Partial Differential Equations in Applied Mathematics 11 (2024) 100836

Contents lists available at ScienceDirect

Partial Differential Equations in Applied Mathematics

journal homepage: www.elsevier.com/locate/padiff

New Grüss's inequalities estimates considering the φ-fractional integrals

Saleh S. Redhwan[a], Tariq A. Aljaaidi[b], Ali Hasan Ali[c,d,e,f,*], Maryam Ahmed Alyami[g], Mona Alsulami[g], Najla Alghamdi[g]

[a] Department of Mathematics, Al-Mahweet University, Al-Mahweet, Yemen
[b] Hajjah University, Hajjah City, Yemen
[c] Department of Mathematics, College of Education for Pure Sciences, University of Basrah, 61001 Basrah, Iraq
[d] Institute of Mathematics, University of Debrecen, Pf. 400, H-4002 Debrecen, Hungary
[e] College of Engineering Technology, National University of Science and Technology, 64001 Dhi Qar, Iraq
[f] Department of Business Management, Al-imam University College, 34011 Balad, Iraq
[g] Department of Mathematics and Statistics, College of Science, University of Jeddah, Jeddah 23218, Saudi Arabia

ARTICLE INFO

MSC:
26D10
26A33

Keywords:
Grüss inequalities
Fractional inequalities
φ-proportional fractional operators

ABSTRACT

Careful study of applied sciences and their development requires us to expand the scope of analytical studies. We aim during introducing the current manuscript to rediscover and present Grüss inequality in a new framework. In order to do that, we use the recently generalized proportional fractional integral operator for a certain function with respect to another continuous and strictly increasing function. Furthermore, we prove some new related inequalities using the current fractional integral operator. Some special cases of the presented results will be discussed.

1. Introduction

Due to its great importance and its wide applications in various applied sciences, fractional calculus is one of the most important new branches in mathematics, where fractional calculus is the extension and generalized case of ordinary calculus in the case of non-integer order.

In mathematics, we call an order relationship which is less than or equal, or less than, and greater than or equal, or greater than, between two quantities or two numbers an inequality. In real life, we note that inequalities are a generalized form of equalities or in other words, we can say that inequalities are including equalities. In almost all fields of mathematics, mathematical inequalities play a vital role as well as other branches of science like physics, economics, statistics, finance, etc.

Historically, the first systemic discipline of the inequalities topic as a book was written by Hardy, Littlewood, and Polya in (1934). After this date, about 27 years later, specifically in 1961, Bechanbach and Bellman issued the second book on this subject. Then, in 1971, Mitrinovic did another systematic work on inequalities. These works contributed to the transfer of mathematical inequalities to the area of systematic fields and became applicable in many fields.

Calculus has gone through many stages during its development, and because of the difficulty of dealing with it in some cases, the authors and writers resorted to the use of mathematical inequalities for the purpose of approximating solutions to differential and integral equations by finding upper limits for some quantities. This point led to the emergence of the concept of differential and integral inequality. One of the most important integration inequalities is the inequality proposed and proved by G. Grüss[1] (see also Ref. [2]) in (1935) as follows:

Theorem 1.1. *Let g, h be two integrable functions on $[a, b]$, satisfying the condition*

$$p \leq g(z) \leq P, q \leq h(z) \leq Q, p, P, q, Q \in \mathbb{R}, z > 0. \tag{1.1}$$

Then, the following inequality holds

$$\left| \frac{1}{b-a} \int_a^b g(z)\, h(z)\, dz - \frac{1}{(b-a)^2} \int_a^b g(z)\, dz \int_a^b h(z)\, dz \right| \leq \frac{1}{4} (P-p)(Q-q). \tag{1.2}$$

This inequality has achieved great importance due to its applications in mathematical fields such as difference equations, integral arithmetic mean, and h-integral arithmetic mean which is association with many modern and applied sciences (see Refs. [3], [4]).

---

* Corresponding author.
E-mail addresses: salehredhwan@zjnu.edu.cn (S.S. Redhwan), tariq10011@gmail.com (T.A. Aljaaidi), ali.hasan@science.unideb.hu (A.H. Ali), maalyami8@uj.edu.sa (M.A. Alyami), mralsolami@uj.edu.sa (M. Alsulami), nmalghamdi1@uj.edu.sa (N. Alghamdi).

https://doi.org/10.1016/j.padiff.2024.100836
Received 28 April 2024; Received in revised form 10 July 2024; Accepted 22 July 2024

Like other branches of mathematics, integral and differential inequalities had their share of enrichment with fractional calculus. Indeed, for the differential and fractional inequalities to become more effective and used, it was necessary to generalize them to the fractional orders. This prompted mathematicians to work on integral and differential inequalities within the framework of fractional calculus. Among these works, Dahmani[5] in (2010), presented the fractional version of inequality (1.2) by using Riemann–Liouville fractional integral as follows:

Theorem 1.2. *Let $g, h : [0, \infty) \longrightarrow \mathbb{R}$ be an integrable functions on $[0, \infty)$ satisfying the condition (1.1). Then, for all $\alpha > 0, y > 0$, the following inequality is holds*

$$\left| \frac{y^{\alpha}}{\Gamma(\alpha+1)} \mathcal{J}^{\alpha}_{0^+} \{gh\}(y) - \mathcal{J}^{\alpha}_{0^+} h(y) \mathcal{J}^{\alpha}_{0^+} g(y) \right|^2 \leq \frac{1}{4} \left(\frac{y^{\alpha}}{\Gamma(\alpha+1)} \right)^2 (Q-q)(P-p). \tag{1.3}$$

Same author in same work, gave the following inequality.

Theorem 1.3. *Let $g, h : [0, \infty) \longrightarrow \mathbb{R}$ be an integrable functions on $[0, \infty)$ satisfying the condition (1.1). Then, for all $\alpha, \delta > 0, y > 0$, the following inequality is holds*

$$\begin{aligned} &\left(\frac{y^{\delta}}{\Gamma(\delta+1)} \mathcal{J}^{\alpha}_{0^+} \{gh\}(y) - \mathcal{J}^{\delta}_{0^+} g(y) \mathcal{J}^{\alpha}_{0^+} h(y) \right. \\ &\left. + \frac{y^{\alpha}}{\Gamma(\alpha+1)} \mathcal{J}^{\delta}_{0^+} \{gh\}(y) - \mathcal{J}^{\delta}_{0^+} h(y) \mathcal{J}^{\alpha}_{0^+} g(y) \right)^2 \\ &\leq \left(\frac{P y^{\delta}}{\Gamma(\delta+1)} - \mathcal{J}^{\delta}_{0^+} g(y) \right) \left(\mathcal{J}^{\alpha}_{0^+} g(y) - \frac{p y^{\alpha}}{\Gamma(\alpha+1)} \right) \\ &+ \left(\frac{P y^{\alpha}}{\Gamma(\alpha+1)} - \mathcal{J}^{\alpha}_{0^+} g(y) \right) \left(\mathcal{J}^{\delta}_{0^+} g(y) - \frac{p y^{\delta}}{\Gamma(\delta+1)} \right) \\ &\times \left(\frac{Q y^{\delta}}{\Gamma(\delta+1)} - \mathcal{J}^{\delta}_{0^+} h(y) \right) \left(\mathcal{J}^{\alpha}_{0^+} h(y) - \frac{q y^{\alpha}}{\Gamma(\alpha+1)} \right) \\ &+ \left(\frac{Q y^{\alpha}}{\Gamma(\alpha+1)} - \mathcal{J}^{\alpha}_{0^+} h(y) \right) \left(\mathcal{J}^{\delta}_{0^+} h(y) - \frac{q y^{\delta}}{\Gamma(\delta+1)} \right). \end{aligned} \tag{1.4}$$

Tariboon et al.,[6] in (2014), introduced a new fractional integral version of inequality (1.2) by replacing the constants p, P, q, Q with four positive integrable functions as follows:

Theorem 1.4. *Let $g, h : [0, \infty) \longrightarrow \mathbb{R}$ be an integrable functions on $[0, \infty)$ satisfying the condition*

$$u_1(z) \leq g(z) \leq u_2(z), v_1(z) \leq h(z) \leq v_2(z), z > 0 \tag{1.5}$$

for the integrable functions u_1, u_2, v_1, v_2 on $[0, \infty)$. Then, for all $\alpha > 0$ the following inequality is holds

$$\left[\frac{y^{\alpha}}{\Gamma(\alpha+1)} \mathcal{J}^{\alpha}_{0^+} \{hg\}(y) - \left(\mathcal{J}^{\alpha}_{0^+} h(y) \mathcal{J}^{\alpha}_{0^+} g(y) \right) \right]^2 \leq T\left(h, u_1, u_2\right) T\left(g, v_1, v_2\right), \tag{1.6}$$

where $T(\zeta, \varsigma, \omega)$ is defined by

$$\begin{aligned} &T(\zeta, \varsigma, \omega) \\ &= \left(\mathcal{J}^{\alpha}_{0^+} \omega(y) - \mathcal{J}^{\alpha}_{0^+} \zeta(y) \right) \left(\mathcal{J}^{\alpha}_{0^+} \zeta(y) - \mathcal{J}^{\alpha}_{0^+} \varsigma(y) \right) \\ &+ \frac{y^{\alpha}}{\Gamma(\alpha+1)} \mathcal{J}^{\alpha}_{0^+} \{\zeta\varsigma\}(y) - \mathcal{J}^{\alpha}_{0^+} \zeta(y) \mathcal{J}^{\alpha}_{0^+} \varsigma(y) \\ &+ \frac{y^{\alpha}}{\Gamma(\alpha+1)} \mathcal{J}^{\alpha}_{0^+} \{\zeta\omega\}(y) - \mathcal{J}^{\alpha}_{0^+} \zeta(y) \mathcal{J}^{\alpha}_{0^+} \omega(y) \\ &- \frac{y^{\alpha}}{\Gamma(\alpha+1)} \mathcal{J}^{\alpha}_{0^+} \{\varsigma\omega\}(y) + \mathcal{J}^{\alpha}_{0^+} \varsigma(y) \mathcal{J}^{\alpha}_{0^+} \omega(y). \end{aligned}$$

Inequality (1.2) also have a fractional integral version for functional bounds had provided by Aljaaidi and Pachpatte[7] (2020) by employing Katugampola fractional integral, same authors at the same year, provided same fractional integral inequality for functional bounds by employing ψ-Riemann–Liouville fractional integral (see Ref. 8). Dragomir[9] (2012), gave some new Grüss' type inequalities for bounded variation functions and some applications in Hilbert spaces for self-adjoint operators. Alomari[10] (2014), presented some new Grüss type inequalities containing double integral and some sharp bounds as well. In same year, Chinchane and Pachpatte,[11] presented some new Grüss-type inequality via Hadamard fractional integral operator. Liu and Tuna[12] (2015), obtained some weighted Grüss type and Ostrowski type inequalities on the frame of time scales by employing the theory of combined dynamic derivatives on the frame of time scales. Sousa et al.[13] (2019), used a Katugampola fractional integral to give new generalized Grüss type inequality. Rashid et al.[14] (2020), established some Grüss-type inequalities by employing generalized proportional fractional integral. Zhou et al.[15] (2020), described the Grüss inequality and they provided some related inequalities employing generalized proportional Hadamard fractional integral. Recently, Naz et al.[16] (2021), proved several Grüss type inequalities via generalized Hilfer–Katugampola k-fractional derivative. Also, Al Qurashi et al.[17] (2021), obtained some discrete dynamical Grüss type inequalities involving h-discrete Atangana–Baleanu fractional operator. For some interesting studies of Grüss inequality, see Refs. 18, 19.

The great importance and the huge number of previous studies on this inequality prompted the researchers in the current manuscript to present a new generalization for Grüss inequality. We develop and present Grü ss inequality in a more general way so that mathematicians can use it better and more effectively. In order to do that, we use the recently generalized proportional fractional integral operator for any function with respect to another continuous and strictly increasing function. Furthermore, we prove some new related inequalities using the current fractional integral operator. Some special cases of the presented results will be discussed.

This paper is organized as follows: The Section 2 is containing some definitions, properties, and facts of the used fractional integral operators. Throughout the Section 3, we present our major results of Grüss inequalities. Other related fractional integral inequalities of the Grüss type are included in the last section.

2. Essential preliminaries

This section is concerned with providing some requisite definitions and some properties of some prime fractional operators, including the current fractional integral operators we applied to present and discuss the new required results.

Definition 2.1 (*Ref. 20*)**.** Assume that the function g is integrable on $[a, b]$ and $a \geq 0$. The notations $\mathcal{J}^{\alpha}_{a^+} g(y)$ and $\mathcal{J}^{\alpha}_{b^-} g(y)$ are called the left and right-sided Riemann–Liouville fractional integrals and defined for all $\alpha > 0$, respectively, as

$$\mathcal{J}^{\alpha}_{a^+} g(y) = \frac{1}{\Gamma(\alpha)} \int_a^y (y-z)^{\alpha-1} g(z)\, dz, z > a \tag{2.1}$$

and

$$\mathcal{J}^{\alpha}_{b^-} g(y) = \frac{1}{\Gamma(\alpha)} \int_y^b (z-y)^{\alpha-1} g(z)\, dz, y < b. \tag{2.2}$$

Definition 2.2 (*Refs. 20, 21*)**.** Assume that the function g is integrable on the interval Λ and the function φ be an increasing function on Λ, where $\varphi(y) \in C^1(\Lambda, \mathbb{R})$ such that $\varphi'(y) \neq 0$, $y \in \Lambda$. The notations ${}^{\varphi}\mathcal{J}^{\alpha}_{a^+} g(y)$ and ${}^{\varphi}\mathcal{J}^{\alpha}_{b^-} g(y)$ are called the left and right-sided Riemann–Liouville fractional integrals with respect to the function φ and defined for all $\alpha > 0$, respectively, as

$${}^{\varphi}\mathcal{J}^{\alpha}_{a^+} g(y) = \frac{1}{\Gamma(\alpha)} \int_a^y \varphi'(z) [\varphi(y) - \varphi(z)]^{\alpha-1} g(z)\, dz \tag{2.3}$$

and

$${}^{\varphi}\mathcal{J}^{\alpha}_{b^-} g(y) = \frac{1}{\Gamma(\alpha)} \int_y^b \varphi'(z) [\varphi(z) - \varphi(y)]^{\alpha-1} g(z)\, dz. \tag{2.4}$$

Definition 2.3 (*Ref.* 22)**.** Assume that the function g is integrable on the interval Λ. Let $\varpi > 0$, the notations $\left(D_{a^+}^{\alpha,\varpi} g\right)(y)$ and $\left(D_{b^-}^{\alpha,\varpi} g\right)(y)$ are called the left and right-sided proportional fractional derivatives and defined for all $\alpha \in \mathbb{C}$, $Re(\alpha) \geq 0$, respectively, as

$$\begin{aligned}\left(D_{a^+}^{\alpha,\varpi} g\right)(y) &= D^{m,\varpi}\ \mathscr{I}_{a^+}^{m-\alpha,\varpi}\ g(y) \\ &= \frac{D_y^{m,\varpi}}{\varpi^{m-\alpha}\Gamma(m-\alpha)} \int_a^y \exp\left[\frac{\varpi-1}{\varpi}(y-z)\right](y-z)^{m-\alpha-1} g(z)\,dz\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\left(D_{b^-}^{\alpha,\varpi} g\right)(y) &= {}_{\gamma}D^{m,\varpi}\ \mathscr{I}_{b^-}^{m-\alpha,\varpi}\ g(y) \\ &= \frac{{}_{\gamma}D_y^{m,\varpi}}{\varpi^{m-\alpha}\Gamma(m-\alpha)} \int_y^b \exp\left[\frac{\varpi-1}{\varpi}(z-y)\right](z-y)^{m-\alpha-1} g(z)\,dz,\end{aligned} \tag{2.6}$$

where

$$D^{m,\varpi} = \underbrace{D^{\varpi} D^{\varpi} \ldots D^{\varpi}}_{m-times}, m = [Re(\alpha)] + 1$$

and

$$\left({}_{\gamma}D^{\varpi} g\right)(y) = (1-\varpi)\, g(y) - \varpi g'(y), {}_{\gamma}D^{m,\varpi} = \underbrace{{}_{\gamma}D_{\gamma}^{\varpi} D^{\varpi} \ldots {}_{\gamma}D^{\varpi}}_{m-times}.$$

Definition 2.4 (*Ref.* 22)**.** Assume that the function g is integrable on the interval Λ. Let $\varpi > 0$, the notations $\left(\mathscr{I}_{a^+}^{\alpha,\varpi}\ g\right)(y)$ and $\left(\mathscr{I}_{b^-}^{\alpha,\varpi}\ g\right)(y)$ are called the left and right-sided proportional fractional integrals and defined for all $\alpha \in \mathbb{C}$, $Re(\alpha) \geq 0$, respectively, as

$$\left(\mathscr{I}_{a^+}^{\alpha,\varpi}\ g\right)(y) = \frac{1}{\varpi^{\alpha}\Gamma(\alpha)} \int_a^y \exp\left[\frac{\varpi-1}{\varpi}(y-z)\right](y-z)^{\alpha-1} g(z)\,dz \tag{2.7}$$

and

$$\left(\mathscr{I}_{b^-}^{\alpha,\varpi}\ g\right)(y) = \frac{1}{\varpi^{\alpha}\Gamma(\alpha)} \int_y^b \exp\left[\frac{\varpi-1}{\varpi}(z-y)\right](z-y)^{\alpha-1} g(z)\,dz. \tag{2.8}$$

Definition 2.5 (*Ref.* 23)**.** Assume that the function g is integrable on the interval Λ and the function φ be a strictly increasing continuous on $[a,b]$. Let $\varpi \in (0,1]$, the notations $\left({}^{\varphi}D_{a^+}^{\alpha,\varpi} g\right)(y)$ and $\left({}^{\varphi}D_{b^-}^{\alpha,\varpi} g\right)(y)$ are called the left and right-sided proportional fractional derivatives and defined for all $\alpha \in \mathbb{C}$, $Re(\alpha) \geq 0$, respectively, as

$$\begin{aligned}\left({}^{\varphi}D_{a^+}^{\alpha,\varpi} g\right)(y) &= {}^{\varphi}D^{m,\varpi}\ {}^{\varphi}\mathscr{I}_{a^+}^{m-\alpha,\varpi}\ g(y) \\ &= \frac{{}^{\varphi}D_y^{m,\varpi}}{\varpi^{m-\alpha}\Gamma(m-\alpha)} \int_a^y \exp\left[\frac{\varpi-1}{\varpi}(\varphi(y)-\varphi(z))\right] \\ &\quad \times (\varphi(y)-\varphi(z))^{m-\alpha-1} \varphi'(z)\, g(z)\,dz\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}\left({}^{\varphi}D_{b^-}^{\alpha,\varpi} g\right)(y) &= {}^{\varphi}_{\gamma}D^{m,\varpi}\ {}^{\varphi}\mathscr{I}_{b^-}^{m-\alpha,\varpi}\ g(y) \\ &= \frac{{}^{\varphi}_{\gamma}D_y^{m,\varpi}}{\varpi^{m-\alpha}\Gamma(m-\alpha)} \int_y^b \exp\left[\frac{\varpi-1}{\varpi}(\varphi(z)-\varphi(y))\right] \\ &\quad \times (\varphi(z)-\varphi(y))^{m-\alpha-1} \varphi'(z)\, g(z)\,dz,\end{aligned} \tag{2.10}$$

where

$${}^{\varphi}D^{m,\varpi} = \underbrace{{}^{\varphi}D^{\varpi\,\varphi}D^{\varpi} \ldots {}^{\varphi}D^{\varpi}}_{m-times}, m = [Ra(\alpha)] + 1$$

and

$$\left({}^{\varphi}_{\gamma}D^{\varpi} g\right)(y) = (1-\varpi)\, g(y) - \varpi\frac{g'(y)}{\varphi'(y)}, {}^{\varphi}_{\gamma}D^{m,\varpi} = \underbrace{{}^{\varphi}_{\gamma}D^{\varpi}\ {}^{\varphi}_{\gamma}D^{\varpi} \ldots {}^{\varphi}_{\gamma}D^{\varpi}}_{m-times}.$$

Definition 2.6 (*Ref.* 23)**.** Assume that the function g is integrable on the interval Λ and the function φ be a strictly increasing continuous on $[a,b]$. Let $\varpi \in (0,1]$, the notations $\left({}^{\varphi}\mathscr{I}_{a^+}^{\alpha,\varpi} g\right)(y)$ and $\left({}^{\varphi}\mathscr{I}_{b^-}^{\alpha,\varpi} g\right)(y)$ are called respectively the left and right-sided proportional fractional integrals and defined for all $\alpha \in \mathbb{C}$, $Re(\alpha) \geq 0$, respectively, as

$$\begin{aligned}\left({}^{\varphi}\mathscr{I}_{a^+}^{\alpha,\varpi} g\right)(y) &= \frac{1}{\varpi^{\alpha}\Gamma(\alpha)} \int_a^y \exp\left[\frac{\varpi-1}{\varpi}(\varphi(y)-\varphi(z))\right] \\ &\quad \times (\varphi(y)-\varphi(z))^{\alpha-1} \varphi'(z)\, g(z)\,dz\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\left({}^{\varphi}\mathscr{I}_{b^-}^{\alpha,\varpi} g\right)(y) &= \frac{1}{\varpi^{\alpha}\Gamma(\alpha)} \int_y^b \exp\left[\frac{\varpi-1}{\varpi}(\varphi(z)-\varphi(y))\right] \\ &\quad \times (\varphi(z)-\varphi(y))^{\alpha-1} \varphi'(z)\, g(z)\,dz.\end{aligned} \tag{2.12}$$

Along this work, we use the following identity as in Ref. 24.

Let $\varpi \in (0,1]$, $\alpha \in \mathbb{C}$, $Re(\alpha) \geq 0$ and φ be a strictly increasing continuous function. Then, for any constant A, we have

$$\left({}^{\varphi}\mathscr{I}_{c^+}^{\alpha,\varpi} A\right)(y) = \frac{(\varphi(y)-\varphi(c))^{\alpha}}{\varpi^{\alpha}\Gamma(\alpha+1)} A. \tag{2.13}$$

In order, to facilitate the display of our new results, we define the functions ${}^{\varphi}\Theta^{\varpi}(\alpha, y)$, ${}^{\varphi}\Xi^{\varpi}(\alpha, z)$ as follows:

Let $\varphi : \Lambda \longrightarrow [0,\infty) \subseteq \mathbb{R}$, be a continuous strictly increasing function. Then, for $\alpha > 0$, $z > 0$ $\varpi \in (0,1]$, we have

$${}^{\varphi}\Theta^{\varpi}(\alpha, c) = \frac{(\varphi(y)-\varphi(c))^{\alpha}}{\varpi^{\alpha}\Gamma(\alpha+1)} \tag{2.14}$$

and

$${}^{\varphi}\Xi^{\varpi}(\alpha, z) = \frac{1}{\varpi^{\alpha}\Gamma(\alpha)} e^{\left[\frac{\varpi-1}{\varpi}(\varphi(y)-\varphi(z))\right]} (\varphi(y)-\varphi(z))^{\alpha-1} \varphi'(z). \tag{2.15}$$

3. φ-proportional fractional integral Grüss inequalities

This section is concerned with giving our new generalization of Grüss inequality for the recent operator the proportional fractional integral with respect to another strictly increasing and continuous function. This work included in this part is motivated by the work presented by Dahmani et al.[5]

The following lemma is required to prove the next result.

Lemma 3.1. *Let* $\varphi : \Lambda \longrightarrow [0,\infty) \subseteq \mathbb{R}$ *be a continuous strictly increasing function and* $\eta : [0,\infty) \longrightarrow \mathbb{R}$ *be an integrable function on* $[0,\infty)$*. If the function* η *satisfying the condition* $p \leq \eta(z) \leq P$*, where* p *and* $P \in \mathbb{R}$*,* $z > 0$*. Then, for all* $\alpha > 0$*, the following identity is holds*

$$\begin{aligned}&{}^{\varphi}\Theta^{\varpi}(\alpha,0)\ {}^{\varphi}\mathscr{I}_{0^+}^{\alpha,\varpi}\eta^2(y) - 2\left({}^{\varphi}\mathscr{I}_{0^+}^{\alpha,\varpi}\eta(y)\right)^2 \\ &= \left({}^{\varphi}\mathscr{I}_{0^+}^{\alpha,\varpi}\eta(y) - p\ {}^{\varphi}\Theta^{\varpi}(\alpha,0)\right)\left(P\ {}^{\varphi}\Theta^{\varpi}(\alpha,0) - {}^{\varphi}\mathscr{I}_{0^+}^{\alpha,\varpi}\eta(y)\right) \\ &\quad - {}^{\varphi}\Theta^{\varpi}(\alpha,0)\ {}^{\varphi}\mathscr{I}_{0^+}^{\alpha,\varpi}(P-\eta(y))(\eta(y)-p).\end{aligned} \tag{3.1}$$

Proof. For any $z_1, z_2 \in [0,\infty)$, we have

$$\begin{aligned}&\left(P-\eta\left(z_2\right)\left(\eta\left(z_1\right)-p\right)\right) + \left(P-\eta\left(z_1\right)\right)\left(\eta\left(z_2\right)-p\right) \\ &\quad - \left(P-\eta\left(z_1\right)\right)\left(\eta\left(z_1\right)-p\right) - \left(P-\eta\left(z_2\right)\right)\left(\eta\left(z_2\right)-p\right) \\ &= \eta^2\left(z_1\right) + \eta^2\left(z_2\right) - 2\eta\left(z_1\right)\eta\left(z_2\right).\end{aligned} \tag{3.2}$$

Taking product on both sides of (3.2) by the function ${}^{\varphi}\Xi^{\varpi}\left(\alpha, z_1\right)$ and integrating the estimating identity with respect to z_1 over $(0, y)$, we get

$$\begin{aligned}&\left(P-\eta\left(z_2\right)\right)\left({}^{\varphi}\mathscr{I}_{0^+}^{\alpha,\varpi}\eta(y) - p^{\varphi}\Theta^{\varpi}(\alpha,0)\right) \\ &\quad + \left(P^{\varphi}\Theta^{\varpi}(\alpha,0) - {}^{\varphi}\mathscr{I}_{0^+}^{\alpha,\varpi}\eta(y)\right)\left(\eta\left(z_2\right)-p\right) \\ &\quad -{}^{\varphi}\mathscr{I}_{0^+}^{\alpha,\varpi}(P-\eta(y))(\eta(y)-p) - {}^{\varphi}\Theta^{\varpi}(\alpha,0)\left(P-\eta\left(z_2\right)\right)\left(\eta\left(z_2\right)-p\right) \\ &= {}^{\varphi}\mathscr{I}_{0^+}^{\alpha,\varpi}\eta^2(y) + {}^{\varphi}\Theta^{\varpi}(\alpha,0)\eta^2\left(z_2\right) - 2\eta\left(z_2\right){}^{\varphi}\mathscr{I}_{0^+}^{\alpha,\varpi}\eta(y).\end{aligned} \tag{3.3}$$

Now, taking product on both sides of (3.3) by the function ${}^{\varphi}\Xi^{\varpi}\left(\alpha, z_2\right)$ and integrating the estimating identity with respect to z_2 over $(0, y)$, we get

$$\left({}^{\varphi}\mathscr{I}_{0^+}^{\alpha,\varpi}\eta(y) - p\ {}^{\varphi}\Theta^{\varpi}(\alpha,0)\right)\int_0^y {}^{\varphi}\Xi^{\varpi}\left(\alpha, z_2\right)\left(P-\eta\left(z_2\right)\right)dz_2$$

$$\begin{aligned}
&+\left(P\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)-{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\eta(y)\right)\int_0^y {}^{\varphi}\Xi^{\varpi}\left(\alpha,z_2\right)\left(\eta\left(z_2\right)-p\right)dz_2\\
&-{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\left\{(P-\eta(y))(\eta(y)-p)\right\}\int_0^y {}^{\varphi}\Xi^{\varpi}\left(\alpha,z_2\right)dz_2\\
&-{}^{\varphi}\Theta^{\varpi}(\alpha,0)\int_0^y {}^{\varphi}\Xi^{\varpi}\left(\alpha,z_2\right)\left(P-\eta\left(z_2\right)\right)\left(\eta\left(z_2\right)-p\right)dz_2\\
=&\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\eta^2(y)\int_0^y {}^{\varphi}\Xi^{\varpi}\left(\alpha,z_2\right)dz_2\\
&+{}^{\varphi}\Theta^{\varpi}(\alpha,0)\int_0^y {}^{\varphi}\Xi^{\varpi}\left(\alpha,z_2\right)\eta^2\left(z_2\right)dz_2\\
&-2\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\,\eta(y)\int_0^y {}^{\varphi}\Xi^{\varpi}\left(\alpha,z_2\right)\eta\left(z_2\right)dz_2, \qquad (3.4)
\end{aligned}$$

which gives

$$\begin{aligned}
&\left({}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\eta(y)-{}^{\varphi}\Theta^{\varpi}(\alpha,0)\,p\right)\left(P\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)-{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\eta(y)\right)\\
&+\left(P\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)-{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\eta(y)\right)\left({}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\eta(y)-p\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)\right)\\
&-{}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\left\{(P-\eta(y))(\eta(y)-p)\right\}\\
&-{}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\left\{(P-\eta(y))(\eta(y)-p)\right\}\\
&={}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\eta^2(y)+{}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\eta^2(y)\\
&-2\left({}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\eta(y)\right)^2, \qquad (3.5)
\end{aligned}$$

and clearly, identity (3.5) is leading to the desired identity (3.1), which completes the proof. □

Theorem 3.1. *Let $\varphi : \Lambda \longrightarrow [0,\infty) \subseteq \mathbb{R}$ be a continuous strictly increasing function and $g,h : [0,\infty) \longrightarrow \mathbb{R}$ be an integrable functions on $[0,\infty)$. Suppose that the functions g,h satisfying the condition*

$$p \le g(z) \le P,\ q \le h(z) \le Q,\ p,P,q,Q \in \mathbb{R},\ z>0.$$

Then, for all $\alpha>0$ the following inequality is holds

$$\begin{aligned}
&\left|{}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\{gh\}(y)-{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}h(y)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}g(y)\right|^2\\
&\le\frac{1}{4}\left({}^{\varphi}\Theta^{\varpi}(\alpha,0)\right)^2(Q-q)(P-p). \qquad (3.6)
\end{aligned}$$

Proof. Define the function $H\left(z_1,z_2\right)$ as follows

$$H\left(z_1,z_2\right)=\left(g\left(z_1\right)-g\left(z_2\right)\right)\left(h\left(z_1\right)-h\left(z_2\right)\right),z_1 \textit{ and } z_2\in(0,y),y>0. \qquad (3.7)$$

So, we have

$$H\left(z_1,z_2\right)=g\left(z_1\right)h\left(z_1\right)-g\left(z_1\right)h\left(z_2\right)-g\left(z_2\right)h\left(z_1\right)+g\left(z_2\right)h\left(z_2\right). \qquad (3.8)$$

Taking product on both sides of (3.8) by the function ${}^{\varphi}\Xi^{\varpi}\left(\alpha,z_1\right)$, then integrate the estimating identity with respect to z_1 over $(0,y)$, we get

$$\begin{aligned}
\int_0^y {}^{\varphi}\Xi^{\varpi}\left(\alpha,z_1\right)H\left(z_1,z_2\right)dz_1 =&\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\{gh\}(y)-h\left(z_2\right){}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}g(y)\\
&-g\left(z_2\right){}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}h(y)+{}^{\varphi}\Theta^{\varpi}(\alpha,0)\,g\left(z_2\right)h\left(z_2\right). \qquad (3.9)
\end{aligned}$$

Taking product on both sides of (3.9) by the function ${}^{\varphi}\Xi^{\varpi}\left(\alpha,z_2\right)$, then integrate the estimating identity with respect to z_2 over $(0,y)$, we get

$$\begin{aligned}
&\int_0^y\int_0^y {}^{\varphi}\Xi^{\varpi}\left(\alpha,z_1\right){}^{\varphi}\Xi^{\varpi}\left(\alpha,z_2\right)H\left(z_1,z_2\right)dz_1dz_2\\
&=2\left({}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\{gh\}(y)-{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}h(y)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}g(y)\right). \qquad (3.10)
\end{aligned}$$

Applying the Cauchy–Schwarz inequality[25] to the Right-hand-side of (3.10), we can certain

$$\begin{aligned}
&\left({}^{\varphi}\Theta^{\varpi}(\alpha,0)^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\{gh\}(y)-{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}h(y)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}g(y)\right)^2\\
&\le\left({}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}g^2(y)-\left({}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}g(y)\right)^2\right)\\
&\times\left({}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}h^2(y)-\left({}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}h(y)\right)^2\right). \qquad (3.11)
\end{aligned}$$

Since $(P-g(z))(g(z)-p)\ge 0$ and $(Q-h(z))(h(z)-q)\ge 0$, we have

$${}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}(P-g(y))(g(y)-p)\ge 0$$

and

$${}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}(Q-h(y))(h(y)-q)\ge 0.$$

Therefore, using Lemma 3.1, we have

$$\begin{aligned}
&{}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}g^2(y)-\left({}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}g(y)\right)^2 \qquad (3.12)\\
&\le\left(P\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)-{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}g(y)\right)\left({}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}g(y)-p\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)\right)
\end{aligned}$$

and

$$\begin{aligned}
&{}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}h^2(y)-\left({}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}h(y)\right)^2 \qquad (3.13)\\
&\le\left(Q\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)-{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}h(y)\right)\left({}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}h(y)-q\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)\right).
\end{aligned}$$

By comparing the inequalities (3.11)–(3.13), we obtain

$$\begin{aligned}
&\left({}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\{gh\}(y)-{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}h(y)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}g(y)\right)^2 \qquad (3.14)\\
&\le\left(P\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)-{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}g(y)\right)\left({}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}g(y)-p\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)\right)\\
&\times\left(Q\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)-{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}h(y)\right)\left({}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}h(y)-q\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)\right).
\end{aligned}$$

Now, Applying the inequality $4cd\le(c+d)^2,\ c,d\in\mathbb{R}$, we can certain

$$\begin{aligned}
&4\left(P\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)-{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}g(y)\right)\left({}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}g(y)-p\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)\right)\\
&\le\left({}^{\varphi}\Theta^{\varpi}(\alpha,0)(P-p)\right)^2 \qquad (3.15)
\end{aligned}$$

and

$$\begin{aligned}
&4\left(Q\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)-{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}h(y)\right)\left({}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}h(y)-q\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)\right)\\
&\le\left({}^{\varphi}\Theta^{\varpi}(\alpha,0)(Q-q)\right)^2. \qquad (3.16)
\end{aligned}$$

Using the inequalities (3.14)–(3.16), we immediately get the desired inequality (3.6). The proof is thus completed. □

The next corollary is a generalized version of Theorem 3.1 for functional bounds.

Corollary 3.1. *Let $\varphi : \Lambda \longrightarrow [0,\infty) \subseteq \mathbb{R}$ be a continuous strictly increasing function and $g,h : [0,\infty) \longrightarrow \mathbb{R}$ be an integrable functions on $[0,\infty)$. Suppose that the functions g,h satisfying the condition (1.5) for the integrable functions u_1,u_2,v_1,v_2 on $[0,\infty)$. Then, for all $\alpha>0$ the following inequality is holds*

$$\begin{aligned}
&\left[{}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\{hg\}(y)-\left({}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}h(y)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}g(y)\right)\right]^2\\
&\qquad\le T\left(h,u_1,u_2\right)T\left(g,v_1,v_2\right), \qquad (3.17)
\end{aligned}$$

where $T(\zeta,\varsigma,\omega)$ is defined by

$$\begin{aligned}
T(\zeta,\varsigma,\omega)=&\left({}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\omega(y)-{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\zeta(y)\right)\left({}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\zeta(y)-{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\varsigma(y)\right)\\
&+{}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\{\zeta\varsigma\}(y)-{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\zeta(y)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\varsigma(y)\\
&+{}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\{\zeta\omega\}(y)-{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\zeta(y)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\omega(y)\\
&-{}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\{\varsigma\omega\}(y)+{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\varsigma(y)\,{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\omega(y).
\end{aligned}$$

Remark 3.1.

(i) Putting $\varphi(y)=y$ and $\varpi=1$, in Theorem 3.1, we get its Riemann–Liouville fractional integral version obtained by Dahmani et al.[5]

(ii) Putting $\varphi(y)=y$ and $\varpi=1$, in Corollary 3.1, we get its Riemann–Liouville fractional integral version obtained by Tariboon et al.[6]

(iii) Applying Theorem 3.1 for $\varphi(y)=y$, $\varpi=1$ and $\alpha=1$, we get the classical Grüss inequality (1.2).

Now, we give the following lemma which is required to prove the next result.

Lemma 3.2. *Let $\varphi : \varLambda \longrightarrow [0,\infty) \subseteq \mathbb{R}$ be a continuous strictly increasing function and $g,h : [0,\infty) \longrightarrow \mathbb{R}$ be an integrable functions on $[0,\infty)$. Suppose that the functions g,h satisfying the condition (1.1). Then, for all $\alpha > 0, \delta > 0$, we have*

$$\begin{aligned}&\Big({}^{\varphi}\Theta^{\varpi}(\delta,0)\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}\{gh\}(y) - {}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}g(y)\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}h(y)\\ &\quad + {}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}\{gh\}(y) - {}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}h(y)\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}g(y)\Big)^{2}\\ &\quad \le \Big({}^{\varphi}\Theta^{\varpi}(\delta,0)\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}g^{2}(y) + {}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}g^{2}(y)\\ &\quad - 2\,{}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}g(y)\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}g(y)\Big)\\ &\quad \times \Big({}^{\varphi}\Theta^{\varpi}(\delta,0)\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}h^{2}(y) + {}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}h^{2}(y)\\ &\quad - 2\,{}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}h(y)\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}h(y)\Big).\end{aligned}\tag{3.18}$$

Proof. Taking product on both sides of (3.9) by the function ${}^{\varphi}\varXi^{\varpi}(\delta,z_2)$, then integrate the estimating identity with respect to z_2 over $(0,y)$, we get

$$\begin{aligned}&\int_0^y\int_0^y \varXi^{\varpi}(\alpha,z_1)\,{}^{\varphi}\varXi^{\varpi}(\delta,z_2)\,H(z_1,z_2)\,dz_1dz_2\\ &= {}^{\varphi}\Theta^{\varpi}(\delta,0)\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}\{gh\}(y) + {}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}\{gh\}(y)\\ &- {}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}g(y)\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}h(y) - {}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}h(y)\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}g(y).\end{aligned}\tag{3.19}$$

Applying the Cauchy–Schwarz inequality[25] to the Right-hand-side of (3.10), we get the desired inequality (3.18). □

Lemma 3.3. *Let $\varphi : \varLambda \longrightarrow [0,\infty) \subseteq \mathbb{R}$ be a continuous strictly increasing function and $\eta : [0,\infty) \longrightarrow \mathbb{R}$ be an integrable functions on $[0,\infty)$. Suppose that the functions η satisfying the condition (1.1). Then, for all $\alpha > 0, \delta > 0$, we have*

$$\begin{aligned}&{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}\eta^{2}(y)\,{}^{\varphi}\Theta^{\varpi}(\delta,0) + {}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}\eta^{2}(y)\\ &-2\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}\eta(y)\,{}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}\eta(y)\\ &= \Big(P\,{}^{\varphi}\Theta^{\varpi}(\delta,0) - {}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}\eta(y)\Big)\Big({}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}\eta(y) - p\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)\Big)\\ &+ \Big(P\,{}^{\varphi}\Theta^{\varpi}(\alpha,0) - {}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}\eta(y)\Big)\Big({}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}\eta(y) - p\,{}^{\varphi}\Theta^{\varpi}(\delta,0)\Big)\\ &- {}^{\varphi}\Theta^{\varpi}(\delta,0)\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}\{(P-\eta(y))(\eta(y)-p)\}\\ &- {}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}\{(P-\eta(y))(\eta(y)-p)\}.\end{aligned}\tag{3.20}$$

Proof. In view of Lemma 3.1, taking product on both sides of (3.3) by the function ${}^{\varphi}\varXi^{\varpi}(\delta,z_2)$ and integrating the estimating identity with respect to z_2 over $(0,y)$, we get

$$\begin{aligned}&\Big({}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}\eta(y) - p\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)\Big)\int_0^y {}^{\varphi}\varXi^{\varpi}(\delta,z_2)\,(P-\eta(z_2))\,dz_2\\ &+ \Big(P\,{}^{\varphi}\Theta^{\varpi}(\alpha,0) - {}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}\eta(y)\Big)\int_0^y {}^{\varphi}\varXi^{\varpi}(\delta,z_2)\,(\eta(z_2)-p)\,dz_2\\ &- {}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}\{(P-\eta(y))(\eta(y)-p)\}\int_0^y {}^{\varphi}\varXi^{\varpi}(\delta,z_2)\,dz_2\\ &- {}^{\varphi}\Theta^{\varpi}(\alpha,0)\int_0^y {}^{\varphi}\varXi^{\varpi}(\delta,z_2)\,(P-\eta(z_2))\,(\eta(z_2)-p)\,dz_2\\ &= {}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}\eta^{2}(y)\int_0^y {}^{\varphi}\varXi^{\varpi}(\delta,z_2)\,dz_2\\ &+ {}^{\varphi}\Theta^{\varpi}(\alpha,0)\int_0^y {}^{\varphi}\varXi^{\varpi}(\delta,z_2)\,\eta^{2}(z_2)\,dz_2\\ &-2\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}\eta(y)\int_0^y {}^{\varphi}\varXi^{\varpi}(\delta,z_2)\,\eta(z_2)\,dz_2,\end{aligned}\tag{3.21}$$

which leads to the required identity (3.20). □

Theorem 3.2. *Let $\varphi : \varLambda \longrightarrow [0,\infty) \subseteq \mathbb{R}$ be a continuous strictly increasing function and $g,h : [0,\infty) \longrightarrow \mathbb{R}$ be an integrable functions on $[0,\infty)$. Suppose that the functions g,h satisfying the condition (1.1). Then, for all $\alpha > 0, \delta > 0$, we have*

$$\begin{aligned}&\Big({}^{\varphi}\Theta^{\varpi}(\delta,0)\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}\{gh\}(y) - {}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}g(y)\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}h(y)\\ &+ {}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}\{gh\}(y) - {}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}h(y)\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}g(y)\Big)^{2}\\ &\le \Big(P\,{}^{\varphi}\Theta^{\varpi}(\delta,0) - {}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}g(y)\Big)\Big({}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}g(y) - p\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)\Big)\\ &+ \Big(P\,{}^{\varphi}\Theta^{\varpi}(\alpha,0) - {}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}g(y)\Big)\Big({}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}g(y) - p\,{}^{\varphi}\Theta^{\varpi}(\delta,0)\Big)\\ &\times \Big(Q\,{}^{\varphi}\Theta^{\varpi}(\delta,0) - {}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}h(y)\Big)\Big({}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}h(y) - q\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)\Big)\\ &+ \Big(Q\,{}^{\varphi}\Theta^{\varpi}(\alpha,0) - {}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}h(y)\Big)\Big({}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}h(y) - q\,{}^{\varphi}\Theta^{\varpi}(\delta,0)\Big).\end{aligned}\tag{3.22}$$

Proof. Since $(P-g(y))(g(y)-p) \ge 0$ and $(Q-h(y))(h(y)-q) \ge 0$. Therefore,

$$\begin{aligned}&- {}^{\varphi}\Theta^{\varpi}(\delta,0)\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}\{(P-g(y))(g(y)-p)\}\\ &- {}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}\{(P-g(y))(g(y)-p)\}\\ &\le 0\end{aligned}\tag{3.23}$$

and

$$\begin{aligned}&- {}^{\varphi}\Theta^{\varpi}(\delta,0)\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}\{(Q-h(y))(h(y)-q)\}\\ &- {}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}\{(Q-h(y))(h(y)-q)\}\\ &\le 0.\end{aligned}\tag{3.24}$$

Applying Lemma 3.3 and using (3.23) and (3.24), we can write

$$\begin{aligned}&{}^{\varphi}\Theta^{\varpi}(\delta,0)\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}g^{2}(y) + {}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}g^{2}(y)\\ &-2\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}g(y)\,{}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}g(y)\\ &\le \Big(P\,{}^{\varphi}\Theta^{\varpi}(\delta,0) - {}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}g(y)\Big)\Big({}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}g(y) - p\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)\Big)\\ &+ \Big(P\,{}^{\varphi}\Theta^{\varpi}(\alpha,0) - {}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}g(y)\Big)\Big({}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}g(y) - p\,{}^{\varphi}\Theta^{\varpi}(\delta,0)\Big)\end{aligned}\tag{3.25}$$

and

$$\begin{aligned}&{}^{\varphi}\Theta^{\varpi}(\delta,0)\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}h^{2}(y) + {}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}h^{2}(y)\\ &-2\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}h(y)\,{}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}h(y)\\ &\le \Big(Q\,{}^{\varphi}\Theta^{\varpi}(\delta,0) - {}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}h(y)\Big)\Big({}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}h(y) - q\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)\Big)\\ &+ \Big(Q\,{}^{\varphi}\Theta^{\varpi}(\alpha,0) - {}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}h(y)\Big)\Big({}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}h(y) - q\,{}^{\varphi}\Theta^{\varpi}(\delta,0)\Big).\end{aligned}\tag{3.26}$$

Applying Lemma 3.2 and employing the inequalities (3.25) and (3.26), we obtain

$$\begin{aligned}&\Big({}^{\varphi}\Theta^{\varpi}(\delta,0)\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}\{gh\}(y) - {}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}g(y)\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}h(y)\\ &+{}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}\{gh\}(y) - {}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}h(y)\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}g(y)\Big)^{2}\\ &\le \Big(P\,{}^{\varphi}\Theta^{\varpi}(\delta,0) - {}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}g(y)\Big)\Big({}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}g(y) - p\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)\Big)\\ &+ \Big(P\,{}^{\varphi}\Theta^{\varpi}(\alpha,0) - {}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}g(y)\Big)\Big({}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}g(y) - p\,{}^{\varphi}\Theta^{\varpi}(\delta,0)\Big)\\ &\times \Big(Q\,{}^{\varphi}\Theta^{\varpi}(\delta,0) - {}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}h(y)\Big)\Big({}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}h(y) - q\,{}^{\varphi}\Theta^{\varpi}(\alpha,0)\Big)\\ &+ \Big(Q\,{}^{\varphi}\Theta^{\varpi}(\alpha,0) - {}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}h(y)\Big)\Big({}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}h(y) - q\,{}^{\varphi}\Theta^{\varpi}(\delta,0)\Big),\end{aligned}$$

which is the required inequality (3.22). The proof is thus completed. □

The next corollary is a generalized version of Theorem 3.2 for functional bounds.

Corollary 3.2. *Let $\varphi : \varLambda \longrightarrow [0,\infty) \subseteq \mathbb{R}$ be a continuous strictly increasing function and $g,h : [0,\infty) \longrightarrow \mathbb{R}$ be an integrable functions on $[0,\infty)$. Suppose that the functions g,h satisfying the condition (1.5), for the integrable functions u_1,u_2,v_1,v_2 on $[0,\infty)$. Then, for all $\alpha > 0$ and $\delta > 0$, the following inequality is holds*

$$\Big|{}^{\varphi}\Theta^{\varpi}(\delta,0)\,{}^{\varphi}\mathscr{J}^{\alpha,\varpi}_{0^{+}}\{gh\}(y) + {}^{\varphi}\Theta^{\varpi}(\alpha,0)\,{}^{\varphi}\mathscr{J}^{\delta,\varpi}_{0^{+}}\{gh\}(y)$$

$$
\begin{aligned}
&- {}^{\varphi}\mathcal{J}_{0+}^{\delta,\varpi} g(y)\ {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} h(y) - {}^{\varphi}\mathcal{J}_{0+}^{\delta,\varpi} h(y)\ {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} g(y) \Big| \\
&\leq \sqrt{K\left(h, u_1, u_2\right) K\left(g, v_1, v_2\right)},
\end{aligned}
$$

where $K(\zeta, \varsigma, \omega)$ *is defined by*

$$
\begin{aligned}
&K(\zeta, \varsigma, \omega) \\
&= \left({}^{\varphi}\mathcal{J}_{0+}^{\delta,\varpi} \omega(y) - {}^{\varphi}\mathcal{J}_{0+}^{\delta,\varpi} \zeta(y)\right)\left({}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} \zeta(y) - {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} \varsigma(y)\right) \\
&\quad + \left({}^{\varphi}\mathcal{J}_{0+}^{\delta,\varpi} \zeta(y) - {}^{\varphi}\mathcal{J}_{0+}^{\delta,\varpi} \varsigma(y)\right)\left({}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} \omega(y) - {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} \zeta(y)\right) \\
&\quad - {}^{\varphi}\mathcal{J}_{0+}^{\delta,\varpi} \omega(y)\ {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} \zeta(y) - {}^{\varphi}\mathcal{J}_{0+}^{\delta,\varpi} \zeta(y)\ {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} \omega(y) \\
&\quad - {}^{\varphi}\mathcal{J}_{0+}^{\delta,\varpi} \zeta(y)\ {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} \varsigma(y) - {}^{\varphi}\mathcal{J}_{0+}^{\delta,\varpi} \varsigma(y)\ {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} \zeta(y) \\
&\quad + {}^{\varphi}\mathcal{J}_{0+}^{\delta,\varpi} \omega(y)\ {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} \varsigma(y) + {}^{\varphi}\mathcal{J}_{0+}^{\delta,\varpi} \varsigma(y)\ {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} \omega(y) \\
&\quad + {}^{\varphi}\Theta^{\varpi}(\delta, 0)\left[{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\{\varsigma\zeta\}(y) + {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\{\omega\zeta\}(y) - {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\{\varsigma\omega\}(y)\right] \\
&\quad + {}^{\varphi}\Theta^{\varpi}(\alpha, 0)\left[{}^{\varphi}\mathcal{J}_{0+}^{\delta,\varpi}\{\varsigma\zeta\}(y) + {}^{\varphi}\mathcal{J}_{0+}^{\delta,\varpi}\{\omega\zeta\}(y) - {}^{\varphi}\mathcal{J}_{0+}^{\delta,\varpi}\{\varsigma\omega\}(y)\right].
\end{aligned}
$$

Remark 3.2.

(i) Putting $\varphi(y) = y$ and $\varpi = 1$, in Theorem 3.2, we get its Riemann–Liouville fractional integral version obtained by Dahmani et al.[5]

(ii) Putting $\varphi(y) = y$ and $\varpi = 1$, in Corollary 3.2, we get its Riemann–Liouville fractional integral version.

(iii) Applying Theorem 3.2 for $\varphi(y) = y$, $\varpi = 1$ and $\alpha = 1$, $\delta = 1$,we get the classical Grüss inequality (1.2).

4. Other related fractional integral inequalities

Throughout this section, we present a novel generalization of some related inequalities of the Grüss type, we use for this generalization the generalized proportional fractional integral operator concerning the certain function. It should be noted that to present our result here, we use only the left-sided proportional fractional operator, and the corresponding results concerned with the right-sided proportional fractional integral can be also obtained by the same arguments.

We start with the following result.

Theorem 4.1. *Let* $\varphi : \Lambda \longrightarrow [0, \infty) \subseteq \mathbb{R}$ *be a continuous strictly increasing function and* $g, h : [0, \infty) \longrightarrow \mathbb{R}$ *be a non-negative integrable functions on* $[0, \infty)$ *and* $P, Q > 1$ *satisfying* $\frac{1}{P} + \frac{1}{Q} = 1$*. Then, for all* $\alpha > 0$ *the following inequalities holds*

$$
(1).\ \frac{{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} g^P(y)}{P} + \frac{{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} h^Q(y)}{Q} \geq \frac{1}{{}^{\varphi}\Theta^{\varpi}(\alpha, 0)}\left\{{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} g(y)\ {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} h(y)\right\}. \tag{4.1}
$$

$$
\begin{aligned}
(2).\ &\frac{{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} g^P(y)\ {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} h^P(y)}{P} + \frac{{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} g^Q(y)\ {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} h^Q(y)}{Q} \qquad (4.2) \\
&\geq \left({}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\{gh\}(y)\right)^2.
\end{aligned}
$$

$$
\begin{aligned}
(3).\ &\frac{{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} g^P(y)\ {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} h^Q(y)}{P} + \frac{{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} g^Q(y)\ {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} h^P(y)}{Q} \\
&\geq {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\left\{gh^{P-1}\right\}(y)\ {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\left\{gh^{Q-1}\right\}(y). \qquad (4.3)
\end{aligned}
$$

$$
(4).\ {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} g^P(y)\ {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} h^Q(y) \geq {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\{gh\}(y)\ {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi}\left\{g^{P-1}h^{Q-1}\right\}(y). \tag{4.4}
$$

Proof. Employing Young's inequality[26]

$$
CD \leq \frac{C^P}{P} + \frac{D^Q}{Q}, \forall C, D \geq 0, \tag{4.5}
$$

we can write

$$
\frac{g^P\left(z_1\right)}{P} + \frac{h^Q\left(z_2\right)}{Q} \geq g\left(z_1\right) h\left(z_2\right), z_1, z_2 \in (0, y). \tag{4.6}
$$

Taking product on both sides of (4.6) by the function ${}^{\varphi}\Xi^{\varpi}\left(\alpha, z_1\right)$ and integrating the estimating inequality with respect to z_1 over $(0, y)$, we get

$$
\begin{aligned}
&\frac{1}{P}\int_0^y {}^{\varphi}\Xi^{\varpi}\left(\alpha, z_1\right) g^P\left(z_1\right) dz_1 + \frac{h^Q\left(z_2\right)}{Q}\int_0^y {}^{\varphi}\Xi^{\varpi}\left(\alpha, z_1\right) dz_1 \\
&\geq h\left(z_2\right)\int_0^y {}^{\varphi}\Xi^{\varpi}\left(\alpha, z_1\right) g\left(z_1\right) dz_1.
\end{aligned}
$$

So, we have

$$
\frac{{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} g^P(y)}{P} + \frac{h^Q\left(z_2\right)}{Q}\ {}^{\varphi}\Theta^{\varpi}(\alpha, 0) \geq h\left(z_2\right)\ {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} g(y). \tag{4.7}
$$

Now, taking product on both sides of (4.7) by the function ${}^{\varphi}\Xi^{\varpi}\left(\alpha, z_2\right)$ and integrating the estimating inequality with respect to z_2 over $(0, y)$, we get

$$
\frac{{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} g^P(y)}{P} + \frac{{}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} h^Q(y)}{Q} \geq \frac{1}{{}^{\varphi}\Theta^{\varpi}(\alpha, 0)}\ {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} h(y)\ {}^{\varphi}\mathcal{J}_{0+}^{\alpha,\varpi} g(y),
$$

which is the first inequality (4.1). To prove the second inequality (4.2), by using the Young's inequality and taking $C = g\left(z_1\right) h\left(z_2\right)$ and $D = g\left(z_2\right) h\left(z_1\right)$, $z_1, z_2 \in (0, y)$, we can write

$$
\frac{g^P\left(z_1\right) h^P\left(z_2\right)}{P} + \frac{g^Q\left(z_2\right) h^Q\left(z_1\right)}{Q} \geq g\left(z_1\right) h\left(z_2\right) g\left(z_2\right) h\left(z_1\right). \tag{4.8}
$$

Easily, by following the same arguments as in the proof of item (1), we can obtain the inequality (4.2). Now, to prove item (3), by taking $C = \frac{g(z_1)}{h(z_1)}$, and $D = \frac{g(z_2)}{h(z_2)}$, $z_1, z_2 \in (0, y)$ in Young's inequality (4.5), where $h\left(z_1\right)$ and $h\left(z_2\right) \neq 0$, we can write

$$
\frac{g^P\left(z_1\right) h^Q\left(z_2\right)}{P} + \frac{g^Q\left(z_2\right) h^P\left(z_1\right)}{Q} \geq g\left(z_1\right) h^{P-1}\left(z_1\right) g\left(z_2\right) h^{Q-1}\left(z_2\right). \tag{4.9}
$$

Clearly, by following the same arguments as in the proof of item (1), we can obtain the inequality (4.3). Finally, to prove item (4), by putting $C = \frac{g(z_1)}{g(z_2)}$, and $D = \frac{h(z_1)}{h(z_2)}$, $z_1, z_2 \in (0, y)$ in Young's inequality (4.5), where $g\left(z_2\right)$ and $h\left(z_2\right) \neq 0$, we can write

$$
\frac{g^P\left(z_1\right) h^Q\left(z_2\right)}{P} + \frac{g^P\left(z_2\right) h^Q\left(z_1\right)}{Q} \geq g\left(z_1\right) h\left(z_1\right) g^{P-1}\left(z_2\right) h^{Q-1}\left(z_2\right). \tag{4.10}
$$

Taking product on both sides of (4.10) by both functions ${}^{\varphi}\Xi^{\varpi}\left(\alpha, z_1\right)$, ${}^{\varphi}\Xi^{\varpi}\left(\alpha, z_2\right)$ and double integrating the estimating inequality with respect to z_1 *and* z_2 over $(0, y)$, we get

$$
\begin{aligned}
&\int_0^y\int_0^y \frac{g^P\left(z_1\right) h^Q\left(z_2\right)}{P}\ {}^{\varphi}\Xi^{\varpi}\left(\alpha, z_1\right)\ {}^{\varphi}\Xi^{\varpi}\left(\alpha, z_2\right) dz_1 dz_2 \\
&+ \int_0^y\int_0^y \frac{g^P\left(z_2\right) h^Q\left(z_1\right)}{Q}\ {}^{\varphi}\Xi^{\varpi}\left(\alpha, z_1\right)\ {}^{\varphi}\Xi^{\varpi}\left(\alpha, z_2\right) dz_1 dz_2 \qquad (4.11) \\
&\geq \int_0^y\int_0^y g\left(z_1\right) h\left(z_1\right) g^{P-1}\left(z_2\right) h^{Q-1}\left(z_2\right) \\
&\times {}^{\varphi}\Xi^{\varpi}\left(\alpha, z_1\right)\ {}^{\varphi}\Xi^{\varpi}\left(\alpha, z_2\right) dz_1 dz_2,
\end{aligned}
$$

which is leads to the required inequality (4.4). The proof is thus completed. □

The next result is as follow

Theorem 4.2. *Let* $\varphi : \Lambda \longrightarrow [0, \infty) \subseteq \mathbb{R}$ *be a continuous strictly increasing function and* $g, h : [0, \infty) \longrightarrow \mathbb{R}$ *be a non-negative integrable* *functions on* $[0,\infty)$ *and* $P,Q>1$ *satisfying* $\frac{1}{P}+\frac{1}{Q}=1$. *Then, for all* $\alpha>0$ *the following inequalities holds*

$$(1).\ \frac{{}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}g^{P}(y)\,{}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}h^{2}(y)}{P}+\frac{{}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}g^{2}(y)\,{}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}h^{Q}(y)}{Q}$$

$$\ge {}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}\{gh\}(y)\ {}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}\left\{h^{\frac{2}{P}}(y)\,g^{\frac{2}{Q}}(y)\right\}. \tag{4.12}$$

$$(2).\ \frac{{}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}g^{2}(y)\,{}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}h^{Q}(y)}{P}+\frac{{}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}h^{2}(y)\,{}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}g^{P}(y)}{Q}$$

$$\ge \left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}\left\{g^{\frac{2}{P}}(y)\,h^{\frac{2}{Q}}(y)\right\}\right)\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}\left\{g^{P-1}(y)\,h^{Q-1}(y)\right\}\right). \tag{4.13}$$

Proof. Putting $C=g(z_1)\,h^{\frac{2}{P}}(z_2)$ and $D=g^{\frac{2}{Q}}(z_2)\,h(z_1)$, $z_1,z_2\in(0,y)$ in inequality (4.5), we can write

$$\frac{g^{P}(z_1)\,h^{2}(z_2)}{P}+\frac{g^{2}(z_2)\,h^{Q}(z_1)}{Q}\ge g(z_1)\,h(z_1)\,h^{\frac{2}{P}}(z_2)\,g^{\frac{2}{Q}}(z_2). \tag{4.14}$$

Taking product on both sides of (4.14) by the functions ${}^{\varphi}\Xi^{\varpi}(\alpha,z_1)$, ${}^{\varphi}\Xi^{\varpi}(\alpha,z_2)$ and double integrating the estimating inequality with respect to z_1 *and* z_2 over $(0,y)$, we obtain

$$\begin{aligned}&\int_0^y\int_0^y\frac{g^{P}(z_1)\,h^{2}(z_2)}{P}\,{}^{\varphi}\Xi^{\varpi}(\alpha,z_1)\,{}^{\varphi}\Xi^{\varpi}(\alpha,z_2)\,dz_1dz_2\\&+\int_0^y\int_0^y\frac{g^{2}(z_2)\,h^{Q}(z_1)}{Q}\,{}^{\varphi}\Xi^{\varpi}(\alpha,z_1)\,{}^{\varphi}\Xi^{\varpi}(\alpha,z_2)\,dz_1dz_2\\&\ge\int_0^y\int_0^y g(z_1)\,h(z_1)\,h^{\frac{2}{P}}(z_2)\,g^{\frac{2}{Q}}(z_2)\,{}^{\varphi}\Xi^{\varpi}(\alpha,z_1)\,{}^{\varphi}\Xi^{\varpi}(\alpha,z_2)\,dz_1dz_2.\end{aligned} \tag{4.15}$$

So, we have

$$\frac{{}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}g^{P}(y)\,{}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}h^{2}(y)}{P}+\frac{{}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}g^{2}(y)\,{}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}h^{Q}(y)}{Q}$$

$$\ge {}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}\{gh\}(y)\ {}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}\left\{h^{\frac{2}{P}}(y)\,g^{\frac{2}{Q}}(y)\right\}$$

which is the required inequality (4.12). Now, we give the proof of item (2). By using the substitution $C=\frac{g^{\frac{2}{P}}(z_1)}{g(z_2)}$ and $D=\frac{h^{\frac{2}{Q}}(z_1)}{h(z_2)}$, $z_1,z_2\in(0,y)$, where $g(z_2)$ *and* $h(z_2)\neq 0$ in inequality (4.5), we can write

$$\frac{g^{2}(z_1)}{Pg^{P}(z_2)}+\frac{h^{2}(z_1)}{Qh^{Q}(z_2)}\ge\frac{g^{\frac{2}{P}}(z_1)}{g(z_2)}\,\frac{h^{\frac{2}{Q}}(z_1)}{h(z_2)},$$

which can be rewritten as

$$\frac{g^{2}(z_1)\,h^{Q}(z_2)}{P}+\frac{h^{2}(z_1)\,g^{P}(z_2)}{Q}\ge\left(g^{\frac{2}{P}}(z_1)\,h^{\frac{2}{Q}}(z_1)\right)\left(g^{P-1}(z_2)\,h^{Q-1}(z_2)\right). \tag{4.16}$$

Taking product on both sides of (4.16) by the functions ${}^{\varphi}\Xi^{\varpi}(\alpha,z_1)$, ${}^{\varphi}\Xi^{\varpi}(\alpha,z_2)$ and double integrating the estimating inequality with respect to z_1 *and* z_2 over $(0,y)$, we obtain

$$\begin{aligned}&\int_0^y\int_0^y\frac{g^{2}(z_1)\,h^{Q}(z_2)}{P}\,{}^{\varphi}\Xi^{\varpi}(\alpha,z_1)\,{}^{\varphi}\Xi^{\varpi}(\alpha,z_2)\,dz_1dz_2\\&+\int_0^y\int_0^y\frac{h^{2}(z_1)\,g^{P}(z_2)}{Q}\,{}^{\varphi}\Xi^{\varpi}(\alpha,z_1)\,{}^{\varphi}\Xi^{\varpi}(\alpha,z_2)\,dz_1dz_2\\&\ge\int_0^y\int_0^y\left(g^{\frac{2}{P}}(z_1)\,h^{\frac{2}{Q}}(z_1)\right)\left(g^{P-1}(z_2)\,h^{Q-1}(z_2)\right)\\&\times{}^{\varphi}\Xi^{\varpi}(\alpha,z_1)\,{}^{\varphi}\Xi^{\varpi}(\alpha,z_2)\,dz_1dz_2.\end{aligned} \tag{4.17}$$

Clearly, the inequality (4.17) leads to the desired inequality (4.13). This completes the proof. □

Theorem 4.3. *Let* $\varphi:\Lambda\longrightarrow[0,\infty)\subseteq\mathbb{R}$ *be a continuous strictly increasing function and* $g,h:[0,\infty)\longrightarrow\mathbb{R}$ *be a non-negative integrable functions on* $[0,\infty)$. *Assume that*

$$P:=\min_{0\le z_1\le y}\frac{g(z_1)}{h(z_1)}\quad\text{and}\quad Q:=\max_{0\le z_1\le y}\frac{g(z_1)}{h(z_1)}. \tag{4.18}$$

Then, for all $\alpha>0$ *the following inequalities holds*

$$(1)0\le\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}g^{2}(y)\right)\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}h^{2}(y)\right)\le\frac{(Q+P)^2}{4}\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}\{gh\}(y)\right)^2. \tag{4.19}$$

$$\begin{aligned}(2)\quad 0&\le\sqrt{\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}g^{2}(y)\right)\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}h^{2}(y)\right)}-{}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}\{gh\}(y)\\&\le\frac{\left(\sqrt{Q}-\sqrt{P}\right)^2}{2\sqrt{PQ}}\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}\{gh\}(y)\right).\end{aligned} \tag{4.20}$$

$$\begin{aligned}(3)0&\le\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}g^{2}(y)\right)\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}h^{2}(y)\right)-\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}\{gh\}(y)\right)^2\\&\le\frac{(Q-P)^2}{4PQ}\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}\{gh\}(y)\right)^2.\end{aligned} \tag{4.21}$$

Proof. Using the assumption (4.18), we can write

$$\left(\frac{g(z_1)}{h(z_1)}-P\right)\left(Q-\frac{g(z_1)}{h(z_1)}\right)h^{2}(z_1)\ge 0,0\le z_1\le y$$

which is equivalent to

$$\left(g(z_1)-Ph(z_1)\right)\left(Qh(z_1)-g(z_1)\right)\ge 0,$$

so, we have

$$(Q+P)\,g(z_1)\,h(z_1)\ge g^{2}(z_1)+PQh^{2}(z_1). \tag{4.22}$$

Taking product on both sides of (4.22) by the functions ${}^{\varphi}\Xi^{\varpi}(\alpha,z_1)$ and integrating the estimating inequality with respect to z_1 over $(0,y)$, we obtain

$$(Q+P)\,{}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}\{gh\}(y)\ge{}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}g^{2}(y)+PQ\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}h^{2}(y)\right). \tag{4.23}$$

Since $PQ>0$, therefore $\left(\sqrt{{}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}g^{2}(y)}-\sqrt{PQ\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}h^{2}(y)\right)}\right)^2\ge 0$. It follows that

$${}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}g^{2}(y)+PQ\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}h^{2}(y)\right)\ge 2\sqrt{{}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}g^{2}(y)}\sqrt{PQ\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}h^{2}(y)\right)}. \tag{4.24}$$

Using the inequalities (4.23) and (4.24), we have

$$\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}g^{2}(y)\right)\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}h^{2}(y)\right)\le\frac{(Q+P)^2}{4}\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}\{gh\}(y)\right)^2, \tag{4.25}$$

which is the required inequality (4.19). To prove item (2), and in view of inequality (4.24), we have

$$\sqrt{\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}g^{2}(y)\right)\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}h^{2}(y)\right)}\le\frac{Q+P}{2\sqrt{PQ}}\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}\{gh\}(y)\right). \tag{4.26}$$

Adding $\left(-\,{}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}\{gh\}(y)\right)$ to both sides of inequality (4.26), we get

$$\begin{aligned}&\sqrt{\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}g^{2}(y)\right)\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}h^{2}(y)\right)}-{}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}\{gh\}(y)\\&\le\frac{Q+P}{2\sqrt{PQ}}\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}\{gh\}(y)\right)-{}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}\{gh\}(y),\end{aligned}$$

which gives

$$\begin{aligned}&\sqrt{\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}g^{2}(y)\right)\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}h^{2}(y)\right)}-{}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}\{gh\}(y)\\&\le\frac{\left(\sqrt{Q}-\sqrt{P}\right)^2}{2\sqrt{PQ}}\left({}^{\varphi}\mathcal{J}_{0^+}^{\alpha,\varpi}\{gh\}(y)\right).\end{aligned}$$

Finally, for the proof of item (3), we take the square of both sides of the inequality (4.26) and then, subtract $\left({}^{\varphi}\mathscr{I}_{0^+}^{\alpha,\varpi}\{gh\}(y)\right)^2$ from both sides of estimating inequality, we obtain the required inequality (4.21). The proof is thus completed. □

Conclusion 4.4. The fractional calculus has been the focus of several authors in recent years. During introducing the current manuscript, we rediscovered and presented Grüss inequality in a new framework. We employed the recently generalized proportional fractional integral operator for certain function with respect to another continuous and strictly increasing function to achieve our goals. Furthermore, we proved some new related inequalities using the current fractional integral operator. Some special cases of the presented results were discussed.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

1. Gruss G. Uber das Maximum des absoluten Betrages von, $1/(b-a)\int_a^b f(t)\,g(t)\,dt-\left(1/(b-a)^2\right)\int_a^b f(t)\,dt\int_a^b g(t)\,dt$. *Math Z.* 1935:215–226.
2. Mitrinovic DS, Pecaric JE, Fink AM. *Classical and New Inequalities in Analysis of Mathematics and Its Applications*. Kluwer Academic; 1993. vol. 61.
3. Akin E, Aslıyüce S, Güvenilir AF, Kaymakçalan B. Discrete grüss type inequality on fractional calculus. *J Inequal Appl.* 2015;17(4). (online).
4. Minculete N, Ciurdariu L. A generalized form of Grüss type inequality and other integral inequalities. *J Inequal Appl.* 2014.
5. Dahmani Z, Tabharit L, Taf S. New generalizations of Gruss inequality using Riemann–Liouville fractional integrals. *Bull Math Anal Appl.* 2010;2:93–99.
6. Tariboon J, Ntouyas, Sudsutad W. Some new Riemann–Liouville fractional integral inequalities. *Int J Math Math Sci.* 2014.
7. Aljaaidi TA, Pachpatte DB. Some Gruss-type inequalities using generalized katugampola fractional integral. *AIMS Math.* 2020;5:1011–1024. http://dx.doi.org/10.3934/math.2020070.
8. Aljaaidi TA, Pachpatte DB. Some gruss-type inequalities via ψ-Riemann–Liouville fractional integral. *Indian J Math.* 2020;62:249–268.
9. Dragomir SS. New Grüss' type inequalities for functions of bounded variation and applications. *Appl Math Lett.* 2012;25(10):1475–1479.
10. Alomari MW. New Grüss type inequalities for double integrals. *Appl Math Comput.* 2014;228:102–107.
11. Chinchane VL, Pachpatte DB. On some new Gruss-type inequality using Hadamard fractional integral operator. *J Frac Calc Appl.* 2014;5:1–10.
12. Liu W, Tuna A. Diamond weighted Ostrowski type and Grüss type inequalities on time scales. *Appl Math Comput.* 2015;270:251–260.
13. Sousa JVd C, Oliveira DS, de Oliveira EC. Grüss-type inequalities by means of generalized fractional integrals. *Bull Braz Math Soc, New Series.* 2019;50:1029–1047. http://dx.doi.org/10.1007/s00574-019-00138-z.
14. Rashid S, Jarad F, Noor MA. Grüss-type integrals inequalities via generalized proportional fractional operators. *RACSAM.* 2020;114:93. http://dx.doi.org/10.1007/s13398-020-00823-5.
15. Zhou SS, Rashid S, Jarad F, et al New estimates considering the generalized proportional Hadamard fractional integral operators. *Adv Differ Equ.* 2020;2020:275. http://dx.doi.org/10.1186/s13662-020-02730-w.
16. Naz S, Naeem MN, Chu YM. Some k-fractional extension of Grüss-type inequalities via generalized Hilfer–Katugampola derivative. *Adv Differ Equ.* 2021;2021:29. http://dx.doi.org/10.1186/s13662-020-03187-7.
17. Al Qurashi M, Rashid S, Sultana S, Ahmad H, Gepreel KA. New formulation for discrete dynamical type inequalities via h-discrete fractional operator pertaining to nonsingular kernel. *MBE.* 2021;18(2):1794–1812.
18. Dragomir SS. A generalization of Grüss inequality in inner product spaces and applications. *J Math Anal Appl.* 1999;31:74–82.
19. Dragomir SS. Some integral inequalities of Gruss type. *Indian J Pur Appl Math.* 2002;31:397–415.
20. Kilbas AA, Srivastava HM, Trujillo JJ. *Theory and Applications of Fractional Differential Equations*. Amsterdam: Elsevier; 2006. In: North-Holland Mathematics Studies.
21. Samko SG, Kilbas AA, Marichev OI, et al Fractional integrals and derivatives. In: *Theory and Applications*. Yverdon: Gordon and Breach; 1993.
22. Jarad F, Abdeljawad T, Alzabut J. Generalized fractional derivatives generated by a class of local proportional derivatives. *Eur Phys J Spec Top.* 2017;226:3457–3471.
23. Jarad F, Alqudah MA, Abdeljawad T. On more generalized form of proportional fractional operators. *Open Math.* 2020;18:167–176.
24. Aljaaidi TA, Pachpatte DB, Shatanawi W, et al Generalized proportional fractional integral functional bounds in Minkowski's inequalities. *Adv Differ Equ.* 2021:419. http://dx.doi.org/10.1186/s13662-021-03582-8.
25. Steele JM. *The Cauchy–Schwarz Master Class: an Introduction to the Art of Mathematical Inequalities*. New York: Cambridge University Press; 2004.
26. Kreyszig E. *Introductory Functional Analysis with Applications*. New York: Wiley; 1989. Vol. 1.