Metrizability of Affine Connections

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Abstract

An affine connection Γ on a vector bundle $\eta = (E, \pi, M, V)$ of a rank r is called Riemann metrizable if there exists on M a Riemann metric which preserves the scalar product of vector fields parallel displaced according to Γ. Γ determines a connection G in a bundle, where M is fibered by the manifold of the ellipsoids of $R^r = \pi^{-1}$, $x \in M$. We prove that Γ is Riemann metrizable iff G is integrable.

An analogous result is deduced in the case, where η is replaced by a Finsler vector bundle, Γ means a Finsler connection, and the metric is a Finsler metric.

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1 Introduction

We consider a vector bundle $\eta = (E, \pi, M, V)$ over the n-dimensional base manifold M with an r-dimensional real vector space V as typical fiber, where E is the total space and $\pi : E \to M$ is the projection operator. An affine connection H_η in η is given by a special splitting $T_zE = V_zE \oplus H_zE$, $z \in E$ and it is determined locally by the connection coefficients $\Gamma_\beta{}^\alpha{}_i(x)$; $\alpha, \beta, \ldots = 1, \ldots, r$; $i, j, \ldots = 1, \ldots, n$, where $x \in M$ has the local coordinates x^i. H_η or Γ is called *Riemann metrizable* if there exists a Euclidean scalar product $\langle\ ,\ \rangle$ in each fiber $\pi^{-1}(x)$, i.e. a symmetrical bilinear form $g(x)$, in local coordinates $\langle \xi, \zeta \rangle = g_{\alpha\beta}(x)\xi^\alpha(x)\zeta^\beta(x)$, such that the length of the parallel translated $\|{}_\Gamma P_C \xi_0\|_g$ of a vector $\xi_0 \in \pi^{-1}(x_0)$ along any curve $C(t) \subset M$, $C(t_0) = x_0$ is constant, i.e. if the connection Γ is compatible with the Riemannian metric g. $g(x_0)$ is equivalent with an ellipsoid $\mathcal{E}(§_\prime) : \}_{\alpha\beta}(§_\prime)\xi^\alpha\xi^\beta = \infty$ in $\pi^{-1}(x_0)$ called *indicatrix*. ${}_\Gamma P_C$ establishes a linear mapping $\pi^{-1}(x(t_0)) \to \pi^{-1}(x(t))$. Γ is *metrizable* if there exists a field $\mathcal{E}(§)$ such that from $\xi_0 \in \mathcal{E}_\prime$ follows ${}_\Gamma P_{C(t)}\xi_0 \in \mathcal{E}(§(\sqcup))$, $\forall \mathcal{C}(\sqcup) \subset \mathcal{M}$. Indicatrices play the role of the unit sphere.

The most simple case is $r = n$. If $\Gamma_j{}^i{}_h(x)$ is symmetrical and metrizable by a $g(x)$, then Γ is the Levi-Civita connection $\overset{g}{\Gamma}$ of the Riemannian manifold $V_n = (M, g)$.

Denoting the set of the Levi-Civita connections for the different g by $\{\overset{g}{\Gamma}\}$ and supposing the symmetry $\Gamma_j{}^i{}_h(x) = \Gamma_h{}^i{}_j(x)$ the question is whether $\Gamma \in \{\overset{g}{\Gamma}\}$. — Riemann metrizability of affine connections has been investigated by many authors from different points of view. I mention here only [1], [4], [5], [6], [9], [12].

A *Finsler space* $F_n = (M, \mathcal{L})$ on the manifold M is given by the smooth fundamental function $\mathcal{L} : \mathcal{TM} \to \mathcal{R}^+$; $(x, y) \mapsto \mathcal{L}(\S, \dagger)$, $y \in T_x M$ which is supposed to be first order positively homogeneous: $\mathcal{L}(\S, \lambda\dagger) = |\lambda|\mathcal{L}(\S, \dagger)$, $\lambda \in R$. Its indicatrix is given by $I(x_0) = \{y \mid \mathcal{L}(\S_/, \dagger) = \infty\} \subset \mathcal{T}_{\S_/}\mathcal{M}$ (the convexity of I is mostly also supposed). Giving of F_n is equivalent to giving of $\{I(x)\}$. Then an affine metrical connection should satisfy that from $y_0 \in I(x_0)$ follows ${}_\Gamma P_C y_0 \in I(x_1)$, $x_1 \in C(t_1)$ (this could be denoted by ${}_\Gamma P_C I(x_0) = I(x_1)$), while ${}_\Gamma P_C$ is an affine mapping. However, this is impossible in general, e.g. if $I(x_0)$ is an ellipsoid and $I(x_1)$ is not so. This necessitates the introduction of the so called Finsler vector fields which are sections of a vector bundle $\zeta = (E, \pi, TM, V^n)$, in components $\xi^i(x, y)$ with the property $\xi^i(x, \lambda y) = \xi^i(x, y)$, $\lambda \in R$, $\lambda y \neq 0$. The set $\{(x_0, \lambda y_0) \mid \lambda \in R,\ \lambda y_0 \neq 0\}$ is geometrically a point x_0 and the direction of y_0 in $T_{x_0}M$; this is called a *line-element.* So Finsler vectors are defined in line-elements. The length (the norm) of such a vector is defined by $g_{ij}(x, y)\xi^i(x, y)\xi^j(x, y) := \|\xi(x, y)\|^2$, where $g_{ij} := \dfrac{1}{2}\dfrac{\partial^2 \mathcal{L}^\in}{\partial y^i \partial y^j}$ and hence $g_{ij}(x, \lambda y) = g_{ij}(x, y)$. In an $F_n = (M, \mathcal{L})$, g_{ij} is derived from $\mathcal{L}$. A more general structure is $F_n = (M, g)$, called *generalized Finsler space*, where we start directly with the metric tensor $g_{ij}(x, y)$.

An affine connection Γ in the Finsler vector bundle ζ can be given locally by the connection coefficients $F_j{}^i{}_k(x, y)$, $V_j{}^i{}_h(x, y)$ in the form $\Gamma\xi = \xi - d_\Gamma\xi$, where

$$d_\Gamma \xi^i(x, y) = F_j{}^i{}_k(x, y)\xi^j(x, y)dx^k + V_j{}^i{}_k(x, y)\xi^j(x, y)dy^k. \tag{1}$$

Γ is metrizable if there exists a scalar product $g_{ij}(x, y)$ in each $\pi^{-1}(x, y)$ such that $\|{}_\Gamma P_C \xi_0\| =$ constant for any curve $C(t) \subset M$.

2 Connection in μ

We want to find a new, geometric condition for the Riemann metrizability of a vector bundle $\eta = (E, \pi, M, V^r)$ endowed with the affine connection H_η given by $\Gamma_\beta{}^\alpha{}_i(x)$. First we derive from H_η an affine connection H_μ in $\mu = (E_\mu, \pi_\mu, M, V^{r^2})$, and then from H_μ a connection H_ν in the bundle $\nu = (E_\nu, \pi_\nu,$
$M, \mathbf{E})$, where $\mathbf{E}$ is the manifold of the ellipsoids in $\pi^{-1}(x) \cong V^r$ centered at the origin O of V^r.

Let us consider a canonical coordinate system (x^i, v^α) in $\pi^{-1}(U) \subset E$, where $U \subset M$ is a coordinate neighbourhood of $x \in M$ and v^α are components of $v \in \pi^{-1} \cong V^r$. Similarly we have local coordinates (x^i, y^a) in $\pi_\mu^{-1}(U) \subset E_\mu$, where y^a, $a = 1, \dots, r^2$ are components of $y \in \pi_\mu^{-1}(x) \cong V^{r^2}$. Let $\overset{\alpha}{v} \in \pi^{-1}(x) \cong V^r$, $\alpha, \beta = 1, \dots, r$ be r vectors with components $(\overset{\alpha}{v})^\beta$. Since any integer a $(1 \leq a \leq r^2)$ can uniquely be represented in the form $a = (\alpha - 1)r + \beta$, and conversely, any pair α, β uniquely determines such an a and thus

$$y^a = (\overset{\alpha}{v})^\beta, \quad a = (\alpha - 1)r + \beta \tag{2}$$

determines a 1:1 mapping between $\pi_\mu^{-1}(x)$ and the vector r-tuples $(\overset{1}{v},\dots,\overset{r}{v})$ which can be considered as elements of $\overset{r}{\oplus}\,\pi^{-1}(x)\cong\overset{r}{\oplus}\,V^r$.

Having an affine connection H_η in η with local connection coefficients $\Gamma_\beta{}^\alpha{}_i(x)$, we obtain for the parallel translated of v from x to $x+dx$

$$_\Gamma P_{x,x+dx}v(x)=v(x)-d_\Gamma v(x),\quad d_\Gamma v^\beta(x)=\Gamma_\sigma{}^\beta{}_i(x)v^\sigma dx^i.$$

Then we define an affine connection H_μ in μ with local coefficients $G_b{}^a{}_i(x)$ by

$$(3)\qquad d_G y:=(d_\Gamma\,\overset{1}{v},\dots,d_\Gamma\,\overset{r}{v}),\qquad \begin{array}{l} y=(\overset{1}{v},\dots,\overset{r}{v})\\ d_\Gamma(\overset{\alpha}{v})^\beta=\Gamma_\sigma{}^\beta{}_i(x)(\overset{\alpha}{v})^\sigma dx^i.\end{array}$$

$G_b{}^a{}_i$ can be expressed explicitely by $\Gamma_\beta{}^\alpha{}_i$ as follows:

$$(4)\qquad \begin{aligned} d_G y^a &= G_b{}^a{}_i(x)y^b dx^i\\ &= d_G y^{(\alpha-1)r+\beta}=G_{(\kappa-1)r+\lambda}{}^{(\alpha-1)r+\beta}{}_i(x)\,y^{(\kappa-1)r+\lambda}dx^i,\end{aligned}$$

since $a=(\alpha-1)r+\beta$, $b=(\kappa-1)r+\lambda$. By (3) and (2) we get

$$(5)\qquad \begin{aligned} d_G y^{(\alpha-1)r+\beta} &= d_\Gamma(\overset{\alpha}{v})^\beta=\Gamma_\sigma{}^\beta{}_i(x)(\overset{\alpha}{v})^\sigma dx^i=\\ &= \Gamma_\sigma{}^\beta{}_i(x)y^{(\alpha-1)r+\sigma}dx^i.\end{aligned}$$

From (4) and (5) we obtain

$$\begin{aligned} G_{(\kappa-1)r+\lambda}{}^{(\alpha-1)r+\beta}{}_i(x)\,y^{(\kappa-1)r+\lambda} &= \Gamma_\sigma{}^\beta{}_i(x)\delta^\alpha_\kappa\delta^\sigma_\lambda y^{(\kappa-1)r+\lambda}=\\ &= \Gamma_\lambda{}^\beta{}_i(x)\delta^\alpha_\kappa y^{(\kappa-1)r+\lambda}\end{aligned}$$

and hence

$$G_{(\kappa-1)r+\lambda}{}^{(\alpha-1)r+\beta}{}_i(x)=\delta^\alpha_\kappa\Gamma_\lambda{}^\beta{}_i(x).$$

3 Connection in ν

An ellipsoid $\mathcal{E}$ in $\pi^{-1}(x)\cong V^r$ centered at the origin O of V^r has the equation $a_{\alpha\beta}v^\alpha v^\beta=1$, $a_{\alpha\beta}=a_{\beta\alpha}$, $\mathrm{Det}|a_{\alpha\beta}|>0$. The set $\{\mathcal{E}\}=\mathbf{E}$ can be given a natural manifold structure, namely each $\mathcal{E}$ can be identified with the coefficients $a_{\alpha\beta}$ which correspond to a point of R^{r^2}. Hence $\mathbf{E}$ can be identified with a variety of the Euclidean space R^{r^2}. Thus $\nu=(E_\nu,\pi_\nu,B,\mathbf{E})$ is a fiber bundle.

Now we want to derive from the H_μ determined by the affine connection H_η a connection H_ν in ν : $H_\eta\Rightarrow H_\mu\Rightarrow H_\nu$. — Let $y=(\overset{1}{v},\dots,\overset{r}{v})\in\pi_\mu^{-1}(x)\subset E_\mu$ be such that $\overset{1}{v},\dots,\overset{r}{v}$ are linearly independent vectors in $\pi^{-1}(x)$. From now on, in this section y denotes elements of E_μ with this independence property. The set of these (x,y)-s will be denoted by $E_{\overset{*}{\mu}}$ and the corresponding bundle by $\overset{*}{\mu}=(E_{\overset{*}{\mu}},\pi_{\overset{*}{\mu}},M,V_*^{r^2})$. We remark that $V_*^{r^2}$ is no vector space, and $\pi_{\overset{*}{\mu}}$ is a restriction of π_μ to $E_\mu^*\subset E_\mu$. H_μ is equivalent with the splitting $T_uE_\mu=V_uE_\mu\oplus H_uE_\mu$, $u\in E_\mu$. The restriction of an affine connection H_μ to $E_{\underset{\mu}{*}} \subset E_\mu$ is also a connection in $E_{\underset{\mu}{*}}$, i.e. $H_\mu \subset E_{\underset{\mu}{*}}$ if $u \in E_{\underset{\mu}{*}} \subset E_\mu$. This is so, because H_η takes by parallel translation linearly independent vectors of $\pi^{-1}(x)$ into linearly independent vectors again. Also, $H_{\underset{\mu}{*}}$ can be extended by continuity to a H_μ, and if $H_{\underset{\mu}{*}}$ is a restriction of an affine connection H_μ, then its extension yields this H_μ.

The vectors $\overset{\alpha}{v}$ of a y can be considered as a system of conjugate axes of an ellipsoid $\mathcal{E} \in \pi_\nu^{-\infty}(\S)$ centered at the origin O, and we order this $\mathcal{E}$ to y. Doing this with every (x, y) we obtain a strong bundle mapping

$$\rho : E_{\underset{\mu}{*}} \to E_\nu, \qquad \pi_{\underset{\mu}{*}}^{-1}(x) \to \pi_\nu^{-1}(x), \quad y \mapsto \mathcal{E}.$$

The inverse $\rho^{-1}(\mathcal{E}) = \{\dagger_\prime, \dagger_\infty, \ldots, \dagger, \ldots\}$ is an infinite set consisting of $y_0 = (\overset{1}{v}_0, \ldots, \overset{r}{v}_0)$, $y_1 = (\overset{1}{v}_1, \ldots, \overset{r}{v}_1), \ldots, y = (\overset{1}{v}, \ldots, \overset{r}{v}), \ldots$ such that every system $\overset{1}{v}_0, \ldots, \overset{r}{v}_0; \overset{1}{v}_1, \ldots, \overset{r}{v}_1; \ldots; \overset{1}{v}, \ldots, \overset{r}{v}; \ldots$ forms conjugate axes of an ellipsoid $\mathcal{E}$. Elements of $\rho^{-1}(\mathcal{E})$ can be generated from a single element, e.g. from y_0 as follows: Let V_0^r be a Euclidean vector space with an orthonormed base $\overset{\alpha}{e}$ and $a : \pi^{-1}(x) \to V_0^r$ an affine mapping taking $\overset{\alpha}{v}_0$ into $\overset{\alpha}{e}$. Then the set $\{\overset{\alpha}{v} = a^{-1} \circ f \circ a \ \overset{\alpha}{v}_0,\ \alpha = 1, \ldots, r \mid f \in O(r)\}$ produces all vector systems $y = (\overset{1}{v}, \ldots, \overset{r}{v})$ of $\rho^{-1}(\mathcal{E})$, where $O(r)$ denotes the group of rotations of V_0^r. This induces a classification of $\pi_{\underset{\mu}{*}}^{-1}(x)$ into equivalence classes, and ρ is a $1:1$ mapping between the equivalence classes and the ellipsoids.

H_μ takes $\pi_\mu^{-1}(x)$ into $\pi_\mu^{-1}(x+dx)$ and so it takes $y \in \pi_{\underset{\mu}{*}}^{-1}(x)$ into $\hat{y} \in \pi_{\underset{\mu}{*}}^{-1}(x+dx)$. However, according to (3), H_μ is defined via H_η, and in such a way that the images $\hat{y}_0, \hat{y}_1, \ldots \hat{y}, \ldots$ by H_μ of the elements of an equivalence class $\{y_0, y_1, \ldots, y, \ldots\}$ (i.e. of conjugate axes systems of an ellipsoid $\mathcal{E}$) form again an equivalence class in $\pi_{\underset{\mu}{*}}^{-1}(x+dx)$ (i.e. $\hat{y}_0, \hat{y}_1, \ldots, \hat{y}, \ldots$ are conjugate axes systems of an ellipsoid again). This is shown on the diagram

$$\begin{array}{ccc} \rho(x)\{y_0, y_1, \ldots, y, \ldots\} & = & \mathcal{E}(\S) \in \pi_\nu^{-\infty}(\S) \\ \downarrow H_\mu & & \downarrow H_\nu \\ \rho(x+dx)\{\hat{y}_0, \hat{y}_1, \ldots, \hat{y}, \ldots\} & = & \hat{\mathcal{E}}(x+dx) \in \pi_\nu^{-1}(x+dx). \end{array} \tag{6}$$

It means that $H_\mu : \pi_{\underset{\mu}{*}}^{-1}(x) \to \pi_{\underset{\mu}{*}}^{-1}(x+dx)$ preserves equivalence classes. Thus

$$\rho \circ H_\mu \circ \rho^{-1} : \pi_\nu^{-1}(x) \to \pi_\nu^{-1}(x+dx)$$

yields a connection H_ν in ν (This fact is discussed in more detail in [10], [11]).

If H_ν is integrable at least for one $\mathcal{E}_\prime \in \pi_\nu^{-\infty}(\S_\prime)$ and $\mathcal{E}(\S), \mathcal{E}(\S_\prime) = \mathcal{E}_\prime$ is the integral manifold, then $\mathcal{E}(\S)$ can be considered as indicatrix $I(x)$ and $g_{\alpha\beta}(x)$ in the equation $g_{\alpha\beta}(x)v^\alpha v^\beta = 1$ of $\mathcal{E}(\S)$ as metric tensor. Any v_0 leading to a point of $\mathcal{E}_\wr : \sqsubseteq_\prime \in \mathcal{E}_\prime$ can be an axe of a conjugate axes system of $\mathcal{E}_\prime$. Then, according to our construction, the parallel translated v of v_0 according to H_η along a curve $C \subset M$ from x_0 to x is an element of $\mathcal{E}(\S)$:

$${}_{H_\eta}P_{C;x_0,x} v_0 = v \in {}_{H_\nu}P_{C;x_0,x} \mathcal{E}_\prime = \mathcal{E}(\S),$$

and hence

$$\|v_0\|_{g(x_0)} = \|v\|_{g(x)}.$$

We remark that v depends on the path C joining x_0 and x, but $\mathcal{E}(\S)$ does not. — This means: if H_ν is integrable, then H_η is metrizable.

The converse is obvious. If H_η is metrical with respect to $g(x)$, then $\mathcal{E}(\S) := \mathcal{I}(\S)$ is an integral manifold of H_ν.

Thus we obtain the

Theorem. *The affine connection H_η of a vector bundle η is Riemann metrizable iff the constructed connection H_ν in a bundle ν fibered with ellipsoids is integrable.*

4 Coefficients of H_ν

We want to determine the connection coefficients of H_ν. H_ν orders to the ellipsoid $\mathcal{E}(\S)$

$$a_{\alpha\beta}(x)v^\alpha v^\beta = 1 \in \pi_\nu^{-1}(x) \tag{7}$$

the ellipsoid $\hat{\mathcal{E}}(x+dx)$

$$a_{\alpha\beta}(x+dx)v^\alpha(x+dx)v^\beta(x+dx) = 1 \in \pi_\nu^{-1}(x+dx). \tag{8}$$

According to the definition (construction) of H_ν this last equation is satisfied by the parallel translated with respect to H_η of $v^\alpha(x)$, i.e. by $v^\alpha(x+dx) = v^\alpha(x) - {\Gamma_\sigma}^\alpha{}_i(x)v^\sigma(x)dx^i + o(dx^i)$. (Since we work with linear connections, $o(dx^i)$, i.e. higher order terms in dx^i, can be omitted.) Then the parallel translated of $a_{\alpha\beta}(x)$ according to H_ν are the $a_{\alpha\beta}(x+dx)$ appearing in (8). Denoting the connection coefficients of H_ν by $M_{\alpha\beta i}(x, a_{\kappa\lambda})$ we obtain from (8)

$$(a_{\alpha\beta} + M_{\alpha\beta i}(x, a_{\kappa\lambda})dx^i)(v^\alpha - {\Gamma_\sigma}^\alpha{}_i v^\sigma dx^i)(v^\beta - {\Gamma_\sigma}^\beta{}_i v^\sigma dx^i) = 1$$

or

$$a_{\alpha\beta}v^\alpha v^\beta + \left[M_{\alpha\beta i} - a_{\kappa\lambda}({\Gamma_\beta}^\lambda{}_i\delta_\alpha^\kappa + {\Gamma_\alpha}^\kappa{}_i\delta_\beta^\lambda)\right] v^\alpha v^\beta dx^i + o(dx^i) = 1.$$

By (7) the right hand side drops out with $a_{\alpha\beta}v^\alpha v^\beta$. The remaining expression must vanish for every $v \in \mathcal{E}(\S)$ and for every dx^i. Thus, omitting $o(dx^i)$, we get

$$M_{\alpha\beta i}(x, a_{\kappa\lambda}) = ({\Gamma_\beta}^\lambda{}_i\delta_\alpha^\kappa + {\Gamma_\alpha}^\kappa{}_i\delta_\beta^\lambda)a_{\kappa\lambda}.$$

This means that $M_{\alpha\beta i}(x, a_{\kappa\lambda})$ is linear in $a_{\kappa\lambda}$, i.e. H_ν is an affine connection and its connection coefficients are

$$M_{\alpha\beta}{}^{\kappa\lambda}{}_i(x) = {\Gamma_\alpha}^\kappa{}_i(x)\delta_\beta^\lambda + {\Gamma_\beta}^\lambda{}_i(x)\delta_\alpha^\kappa. \tag{9}$$

We remark that these coefficients are symmetric in the sense that $M_{\alpha\beta}{}^{\kappa\lambda}{}_i = M_{\beta\alpha}{}^{\lambda\kappa}{}_i$. Thus the symmetry of $a_{\alpha\beta}(x)$ implies the symmetry of $a_{\alpha\beta}(x+dx) = a_{\alpha\beta}(x) + M_{\alpha\beta}{}^{\kappa\lambda}{}_i(x)a_{\kappa\lambda}dx^i$ too, which are the coefficients of $\hat{\mathcal{E}}(x+dx)$.

The condition of the absolute parallelism of $a_{\alpha\beta}(x)$ with respect to H_ν is

$$\frac{\partial a_{\alpha\beta}}{\partial x^i} = -M_{\alpha\beta}{}^{\kappa\lambda}{}_i(x)a_{\kappa\lambda}(x).$$

This is integrable iff

$$T_{\alpha\beta}{}^{\kappa\lambda}{}_{ij}(x)a_{\kappa\lambda}(x) = 0$$

$$T_{\alpha\beta}{}^{\kappa\lambda}{}_{ij} \equiv \left(\frac{\partial M_{\alpha\beta}{}^{\kappa\lambda}{}_i}{\partial x^j} - M_{\alpha\beta}{}^{\mu\nu}{}_i M_{\mu\nu}{}^{\kappa\lambda}{}_j\right)_{[i,j]}$$

has a solution for $a_{\kappa\lambda}$ with positive determinant. We find that

$$T_{\alpha\beta}{}^{\kappa\lambda}{}_{ij} = R_\alpha{}^\kappa{}_{ij}\delta^\lambda_\beta + R_\beta{}^\lambda{}_{ij}\delta^\kappa_\alpha,$$

where R is the curvature tensor of $\Gamma_\beta{}^\alpha{}_i(x)$.

5 Finsler vector bundles

Considering a Finsler vector bundle $\zeta = (E, \pi, TM, V^n)$ and a connection Γ with connection coefficients $F_j{}^i{}_h(x,y)$, $V_j{}^i{}_h(x,y)$ we have (1). In this case the base manifold TM has dimension $2n$. Its coordinates can be denoted by u^A, $A = 1, \dots, 2n$; $u^i = x^i$, $u^{n+i} = y^i$. $\mathcal{E}(\S, \dagger)$ has the equation $a_{ij}(x,y)\xi^i\xi^j = 1$, and the equation of $\hat{\mathcal{E}}(x+dx)$ is

$$a_{ij}(x+dx, y+dy)\xi^i(x+dx, y+dy)\xi^j(x+dx, y+dy) = 1.$$

Here

$$a_{ij}(x+dx, y+dy) = a_{ij}(x) + M_{ij}{}^{rs}{}_h(x,y)a_{rs}(x,y)dx^h + M_{ij}{}^{rs}{}_{n+k}(x,y)a_{rs}dy^h.$$

Contrasting with (9), here the last index of M runs from 1 to $2n$ the other indices from 1 to n. Considerations and calculations similar to those done above yield

$$\begin{aligned} M_{ij}{}^{rs}{}_h &= F_j{}^s{}_h\delta^r_i + F_i{}^r{}_h\delta^s_j \\ M_{ij}{}^{rs}{}_{n+k} &= V_j{}^s{}_k\delta^r_i + V_i{}^r{}_k\delta^s_j, \end{aligned}$$

and furthermore

$$\begin{aligned} T_{ij}{}^{rs}{}_{kh} &= {}^F R_i{}^r{}_{kh}\delta^s_j + {}^F R_j{}^s{}_{kh}\delta^r_i \\ T_{ij}{}^{rs}{}_{n+k\, n+h} &= {}^V R_i{}^r{}_{kh}\delta^s_j + {}^V R_j{}^s{}_{kh}\delta^r_i, \end{aligned}$$

where ${}^F R$ and ${}^V R$ are formed from $F_j{}^s{}_i$ and $V_j{}^s{}_i$ resp. like common curvature tensors. Finally

$$\begin{aligned} T_{ij}{}^{rs}{}_{n+h\,k} = {}& \frac{\partial M_{ij}{}^{rs}{}_{n+h}}{\partial x^k} - \frac{\partial M_{ij}{}^{rs}{}_k}{\partial y^h} + (V_j{}^s{}_k F_s{}^c{}_h - F_j{}^s{}_k V_s{}^c{}_h)\delta^b_i + \\ & + V_j{}^c{}_k F_i{}^b{}_h - F_j{}^c{}_k V_i{}^b{}_h + V_i{}^b{}_k F_j{}^c{}_h - F_i{}^b{}_k V_j{}^c{}_h + (V_i{}^r{}_k F_r{}^b{}_h - F_i{}^r{}_k V_r{}^b{}_h)\delta^c_j. \end{aligned}$$

One can use other connections, e.g. a pre-Finsler connection $F\Gamma(F_j{}^i{}_k,\ N^i{}_j, V_j{}^i{}_h)$ and h- and v-covariant derivatives

$$\xi^i{}_{|k} = \frac{\partial \xi^i}{\partial x^k} - \frac{\partial \xi^i}{\partial y^r}N^r{}_k + F_j{}^i{}_k\xi^j$$

$$\xi^i|_k = \frac{\partial \xi^i}{\partial y^k} + V_j{}^i{}_k\xi^j.$$

In this case (1) becomes

$$d_\Gamma \xi^i = (F_j{}^i{}_k - V_j{}^i{}_r N^r{}_k)\xi^j dx^k + V_j{}^i{}_k \xi^j dy^k,$$

or

$$d_\Gamma \xi^i = \left[(F_j{}^i{}_k - V_j{}^i{}_r F_s{}^r{}_k y^s)dx^k + V_j{}^i{}_k dy^k\right]\xi^j$$

if $F\Gamma$ is without deflection. These lead to other formulae for $M_{ij}{}^{rs}{}_A$ and $T_{ij}{}^{rs}{}_{AB}$. If $F_j{}^i{}_k$ and $V_j{}^i{}_k$ are symmetric, $F\Gamma$ is without deflection and metrizable, then $F\Gamma$ is the Cartan connection.

Finally we mention still another affine connection introduced by M. Matsumoto [7], [8] (see also [2], [3]) which is an ordinary affine connection derived from a Finsler connection $F\Gamma(F_j{}^i{}_k, N^i{}_j, V_j{}^i{}_k)$. Starting with an $F\Gamma$ and a nonvanishing vector field $Y(x)$ which depends on the point x only

$$\underline{\Gamma}_j{}^i{}_k(x) := F_j{}^i{}_k(x, Y(x)) + V_j{}^i{}_r(x, Y(x))\left(\frac{\partial Y^r}{\partial x^k} + Y^s(x) F_s{}^r{}_k(x, Y(x))\right) \tag{10}$$

turn out to be connection coefficients of an ordinary affine connection. Using the vector field $Y(x)$ one can associate to any Finsler vector field $\xi^i(x,y)$ an ordinary vector field $\underline{\xi}^i(x) := \xi^i(x, Y(x))$. Then there exists a nice relation among the covariant derivative $\underline{\xi}^i{}_{;k}$ constructed with $\underline{\Gamma}$, and the h- and v-covariant derivatives with respect to $F\Gamma$, namely

$$\underline{\xi}^i{}_{;k} = \left[\xi^i{}_{|k} + \xi^i|_k\left(\frac{\partial Y^r}{\partial x^k} + Y^s F_s{}^r{}_k\right)\right]\Big|_{y=Y(x)}.$$

Given a $\underline{\Gamma}$ and a $Y(x)$, there are many $F\Gamma$ which satisfy (10). Then we can use our method to search for metrizable ones among these $F\Gamma$, e.g. for such, where $F\Gamma$ satisfies (10) with the given $\underline{\Gamma}$ and $Y(x)$ and $g_{ij|k} = g_{ij}|_k = 0$ with respect to this $F\Gamma$.

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