f(P(x)) = Q(f(x))$ for the monomial function $f\colon \mathbb{F} \to \mathbb{C}$ was considered.

A result for additive functions. The simplest special case of the problem we are interested in is when the generalized monomial $f\colon \mathbb{F} \to \mathbb{C}$ is of degree 1, i.e., when f is an additive function. In this regard, we have the following statement.

Let k be a positive integer, $\mathbb{F} \subset \mathbb{C}$ be a field and $a\colon \mathbb{F} \to \mathbb{C}$ be an additive function. If the mapping

$$\mathbb{F} \ni x \longmapsto a(x^k)$$

is a (normal) polynomial, then a is a generalized higher order derivation.

Results for quadratic functions. As a continuation, the question arises what happens when the generalized monomial $f\colon \mathbb{F} \to \mathbb{C}$ is of degree 2, i.e., when f is a quadratic function. In this case, we have the following results.

Let $n \in \mathbb{N}$, $n \geq 2$ and $\mathbb{F} \subset \mathbb{C}$ be a field. Assume that $f\colon \mathbb{F} \to \mathbb{C}$ is a quadratic function, while $a\colon \mathbb{F} \to \mathbb{C}$ is additive and we have

$$f(x^n) = a(x)^{2n} \qquad (x \in \mathbb{F}). \tag{15}$$

Then there exists a complex constant $\alpha \in \mathbb{C}$ and a homomorphism $\varphi\colon \mathbb{F} \to \mathbb{C}$ such that

$$a(x) = \alpha\varphi(x) \qquad \text{and} \qquad f(x) = \alpha^{2n}\varphi(x)^2 \qquad (x \in \mathbb{F}).$$

And also conversely, if we define the functions a and f through the above formula then they satisfy equation (15) for all $x \in \mathbb{F}$.

The second result states the following.

Let $\mathbb{F} \subset \mathbb{C}$ be a field, $f\colon \mathbb{F} \to \mathbb{C}$ be a quadratic function, while $a_1, a_2\colon \mathbb{F} \to \mathbb{C}$ be additive functions. Then equation

$$f(x^2) = a_1(x)^2 a_2(x)^2$$

holds for all $x \in \mathbb{F}$ if and only if there exist homomorphisms $\varphi_1, \varphi_2 \colon \mathbb{F} \to \mathbb{C}$ such that

$$f(x) = f(1) \cdot \varphi_1(x)\varphi_2(x) \qquad \text{and} \qquad a_i(x) = a_i(1)\varphi_i(x)\,(x \in \mathbb{F}, i = 1, 2)\,.$$

Qudaratic functions as the solutions of polynomial equations. Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F} \subset \mathbb{K}$ be a subfield of $\mathbb{K}$. The main objective of Chapter 3 is to determine all those quadratic functions $q \colon \mathbb{F} \to \mathbb{K}$ that satisfy a Levi-Civita equation on the multiplicative structure, i.e., that can be written as

$$q(xy) = \sum_{i=1}^{k} g_i(x)h_i(y) \qquad \left(x, y \in \mathbb{F}^\times\right)$$

with some positive integer k and with some appropriate functions g_i, h_i, $i = 1, \dots, k$. For this, those quadratic functions q that satisfy the equations

$$q(xy) = q(x)q(y) \qquad \left(x, y \in \mathbb{F}^\times\right)$$

and

$$q(xy) = x^2 q(y) + q(x)y^2 \qquad \left(x, y \in \mathbb{F}^\times\right),$$

respectively, must first be determined.

Let us endow $\mathbb{F}^\times$ and also $\mathbb{K}$ with the discrete topology and let $\mathscr{C}(\mathbb{F}^\times, \mathbb{K})$ denote the linear space of all those functions $f \colon \mathbb{F}^\times \to \mathbb{K}$ that are continuous. Then the set

$$\mathscr{V} = \{f|_{\mathbb{F}^\times} \mid f \colon \mathbb{F} \to \mathbb{K} \text{ is quadratic}\}$$

is a closed, translation-invariant linear subspace of $\mathscr{C}(\mathbb{F}^\times, \mathbb{K})$. So $\mathscr{V} \subset \mathscr{C}(\mathbb{F}^\times, \mathbb{K})$ is a *variety.* Note that the translate of a function $f \in \mathscr{C}(\mathbb{F}^\times, \mathbb{K})$ by an element $y \in \mathbb{F}^\times$ is defined as

$$(\tau_y f) = f(x \cdot y) \qquad \left(x \in \mathbb{F}^\times\right).$$

We also have some illustrative examples

(1) Let $\varphi_1, \varphi_2 \colon \mathbb{F} \to \mathbb{K}$ be homomorphisms and let us consider the function $q \colon \mathbb{F} \to \mathbb{K}$ defined by

$$q(x) = \varphi_1(x)\varphi_2(x) \qquad (x \in \mathbb{F})\,.$$

Then q is the trace of the symmetric and bi-additive mapping

$$B(x, y) = \varphi_1(x)\varphi_2(y) + \varphi_1(y)\varphi_2(x) \qquad (x, y \in \mathbb{F})\,.$$

So q is a quadratic function. Further, we have

$$\begin{aligned} q(xy) &= \varphi_1(xy)\varphi_2(xy) \\ &= (\varphi_1(x)\varphi_1(y)) \cdot (\varphi_2(x)\varphi_2(y)) \\ &= (\varphi_1(x)\varphi_2(x)) \cdot (\varphi_1(y)\varphi_2(y)) = q(x) \cdot q(y) \qquad (x, y \in \mathbb{F})\,. \end{aligned}$$

(2) Let $d\colon \mathbb{F} \to \mathbb{K}$ be a derivation and define the quadratic mapping $q\colon \mathbb{F} \to \mathbb{K}$ by

$$q(x) = d(x^2) \qquad (x \in \mathbb{F})\,.$$

An easy computation shows that in this case, q determines uniquely a symmetric and bi-additive mapping, namely

$$B(x,y) = d(xy) \qquad (x, y \in \mathbb{F})\,.$$

Moreover, we have

$$\begin{aligned}\frac{q(xy)}{x^2y^2} &= \frac{1}{x^2y^2}d((xy)^2) = \frac{1}{x^2y^2}\cdot 2xyd(xy) = \frac{2}{xy}\left[xd(y) + d(x)y\right] \\ &= 2\frac{d(x)}{x} + 2\frac{d(y)}{y} = \frac{d(x^2)}{x^2} + \frac{d(y^2)}{y^2} \\ &= \frac{q(x)}{x^2} + \frac{q(y)}{y^2}\end{aligned}$$

for all $x, y \in \mathbb{F}^{\times}$.

With the above notations, the first example shows that the mapping q defined with the aid of the homomorphisms φ_1 and φ_2 by the formula

$$q(x) = \varphi_1(x)\varphi_2(x) \qquad (x \in \mathbb{F})\,,$$

is an exponential in the variety $\mathscr{V}$.

Similarly, the second example shows that the mapping q defined with the help of the derivation $d\colon \mathbb{F} \to \mathbb{K}$ by

$$q(x) = d(x^2) \qquad (x \in \mathbb{F})$$

is a moment function of degree 1 corresponding to the exponential $q_0(x) = x^2$ $(x \in \mathbb{F})$.

In Chapter 3, we are mainly interested in converse direction.

Quadratic functions that are multiplicative and additive on the multiplicative group $\mathbb{F}^{\times}$. First we have a result for the quadratic functions that are multiplicative on $\mathbb{F}^{\times}$. The theorem states the following.

(1) Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F} \subset \mathbb{K}$ be a subfield. Let further $q\colon \mathbb{F} \to \mathbb{K}$ be a quadratic function. If the mapping q is multiplicative on $\mathbb{F}^{\times}$, that is,

$$q(xy) = q(x)q(y)$$

holds for all $x, y \in \mathbb{F}^{\times}$, then there exist homomorphisms $\varphi_1, \varphi_2\colon \mathbb{F} \to \mathbb{K}$ such that

$$q(x) = c\cdot\varphi_1(x)\varphi_2(x) \qquad (x \in \mathbb{F})\,,$$

where $c \in \{0, 1\}$.

The second theorem for the quadratic functions that are additive on $\mathbb{F}^{\times}$ states the following.

(2) Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F} \subset \mathbb{K}$ be a subfield. Let further $q\colon \mathbb{F} \to \mathbb{K}$ be a quadratic function. If the mapping $\dfrac{q}{\pi_2}$ is additive on $\mathbb{F}^{\times}$, that is,

$$q(xy) = q(x)y^2 + x^2 q(y)$$

holds for all $x, y \in \mathbb{F}^{\times}$ and $\pi_2(x) = x^2\,(x \in \mathbb{F})$, then there exists a second order derivation $d\colon \mathbb{F} \to \mathbb{K}$ such that

$$q(x) = 4xd(x) - d(x^2) \qquad (x \in \mathbb{F})\,.$$

Note that the quadratic mapping that appears in Example 2 is covered in Theorem 2. Indeed, if $d\colon \mathbb{F} \to \mathbb{K}$ is a derivation, then $d \in \mathscr{D}_1(\mathbb{F}, \mathbb{K}) \subset \mathscr{D}_2(\mathbb{F}, \mathbb{K})$. So d is a derivation of order two, too. Thus

$$q(x) = 4xd(x) - d(x^2) = 2d(x^2) - d(x^2) = d(x^2) \qquad (x \in \mathbb{F})\,,$$

showing that mappings appearing in Example 2 can indeed be written as mappings appearing in Theorem 2.

However, Theorem 2 shows that the equation

$$q(xy) = x^2 q(y) + y^2 q(x) = 0 \qquad (x, y \in \mathbb{F})$$

has other quadratic solutions than those found in Example 2. Indeed, if $d_1, d_2\colon \mathbb{F} \to \mathbb{K}$ are non-identically zero derivations, then $d_1 \circ d_2 \in \mathscr{D}_2(\mathbb{F}, \mathbb{K}) \setminus \mathscr{D}_1(\mathbb{F}, \mathbb{K})$. Then the mapping $q\colon \mathbb{F} \to \mathbb{K}$ defined by

$$q(x) = 4xd_1 \circ d_2(x) - d_1 \circ d_2(x^2) \qquad (x \in \mathbb{F})$$

is quadratic. Further, we have

$$\begin{aligned}
q(x^2) &= 4x^2 d_1 \circ d_2(x^2) - d_1 \circ d_2(x^4) \\
&= 4x^2 \left(2xd_1 \circ d_2(x) + 2d_1(x)d_2(x)\right) \\
&\quad - \big(4x^3 d_1 \circ d_2(x) + 12x^2 d_1(x)d_2(x)\big) \\
&= 4x^3 d_1 \circ d_2(x) - 4x^2 d_1(x)d_2(x)
\end{aligned}$$

and

$$\begin{aligned}
2x^2 q(x) &= 2x^2 \big(4xd_1 \circ d_2(x) - d_1 \circ d_2(x^2)\big) \\
&= 8x^3 d_1 \circ d_2(x) - 2x^2 \left(2xd_1 \circ d_2(x) + 2d_1(x)d_2(x)\right) \\
&= 4x^3 d_1 \circ d_2(x) - 4x^2 d_1(x)d_2(x)
\end{aligned}$$

for all $x \in \mathbb{F}$. Thus

$$q(x^2) = 2x^2 q(x) \qquad (x \in \mathbb{F})\,.$$

Note, however, that due to the methods described in the proof of Theorem 2, this equation is equivalent to

$$q(xy) = x^2q(y) + y^2q(x) \qquad (x, y \in \mathbb{F})\,.$$

In view of Theorem 1, if $\varphi\colon \mathbb{F} \to \mathbb{K}$ is a homomorphism then the mapping q defined on $\mathbb{F}$ by

$$q(x) = \varphi(x)^2 \qquad (x \in \mathbb{F})$$

is quadratic and is also an exponential on the multiplicative group $\mathbb{F}^{\times}$ Thus, using the previous theorem, we obtain the following statement.

Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F} \subset \mathbb{K}$ be a subfield. Let further $q\colon \mathbb{F} \to \mathbb{K}$ be a quadratic function, while $\varphi\colon \mathbb{F} \to \mathbb{K}$ be a homomorphism such that

$$q(xy) = \varphi(x)^2q(y) + q(x)\varphi(y)^2 \qquad (x, y \in \mathbb{F})\,.$$

Then there exists a second-order derivation $d \in \mathscr{D}_2(\mathbb{F}, \mathbb{K})$ such that

$$q(x) = \varphi\left(4xd(x) - d(x^2)\right) \qquad (x \in \mathbb{F})\,.$$

And vice versa, that is, if d is a second-order derivation and φ is a homomorphism, then the function q given by the above formula is a quadratic function that also satisfies the above equation.

We also obtain some extension of these results, in which instead of the identity mapping, homomorphisms appear.

Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F} \subset \mathbb{K}$ be a subfield. Let further $q\colon \mathbb{F} \to \mathbb{K}$ be a quadratic function, while $\varphi_1, \varphi_2\colon \mathbb{F} \to \mathbb{K}$ be homomorphisms such that

$$q(xy) = \varphi_1(x)\varphi_2(x)q(y) + \varphi_1(y)\varphi_2(y)q(x) \qquad (x, y \in \mathbb{F})\,.$$

Then there exists an additive function $a\colon \mathbb{F} \to \mathbb{K}$ such that

$$q(x) = 2(\varphi_1(x) + \varphi_2(x))a(x) - a(x^2) \qquad (x \in \mathbb{F})\,,$$

where the additive function also fulfills

$$(\spadesuit) \qquad \begin{aligned} &2\,a\,(x\,y\,z) - (\varphi_1(x) + \varphi_2(x))\;a\,(y\,z) - (\varphi_1(y) + \varphi_2(y))\;a\,(x\,z) \\ &\quad - (\varphi_1(z) + \varphi_2(z))\;a\,(x\,y) + (\varphi_1(x)\,\varphi_2(y) + \varphi_2(x)\,\varphi_1(y))\;a\,(z) \\ &\quad + (\varphi_1(x)\,\varphi_2(z) + \varphi_2(x)\,\varphi_1(z))\;a\,(y) \\ &\quad + (\varphi_1(y)\,\varphi_2(z) + \varphi_2(y)\,\varphi_1(z))\;a\,(x) = 0 \end{aligned}$$

for all $x, y, z \in \mathbb{F}$. And also conversely, if a is an additive function fulfilling ($\spadesuit$) with the homomorphisms φ_1, φ_2, then the function q given by the above formula is a quadratic function that also satisfies the above equation.

If $\varphi_1 = \varphi_2 = \mathrm{id}$ in equation ($\spadesuit$), then we have

$$a(x^3) - 6xa(x^2) + 3x^2 a(x) = 0 \qquad (x \in \mathbb{F})$$

and $a(1) = 0$ for the additive function $a\colon \mathbb{F} \to \mathbb{K}$. In view of [**20**, Corollary 2] we deduce that $a \in \mathscr{D}_2(\mathbb{F}, \mathbb{K})$.

Similarly, if $\varphi\colon \mathbb{F} \to \mathbb{K}$ is a non-identically zero homomorphism and we take $\varphi_1 = \varphi_2 = \varphi$, then equation ($\spadesuit$) reduces to

$$a(x^3) - 6\varphi(x)a(x^2) + 3\varphi(x)^2 a(x) = 0 \qquad (x \in \mathbb{F}) \,.$$

This means that the mapping $d\colon \mathbb{F} \to \mathbb{K}$ defined by

$$d(x) = \varphi^{-1}(a(x)) \qquad (x \in \mathbb{F})$$

satisfies

$$d(x^3) - 6xd(x^2) + 3x^2 d(x) = 0 \qquad (x \in \mathbb{F}) \,.$$

So $a = \varphi \circ d$ with an appropriate $d \in \mathscr{D}_2(\mathbb{F}, \mathbb{K})$.

These two special cases allow us to conclude that in the general case (see identity ($\spadesuit$)), the function a can be represented with the help of a second-order derivation and homomorphisms.

Quadratic functions as solutions of polynomial equations. Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F} \subset \mathbb{K}$ be a subfield. Let r be a positive integer and suppose that for all multi-index $\alpha \in \mathbb{N}^r$ the mapping $q_\alpha\colon \mathbb{F} \to \mathbb{K}$ is quadratic such that

$$q_0(x) = x^2 \qquad (x \in \mathbb{F}) \,.$$

Assume further that we have

$$q_\alpha(xy) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} q_\beta(x) q_{\alpha-\beta}(y) \qquad \big(x, y \in \mathbb{F}^\times\big)$$

for all multi-index $\alpha \in \mathbb{N}^r$. Then there exists a sequence of $\mathbb{K}$-valued 'additive functions' $a = (a_\alpha)_{\alpha \in \mathbb{N}^r}$ on $\mathbb{F}^\times$ such that

$$q_\alpha(x) = B_\alpha(a(x))x^2$$

holds for all $x \in \mathbb{F}$ and for each multi-index $\alpha \in \mathbb{N}^r$. Here the fact that $a_\alpha\colon \mathbb{F} \to \mathbb{K}$ is an 'additive function' means that there exists $d_\alpha \in \mathscr{D}_2(\mathbb{F}, \mathbb{K})$ such that

$$a_\alpha(x) = 4\frac{d_\alpha(x)}{x} - \frac{d_\alpha(x^2)}{x^2} \qquad \big(x \in \mathbb{F}^\times\big) \,.$$

A functional equation for monomials. In Chapter 4 our study, inspired by [**7**], seeks to build upon the work of the authors who investigated real quadratic functions f that satisfy the equation

$$f(x^2) = K \cdot x^2 f(x) \qquad (x \in \mathbb{R}).$$

Their research laid the foundation for our further exploration of functional equations involving monomial functions. Assuming that the functions involved are monomials, we started the systematic examination of polynomial equation in the papers [**17, 18**]. Our research work delves into a natural extension of the aforementioned equation, as we set out to uncover the properties of monomial functions $f\colon \mathbb{F} \to \mathbb{K}$ of degree n that satisfy the equation

$$f(x^2) = \kappa \cdot x^n f(x) \qquad (x \in \mathbb{F}).$$

As we will see, similarly to additive and quadratic functions, monomial functions in the equation in some cases can be represented with the aid of homomorphisms and higher-order derivations.

Let $\mathbb{F}, \mathbb{K}$ be fields with characteristic zero such that $\mathbb{F} \subset \mathbb{K}$, and n be a positive integer. Define

$$\mathscr{M}_n(\mathbb{F}, \mathbb{K}) = \{f\colon \mathbb{F} \to \mathbb{K} \mid f \text{ is a monomial of degree } n\}.$$

At first instance, the above problem may seem quite specific, and naturally the question arises whether we should consider the more general equation

$$f(x^2) = \kappa \cdot x^l f(x) \qquad (x \in \mathbb{F})$$

with a fixed positive integer l. In this regard, we prove firstly the following statement.

Let n, l be positive integers, κ be a nonzero element of the field $\mathbb{K}$ and $f \in \mathscr{M}_n(\mathbb{F}, \mathbb{K})$ such that

$$f(x^2) = \kappa \cdot x^l f(x) \qquad (x \in \mathbb{F})$$

holds. If $n \neq l$, then the function f is identically zero.

This shows that the above equation can have nonidentically zero solutions only when $n = l$. So it is enough to restrict ourselves to the case $n = l$. In this direction we have the following result.

Let n be a given positive integer, $\kappa \in \mathbb{K}$ and $f \in \mathscr{M}_n(\mathbb{F}, \mathbb{K})$ such that

$$f(x^2) = \kappa \cdot x^n f(x) \tag{16}$$

holds for all $x \in \mathbb{F}$. Then

(i) if $\kappa \notin \left\{2^k \mid k = 0, 1, \dots, n\right\}$, then f is identically zero,

(ii) if $\kappa = 1$, then

$$f(x) = f(1) \cdot x^n \qquad (x \in \mathbb{F}),$$

(iii) if $\kappa = 2$, then there exists an $a \in \mathscr{D}_{2n-1}(\mathbb{F}, \mathbb{K})$ such that

$$f(x) = \sum_{j=1}^{n} \lambda_{n,j} x^{n-j} a(x^j) \qquad (x \in \mathbb{F})$$

holds with appropriate constants $\lambda_1, \dots, \lambda_n \in \mathbb{K}$,

(iv) if $\kappa = 2^n$, then there exists a symmetric and n-additive mapping $A_n : \mathbb{F}^n \to \mathbb{K}$ such that

$$\sum_{\sigma \in \mathscr{S}_{n+1}} \Big\{ A_n(x_{\sigma(1)} \cdot x_{\sigma(2)}, x_{\sigma(3)}, ..., x_{\sigma(n+1)}) \\ - x_{\sigma(1)} \cdot A_n(x_{\sigma(2)}, \dots, x_{\sigma(n+1)}) - x_{\sigma(2)} \cdot A_n(x_{\sigma(1)}, \dots, x_{\sigma(n+1)}) \Big\} = 0 \\ (x_1, \dots, x_{n+1} \in \mathbb{F})$$

and

$$f(x) = A_n(x, \dots, x) \qquad (x \in \mathbb{F})$$

holds.

We also deal with the special case $n = 3$ separately. Compared to the above result, for $n = 3$ we can prove two more things. On the one hand, we show that in the case $\kappa = 2$, the order of the higher-order derivation appearing in the representation of the function f is at most 3, not at most 5. On the other hand, we show that in the case $\kappa = 4$ the function f is identically zero. In this regard, we have the following outcome:

Let $\kappa \in \mathbb{K}$ be arbitrarily fixed and $f \in \mathscr{M}_3(\mathbb{F}, \mathbb{K})$ be a monomial for which

$$f(x^2) = \kappa \cdot x^3 f(x) \tag{17}$$

holds for all $x \in \mathbb{F}$. Then the following cases are possible

(i) $\kappa = 1$ and then

$$f(x) = f(1) \cdot x^3 \qquad (x \in \mathbb{F}).$$

(ii) $\kappa = 2$ and then there exists a $d \in \mathscr{D}_3(\mathbb{F}, \mathbb{K})$ such that

$$f(x) = d\left(x^3\right) - \frac{9\,x\,d\,(x^2)}{2} + 9\,x^2\,d\,(x) \qquad (x \in \mathbb{F}).$$

(iii) $\kappa = 8$ and then there exists a symmetric and 3-additive mapping D_3 for which

$$\sum_{\sigma \in \mathscr{S}_4} \Big[D_3(x_{\sigma(1)} x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}) - x_{\sigma(1)} D_3(x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}) \\ - x_{\sigma(2)} D_3(x_{\sigma(1)}, x_{\sigma(3)}, x_{\sigma(4)}) \Big] = 0 \\ (x_1, x_2, x_3, x_4 \in \mathbb{F})$$

such that

$$f(x) = D_3(x, x, x) \qquad (x \in \mathbb{F}) .$$

(iv) $\kappa \in \mathbb{K} \setminus \{1, 2, 8\}$ and f is identically zero.

Bibliography

[1] Erhard Aichinger and Jakob Moosbauer. Chevalley–Warning type results on abelian groups. *J. Algebra*, 569:30–66, 2021.

[2] José M. Almira. Using Aichinger's equation to characterize polynomial functions. *Aequationes Math.*, 2022.

[3] Masaaki Amou. Quadratic functions satisfying an additional equation. *Acta Math. Hungar.*, 162:40–51, 2020.

[4] Germán Ancochea. Le théorème de von Staudt en géométrie projective quaternionienne. *J. Reine Angew. Math.*, 184:193–198, 1942.

[5] Zoltán Boros and Edit Garda-Mátyás. Conditional equations for monomial functions. *Publ. Math. Debr.*, 100(3-4):263–276, 2022.

[6] Zoltán Boros and Edit Garda-Mátyás. Quadratic functions fulfilling an additional condition along the hyperbola $xy = 1$. *Aequationes Math.*, 97(5-6):1141–1155, 2023.

[7] Zoltán Boros and Edit Garda-Mátyás. Conditional equations for quadratic functions. *Acta Math. Hungar.*, 154:389–401, 2018.

[8] Zoltán Boros and Rayene Menzer. An alternative equation for generalized monomials. *Aequationes Math.*, 97(1):113–120, 2023.

[9] Zoltán Boros and Rayene Menzer. An alternative equation for generalized polynomials of degree two. *Ann. Math. Silesianae*, 2023.

[10] Bruce Ebanks. Characterizing ring derivations of all orders via functional equations: results and open problems. *Aequationes Math.*, 89(3):685–718, 2015.

[11] Bruce Ebanks. Polynomially linked additive functions. *Aequationes Math.*, 91(2):317–330, 2017.

[12] Bruce Ebanks. Polynomially linked additive functions—II. *Aequationes Math.*, 92(3):581–597, 2018.

[13] Bruce Ebanks, Thomas Riedel, and Prasanna K. Sahoo. On the order of a derivation. *Aequationes Math.*, 90(2):335–340, 2016.

[14] Żywilla Fechner, Eszter Gselmann, and László Székelyhidi. Moment functions on groups. *Results Math.*, 76(4):Paper No. 171, 16, 2021.

[15] Zbigniew Gajda. On multiplicative solutions of the parallelogram functional equation. *Abh. Math. Semin. Univ. Hamb.*, 63:59–66, 1993.

[16] Eszter Gselmann. Polynomial identities satisfied by generalized polynomials. *Publ. Math. Debr.*, 101(3-4):421–450, 2022.

[17] Eszter Gselmann and Mehak Iqbal. Monomial functions, normal polynomials and polynomial equations. *Aequationes Math.*, 97(5-6):1059–1082, 2023.

[18] Eszter Gselmann and Mehak Iqbal. Quadratic functions as solutions of polynomial equations, 2024.

[19] Eszter Gselmann and Mehak Iqbal. A functional equation for monomial functions. *J. Appl. Anal.*, 2025.

[20] Eszter Gselmann, Gergely Kiss, and Csaba Vincze. On functional equations characterizing derivations: methods and examples. *Results Math.*, 73(2):Art. 74, 27, 2018.

[21] Eszter Gselmann, Gergely Kiss, and Csaba Vincze. Characterization of field homomorphisms through Pexiderized functional equations. *Journal of Difference Equations and Applications*, 25(12):1645–1679, 2019.

[22] Eszter Gselmann, Csaba Vincze, and Gergely Kiss. On a class of linear functional equations without range condition. *Aequat. Math.*, 94:473–509, 2020.

[23] Israel Nathan Herstein. Jordan homomorphisms. *Trans. Amer. Math. Soc.*, 81:331–341, 1956.

[24] Nathan Jacobson and Charles E. Rickart. Jordan homomorphisms of rings. *Trans. Amer. Math. Soc.*, 69:479–502, 1950.

[25] Irving Kaplansky. Semi-automorphisms of rings. *Duke Math. J.*, 14:521–525, 1947.

[26] Gergely Kiss and Miklós Laczkovich. Derivations and differential operators on rings and fields. *Annales Univ. Sci. Budapest., Sect. Comp.*, 48(31–43), 2018.

[27] Marek Kuczma. *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality.* Basel: Birkhäuser, 2$^{\text{nd}}$ edition, 2009.

[28] Miklós Laczkovich. Polynomial mappings on abelian groups. *Aequationes Math.*, 68(3):177–199, 2004.

[29] Miklós Laczkovich. A characterization of generalized exponential polynomials in terms of decomposable functions. *Acta Math. Hung.*, 158(2):338351, 2019.

[30] Ludwig Reich. Derivationen zweiter Ordnung als Lösungen von Funktionalgleichungen—ein überblick. In *Gyula Maurer zum 70. Geburtstag*, volume 337 of *Grazer Math. Ber.*, pages 45–65. Karl-Franzens-Univ. Graz, Graz, 1998.

[31] Ekaterina Shulman. Decomposable functions and representations of topological semigroups. *Aequationes Math.*, 79(1-2):13–21, 2010.

[32] László Székelyhidi. *Convolution Type Functional Equations on Topological Abelian Groups.* World Scientific Publishing Co., Inc., Teaneck, NJ, 1991.

[33] László Székelyhidi. *Harmonic and spectral analysis.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2014.

[34] Josef Unger and Ludwig Reich. *Derivationen höherer Ordnung als Lösungen von Funktionalgleichungen*, volume 336 of *Grazer Mathematische Berichte [Graz Mathematical Reports].* Karl-Franzens-Universität Graz, Graz, 1998.

[35] Oscar Zariski and Pierre Samuel. *Commutative algebra. Vol. 1.* Springer-Verlag, New York-Heidelberg-Berlin, 1975. With the cooperation of I. S. Cohen, Corrected reprinting of the 1958 edition, Graduate Texts in Mathematics, No. 28.

[36] Wiesław Żelazko. A characterization of multiplicative linear functionals in complex Banach algebras. *Studia Math.*, 30:83–85, 1968.

List of talks

[1] A functional equation for monomial functions, *The* 24^{th} *Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities, Hajdúszoboszló, Debrecen, Hungary, February 6-9, 2025.*

[2] Monomial functions, normal polynomials and polynomial equations, *Dr. Varecza Árpád Memorial Conference, Nyíregyháza, Hungary, November 14–15, 2024*

[3] A functional equation for monomial functions, *The* 60^{th} *International Symposium on Functional Equations and Inequalities, Kościelisko, Poland, June 9-15, 2024*

[4] Quadratic functions as solutions of polynomial equations, *The* 23^{rd} *Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities, Brenna, Poland, January 31 – February 3, 2024.*

[5] Generalised polynomials and polynomial equations, *Islamic university of Science and technology, Awantipora, Kashmir, India, August 16, 2023.*

[6] Monomial functions, normal polynomials and polynomial equations, *The* 59^{th} *International Symposium on Functional Equations and Inequalities, Hajdúszoboszló, Debrecen, Hungary, June 18-25, 2023.*

[7] Monomial functions, normal polynomials and polynomial equations, *The* 22^{nd} *Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities, Hajdúszoboszló, Debrecen, Hungary, February 1-4 2023.*

[8] On the existence of monotone sequences in generalised ordered sets, *The* 21^{st} *Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities, Brenna, Poland, February 2-5, 2022.*