Egyetemi doktori (PhD) értekezés tézisei

Nullity distribution in Finsler geometry

Salah Gomaa Ahmed Ali

Témavezető: Dr. Zoltan Muzsnay, Dr. Nabil L. Youssef

Debreceni Egyetem
Matematika- és Számítástudományok Doktori Iskola
Debrecen, 2016.

1. ND- of Cartan connection

1.1. Definition. *Let R be the h-curvature tensor of Cartan connection. The nullity space of R at a point $z \in TM$ is the subspace of $H_z(TM)$ defined by*

$$\mathcal{N}_R(z) := \{v \in H_z(TM)|\, R_z(v, w) = 0, \text{ for all } w \in H_z(TM)\}.$$

The dimension of $\mathcal{N}_R(z)$, denoted by $\mu_R(z)$, is the index of nullity of R at z.

If the index of nullity is constant, then the map $\mathcal{N}_R : z \mapsto \mathcal{N}_R(z)$ defines a distribution $\mathcal{N}_R$ of dimension μ_R called nullity distribution of R.

Any smooth section in the nullity distribution $\mathcal{N}_R$ is called a nullity vector field*.*

1.2. Theorem. *Let μ_R be constant on an open subset U of TM. Then, the nullity distribution $z \mapsto \mathcal{N}_R(z)$ is completely integrable on U.*

1.3. Definition. *[1, 55] A Finsler space (M, E), where $\dim M \geq 3$, is said to be h-isotropic if there exists a scalar function k_o such that the h-curvature tensor R of Cartan connection has the form*

$$R(X, Y)Z = k_o\{g(X, Z)Y - g(Y, Z)X\}, \ \forall X, Y, Z \in \mathfrak{X}(TM).$$

1.4. Theorem. *For an h-isotropic Finsler space, the index of nullity μ_R is either 0 or n.*

1.5. Definition. *[47, 55] A Finsler space (M, E), is said to be Berwald space if the hv-curvature tensor $\overset{\circ}{P}$ of Berwald connection vanishes or, equivalently, $D_{hX}\mathcal{C} = 0$ for all $X \in \mathfrak{X}(TM)$.*

1.6. Theorem. *For a Berwald space, the index of nullity $\mu_{\mathfrak{R}}$ of $\mathcal{N}_{\mathfrak{R}}$ takes its maximal value if and only if the index of nullity μ_R of $\mathcal{N}_R$ takes its maximal value.*

1.7. Theorem. *For a Landsberg space, the nullity distributions $\mathcal{N}_R$ coincides with the nullity distribution $\mathcal{N}_{\overset{\circ}{R}}$ of the h-curvature $\overset{\circ}{R}$ of Berwald connection.*

The nullity distribution $\mathcal{N}_P$ is not completely integrable in general as shown by the following example.

1.8. Example. *Let* $M = \{(x^1, x^2, x^3) \in \mathbb{R}^3 : x^2 \neq 0\}$, $U = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : x^2 \neq 0;\ y^1, y^2 \neq 0\} \subset TM$. *Let* L *be defined on* U *by:*

$$L(x, y) = \sqrt{e^{-x^1}(e^{-x^1 x^3}(y^1)^2 y^3 + x^2 (y^2)^3)^{2/3}}.$$

Nevertheless, we have

1.9. Theorem. *Let μ_P be constant on an open subset U of TM. The nullity distribution $\mathcal{N}_P$ is completely integrable on U if and only if $\mathfrak{R}(X, Y) = 0$ and $(D_{JZ}R)(X, Y) = R(Y, F\mathcal{C}(X, Z)) - R(X, F\mathcal{C}(Y, Z))$, $\forall X, Y \in Sec(\mathcal{N}_P)$.*

The nullity distribution $\mathcal{N}_Q$ is not completely integrable in general as shown by the following example.

1.10. Example. $M = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 : x^2 \neq 0\}$, $U = \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : x^2 \neq 0;\ y^1, y^3, y^4 \neq 0\} \subset TM$. *Let* L *be defined on* U *by*

$$L(x, y) = \sqrt{x^2 (y^1)^2 e^{-y^3/y^4} + (y^2)^2}.$$

Nevertheless, we have

1.11. Theorem. *Let μ_Q be constant on an open subset U of TM. The nullity distribution $\mathcal{N}_Q$ is completely integrable on U if and only if for all $X, Y \in Sec(\mathcal{N}_Q)$, $\mathfrak{R}(X,Y) = 0$ and the tensor*

$$A(X,Y,Z) := P(F\mathcal{C}(Z,X),Y) - (D_{JX}P)(Y,Z) - (D_{JZ}P)(X,Y),$$

is symmetric in X and Y, $\forall Z \in \mathfrak{X}(TM)$.

1.12. Definition. *Let (M,L) be a Finsler manifold. The angular metric $\hbar$ on TM is defined by*

$$\hbar(X,Y) = g(X,Y) - \ell(X)\ell(Y),$$

where g is the metric tensor on TM and $\ell(X) := \frac{1}{L}g(X,C)$.

1.13. Definition. *A Finsler space (M,L), with $\dim M \geq 4$, is said to be S_3-like if*

$$Q(X,Y,Z,W) = r\{\hbar(JX,JZ)\hbar(JY,JW) - \hbar(JX,JW)\hbar(JY,JZ)\},$$

where $Q(X,Y,Z,W) = g(Q(X,Y)Z,JW)$ and r is a scalar function.

1.14. Theorem. *Let (M,L) be an S_3-like space. Then, the index of nullity μ_Q takes its maximal value.*

2. ND- of Chern connection

2.1. Theorem. *For a Finsler manifold* (M, L) *there exists a unique normal lift* $\overset{*}{D}$ *of Barthel connection* $\Gamma = [J, S]$ *such that:*

(a) $\overset{*}{D}$ *is horizontally metric:* $\overset{*}{D}_{hX} g = 0, \ \forall X \in \mathfrak{X}(TM)$.

(b) *The classical torsion* $\overset{*}{T}$ *has the property that:* $J\overset{*}{T}(hX, hY) = 0, \forall X, Y \in \mathfrak{X}(TM)$.

This connection is called Chern connection.

2.2. Corollary. *The Chern connection* $\overset{*}{D}$ *is completely determined by:*

(a) $\overset{*}{D}_{JX} JY = J[JX, Y] = \overset{\circ}{D}_{JX} JY$.

(b) $\overset{*}{D}_{hX} JY = v[hX, JY] + \mathcal{C}'(X, Y) = D_{hX} JY$.

(c) $\overset{*}{D} F = 0$.

2.3. Proposition. *The h-curvature* $\overset{*}{R}$, *hv-curvature* $\overset{*}{P}$ *and v-curvature* $\overset{*}{Q}$ *of the Chern connection are given by:*

(a) $\overset{*}{R}(X, Y)Z = R(X, Y)Z - \mathcal{C}(F\mathfrak{R}(X, Y), Z)$.

(b) $\overset{*}{P}(X, Y)Z = \overset{\circ}{P}(X, Y)Z - (\overset{*}{D}_{JY} \mathcal{C}')(X, Z)$.

(c) $\overset{*}{Q}(X,Y)Z = 0.$

2.4. Proposition. *The h-curvature* $\overset{*}{R}$ *and hv-curvature* $\overset{*}{P}$ *of Chern connection have the following properties:*

(a) $\overset{*}{R}(X,Y)S = \mathfrak{R}(X,Y).$

(b) $\overset{*}{P}(X,Y)S = \overset{*}{P}(S,Y)X = \mathcal{C}'(X,Y).$

(c) $\overset{*}{P}(X,S)Z = 0.$

2.5. Theorem. *The nullity distribution* $\mathcal{N}_{R^*}$ *of the Chern h-curvature and the nullity distribution* $\mathcal{N}_R$ *of the Cartan h-curvature coincide.*

2.6. Definition. *The conullity space of the h-curvature tensor at* z*, denoted by* ${\mathcal{N}_{R^*}}^{\perp}(z)$*, is the orthogonal complement of* $\mathcal{N}_{R^*}$ *in* $H_z(TM)$*, where the orthogonality is taken with respect to the metric* g *defined on* TM*.*

2.7. Proposition. *For each point* $z \in TM$*, either* $\mu_{R^*}(z) = n$ *or* $\mu_{R^*}(z) \leq n-2$*. Consequently,* $\dim \ker_{R^*} > n-2$*.*

2.8. Proposition. *If* $\mathfrak{R} = 0$*, then* $\mathrm{Im}(\overset{*}{R}) = (J\mathcal{N}_{R^*})^{\perp}$. Consequently, $\mathrm{rank}\,(\overset{*}{R}) = n - \mu_{R^*}$.

2.9. Corollary. *Let* μ_{R^*} *be constant on an open subset* U *of* TM*. The nullity distribution* $z \mapsto \mathcal{N}_{R^*}(z)$ *is completely integrable on* U*.*

2.10. Theorem. *If $\mathfrak{R} = 0$, then the two distributions $\mathcal{N}_{R^*}$ and* $\ker_{R^*}$ *coincide.*

We have seen that if the index of nullity μ_{R^*} is constant, then the nullity distribution $\mathcal{N}_{R^*}$ is completely integrable. According to the Frobenius theorem, there exists a foliation of M by μ_{R^*}-dimensional maximal connected submanifolds as leaves, such that the nullity space at a point $x \in M$ is the tangent space to the leaf at x. We call the foliation induced by the nullity distribution $\mathcal{N}_{R^*}$ the nullity foliation and denote it again by $\mathcal{N}_{R^*}$.

2.11. Theorem. *Let (M, L) be a complete Finsler manifold and U the open subset of M on which μ_{R^*} takes its minimum. If $\mathfrak{R}$ vanishes, then every totally geodesic integral manifold of the nullity foliation $\mathcal{N}_{R^*}$ in U is complete.*

2.12. Theorem. *A Finsler manifold (M, L) is Landsbergian if and only if the canonical spray S is a nullity vector field for the the distribution $\mathcal{N}_{P^*}$.*

The nullity distribution $\mathcal{N}_{P^*}$ is not completely integrable in general, as is illustrated by the following example.

2.13. Example. *Let $U = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : y^1, y^2, y^3 \neq 0, y^3 \neq 4y^2\} \subset TM$, where $M := \mathbb{R}^3$. Define L on U by*

$$L(x, y) := \sqrt[4]{e^{-x^1 x^2} (y^1)^2 (y^3)^2 e^{-\frac{y^3}{y^2}}}.$$

2.14. Theorem. *Let $\mu_{\overset{*}{P}}$ be constant on an open subset U of TM. The nullity distribution $\mathcal{N}_{\overset{*}{P}}$ is completely integrable on U if and only if $\mathfrak{R}(X,Y)=0$ and $(\overset{*}{D}_{JZ}\overset{*}{R})(X,Y)=\ 0$, for all $X,Y\in Sec(\mathcal{N}_{\overset{*}{P}})$.*

2.15. Theorem. *The nullity distribution $\mathcal{N}_{\overset{*}{P}}$ and the kernel distribution* $\ker_{\overset{*}{P}}$ *coincide.*

A Finsler manifold in which the Chern hv-curvature tensor $\overset{*}{P}$ vanishes is called a Berwald space [47]. It is well known that every Berwald space is a Landsberg space, but it is not known whether the converse is true. In [42], Shen introduced a class of non-regular Finsler metrics which is Landsbergian and not Berwaldian. The calculations are not easy, especially, if one wants to study some concrete examples. Here, by using Maple program together with the results of [42] and [60], we give a simple class of proper non-regular non Berwaldian Landsbergian spaces.

2.16. Example. *Let $M=\mathbb{R}^3$, $U=\{(x,y)\in\mathbb{R}^3\times\mathbb{R}^3 : y^2>0, y^3>0\}\subset TM$. Define L on U by*

$$L(x,y):=f(x^1)\sqrt{(y^1)^2+y^2y^3+y^1\sqrt{y^2y^3}}\,e^{\frac{1}{\sqrt{3}}\arctan\left(\frac{2y^1}{\sqrt{3y^2y^3}}+\frac{1}{\sqrt{3}}\right)}.$$

3. ND- in the pull-back approach

Akbar-Zadeh has obtained the following results.

3.1. Theorem. *The nullity spaces* $\mathcal{N}_{\mathbf{K}}(x)$ *and the kernel space* $\ker_{\mathbf{K}}(x)$ *coincide.*

We show by a counterexample that the above mentioned spaces do not coincide.

3.2. Theorem. *The nullity space* $\mathcal{N}_{\mathcal{R}}(x)$ *and the kernel space* $\ker_{\mathcal{R}}(x)$ *do not coincide.*

The proof can be obtained by the following counter example.

3.3. Example. *Let* $M = \mathbb{R}^3$ *and* $U = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : y^i \neq 0;\ i = 1, 2, 3\} \subset TM$.
Let L *be defined on* U *by:*

$$L(x, y) = e^{-x^1 x^2} (y^1 y^2 y^3)^{1/3}.$$

3.4. Theorem. *Let* (M, L) *be a Finsler manifold and* $\mathcal{R}$ *the h-curvature of Cartan connection. If*

$$\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}} \mathcal{R}(\overline{X}, \overline{Y})\overline{Z} = 0, \tag{1}$$

then the two distributions $\mathcal{N}_{\mathcal{R}}$ *and* $\ker_{\mathcal{R}}$ *coincide.*

The following corollary shows that there are nontrivial cases in which (1) is verified and consequently the two distributions coincide.

3.5. Corollary. *Let* (M, L) *be a Finsler manifold and* $\overline{g}$ *the associated Finsler metric. If one of the following conditions holds:*

(a) $\widehat{R} = 0$ *(the integrability condition for the horizontal distribution),*

(b) $\widehat{R}(\overline{X}, \overline{Y}) = \lambda L(\ell(\overline{X})\overline{Y} - \ell(\overline{Y})\overline{X})$, *where* $\lambda(x, y)$ *is a homogenous function of degree* 0 *in* y *and* $\ell(\overline{X}) := L^{-1}\overline{g}(\overline{X}, \overline{\eta})$ *(the isotropy condition),*

then the two distributions $\mathcal{N}_{\mathcal{R}}$ *and* $\ker_{\mathcal{R}}$ *coincide.*

4. Metric freedom of a spray

In this chapter, the question of how many essentially different metrics metricize a spray is discussed. The notion of metric freedom of a spray is introduced and investigated. We show that in the regular case the holonomy distribution can be used to calculate the metric freedom of a spray. The metric freedom of the isotropic spray is characterized. Different examples are given.

4.1. Definition. *A spray* S *on a manifold* M *is called* Finsler metrizable *if there exists a Finsler function* L *such that the geodesic spray of the Finsler manifold* (M, L) *is* S*.*

The set of 2-homogeneous Euler-Lagrange functions is denoted by

$$\mathcal{E}_{S,2} = \{E \in C^\infty(\mathcal{T}M) \mid \omega_E = 0,\ \mathcal{L}_C E = 2E\}. \tag{2}$$

4.2. Property. *A spray* S *is metrizable if and only if there exists a 2-homogeneous Euler-Lagrange function* $E \in \mathcal{E}_{S,2}$ *such that the matrix field* $g_{ij} = \frac{\partial^2 E}{\partial y^i \partial y^j}$ *is positive definite at any point of* $\mathcal{T}M$*.*

4.3. Definition. *We say that the* metric freedom *of a metrizable spray S is $m_S \in \mathbb{N}$ if $\mathcal{E}_{S,2}$ can be locally generated by its m_S functionally independent elements. If the spray S is non-metrizable, then we set $m_S = 0$.*

In other words, if the *metric freedom* of a spray S is $m_S > 1$, then for every $E \in \mathcal{E}_{S,2}$ and $v_0 \in \mathcal{T}M$ there exists a neighbourhood $U \subset \mathcal{T}M$ of v_0, a function $\varphi\colon \mathbb{R}^{m_S} \to \mathbb{R}$ and $E_1, ..., E_{m_S} \in \mathcal{E}_{S,2}$ functionally independent on U such that

$$E(v) = \varphi\big(E_1(v), \ldots, E_{m_S}(v)\big), \qquad \forall\, v \in U.$$

4.4. Theorem. *Let S be a metrizable spray on a manifold M. If its parallel translation is regular, then the metric freedom of S is*

$$m_S = \operatorname{codim} \mathcal{D}_{\mathcal{H}}.$$

4.5. Proposition. *Let S be an isotropic spray on the n-dimensional manifold M with regular parallel translation. Then we have $m_S \in \{0, 1, n\}$. More precisely we have the following possibilities:*

(a) *$m_S = 0$ if and only if S is not metrizable; (in this case $\mathfrak{R} \neq 0$)*

(b) *$m_S = 1$ if and only if $\mathfrak{R} \neq 0$ and S is metrizable;*

(c) *$m_S = n$, that is maximal, if and only if $\mathfrak{R} = 0$.*

4.6. Example ($m_S = 0$, $\operatorname{codim} \mathcal{D}_{\mathcal{H}} = 0$).
Let $M = \{(x^1, x^2) \in \mathbb{R}^2 : x^2 > 0\}$ and S is given by the coefficients

$$G^1 := y^1 \sqrt{x^2 (y^1)^2 + (y^2)^2} + \frac{y^1 y^2}{2x^2},$$
$$G^2 := y^2 \sqrt{x^2 (y^1)^2 + (y^2)^2} - \frac{(y^1)^2}{4}.$$

4.7. Example ($m_S = 0$, $\operatorname{codim} \mathcal{D}_{\mathcal{H}} > 0$).
Let $M = \{(x^1, x^2) \in \mathbb{R}^2 : x^2 > 0\}$ and S is given by the coefficients $G^1 = \frac{(y^1)^2}{2x^2}$, $G^2 = 0$.

4.8. Example ($m_S = 1$).
Let us consider on the unite disk $\mathbb{D} \subset \mathbb{R}^n$ and the spray is given by $G^i = -\dfrac{\mu \langle x, y \rangle}{1 + \mu |x|^2} y^i$ with $\mu \in \mathbb{R} \setminus \{0\}$.

4.9. Example (m_S is maximal).
One can consider the trivial example where $M = \mathbb{R}^n$ and $G^i = 0$. In this case the parallel translation is regular and the holonomy group is trivial. Hence we have $m_S = n$.
We prefer to give also another, not so obvious example: Let $\mathbb{B}^n \subset \mathbb{R}^n$ be the standard unit ball and S the spray with

$$G^i = -\frac{\langle a, y \rangle}{1 + \langle a, x \rangle} y^i, \tag{3}$$

where $a \in \mathbb{R}^n$ is a constant vector with $|a| < 1$. Since $\mathfrak{R} = 0$, then $\mathcal{D}_{\mathcal{H}} = H\mathcal{T}M$, the horizontal distribution. Hence, by Theorem 4.4, the metric freedom is maximal.

We remark, that S.S. Chern and Z. Shen investigated in [18] the family of Riemannian metrics associated with the norms

$$L_a = \frac{\sqrt{1-|a|^2}}{(1+\langle a,x\rangle)^2}\sqrt{|y|^2 - \frac{2\langle a,y\rangle\langle x,y\rangle}{1+\langle a,x\rangle} - \frac{(1-|x|^2)\langle a,y\rangle^2}{1+\langle a,x\rangle}}. \tag{4}$$

The geodesic equation of (4) *is* (3), *but one can find other generating Finsler metrics too. Indeed, putting $z^i = ((1+\langle a,x\rangle)y^i - \langle a,y\rangle x^i)/(1+\langle a,x\rangle)^2$ and considering a 1-homogeneous function $\phi\colon \mathbb{R}^n \to \mathbb{R}$, we get*

$$L_\phi(x,y) = \phi\big(z^1(x,y),\ldots,z^n(x,y)\big) \tag{5}$$

such that $E_\phi = \frac{1}{2}L_\phi^2$ is a (not necessarily regular) element of $\mathcal{E}_{S,2}$. Therefore, if L_ϕ satisfies the regularity condition, then it is a projectively flat Finsler metric of zero flag curvature with geodesic spray given by (3). *The family (4) can be considered as a special case of (5) by choosing $\phi(z) = \big(\langle z,z\rangle - \langle a,z\rangle^2\big)^{1/2}$.*

5. New Finsler Package

In [39], Rutz and Portugal discussed and applied the FINSLER package they introduced in [38]. This package is an extension of the RIEMANN package [37]. The FINSLER package is included in a CD with the book [7].

When performing some applications using the FINSLER package, we have encountered some problems. To show one of these problems, let us consider the following example. Let $M = \mathbb{R}^4$, $U = \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : x^1 \neq 0\}$. Let L be defined on U by:

$$L(x, y) = \sqrt{x^1 y^4 \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}}.$$

Based on this package, the non-vanishing coefficients of Berwald connection are as follows:

$$G^1_{11} = G^2_{12} = G^3_{13} = \frac{1}{x^1}, \quad G^1_{22} = G^1_{33} = -\frac{1}{x^1}.$$

This shows that the coefficients of Berwald connection are functions of the positional argument x^i only. Hence, the space under consideration is Berwaldian and is thus Landsbergian. Consequently, the hv-curvature P^h_{ijk} of Cartan connection should vanish identically. However, the FINSLER package calculated non-vanishing components of P^h_{ijk}.

After a deep study of the source code (Finsler.mpl), we have discovered some wrong indices in the definition of P^h_{ijk}. (Similar error is found in [7], page 1154). Another problem with this package is the problem of dimension. If one considers a three dimensional Finsler space, the package can not compute the components of the hh-curvature R^h_{ijk} and hv-curvature P^h_{ijk} of Cartan connection. The package response is that these objects are outside dimension.

Summing up, we have two problems with the Rutz and Portugal's package. The first is the wrong calculations of the curvature P^h_{ijk}. The second is the disability of computing R^h_{ijk} and P^h_{ijk} in dimensions different from 4.

We solve the two above mentioned problems. Moreover, we extend the package in order to compute various geometric objects associated not only with Cartan connection but also with the other fundamental connections in Finsler geometry. And this is for any dimension. Other geometric objects can be similarly added to the package.

6. Explicit examples

Based on the new Finsler package, we introduce a computational technique to calculate the nullity and kernel vectors. We give some counter examples.

The nullity distributions associated with Cartan connection are studied in Chapter 2. The following example shows that *the nullity space* $\mathcal{N}_R$ *of the h-curvature* R *of Cartan connection and the kernel* Ker_R *do not coincide.*

6.1. Example. *Let* $U = \{(x^1,...,x^4;y^1,...,y^4) \in \mathbb{R}^4 \times \mathbb{R}^4 : y^2 \neq 0\ , y^4 \neq 0\} \subset TM$, $M = \{(x^1,...,x^4) \in \mathbb{R}^4 | \, x^2 > 0\}$. *Let* L *be defined on* U *by*

$$L(x,y) := \sqrt[4]{(x^2)^2(y^1)^4 + (y^2)^4 + (y^3)^4 + (y^4)^4}.$$

In [53] Youssef proved that the nullity distribution $\mathcal{N}_{\overset{\circ}{R}}$ associated with the h-curvature $\overset{\circ}{R}$ of Berwald connection is completely integrable. He conjectured that the nullity distribution $\mathcal{N}_{\overset{\circ}{P}}$

of the hv-curvature $\overset{\circ}{P}$ of Berwald connection is not completely integrable. In the next example, *we show that his conjecture is true.*

6.2. Example. *Let* $U = \{(x^1, x^2, x^3; y^1, y^2, y^3) \in \mathbb{R}^3 \times \mathbb{R}^3 : y^1 \neq 0\} \subset TM$ *and* $M = \mathbb{R}^3$*. Let* L *be defined on* U *by*

$$L := \mathrm{e}^{-x^1} \left(y^{2\,3} + \mathrm{e}^{-x^1 x^3} y^3\, y^{1\,2} \right)^{1/3}.$$

Let $\mathcal{N}_{R^\circ}$ and $\mathcal{N}_{\mathfrak{R}}$ be the nullity distributions associated with the h-curvature $\overset{\circ}{R}$ of Berwald connection and the curvature $\mathfrak{R}$ of the Barthel connection respectively. In [51], Youssef proved that $\mathcal{N}_{R^\circ} \subseteq \mathcal{N}_{\mathfrak{R}}$. The following example shows that *the converse is not true: that is* $\mathcal{N}_{R^\circ}$ *is a proper sub-distribution of* $\mathcal{N}_{\mathfrak{R}}$.

6.3. Example. *Let* $U = \{(x^1, \cdots, x^4; y^1, \cdots, y^4) \in \mathbb{R}^4 \times \mathbb{R}^4 : y^2 \neq 0,\ y^4 \neq 0\} \subset TM$ *and* $M = \mathbb{R}^4$*. Let* L *be defined on* U *by*

$$L := \sqrt{\mathrm{e}^{-x^2} y^1 \sqrt[3]{y^{2\,3} + y^{3\,3} + y^{4\,3}}}.$$

7. English summary

Chern and Kuiper [17] in 1952 defined a distribution on a Riemannian manifold M which assigns to each point $x \in M$ the subspace

$$\mathcal{N}_R(x) = \{X \in T_xM : R(X,Y) = 0,\ \forall\, Y \in T_xM\},$$

where R is the curvature of the Riemannian connection on M. It is called the nullity space at x. The distribution defined by the subspace $\mathcal{N}_R(x)$ at each point x of M is called the nullity distribution $\mathcal{N}_R$ of the Riemannian manifold M. The dimension $\mu_R(x)$ of $\mathcal{N}_R(x)$ is called the index of nullity at x. Chern and Kuiper showed that, if $\mu_R(x)$ is constant in a neighborhood, then $\mathcal{N}_R$ constitutes a completely integrable distribution there, and that the leaves of the resulting foliation are flat. Later, Maltz and others developed this point in different papers, for example, [19, 21, 23, 32, 33, 41, 49, 50].

In 1972, Akbar Zadeh [3, 4] extended this work to Finsler geometry adopting the *pullback approach* (PB-) approach to Finsler geometry. He studied the nullity distribution of the (classical) curvature of Cartan connection. Recently, Bidabad and Refie-Rad [11] studied a more general case called k-nullity distribution in Finsler geometry.

On the other hand, in 1982, Youssef [51, 53] studied the nullity distributions of the curvature tensors of Barthel connection and Berwald connection, adopting the *Klein-Grifone approach*(KG-) approach to Finsler geometry.

In the PB-approach, the existence and uniqueness theorems for the four fundamental linear connections (Berwald, Cartan, Chern and Hashiguchi connections) on a Finsler manifold have been satisfactorily established [56, 57]. In the KG-approach, Grifone [25] has investigated Cartan and Berwald connections. Szilasi and Vincze [47] have studied Chern and Hasiguchi connections using the technique of lifting vector fields to the tangent bundle.

Adopting the Klein-Grifone formalism of Finsler geometry, we investigated the nullity distributions of the h-curvature R, hv-curvature P and v-curvature Q tensors of Cartan connection. We showed that the nullity distribution $\mathcal{N}_R$ is included in $\mathcal{N}_{\mathfrak{R}}$ of the curvature of Barthel connection and we showed, by an example, that this inclusion is proper. We proved that $\mathcal{N}_R$ is completely integrable. Through examples, we show that the distributions $\mathcal{N}_P$ and $\mathcal{N}_Q$ are not completely integrable. Nevertheless, we investigated the necessary and sufficient conditions for these distributions to be completely integrable. A coordinate-free existence and uniqueness theorem for Chern connection is formulated and proved. The torsion and curvature tensors of Chern connection are derived. Some properties and the Bianchi identities for this connection are derived. The nullity distributions of the two curvature tensors $\overset{*}{R}$ and $\overset{*}{P}$ of Chern connection are investigated. The completely integrable property of $\mathcal{N}_{R^*}$ and the completeness of the nullity foliation associated with $\mathcal{N}_{R^*}$ are proved. Two counterexamples are given. The first shows that $\mathcal{N}_{R^*}$ does not coincide with the kernel distribution of $\overset{*}{R}$. The second shows that $\mathcal{N}_{P^*}$ is not completely integrable. An example of non regular Landesbergian non Berwaldian metric is given.

Adopting the pullback formalism of Finsler geometry, we show by a counterexample that the kernel distribution $\ker_{\mathcal{R}}$ of the h-curvature $\mathcal{R}$ of Cartan connection and the associated nullity distribution $\mathcal{N}_{\mathcal{R}}$ do not coincide, contrary to Akbar-Zadeh's result [2]. We give sufficient conditions for $\ker_{\mathcal{R}}$ and $\mathcal{N}_{\mathcal{R}}$ to coincide.

The question of how many essentially different metrics metricize a spray is discussed. The notion of metric freedom of a spray is introduced and investigated. We show that in the regular case, the holonomy distribution can be used to calculate the metric freedom of a spray. The metric freedom of isotropic sprays is characterized. Different examples are given.

Some modifications of the Maple package, FINSLER, (for calculations in Finsler geometry) included in the book "Handbook of Finsler geometry [7]" are performed. A technique for simplifying tensor expressions is proposed. A computational technique for calculating nullity vectors and kernel vectors, using the new Finsler package, is introduced. Three interesting examples are given.

8. Hungarian summary

Chern és Kuiper [17] 1952-ben definiálta egy M Riemann-sokaság nullitás-disztribúcióját az alábbi módon. Jelölje minden $x \in M$ esetén

$$\mathcal{N}_R(x) = \{X \in T_xM : R(X,Y) = 0, \forall\, Y \in T_xM\}$$

az x-beli nullitás teret, ahol R a Riemann-konnexió görbületi tenzora. Ezen alterek összességét hívjuk nullitás-disztribúciónak. Az $\mathcal{N}_R(x)$ dimenzióját az x-beli nullitás indexnek nevezzük, és $\mu_R(x)$-szel jelöljük. Chern és Kuiper megmutatta, hogy ha $\mu_R(x)$ konstans egy x_0 pont egy környezetén, akkor $\mathcal{N}_R$ integrálható disztribúció ezen a környezeten, és az integrálsokaságai laposak. Később Maltz és mások további vizsgálatokat végeztek a témakörben [19, 21, 23, 32, 33, 41, 49, 50].

1972-ben Akbar-Zadeh [3, 4] kiterjesztette ezeket a vizsgálatokat a Finsler-geometriára, a *pull-back* megközelítést alkalmazva. A Cartan-konnexió (klasszikus) görbületének nullitás-disztribúcióját vizsgálta. A közelmúltban Bidabad és Refie-Rad [11] vizsgálta az általánosabb k-nullitás-disztribúciót a Finsler esetben.

Ezek mellett 1982-ben, Youssef [51, 53] tanulmányozta a Barthel- és Berwald-konnexiók görbületi tenzorainak nullitás-disztribúcióját, a Klein–Grifone megközelítést alkalmazva.

A PB-megközelítésben Finsler-sokaságok négy alapvető konnexiójának (Berwald-, Cartan-, Chern- és Hashiguchi-konnexiók) létezése és egyértelműsége már ki van dolgozva [56, 57]. A KG-megközelítésben Grifone [25] vizsgálta a Cartan- és Berwald-konnexiókat. Szilasi és Vincze [47] tanulmányozta a Chern- és Hasiguchi-konnexiókat a vektormezők érintősokaságra való liftelésének technikájával.

A disszertáció 2. fejezetében Klein–Grifone formalizmust alkalmazva vizsgáltuk a Cartan-konnexió h-görbületének, hv-görbületének és v-görbületének (jelölésük rendre R, P és Q) nullitás-disztribúcióját. Megmutattuk, hogy az $\mathcal{N}_R$ nullitás-disztribúcióját tartalmazza a Barthel-konnexió $\mathcal{N}_{\mathfrak{R}}$ nullitás-disztribúciója, és egy példával megmutattuk, hogy a tartalmazás valódi. Megmutattuk, hogy $\mathcal{N}_R$ integrálható. Példákkal demonstráltuk, hogy az $\mathcal{N}_P$ és $\mathcal{N}_Q$ disztribúciók nem integrálhatók. Mindazonáltal vizsgáltuk ezen disztribúciók integrálhatóságának szükséges és elégséges feltételeit.

A 3. fejezetben koordinátamentes bizonyítást adtunk a Chern-konnexió létezésére és egyértelműségére, és levezettük a torzió és görbületi tenzorait, a konnexió további tulajdonságait és Bianchi-azonosságokat. Vizsgáltuk a Chern-konnexió $\overset{*}{R}$ és $\overset{*}{P}$ görbületi tenzorainak a nullitás-disztribúcióit. Megmutattuk, hogy $\mathcal{N}_{R^*}$ integrálható. Két ellenpéldával rámutattunk, hogy $\mathcal{N}_{R^*}$ nem esik egybe az $\overset{*}{R}$ kernel-disztribúciójával, és hogy $\mathcal{N}_{P^*}$ nem integrálható. Példát adtunk olyan nem reguláris Landsberg-sokaságra, ami nem Berwald-sokaság.

A 4. fejezetben a pull-back formalizmusban ellenpéldával megmutattuk, hogy a Chern-konnexió $\mathcal{R}$ h-görbületének $ker_{\mathcal{R}}$ magja és a hozzá tartozó $\mathcal{N}_{\mathcal{R}}$ nullitás-disztribúció nem esnek egybe, cáfolva ezzel Akbar-Zadeh eredményét [2]. Elégséges feltételt adtunk $\ker_{\mathcal{R}}$ és $\mathcal{N}_{\mathcal{R}}$ egybeesésére.

Az 5. fejezetben azt vizsgáltuk, hogy hány különböző metrikából származhat egy adott spray. Bevezettük és vizsgáltuk egy spray metrikus szabadságának fogalmát. Megmutattuk, hogy a reguláris esetben a holonómia disztribúció segítségével meghatározható a spray metrikus szabadsága. Megadtuk az izotropikus sprayk lehetséges metrikus szabadságait. Több konkrét példával is szolgáltunk.

Az appendixben továbbfejlesztettük a "Handbook of Finsler geometry" [7] című könyvben található, FINSLER nevű Maple csomagot, többek közt egy tenzor-kifejezések egyszerűsítésére szolgáló technikával Az új csomagot NFP-nek (new Finsler package) neveztük el.

Az NFP csomagban módszert adtunk a nullitás és kernel vektorok kiszámítására. Végül példákon át megmutattuk a következőket: a Ker_R és $\mathcal{N}_R$ disztribúciók nem egyenlőek, a $\mathcal{N}_{P^\circ}$ disztribúció nem integrálható, és a $\mathcal{N}_{\mathfrak{R}}$ disztribúciót nem tartalmazza $\mathcal{N}_{R^\circ}$.

Megjegyezzük, hogy a disszertációban található eredmények többsége publikálásra vagy benyújtásra került ([35, 59, 60, 61, 63, 62, 64]).

Hivatkozások

[1] H. Akbar-Zadeh, *Sur les espaces de Finsler isotropes*, C. R. Acad. Sci. Paris, Sér. A, **252** (1961), 2061–2063.

[2] H. Akbar-Zadeh, *Sur le noyau de l'opérateure de courbure d'une variété finslérienne*, C. R. Acad. Sci. Paris, Sér. A, **272** (1971), 807–810.

[3] H. Akbar-Zadeh, *Espaces de nullité de cértains opérateure en géométrie des sous-variétés*, C. R. Acad. Sci. Paris, Sér. A, **274** (1972), 490–493.

[4] H. Akbar-Zadeh, *Espaces de nullité en géométrie Finslérienne*, Tensor, N. S., **26** (1972), 89–101.

[5] H. Akbar-Zadeh, *Initiation to global Finsler geometry*, Elsevier, 2006.

[6] M. Anastasiei, *Finsler connections in generalized Lagrange spaces*, Balkan J. Geom. Appl., **1** (1996), 1–9.

[7] P. L. Antonelli (Ed.), *Handbook of Finsler geometry* I, II, Kluwer Acad. publ., 2003.

[8] P. L. Antonelli, I. Bucataru and S. F. Rutz, *Computer algebra and two and three dimensional Finsler geometry*, Publ. Math. Debrecen, **62** (2003), 289–313.

[9] P. L. Antonelli, S. F. Rutz and K. T. Fonseca, *The mathematical theory of endosymbiosis, II: Models of the fungal fusion hypothesis*, Nonlinear Anal., Real World Appl., **13** (2012), 2096–2103.

[10] D. Bao, S. S. Chern and Z. Shen, *An introduction to Riemann-Finsler geometry*, Springer-Verlag, Berlin, 2000.

[11] B. Bidabad and M. Refie-Rad, *On the k-nullity foliations in Finsler geometry and completeness*, Bull. Iranian Math. Soc., **37**, **4** (2011), 1-18.

[12] I. Bucataru and R. Miron, *Finsler-Lagrange geometry. Applications to dynamical systems*, Editura Acad. Române, 2007.

[13] I. Bucataru and Z. Muzsnay, *Projective metrizability and formal integrability*, Symmetry, Integrability and Geometry: Methods and Applications, **7** (2011) 114.

[14] I. Bucataru and Z. Muzsnay, *Projective and Finsler metrizability: parameterization-rigidity of the geodesics*, Int. J. Math. **23**, 9 (2012). ArXiv: 1108.4628 [math. DG].

[15] I. Bucataru and Z. Muzsnay, *Metrizable isotropic second-order differential equations and Hilbert's fourth problem*, Journal of the Aust Math. Soc., 97, 1, 2014, pp 27–47.

[16] E. Cartan, *Leçons sur la géométrie des espaces de Riemann*, Paris, Gauthier-Villars, 1928; 2nd. Ed., 1946.

[17] S. S. Chern and N. H. Kuiper, *On some theorems on isometric imbedding of compact Riemann manifolds in Euclidean space*, Ann. Math., **56** (1952), 313-316.

[18] S. S. Chern and Z. Shen, *Riemann-Finsler Geometry*, Singapore: World Scientific, 2005.

[19] Y. H. Clifton and R. Maltz, *The k-nullity spaces of the curvature operator*, Michigan Math. J., **17** (1970), 85–89.

[20] M. Crampin, *On the inverse problem for sprays*, Publ. Math. Debrecen, **70** (2007), 319–335.

[21] D. Ferus, *On the completeness of nullity foliations*, Michigan Math. J., **18** (1971), 61–64.

[22] A. Frölicher and A. Nijenhuis, *Theory of vector-valued differential forms*, I, Ann. Proc. Kon. Ned. Akad., A, **59** (1956), 338–359.

[23] A. Gray, *Spaces of constancy of curvature operators*, Proc. of the Amer. Mth. Soc., **17** (1966), 897–902.

[24] J. Grifone, *Structure presque-tangente et connexions,* I, Ann. Inst. Fourier, Grenoble, **22, 1** (1972), 287–334.

[25] J. Grifone, *Structure presque-tangente et connexions,* II, Ann. Inst. Fourier, Grenoble, **22, 3** (1972), 291–338.

[26] J. Grifone and Z. Muzsnay, *Variational Principles for Second Order Differential Equations: Application of the Spencer Theory to Characterize Variational Sprays*, World Scientific, 2000.

[27] M. Hashiguchi, *On conformal transformations of Finsler metrics*, J. Math. Kyoto Univ., **16** (1976), 25–50.

[28] D. Krupka and A. E. Sattarov, *The inverse problem of the calculus of variations for Finsler structures*, Math. Slovaca, **35** (1985), 217–222.

[29] O. Krupková, *Variational metric structures* Publ. Math. Debrecen **62** (2003), 461–495.

[30] J. Klein and A. Voutier, *Formes extérieures génératrices de sprays*, Ann. Inst. Fourier, Grenoble, **18, 1** (1968), 241–260.

[31] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, II, Interscience, New York, 1969.

[32] R. Maltz, *The nullity space of the curvature operator*, Cahiers Top. Géom. Diff., **8** (1965), 1–20.

[33] R. Maltz, *The nullity spaces of the curvature-like tensors*, J. Diff. Geom, **7** (1972), 519–523.

[34] Z. Muzsnay, *The Euler-Lagrange PDE and Finsler metrizability*, Houston J. Math., **32** (2006), 79–98.

[35] Z. Muzsnay and **S. G. Elgendi**, *Metric freedom of a spray*, submitted (2016).

[36] R. Portugal, *An algorithm to simplify tensor expressions*, Comp. Phys. Commun., **115** (1998) 215–230.

[37] R. Portugal, S. L. Sautu, *Applications of Maple to general relativity*, Comp. Phys. Commun., **105** (1997), 233–253.

[38] S. F. Rutz and R. Portugal, *FINSLER: A computer algebra package for Finsler geometries*, Nonlinear Analysis, **47** (2001), 6121–6134.

[39] S. F. Rutz and R. Portugal, *Symbolic Finsler Geometry.* In: P. L. Antonelli (Ed.), Handbook of Finsler geometry II, Kluwer Acad. publ., 2003.

[40] W. Sarlet, G. Thompson and G. Prince, *The inverse problem of the calculus of variations: the use of geometrical calculus in Douglas's analysis*, Trans. Amer. Math. Soc. **354** (2002), 2897–2919.

[41] M. Sekizawa, *The nullity spaces of the conformal curvature tensor*, J. Math. Soc. Japan, **25** (1973), 125–131.

[42] Z. Shen, *On a class of Landsberg metrics in Finsler geometry*, Canad. J. Math. **61** (2009), 1357–1374.

[43] Z. Shen, *Differential geometry of spray and Finsler spaces*, Springer, 2001.

[44] A. Soleiman, *On Akbar-Zadeh's theorem on a Finsler space of constant curvature*, J. Egypt. Math. Soc., **23(2)** (2015), 518–522. ArXiv: 1201.2012 [math.DG].

[45] J. Szilasi, R. L. Lovas and D. Cs. Kertész, *Connections, Sprays and Finsler Structures*, World Scientific, 2014.

[46] J. Szilasi and S. Vattamány, *On the Finsler-metrizabilities of spray manifolds*, Period. Math. Hungar., **44** (2002), 81–100.

[47] J. Szilasi and Cs. vincze, *A new look at Finsler connections and special Finsler manifolds*, Acta Math. Acad. Paedagog. Nyházi, N. S., **16** (2000), 33–63.

[48] A. A. Tamim, *General theory of Finsler spaces with applications to Randers spaces*, Ph. D. Thesis, Cairo Univ., 1991.

[49] S. Tan, *On nullity distributions*, Trans. Amer. Math. Soc., **223** (1976), 323–335.

[50] F. Vittone, *On nullity distribution of a submanifold of a space form*, Math. Z., **272** (2012), 1–16. ArXiv: 1103.0918 [math. DG].

[51] Nabil L. Youssef, *Distribution de nullité du tensor de courbure d'une connexion*, C. R. Acad. Sci. Paris, Sér. A, **290** (1980), 653–656.

[52] Nabil L. Youssef, *Etude de certaines connexions linéaires sur le fibré tangent d'une variété finslérienne*, Ph. D. Thesis, Cairo University and Grenoble University, 1981.

[53] Nabil L. Youssef, *Sur les tenseurs de courbure de la connexion de Berwald et ses distributions de nullité.* Tensor, N. S., **36** (1982), 275–280.

[54] Nabil L. Youssef, S. H. Abed and S. G. Elgendi, *Generalized beta conformal changes: General theory of (α, β)-metric with applications to special Finsler spaces*, Lambert Academic Publishing (LAP), 2012.

[55] Nabil L. Youssef, S. H. Abed and A. Soleiman, *A global approach to the theory of special Finsler manifolds*, J. Math. Kyoto Univ., **48**, **4** (2008), 857–893. ArXiv: 0704.0053 [math. DG].

[56] Nabil L. Youssef, S. H. Abed and A. Soleiman, *Cartan and Berwald connections in the pullback formalism*, Algebras, Groups and Geometries, **25**, **4** (2008), 363-386. arXiv: 0707.1320 [math. DG].

[57] Nabil L. Youssef, S. H. Abed and A. Soleiman, *A global approach to the theory of connections in Finsler geometry*, Tensor, N. S., **71**,**3** (2009), 187–208. ArXiv: 0801.3220 [math.DG].

[58] Nabil L. Youssef, S. H. Abed and A. Soleiman, *Geometric objects associated with the fundamental connections in Finsler geometry*, J. Egypt. Math. Soc., **18(1)** (2010), 67-90.

[59] Nabil L. Youssef and **S. G. Elgendi**, *A note on "Sur le noyau de l'opérateur de courbure d'une variété finslérienne, C. R. Acad. Sci. Paris, sér. A, t. 272 (1971), 807-810"*, C. R. Math., Ser. I, **351** (2013), 829–832. ArXiv: 1305.4498 [math. DG].

[60] Nabil L. Youssef and **S. G. Elgendi**, *New Finsler package*, Comput. Phys. Commun., **185**, 3 (2014), 986–997. ArXiv: 1306.0875 [math. DG].

[61] Nabil L. Youssef and **S. G. Elgendi**, *Computing nullity and kernel vectors using NF-package: Counterexamples*, Comput. Phys. Commun., **185** (2014), 2859–2864. ArXiv: 1401.0133 [math. DG].

[62] Nabil L. Youssef, and **S. G. Elgendi**, *Nullity distributions associated with Chern connection*, Publ. Math. Debrecen, **88** (2016), 235–248. ArXiv: 1410.0193 [math. DG].

[63] Nabil L. Youssef and **S. G. Elgendi**, *Existence and uniqueness of Chern connection in the Klein-Grifone approach*, Submitted. ArXiv: 1402.0317 [math. DG].

[64] Nabil L. Youssef, A. Soleiman and **S. G. Elgendi**, *Nullity distributions associated to Cartan connection*, Indian J. Pure Appl. Math., **45**, **2** (2014), 213-238. ArXiv: 1210.8359 [math. DG].

Registry number: DEENK/146/2016.PL
Subject: Ph.D. List of Publications

Candidate: Salah Gomaa Ahmed Ali Elgendi
Neptun ID: YS5QP3
Doctoral School: Doctoral School of Mathematical and Computational Sciences

List of publications related to the dissertation

Foreign language scientific article(s) in Hungarian journal(s) (1)

1. Youssef, N.L., **Elgendi, S.G.**: Nullity distributions associated with Chern connection.
   Publ. Math.-Debr. 88 (1-2), 235-248, 2016. ISSN: 0033-3883.
   DOI: http://dx.doi.org/10.5486/PMD.2016.7414
   IF:0.503 (2014)

Foreign language scientific article(s) in international journal(s) (4)

2. Youssef, N.L., Soleiman, A., **Elgendi, S.G.**: Nullity distribution associated to Cartan connection.
   Indian J. Pure Appl. Math. 45 (2), 213-238, 2014. ISSN: 0019-5588.
   DOI: http://dx.doi.org/10.1007/s13226-014-0060-0
   IF:0.224

3. Youssef, N.L., **Elgendi, S.G.**: Computing nullity and kernel vectors using NF-package:Counterexamples.
   Comput. Phys. Commun. 185 (11), 2859-2864, 2014. ISSN: 0010-4655.
   DOI: http://dx.doi.org/10.1016/j.cpc.2014.07.005
   IF:3.112

4. Youssef, N.L., **Elgendi, S.G.**: New Finsler package.
   Comput. Phys. Commun. 185 (3), 986-997, 2014. ISSN: 0010-4655.
   DOI: http://dx.doi.org/10.1016/j.cpc.2013.10.024
   IF:3.112

5. Youssef, N.L., **Elgendi, S.G.**: A note on "Sur le noyau de l'opérateur de courbure d'une variété finslérienne" [C. R. Acad. Sci. Paris, Ser. A 272 (1971) 807-810].
   C. R. Math. 351 (21-22), 829-832, 2013. ISSN: 1631-073X.
   DOI: http://dx.doi.org/10.1016/j.crma.2013.09.025
   IF:0.425

List of other publications

Foreign language scientific article(s) in international journal(s) (2)

6. Youssef, N.L., Abed, S.H., **Elgendi, S.G.**: Generalized [beta]-conformal change and special Finsler spaces.
   Int. J. Geom. Methods Mod. Phys. 9 (3), 1250016-1 - 1250016-25, 2012. ISSN: 0219-8878.
   DOI: http://dx.doi.org/10.1142/S0219887812500168
   IF:0.951

7. Youssef, N.L., Abed, S.H., **Elgendi, S.G.**: Generalized [beta]-conformal change of Finsler metrics.
   Int. J. Geom. Methods Mod. Phys. 7 (4), 565-582, 2010. ISSN: 0219-8878.
   DOI: http://dx.doi.org/10.1142/S0219887810004440
   IF:0.757

Total IF of journals (all publications): 9,084
Total IF of journals (publications related to the dissertation): 7,376

The Candidate's publication data submitted to the iDEa Tudóstér have been validated by DEENK on the basis of Web of Science, Scopus and Journal Citation Report (Impact Factor) databases.

13 June, 2016

DEBRECENI EGYETEM
EGYETEMI ÉS NEMZETI KÖNYVTÁR

Nyilvántartási szám: DEENK/146/2016.PL
Tárgy: PhD Publikációs Lista

Jelölt: Elgendi Salah Gomaa Ahmed Ali
Neptun kód: YS5QP3
Doktori Iskola: Matematika- és Számítástudományok Doktori Iskola

A PhD értekezés alapjául szolgáló közlemények

Idegen nyelvű tudományos közlemény(ek) hazai folyóiratban (1)

1. Youssef, N.L., **Elgendi, S.G.**: Nullity distributions associated with Chern connection.
   Publ. Math.-Debr. 88 (1-2), 235-248, 2016. ISSN: 0033-3883.
   DOI: http://dx.doi.org/10.5486/PMD.2016.7414
   IF:0.503 (2014)

Idegen nyelvű tudományos közlemény(ek) külföldi folyóiratban (4)

2. Youssef, N.L., Soleiman, A., **Elgendi, S.G.**: Nullity distribution associated to Cartan connection.
   Indian J. Pure Appl. Math. 45 (2), 213-238, 2014. ISSN: 0019-5588.
   DOI: http://dx.doi.org/10.1007/s13226-014-0060-0
   IF:0.224

3. Youssef, N.L., **Elgendi, S.G.**: Computing nullity and kernel vectors using NF-package:Counterexamples.
   Comput. Phys. Commun. 185 (11), 2859-2864, 2014. ISSN: 0010-4655.
   DOI: http://dx.doi.org/10.1016/j.cpc.2014.07.005
   IF:3.112

4. Youssef, N.L., **Elgendi, S.G.**: New Finsler package.
   Comput. Phys. Commun. 185 (3), 986-997, 2014. ISSN: 0010-4655.
   DOI: http://dx.doi.org/10.1016/j.cpc.2013.10.024
   IF:3.112

5. Youssef, N.L., **Elgendi, S.G.**: A note on "Sur le noyau de l'opérateur de courbure d'une variété finslérienne" [C. R. Acad. Sci. Paris, Ser. A 272 (1971) 807-810].
   C. R. Math. 351 (21-22), 829-832, 2013. ISSN: 1631-073X.
   DOI: http://dx.doi.org/10.1016/j.crma.2013.09.025
   IF:0.425

További közlemények

Idegen nyelvű közlemény(ek) külföldi folyóiratban (2)

6. Youssef, N.L., Abed, S.H., **Elgendi, S.G.**: Generalized [beta]-conformal change and special Finsler spaces.
   Int. J. Geom. Methods Mod. Phys. 9 (3), 1250016-1 - 1250016-25, 2012. ISSN: 0219-8878.
   DOI: http://dx.doi.org/10.1142/S0219887812500168
   IF:0.951

7. Youssef, N.L., Abed, S.H., **Elgendi, S.G.**: Generalized [beta]-conformal change of Finsler metrics.
   Int. J. Geom. Methods Mod. Phys. 7 (4), 565-582, 2010. ISSN: 0219-8878.
   DOI: http://dx.doi.org/10.1142/S0219887810004440
   IF:0.757

A közlő folyóiratok összesített impakt faktora: 9,084
A közlő folyóiratok összesített impakt faktora (az értekezés alapjául szolgáló közleményekre): 7,376

A DEENK a Jelölt által az iDEa Tudóstérbe feltöltött adatok bibliográfiai és tudománymetriai ellenőrzését a tudományos adatbázisok és a Journal Citation Reports Impact Factor lista alapján elvégezte.

Debrecen, 2016.06.13.