Short thesis for the degree
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Almost Everywhere Summability of Vilenkin-Fourier Series

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PREFACE

In mathematics, Fourier analysis is a method in which general functions can be represented or approximated by sums of simpler trigonometric functions. Fourier analysis originated from the study of Fourier series and obtained its name after Joseph Fourier, who showed that representing a function as a sum of trigonometric functions greatly simplifies the study of heat transfer.

Now a days, the topic Fourier analysis covers a wide area of mathematics. Besides, it has different applications in sciences and engineering. The process of decomposing a function into oscillatory components is often said to be Fourier analysis while the operation of reconstructing the function from these pieces is called Fourier synthesis. For example, determining which component frequencies exist in a musical note would involve calculating the Fourier transform of a sampled musical note. Therefore, the same sound can be synthesized again by including the frequency components as revealed in Fourier analysis. In mathematics, the term Fourier analysis refers to the study of both operations.

Fourier transformation is the decomposition process and its result is called the Fourier transform. Based on the domain and other properties of the function being transformed, Fourier transform is usually gotten a more particular name. The original concept of Fourier analysis has become grown rapidly and applied broadly to different abstract and general situations. The general field is often mentioned as harmonic analysis.

The term ”harmonics” originated from the ancient Greek word ”harmonikos” and means ”experienced” in music [15]. Harmonic analysis is a branch of mathematics and it is an expanded form of Fourier analysis that has dealt with representing functions or signals as the superposition of fundamental waves, studying and generalizing the terms Fourier series and Fourier transforms. In the last two centuries it has been developed into different fields with wide range of applications. For instance, some of its applications are related to number theory, representation theory, signal processing, quantum mechanics, tidal analysis and neuroscience.

One of the most modern branches of harmonic analysis, which is introduced in the mid-20th century, is analysis on topological groups. The fundamental motivational ideas are the different Fourier transforms which can be generalized to a transform of functions that are defined on locally compact topological Hausdorff groups.

The theory of locally compact Abelian groups is called Pontryagin duality. Harmonic analysis investigates the properties of this duality and the Fourier transform. Moreover, it tries to extend these properties to different settings, for instance to the case of non-Abelian Lie groups.

Many applications of harmonic analysis in science and technology are based on the idea or hypothesis that a phenomenon or signal consists of a sum of individual vibration components. Oceans, tides and vibrating strings are common and simple examples. The theoretical approach is often to describe the system by a differential equation or system of equations in order to predict the important characteristics including the amplitude, frequency and phase of the vibration components.

In this thesis we summarize results of our PhD dissertation which is written as a monograph and is based on the following two published papers in peer reviewed journals:

György Gát and Anteneh Tilahun[6], Multi-parameter setting Cesàro means with respect to one dimensional Vilenkin system, FILOMAT, Vol.35(2021), No.12, pp. 4121–4133.

György Gát and Anteneh Tilahun[5], On almost everywhere convergence of the generalized Marcienkiwicz means with respect to two dimensional Vilenkin-like systems, Miskolc Math. Notes, Vol.21 (2020), No. 2, pp. 823-840.

Preliminaries

In this part, we mainly introduce the basic concepts of the three orthonomal systems: Walsh systems, Vilenkin systems and Vilenkin-like systems. Besides, we also include lemmas and definitions which are essential for discussing our main results.

Walsh systems

Walsh system is an orthonormal system which is formed from Walsh functions. It is the representation of dyadic group ordered in the Paley sense.

From the practical and theoretical point of view, Walsh system can be applied in many situations. All the usual applications of orthogonal systems (e.g. data transmission, multiplexing, filtering, image enhancement, and pattern recognition) can be performed in the Walsh system more efficiently. Due to the fact that Walsh functions take only the values +1 and -1, they are not difficult to be applicable on high speed computers and can be used with very little storage space. Moreover, as early as the late 1800's, transposition of conductors in open wire lines used Walsh functions. It is also interesting from a theoretical point of view since it is the simplest non-trivial model for harmonic analysis.

In 1923 Walsh[45] introduced the original Walsh system. He also showed that the Walsh and Haar systems are Hadamard transforms of each other. Although Rademacher functions were introduced in 1922 by Rademacher[43], they were probably unknown to Walsh. In 1932 R.E.A.C.Paley[42] introduced Walsh system which is often referred to as the Walsh-Paley system. He was also the first to recognize that Walsh functions are products of Rademacher functions.

Moreover, important observation were made by Fine [24] in 1949 that the Walsh functions can be viewed as characters of the dyadic group ([44]). Vilenkin in 1947[18] extended this fact in a more general form.

One of the three orthonormal systems (Walsh-Paley system, the original Walsh system, or the Walsh-Kaczmarz system) are referred to "Walsh functions" and differed only in enumeration. Besides, they are complete orthonormal systems on $[0,1)$ and share many properties with classical trigonometric, Sturm-Liouville, and Legendre systems[44].

By following the standard notions of dyadic analysis introduced in the book of F. Schipp, P. Simon, W. R. Wade (see e.g.[44]), we introduce basic notions on Walsh systems as follows. Denote the set of non-negative integers by $\mathbb{N}$, the set of positive integers by $\mathbb{P}$, the set of real numbers by $\mathbb{R}$, the complex plane by $\mathbb{C}$, and the set of dyadic rationals in the unit interval $[0, 1)$ by $\mathbb{Q}$. In particular, each element of $\mathbb{Q}$ has the form $\frac{p}{2^n}$ for some $p, n \in \mathbb{N}$, $0 \leq p < 2^n$.

Let the notation $x = (x_\alpha,\, \alpha \in \mathbb{A})$ represent a collection x indexed by a set $\mathbb{A}$. Thus, a sequence will be represented in the form $(x_n,\, n \in \mathbb{N})$.

Let r be the function defined on $[0, 1)$ by

$$r(x) := \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}), \\ -1, & \text{if } x \in [\frac{1}{2}, 1) \end{cases}$$

extended to $\mathbb{R}$ by periodicity of period 1. The Rademacher system $r := (r_n,\, n \in \mathbb{N})$ is defined by

$$r_n(x) := r(2^n x) \quad (x \in \mathbb{N}).$$

It is possible to write any $n \in \mathbb{N}$ uniquely as

$$n = \sum_{k=0}^{\infty} n_k 2^k,$$

where $n_k = 0$ or 1 for $k \in \mathbb{N}$. This expression will be called the binary expansion of n, and the numbers n_k are called the binary coefficients of n.

The elements of the Walsh-Paley system $\omega := (\omega_n, n \in \mathbb{N})$ were introduced as products of Rademacher functions in the following way. If $n \in \mathbb{N}$ has binary coefficients $(n_k,\, n \in \mathbb{N})$ then

$$\omega_n := \prod_{k=0}^{\infty} r_k^{n_k}.$$

Vilenkin systems

A natural generalization of the Walsh Paley system is called Vilenkin system. These are orthonormal systems which were introduced by N.Ya. Vilenkin in 1947 (see [18]). First we give a brief introduction to the theory of Vilenkin orthonormal systems.

Denote by $m := (m_k : k \in \mathbb{N})$ a sequence of positive integers such that $m_k \geq 2$, $k \in \mathbb{N}$ and Z_{m_k} the discrete cyclic group of order m_k. That is, Z_{m_k} can be represented by the set $\{0, 1, 2, ..., m_k - 1\}$, with the group operation being the $\mod m_k$ addition. Since the group is discrete, every subset is open.

Let $M_0 := 1$ and $M_{k+1} := m_k M_k$, for $k \in \mathbb{N}$ be the so-called generalized powers. Then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, $0 \leq n_k < m_k$, $n_k \in \mathbb{N}$. This allows one to say that the sequence $(n_0, n_1, ...)$ is the expansion of n with respect to m. We often use the following notation.

Let $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$ (that is, $M_{|n|} \leq n < M_{|n|+1}$, for any $n > 0$) and $n^{(k)} = \sum_{j=k}^{\infty} n_j M_j$.

The normalized Haar measure μ_k on Z_{m_k} is defined by $\mu_k(\{j\}) := \frac{1}{m_k}(j \in \{0, 1, ..., m_k - 1\})$. Let

$$G_m := \mathop{\times}_{k=0}^{\infty} Z_{m_k}.$$

Then, every $x \in G_m$ can be represented by a sequence $x = (x_i,\, i \in \mathbb{N})$, where $x_i \in Z_{m_i}$ $(i \in \mathbb{N})$.

The group operation on G_m (denoted by $+$) is the coordinate-wise addition (the inverse operation is denoted by $-$), the measure (denoted by μ), is the normalized Haar measure, and the topology is the product topology. Consequently, G_m is a compact Abelian group. If $\sup_{n \in \mathbb{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group. In this dissertation we discuss bounded Vilenkin groups, only.

The Vilenkin group is metrizable in the following way:

$$d(x, y) := \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{M_{i+1}} \quad (x, y \in G_m).$$

The topology induced by this metric, the product topology, and the topology given by intervals defined below, are the same. A base for the neighborhoods in G_m can be given by the intervals: $I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbb{N}) \in G_m : y_i = x_i \text{ for } i < n\}$ for $x \in G_m$, $n \in \mathbb{P}$. Let $0 = (0,\, i \in \mathbb{N}) \in G_m$ denote the null element of G_m and $I_n = I_n(0)$, $\bar{I}_n = G_m \backslash I_n$.

Denote by $L^p(G_m)$ the usual Lebesgue spaces ($\|.\|_p$ the corresponding norms) $(1 \leq p \leq \infty)$, $\mathcal{A}_n$ the σ-algebra generated by the sets $I_n(x)\,(x \in G_m)$ and E_n the conditional expectation operator with respect to $\mathcal{A}_n$ $(n \in \mathbb{N})$. We say that an operator $T : L^1 \to L^0$ ($L^0(G_m)$ is the space of measurable functions on G_m) is of type (L^p, L^p) (for $1 \leq p \leq \infty$) if $\|Tf\|_p \leq C_p \|f\|_p$ for all $f \in L^p(G_m)$ and the constant C_p depends only on p. We say that T is of weak type (L^1, L^1) if $\mu(|Tf| > \lambda) \leq C\|f\|_1/\lambda$ for all $f \in L^1(G_m)$ and $\lambda > 0$.

Next, we introduce an orthonormal system that we call the Vilenkin system on G_m.

Definition

For $k \in \mathbb{N}$ and $x \in G_m$ denote by r_k the k-th generalized Rademacher function:

$$r_k(x) := \exp(2\pi\imath \frac{x_k}{m_k}) \quad (x \in G_m,\ \imath := \sqrt{-1},\ k \in \mathbb{N}).$$

The n^{th} Vilenkin function is defined as

$$\psi_n := \prod_{j=0}^{\infty} r_j^{n_j} (n \in \mathbb{N}).$$

The system $\psi := (\psi_n,\, n \in \mathbb{N})$ is called a Vilenkin system.

Each ψ_n is a character of G_m and all the characters of G_m are of this form. Define the m-adic addition as

$$k \oplus n := \sum_{j=0}^{\infty} (k_j + n_j (\bmod m_j)) M_j \quad (k, n \in \mathbb{N}).$$

Then $\psi_{k\oplus n} = \psi_k \psi_n$, $\psi_n(x+y) = \psi_n(x)\psi_n(y)$, $\psi_n(-x) = \bar{\psi}_n(x)$, $|\psi_n| = 1 \quad (k,\, n \in \mathbb{N},\, x,\, y \in G_m)$.

Definition

The Dirichlet and the Fejér or (C,1) kernels on the Vilenkin system are defined and denoted as

$$D_n := \sum_{k=0}^{n-1} \psi_k \quad \text{and} \quad K_n := \frac{1}{n+1} \sum_{k=0}^{n} D_k, \text{ respectively.}$$

Definition

For $f\in L^1(G_m),$ the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels, the (C,α) kernels and means with respect to the Vilenkin system ψ are defined as follows

$$\begin{aligned}
\hat{f}(n) &:= \int_{G_m} f\bar{\psi}_n d\mu, \\
S_n f &:= \sum_{k=0}^{n-1} \hat{f}(k)\psi_k, \\
\sigma_n^\alpha f &:= \frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k f, \\
K_n^\alpha &:= \frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1} D_k, \\
\sigma_n f &:= \sigma_n^1 f,\ K_n := K_n^1.
\end{aligned}$$

It is known [20] that,

$$A_n^\alpha=\sum_{k=0}^{n}A_k^{\alpha-1},\ A_k^\alpha-A_{k+1}^\alpha=\frac{-\alpha A_k^\alpha}{k+1} \tag{1}$$

where A_k^α is defined for all possible values of $\alpha\in\mathbb{R}\setminus\{-1,\ -2,\ ...,\ -k\}$ as

$$A_k^\alpha=\frac{(\alpha+1)(\alpha+2)...(\alpha+k)}{k!},$$

α may also be a sequence $\alpha=(\alpha_n)$.
It is known that

$$S_n f(y)=\int_{G_m} f(x)D_n(y-x)d\mu(x) \quad (n\in\mathbb{N},\ f\in L^1(G_m)).$$

It is also well-known that(see [4], [7])

$$\begin{aligned}
D_{M_n}(y,x) &= \begin{cases} M_n, & \text{if } y\in I_n(x), \\ 0, & \text{if } y\notin I_n(x) \end{cases} \\
S_{M_n}f(x) &= M_n\int_{I_n(x)} f d\mu = E_n f(x)
\end{aligned} \tag{2}$$

$f \in L^1(G_m)$, $n \in \mathbb{N}$,

$$\begin{aligned} D_{sM_n} &= D_{M_n} \sum_{k=0}^{s-1} \psi_{kM_n} \\ &= D_{M_n} \sum_{k=0}^{s-1} r_n^k, \quad \text{for} \quad s \le m_n. \end{aligned} \tag{3}$$

Vilenkin-like systems

Next on a Vilenkin space G_m we introduce an orthonormal system called a Vilenkin-like system (or $\psi\alpha$ system).

Vilenkin-like orthonormal systems were introduced by György Gát in 1991 (see [30]) and they are defined as follows.

Definition

Let the functions

$$\psi_n, \alpha_n, \alpha_k^j : G_m \to \mathbb{C}\ (n, j, k \in \mathbb{N})$$

satisfy :

$$\alpha_k^j \text{ is measurable with respect to } \mathcal{A}_k\ (j, k \in \mathbb{N}), \tag{4}$$

$$|\alpha_k^j| = \alpha_k^j(0) = \alpha_0^j = \alpha_k^0 = 1\ (j, k \in \mathbb{N}), \tag{5}$$

$$\alpha_n := \prod_{k=0}^{\infty} \alpha_k^{n^{(k)}}, \psi_n := \prod_{k=0}^{\infty} r_k^{n_k},\ n^{(k)} := \sum_{i=k}^{\infty} n_i M_i\ (n \in \mathbb{N}). \tag{6}$$

Let $\chi_n := \psi_n \alpha_n (n \in \mathbb{N})$. The system $\chi := \{\chi_n : n \in \mathbb{N}\}$ is called a Vilenkin-like (or $\psi\alpha$) system (see [30], [22]).

We also introduce the two-variable functions:

$$\chi_n(y, x) := \chi_n(y)\bar{\chi}_n(x), \quad r_n(y, x) := r_n(y)\bar{r}_n(x) \quad (n \in \mathbb{N}, y, x \in G_m).$$

This will not cause misunderstanding by clearly making a difference between $\chi_n(x)$ and $\chi_n(y, x)$.

Example A: the Vilenkin and the Walsh system

Let $\alpha_k^j(x) := 1$, where $j, k \in \mathbb{N}, x \in G_m$ where G_m is the Vilenkin group. The system $\chi := (\chi_n,\ n \in \mathbb{N})$ is the Vilenkin system, where $\chi_n := \prod_{k=0}^{\infty} r_k^{n_k} \alpha_k^{n^{(k)}} = \prod_{k=0}^{\infty} r_k^{n_k}$. In the case of the Vilenkin group, if $m_k = 2$ for all $k \in \mathbb{N}$, we get the Walsh-Paley system. Properties (4), (5), (6) are trivially fulfilled. For more on Vilenkin and Walsh systems and groups see e.g. [21] and [27].

Example B: the group of 2-adic (m-adic) integers

Let $G_{m_k} := \{0, 1, ..., m_k - 1\}$ for all $k \in \mathbb{N}$. On G_m define the following (commutative) addition: Let $x, y \in G_m$. Then $x + y = z \in G_m$ is defined in a recursive way. $x_0 + y_0 = t_0 m_0 + z_0$, where (of course) $z_0 \in \{0, 1, ..., m_0 - 1\}$ and $t_0 \in \mathbb{N}$. Suppose that $z_0, ..., z_k$ and $t_0, ..., t_k$ have been defined. Then write $x_{k+1} + y_{k+1} + t_k = t_{k+1} m_{k+1} + z_{k+1}$, where $z_{k+1} \in \{0, 1, ..., m_{k+1} - 1\}$ and $t_{k+1} \in \mathbb{N}$. Then G_m is called the group of m-adic integers (if $m_k = 2$ for all $k \in \mathbb{N}$, then 2-adic integers). In this case let

$$\alpha_k^j(x) := \left(\exp\left(2\pi\imath\left(\frac{x_{k-1}}{m_k m_{k-1}} + \cdots + \frac{x_0}{m_k m_{k-1} ... m_0}\right)\right)\right)^j.$$

Let $\chi_n := \prod_{k=0}^{\infty} r_k^{n_k} \alpha_k^{n^{(k)}}$. Then the system $\chi := (\chi_n,\, n \in \mathbb{N})$ is the character system of the group of m-adic (if $m_k = 2$ for each $k \in \mathbb{N}$ then 2-adic) integers. Conditions (4), (5), (6) are trivially fulfilled. For more on the group of m-adic (if $m_k = 2$ for each $k \in \mathbb{N}$ then 2-adic) integers see e.g. [22] or [12]. For the case when $m_k = 2\,(k \in \mathbb{N})$ the a.e. convergence of the ordinary Marcinkiewicz means were discussed by Blahota and Gát in [22]. That is, the results included in Chapter 3 of this Dissertation are new on the two-dimensional group of m-adic integers. Not only with respect to the general case $\alpha : \mathbb{N}^2 \to \mathbb{N}^2$ but also for $\alpha_1(n) = \alpha_2(n) = n$. Besides, the same can be said in the situation of Example C below.

Example C: a system in the field of number theory

Let

$$\alpha_n(x) := \exp\left(2\pi\imath \sum_{j=0}^{\infty} \frac{n_j}{M_{j+1}} \sum_{i=0}^{\infty} x_i M_i\right)$$

for $n \in \mathbb{N}$ and $x \in G_m$. Then

$$\chi_n(x) = \exp\left(2\pi\imath\left(\sum_{k=0}^{\infty} \frac{n_k x_k}{m_k} + \sum_{k=0}^{\infty} \frac{n_k}{M_{k+1}} \sum_{i=0}^{k-1} x_i M_i\right)\right) = \psi_n(x)\alpha_n(x),$$

where $\alpha_k^{n^{(k)}}(x) = \exp\left(2\pi\imath \frac{n_k}{M_{k+1}} \sum_{i=0}^{k-1} x_i M_i\right)$. Then, $\chi := (\chi_n,\, n \in \mathbb{N})$ is a Vilenkin-like system (introduced in [30]) which is a useful tool in the approximation theory of limit periodic, almost even arithmetical functions [30] and [31]. Again, properties (4), (5), (6) are trivially fulfilled. This system (on

Vilenkin groups) was a new tool in order to investigate limit periodic arithmetical functions. For the definition of these arithmetical functions see also the book of Mauclaire [39].

Definition

For $n \in \mathbb{N}, \quad y, x \in G_m, f \in L^1(G_m),$ the Fourier coefficients, the partial sums of the Fourier series and the Dirichlet kernels with respect to the Vilenkin-like system χ are defined respectively as follows

$$\begin{aligned}
\hat{f}(n) &:= \int_{G_m} f\bar{\chi}_n d\mu, \\
S_n f &:= \sum_{k=0}^{n-1} \hat{f}(k)\chi_k, \\
D_n(y,x) &:= \sum_{k=0}^{n-1} \chi_k(y)\bar{\chi}_k(x) = \sum_{k=0}^{n-1} \chi_k(y,x).
\end{aligned}$$

For $n \in \mathbb{N}, \quad y \in G_m, f \in L^1(G_m),$ it is well-known that

$$S_n f(y) = \int_{G_m} f(x)D_n(y,x)d\mu(x).$$

It is also well-known [30] that

$$\begin{aligned}
&D_{M_n}(y,x) = \begin{cases} M_n, & \text{if } y \in I_n(x), \\ 0, & \text{if } y \notin I_n(x) \end{cases} \\
&D_n(y,x) = \chi_n(y)\bar{\chi}_n(x) \sum_{j=0}^{\infty} D_{M_j}(y,x) \sum_{p=m_j-n_j}^{m_j-1} r_j^p(y)\bar{r}_j^p(x), \\
&S_{M_n} f(x) = M_n \int_{I_n(x)} f d\mu = E_n f(x) \quad (f \in L^1(G_m),\, n \in \mathbb{N}), \\
&D_n(y,x) = \chi_n(y)\bar{\chi}_n(x) \left(\sum_{j=0}^{t-1} n_j M_j + M_t \sum_{i=m_t-n_t}^{m_t-1} r_t^i(y)\bar{r}_t^i(x) \right), \\
&y \in I_t(x) \setminus I_{t+1}(x),\, t \in \mathbb{N}.
\end{aligned} \tag{7}$$

Next, we introduce some notation used in the theory of two-dimensional Vilenkin-like systems. Let $\tilde{m}$ be a sequence like m. The relation between the sequence $(\tilde{m}_n)$ and $(\tilde{M}_n)$ is the same as between sequence (m_n) and (M_n). The group $G_m \times G_{\tilde{m}}$ is called a two-dimensional Vilenkin group and $\chi_{k,l}(x,y) = \chi_k(x)\chi_l(y) \quad (k,l \in \mathbb{N},\, x \in G_m,\, y \in G_{\tilde{m}})$ is called a two dimensional Vilenkin-like system. The normalized Haar measure is denoted by μ, just as in the one-dimensional case. It will not cause any misunderstanding. In this thesis we also suppose that $m = \tilde{m}$.

Definition

For $y = (y^1, y^2),\ x = (x^1, x^2) \in G_m \times G_m, n \in \mathbb{N},$ the two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, the Dirichlet kernels, the Marcinkiewicz means, and the Marcinkiewicz kernels with respect to a two-dimensional Vilenkin-like system are defined respectively as follows:

$$\hat{f}(n_1, n_2) := \int_{G_m \times G_{\tilde{m}}} f(x^1, x^2)\bar{\chi}_{n_1}(x^1)\bar{\chi}_{n_2}(x^2)d\mu(x^1, x^2),$$

$$S_{n_1,n_2}f(y^1, y^2) := \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1} \hat{f}(k_1, k_2)\chi_{k_1}(y^1)\chi_{k_2}(y^2),$$

$$D_{n_1,n_2}(y, x) = D_{n_1}(y^1, x^1)D_{n_2}(y^2, x^2)$$

$$:= \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1} \chi_{k_1}(y^1)\chi_{k_2}(y^2)\bar{\chi}_{k_1}(x^1)\bar{\chi}_{k_2}(x^2),$$

$$\sigma_n f := \frac{1}{n}\sum_{j=0}^{n-1} S_{j,j}f,$$

$$K_n(y, x) := \frac{1}{n+1}\sum_{j=0}^{n} D_{j,j}(y, x).$$

It is also well-known that

$$\sigma_n f(y) = \int_{G_m \times G_m} f(x)K_n(y, x)d\mu(x) =: f * K_n(y).$$

The next well-known Lemmas concerning Dirichlet kernels will be used many times in the proofs of our main results.

Lemma 1. [3] If k and n are natural numbers, then

$$a). \quad C_1(1+\alpha_n)(2+\alpha_n)k^{\alpha_n} < A_k^{\alpha_n} < C_2(1+\alpha_n)(2+\alpha_n)k^{\alpha_n},$$
$$-2 < \alpha_n < -1,$$
$$b). \quad C_1(1+\alpha_n)k^{\alpha_n} < A_k^{\alpha_n} < C_2(1+\alpha_n)k^{\alpha_n},$$
$$-1 < \alpha_n < 0,$$
$$c). \quad C_1(d)k^{\alpha_n} < A_k^{\alpha_n} < C_2(d)k^{\alpha_n},$$
$$0 < \alpha_n \leq d.$$

where C_1, C_2 are positive absolute constants(though in case (c) they depend on d).

Lemma 2. [7] Let $0 \leq j < n_t M_t$ and $0 \leq n_t < m_t$. Then,

$$D_{n_t M_t - j} = D_{n_t M_t} - \psi_{n_t M_t - 1} \bar{D}_j.$$

Lemma 3. [18]

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n := I_n(0), \\ 0, & \text{if } x \notin I_n. \end{cases}$$

Lemma 4. [18]

$$D_n(x) = \psi_n(x) \sum_{j=0}^{\infty} D_{M_j}(x) \sum_{p=m_j-n_j}^{m_j-1} r_j^p(x).$$

Lemma 5. [18]

$$D_n(x) = D_n(z) = \psi_n(z)(\sum_{j=0}^{t-1} n_j M_j + M_t \sum_{i=m_t-n_t}^{m_t-1} r_t^i(z));$$
$$z \in I_t \setminus I_{t+1},\ t \in \mathbb{N}.$$

The following Lemmas concerning to Vilenkin-like systems are also well-known [30]

Lemma 6. [30] Let $t, n, l \in \mathbb{N}, u \in G_m$. Then we have that

$$\int_{I_{t+1}(u)} \chi_n(x)\bar{\chi}_l(x)d\mu(x) \neq 0$$

implies $n^{(t+1)} = l^{(t+1)}$.

Lemma 7. [30]

$$D_{M_n}(y,x) = \begin{cases} M_n, & \text{if } y \in I_n(x), \\ 0, & \text{if } y \notin I_n(x). \end{cases}$$

Lemma 8. [30]

$$D_n(y,x) = \chi_n(y)\bar{\chi}_n(x)\sum_{j=0}^{\infty} D_{M_j}(y,x)\sum_{p=m_j-n_j}^{m_j-1} r_j^p(y)\bar{r}_j^p(x).$$

Lemma 9. [30]

$$D_n(y,x) = \chi_n(y)\bar{\chi}_n(x)\left(\sum_{j=0}^{t-1} n_j M_j + M_t \sum_{i=m_t-n_t}^{m_t-1} r_t^i(y)\bar{r}_t^i(x)\right),$$

$y \in I_t(x) \setminus I_{t+1}(x),\ t \in \mathbb{N}$.

Besides to the above lemmas, the following lemma are also played the vital role to prove our main results.

Lemma 10. (Calderon-Zygmund decomposition lemma)
Let $f \in L^1(I),\ \lambda > \|f\|_1$. Then there exists a decomposition

$$f = \sum_{j=0}^{\infty} f_j \text{ and disjoint intervals } I^j := I_{k_j}(u^j)$$

of I for which

$$\operatorname{supp} f_j \subset I^j,\ \int_{I^j} f_j = o,\ \lambda < \left|I^{j}\right|^{-1} \int_{I^j} |f_j| \leq c\lambda,$$

$$(u^j \in I,\ k_j \in \mathbb{N},\ j \in \mathbb{P}),\ \|f_0\|_\infty \leq c\lambda,\ |F| \leq \frac{c\,\|f\|_1}{\lambda},$$

where $F = \cup_{j\in\mathbb{P}} I^j$.

We can get this lemma in [44] and in the dyadic case it is proved by Gát[32]. It is well-known that the Calderon-Zygmund decomposition lemma plays a prominent role in the theory of harmonic analysis. This famous lemma is mainly used to prove weak type (L^1, L^1) estimations for the maximal operators of the summability methods.

Cesàro means in the variable parameter setting

The idea of Cesàro means with variable parameters of numerical sequences is due to Kaplan [13]. In 2007 Akhobadze [3] introduced the notion of (C,α) means of trigonometric Fourier series with variable parameter setting. Fine [24] introduced this for Walsh-Paley system for constant sequences. On the rate of convergence of (C,α) means in the constant sequences case see the paper of Fridli [8]. For the two dimensional case see the papers of [48] and Goginava [11]. The almost everywhere convergence of this summability method for a constant parameter in the quadrilateral partial sums of double Vilenkin-Fourier series was proved by Gát and Goginiva in 2006 [7]. In 2008 Abu Joudeh and Gát [1] proved for varying-parameter setting in the case of Walsh-Paley system. The a.e. divergence of Cesàro means with varying-parameter of Walsh-Fourier series was investigated by Tetunashvili [17].
In [10] Lemma 8 about the Fejér kernel with a constant $\alpha = 1$ is proved with respect to Vilenkin system.

In this part, we prove the almost everywhere convergence of (C,α) means in the variable parameter setting with respect to the one dimensional bounded Vilenkin system. That is,

$$\sigma_n^{\alpha_n} f \to f \quad \text{when} \quad n \to \infty,$$

where $\alpha = (\alpha_n)$ is not constant but it is varying in the open interval $(0,1)$ for all $n \in \mathbb{N}_{\alpha,q},$ where $\mathbb{N}_{\alpha,q}$ will be defined later.

In the first part of this chapter, we prove some lemmas on (C,α) kernel functions which are important tools. The first and important lemma is about the kernel function, when $n, a \in \mathbb{N}$, $M_B \leq n < M_{B+1}$, $|n| = B$, $\alpha_a \in (0,1)$. Then, it is showed that the inequality

$$|T_n^{\alpha_a}| \leq \tilde{T}_n^{\alpha_a}$$

holds true. When $0 < \alpha_n < 1$, $M_B \leq n < M_{B+1}$, $|n| = B$, $n \in \mathbb{N},$ it is also proved that the following inequality holds true,

$$|K_n^{\alpha_n}| \leq \tilde{K}_n^{\alpha_n}.$$

In the second part of this Chapter, we focus on the maximal operators of (C,α) means. We prove the quasi-locality property of the maximal operators $\tilde{t}_*^{\alpha_a}$ and $\tilde{\sigma}_*^{\alpha_a}$ (for the notion of quasi-locality see [14]). The other very important lemma states that σ_*^{α} is of weak type (L^1, L^1). We prove this after we give the proof of the statement that the maximal operator $\tilde{\sigma}_*^{\alpha_a}$ is of type (L^∞, L^∞) and of weak type (L^1, L^1). Having all these important tools proved, in the third part of the chapter we finally give the proof of the main result.

Where $T_n^{\alpha_a}$, $\tilde{T}_n^{\alpha_a}$, $K_n^{\alpha_n}$, $\tilde{K}_n^{\alpha_n}$, $\tilde{t}_*^{\alpha_a}$, $\tilde{\sigma}_*^{\alpha_a}$ and $\sigma_*^{\alpha_a}$ will be defined latter.

Cesàro kernel in the variable parameter setting

The following notations as well as definitions of functions and operators are used through the proofs of this part.
For $a,\ s,\ n \in \mathbb{N}$ let $n_{(s)} := \sum_{j=0}^{s-1} n_j M_j,$ that is, $n_{(0)} = 0,\ n_{(1)} = n_0$ and for $M_B \leq n < M_{B+1},\ |n| := B,\ n_{(B+1)} = n.$
Define two variable function $P(n, \alpha) := \sum_{i=0}^{\infty} n_i M_i^{\alpha}$ for $n \in \mathbb{N},\ \alpha \in \mathbb{R}$. For example $P(n, 1) = n$. Besides, set for sequences $\alpha = (\alpha_n)$ and positive reals $q,$ the subset of natural numbers

$$\mathbb{N}_{\alpha,q} := \left\{ n \in \mathbb{N} : \frac{P(n, \alpha_n)}{n^{\alpha_n}} \leq q \right\}. \tag{8}$$

For sequence α such that $0 < \alpha_0 \leq \alpha_n < 1$ we have $\mathbb{N}_{\alpha,\,q} = \mathbb{N}$ for some q depending only on α_0. We remark that $M_n \in \mathbb{N}_{\alpha,\,q}$ for every $\alpha = (\alpha_n),\ 0 < \alpha_n < 1$ and $q \geq 1$.
In this thesis, C denotes an absolute constant and C_q another one which may depend only on q.
We also use lemma 11 (see[10]) in the proof of our results. Thus, it is included as follows without its proof.

Lemma 11. $\int_{\bar{I}_k} \sup_{j \geq M_k} \left| K_j \right| d\mu \leq C.$
We introduce the following functions and operators in the variable parameter setting $(n,\ a \in \mathbb{N},\ 0 < \alpha_a < 1)$.

$$T_n^{\alpha_a} := \frac{1}{A_n^{\alpha_a}} \sum_{k=0}^{n_B M_B - 1} A_{n-k}^{\alpha_a - 1} D_k,$$

$$\tilde{T}_n^{\alpha_a} := \frac{n_B D_{M_B}}{A_n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 1} A_{n_{(B)}+j}^{\alpha_a - 1}$$

$$+ \frac{\alpha_a(1 - \alpha_a)}{n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 2} \frac{j+1}{(n_{(B)} + j)^{2-\alpha_a}} \left| K_j \right|$$

$$+ \alpha_a \left| K_{n_B M_B - 1} \right|,$$

$$t_n^{\alpha_a} f(y) := \int_{G_m} f(x) T_n^{\alpha_a}(y - x) d\mu(x),$$

$$\tilde{t}_n^{\alpha_a} f(y) := \int_{G_m} f(x) \tilde{T}_n^{\alpha_a}(y - x) d\mu(x).$$

Lemma 12. [6] For $n, a \in \mathbb{N}$, $M_B \leq n < M_{B+1}$, $|n| = B$, $\alpha_a \in (0,1)$. Then,

$$|T_n^{\alpha_a}| \leq \tilde{T}_n^{\alpha_a}.$$

The following Lemma plays a central role in the proof of the next lemmas and the main theorem too.
For $f \in L^1(G_m)$ and for all real number $\alpha_n \neq -1, -2, -3, ...$, define the (C, α_n) kernel as follows

$$K_n^{\alpha_n} = \frac{1}{A_n^{\alpha_n}} \sum_{t=0}^{n} A_{n-t}^{\alpha_n - 1} D_t, \tag{9}$$

where $A_k^{\alpha_n}$ is defined in (1). Besides, introduce the following Kernel functions and operators where $0 < \alpha_n < 1$.

$$\tilde{K}_n^{\alpha_n} := \left|\tilde{T}_n^{\alpha_n}\right| + \sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} n_l D_{M_l} + \sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} |T_{n_{(l-1)}}^{\alpha_n}|,$$
$$\tilde{\sigma}_n^{\alpha_n} := \int_{G_m} f(x) \tilde{K}_n^{\alpha_n}(y - x) d\mu(x).$$

Lemma 13. [6] Let $0 < \alpha_n < 1$, $n \in \mathbb{N}$, $M_B \leq n < M_{B+1}$, $|n| = B$. Then,

$$|K_n^{\alpha_n}| \leq \tilde{K}_n^{\alpha_n}.$$

Maximal operators of Cesàro means in the variable parameter setting

In the variable parameter setting α, the following results about the maximal operator are investigated. We prove lemma below, which means that the maximal operator $\tilde{t}_*^{\alpha_a} := \sup_{n,\, a\in\mathbb{N}} |\tilde{t}_n^{\alpha_a}|$ is quasi-local.

Lemma 14. [6] Let $1 > \alpha_a > 0,\, a \in \mathbb{N},\, f \in L^1(G_m)$ such that $suppf \subset I_k(u),\, \int_{I_k(u)} f d\mu(x) = 0$ for some m-adic interval $I_k(u)$. Then, we have

$$\int_{\bar{I}_k(u)} \sup_{n,\, a\in N} |\tilde{t}_n^{\alpha_a} f| d\mu(x) \leq C\|f\|_1.$$

$$= \int_{\bar{I}_k(u)} \sup_{n\geq M_k,\, a\in N} \left|\tilde{t}_n^{\alpha_a} f\right| d\mu.$$

In the following corollary, it is also proved that operators $t_n^{\alpha_a}$, $\tilde{t}_n^{\alpha_a}$ are of type $(L^1,\, L^1)$ and $(L^\infty,\, L^\infty)$ uniformly in n.

Corollary 15. [6] Let $1 > \alpha_a > 0,\, a \in \mathbb{N}$. Then, we have

$$\|T_n^{\alpha_a}\|_1 \leq \|\tilde{T}_n^{\alpha_a}\|_1 \leq C,$$
$$\|t_n^{\alpha_a} f\|_1,\, \|\tilde{t}_n^{\alpha_a} f\|_1 \leq C\|f\|_1$$

and

$$\|t_n^{\alpha_a} g\|_\infty,\, \|\tilde{t}_n^{\alpha_a} g\|_\infty \leq C\|g\|_\infty$$

for all natural numbers a, n and where C is some absolute constant and $f \in L^1(G_m),\, g \in L^\infty(G_m)$.

That is, operator $t_n^{\alpha_a}$, $\tilde{t}_n^{\alpha_a}$ are of type $(L^1,\, L^1)$ and $(L^\infty,\, L^\infty)$ and uniformly in n. For the general case by considering $n \in \mathbb{N}_{\alpha,q},$ where $\mathbb{N}_{\alpha,q}$ is defined in (8), we proved the following lemma. That is, the maximal operator

$$\tilde{\sigma}_{*,\,q}^{\alpha} := \sup_{n\in\mathbb{N}_{\alpha,\,q}} |\tilde{\sigma}_n^{\alpha_n}|$$

is quasi-local. We get this by the investigation of kernel functions, its maximal function on the Vilenkin group by making a hole around zero and some quasi-locality issues.

Lemma 16. [6] Let $0 < \alpha_n < 1$, $f \in L^1(G_m)$ such that $suppf \subset I_k(u)$, $\int_{I_k(u)} fd\mu = 0$ for some m-adic interval $I_k(u)$. Then we have

$$\int_{G_m \setminus I_k(u)} \tilde{\sigma}^{\alpha}_{*,\,q} fd\mu \leq C_q \|f\|_1.$$

Where constants C_q can depend only on q. For the general case of $n \in \mathbb{N}_{\alpha,q}$. Define (C, α_n) mean as follows

$$\sigma_n^{\alpha_n} f(x) := \frac{1}{A_n^{\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k(x) = \int_{G_m} f(y) K_n^{\alpha_n}(x-y) d\mu(y). \tag{10}$$

Considering the definition in (10), we define maximal operators as follows.

$$\sigma^{\alpha}_{*,\,q} f := \sup_{n \in N_{\alpha,\,q}} |\sigma_n^{\alpha_n} f|.$$

Lemma 17. [6] The operator $\tilde{\sigma}^{\alpha}_{*}$ is of type $(L^{\infty},\, L^{\infty})$ and weak type $(L^1,\, L^1)$; σ^{α}_{*} is of weak type $(L^1,\, L^1)$.

Almost everywhere convergence of the Cesàro means

The almost everywhere convergence of the (C, α_n) means is proved in the following theorem.

Theorem 18. [6] Let $0 < \alpha_n < 1$. Let $f \in L^1(G_m)$. Then, $\sigma_n^{\alpha_n} f \longrightarrow f \quad a.e.$ if $n \longrightarrow \infty$, $n \in \mathbb{N}_{\alpha,\,q}$.

Generalized Marcinkiewicz means

In 1939 for the two-dimensional trigonometric Fourier partial sums $S_{j,j}f$ Marcinkiewicz[38] proved that for all $f \in L\log L([0,2\pi]^2)$ the a.e. relation

$$\frac{1}{n}\sum_{j=1}^{n} S_{j,j}f \to f \tag{11}$$

holds as $n \to \infty$. Zhizhiashvili [49] improved this result for $f \in L([0,2\pi]^2)$. Dyachenko [23] proved this result for dimensions greater than 2.
In 2003 Goginava [33] proved this result with respect to the multiple Walsh-Paley system. The case $d = 2$ is due to Weisz [19]. This result for bounded Vilenkin systems due to Gát [27]. In 2012 Gát [28] proved this result for generalized Marcinkiewicz means with respect to the two dimensional Walsh system and in 2016 [29] for bounded two-dimensional Vilenkin systems.
We generalized the result of Gát [29] with respect to two-dimensional generalized Vilenkin-like systems. Besides, we give an application of the main result. That is, theorem with respect to triangular summability of Vilenkin-like-Fourier series.
The two-dimensional generalized Marcinkiewicz kernels and Marcinkiewicz means, with respect to the two-dimensional Vilenkin-like system are defined as follows: Let $\alpha = (\alpha_1, \alpha_2) : \mathbb{N}^2 \to \mathbb{N}^2$ be a function. From now functions α_1, α_2 play the role of indices. We know that in the preliminary part of the dissertation the function α_n appeared in the definition of the Vilenkin-like (or $\psi\alpha$ system), but this will not cause any misunderstanding. Define the following generalized Marcinkiewicz kernels and means respectively:

$$M_n^\alpha(y,x) := \frac{1}{n}\sum_{k=0}^{n-1} D_{\alpha_1(|n|,k)}(y^1,x^1)D_{\alpha_2(|n|,k)}(y^2,x^2),$$
$$t_n^\alpha f := f * M_n^\alpha \quad (f \in L^1(G_m^2),\, n \in \mathbb{P}).$$

This concept of Marcinkiewicz-like kernels and means is due to Gát [28].

The main aim of this part is to give a class of functions α for which we have the a.e. convergence relation $t_n^\alpha f \to f$ for each integrable two variable function with respect to two dimensional bounded Vilenkin-like systems.

To investigate this the following properties play a prominent role ($Car(B)$ denotes cardinality of the set B),

$$Car\{l \in \mathbb{N} : \alpha_j(|n|,l) = \alpha_j(|n|,k),\, l < n\} \le C \tag{12}$$

$$(k < n,\, n \in \mathbb{P},\, j = 1,2),$$

$$\max \{\alpha_j(|n|, k) :\ k < n\} \leq Cn \quad (n \in \mathbb{P},\, j = 1,2). \tag{13}$$

Our first aim is to prove that the operator $t_*^\alpha := \sup_{n\in\mathbb{P}} \mid t_n^\alpha \mid$ is of weak type (L^1, L^1). In order to do this we need a sequence of Lemmas which are base for the proof of Theorems . The Walsh-Paley version of Theorems 24 and 25 are due to Gát [28]. That is, we generalize a result of Gát. Moreover, techniques of papers [28] and [29] will also be used in the proof of the forthcoming lemmas. Denote for $k \in \mathbb{N}, x \in G_m,\ J_k(x) = I_k(x) \setminus I_{k+1}(x)$ and recall also that

$$n^{(s)} = \sum_{k=s}^{\infty} n_k M_k,\ (n,\ s \in \mathbb{N}),\ n^{(0)} = n,\ n^{(|n|+1)} = 0.$$

Generalized Marcinkiewicz kernels and their maximal operators

For $x, y \in G_m^2, A, n, s, j, k \in \mathbb{N}$ let

$$\Phi(A, n^{(s+1)} + jM_s + k,\, y, x) = D_{\alpha_1(A,\, n^{(s+1)}+jM_s+k)}(y^1, x^1)$$
$$D_{\alpha_2(A,\, n^{(s+1)}+jM_s+k)}(y^2, x^2).$$

Lemma 19. [5] Let $t^1, t^2, A, s \in \mathbb{N}, s \leq A$ and $y \in G_m^2$. Then,

$$\int_{J_{t^1}(y^1)\times J_{t^2}(y^2)} \sup_{|n|=A} \Big| \sum_{j=0}^{n_s-1} \sum_{k=0}^{M_s-1} \Phi(A,\, n^{(s+1)} + jM_s + k,\, y, x)\Big| d\mu(x)$$
$$\leq C(M_A M_{t^1})^{\frac{1}{2}}.$$

Lemma 20. [5] Let $a \in \mathbb{N}, y \in G_m^2$. Then,

$$\sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \int_{J_{t^1}(y^1)\times J_{t^2}(y^2)} \sup_{A\geq a} \sup_{|n|=A} \frac{1}{M_A} \times \sum_{s=t^1}^{A} \sum_{j=0}^{n_s-1}$$
$$\Big| \sum_{k=0}^{M_s-1} \Phi(A, n^{(s+1)} + jM_s + k, y, x) \Big| \, d\mu(x) \leq C.$$

In the sequel we step further and with the application of Lemma 20, we prove the main tool with respect to the maximal generalized Marcinkiewicz kernel in order to prove that the maximal operator $t_*^{\alpha} := \sup_{n\in\mathbb{P}} |t_n^{\alpha} f|$ is quasi-local and then it is of weak type (L^1, L^1).

Lemma 21. [5] Let $u \in G_m^2, a \in \mathbb{N}, y \in I_a(u^1) \times I_a(u^2)$. Then we have

$$\int_{G_m^2\setminus(I_a(u^1)\times I_a(u^2))} \sup_{n\geq M_{a-c}} |M_n^{\alpha}(y, x)| d\mu(x) \leq C.$$

Corollary 22. [5] Let $y \in G_m, n \in \mathbb{P}$. Then,

$$||M_n^{\alpha}(y, \cdot)||_1 \leq C.$$

Generalized Marcinkiewicz means and their maximal operators

In this section, we check the quasi-locality of the maximal operator t_*^α is quasi-local.

Lemma 23. [5] Let $f \in L^1(G_m^2)$ such that $\operatorname{supp} f \subset I_a(u^1) \times I_a(u^2), \int f d\mu(x) = 0$ for some $u \in G_m^2$ and $a \in \mathbb{N}$. Then,

$$\int_{G_m^2 \setminus (I_a(u^1) \times (I_a(u^2))} t_*^\alpha f(x) d\mu(x) \leq C||f||_1.$$

Theorem 24. [5] The operator t_*^α is of weak type (L^1, L^1) and it is also of type (L^p, L^p) for all $1 < p \leq \infty$.

Almost everywhere convergence of the generalized Marcinkiewicz means

Using the Lemmas in sections above, we get the following almost everywhere convergence result.

Theorem 25. [5] Let α satisfy conditions (12) and (13). Then, we have $t_n^\alpha f \to f$ for each $f \in L^1(G_m^2)$ a.e. with respect to every bounded Vilenkin-like system.

Finally, we give an application of Theorem 25 . Before this, the following Corollary is given.

Corollary 26. [5] Let (a_n) be a lacunary sequence of positive reals, i.e. $a_{n+1} \geq a_n q$ for some $q > 1$ ($n \in \mathbb{N}$ and $\alpha_j(n,k) \leq Ca_n$ ($k < a_n$, $j = 1,2$)) (modified version of condition (13). Then for every integrable function $f \in L^1(G_m^2)$ we have

$$\frac{1}{a_n} \sum_{k=0}^{a_n - 1} S_{\alpha_1(n,k), \alpha_2(n,k)} f(x) \to f(x)$$

for a.e. $x \in G_m^2$.

In the sequel we give an application of the Corollary above. The triangular partial sums of the 2-dimensional Fourier series and the triangular Dirichlet kernels (with respect to the Vilenkin-like system χ) are defined as $S_k^{\triangle} f(x^1, x^2) := \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} \hat{f}(i,j)\chi_i(x^1)\chi_j(x^2)$ and $D_k^{\triangle}(x^1, x^2) := \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} \chi_i(x^1)\chi_j(x^2)$. respectively. The Fejér means of the triangular partial sums of the two-dimensional integrable function f (see e.g. [34]) are

$$\sigma_n^{\triangle} f := \frac{1}{n} \sum_{k=0}^{n-1} S_k^{\triangle} f.$$

For the trigonometric system Herriot proved [36] the a.e. (and norm) convergence $\sigma_n^{\triangle} f \to f$ ($f \in L^1$). His method can not be adopted for the Vilenkin system, since for the time being there is no kernel formula available for these systems. The first result in this a.e. convergence issue of triangular means is due to Goginava and Weisz [34]. They proved for the Walsh-Paley system and each integrable function the a.e. convergence relation $\sigma_{2^n}^{\triangle} f \to f$. This result for the whole sequence of the triangular mean operators in the Walsh case is given by Gát [25].
In the Vilenkin-like situation there is nothing proved yet. By the Corollary above, we prove for bounded Vilenkin-like systems:

Theorem 27. [5] For every lacunary sequence (a_n) (that is, $a_{n+1} \geq q a_n$, $q > 1$) we have the a.e. convergence $\sigma_{a_n}^{\triangle} f \to f$ for each $f \in L^1(G_m^2)$.

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RESEARCH CONFERENCE PARTICIPATION

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7. The 14th International Student Conference on Analysis: in Sikfőkút, Hungary, 3-6 of February, 2018.
8. Regular Department Research Seminars: University of Debrecen, Hungary, 2017-2021.
9. The 19th International Conference on Functional Equations and Inequalities, Pedagogical University of Krakow and Stefan Banach International Mathematical Center, Poland, September 12-18, 2021(online).

List of symbols

A_n- σ algebra generated by the sets of I_n......5
D_n- n^{th} Dirichlet kernel......5
E_n- Conditional expectation operator with respect to A_n......55
$\hat{f}(n_1, n_2)-$Two dimensional Fourier coefficients......11
G_m- Vilenkin group......4
$G_m \times G_{\tilde{m}}-$ Two dimensional Vilenkin group......11
K_n- n^{th} Fejér kernel......5
$m := (m_k : k \in \mathbb{N})-$ Sequence of positive integers such that $m_k \geq 2$......32
$k \oplus n-$ m-adic addition of $k,\ n \in \mathbb{N}$ 1
$Z_{m_k}-$ Discrete cyclic group of order m_k......50
$\hat{f}(n)-$ n^{th} Fourier coefficient......6,10
$I_n(x)-$ n^{th} Interval of the Vilenkin Group G_m 32
$L^p(G_m)-$ usual Lebesgue space on the Vilenkin group G_m......5
$M_n^{\alpha}-$ Marcinkiewicz kernel......11
r_k- k^{th} Rademacher functions......2,5
S_nf- n^{th} Partial sums of Fourier series......6,10
$S_{n_1,n_2}-$ Rectangular partial sums of the Fourier series......11
$t_n^{\alpha}-$ Marcinkiewicz means......11
μ_k- Normalized Haar Measure on Z_{m_k}......5
ψ_n- n^{th} Vilenkin function (character of G_m)......5
$\psi-$ Vilenkin system......4
$\sigma_n^{\alpha}-$ n^{th} (C, α) mean......6,10
$\sigma_\star^{\alpha}f-$ Maximal operator of (C, α) mean......2,5
$\Psi\psi-$ Vilenkin-like system......8

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