Short Thesis for The Degree of Doctor of
Philosophy (PhD)

Generalized Open Sets, Minimality and Connectedness Properties in Relator Spaces

by

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Motivated by corresponding definitions of the various generalized open sets in topological spaces, we introduce and study ten kinds of generalized topologically open sets in relator spaces.

Moreover, having in mind the various connectedness properties considered in topological spaces we introduce and study seventeen reasonable connectedness properties of relator spaces.

The results contained in the dissertation have been presented in two papers [35, 38] and one chapter [36].

In this thesis we summarize the main results that are touched by the three chapters of the dissertation.

Introduction

The dissertation consists of an introduction and three chapters. The Introduction contains several historical facts on the investigations of these two enormous topics. Moreover, it indicates that by using *relators* (families of relations) instead of topologies we can get some substantial generalizations.

Actually, by the results of Pervin [34] and Száz [47], each minimal structure and generalized topology can be easily derived from families of preorder relations. Thus, in contrast to a common belief, they should not also be studied separately.

Chapter 1 is devoted to collect some relevant facts on relators and their induced basic tools, such as proximal and topological interiors, open and fat sets, for instance. Moreover, here some primary classifications of relators are also included.

In Chapter 2, ten types of generalized open sets are introduced and investigated. For instance, a subset A of a relator space $X(\mathcal{R})$ is called *semi-open* if $A \subseteq \operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big)$, and *quasi-open* if $V \subseteq A \subseteq \operatorname{cl}_{\mathcal{R}}(V)$ for some open subset V of $X(\mathcal{R})$.

Thus, for instance, it is shown that if in particular the relator $\mathcal{R}$ is topological, then A is a semi-open (quasi-open) subset of $X(\mathcal{R})$ if and only if there exist an open subset V of $X(\mathcal{R})$ and a subset B of $\operatorname{res}_{\mathcal{R}}(A) = \operatorname{cl}_{\mathcal{R}}(A) \setminus A$ such that $A = V \cup B$.

While, in Chapter 3, a relator $\mathcal{R}$ on X is, for instance, called *quasi-proximally minimal* if $\tau_{\mathcal{R}} \subseteq \{\emptyset, X\}$, and *quasi-topologically connected* if $\mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} \subseteq \{\emptyset, X\}$, where $\tau_{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{R}}$ denote the families of all proximally and topologically open subsets of $X(\mathcal{R})$, respectively.

There, for instance, it is shown that $\mathcal{R}$ is quasi-topologically connected if the relator $\mathcal{R}^{\wedge} \vee \mathcal{R}^{\wedge -1} = \{ R \cup S^{-1} : \ \mathcal{R}, S \in \mathcal{R}^{\wedge} \}$, where $\mathcal{R}^{\wedge} = \{ S \subseteq X^{2} : \ \forall \ x \in X : \ x \in \operatorname{int}_{\mathcal{R}}\big(S(x) \big) \}$, is quasi-proximally minimal.

The latter statement shows that the properties of quasi-topologically connected relators can, in principle, be immediately derived from those of the quasi-proximally minimal ones. Hence, it can be seen that connectedness is a particular case of well-chainedness.

At the end of the dissertation, several possibilities for some further, more general investigations are suggested.

A family $\mathcal{R}$ of relations on one set X to another Y is called a *relator on X to Y*, and the ordered pair $(X, Y)(\mathcal{R}) = \big((X, Y), \mathcal{R} \big)$ is called a *relator space*. For the origins of this notion see [**41, 46**] and the references in [**41**].

If in particular $\mathcal{R}$ is a relator on X to itself, then $\mathcal{R}$ is simply called a *relator on* X. Thus, by identifying singletons with their elements, we may naturally write $X(\mathcal{R})$ instead of $(X, X)(\mathcal{R})$. Namely, $(X, X) = \{\{X\}, \{X, X\}\} = \{\{X\}\}$.

Relator spaces of this simpler homogeneous type are already substantial generalizations of the various *ordered sets* [**5**] and *uniform spaces* [**10**]. However, they are insufficient for some important purposes. (See, for instance, [**13**] and [**45**].)

A relator $\mathcal{R}$ on X to Y, or the relator space $(X, Y)(\mathcal{R})$, is called *simple* if $\mathcal{R} = \{R\}$ for some relation R on X to Y. Simple relator spaces $(X, Y)(R)$ and $X(R)$ were called *formal contexts* and *gosets* (generalized ordered sets) in [**13**] and [**49**], respectively.

In the dissertation, we shall mainly be considering relators on X. A relator $\mathcal{R}$ on X, or the relator space $X(\mathcal{R})$, will, for instance, be called *reflexive* if each member of $\mathcal{R}$ is reflexive on X. Thus, we may also naturally speak of *preorder, tolerance, and equivalence relators*.

For instance, for a family $\mathcal{A}$ of subsets of X, the family $\mathcal{R}_{\mathcal{A}} = \{ R_A : A \in \mathcal{A} \}$, where $R_A = A^2 \cup (A^c \times X)$, is an important preorder relator on X. Such relators were first used by Pervin [**34**], and later also by Levine [**28**] and Száz [**47**].

While, for a family $\mathcal{D} \in X^2$ of *pseudo-metrics* on X, the family $\mathcal{R}_{\mathcal{D}} = \{ B_r^d : r > 0, d \in \mathcal{D} \}$, where $B_r^d = \{ (x, y) : \ d(x, y) < r \}$, is an important tolerance relator on X. Such relators were first considered by Weil [**53**].

Moreover, if $\mathfrak{S}$ is a family of *covers (partitions)* of X, then the family $\mathcal{R}_{\mathfrak{S}} = \{ S_{\mathcal{A}} : \ \mathcal{A} \in \mathfrak{S} \}$, where $S_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$, is a tolerance (equivalence) relator on X. Equivalence relators were first investigated by Levine [27].

Relator Spaces

In this chapter, we present some necessary prerequisites on relators.

NOTATION. ***Throughout the sequel,*** *for the readers convenience and the requirements of most of the forthcoming sections,* ***we shall assume that*** $\mathcal{R}$ ***is a relator on*** X, *not a relator on* X *to* Y.

DEFINITION. For any $A, B \subseteq X$ and $x, y \in X$, we define
(1) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ if $R[A] \subseteq B$ for some $R \in \mathcal{R}$;
(2) $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ if $R[A] \cap B \neq \emptyset$ for all $R \in \mathcal{R}$;
(3) $x \in \operatorname{int}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Int}_{\mathcal{R}}(B)$;
(4) $x \in \operatorname{cl}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Cl}_{\mathcal{R}}(B)$;
(5) $A \in \tau_{\mathcal{R}}$ if $A \in \operatorname{Int}_{\mathcal{R}}(A)$; (6) $A \in \digamma_{\mathcal{R}}$ if $A^c \notin \operatorname{Cl}_{\mathcal{R}}(A)$;
(7) $A \in \mathcal{T}_{\mathcal{R}}$ if $A \subseteq \operatorname{int}_{\mathcal{R}}(A)$; (8) $A \in \mathcal{F}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(A) \subseteq A$;
(9) $A \in \mathcal{E}_{\mathcal{R}}$ if $\operatorname{int}_{\mathcal{R}}(A) \neq \emptyset$; (10) $A \in \mathcal{D}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(A) = X$;
(11) $A \in \mathcal{N}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(A) \notin \mathcal{E}_{\mathcal{R}}$; (12) $A \in \mathcal{M}_{\mathcal{R}}$ if $\operatorname{int}_{\mathcal{R}} \in \mathcal{D}_{\mathcal{R}}$.

The relations $\operatorname{Int}_{\mathcal{R}}$ and $\operatorname{int}_{\mathcal{R}}$ are called *the proximal and topological interiors* generated by $\mathcal{R}$, respectively. While, the members of the families $\tau_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}}$ are called the *proximally open, topologically open and fat subsets* of the relator space $X(\mathcal{R})$, respectively.

DEFINITION. The relators

$$\mathcal{R}^{*} = \big\{ S \subseteq X^2 : \ \exists \ R \in \mathcal{R} : \ R \subseteq S \big\};$$

$$\mathcal{R}^{\#} = \big\{ S \subseteq X^2 : \ \forall \ A \subseteq X : \ \exists \ R \in \mathcal{R} : \ R[A] \subseteq S[A] \big\};$$

$$\mathcal{R}^{\wedge} = \big\{ S \subseteq X^2 : \ \forall \ x \in X : \ \exists \ R \in \mathcal{R} : \ R(x) \subseteq S(x) \big\};$$

$$\mathcal{R}^{\triangle} = \big\{ S \subseteq X^2 : \ \forall \ x \in X : \exists \ u \in X : \ \exists R \in \mathcal{R} : \ R(u) \subseteq S(x) \big\},$$

are called the *uniform, proximal, topological and paratopological closures (or refinements)* of the relator $\mathcal{R}$, respectively.

Thus, $\mathcal{R}^{\#}$, $\mathcal{R}^{\wedge}$ and $\mathcal{R}^{\triangle}$ are the largest relators on X such that $\operatorname{Int}_{\mathcal{R}} = \operatorname{Int}_{\mathcal{R}^{\#}}$, $\operatorname{int}_{\mathcal{R}} = \operatorname{int}_{\mathcal{R}^{\wedge}}$ and $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}^{\triangle}}$. However, in general there is no largest relator $\mathcal{S}$ on X such that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{S}}$. This is a serious disadvantage of the topologically open sets compared to the fat and proximally open ones.

DEFINITION. A relator $\mathcal{R}$ on X, or the relator space $X(\mathcal{R})$, will be called *reflexive* if each member of $\mathcal{R}$ is reflexive on X.

The importance of reflexive relators is apparent from the following.

THEOREM. The following assertions are equivalent:
(1) $\mathcal{R}$ is reflexive;
(2) $A \subseteq \operatorname{cl}_{\mathcal{R}}(A)$ for all $A \subseteq X$; (3) $\operatorname{int}_{\mathcal{R}}(A) \subseteq A$ for all $A \subseteq X$.

THEOREM. The following assertions are equivalent:
(1) $\mathcal{R}$ is reflexive;
(2) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ implies $A \subseteq B$ for all $A, B \subseteq X$;
(3) $A \cap B \neq \emptyset$ implies $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ for all $A, B \subseteq X$.

Also, we may also naturally have the following

DEFINITION. The relator $\mathcal{R}$ is called *non-partial* if each member R of $\mathcal{R}$ is a non-partial relation on X.

The importance of non-partial relators is apparent from the following

THEOREM. The following assertions are equivalent:
(1) $\mathcal{R}$ is non-partial;
(2) $\emptyset \notin \mathcal{E}_{\mathcal{R}}$; (3) $\mathcal{D}_{\mathcal{R}} \neq \emptyset$; (4) $X \in \mathcal{D}_{\mathcal{R}}$; (5) $\mathcal{E}_{\mathcal{R}} \neq \mathcal{P}(X)$.

DEFINITION. The relator $\mathcal{R}$ is called *locally non-partial* if for each $x \in X$ there exists $R \in \mathcal{R}$ such that for any $y \in R(x)$ and $S \in \mathcal{R}$ we have $S(y) \neq \emptyset$.

Moreover, by using the corresponding definitions, we state

THEOREM. The following assertions are equivalent:
(1) $\mathcal{R}$ is locally non-partial; (2) $X = \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(X)\big)$.

It is also worth introducing the following

DEFINITION. The relator $\mathcal{R}$ is called *non-degerated* if $X \neq \emptyset$ and $\mathcal{R} \neq \emptyset$.

Thus, we can also easily establish the following

THEOREM. The following assertions are equivalent:
(1) $\mathcal{R}$ is non-degenerated;
(2) $\emptyset \notin \mathcal{D}_{\mathcal{R}}$; (3) $\mathcal{E}_{\mathcal{R}} \neq \emptyset$; (4) $X \in \mathcal{E}_{\mathcal{R}}$; (5) $\mathcal{D}_{\mathcal{R}} \neq \mathcal{P}(X)$.

The following improvement of [42, Definition 2.1] was first considered in [43].

DEFINITION. The relator $\mathcal{R}$ is called:

(1) *quasi-topological* if $x \in \operatorname{int}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}\big(R(x)\big)\big)$ for all $x \in X$ and $R \in \mathcal{R}$;

(2) *topological* if for any $x \in X$ and $R \in \mathcal{R}$ there exists $V \in \mathcal{T}_{\mathcal{R}}$, such that $x \in V \subseteq R(x)$.

The appropriateness of these definitions is already quite obvious from the following theorems.

THEOREM. The following assertions are equivalent:

(1) $\mathcal{R}$ is quasi-topological;
(2) $\operatorname{int}_{\mathcal{R}}\big(R(x)\big) \in \mathcal{T}_{\mathcal{R}}$ for all $x \in X$ and $R \in \mathcal{R}$;
(3) $\operatorname{cl}_{\mathcal{R}}(A) \in \mathcal{F}_{\mathcal{R}}$ $\big(\operatorname{int}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}\big)$ for all $A \subseteq X$.

THEOREM. The following assertions are equivalent:

(1) $\mathcal{R}$ is topological; (2) $\mathcal{R}$ is reflexive and quasi-topological.

DEFINITION. The relator $\mathcal{R}$ is called

(1) *properly filtered* if for any $R, S \in \mathcal{R}$ we have $R \cap S \in \mathcal{R}$;
(2) *uniformly filtered* if for any $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T \subseteq R \cap S$;
(3) *proximally filtered* if for any $A \subseteq X$ and $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T[A] \subseteq R[A] \cap S[A]$;
(4) *topologically filtered* if for any $x \in X$ and $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T(x) \subseteq R(x) \cap S(x)$.

We can easily prove the following theorem which shows the appropriateness of the above proximal filteredness property.

THEOREM. The following assertions are equivalent:

(1) $\mathcal{R}$ is proximally filtered;
(2) $\operatorname{Cl}_{\mathcal{R}}(A \cup B) = \operatorname{Cl}_{\mathcal{R}}(A) \cup \operatorname{Cl}_{\mathcal{R}}(B)$ for all $A, B \subseteq X$;
(3) $\operatorname{Int}_{\mathcal{R}}(A \cap B) = \operatorname{Int}_{\mathcal{R}}(A) \cap \operatorname{Int}_{\mathcal{R}}(B)$ for all $A, B \subseteq X$.

COROLLARY. If $\mathcal{R}$ is proximally filtered, then the families $\digamma_{\mathcal{R}}$ and $\tau_{\mathcal{R}}$ are closed under binary unions and intersections, respectively.

From the above theorem, we can also easily derive the following

THEOREM. The following assertions are equivalent:

(1) $\mathcal{R}$ is topologically filtered;
(2) $\operatorname{cl}_{\mathcal{R}}(A \cup B) = \operatorname{cl}_{\mathcal{R}}(A) \cup \operatorname{cl}_{\mathcal{R}}(B)$ for all $A, B \subseteq X$;
(3) $\operatorname{int}_{\mathcal{R}}(A \cap B) = \operatorname{int}_{\mathcal{R}}(A) \cap \operatorname{int}_{\mathcal{R}}(B)$ for all $A, B \subseteq X$.

COROLLARY. If $\mathcal{R}$ is topologically filtered, then the families $\mathcal{F}_{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{R}}$ are closed under binary unions and intersections, respectively.

THEOREM. If $\mathcal{R}$ is topologically filtered , then for any $A, B \subseteq X$ we have $\operatorname{cl}_{\mathcal{R}}(A) \setminus \operatorname{cl}_{\mathcal{R}}(B) = \operatorname{cl}_{\mathcal{R}}(A \setminus B) \setminus \operatorname{cl}_{\mathcal{R}}(B)$.

COROLLARY. If $\mathcal{R}$ is topologically filtered, then for any $A, B \subseteq X$ we have $\operatorname{cl}_{\mathcal{R}}(A) \setminus \operatorname{cl}_{\mathcal{R}}(B) \subseteq \operatorname{cl}_{\mathcal{R}}(A \setminus B)$.

The importance of topologically filtered relators is apparent from

THEOREM. If $\mathcal{R}$ is topologically filtered, then for any $A, B \subseteq X$ we have
(1) $\operatorname{cl}_{\mathcal{R}}(A) \cap \operatorname{int}_{\mathcal{R}}(B) \subseteq \operatorname{cl}_{\mathcal{R}}(A \cap B)$;
(2) $\operatorname{int}_{\mathcal{R}}(A \cup B) \subseteq \operatorname{int}_{\mathcal{R}}(A) \cup \operatorname{cl}_{\mathcal{R}}(B)$.

COROLLARY. If $\mathcal{R}$ is topologically filtered, then
(1) $\operatorname{cl}_{\mathcal{R}}(A) \cap B \subseteq \operatorname{cl}_{\mathcal{R}}(A \cap B)$ for all $A \subseteq X$ and $B \in \mathcal{T}_{\mathcal{R}}$;
(2) $\operatorname{int}_{\mathcal{R}}(A \cup B) \subseteq \operatorname{int}_{\mathcal{R}}(A) \cup B$ for all $A \subseteq X$ and $B \in \mathcal{F}_{\mathcal{R}}$.

Now, as some improvements of the above theorem and its corollary, we can also prove the following theorem and its corollary.

THEOREM. If $\mathcal{R}$ is topologically filtered, then for any $A, B \subseteq X$ we have
(1) $\operatorname{cl}_{\mathcal{R}}(A) \cap \operatorname{int}_{\mathcal{R}}(B) = \operatorname{cl}_{\mathcal{R}}(A \cap B) \cap \operatorname{int}_{\mathcal{R}}(B)$;
(2) $\operatorname{int}_{\mathcal{R}}(A \cup B) \cup \operatorname{cl}_{\mathcal{R}}(B) = \operatorname{int}_{\mathcal{R}}(A) \cup \operatorname{cl}_{\mathcal{R}}(B)$.

COROLLARY. If $\mathcal{R}$ is topologically filtered, then
(1) $\operatorname{cl}_{\mathcal{R}}(A) \cap B = \operatorname{cl}_{\mathcal{R}}(A \cap B) \cap B$ for all $A \subseteq X$ and $B \in \mathcal{T}_{\mathcal{R}}$;
(2) $\operatorname{int}_{\mathcal{R}}(A \cup B) \cup B = \operatorname{int}_{\mathcal{R}}(A) \cup B$ for all $A \subseteq X$ and $B \in \mathcal{F}_{\mathcal{R}}$.

Also, we state some more particular theorems on topologically filtered relators

THEOREM. If $\mathcal{R}$ is quasi-topological and topologically filtered, then for any $A, B \in \mathcal{N}_{\mathcal{R}}$ we have $A \cup B \in \mathcal{N}_{\mathcal{R}}$.

COROLLARY. If $\mathcal{R}$ is nonvoid, non-partial, quasi-topological and topologically filtered, then $\mathcal{N}_{\mathcal{R}}$ is an ideal on X.

THEOREM. If $\mathcal{R}$ is topological and topologically filtered, then for any $A \in \mathcal{T}_{\mathcal{R}}$ we have $\operatorname{res}_{\mathcal{R}}(A) \in \mathcal{F}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}}$.

COROLLARY. If $\mathcal{R}$ is topological and topologically filtered, then $\operatorname{res}_{\mathcal{R}}(A) \in \mathcal{N}_{\mathcal{R}}$ for all $A \in \mathcal{T}_{\mathcal{R}}$.

DEFINITION. The relator $\mathcal{R}$ is called *properly simple* if it is a singleton relator. That is, there exists a relation R on X such that $\mathcal{R} = \{R\}$.

Now, we can also easily prove the following

THEOREM. Under the notation $R = \bigcap \mathcal{R}$, the following assertions are equivalent:

(1) $\mathcal{R}$ is topologically simple; (2) $R \in \mathcal{R}^{\wedge}$; (3) $\mathcal{R}^{\wedge} = \{ R \}^{\wedge}$.

THEOREM. If $\mathcal{R}$ is nonvoid, then under the notation $R = \bigcap \mathcal{R}$, we have $\mathcal{R}^{\vee \wedge} = \left\{ R^{-1} \right\}^{\wedge}$.

In contrast to the reflexivity property of the relator $\mathcal{R}$, we may naturally introduce a great abundance of important symmetry and transitivity properties of $\mathcal{R}$ [**43**, **44**].

DEFINITION. The relator $\mathcal{R}$ is called *topologically symmetric* if for each $x \in X$ and $R \in \mathcal{R}$ there exists $S \in \mathcal{R}$ such that $S(x) \subseteq R^{-1}(x)$.

We can state the following two theorems.

THEOREM. The following assertions are equivalent:

(1) $\mathcal{R}$ is topologically symmetric;

(2) $\operatorname{cl}_{\mathcal{R}} \subseteq \operatorname{cl}_{\mathcal{R}^{-1}}$; (3) $\operatorname{int}_{\mathcal{R}^{-1}} \subseteq \operatorname{int}_{\mathcal{R}}$.

THEOREM. If $\mathcal{R}$ is nonvoid, then the following assertions are equivalent:

(1) $\mathcal{R}$ is quasi-topologically symmetric;

(2) $\mathcal{T}_{\mathcal{R}^{-1}} \subseteq \mathcal{T}_{\mathcal{R}}$; (3) $\mathcal{F}_{\mathcal{R}^{-1}} \subseteq \mathcal{F}_{\mathcal{R}}$.

DEFINITION. For a relator $\mathcal{S}$ on X the relator

$$\mathcal{R} \vee \mathcal{S} = \left\{ R \cup S : \quad R \in \mathcal{R}, \quad S \in \mathcal{S} \right\}$$

is called the *elementwise union* of the relators $\mathcal{R}$ and $\mathcal{S}$.

If somewhat more generally $\mathcal{R} = \left(R_i \right)_{i \in I}$ and $\mathcal{S} = \left(S_i \right)_{i \in I}$, where R_i and S_i are relations on X, then we may also naturally define $\mathcal{R} \triangledown \mathcal{S} = \left(R_i \cup S_i \right)_{i \in I}$.

Thus, in particular for any relator $\mathcal{R}$, we may also naturally write

$$\mathcal{R} \triangledown \mathcal{R}^{-1} = \left\{ R \cup R^{-1} : \ R \in \mathcal{R} \right\}$$

and

$$\mathcal{R} \vee \mathcal{R}^{-1} = \left\{ R \cup S^{-1} : \ R, S \in \mathcal{R} \right\}.$$

The importance of $\mathcal{R} \vee \mathcal{S}$ is apparent from the following

THEOREM. We have

(1) $\operatorname{Int}_{\mathcal{R} \vee \mathcal{S}} = \operatorname{Int}_{\mathcal{R}} \cap \operatorname{Int}_{\mathcal{S}}$; (2) $\operatorname{Cl}_{\mathcal{R} \vee \mathcal{S}} = \operatorname{Cl}_{\mathcal{R}} \cup \operatorname{Cl}_{\mathcal{S}}$.

COROLLARY. We have

(1) $\mathcal{T}_{\mathcal{R} \vee \mathcal{S}} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{T}_{\mathcal{S}}$; (2) $\mathcal{F}_{\mathcal{R} \vee \mathcal{S}} = \mathcal{F}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{S}}$.

We also immediately derive

THEOREM. We have

(1) $\operatorname{int}_{\mathcal{R}\vee\mathcal{S}} = \operatorname{int}_{\mathcal{R}} \cap \operatorname{int}_{\mathcal{S}}$; (2) $\operatorname{cl}_{\mathcal{R}\vee\mathcal{S}} = \operatorname{cl}_{\mathcal{R}} \cup \operatorname{cl}_{\mathcal{S}}$.

COROLLARY. We have

(1) $\mathcal{T}_{\mathcal{R}\vee\mathcal{S}} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{T}_{\mathcal{S}}$; (2) $\mathcal{F}_{\mathcal{R}\vee\mathcal{S}} = \mathcal{F}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{S}}$;

(3) $\mathcal{E}_{\mathcal{R}\vee\mathcal{S}} \subseteq \mathcal{E}_{\mathcal{R}} \cap \mathcal{E}_{\mathcal{S}}$; (4) $\mathcal{D}_{\mathcal{R}} \cup \mathcal{D}_{\mathcal{S}} \subseteq \mathcal{D}_{\mathcal{R}\vee\mathcal{S}}$.

We may also naturally consider the *elementwise intersection*

$$\mathcal{R} \wedge \mathcal{S} = \left\{ R \cap S : \; R \in \mathcal{R}, \; S \in \mathcal{S} \right\}.$$

Now, by using the above definition, we also prove the following

THEOREM. If $\mathcal{R}$ is uniformly filtered, then for any $\square \in \{ *, \#, \wedge, \vartriangle \}$, we have $\left(\mathcal{R} \triangledown \mathcal{R}^{-1}\right)^{\square} = \left(\mathcal{R} \vee \mathcal{R}^{-1}\right)^{\square}$.

Concerning the relator $\mathcal{R} \vee \mathcal{S}$, we also state the following

THEOREM. If $\square \in \{ *, \#, \wedge \}$, then $\left(\mathcal{R} \vee \mathcal{S}\right)^{\square} = \mathcal{R}^{\square} \cap \mathcal{S}^{\square}$.

COROLLARY. If $\square \in \{ *, \#, \wedge \}$, then

(1) $\left(\mathcal{R} \vee \mathcal{S}\right)^{\square} = \left(\mathcal{R}^{\square} \vee \mathcal{S}^{\square}\right)^{\square}$; (2) $\mathcal{R}^{\square} \cap \mathcal{S}^{\square} = \left(\mathcal{R}^{\square} \cap \mathcal{S}^{\square}\right)^{\square}$.

We also state the following

THEOREM. If $\square \in \{ *, \#, \wedge \}$, then the following assertions are equivalent:

(1) $\mathcal{R} \vee \mathcal{S} \subseteq \left(\mathcal{R} \cap \mathcal{S}\right)^{\square}$; (2) $\left(\mathcal{R} \vee \mathcal{S}\right)^{\square} \subseteq \left(\mathcal{R} \cap \mathcal{S}\right)^{\square}$;

(3) $\left(\mathcal{R} \vee \mathcal{S}\right)^{\square} = \left(\mathcal{R} \cap \mathcal{S}\right)^{\square}$; (4) $\mathcal{R}^{\square} \cap \mathcal{S}^{\square} \subseteq \left(\mathcal{R} \cap \mathcal{S}\right)^{\square}$;

(5) $\mathcal{R}^{\square} \cap \mathcal{S}^{\square} = \left(\mathcal{R} \cap \mathcal{S}\right)^{\square}$.

Generalized Topologically Open Sets in Relator Spaces

Parts (2) and (3) of the following definition have been suggested by [**24**, Theorem 1 and Definition 1] of Levine.

For the motivations of parts (1) and (4), see Mashhour et al. [**30**, p. 47] and Jun et al. [**17**, Lemma 4.21].

DEFINITION. A subset A of the relator space $X(\mathcal{R})$ will be called *topologically*

(1) *preopen* if $A \subseteq \operatorname{int}_{\mathcal{R}}\left(\operatorname{cl}_{\mathcal{R}}(A)\right)$;

(2) *semi-open* if $A \subseteq \mathrm{cl}_{\mathcal{R}}\big(\mathrm{int}_{\mathcal{R}}(A)\big)$;
(3) *quasi-open* if there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $V \subseteq A \subseteq \mathrm{cl}_{\mathcal{R}}(V)$;
(4) *pseudo-open* if there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $A \subseteq V \subseteq \mathrm{cl}_{\mathcal{R}}(A)$.

And, the families of all such subsets A of $X(\mathcal{R})$ will be denoted by $\mathcal{T}_{\mathcal{R}}^{\kappa}$ with $\kappa = p$, s, q and ps, respectively.

We can prove the following

THEOREM. If $\mathcal{R}$ is topological, then
(1) $\mathcal{T}_{\mathcal{R}}^{q} = \mathcal{T}_{\mathcal{R}}^{s}$; (2) $\mathcal{T}_{\mathcal{R}}^{ps} = \mathcal{T}_{\mathcal{R}}^{p}$.

By using our former results on topological relators, we can also prove

THEOREM. If $\mathcal{R}$ is topological, then for any $A \subseteq X$, the following assertions are equivalent
(1) $A \in \mathcal{T}_{\mathcal{R}}^{s}$; (2) $\mathrm{cl}_{\mathcal{R}}(A) = \mathrm{cl}_{\mathcal{R}}\big(\mathrm{int}_{\mathcal{R}}(A)\big)$;
(3) $\mathrm{cl}_{\mathcal{R}}(A) = \mathrm{cl}_{\mathcal{R}}(V)$ for some $V \in \mathcal{T}_{\mathcal{R}} \cap \mathcal{P}(A)$.

Now, as an immediate consequence of this theorem, we also state

COROLLARY. If $\mathcal{R}$ is topological, then for any $A \subseteq X$, the following assertions are equivalent:
(1) $A \in \mathcal{F}_{\mathcal{R}} \cap \mathcal{T}_{\mathcal{R}}^{s}$; (2) $A = \mathrm{cl}_{\mathcal{R}}\big(\mathrm{int}_{\mathcal{R}}(A)\big)$;
(3) $A = \mathrm{cl}_{\mathcal{R}}(V)$ for some $V \in \mathcal{T}_{\mathcal{R}} \cap \mathcal{P}(A)$.

We also state the following two generalizations of [24, Theorem 3] of Levine.

THEOREM. If $\mathcal{R}$ is quasi-topological, $A \in \mathcal{T}_{\mathcal{R}}^{s}$ and $A \subseteq B \subseteq \mathrm{cl}_{\mathcal{R}}(A)$, then $B \in \mathcal{T}_{\mathcal{R}}^{s}$ also holds.

THEOREM. If $\mathcal{R}$ is quasi-topological, $A \in \mathcal{T}_{\mathcal{R}}^{q}$ and $A \subseteq B \subseteq \mathrm{cl}_{\mathcal{R}}(A)$, then $B \in \mathcal{T}_{\mathcal{R}}^{q}$ also holds.

Now, as an immediate consequence of the above theorem, we can state

THEOREM. If $\mathcal{R}$ is topological, then for every $A \in \mathcal{T}_{\mathcal{R}}^{s}$ we have $\mathrm{cl}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}^{s}$.

Now, we state the following generalization of [24, Theorem 5] of Levine.

THEOREM. If $\mathcal{R}$ is topological, then $\mathcal{A} = \mathcal{T}_{\mathcal{R}}^{s}$ is the smallest family of subsets of X such that
(1) $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{A}$; (2) $A \in \mathcal{A}$ and $A \subseteq B \subseteq \mathrm{cl}_{\mathcal{R}}(A)$ imply $B \in \mathcal{A}$.

We also state the following generalization of [24, Lemma 2] of Levine.

THEOREM. If $\mathcal{R}$ is topological, then $\mathcal{T}_{\mathcal{R}} = \{ \operatorname{int}_{\mathcal{R}}(A) : \quad A \in \mathcal{T}_{\mathcal{R}}^{s} \}$.

Moreover, we also prove the following two theorems.

THEOREM. If $\mathcal{R}$ is quasi-topological, $A \in \mathcal{T}_{\mathcal{R}}^{p}$ and $B \subseteq A \subseteq \operatorname{cl}_{\mathcal{R}}(B)$, then $B \in \mathcal{T}_{\mathcal{R}}^{p}$ also holds.

THEOREM. If $\mathcal{R}$ is quasi-topological, $A \in \mathcal{T}_{\mathcal{R}}^{ps}$ and $B \subseteq A \subseteq \operatorname{cl}_{\mathcal{R}}(B)$, then $B \in \mathcal{T}_{\mathcal{R}}^{ps}$ also holds.

THEOREM. If $\mathcal{R}$ is nonvoid, then $\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{p}$.

THEOREM. If $\mathcal{R}$ is topologically filtered and $A = V \cap B$ for some $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{D}_{\mathcal{R}}$, then $A \in \mathcal{T}_{\mathcal{R}}^{ps}$.

Now, we also prove the following two theorems.

THEOREM. If $\mathcal{R}$ is topological, then $\mathcal{A} = \mathcal{T}_{\mathcal{R}}^{p}$ is the smallest family of subsets of X such that

(1) $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{A}$; (2) $A \in \mathcal{A}$ and $A \subseteq B \subseteq \operatorname{cl}_{\mathcal{R}}(A)$ imply $B \in \mathcal{A}$.

THEOREM. If $\mathcal{R}$ is topological, then $\mathcal{T}_{\mathcal{R}} = \{ \operatorname{int}_{\mathcal{R}}(A) : \quad A \in \mathcal{T}_{\mathcal{R}}^{p} \}$.

To introduce the corresponding generalized topologically closed sets, we shall use the following plausible notation.

DEFINITION. For any $\kappa = s, p, q$ and ps, we define

$$\mathcal{F}_{\mathcal{R}}^{\kappa} = \{ A \subseteq X : \quad A^{c} \in \mathcal{T}_{\mathcal{R}}^{\kappa} \}.$$

We can prove the following theorems

THEOREM. For any $A \subseteq X$ the following assertions are equivalent:

(1) $A \in \mathcal{F}_{\mathcal{R}}^{q}$; (2) there exists $W \in \mathcal{F}_{\mathcal{R}}$ such that $\operatorname{int}_{\mathcal{R}}(W) \subseteq A \subseteq W$.

THEOREM. For any $A \subseteq X$ the following assertions are equivalent:

(1) $A \in \mathcal{F}_{\mathcal{R}}^{ps}$; (2) there exists $W \in \mathcal{F}_{\mathcal{R}}$ such that $\operatorname{int}_{\mathcal{R}}(A) \subseteq W \subseteq A$.

THEOREM. For any $A \subseteq X$, the following assertions are equivalent:

(1) $A \in \mathcal{F}_{\mathcal{R}}^{s}$; (2) $\operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big) \subseteq A$.

THEOREM. For any $A \subseteq X$, the following assertions are equivalent:

(1) $A \in \mathcal{F}_{\mathcal{R}}^{p}$; (2) $\operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big) \subseteq A$.

Regular open sets were first introduced by Kuratowski [19] with reference to a paper of Henri Lebesgue. However, their importance became completely clear only after the considerations of Stone [40].

Following Kuratowski's definition, we have also introduced

DEFINITION. A subset A of the relator space $X(\mathcal{R})$ will be called *topologically regular open* if

$$A = \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big).$$

Also, the family of all such subsets of $X(\mathcal{R})$ will be denoted by $\mathcal{T}_{\mathcal{R}}^{r}$.

Thus, in contrast to the topological case, $\mathcal{T}_{\mathcal{R}}^{r}$ need not be a subfamily of $\mathcal{T}_{\mathcal{R}}$. To show this, we can use the following

EXAMPLE. If $X = \{1, 2\}$ and R is a relation on X such that $R(1) = \{2\}$ and $R(2) = \{1\}$, then $\mathcal{R} = \{R\}$ is a symmetric relator on X such that $\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\}$ and $\mathcal{T}_{\mathcal{R}}^{r} = \mathcal{P}(X)$.

We also can prove the following generalization of Kuratowski [19].

THEOREM. If $\mathcal{R}$ is topological, then for any $A \in \mathcal{T}_{\mathcal{R}}^{s}$ we have $\operatorname{cl}_{\mathcal{R}}(A)^{c} \in \mathcal{T}_{\mathcal{R}}^{r}$.

From this theorem, by using that $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{s}$ whenever $\mathcal{R}$ is reflexive, we can easily derive the following

COROLLARY. If $\mathcal{R}$ is topological and $\mathcal{A} = \mathcal{T}_{\mathcal{R}}$ or $\mathcal{T}_{\mathcal{R}}^{s}$, then

$$\mathcal{T}_{\mathcal{R}}^{r} = \big\{ \operatorname{cl}_{\mathcal{R}}(A)^{c} : \quad A \in \mathcal{A} \big\}.$$

Now, we can also easily establish the following

THEOREM. If $\mathcal{R}$ is topological, then for any $A \subseteq X$ we have $\operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big) \in \mathcal{T}_{\mathcal{R}}^{r}$.

COROLLARY. If $\mathcal{R}$ is topological, then

$$\mathcal{T}_{\mathcal{R}}^{r} = \big\{ \operatorname{int}_{\mathcal{R}}(A) : \quad A \in \mathcal{F}_{\mathcal{R}} \big\} = \big\{ \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big) : \quad A \subseteq X \big\}.$$

Hence, it is clear that Stone's definition [40, p. 376] of a regular open set coincides with that of Kuratowski [19, p. 9].

We can also prove the following

THEOREM. For any $A \subseteq X$, the following assertions are equivalent:

(1) $A \in \mathcal{F}_{\mathcal{R}}^{r}$; (2) $A = \operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big)$.

Now, as a generalization of [8, Lemma 1] of Duszyński and Noiri, we state the following characterized theorems of semi-open and quasi-open sets

THEOREM. If $\mathcal{R}$ is reflexive, then for $A \subseteq X$ the following assertions are equivalent:

(1) $A \in \mathcal{T}_{\mathcal{R}}^{s}$;

(2) there exists $B \subseteq X$ such that $A = \operatorname{int}_{\mathcal{R}}(A) \cup B$ and $B \subseteq \operatorname{res}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big)$.

Now, by using our former results, we also state the following generalization of an observation of Dlaska, Ergun and Ganster [6].

THEOREM. If $\mathcal{R}$ is topological, then for $A \subseteq X$ the following assertions are equivalent:

(1) $A \in \mathcal{T}_{\mathcal{R}}^{s}$;
(2) there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \subseteq X$ such that $A = V \cup B$ and $B \subseteq \operatorname{res}_{\mathcal{R}}(V)$.

Moreover, as analogously to [24, Theorem 7] of Levine, we can also state the following stability type theorem.

THEOREM. If $\mathcal{R}$ is topologically filtered, topological and $A \in \mathcal{T}_{\mathcal{R}}^{s}$, then there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{N}_{\mathcal{R}}$ such that $A = V \cup B$ and $V \cap B = \emptyset$.

The following obvious reformulation of [24, Example 2] of Levine shows that the converse of this theorem is false.

Define $X = \mathbb{R}$ and $R_n = \big\{ (x, y) \in X^2 : \quad d(x, y) < n^{-1} \big\}$ for all $n \in \mathbb{N}$. Then, $\mathcal{R} = \{ R_n : n \in \mathbb{N} \}$ is a properly filtered, strongly topological relator on X such that, under the notations $V = \,]0, 1[$ and $B = \{2\}$, we have $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{N}_{\mathcal{R}}$ such that $A = V \cup B \notin \mathcal{T}_{\mathcal{R}}^{s}$.

As a conterpart of [11, Proposition 2.1] of Ganster, we can also prove the following characterized theorems of preopen and pseudo-open sets

THEOREM. If $\mathcal{R}$ is topologically filtered and topological, then for any $A \subseteq X$ the following assertions are equivalent:

(1) $A \in \mathcal{T}_{\mathcal{R}}^{p}$;
(2) there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{D}_{\mathcal{R}}$ such that $A = V \cap B$.

In addition to the above theorem, we also prove the following

THEOREM. If $\mathcal{R}$ is topologically filtered and topological, then for any $A \subseteq X$ the following assertions are equivalent:

(1) $A \in \mathcal{T}_{\mathcal{R}}^{p}$;
(2) there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $A \subseteq V$ and $\operatorname{cl}_{\mathcal{R}}(A) = \operatorname{cl}_{\mathcal{R}}(V)$.

Note that, in the above two theorems we may again write $\mathcal{T}_{\mathcal{R}}^{ps}$ in place of $\mathcal{T}_{\mathcal{R}}^{p}$.

Moreover, we state an important property of topologically semi-open and preopen sets. we prove the following improvement of [32, Lemma 2.5] of Noiri.

THEOREM. If $\mathcal{R}$ is topologically filtered and quasi-topological, and

$$A \in \mathcal{T}_{\mathcal{R}}^{s}, \qquad B \in \mathcal{T}_{\mathcal{R}}^{p}, \qquad \operatorname{int}_{\mathcal{R}}(A) \subseteq C \subseteq \operatorname{cl}_{\mathcal{R}}(A), \qquad B \subseteq D \subseteq X,$$

then
$$\operatorname{cl}_{\mathcal{R}}(A) \cap B = \operatorname{cl}_{\mathcal{R}}(C \cap D) \cap B .$$

COROLLARY. If $\mathcal{R}$ is topologically filtered and quasi-topological , and $A \in \mathcal{T}_{\mathcal{R}}^{s}$ and $B \in \mathcal{T}_{\mathcal{R}}^{p}$, then

$$\operatorname{cl}_{\mathcal{R}}(A) \cap B = \operatorname{cl}_{\mathcal{R}}\Big(\operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big) \cap \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(B)\big)\Big) \cap B .$$

Now, we define some further generalized topologically open sets. Parts (1) and (2) of the following definition have been suggested by Njåstad [31] and Abd El-Monsef et al. [1] .

While, for some motivations of parts (3) and (4), see [2, Definition 2.1] of Andrijević.

DEFINITION. A subset A of the relator space $X(\mathcal{R})$ will be called *topologically*

(1) *α–open* if $A \subseteq \operatorname{int}_{\mathcal{R}}\Big(\operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big)\Big)$;

(2) *β–open* if $A \subseteq \operatorname{cl}_{\mathcal{R}}\Big(\operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big)\Big)$;

(3) *γ–open* if there exists $V \in \mathcal{T}_{\mathcal{R}}^{s}$ such that $A \subseteq V \subseteq \operatorname{cl}_{\mathcal{R}}(A)$;

(4) *δ–open* if there exists $V \in \mathcal{T}_{\mathcal{R}}^{p}$ such that $V \subseteq A \subseteq \operatorname{cl}_{\mathcal{R}}(V)$.

And, the family of all such subsets of $X(\mathcal{R})$ will be denoted by $\mathcal{T}_{\mathcal{R}}^{\kappa}$ with $\kappa = \alpha,\ \beta,\ \gamma$ and δ, respectively.

Note that if $\mathcal{R}$ is not topological, then by using the families $\mathcal{T}_{\mathcal{R}}^{q}$ and $\mathcal{T}_{\mathcal{R}}^{ps}$ instead of $\mathcal{T}_{\mathcal{R}}^{s}$ and $\mathcal{T}_{\mathcal{R}}^{p}$, respectively, we can get some stronger forms of generalized topologically open sets.

Moreover, we can also prove the following characterized theorems of topologically α–open and β–open Sets

As an improvement of [31, Proposition 4] of Njåstad, we prove

THEOREM. If $\mathcal{R}$ is topologically filtered and topological, then for any $A \subseteq X$ the following assertions are equivalent:

(1) $A \in \mathcal{T}_{\mathcal{R}}^{\alpha}$;

(2) there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{N}_{\mathcal{R}}$ such that $A = V \setminus B$.

Now, by using the above theorem, we also prove the following improvement of [31, Corollary] of Njåstad.

COROLLARY. If $\mathcal{R}$ is nonvoid, topologically filtered and topological, then the following assertions are equivalent:

(1) $\mathcal{T}_{\mathcal{R}}^{\alpha} \subseteq \mathcal{T}_{\mathcal{R}}$; (2) $\mathcal{T}_{\mathcal{R}}^{\alpha} = \mathcal{T}_{\mathcal{R}}$;

(3) $\mathcal{N}_{\mathcal{R}} \subseteq \mathcal{F}_{\mathcal{R}}$; (4) $\mathcal{N}_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}}$.

Now, by using the plausible notation $\mathcal{F}_{\mathcal{R}}^{r} = \big\{ A^{c} : \ A \in \mathcal{T}_{\mathcal{R}}^{r} \big\}$, as a partial counterpart of [9, Theorem 26] of Ekici and [16, Theorem 3.7] of Jamunarani et al., we can also prove the following

THEOREM. If $\mathcal{R}$ is topological, then for any $A \subseteq X$ the following assertions are equivalent:

(1) $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$; (2) $\operatorname{cl}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}^{s}$;
(3) $\operatorname{cl}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}^{q}$; (4) $\operatorname{cl}_{\mathcal{R}}(A) \in \mathcal{F}_{\mathcal{R}}^{r}$.

Also, we can immediately derive the following

THEOREM. If $\mathcal{R}$ is topological, then for any $A \subseteq X$ the following assertions are equivalent:

(1) $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$;
(2) there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \subseteq X$ such that $\operatorname{cl}_{\mathcal{R}}(A) = V \cup B$ and $B \subseteq \operatorname{res}_{\mathcal{R}}(V)$.

Hence, we easily derive the following counterpart of [9, Theorem 27] of Ekici and [16, Theorem 3.8] of Jamunarani et al.

COROLLARY. If $\mathcal{R}$ is topological, $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$ and $A \subseteq B \subseteq \operatorname{cl}_{\mathcal{R}}(A)$, then $B \in \mathcal{T}_{\mathcal{R}}^{\beta}$ also holds.

Moreover, we also easily derive

THEOREM. If $\mathcal{R}$ is topologically filtered, topological and $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$, then there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{N}_{\mathcal{R}}$ such that $\operatorname{cl}_{\mathcal{R}}(A) = V \cup B$ and $V \cap B = \emptyset$.

Now, analogously to [9, Theorem 23] of Ekici and [16, Theorem 3.5] of Jamunarani et al., we can also prove the following

THEOREM. If $\mathcal{R}$ is topological and $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$, then there exist $V \in \mathcal{T}_{\mathcal{R}}^{s}$ and $B \in \mathcal{D}_{\mathcal{R}}$ such that $A = V \cap B$.

Moreover, we can also easily prove the following counterpart of [9, Theorem 7] of Ekici and [16, Theorem 2.1] of Jamunarani et al.

THEOREM. If $\mathcal{R}$ is topological, then $\mathcal{T}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}}^{\alpha} \cap \mathcal{F}_{\mathcal{R}}^{\beta}$.

Now, we can also easily prove the following two counterparts of [2, Theorem 2.4] of Andrijević.

THEOREM. If $\mathcal{R}$ is topological, then $\mathcal{T}_{\mathcal{R}}^{\gamma} = \mathcal{T}_{\mathcal{R}}^{\beta}$.

THEOREM. If $\mathcal{R}$ is topologically filtered and topological, then $\mathcal{T}_{\mathcal{R}}^{\delta} = \mathcal{T}_{\mathcal{R}}^{\beta}$.

In topological spaces, β–open sets were actually called semi-preopen by Andrijević [2]. Later, this terminology was used by Ganster and Andrijević [12] and Dontchev [7].

In a subsequent paper [3], Andrijević also introduced the notion of a b–open subset of a topological space. Motivated by his definition, we may also naturally introduce the following

DEFINITION. A subset A of a relator space $X(\mathcal{R})$ will be called *topologically*

(1) *a–open* if $A \subseteq \operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big) \cap \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big)$;

(2) *b–open* if $A \subseteq \operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big) \cup \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big)$.

And, the family of all such subsets of $X(\mathcal{R})$ will be denoted by $\mathcal{T}_{\mathcal{R}}^{\kappa}$ with $\kappa = a$ and b, respectively.

Now, in accordance with [3, Remark 1] of Adrijević, we state

THEOREM. If $\mathcal{R}$ is topologically filtered and topological, then for any $A \subseteq X$ the following assertions are equivalent:

(1) $A \in \mathcal{T}_{\mathcal{R}}^{b}$;

(2) there exist $B \in \mathcal{T}_{\mathcal{R}}^{s}$ and $C \in \mathcal{T}_{\mathcal{R}}^{p}$ such that $A = B \cup C$.

We also introduce some further corresponding generalized topologically closed sets, we also use the following plausible notation.

DEFINITION. For any $\kappa = \alpha, \beta, \gamma, \delta, a$ and b, we define

$$\mathcal{F}_{\mathcal{R}}^{\kappa} = \big\{ A \subseteq X : \quad A^{c} \in \mathcal{T}_{\mathcal{R}}^{\kappa} \big\}.$$

Analogously to our former results, we also prove the following two theorems.

THEOREM. For any $A \subseteq X$ the following assertions are equivalent:

(1) $A \in \mathcal{F}_{\mathcal{R}}^{\gamma}$;

(2) there exists $W \in \mathcal{F}_{\mathcal{R}}^{s}$ such that $\operatorname{int}_{\mathcal{R}}(A) \subseteq W \subseteq A$.

THEOREM. For any $A \subseteq X$ the following assertions are equivalent:

(1) $A \in \mathcal{F}_{\mathcal{R}}^{\delta}$;

(2) there exists $W \in \mathcal{F}_{\mathcal{R}}^{p}$ such that $\operatorname{int}_{\mathcal{R}}(W) \subseteq A \subseteq W$.

Moreover, analogously to our former results, we also prove the following two theorems.

THEOREM. For any $A \subseteq X$, the following assertions are equivalent:

(1) $A \in \mathcal{F}_{\mathcal{R}}^{\alpha}$; (2) $\operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(\operatorname{cl}_{\mathcal{R}}(A))\big) \subseteq A$.

THEOREM. For any $A \subseteq X$ the following assertions are equivalent:
(1) $A \in \mathcal{F}_{\mathcal{R}}^{\beta}$; (2) $\operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big)\big) \subseteq A$.

Now, by using our former results, we also prove the following

THEOREM. We have $\mathcal{F}_{\mathcal{R}}^{a} = \mathcal{F}_{\mathcal{R}}^{s} \cap \mathcal{F}_{\mathcal{R}}^{p}$.

COROLLARY. For any $A \subseteq X$, the following assertions are equivalent:
(1) $A \in \mathcal{F}_{\mathcal{R}}^{a}$; (2) $\operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big) \cup \operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big) \subseteq A$.

The latter statement can also be easily proved directly, by using only the corresponding definitions.

Moreover, by using a direct argument, we can also easily prove the following counterpart of this corollary.

THEOREM. For any $A \subseteq X$, the following assertions are equivalent:
(1) $A \in \mathcal{F}_{\mathcal{R}}^{b}$; (2) $\operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big) \cap \operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big) \subseteq A$.

DIAGRAM. For a reflexive relator $\mathcal{R}$, the following implications hold:

$$\begin{tikzcd}
 & & A \in \mathcal{T}_{\mathcal{R}}^{\beta} & & \\
A \in \mathcal{T}_{\mathcal{R}}^{\delta} \arrow[urr, Rightarrow] & & A \in \mathcal{T}_{\mathcal{R}}^{b} \arrow[u, Rightarrow] & & A \in \mathcal{T}_{\mathcal{R}}^{\gamma} \arrow[ull, Rightarrow] \\
 & A \in \mathcal{T}_{\mathcal{R}}^{s} \arrow[ur, Rightarrow] \arrow[urrr, Rightarrow] & A \in \mathcal{T}_{\mathcal{R}}^{r} \arrow[r, Rightarrow] & A \in \mathcal{T}_{\mathcal{R}}^{p} \arrow[ulll, Rightarrow] \arrow[ul, Rightarrow] & \\
 & & A \in \mathcal{T}_{\mathcal{R}}^{a} \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] & & \\
 & A \in \mathcal{T}_{\mathcal{R}}^{q} \arrow[u, Rightarrow] \arrow[uuul, Rightarrow] & A \in \mathcal{T}_{\mathcal{R}}^{\alpha} \arrow[u, Rightarrow] & A \in \mathcal{T}_{\mathcal{R}}^{ps} \arrow[u, Rightarrow] \arrow[uuur, Rightarrow] & \\
 & & A \in \mathcal{T}_{\mathcal{R}} \arrow[ul, Rightarrow] \arrow[u, Rightarrow] \arrow[ur, Rightarrow] & &
\end{tikzcd}$$

The following simple example will show only that eight implications in the above diagram are not reversible.

EXAMPLE. If $X = \{1, 2, 3\}$ and R is a relation on X such that

$$R(1) = \{1, 2\} \qquad \text{and} \qquad R(2) = R(3) = X,$$

then $\mathcal{R} = \{R\}$ is a reflexive relator on X such that:

(1) $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}}^{q} = \{\emptyset, X\}$;

(2) $\mathcal{T}_{\mathcal{R}}^{s} = \mathcal{T}_{\mathcal{R}}^{a} = \mathcal{T}_{\mathcal{R}}^{\alpha} = \big\{\emptyset, \{1, 2\}, X\big\}$;

(3) $\mathcal{T}_{\mathcal{R}}^{ps} = \mathcal{T}_{\mathcal{R}}^{p} = \mathcal{T}_{\mathcal{R}}^{b} = \mathcal{T}_{\mathcal{R}}^{\beta} = \mathcal{T}_{\mathcal{R}}^{\gamma} = \mathcal{T}_{\mathcal{R}}^{\delta} = \mathcal{P}(X) \setminus \{\{3\}\}$.

The following, more difficult example will already show that sixteen implications in the above diagram are not reversible.

EXAMPLE. If $X = \{1, 2, 3, 4\}$ and R_1 and R_2 are relations on X such that

$R_1(1) = R_1(2) = \{1, 2, 3\}$, $\qquad R_1(3) = R_1(4) = \{1, 3, 4\}$;

$R_2(1) = \{1, 2, 3\}$, $\qquad R_2(2) = \{1, 2\}$, $\qquad R_2(3) = R_2(4) = \{3, 4\}$;

then $\mathcal{R} = \{R_1, R_2\}$ is a reflexive relator on X such that:

(1) $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}}^{\alpha} = \{\emptyset, \{3, 4\}, X\}$; $\qquad$ (2) $\mathcal{T}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}} \cup \{\{2\}\}$;

(3) $\mathcal{T}_{\mathcal{R}}^{q} = \mathcal{T}_{\mathcal{R}}^{a} = \mathcal{T}_{\mathcal{R}} \cup \{\{1, 3, 4\}\}$; $\qquad$ (4) $\mathcal{T}_{\mathcal{R}}^{s} = \mathcal{T}_{\mathcal{R}}^{q} \cup \{\{1, 2\}\}$;

(5) $\mathcal{T}_{\mathcal{R}}^{p} = \mathcal{P}(X) \setminus \{\{1\}, \{1, 2\}\}$; $\qquad$ (6) $\mathcal{T}_{\mathcal{R}}^{ps} = \mathcal{T}_{\mathcal{R}}^{p} \setminus \{\{2\}\}$;

(7) $\mathcal{T}_{\mathcal{R}}^{b} = \mathcal{T}_{\mathcal{R}}^{\delta} = \mathcal{P}(X) \setminus \{\{1\}\}$; $\qquad$ (8) $\mathcal{T}_{\mathcal{R}}^{\beta} = \mathcal{T}_{\mathcal{R}}^{\gamma} = \mathcal{P}(X)$.

Note that, the two above examples, together, already show that seventeen implications in the above diagram are not reversible.

However, unfortunately, they cannot be used to show that the remaining implication $A \in \mathcal{T}_{\mathcal{R}}^{\gamma} \implies A \in \mathcal{T}_{\mathcal{R}}^{\beta}$ is also not reversible.

Minimality and Connectedness Properties in Relator Spaces

Analogously to the definition of a minimal topology, we may naturally introduce

DEFINITION. The relator $\mathcal{R}$ will be called

(1) *quasi-proximally minimal* if $\tau_{\mathcal{R}} \subseteq \{\emptyset, X\}$;

(2) *quasi-topologically minimal* if $\mathcal{T}_{\mathcal{R}} \subseteq \{\emptyset, X\}$.

Moreover, by the above definition, we also prove the following

THEOREM. The following assertions are equivalent:

(1) $\mathcal{R}$ is quasi-topologically minimal;

(2) $\mathcal{R}^{\wedge}$ is quasi-proximally minimal.

As an immediate consequence of the above theorem, we prove

COROLLARY. If $\mathcal{R}$ is topologically fine, then $\mathcal{R}$ is quasi-proximally minimal if and only if $\mathcal{R}$ is quasi-topologically minimal.

In addition to this corollary, it is also worth proving the following

THEOREM. If $\mathcal{R}$ is proximally simple, then $\mathcal{R}$ is quasi-proximally minimal if and only if $\mathcal{R}$ is quasi-topologically minimal.

Concerning quasi-minimal relators, we prove the following theorems.

THEOREM. The relator $\mathcal{R}$ is quasi-proximally minimal if and only if any one of the relators $\mathcal{R}^{\infty}$, $\mathcal{R}^{*}$, $\mathcal{R}^{\#}$ and $\mathcal{R}^{-1}$ is quasi-proximally minimal.

THEOREM. The relator $\mathcal{R}$ is quasi-topologically minimal if and only if any one of the relators $\mathcal{R}^{*}$, $\mathcal{R}^{\#}$, $\mathcal{R}^{\wedge}$ and $\mathcal{R}^{\wedge\infty}$ is quasi-topologically minimal.

Note that $\mathcal{R}^{\infty} \subseteq \mathcal{R}^{\wedge}$, and thus $\mathcal{T}_{\mathcal{R}^{\infty}} \subseteq \mathcal{T}_{\mathcal{R}}$. Therefore, if $\mathcal{R}$ is quasi-topologically minimal, then $\mathcal{R}^{\infty}$ is also quasi-topologically minimal.

Now, we can see that the properties of the quasi-topologically minimal relators, in priciple, is immediately derived from those of the quasi-proximally minimal ones.

Therefore, it is of importance to prove the following basic characterization theorem of quasi-proximally minimal relators.

THEOREM. The following assertions are equivalent:

(1) $\mathcal{R}$ is quasi-proximally minimal;

(2) $\mathcal{R} \subseteq \{X^2\}^{\partial}$; (3) $\mathcal{R}^{\infty} \subseteq \{X^2\}$;

(4) $\mathcal{R}^{\#} \subseteq \{X^2\}^{\partial}$; (5) $\mathcal{R}^{\#\infty} \subseteq \{X^2\}$.

Detailed reformulations of assertion (3) of the above theorem give the following

COROLLARY. The following assertions are equivalent:

(1) $\mathcal{R}$ is quasi-proximally minimal;

(2) for each $R \in \mathcal{R}$ we have $R^{\infty} = X^2$;

(3) for each $R \in \mathcal{R}$ and $a, b \in X$ $\exists n \in \{0\} \cup \mathbb{N}$ and a family $(x_i)_{i=0}^{n}$ in X such that $x_0 = a$, $x_n = b$ and $(x_{i-1}, x_i) \in R$ for all $i = 1, 2, \dots, n$.

From the equivalence of assertions (1) and (3) in this corollary, we see that, for Euclidean and metric spaces, our quasi-proximal minimalness coincides with the well-chainedness (chain-connectedness) studied by G. Cantor in 1883. (See Thron [**52**, p. 29] , and also Wilder [**54**] .)

While, from the equivalence of assertions (1) and (3) in the above theorem, we can see that, for uniformities and nonvoid relators, our quasi-proximal minimalness coincides with the *well-chainedness* and *proper well-chainedness* studied mainly by Levine [**29**] and Kurdics, Pataki and Száz [**20, 21, 22, 33**] .

Now, as an immediate consequence of above results, we also state

THEOREM. The following assertions are equivalent :

(1) $\mathcal{R}$ is quasi-topologically minimal ;

(2) $\mathcal{R}^{\wedge} \subseteq \{ X^2 \}^{\partial}$; (3) $\mathcal{R}^{\wedge\,\infty} \subseteq \{ X^2 \}$.

By the two above theorems, the relator $\mathcal{R}$ may be naturally called $\square$–*minimal*, for some unary operation $\square$ for relators, if $\mathcal{R}^{\square} \subseteq \{ X^2 \}$.

Moreover, in particular the relator $\mathcal{R}$ may be naturally called *quasi–*$\square$*–minimal*, for some unary operation $\square$ for relators on X, if it is $\square\,\infty$–minimal. That is, $\mathcal{R}^{\square\,\infty} \subseteq \{ X^2 \}$.

A simple reformulation of the above definition gives the following

THEOREM. The following assertions hold :

(1) $\mathcal{R}$ is quasi-proximally minimal if and only if $\mathcal{T}_{\mathcal{R}} \subseteq \{ \emptyset, X \}$;

(2) $\mathcal{R}$ is quasi-topologically minimal if and only if $\mathcal{F}_{\mathcal{R}} \subseteq \{ \emptyset, X \}$.

Concerning quasi-proximally minimal relators, we also prove

THEOREM. If $\emptyset \notin \mathcal{R}$, then the following assertions are equivalent :

(1) $\mathcal{R}$ is quasi-proximally minimal ;

(2) $X = A \cup B$ implies $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ for all $A , B \subseteq X$ with $A, B \neq \emptyset$;

(3) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ implies $B \not\subseteq A$ for all $A , B \subseteq X$ with $A \neq \emptyset$ and $B \neq X$.

THEOREM. If $\operatorname{card}(X) > 1$, then the following assertions are equivalent :

(1) $\mathcal{R}$ is quasi-topologically minimal ;

(2) $X = A \cup B$ implies $A \cap \operatorname{cl}_{\mathcal{R}}(B) \neq \emptyset$ for all $A, B \subseteq X$ with $A, B \neq \emptyset$;

(3) $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$ implies $B \not\subseteq A$ for all $A , B \subseteq X$ with $A \neq \emptyset$ and $B \neq X$.

Note that in this theorem, instead of $\operatorname{card}(X) > 1$ we may assume that $\varnothing \notin \mathcal{R}^{\wedge}$. That is, there exists $x \in X$ such that for any $R \in \mathcal{R}$ we have $R(x) \nsubseteq \varnothing$, and thus $R(x) \neq \varnothing$.

Analogously to the definition of a minimal topology, a stack (ascending family) $\mathcal{A}$ of subsets of a set X may be naturally called minimal if $\mathcal{A} \subseteq \{X\}$.

Therefore, having in mind the family $\mathcal{E}_{\mathcal{R}}$ of all fat sets generated by a relator $\mathcal{R}$, we may also naturally introduce the following

DEFINITION. The relator $\mathcal{R}$ will be called *paratopologically minimal* if

$$\mathcal{E}_{\mathcal{R}} \subseteq \{X\}.$$

THEOREM. If $\mathcal{R}$ is non-partial, then the following assertions are equivalent:

(1) $\mathcal{R}$ is paratopologically minimal;

(2) $\mathcal{R}^{\triangle} \subseteq \{X^2\}^{\partial}$; (3) $\mathcal{R}^{\triangle\infty} \subseteq \{X^2\}$.

By using the above definition, we also prove the following

THEOREM. The following assertions are equivalent:

(1) $\mathcal{R}$ is paratopologically minimal; (2) $\mathcal{R} \subseteq \{X^2\}$.

Now, we can also easily prove

THEOREM. The relator $\mathcal{R}$ is paratopologically minimal if and only if any of the relators $\mathcal{R}^{*}$, $\mathcal{R}^{\#}$, $\mathcal{R}^{\wedge}$, $\mathcal{R}^{\triangle}$ and $\mathcal{R}^{-1}$ is paratopologically minimal.

Moreover, as some useful reformulation of definition of paratopologically minimal, we also prove the following two theorems.

THEOREM. The following assertions are equivalent:

(1) $\mathcal{R}$ is paratopologically minimal; (2) $\mathcal{P}(X) \setminus \{\varnothing\} \subseteq \mathcal{D}_{\mathcal{R}}$.

THEOREM. The following assertions are equivalent:

(1) $\mathcal{R}$ is paratopologically minimal;

(2) $\operatorname{Int}_{\mathcal{R}}(B) \subseteq \{\varnothing\}$ for all $B \subseteq X$ with $B \neq X$;

(3) $\mathcal{P}(X) \setminus \{\varnothing\} \subseteq \operatorname{Cl}_{\mathcal{R}}(B)$ for all $B \subseteq X$ with $B \neq \varnothing$.

Now, as an immediate consequence of this theorem, we state

THEOREM. The following assertions are equivalent:

(1) $\mathcal{R}$ is paratopologically minimal;

(2) $\operatorname{int}_{\mathcal{R}}(B) = \varnothing$ for all $B \subseteq X$ with $B \neq X$;

(3) $X = \operatorname{cl}_{\mathcal{R}}(B)$ for all $B \subseteq X$ with $B \neq \varnothing$.

Analogously to the definition of a connected topology, we naturally introduce the following

DEFINITION. The relator $\mathcal{R}$ will be called

(1) *quasi-proximally connected* if $\tau_{\mathcal{R}} \cap \digamma_{\mathcal{R}} \subseteq \{\emptyset, X\}$;

(2) *quasi-topologically connected* if $\mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} \subseteq \{\emptyset, X\}$.

If in particular $\mathcal{R} \neq \emptyset$, then by using our former results we have $\{\emptyset, X\} \subseteq \tau_{\mathcal{R}} \cap \digamma_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}$. Therefore, in this case, we may write equalities instead of inclusions in the above definition.

THEOREM. The following assertions are equivalent :

(1) $\mathcal{R}$ is quasi-topologically connected ;

(2) $\mathcal{R}^{\wedge}$ is quasi-proximally connected .

COROLLARY. If $\mathcal{R}$ is topologically fine, then $\mathcal{R}$ is quasi-proximally connected if and only if $\mathcal{R}$ is quasi-topologically connected.

THEOREM. If $\mathcal{R}$ is proximally simple, then $\mathcal{R}$ is quasi-proximally connected if and only if $\mathcal{R}$ is quasi-topologically connected.

THEOREM. The relator $\mathcal{R}$ is quasi-proximally connected if and only if any one of the relators $\mathcal{R}^{\infty}$, $\mathcal{R}^{*}$, $\mathcal{R}^{\#}$ and $\mathcal{R}^{-1}$ is quasi-proximally connected.

From this theorem, for instance, we can see that $\mathcal{R}$ is quasi-proximally connected if and only if any one of the relators $\mathcal{R}^{\#\infty}$ and $\mathcal{R}^{\infty\#}$ is quasi-proximally connected.

THEOREM. The relator $\mathcal{R}$ is quasi-topologically connected if and only if any one of the relators $\mathcal{R}^{*}$, $\mathcal{R}^{\#}$ and $\mathcal{R}^{\wedge}$ is quasi-topologically connected.

We already know that $\mathcal{T}_{\mathcal{R}^{\infty}} \subseteq \mathcal{T}_{\mathcal{R}}$. Hence, it follows that $\mathcal{F}_{\mathcal{R}^{\infty}} \subseteq \mathcal{F}_{\mathcal{R}}$, and thus also $\mathcal{T}_{\mathcal{R}^{\infty}} \cap \mathcal{F}_{\mathcal{R}^{\infty}} \subseteq \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}$. Therefore, if $\mathcal{R}$ is quasi-topologically connected, then $\mathcal{R}^{\infty}$ is also quasi-topologically connected.

From the above, we can see that the properties of quasi-topologically connected relators can, in principle, be immediately derived from those of the quasi-proximally connected ones.

Therefore, it is of major importance to note that, by using the relator

$$\mathcal{R} \vee \mathcal{R}^{-1} = \left\{ R \cup S^{-1} : \quad R, S \in \mathcal{R} \right\},$$

we can also prove the following

THEOREM. The following assertions are equivalent :

(1) $\mathcal{R}$ is quasi-proximally connected ;

(2) $\mathcal{R} \vee \mathcal{R}^{-1}$ is quasi-proximally minimal .

As an immediate consequence of the above theorems, we state

THEOREM. The following assertions are equivalent:

(1) $\mathcal{R}$ is quasi-topologically connected;

(2) $\mathcal{R}^{\wedge} \vee \mathcal{R}^{\vee}$ is quasi-proximally minimal.

The latter two theorems show that the properties of the quasi-proximally and quasi-topologically connected relators can, in principle, be also immediately derived from those of the quasi-proximally minimal ones.

The fact that minimalness is a more important notion than connectedness was first established by Kurdics, Pataki and Száz [**20**, **22**, **33**] by using well-chainedness instead of minimalness and the relator $\mathcal{R} \triangledown \mathcal{R}^{-1}$ instead of $\mathcal{R} \vee \mathcal{R}^{-1}$.

In addition to the above theorems, we prove the following

THEOREM. If $\emptyset \notin \mathcal{R}$, then the following assertions are equivalent:

(1) $\mathcal{R}$ is quasi-proximally connected;

(2) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ and $B^c \in \operatorname{Int}_{\mathcal{R}}(A^c)$ imply $B \nsubseteq A$ for all $A, B \subseteq X$ with $A \neq \emptyset$ and $B \neq X$;

(3) $X = A \cup B$ implies that either $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ or $B \in \operatorname{Cl}_{\mathcal{R}}(A)$ for all $A, B \subseteq X$ with $A, B \neq \emptyset$.

Moreover, we can also prove the following

THEOREM. If $\operatorname{card}(X) > 1$, then the following assertions are equivalent:

(1) $\mathcal{R}$ is quasi-topologically connected;

(2) $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$ and $B^c \subseteq \operatorname{int}_{\mathcal{R}}(A^c)$ imply $B \nsubseteq A$ for all $A, B \subseteq X$ with $A \neq \emptyset$ and $B \neq X$;

(3) $X = A \cup B$ implies that either $A \cap \operatorname{cl}_{\mathcal{R}}(B) \neq \emptyset$ or $\operatorname{cl}_{\mathcal{R}}(A) \cap B \neq \emptyset$ for all $A, B \subseteq X$ with $A, B \neq \emptyset$.

The following theorem shows that the relationships between quasi-connectedness and mild continuity.

THEOREM. The following assertions are equivalent:

(1) $\mathcal{R}$ is quasi-proximally connected;

(2) $f^{-1} \circ f \notin \mathcal{R}^{\#}$ for any function f of X onto $\{0, 1\}$;

(3) $f^{-1} \circ f \notin \mathcal{R}^{\# \infty}$ for any function f of X onto $\{0, 1\}$.

From this theorem, we can immediately derive

THEOREM. The following assertions are equivalent:

(1) $\mathcal{R}$ is quasi-topologically connected ;
(2) $f^{-1} \circ f \notin \mathcal{R}^{\wedge}$ for any function f of X onto $\{0, 1\}$;
(3) $f^{-1} \circ f \notin \mathcal{R}^{\wedge\infty}$ for any function f of X onto $\{0, 1\}$.

Because of last two theorems, the relator $\mathcal{R}$ may be naturally called $\square$–*connected*, for some unary operation $\square$ for relators on X, if $f^{-1} \circ f \notin \mathcal{R}^{\square}$ for any function f of X onto $\{0, 1\}$. Moreover, in particular the relator $\mathcal{R}$ may be naturally called quasi–$\square$–connected if it is $\square\infty$–connected. Hence, by noticing that $f^{-1} \circ f = f^{-1} \circ \Delta_{\{0,1\}} \circ f$, we can see that the relator $\mathcal{R}$ is $\square$–connected (quasi–$\square$–connected) if and only if the constant functions of X to $\{0, 1\}$ can be mildly $\square$–continuous (quasi– $\square$–continuous) with respect to the relators $\mathcal{R}$ and $\left\{\Delta_{\{0,1\}}\right\}$. (Concerning continuity properties, see [50] .)

Analogously to the definition of a hyperconnected topology, we may also naturally introduce the following

DEFINITION. The relator $\mathcal{R}$ will be called

(1) *quasi-proximally hyperconnected* if $A \cap B \neq \emptyset$ for all $A, B \in \tau_{\mathcal{R}} \setminus \{\emptyset\}$;
(2) *quasi-topologically hyperconnected* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$.

REMARK. Thus, $\mathcal{R}$ is quasi-proximally (quasi-topologically) hyperconnected if and only if the family $\tau_{\mathcal{R}} \setminus \{\emptyset\} \left(\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \right)$ has a certain pairwise intersection property.

THEOREM. The following assertions are equivalent :

(1) $\mathcal{R}$ is quasi-topologically hyperconnected ;
(2) $\mathcal{R}^{\wedge}$ is quasi-proximally hyperconnected .

Moreover, we can also prove the following two theorems.

THEOREM. The relator $\mathcal{R}$ is quasi-proximally hyperconnected if and only if any one of the relators $\mathcal{R}^{\infty}$, $\mathcal{R}^{*}$ and $\mathcal{R}^{\#}$ is quasi-proximally hyperconnected.

From this theorem, for instance, we can see that $\mathcal{R}$ is quasi-proximally hyperconnected if and only if any one of the relators $\mathcal{R}^{\#\infty}$ and $\mathcal{R}^{\infty\#}$ is quasi-proximally hyperconnected.

THEOREM. The relator $\mathcal{R}$ is quasi-topologically hyerconnected if and only if any one of the relators $\mathcal{R}^{*}$, $\mathcal{R}^{\#}$, $\mathcal{R}^{\wedge}$ and $\mathcal{R}^{\wedge\infty}$ is quasi-topologically hyperconnected.

From our former results, we already know that $\mathcal{T}_{\mathcal{R}^\infty} \subseteq \mathcal{T}_{\mathcal{R}}$, and thus $\mathcal{T}_{\mathcal{R}^\infty} \setminus \{\emptyset\} \subseteq \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$. Therefore, if $\mathcal{R}$ is quasi-topologically hyperconnected, then $\mathcal{R}^\infty$ is also quasi-topologically hyperconnected.

From the above definition, we can derive the following two theorems.

THEOREM. The following assertions are equivalent:
(1) $\mathcal{R}$ is quasi-proximally hyperconnected;
(2) $A \cup B \neq X$ for all $A, B \in \digamma_{\mathcal{R}} \setminus \{X\}$;
(3) $A \setminus B \neq \emptyset$ for all $A \in \tau_{\mathcal{R}} \setminus \{\emptyset\}$ and $B \in \digamma_{\mathcal{R}} \setminus \{X\}$.

THEOREM. The following assertions are equivalent:
(1) $\mathcal{R}$ is quasi-topologically hyperconnected;
(2) $A \cup B \neq X$ for all $A, B \in \mathcal{F}_{\mathcal{R}} \setminus \{X\}$;
(3) $A \setminus B \neq \emptyset$ for all $A \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ and $B \in \mathcal{F}_{\mathcal{R}} \setminus \{X\}$.

Analogously to the definition of an ultraconnected topology, we may also naturally introduce the following

DEFINITION. The relator $\mathcal{R}$ will be called
(1) *quasi-proximally ultraconnected* if $A \cap B \neq \emptyset$ for all $A, B \in \digamma_{\mathcal{R}} \setminus \{\emptyset\}$;
(2) *quasi-topologically ultraconnected* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$.

REMARK. Thus, $\mathcal{R}$ is quasi-proximally (quasi-topologically) hyperconnected if and only if the family $\digamma_{\mathcal{R}} \setminus \{\emptyset\}$ $\big(\mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\} \big)$ has a certain pairwise intersection property.

Now, analogously to our former statements, we can also easily prove the following assertions.

THEOREM. The following assertions are equivalent:
(1) $\mathcal{R}$ is quasi-topologically ultraconnected;
(2) $\mathcal{R}^\wedge$ is quasi-proximally ultraconnected.

THEOREM. The relator $\mathcal{R}$ is quasi-proximally ultraconnected if and only if any one of the relators $\mathcal{R}^\infty$, $\mathcal{R}^*$ and $\mathcal{R}^\#$ is quasi-proximally ultraconnected.

From this theorem, we can see that $\mathcal{R}$ is quasi-proximally connected if, for instance, any one of the relators and $\mathcal{R}^{\#\infty}$ and $\mathcal{R}^{\infty\#}$ is quasi-proximally ultraconnected.

THEOREM. The relator $\mathcal{R}$ is quasi-topologically ultraconnected if and only if any one of the relators $\mathcal{R}^{*}$, $\mathcal{R}^{\#}$, $\mathcal{R}^{\wedge}$ and $\mathcal{R}^{\wedge\infty}$ is quasi-topologically ultraconnected.

From our former results, we already know that $\mathcal{T}_{\mathcal{R}^{\infty}} \subseteq \mathcal{T}_{\mathcal{R}}$. Hence, we can infer that $\mathcal{F}_{\mathcal{R}^{\infty}} \subseteq \mathcal{F}_{\mathcal{R}}$, and thus $\mathcal{F}_{\mathcal{R}^{\infty}} \setminus \{\emptyset\} \subseteq \mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$. Therefore, if $\mathcal{R}$ is quasi-topologically ultraconnected, then $\mathcal{R}^{\infty}$ is also quasi-topologically ultraconnected.

THEOREM. The following assertions are equivalent:

(1) $\mathcal{R}$ is quasi-proximally ultraconnected;

(2) $A \cup B \neq X$ for all $A, B \in \tau_{\mathcal{R}} \setminus \{X\}$;

(3) $A \setminus B \neq \emptyset$ for all $A \in \mathscr{F}_{\mathcal{R}} \setminus \{\emptyset\}$ and $B \in \tau_{\mathcal{R}} \setminus \{X\}$.

THEOREM. The following assertions are equivalent:

(1) $\mathcal{R}$ is quasi-topologically ultraconnected;

(2) $A \cup B \neq X$ for all $A, B \in \mathcal{T}_{\mathcal{R}} \setminus \{X\}$;

(3) $A \setminus B \neq \emptyset$ for all $A \in \mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$ and $B \in \mathcal{T}_{\mathcal{R}} \setminus \{X\}$.

This theorem shows that our quasi-topologically ultraconnectedness also extends the *strong connectedness* of Levine [**25**] studied also by Leuschen and Sims [**23**].

Namely, it can be easily seen that assertion (2) of the last theorem can be reformulated in the form that $X = A \cup B$, together with $A, B \in \mathcal{T}_{\mathcal{R}}$, implies that either $A = X$ or $B = X$.

Now, in addition to the above theorems, we can also easily prove the following

THEOREM. The following assertions are equivalent:

(1) $\mathcal{R}$ is quasi-proximally ultraconnected;

(2) $\mathcal{R}^{-1}$ is quasi-proximally hyperconnected.

This theorem shows that, in contrast to the independence of quasi-topological ulraconnectedness and quasi-topological hyperconnedtedness [**39**, p. 29], the quasi-proximal ultraconnectedness is not completely independent of the quasi-proximal hyperconnectedness.

Furthermore, because of a reformulation of the definition of a hyperconnected topology mentioned earlier in the introduction, we may also naturally introduce the following

DEFINITION. The relator $\mathcal{R}$ will be called *hyperconnected* if

$$\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}.$$

REMARK. This property can be expressed in a more instructive form that the identity function Δ_X of X is *fatness reversing*.
Therefore, some of the forthcoming results can be greatly generalized according to the ideas of a former paper [**51**] of Száz.

Note that if in particular $\mathcal{R}$ is a relator to $\emptyset$, then because of $\mathcal{R} \subseteq \mathcal{P}(X^2) = \mathcal{P}(\emptyset) = \{\emptyset\}$ we have either $\mathcal{R} = \emptyset$ or $\mathcal{R} = \{\emptyset\}$.

Hence, by using our former results, we can see that either $\mathcal{E}_{\mathcal{R}} = \emptyset$ and $\mathcal{D}_{\mathcal{R}} = \{\emptyset\}$, or $\mathcal{E}_{\mathcal{R}} = \{\emptyset\}$ and $\mathcal{D}_{\mathcal{R}} = \emptyset$. Therefore, in the latter case $\mathcal{R}$ is not hyperconnected despite that in both cases it is paratopologically minimal.

By using the corresponding definitions, we can prove the following

THEOREM. The following assertions are equivalent:
(1) $\mathcal{R}$ is hyperconnected;
(2) $R(x) \in \mathcal{D}_{\mathcal{R}}$ for all $x \in X$ and $R \in \mathcal{R}$;
(3) $R(x) \cap S(y) \neq \emptyset$ for all $x, y \in X$ and $R, S \in \mathcal{R}$.

According to [**43**], the relator $\mathcal{R}$ may be called *semi-directed* if (3) holds. Thus, a relator is hyperconnected if and only if it is semi-directed.

Moreover, the relator $\mathcal{R}$ may be called *quasi-directed* if $R(x) \cap S(y) \in \mathcal{E}_{\mathcal{R}}$ holds for all $x, y \in X$ and $R, S \in \mathcal{R}$. Thus, a non-partial, quasi-directed relator is semi-directed.

From the above theorem, we can immediately derive

COROLLARY. If $\mathcal{R}$ is hyperconnected, then $\mathcal{R}$ is non-partial.

Moreover, for instance if we have $R = \{(x, y) \in \mathbb{R}^2 : \ x \leq y\}$ and $S = \{(x, y) \in \mathbb{R}^2 : \ |x - y| < r\}$ for some $r > 0$, then by using Theorem we can also at once see that $\mathcal{R} = \{R\}$ is hyperconnected, but $\mathcal{S} = \{S\}$ is not hyperconnected.

However, it is now more important to note that, by using Theorem and the plausible notation $\mathcal{R}^{-1} \circ \mathcal{R} = \{S^{-1} \circ R : \ R, S \in \mathcal{R}\}$, we can also easily prove some more instructive characterizations of hyperconnected relators.

THEOREM. The following assertions are equivalent:
(1) $\mathcal{R}$ is hyperconnected; (2) $X^2 = S^{-1} \circ R$ for all $R, S \in \mathcal{R}$;
(3) $\mathcal{R}^{-1} \circ \mathcal{R} \subseteq \{X^2\}$; (4) $X^2 = \bigcap \mathcal{R}^{-1} \circ \mathcal{R}$.

By using the equality $\rho_{\mathcal{R}} = \bigcap \mathcal{R}^{-1}$, assertion (4) can be written in the shorter form that $X^2 = \rho_{\mathcal{R}^{-1} \circ \mathcal{R}}$.

Moreover, by using the cross product of relations [48], assertion (4) can also reformulated in the shorter form that $\Delta_X \in \mathcal{E}_{\mathcal{R} \boxtimes \mathcal{R}}$.

Now, analogously to our former we can also easily prove

THEOREM. The relator $\mathcal{R}$ is hyperconnected if and only if any one of the relators $\mathcal{R}^*$, $\mathcal{R}^\#$, $\mathcal{R}^\wedge$ and $\mathcal{R}^\triangle$ is hyperconnected.

However, it is now more important to note that by using our former results, we can also prove the following

THEOREM. If $\mathcal{R}$ is non-partial, then the following assertions are equivalent:

(1) $\mathcal{R}$ is hyperconnected;

(2) $\mathcal{R}^\triangle$ is quasi-proximally connected;

(3) $\mathcal{R}^\triangle$ is quasi-topologically connected.

This theorem shows that the properties of non-partial hyperconnected relators can, in principle, be immediately derived from those of the quasi-proximally connected ones.

For instance, from our former results, we can derive the following

THEOREM. If $\mathcal{R}$ is non-partial, then the following assertions are equivalent:

(1) $\mathcal{R}$ is hyperconnected; (2) $\mathcal{R}^\triangle \vee \mathcal{R}^\triangledown$ is quasi-proximally minimal;

(3) $\mathcal{R}^\triangle \vee \mathcal{R}^\triangledown \subseteq \{X^2\}^\partial$; (4) $\big(\mathcal{R}^\triangle \vee \mathcal{R}^\triangledown\big)^\infty \subseteq \{X^2\}$.

By using some basic properties of the families $\mathcal{E}_\mathcal{R}$ and $\mathcal{D}_\mathcal{R}$, we can also easily prove the following two theorems.

THEOREM. The following assertions are equivalent:

(1) $\mathcal{R}$ is hyperconnected;

(2) $A^c \notin \mathcal{E}_\mathcal{R}$ for all $A \in \mathcal{E}_\mathcal{R}$; (3) $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{E}_\mathcal{R}$.

THEOREM. The following assertions are equivalent:

(1) $\mathcal{R}$ is hyperconnected;

(2) $A \in \mathcal{D}_\mathcal{R}$ or $A^c \in \mathcal{D}_\mathcal{R}$ for all $A \subseteq X$;

(3) $A \in \mathcal{D}_\mathcal{R}$ or $B \in \mathcal{D}_\mathcal{R}$ whenever $X = A \cup B$.

In addition to our former results, we proves some particular theorems on minimal and connected relators

THEOREM. If $\mathcal{R}$ is weakly proximal, then the following assertions are equivalent:

(1) $\mathcal{R}$ is paratopologically minimal;

(2) $\mathcal{R}$ is quasi-proximally minimal;
(3) $\mathcal{R}$ is quasi-topologically minimal.

Quite similarly, we can also prove the following theorem which will now be rather proved as a consequence of the above theorem.

THEOREM. If $\mathcal{R}$ is topological, then the following assertions are equivalent:
(1) $\mathcal{R}$ is paratopologically minimal;
(2) $\mathcal{R}$ is quasi-topologically minimal.

However, it is now more important to note that, in addition, to the above theorem, we can also prove the following

THEOREM. If $\mathcal{R}$ is topological, then the following assertions are equivalent:
(1) $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{D}_{\mathcal{R}}$;
(2) $\mathcal{R}$ is hyperconnected; (3) $\mathcal{R}$ is quasi-topologically hyperconnected .

The following two theorems show that quasi-ultraconnected relators are less important than the quasi-hyperconnected ones.

THEOREM. If $\mathcal{R}$ is T_1–separating and $\operatorname{card}(X) > 1$, then $\mathcal{R}$ is not quasi-topologically ultraconnected.

THEOREM. If $\mathcal{R}$ is weakly topological, then the following assertions are equivalent:
(1) $\mathcal{R}$ is quasi-topologically ultraconnected;
(2) $\operatorname{cl}_{\mathcal{R}}(x) \cap \operatorname{cl}_{\mathcal{R}}(y) \neq \emptyset$ for all $x, y \in X$;
(3) $\operatorname{cl}_{\mathcal{R}}\big(A\big) \cap \operatorname{cl}_{\mathcal{R}}(B) \neq \emptyset$ for all $\emptyset \neq A, B \subseteq X$.

Analogously to the definition of a door topology by Kelley [18], we may naturally introduce the following

DEFINITION. The relator $\mathcal{R}$ will be called
(1) a *quasi-proximally door relator* if $\mathcal{P}(X) = \tau_{\mathcal{R}} \cup \digamma_{\mathcal{R}}$;
(2) a *quasi-topologically door relator* if $\mathcal{P}(X) = \mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}}$.

Now, we establish the following two theorems.

THEOREM. The following assertions are equivalent :
(1) $\mathcal{R}$ is a quasi-proximally door ;
(2) $\mathcal{P}(X) \setminus \tau_{\mathcal{R}} \subseteq \digamma_{\mathcal{R}}$; (3) $\mathcal{P}(X) \setminus \digamma_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$.

THEOREM. The following assertions are equivalent :
(1) $\mathcal{R}$ is a quasi-topologically door ;
(2) $\mathcal{P}(X) \setminus \mathcal{T}_{\mathcal{R}} \subseteq \mathcal{F}_{\mathcal{R}}$; (3) $\mathcal{P}(X) \setminus \mathcal{F}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$.

Now, we also easily see that $\mathcal{R}$ is quasi-topologically door if and only if, for any $A \subseteq X$, we have either $A \in \mathcal{T}_{\mathcal{R}}$ or $A^c \in \mathcal{T}_{\mathcal{R}}$.

Because of a reformulation of the definition of a superset topology by Levine [26], we may also naturally introduce the following

DEFINITION. The relator $\mathcal{R}$ will be called
(1) *quasi-proximally superset relator* if $\mathcal{E}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$;
(2) *quasi-topologically superset relator* if $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$.

We state the following two theorems.

THEOREM. The following assertions are equivalent:
(1) $\mathcal{R}$ is quasi-proximally superset;
(2) $\mathcal{E}_{\mathcal{R}} \setminus \tau_{\mathcal{R}} = \emptyset$;
(3) $\mathcal{P}(X) = \digamma_{\mathcal{R}} \cup \mathcal{D}_{\mathcal{R}}$;
(4) $\mathcal{P}(X) \setminus \digamma_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$;
(5) $\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}} \subseteq \digamma_{\mathcal{R}}$.

THEOREM. The following assertions are equivalent:
(1) $\mathcal{R}$ is quasi-topologically superset;
(2) $\mathcal{E}_{\mathcal{R}} \setminus \mathcal{T}_{\mathcal{R}} = \emptyset$;
(3) $\mathcal{P}(X) = \mathcal{F}_{\mathcal{R}} \cup \mathcal{D}_{\mathcal{R}}$;
(4) $\mathcal{P}(X) \setminus \mathcal{F}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$;
(5) $\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}} \subseteq \mathcal{F}_{\mathcal{R}}$.

Concerning superset relators, we also prove the following

THEOREM. If $\mathcal{R}$ is non-partial, then the following assertions hold:
(1) $\mathcal{R}$ is quasi-proximally superset if and only if $\mathcal{E}_{\mathcal{R}} = \tau_{\mathcal{R}} \setminus \{\emptyset\}$;
(2) $\mathcal{R}$ is quasi-topologically superset if and only if $\mathcal{E}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$.

Analogously to the definition of a submaximal topology by Bourbaki [4], we may also naturally introduce the following

DEFINITION. The relator $\mathcal{R}$ will be called
(1) *quasi-proximally submaximal* if $\mathcal{D}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$;
(2) *quasi-topologically submaximal* if $\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$.

Thus, we can also easily prove the following two theorems.

THEOREM. The following assertions are equivalent:
(1) $\mathcal{R}$ is quasi-proximally submaximal;
(2) $\mathcal{D}_{\mathcal{R}} \setminus \tau_{\mathcal{R}} = \emptyset$;
(3) $\mathcal{P}(X) = \digamma_{\mathcal{R}} \cup \mathcal{E}_{\mathcal{R}}$;
(4) $\mathcal{P}(X) \setminus \digamma_{\mathcal{R}} \subseteq \mathcal{E}_{\mathcal{R}}$;
(5) $\mathcal{P}(X) \setminus \mathcal{E}_{\mathcal{R}} \subseteq \digamma_{\mathcal{R}}$.

THEOREM. The following assertions are equivalent:
(1) $\mathcal{R}$ is quasi-topologically submaximal;

(2) $\mathcal{D}_{\mathcal{R}} \setminus \mathcal{T}_{\mathcal{R}} = \emptyset$; (3) $\mathcal{P}(X) = \mathcal{F}_{\mathcal{R}} \cup \mathcal{E}_{\mathcal{R}}$;

(4) $\mathcal{P}(X) \setminus \mathcal{F}_{\mathcal{R}} \subseteq \mathcal{E}_{\mathcal{R}}$; (5) $\mathcal{P}(X) \setminus \mathcal{E}_{\mathcal{R}} \subseteq \mathcal{F}_{\mathcal{R}}$.

Because of a reformulation of the definition of a resolvable topology by Hewitt [**15**], we may also naturally introduce the following

DEFINITION. The relator $\mathcal{R}$ will be called *resolvable* if $\mathcal{D}_{\mathcal{R}} \not\subseteq \mathcal{E}_{\mathcal{R}}$.

The above definition is reformulated in the following

THEOREM. The following assertions are equivalent:

(1) $\mathcal{R}$ is resolvable; (2) $\mathcal{D}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}} \neq \emptyset$;

(3) there exists $A \in \mathcal{D}_{\mathcal{R}}$ such that $A^c \in \mathcal{D}_{\mathcal{R}}$.

Now, by calling the relator $\mathcal{R}$ to be *irresolvable* if it is not resolvable, we also easily establish the following

THEOREM. The following assertions are equivalent:

(1) $\mathcal{R}$ is irresolvable; (2) $\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{E}_{\mathcal{R}}$; (3) $\mathcal{D}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}} = \emptyset$.

COROLLARY. The following assertions are equivalent:

(1) $\mathcal{D}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}}$; (2) $\mathcal{R}$ is irresolvable and hyperconnected.

Now, by using the above definition, we also prove the following theorem

THEOREM. The following assertions are equivalent:

(1) $\mathcal{R}$ is irresolvable;

(2) $A^c \notin \mathcal{D}_{\mathcal{R}}$ for all $A \in \mathcal{D}_{\mathcal{R}}$; (3) $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{D}_{\mathcal{R}}$.

Moreover, we also prove the following two theorems.

THEOREM. The following assertions are equivalent:

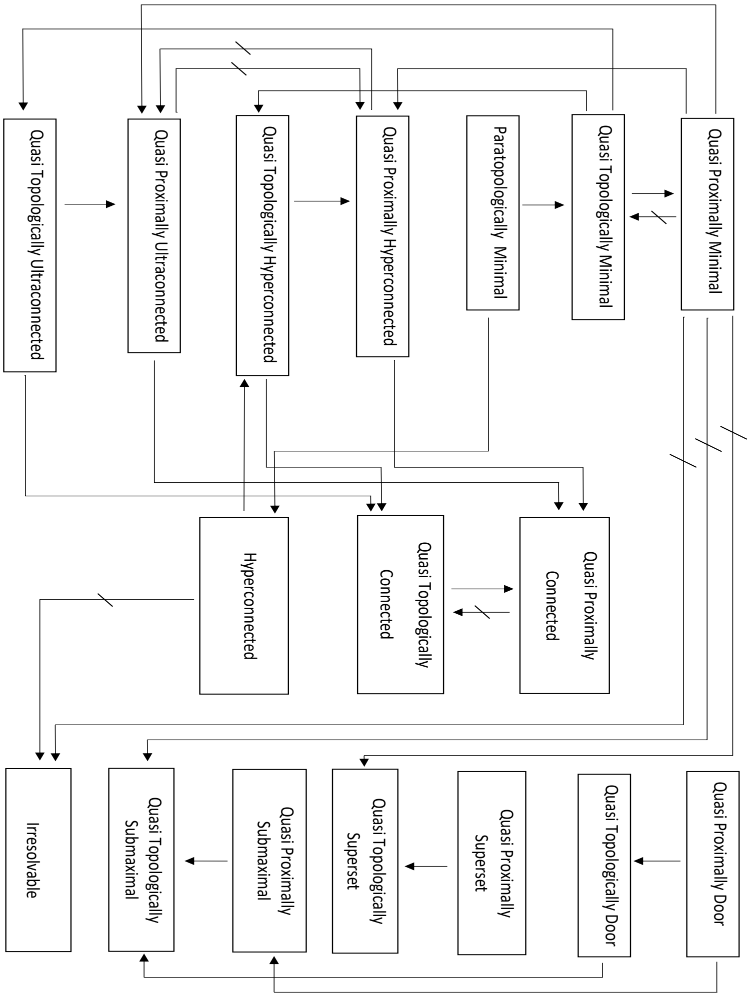
(1) $\mathcal{R}$ is irresolvable;

(2) $A \in \mathcal{E}_{\mathcal{R}}$ or $A^c \in \mathcal{E}_{\mathcal{R}}$ for all $A \subseteq Y$;

(3) $A \in \mathcal{E}_{\mathcal{R}}$ or $B \in \mathcal{E}_{\mathcal{R}}$ whenever $Y = A \cup B$.

THEOREM. The relator $\mathcal{R}$ is resolvable (irresolvable) if and only if any one of the relators $\mathcal{R}^{*}$, $\mathcal{R}^{\#}$, $\mathcal{R}^{\wedge}$ and $\mathcal{R}^{\triangle}$ is resolvable (irresolvable).

Diagram. For a nonvoid relator $\mathcal{R}$ on a nonvoid set X, the following implications are true:



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List of Talks

(1) *Two Important Operations in Generalized Uniform Spaces*, 2^{nd} International Hazar Scientific Researches Conference, Khazar University, Baku, Azerbaijan, April 10-12, 2021.

(2) *Minimality Properties of the Family $\mathcal{T}_{\mathcal{R}}^{p}$ in Relator Spaces*, 64^{th} Annual Online Meeting of the Australian Mathematical Society, University of New England, Australia, December 8–11, 2020.

(3) *Closure and Interior Operations in Relator Spaces*, 9^{th} Interdisciplinary Doctoral Conference, Doctoral Student Association of the University of Pécs, Hungary, November 27–28, 2020.

(4) *An Important Property of Topologically Semi-open and Preopen sets*, 34^{th} International Summer Conference on Real Functions Theory, Slovak Academy of Sciences, Slovakia, September 7–13, 2020.

(5) *Two Important Operations in Relator Spaces*, Síkfőkút Seminar of the Department of Analysis, University of Debrecen, Síkfőkút, Hungary, August 28–30, 2020.

(6) *Generalized Open Sets Should Not Also Be Studied Without Generalized Uniform Spaces*, The 16^{th} International Students' Conference on Analysis, Síkfőkút, Hungary, February 1–4, 2020.

(7) *Ordinary and Extreme Connectedness Properties of Relator Spaces*, Analysis Research Seminar, Institute of Mathematics, University of Debrecen, Hungary, March 20, 2019.

(8) *A New Type of Weakly Commutative Groups*, The 2^{nd} International Scientific Conference, The University of Zakho, Kurdistan Region of Iraq, April 18 – 20, 2017.

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Foreign language scientific articles in international journals (2)

2. Rassias, T. M., **Salih, M. M.**, Száz, Á.: Characterizations of Generalized Topologically Open Sets in Relator Spaces.
   Montes Taurus J. Pure Appl. Math. 3 (3), 39-94, 2021. ISSN: 2687-4814.
3. **Salih, M. M.**, Száz, Á.: Generalizations of some ordinary and extreme connectedness properties of topological spaces to relator spaces.
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List of other publications

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   In: Mathematical Analysis, Optimization, Approximation and Applications, World Scientific, Singapore, [Accepted by publisher], 1-96, 2022.

<u>Foreign language scientific articles in international journals</u> (3)

5. Abdullah, H. N., Ahmed, D., **Salih, M. M.**: Using fibonacci number to integrate 2x2 and 3x3 matrices.
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