Spectral Synthesis Problems on Locally Compact Groups

László Székelyhidi
Institute of Mathematics, University of Debrecen,
e-mail: `lszekelyhidi@gmail.com`

Abstract

Spectral analysis and spectral synthesis problems are formulated on noncommutative locally compact groups and solved on compact groups.

1 Introduction

In this paper $\mathbb{C}$ denotes the set of complex numbers. If G is a locally compact group with identity e, then $\mathcal{C}(G)$ denotes the locally convex topological vector space of all continuous complex valued functions defined on G equipped with the point-wise operations and with the topology of uniform convergence on compact sets.

For each y in G the symbol τ_y denotes the *right translation operator by y* which is defined on each f in $\mathcal{C}(G)$ by the formula

$$\tau_y f(x) = f(xy)\,,$$

whenever x is in G. The operator Δ_y is defined by

$$\Delta_y = \tau_y - 1\,,$$

where 1 is the identity operator τ_e. It is called the *right difference operator by y*. The iterates $\Delta_{y_1,y_2,\dots,y_n}$ are defined by the obvious way:

$$\Delta_{y_1,y_2,\dots,y_n} = \Delta_{y_n}\Delta_{y_{n-1}}\dots\Delta_{y_1}\,,$$

further we write Δ_y^n for $\Delta_{y,y,\dots,y}$, which is the n-th iterate of Δ_y.

The research was supported by the Hungarian National Foundation for Scientific Research (OTKA), Grant No. T-043080.

Keywords and phrases: spectral analysis, spectral synthesis, locally compact groups

AMS (2000) Subject Classification: 43A45, 43A60, 43A65

A linear subspace V of $\mathcal{C}(G)$ is called *right invariant*, if $\tau_y f$ belongs to V, whenever f is in V. A right invariant closed linear subspace of $\mathcal{C}(G)$ is called a *right variety*. We can analogously define the concepts of *left translation operator, left invariant subspace, left variety.* A right variety, which is also a left variety is called a *two-sided variety*, or simply a *variety.* As each finite dimensional subspace of $\mathcal{C}(G)$ is closed, hence each finite dimensional translation invariant subspace is a variety.

If f is any function in $\mathcal{C}(G)$, then $\tau(f)$ denotes the smallest variety containing f, the *variety generated by f*.

A nonzero right (left, or two-sided) variety in $\mathcal{C}(G)$ is called *decomposable*, if it is the sum of two subvarieties, both of them different from it. Otherwise it is called *indecomposable.* Clearly, if V is a finite dimensional variety, which is the sum of two subvarieties, both of them different from it, then both summands have smaller dimension than that of V.

Spectral analysis and spectral synthesis deal with the description of different varieties. Recently several new results on spectral analysis and spectral synthesis have been found on discrete Abelian groups. In this paper we make an attempt to formulate and study the basic problems of spectral analysis and spectral synthesis in the noncommutative non-discrete setting. In particular, we prove that spectral synthesis holds over compact groups.

2 Exponential monomials on Abelian groups

If G is a locally compact Abelian group, then the building blocks of spectral analysis and spectral synthesis are the exponential monomials. A continuous homomorphism of G into the multiplicative group of nonzero complex numbers is called an *exponential*, and a continuous homomorphism of G into the additive group of complex numbers is called an *additive function.* A complex valued function on G having the form $x \mapsto P\big(a_1(x), a_2(x), \dots, a_n(x)\big)$ is called a *polynomial*, if $P : \mathbb{C}^n \to \mathbb{C}$ is a complex polynomial and $a_1, a_2, \dots, a_n : G \to \mathbb{C}$ are additive functions. Hence polynomials are the elements of the function algebra generated by the constants and the additive functions. It is well-known (see e.g. [16], Section 3.2) that for any nonzero polynomial p there exists a nonnegative integer n such that

$$\Delta_y^{n+1} p(x) = 0$$

holds for each x, y in G. The smallest n with this property is called the *degree* of the polynomial p. Clearly, p is constant if and only if $\Delta_y p(x) = 0$ for each x, y in G, an if p is nonconstant and its degree is n, then $\Delta_y p$ is of degree at most $n-1$ for each y in G. If p is of degree $n \geq 1$, then $\Delta_y^n p$ is a nonzero constant.

A function which is a product of a polynomial and an exponential is called an *exponential monomial.* Therefore the general form of exponential monomials is

$$\varphi(x) = p(x)m(x)\,, \tag{1}$$

where $m : G \to \mathbb{C}$ is an exponential and $p : G \to \mathbb{C}$ is a polynomial.

Theorem 1. *Let G be a locally compact Abelian group and let V be a variety in $\mathcal{C}(G)$. If the exponential monomial* (1) *belongs to V and y is in G, then $\Delta_y p \cdot m$ belongs to V, too.*

Proof. For any y in G we have

$$\Delta_y p(x)m(x) = m(y^{-1})\varphi(xy) - \varphi(x)\,,$$

which implies our statement. □

Corollary 1. *Let G be a locally compact Abelian group and let V be a variety in $\mathcal{C}(G)$. If the nonzero exponential monomial* (1) *belongs to V, then the exponential m belongs to V, too.*

Proof. The statement follows from the previous theorem by iterating Δ_y. □

Linear combinations of exponential monomials are called *exponential polynomials.* Hence the general form of an exponential polynomial is

$$\varphi(x) = \sum_{i=1}^{n} p_i(x)m_i(x)\,, \tag{2}$$

where $m_i : G \to \mathbb{C}$ is an exponential and $p_i : G \to \mathbb{C}$ is a polynomial for $i = 1, 2, \dots, n$. From Lemma 4.3 on p.41. in [16] it follows that if the exponentials m_i in (2) are different, then this representation is unique. Moreover, it follows from Lemma 4.2 on p. 40. in [16] that if φ in (2) belongs to a variety V and the exponentials m_i in (2) are different and the polynomials p_i are different from zero, then all these exponentials belong to V, too.

Theorem 2. *Let G be a locally compact Abelian group and let V be a variety in $\mathcal{C}(G)$. If the exponential polynomial* (2) *belongs to V, where the p_i's are nonzero polynomials and the m_i's are different exponentials, then all the exponential monomials $p_i \cdot m_i$ belong to V, too ($i = 1, 2, \dots, n$).*

Proof. In the proof we shall use multi-index notation. We recall that if n is a positive integer, then an n-dimensional multi-index α is an element of $\mathbb{N}^n$, that is,

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)\,,$$

where the α_i's are nonnegative integers. Ordering of multi-indices is defined componentwise.

First of all we fix a linearly independent set of additive functions on G and we always suppose that all polynomials of the form

$$p(x) = P\big(a_1(x), a_2(x), \dots, a_k(x)\big)$$

are built up from these additive functions.

It is easy to see that for any linearly independent additive functions $a_1, a_2, \dots, a_k$, for any different exponentials $m_1, m_2, \dots, m_n$ and for any k-dimensional multi-index α the set of functions

$$\{a_1^{\alpha_1}\, a_2^{\alpha_2} \dots a_k^{\alpha_k} \cdot m_i \,|\, \alpha \in \mathbb{N}^k, i = 1, 2, \dots, n\}$$

is linearly independent.

For any complex polynomial P in k variables let $\deg_i P$ denote the degree of P in the i-th variable for $i = 1, 2, \dots, k$ and let $\deg_P$ denote the multi-index

$$\deg_P = (\deg_1 P, \deg_2 P, \dots, \deg_k P)\,.$$

We call $\deg_P$ the *multi-degree* of the polynomial P.

Now let φ be an exponential polynomial of the form

$$\varphi(x) = \sum_{i=1}^{n} p_i(x) m_i(x) = \sum_{i=1}^{n} P_i\big(a_1(x), a_2(x), \dots, a_k(x)\big) m_i(x)\,,$$

where the P_i's are nonzero complex polynomials in k variables, the m_i's are different exponentials and the a_j's are linearly independent additive functions. Suppose that φ belongs to the variety V and let y be in G arbitrary. Then, by the Taylor–formula we have

$$\varphi(x+y) = \tag{3}$$

$$= \sum_{i=1}^{n} \sum_{\alpha \leq \deg_{P_i}} \frac{1}{\alpha!} \partial^\alpha P_i\big(a_1(x), \dots, a_k(x)\big) m_i(x) a_1(y)^{\alpha_1} \dots a_k(y)^{\alpha_k} m_i(y)$$

for each x in G. Here $\alpha! = \alpha_1!\alpha_2! \dots \alpha_k!$ and $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_k^{\alpha_k}$. As the functions

$$y \mapsto a_1(y)^{\alpha_1} \dots a_k(y)^{\alpha_k} m_i(y)$$

for different multi-indices α and $i = 1, 2, \dots, n$ are linearly independent, hence there exist elements y_j for $j = 1, 2, \dots, N$ in G such that the matrix

$$\big(a_1(y)^{\alpha_1} \dots a_k(y)^{\alpha_k} m_i(y)\big)_{i=1,2,\dots,n; \alpha \leq \deg_{P_i}}$$

is regular. Here N is the number of these functions for $i = 1, 2, \dots, n$ and $\alpha \leq \deg_{P_i}$. Substituting y_j for y in (3) we get a system of linear equations with the previous matrix for the unknowns $\partial^\alpha P_i\big(a_1(x), \dots, a_k(x)\big) m_i(x)$ for any fixed x in G. This means, by Cramer's rule, that these functions are linear combinations of translates of φ, hence they belong to V. In particular, with $\alpha = (0, 0, \dots, 0)$ we have that $x \mapsto P_i\big(a_1(x), \dots, a_k(x)\big) m_i(x)$ belongs to V and the theorem is proved. $\square$

Now we characterize exponential monomials on Abelian groups.

Theorem 3. *Let G be a locally compact Abelian group and φ a function in $\mathcal{C}(G)$. The function φ is an exponential monomial if and only if $\tau(\varphi)$ is finite dimensional and indecomposable.*

Proof. By the results of [14] (see also [13], [10], [11], [16]) it follows that if $\tau(\varphi)$ is finite dimensional, then φ is an exponential polynomial, that is, it has the form

$$\varphi(x) = \varphi_1(x) + \varphi_2(x) + \cdots + \varphi_n(x)$$

for each x in G with some exponential monomials $\varphi_1, \varphi_2, \ldots, \varphi_n$ of the form

$$\varphi_i(x) = p_i(x) m_i(x) \qquad i = 1, 2, \ldots, n\,,$$

where the p_i's are nonzero polynomials and the m_i's are different exponentials. If $n \geq 2$ and $V_1 = \tau(\varphi_1)$ and $V_2 = \tau(\varphi_2) + \cdots + \tau(\varphi_n)$, then clearly V_2 is a translation invariant subspace, thus — by finite dimensionality — it is closed, hence it is a variety. Further, φ belongs to $V_1 + V_2$, therefore

$$\tau(\varphi) = V_1 + V_2\,,$$

where both V_1 and V_2 are different from $\tau(\varphi)$, which contradicts to the indecomposability of $\tau(\varphi)$. Hence φ is an exponential monomial.

Conversely, let φ be an exponential monomial of the form

$$\varphi(x) = p(x) m(x)$$

for each x in G, with some nonzero polynomial p and exponential m. In this case p has the form $p(x) = P\big(a_1(x), a_2(x), \ldots, a_k(x)\big)$ for each x in G with some complex polynomial P and additive functions a_i ($i = 1, 2, \ldots, k$). By (3) one can see that $\tau(\varphi)$ is linearly generated by the functions

$$x \mapsto \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_k^{\alpha_k} P\big(a_1(x), a_2(x), \ldots, a_k(x)\big) m(x)$$

for each multi-indices $\alpha \leq \deg_P$, hence it is of finite dimension.

Suppose that $\tau(\varphi) = V_1 + V_2$, where V_1, V_2 are subvarieties of $\tau(\varphi)$, both different from it. Clearly φ cannot belong to either V_1 or V_2. This means that for any exponential monomial $q \cdot m$ in V_1 and V_2 the degree of q is less than the degree of p, hence it is impossible to represent $p \cdot m$ as a sum of an element of V_1 and one element of V_2. This contradiction shows that $\tau(\varphi)$ is indecomposable and the theorem is proved. $\square$

We can prove another characterization of exponential monomials.

Theorem 4. *Let G be a locally compact Abelian group and φ a function in $\mathcal{C}(G)$. The function φ is an exponential monomial if and only if φ belongs to a finite dimensional indecomposable variety.*

Proof. The necessity is obvious by the previous theorem. Suppose that φ belongs to the finite dimensional and indecomposable variety V. Then, exactly like in the previous theorem, φ is an exponential polynomial, that is, it has the form

$$\varphi(x) = \varphi_1(x) + \varphi_2(x) + \cdots + \varphi_n(x)$$

for each x in G with some exponential monomials $\varphi_1, \varphi_2, \ldots, \varphi_n$, and the proof can be finished like in the previous theorem. □

3 Spectral analysis and spectral synthesis over Abelian groups

Let G be a locally compact Abelian group and let V be a variety in $\mathcal{C}(G)$. We say that *spectral analysis* holds in V, if V contains an exponential. We say that *spectral synthesis* holds in V, if all exponential monomials in V span a dense subvariety in V. If V is a nonzero variety, then, by Corollary 1, spectral synthesis in V implies spectral analysis in V. We say that *spectral analysis*, respectively, *spectral synthesis* holds over G, if spectral analysis, respectively, spectral synthesis holds in every nonzero variety in $\mathcal{C}(G)$. Clearly, the locally compactness of the group is not necessary to formulate the above concepts, but in this paper we consider this setting only.

Now we shortly summarize the most relevant results on spectral analysis and spectral synthesis over discrete Abelian groups.

The first important result on spectral synthesis is due to L. Schwartz, who proved his celebrated theorem in his 1947 paper [15] (see also [5] and [6]).

Theorem 5. *Spectral synthesis holds over the reals with the usual topology.*

Actually this is the only general result on spectral synthesis over non-discrete locally compact groups. On discrete Abelian groups the first general result is due to M. Lefranc from 1958 in [9].

Theorem 6. *Spectral synthesis holds over finitely generated free Abelian groups.*

Finitely generated free Abelian groups have the form $\mathbb{Z}^k$. In 1965 R. J. Elliot published the paper [2] including a theorem on spectral synthesis over arbitrary discrete Abelian groups, but in 1986 Z. Gajda observed that the proof of Elliot's theorem was defective.

In his 1975 paper [3] D. I.Gurevič showed that spectral synthesis fails to hold over $\mathbb{R}^2$.

In 1991 the present author published a monograph about the possible applications of spectral analysis and spectral synthesis over discrete Abelian groups. In 2001 we proved the following theorem (see [17]).

Theorem 7. *Spectral analysis holds over discrete commutative torsion groups.*

The following 2001 result in [18] extends Lefranc's result.

Theorem 8. *Spectral synthesis holds over finitely generated discrete Abelian groups.*

In our 2004 paper [19] we presented a counterexample for the above mentioned result of R. J. Elliot and we proved the following theorem.

Theorem 9. *Spectral synthesis fails to hold over any discrete Abelian group with infinite torsion free rank.*

In the same paper we formulated the following conjecture: spectral synthesis holds over a discrete Abelian group if and only if its torsion free rank is finite.

In 2005 we proved the following theorem in [1].

Theorem 10. *Spectral synthesis holds over discrete commutative torsion groups.*

Concerning spectral analysis over discrete Abelian groups in 2005 M. Laczkovich and G. Székelyhidi settled the problem (see [7]).

Theorem 11. *Spectral analysis holds over a discrete Abelian group if and only if its torsion free rank is less than the continuum.*

In our paper [8] we have verified the above mentioned conjecture by proving the following theorem.

Theorem 12. *Spectral synthesis holds over a discrete Abelian group if and only if its torsion free rank is finite.*

Now we reformulate the concepts of spectral analysis and spectral synthesis in order to extend their meaning for noncommutative groups. In fact, it turns out that exponentials and exponential monomials - defined in the same way as in the commutative case - are not the adequate building blocks for spectral analysis and spectral synthesis over noncommutative groups.

Theorem 13. *Let G be a locally compact Abelian group and V a nonzero variety in $\mathcal{C}(G)$. Then spectral analysis holds in V if and only if there is a nonzero exponential monomial in V.*

Proof. The necessity is obvious, and the sufficiency follows from Theorem 1. $\square$

Let G be a locally compact Abelian group and V a nonzero variety in $\mathcal{C}(G)$. We say that there are *sufficiently many* exponential monomials in V if the linear hull of all exponential monomials in V is dense in V. The following theorem is just a reformulation of the definition in order to see clearly the relation to and the contrast with Theorem 13.

Theorem 14. *Let G be a locally compact Abelian group and V a nonzero variety in $\mathcal{C}(G)$. Then spectral synthesis holds in V if and only if there are sufficiently many exponential monomials in V.*

Theorem 15. *Let G be a locally compact Abelian group. Then spectral synthesis — hence also spectral analysis — holds in every finite dimensional nonzero variety in $\mathcal{C}(G)$.*

Proof. This is a consequence of Theorem 2. □

The following theorem is an immediate consequence of our previous results. We note that by the sum of a set of subvarieties we mean the closure of the set of all finite sums, where the summands are taken from the given subvarieties.

Theorem 16. *Let G be a locally compact Abelian group and let V be a variety in $\mathcal{C}(G)$. Spectral analysis holds in V if and only if V has a nonzero finite dimensional subvariety. Spectral synthesis holds in V if and only if V is the sum of its finite dimensional subvarieties.*

Another corollary is formulated below.

Corollary 2. *Let G be a locally compact Abelian group. Spectral analysis holds over G if and only if each variety in $\mathcal{C}(G)$ has a nonzero finite dimensional subvariety. Spectral synthesis holds over G if and only if each variety in $\mathcal{C}(G)$ is the sum of finite dimensional varieties.*

4 Spectral analysis and spectral synthesis over locally compact groups

After the previous considerations we can define exponential monomials on any — not necessarily commutative — locally compact groups. We call the function φ in $\mathcal{C}(G)$ an *exponential monomial*, if φ belongs to a finite dimensional indecomposable variety. By Theorem 4 in the commutative case this coincides with the previous concept of exponential monomials. Using this definition, we say that *spectral analysis holds in a variety*, if there is a nonzero exponential monomial in the variety, and *spectral synthesis holds in a variety*, if the linear hull of the set of all exponential monomials in the variety is dense in the variety, or, in other words, there are sufficiently many exponential monomials in the variety. The analogue of Theorem 15 follows.

Theorem 17. *Let G be a locally compact group. Then spectral synthesis — hence also spectral analysis — holds for each finite dimensional variety in $\mathcal{C}(G)$.*

Proof. We have to show that if V is a finite dimensional variety, then it is the sum of indecomposable subvarieties. Indeed, if V is decomposable, say $V = V_1 + V_2$, and V_1, V_2 are indecomposable, then we are ready. If any of them is decomposable, then we apply the same process for it. As the dimensions decrease at each step, finally we arrive at a decomposition of V into the sum of indecomposable subvarieties. Together with the second statement this proves or theorem. $\square$

Now the analogue of Theorem 16 can be stated.

Theorem 18. *Let G be a locally compact group and let V be a variety in $\mathcal{C}(G)$. Spectral analysis holds in V if and only if V has a nonzero finite dimensional subvariety. Spectral synthesis holds in V if and only if V is the sum of its finite dimensional subvarieties.*

Proof. This is a consequence of the previous theorem. $\square$

Corollary 3. *Let G be a locally compact group. Spectral analysis holds over G if and only if each variety in $\mathcal{C}(G)$ has a nonzero finite dimensional subvariety. Spectral synthesis holds over G if and only if each variety in $\mathcal{C}(G)$ is the sum of finite dimensional varieties.*

5 Spectral synthesis on compact groups

In this section we prove that spectral synthesis holds on compact groups. Our method is based on the theory of almost periodic functions.

Following [4], given a group G the function $f : G \to \mathbb{C}$ is called *almost periodic*, if the set $\{\tau_y f : y \in G\}$ is relatively compact in the Banach space $\mathcal{B}(G)$ of all bounded complex valued functions, equipped with the sup-norm. If G is a locally compact topological group, then the set of all continuous almost periodic functions $\mathcal{A}(G)$ on G forms a translation invariant closed subspace of $\mathcal{C}(G) \cap \mathcal{B}(G)$.

In [12] in paragraph 13. the author deals with *modules* of almost periodic functions. Actually, by a module he means a linear subspace of $\mathcal{A}(G)$. An *invariant module* is a translation invariant subspace and a closed invariant module is exactly a variety. A module is called *finite* if it is finite dimensional and it is called *irreducible*, if it has no proper submodule. The fundamental theorem of almost periodic functions follows (see e.g. [12], Hauptsatz on p.47.).

Theorem 19. *Each closed invariant submodule in $\mathcal{A}(G)$ is the sum of finite irreducible invariant submodules.*

In our terminology this theorem reads as follows.

Theorem 20. *Each variety in $\mathcal{A}(G)$ is the sum of finite dimensional varieties, which have no proper subvarieties.*

Now we can easily derive the following result.

Theorem 21. *Spectral synthesis — hence also spectral analysis — holds over compact groups.*

Proof. If G is a compact group, then every continuous complex valued function on G is almost periodic (see [12], Satz 1. on p.154), that is, $\mathcal{A}(G) = \mathcal{C}(G)$. Hence, by the previous theorem, the proof is complete. $\square$

References

[1] Á. Bereczky and L. Székelyhidi, *Spectral synthesis on torsion groups*, Jour. Math. Anal. Appl. **304(2)**(2005), 607–613.

[2] R. J. Elliot, *Two notes on spectral synthesis for discrete Abelian groups*, Proc. Cambridge Phil. Soc. **61**(1965), 617–620.

[3] D. I. Gurevič, Counterexamples to a problem of L. Schwartz, *Funkcional. Anal. i Priložen* **9** (1975), no. 2, 29-35. (English translation: Funkcional Anal. Appl. **9**(1975), no. 2, 116-120.)

[4] E. Hewitt and K. Ross, *Abstract Harmonic Analysis I., II.*, Die Grundlehren der Mathematischen Wissenschaften, vol. 115, Springer Verlag, Berlin, Göttingen, Heidelberg, 1963.

[5] J.-P. Kahane, Sur quelques problèmes d'unicité et de prolongement, relatifs aux fonctions approchables par des sommes d'exponentielles, *Ann. Inst. Fourier* (Grenoble) **5**(1953-54), 39–130.

[6] J.-P. Kahane: *Lectures on Mean Periodic Functions.* Tata Institute, 1956.

[7] M. Laczkovich and G. Székelyhidi, *Harmonic analysis on discrete Abelian groups*, Proc. Amer. Math. Soc. **133**(6)(2005), 1581–1586.

[8] M. Laczkovich and L. Székelyhidi, *Spectral synthesis on discrete Abelian groups*, Math. Proc. Camb. Phil. Soc., **143**(01), pp. 103–120, 2007.

[9] M. Lefranc, *L'analyse harmonique dans* $\mathbb{Z}^n$, C. R. Acad. Sci. Paris **246**(1958), 1951–1953.

[10] J. A. Lester,*A canonical form for a system of quadratic functional equations*, Ann. Polon. Math. **35**(1976), 105.

[11] J. A. Lester,*The solution of a system of quadratic functional equations*, Ann. Polon. Math. **37**(1980), 113–117.

[12] W. Maak, *Fastperiodische Funktionen*, Die Grundlehren der Mathematischen Wissenschaften, vol. 61, Springer Verlag, Berlin, Göttingen, Heidelberg, 1950.

[13] M. A. McKiernan,*The matrix equation* $a(x \circ y) = a(x) + a(x)a(y) + a(y)$, Aequationes Math. **15**(1977), 213–223.

[14] M. A. McKiernan,*Equations of the form* $H(x \circ y) = \sum_i f_i(x)g_i(y)$, Aequationes Math. **16**(1977), 51–58.

[15] L. Schwartz, Théorie génerale des fonctions moyenne-périodiques, *Ann. of Math.* **48**(4)(1947), 857–929.

[16] L. Székelyhidi, *Convolution type functional equations on topological Abelian groups*, World Scientific Publishing Co. Pte. Ltd., Singapore, New Jersey, London, Hong Kong, 1991.

[17] L. Székelyhidi, *A Wiener Tauberian theorem on discrete Abelian torsion groups*, Annales Acad. Paedag. Cracov., Studia Mathematica I. **4**(2001), 147–150.

[18] L. Székelyhidi, *On discrete spectral synthesis*, Functional Equations–Results and Advances (Z. Daróczy and Zs. Páles), Kluwer Academic Publishers, Boston, Dordrecht, London, 2001.

[19] L. Székelyhidi, *The failure of spectral synthesis on some types of discrete Abelian groups*, Jour. Math. Anal. Appl. **291**(2004), 757–763.