J. Math. Anal. Appl. 543 (2025) 128917

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

Generalized Mittag-Leffler-confluent hypergeometric functions in fractional calculus integral operator with numerical solutions

Firas Ghanim[a], Fareeha Sami Khan[b], Ali Hasan Ali[c,d,e,f,*], Abdon Atangana[g]

[a] *Department of Mathematics, College of Sciences, University of Sharjah, Sharjah, United Arab Emirates*
[b] *Department of Mathematical Sciences, Federal Urdu University of Arts, Science and Technology, Karachi-75300, Pakistan*
[c] *Department of Mathematics, College of Education for Pure Sciences, University of Basrah, 61001 Basrah, Iraq*
[d] *Institute of Mathematics, University of Debrecen, Pf. 400, H-4002 Debrecen, Hungary*
[e] *Department of Business Management, Al-imam University College, Balad, Iraq*
[f] *Technical Engineering College, Al-Ayen University, 64001 Dhi Qar, Iraq*
[g] *Faculty of Natural and Agricultural Sciences, University of the Free State, South Africa*

ARTICLE INFO

Article history:
Received 29 January 2024
Available online 1 October 2024
Submitted by A.A.J. Durán

Keywords:
Laplace transform
Fractional calculus
Confluent hypergeometric function
Mittag-Leffler function
Integral operator
Power series

ABSTRACT

The Mittag-Leffler and confluent hypergeometric functions were originally developed to extend the exponential function and its area of applications. This study aims to examine some operators involving generalized Mittag-Leffler-type functions in the kernels, employing the generalized Fox-Wright function in specific circumstances. Furthermore, we investigate some of the commonly utilized generalized fractional integral operators in fractional calculus. Moreover, a numerical technique is developed to solve fractional differential equations of both kinds, linear and nonlinear. The graphic results of the examples show how effective this method is at solving fractional differential equations. Lastly, various effects and implications of these results are thoroughly examined.

1. Introduction

Throughout the 20th century, fractional calculus and its associated special functions emerged as powerful tools in analysis and mathematical models. These concepts found wide-ranging applications in various fields of physics, mathematics, and engineering. A fundamental branch of applied mathematics, fractional calculus employs non-integer order integrals and derivatives in both real and complex domains. In essence, it extends the traditional integer-order calculus. This field employs both left-sided and right-sided differential integrals, corresponding to left and right derivatives. The two most well-known fractional operators are the Riemann-

* Corresponding author.
E-mail addresses: fgahmed@sharjah.ac.ae (F. Ghanim), fareeha.samikhan@fuuast.edu.pk (F.S. Khan), ali.hasan@science.unideb.hu (A.H. Ali), AtanganaA@ufs.ac.za (A. Atangana).

https://doi.org/10.1016/j.jmaa.2024.128917

Liouville integral and derivative operators, both on the right and left sides, as well as the Caputo fractional derivative operators.

Interestingly, the history of fractional calculus dates back to the same era as the development of classical calculus by Newton and Leibniz in the seventeenth century. It was in a letter from Leibniz to L'Hospital in 1695 that the concept of a semi-derivative was first discussed. More precisely, Leibniz and L'Hospital's correspondence revolved around the idea of extending the principle of differentiation to non-integer orders, particularly to orders such as one-half.

Numerous mathematicians, including Euler, Lagrange, Laplace, Fourier, Abel, Liouville, Riemann, Grünwald, and others, played pivotal roles in establishing the initial systematic foundations of fractional calculus. For more details, please refer to [33] and [49].

Advanced calculus and virtually all fields of mathematics rely on special functions which represent specific mathematical processes. There exist several well-known types of special functions that prove to be invaluable in solving a wide range of problems related to fractional differential equations. Indeed, the application of special functions is fundamental in the context of fractional calculus [24]. Specifically, this study explores the concepts and techniques for solving fractional differential equations, also known as differential equations involving fractional derivatives.

One of the first and most fundamental fractional calculus function is the Gamma function, originally introduced by Euler in 1729. It serves as a factorial formula that has been extended to cover positive integers as well as complex numbers. The exponential function, denoted as e^z, plays a key role in constructing this function, as discussed in [36]:

$$\Gamma(\psi) = \int_0^\infty t^{\psi-1}\, e^t\, dt, \qquad (\psi \in \mathcal{C},\ \mathcal{R}(\psi) > 0). \tag{1.1}$$

Following that, Legendre and Euler established the Beta function, a crucial function that is closely connected to the Gamma function, [36] as follows:

$$\mathfrak{B}(\psi,\omega) = \int_0^1 t^{\psi-1}\,(1-t)^{\omega-1}\, dt, \qquad (\psi,\omega \in \mathcal{C},\ \mathcal{R}(\psi) > 0,\ \mathcal{R}(\omega) > 0). \tag{1.2}$$

Further,

$$\mathfrak{B}(\psi,\omega) = \frac{\Gamma(\psi)\,\Gamma(\omega)}{\Gamma((\psi+\omega)}, \qquad (\psi,\omega \in \mathcal{C}\backslash\mathbb{Z}_0^-). \tag{1.3}$$

On the other hand, the Pochhammer symbol (rising factorial), related to $\Gamma(\psi)$, is denoted by $(\psi)_\ell$, and defined as:

$$(\psi)_\ell = \frac{\Gamma(\psi+\ell)}{\Gamma(\psi)}, \qquad (\psi,\ell \in \mathcal{C}\backslash\mathbb{Z}_0^-). \tag{1.4}$$

Since then, there has been a substantial increase in interest in applying the Gamma function in all special functions up to the present. More specifically, solutions to fractional differential equations are primarily discussed using the Laplace transform approach. The following research is motivated by this generalization, aiming to pursue more creative solutions that result in various formulations of Mittag-Leffler-type functions and fractional operators, see [3], [6], [27] and [38]. Physical rules might also be used to derive physical phenomena of an exponential nature using Mittag-Leffler type functions (power-law) ([4] and [8]).

Riemann-Liouville fractional integral and derivative formulas are the most fundamental concepts in fractional calculus. These are defined as follows:

$$ {}_{\imath}^{RL}I_{x}^{\mathrm{a}}f(x)=\frac{1}{\Gamma(\mathrm{a})}\int_{\imath}^{x}(x-t)^{\mathrm{a}-1}f(t)\,dt, \qquad \mathcal{R}(\mathrm{a})>0, \tag{1.5} $$

and

$$ {}_{\imath}^{RL}D_{x}^{\mathrm{a}}f(x)=\frac{d^{m}}{dx^{m}}\left({}_{\imath}^{RL}D_{x}^{\mathrm{a}-m}f(x)\right), \qquad \mathcal{R}(\mathrm{a})\geq 0,\ m:=[\mathcal{R}(\mathrm{a})]+1. \tag{1.6} $$

The expansions of the exponential function in relation to the Gamma function, also known as Mittag-Leffler type functions, have a fascinating role to play in the applications of fractional calculus. Many scholars have focused on the behavior of Mittag-Leffler-type functions in physics and math-related difficulties and have expanded their results into the complex domain due to the successful and diversified applications of Mittag-Leffler-type functions correlated with fractional calculus, [12], [15], [32], [40] and [43].

The distinguished Mittag-Leffler function $E_{\mathrm{a}}(z)$, proposed and investigated by Mittag-Leffler, is defined as

$$ E_{\mathrm{a}}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\mathrm{a}k+1)} \qquad (z\in\mathcal{C};\ \mathcal{R}(\mathrm{a})>0). \tag{1.7} $$

Following that, Wiman [52] and [53] developed a 2-parameter generalization of $E_{\mathrm{a}}(z)$ as stated below:

$$ E_{\mathrm{a},\lambda}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\mathrm{a}k+\lambda)} \qquad (\mathrm{a},\lambda\in\mathcal{C};\ \mathcal{R}(\mathrm{a})>0). \tag{1.8} $$

Since then, the Mittag-Leffler functions $E_{\mathrm{a}}(z)$ in (1.7) and $E_{\mathrm{a},\lambda}(z)$ in (1.8) have been thoroughly explored and studied in numerous forms and situations. Mittag-Leffler-type functions and their implementations have been studied in a variety of fields, among which are mathematics, physics, engineering, and statistics. In Wright [56], the distributions associated with the Mittag-Leffler-type functions are investigated. Additionally, the expanded Mittag-Leffler pathways and model functionality have also been connected by Pillai [35].

The function with the a and λ parameters in (1.8) has been extended to more parameters, such as the Prabhakar type functions,

$$ E_{\mathrm{a},\lambda}^{\gamma}(z)=\sum_{k=0}^{\infty}\frac{\Gamma(k+\gamma)\,z^{k}}{\Gamma(\gamma)\,\Gamma(\mathrm{a}k+\lambda)\,k!}, \qquad (\mathrm{a},\lambda,\gamma\in\mathcal{C};\ \mathcal{R}(\mathrm{a})>0), \tag{1.9} $$

and

$$ E_{\mathrm{a},\lambda}^{\mathrm{b},\gamma}(z)=\sum_{k=0}^{\infty}\frac{\Gamma(\mathrm{b}k+\gamma)\,z^{k}}{\Gamma(\gamma)\,\Gamma(\mathrm{a}k+\lambda)\,k!}, \qquad (\mathrm{a},\lambda,\mathrm{b},\gamma\in\mathcal{C};\ \mathcal{R}(\mathrm{a})>0), \tag{1.10} $$

see for example [14] and [45].

The advantages of confluent hypergeometric functions have been demonstrated in various physical fields. They have been used to solve problems related to diffusion and sedimentation, like separating isotopes and finding the molecular weight of a protein in an ultra-centrifuge. These functions can also be used to express solutions to problems regarding high-frequency gas discharges and the representation of electron velocity distribution.

The second-order linear homogeneous differential equation

$$ z\frac{d^{2}M}{dz^{2}}+(\lambda-z)\frac{dM}{dz}-\gamma M=0, \qquad (\gamma,\lambda,z\in\mathcal{C}) \tag{1.11} $$

has been associated with the confluent hypergeometric function $M(\gamma;\lambda;z)$ as its solution. According to MacDonald [31], equation (1.11) exhibits an irregular singularity at infinity and a regular singularity at the origin.

If λ is not integral, a second equation solution (1.11) may be shown by

$$\Lambda(\gamma;\lambda;z) = z^{1-\lambda} M(\gamma-\lambda+1;2-\lambda;z). \tag{1.12}$$

If λ is integral, the solution could be demonstrated by

$$\Lambda(\mathrm{a};\lambda;z) = M(\mathrm{a};\lambda;z)\{\ln z + \mho(1-\mathrm{a}) - \mho(\lambda) + \mathbb{C}\} + \sum_{k=1}^{\infty} \frac{\Gamma(k+\gamma)\Gamma(\lambda) B_k z^k}{\Gamma(\gamma)\Gamma(k+\lambda) k!}$$
$$+(-1)^{\lambda} \sum_{k=0}^{\infty} \frac{\Gamma(\lambda)\Gamma(k+\gamma-\lambda+1)\Gamma(\lambda-k-1)(-1)^k}{\Gamma(\gamma) k!\, z}, \tag{1.13}$$

where

$$\mho(\gamma) = \frac{\Gamma'(\gamma)}{\mathrm{T}(\gamma)}.$$

The value of $\mathbb{C}$, which is Euler's constant, is 0.577216..., and

$$B_k = \left(\frac{1}{\gamma} + \frac{1}{\gamma+1} + \dots + \frac{1}{\gamma+k-1}\right)$$
$$- \left(\frac{1}{\lambda} + \frac{1}{\lambda+1} + \dots + \frac{1}{\lambda+k-1}\right) - \left(1 + \frac{1}{2} + \dots + \frac{1}{k}\right).$$

The $\mho$ function was used in [7] to exhibit the extensive tables. The confluent hypergeometric function $M(\gamma;\lambda;z)$ is represented as a series by

$$M(\gamma;\lambda;z) = 1 + \frac{\gamma}{\lambda} z + \frac{\gamma(\gamma+1)}{\lambda(\lambda+1)} \frac{z^2}{2} + \dots = \sum_{k=0}^{\infty} \frac{\Gamma(\lambda)\Gamma(k+\gamma) z^k}{\Gamma(\gamma)\Gamma(k+\lambda) k!}. \tag{1.14}$$

This series absolutely converges for all values of z. The following relations hold:

$$\frac{d}{dz} M(\gamma;\lambda;z) = \frac{\gamma}{\lambda} M(\gamma+1;\lambda+1;z) \tag{1.15}$$

$$\gamma M(\gamma+1;\lambda+1;z) = (\gamma-\lambda) M(\gamma;\lambda+1;z) + \lambda M(\gamma;\lambda;z) \tag{1.16}$$

$$\gamma M(\gamma+1;\lambda;z) = (z+2\gamma-\lambda) M(\gamma;\lambda;z) + (\lambda-\gamma) M(\gamma-1;\lambda;z). \tag{1.17}$$

The Wronskian equation (1.11) and a functional association between its value at any point on a plane versus its value at that particular point may be revealed between $M(\gamma;\lambda;z)$ and $\Lambda(\mathrm{a};\lambda;z)$. See [1], [7], and [31] for more details.

Now, we emphasize on the recently studied Mittag-Leffler-Confluent hypergeometric function (MLCHF), denoted by $M_{\mathrm{a},\lambda}^{\mathrm{b},\gamma}(z)$, as follows:

$$M_{\mathrm{a},\lambda}^{\mathrm{b},\gamma}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\lambda)\Gamma(\mathrm{b}k+\gamma) z^k}{\Gamma(\gamma)\Gamma(\mathrm{a}k+\lambda) k!}, \qquad (\mathrm{a},\lambda,\mathrm{b},\gamma \in \mathcal{C};\ \mathcal{R}(\mathrm{a}) > 0). \tag{1.18}$$

The confluent hypergeometric function, as stated in equation (1.14), is a noteworthy consequence of this function if $\mathrm{a} = \mathrm{b} = 1$. Furthermore, if b, λ and γ, are all equal to one, we get the initial Mittag-Leffler function as stated in equation (1.7).

Thus, it is evident that

$$\begin{aligned} M_{\mathrm{a},\lambda}^{\mathrm{b},\gamma}(z) &= \sum_{k=0}^{\infty} \frac{\Gamma(\lambda)\Gamma(\mathrm{b}k+\gamma) z^k}{\Gamma(\gamma)\Gamma(\mathrm{a}k+\lambda) k!} \\ &= \frac{(-z)^{-\lambda/\mathrm{a}}\lambda\left(\gamma\Gamma\left(\frac{\lambda}{\mathrm{a}},0,-z\right) - \mathrm{b}\Gamma\left(\frac{\lambda}{\mathrm{a}}+1,0,-z\right)\right)}{\mathrm{a}\gamma}, \end{aligned} \tag{1.19}$$

where $\Gamma(a, z_0, z_1)$ is the generalized incomplete Gamma function.

The generalized hypergeometric function ${}_s\Psi_t$ has been defined in the fundamental works of Fox [11] and Wright's publications [55] for any positive integer indices s and t, see also, [2] and [28].

The fundamental studies by Fox [11] and Wright from 1934 to 1940 [55] described the generalized hypergeometric function ${}_s\Psi_t$ for any indices $s \le t$ or $s = t+1$, see also, [2], [34] and [28]. Subsequently, Gorenflo et al. [14], Kilbas et al. ([18], [20], and [21]), and Kiryakova ([22], [25] and [26]) have conducted substantial research on the so-called Fox-Wright function.

Definition 1. The series

$${}_s\Psi_t(z) = {}_s\Psi_t(z)\left({}^{(\gamma_i,\mathrm{b}_i)_1^s}_{(\lambda_j,\mathrm{a}_j)_1^t}; z\right) = \sum_{k\ge 0} \frac{\prod_{i=1}^{s}\Gamma(\gamma_i+\mathrm{b}_i k)}{\prod_{j=1}^{t}\Gamma(\lambda_j+\mathrm{a}_j k)} \frac{z^k}{k!}, \tag{1.20}$$

defines the generalized Fox-Wright psi function of $z \in \mathcal{C}$, where $\gamma_i,\ \lambda_j \in \mathcal{C}, \mathrm{a}_j,\ \mathrm{b}_i \in R$, $(i = 1, ..., s$ and $j = 1, ..., t)$.

Wright demonstrated a number of results on the asymptotic expansion of ${}_s\Psi_t(z)$ for any argument values z when the property $\sum_{j=1}^{t}\mathrm{a}_j - \sum_{i=1}^{s}\mathrm{b}_i > -1$ is satisfied. According to this restriction, it was shown in [19] that the series is an entire function of $z \in \mathcal{C}$. The following lemma provides conditions for convergence:

Lemma 1. [[21], Thm. 1]. *Assume the conditions in Definition 1, and let*

$$\Omega := \sum_{j=1}^{t}\mathrm{a}_j - \sum_{i=1}^{s}\mathrm{b}_i,\ \ \Delta := \prod_{j=1}^{t}|\mathrm{a}_j|^{\mathrm{a}_j}\prod_{i=1}^{s}|\mathrm{b}_i|^{-\mathrm{b}_i}\ \ \textit{and}\ \ \psi := \sum_{j=1}^{t}\lambda_j - \sum_{i=1}^{s}\gamma_i + \frac{s-t}{2}. \tag{1.21}$$

Then, the following are the convergence conditions:
(i) If $\Omega > -1$, then for all $z \in \mathcal{C}$, the series in (1.20) is absolutely convergent.
(ii) If $\Omega = -1$, then for $z \le \Delta$, and $z = \Delta$ and $\mathcal{R}(\psi) > \frac{1}{2}$, the series in (1.20) is absolutely convergent.

When $\gamma_i,\ \lambda_j \in \mathbb{R}\,(i = 1, ..., s$ and $j = 1, ..., t)$, the following integral representation of the Fox-Wright function ${}_s\Psi_t(z)$ is a Mellin-Barnes contour integral ([21], Eq. (1.11.21), [15]):

$${}_s\Psi_t(z)\left({}^{(\gamma_i,\mathrm{b}_i)_1^s}_{(\lambda_j,\mathrm{a}_j)_1^t}; z\right) = \frac{1}{2\pi i}\int_{\Upsilon} \frac{\prod_{i=1}^{s}\Gamma(\gamma_i+\mathrm{b}_i\tau)}{\prod_{j=1}^{t}\Gamma(\lambda_j+\mathrm{a}_j\tau)}\Gamma(\tau)(-z)^{-\tau}d\tau, \tag{1.22}$$

where Υ (the contour integration) separates all the poles of $\Gamma(\tau)$ at $\tau = -p\,(p \in N_0)$ to the left from all the poles of $\Gamma(\gamma_i + \mathrm{b}_i\tau)$ at $\tau = (\gamma_i + p_i)/\mathrm{b}_i$ ($i = 1, ..., s$ and $b_i \in N$) to the right. As $\Upsilon = (a - i\infty, a + i\infty)\,(a \in \mathbb{R})$, if any of the following two conditions hold, then representation (1.22) is valid:
(i) $\Omega < 1$, $|\arg(-z)| < (1-\Omega)\,\pi/2$, $z \neq 0$, or
(ii) $\Omega = 1$, $\mathcal{R}(\vartheta) > (1-\Omega)\,a + 1/2$ and $\arg(-z) = 0$, $z \neq 0$.

Requirements for the representation (1.22) are also provided in instances where $\Upsilon = L_{-\infty}(\Upsilon = L_{+\infty}$, respectively) is a loop situated in a horizontal strip beginning at the point $-\infty + ih_1\,(+\infty + ih_1$, respectively) and ending at the point $-\infty + ih_2\,(+\infty + ih_2$, respectively) with $-\infty < h_1 < h_2 < +\infty$, [28].

The normalized generalized hypergeometric function ${}_s\Psi_t^*(z)$ is defined as (see, for details [42]):

$$ {}_s\Psi_t^*(z) = {}_s\Psi_t^*(z)\left({(\gamma_i,\mathrm{b}_i)_1^s \atop (\lambda_j,\mathrm{a}_j)_1^t}; z\right) = \frac{\Gamma(\lambda_t)}{\Gamma(\gamma_s)}\,{}_s\Psi_t(z)\left({(\gamma_i,\mathrm{b}_i)_1^s \atop (\lambda_j,\mathrm{a}_j)_1^t}; z\right). \tag{1.23} $$

If $s = t = 1$, then we have

$$ {}_1\Psi_1^*(z)\left({(\gamma,\mathrm{b}) \atop (\lambda,\mathrm{a})}; z\right) = \frac{\Gamma(\lambda)}{\Gamma(\gamma)}\sum_{k=0}^{\infty}\frac{\Gamma(\gamma + \mathrm{b}k)}{\Gamma(\lambda + \mathrm{a}k)}\frac{z^k}{k!} = M_{\mathrm{a},\lambda}^{\mathrm{b},\gamma}(z). \tag{1.24} $$

The significance of ${}_1\Psi_1^*(z)$ is highlighted in Fermat's last theorem and in practical issues like solving trinomial equations.

Now, if we write $\mathrm{a}_j = \mathrm{b}_i = 1$ for every $1 \leq i \leq s$ and $1 \leq j \leq t$ in Definition 1, we get a relationship between the generalized Fox-Wright function ${}_s\Psi_t(z)$ and the generalized hypergeometric function ${}_sF_t(z)$ as follows:

$$ {}_s\Psi_t(z)\left({(\gamma_i,1)_1^s \atop (\lambda_j,1)_1^t}; z\right) = \frac{\Gamma(\lambda_t)}{\Gamma(\gamma_s)}\,{}_sF_t\left({\gamma_s \atop \lambda_t}; z\right). \tag{1.25} $$

The more complicated Fox $H-$ and Meijer G-functions include as special cases the ${}_s\Psi_t(z)$ and ${}_sF_t(z)$ functions, [14], [22], [21], [28] and others.

Remark 1. When $s \leq t$ and $\Omega \geq 0$, ${}_sF_t(z)$ is an entire function of $z \in \mathcal{C}$, see [10].

2. Main results

Theorem 1. *The generalized Mittag-Leffler function* $M_{\mathrm{a},\lambda}^{\mathrm{b},\gamma}(z)$ *defined by* (1.18) *is an entire function in the complex* z*-plane of order* ν *and type* η *provided by*

$$ \upsilon := \frac{1}{\mathcal{R}(\mathrm{a}-\gamma)+1} \quad \textit{and} \quad \eta := \frac{1}{\upsilon}\left(\frac{(\mathcal{R}(\gamma))^{\mathcal{R}(\gamma)}}{(\mathcal{R}(\mathrm{a}))^{\mathcal{R}(\mathrm{a})}}\right)^{\upsilon}. \tag{2.1} $$

Additionally, the power series in the defining equation (1.18) converges absolutely in this case when

$$ \mathcal{R}(\mathrm{a}) = \mathcal{R}(\gamma) - 1 > 0 \quad \textit{and} \quad |z| < \frac{(\mathcal{R}(\mathrm{a}))^{\mathcal{R}(\mathrm{a})}}{(\mathcal{R}(\gamma))^{\mathcal{R}(\gamma)}}. \tag{2.2} $$

Proof. The proof of this theorem can be obtained from the representation of (1.24) and the basic theory of the Fox-Wright function when $s = t = 1$; therefore, it is omitted. □

We introduce an integral operator $\mathfrak{E}_{a,\lambda}^{b,\gamma}(\mu)$ on a space ϖ by

$$\mathfrak{E}_{a,\lambda}^{b,\gamma}(\mu) f(x) \equiv \int_{\imath}^{x} (x-t)^{\lambda-1} M_{a,\lambda}^{b,\gamma} \mu(x-t)^{a} f(t)\, dt = g(x) \qquad \mathcal{R}(\lambda) > 0, \tag{2.3}$$

and use an operator of fractional integration $I^{\mu} : \varpi \to \varpi$ to show results on $\mathfrak{E}_{a,\lambda}^{b,\gamma}(\mu)$. ϖ indicates the linear space of complex-value functions f which are integrable on a finite $[a,b]$, $a \geq 0$ with the norm $\|f\| = \int_{\imath}^{b} |f(t)|\, dt$, see [37], [47] and [48].

Several authors, see for example, [5], [17], [37], [39], [51] and [54] have employed the Laplace transform to solve convolution equations, which are cases of (2.3). The target of this study is to investigate the integral operator $\mathfrak{E}_{a,\lambda}^{b,\gamma}(\mu)$ for every real constant $\imath > 0$, where $M_{a,\lambda}^{b,\gamma}(z)$ is the function in (1.18) that includes a number of other special functions.

Theorem 2. *Let the function Θ be in the space $\mathbb{L}(\imath,\jmath)$ of Lebesgue measurable functions on a finite interval of the real line $\mathbb{R}$ defined by*

$$\mathbb{L}(\imath,\jmath) = \left\{ h : \|h\|_1 := \int_{\imath}^{\jmath} |h(x)|\, dx < \infty \right\}. \tag{2.4}$$

The integral operator $\mathfrak{E}_{a,\lambda}^{b,\gamma}(\mu)$ is then bounded by $\mathbb{L}(\imath,\jmath)$ and

$$\left\| \mathfrak{E}_{a,\,\lambda}^{b,\,\gamma}(\mu)\, \Theta \right\|_1 \leqslant E.\|\Theta\|_1, \tag{2.5}$$

where the constant $E\,(0 < E < \infty)$ is given by

$$E := (\jmath-\imath)^{\mathcal{R}(\lambda)} \sum_{k=0}^{\infty} \frac{\Gamma(\lambda)}{\Gamma(\gamma)} \frac{\Gamma(\gamma + bk)}{\Gamma(\lambda + ak)\, k!} \cdot \frac{\left|\mu(\jmath-\imath)^{\mathcal{R}(a)}\right|^{k}}{(\mathcal{R}(a)\, k + \lambda)}. \tag{2.6}$$

Proof. To prove Theorem 2, it is sufficient to show that

$$\left\| \mathfrak{E}_{a,\,\lambda}^{b,\,\gamma}(\mu)\, \Theta \right\|_1 = \int_{\imath}^{\jmath} \left| \int_{\imath}^{x} (t-x)^{\lambda-1} M_{a,\,\lambda}^{b,\,\gamma} \mu(t-x)^{a} \Theta(t)\, dt \right| dx < \infty, \tag{2.7}$$

$(b, \mu \in \mathcal{C};\ \max\{0, \mathcal{R}(\gamma) - 1\} < \mathcal{R}(a);\ \min\{\mathcal{R}(\lambda), \mathcal{R}(\gamma)\} > 0)$. By Fubini's Theorem, we do in fact get

$$\begin{aligned}
\left\| \mathfrak{E}_{a,\,\lambda}^{b,\,\gamma}(\mu)\, \Theta \right\|_1 &\leqslant \int_{\imath}^{\jmath} |\Theta(t)| \left(\int_{t}^{\jmath} (t-x)^{\mathcal{R}(\lambda)-1} \left| M_{a,\,\lambda}^{b,\,\gamma} \mu(t-x)^{a} \right| dx \right) dt \\
&= \int_{\imath}^{\jmath} |\Theta(t)| \left(\int_{0}^{\jmath-t} (t-x)^{\mathcal{R}(\lambda)-1} \left| M_{a,\,\lambda}^{b,\,\gamma} \mu(t-x)^{a} \right| dx \right) dt \\
&\leq \int_{\imath}^{\jmath} |\Theta(t)| \left(\int_{0}^{\jmath-\imath} (t-x)^{\mathcal{R}(\lambda)-1} \left| M_{a,\,\lambda}^{b,\,\gamma} \mu(t-x)^{a} \right| dx \right) dt
\end{aligned}$$

$$\leqslant \left(\sum_{k=0}^{\infty} \frac{\Gamma(\lambda)}{\Gamma(\gamma)} \frac{\Gamma(\gamma+\mathrm{b}k)|\mu|^k}{\Gamma(\lambda+\mathrm{a}k)\, k!} \cdot \int_0^{\jmath-\imath} (t-x)^{\mathcal{R}(\mathrm{a})n+\mathcal{R}(\lambda)-1} dx\right) \cdot \|\Theta\|_1$$

$$= E \cdot \|\Theta\|_1, \qquad (\mathcal{R}(\gamma) > 0) \tag{2.8}$$

where the value E is determined by (2.6) and is finite according to Theorem 1. Hence, the boundedness of the integral operator $\mathfrak{E}_{\mathrm{a},\lambda}^{\mathrm{b},\gamma}(\mu)$ is now fully proven. □

Theorem 3. *Let*

$$x > \imath \ \left(\imath \in \mathbb{R}^+ := [0,\infty)\right), \quad 0 < \mathrm{a} < 1, \quad 0 \leqslant \kappa \leqslant 1,$$

and

$$\mathcal{R}(\mathrm{a}) > \max\{0, \mathcal{R}(\gamma) - 1\}; \quad \min\{\mathcal{R}(\lambda), \mathcal{R}(\gamma), \mathcal{R}(\phi)\} > 0 \ \textit{and} \ \mathrm{b}, \mu \in \mathcal{C}.$$

Then

$${}_{\imath}^{RL}I_x^{\sigma}\left[(t-\imath)^{\lambda-1} M_{\mathrm{a},\lambda}^{\mathrm{b},\gamma} \mu (t-\imath)^{\mathrm{a}}\right](x) = (x-\imath)^{\lambda+\sigma-1} M_{\mathrm{a},\lambda+\sigma}^{\mathrm{b},\gamma}[\mu(x-\imath)^{\mathrm{a}}], \tag{2.9}$$

$${}_{\imath}^{RL}D_x^{\sigma}\left[(t-\imath)^{\lambda-1} M_{\mathrm{a},\lambda}^{\mathrm{b},\gamma} \mu (t-\imath)^{\mathrm{a}}\right](x) = (x-\imath)^{\lambda-\sigma-1} M_{\mathrm{a},\lambda-\sigma}^{\mathrm{b},\gamma}[\mu(x-\imath)^{\mathrm{a}}], \tag{2.10}$$

and

$${}_{\imath}^{RL}D_x^{\mathrm{a},\kappa}\left[(t-\imath)^{\lambda-1} M_{\mathrm{a},\lambda}^{\mathrm{b},\gamma} [\mu (t-\imath)^{\mathrm{a}}]\right](x) = (x-\imath)^{\lambda-\mathrm{a}-1} M_{\mathrm{a},\lambda-\mathrm{a}}^{\mathrm{b},\gamma}[\mu(x-\imath)^{\mathrm{a}}]. \tag{2.11}$$

Proof. Our proofs of the assumptions (2.9) and (2.10) would coincide with those of the associated established findings ([20] and [45]). Accordingly, we can assert the assumption (2.11) of the theorem. We have

$$
\begin{aligned}
&{}_{\imath}^{RL}D_x^{\mathrm{a},\kappa}\left[(t-\imath)^{\lambda-1} M_{\mathrm{a},\lambda}^{\mathrm{b},\gamma} [\mu (t-\imath)^{\mathrm{a}}]\right](x) \\
&= \left({}_{\imath}^{RL}D_x^{\mathrm{a},\kappa}\left(\sum_{k=0}^{\infty} \frac{\Gamma(\lambda)}{\Gamma(\gamma)} \frac{\Gamma(\gamma+\mathrm{b}k)\,\mu^k}{\Gamma(\lambda+\mathrm{a}k)\, k!} (t-\imath)^{\mathrm{a}k+\lambda-1}\right)\right)(x) \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(\lambda)}{\Gamma(\gamma)} \frac{\Gamma(\gamma+\mathrm{b}k)\,\mu^k}{\Gamma(\lambda+\mathrm{a}k)\, k!} \left({}_{\imath}^{RL}D_x^{\mathrm{a},\kappa}\left((t-\imath)^{\mathrm{a}k+\lambda-1}\right)\right)(x) \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(\lambda)}{\Gamma(\gamma)} \frac{\Gamma(\gamma+\mathrm{b}k)\,\mu^k}{\Gamma(\lambda+\mathrm{a}k)\, k!} \frac{\Gamma(\lambda+\mathrm{a}k)}{\Gamma(\lambda+\mathrm{a}k-\mathrm{a})} (t-\imath)^{\mathrm{a}k+\lambda-\mathrm{a}-1} \\
&= (x-\imath)^{\lambda-\mathrm{a}-1} \sum_{k=0}^{\infty} \frac{\Gamma(\lambda)}{\Gamma(\gamma)} \frac{\Gamma(\gamma+\mathrm{b}k)\,\mu^k}{\Gamma(\lambda+\mathrm{a}k)\, k!} \left([\mu(x-\imath)^{\mathrm{a}}]\right)^k \\
&= (x-\imath)^{\lambda-\mathrm{a}-1} M_{\mathrm{a},\lambda-\mathrm{a}}^{\mathrm{b},\gamma}[\mu(x-\imath)^{\mathrm{a}}],
\end{aligned}
$$

which is exactly the conclusion (2.11) that Theorem 3 states. □

The theorem mentioned above is an exceptional instance; readers can learn more about the ${}_s\Psi_t$ function and related topics from a variety of prestigious articles and books, such as [22], [23] and [42]. Numerous researchers have provided integral representations, not only for the result of two orthogonal hypergeometric polynomials but also for additional hypergeometric polynomials that are connected to it. For instance, Lin et al. [30] recently conducted a similar analysis involving a number of generic classes of polynomials that are found in Srivastava's $E_k^K(z)$ polynomials, which were first described by Srivastava [44]:

$$E_n^K(z) = \sum_{k=0}^{(n/K)} \frac{(-n)_{Kn}}{n!} a_{n,k} z^k, \qquad (n \in k_0; K \in N), \tag{2.12}$$

using an appropriately bounded double sequence $\{a_{n,k}\}_{n=0}^{\infty}$ with practically arbitrary parameters (real or complex), see also, [13], [29], and [45].

We will need the integral formulae for the Beta function and the Gamma function (1.1)-(1.3) to derive the aforementioned integral representations, for more information see [50].

Theorem 4. *The following integral representations hold:*

$$M_{\mathrm{a},\lambda}^{\mathrm{b},\gamma}(z) \cdot M_{\mathrm{a},\varepsilon}^{\phi,\gamma}(z) = -\frac{\Gamma(\lambda)\Gamma(\phi)}{4(\pi\Gamma(\gamma))^2} \int_0^1 t^{\mathrm{b}-1}(1-t)^{\phi-1}$$

$$\cdot \left(\int_{-\infty}^{(0+)} \int_{-\infty}^{(0+)} e^{\mathrm{A}+\mathrm{B}} \mathrm{A}^{-\lambda} \mathrm{B}^{-\varepsilon} {}_1\Psi_1 \left[{}^{(\gamma,\mathrm{b}+\phi)}_{(\lambda+\varepsilon,\mathrm{a})}; \left[t^\gamma \mathrm{A}^{-\mathrm{a}} + (1-t)^\gamma \mathrm{B}^{-\mathrm{a}} \right] z \right] d\mathrm{A} d\mathrm{B} \right) dt \tag{2.13}$$

$$(|\arg(\mathrm{A})|, |\arg(\mathrm{B})| \leq \pi; \max\{0, \mathcal{R}(\gamma) - 1\} < \min\{\mathcal{R}(\mathrm{a})\};$$
$$\min\{\mathcal{R}(\mathrm{b}), \mathcal{R}(\gamma), \mathcal{R}(\phi)\} > 0),$$

$$M_{\mathrm{a},\lambda}^{\mathrm{b},\gamma}(z) \cdot M_{\mathrm{a},\varepsilon}^{\phi,\gamma}(z) = -\frac{\Gamma(\lambda)\Gamma(\phi)}{4(\pi\Gamma(\gamma))^2} \int_0^1 t^{\mathrm{b}-1}(1-t)^{\phi-1}$$

$$\cdot \left(\int_{-\infty}^{(0+)} \int_{-\infty}^{(0+)} e^{\mathrm{A}+\mathrm{B}} \mathrm{A}^{-\lambda} \mathrm{B}^{-\varepsilon} \left[\int_0^\infty \tau^{\mathrm{b}+\phi-1} e^{\left[t^\gamma \mathrm{A}^{-\mathrm{a}} + (1-t)^\gamma \mathrm{B}^{-\mathrm{a}} \right] z\tau^\gamma - \tau} d\tau \right] d\mathrm{A} d\mathrm{B} \right) dt \tag{2.14}$$

$$(|\arg(\mathrm{A})|, |\arg(\mathrm{B})| \leq \pi; \min\{\mathcal{R}(\alpha), \mathcal{R}(\mathrm{b}), \mathcal{R}(\phi)\} > 0;\ 0 < \mathcal{R}(\gamma) < 1),$$

$${}_1\Psi_1^*(z) \left({}^{(\gamma,\mathrm{b})}_{(\lambda,\mathrm{a})}; z \right) \cdot {}_1\Psi_1^*(z) \left({}^{(\gamma,\phi)}_{(\varepsilon,\mathrm{a})}; z \right)$$

$$= \int_0^1 t^{\mathrm{b}-1}(1-t)^{\phi-1} {}_1\Psi_1^*(z) \left[{}^{(\gamma,\mathrm{b}+\phi);}_{(\lambda+\varepsilon,\mathrm{a})} \left[t^\gamma + (1-t)^\gamma \right] z \right] dt \tag{2.15}$$

$$(\min\{\mathcal{R}(\mathrm{b}), \mathcal{R}(\phi)\} > 0;\ 0 < \mathcal{R}(\gamma) < 1),$$

and

$${}_1\Psi_1^*(z) \left({}^{(\gamma,\mathrm{b})}_{(\lambda,\mathrm{a})}; z \right) \cdot {}_1\Psi_1^*(z) \left({}^{(\gamma,\phi)}_{(\varepsilon,\mathrm{a})}; z \right)$$

$$= \int_0^1 t^{\mathrm{b}-1}(1-t)^{\phi-1} \left(\int_0^\infty \tau^{\mathrm{b}+\phi-1} e^{\left[t^\gamma \mathrm{A}^{-\mathrm{a}} + (1-t)^\gamma \mathrm{B}^{-\mathrm{a}} \right] z\tau^\gamma - \tau} d\tau \right) dt \tag{2.16}$$

$$(\min\{\mathcal{R}(\mathrm{b}), \mathcal{R}(\phi)\} > 0;\ 0 < \mathcal{R}(\gamma) < 1).$$

Proof. Using the definition (1.24), we get the following for the left-hand side of (2.13)

$$M_{\mathrm{a},\lambda}^{\mathrm{b},\gamma}(z)\cdot M_{\mathrm{a},\varepsilon}^{\phi,\gamma}(z)=\left(\sum_{k=0}^{\infty}\frac{\Gamma(\lambda)}{\Gamma(\gamma)}\frac{\Gamma(\gamma+\mathrm{b}k)\,z^k}{\Gamma(\lambda+\mathrm{a}k)\,k!}\right)\cdot\left(\sum_{k=0}^{\infty}\frac{\Gamma(\varepsilon)}{\Gamma(\gamma)}\frac{\Gamma(\gamma+\phi k)\,z^k}{\Gamma(\varepsilon+\mathrm{a}k)\,k!}\right)$$
$$=\frac{\Gamma(\lambda)\Gamma(\phi)}{(\Gamma(\gamma))^2}\sum_{k=0}^{\infty}\frac{\Gamma(\gamma+\varepsilon+\mathrm{b}k)\,z^k}{k!}\sum_{i=0}^{k}\binom{k}{i}\frac{\mathrm{b}\,(\mathrm{b}i+\gamma,\phi+\gamma\,(k-i))}{\Gamma(\lambda+\alpha i)\,\Gamma(\varepsilon+\mathrm{a}\,(k-i))},\tag{2.17}$$

based on the Beta function $\mathfrak{B}$ defined as (1.3).

Now, by appropriately using the last integral formulae (1.2) and

$$\frac{1}{\Gamma(z)}=\frac{1}{2\pi i}\int_{-\infty}^{(0+)}e^{\mathrm{B}}\mathrm{B}^{-z}d\mathrm{B},\qquad(|\arg(\mathrm{B})|\leq\pi),\tag{2.18}$$

we easily get the integral representation (2.13) as proven by Theorem 4 in the last part of (2.17).

Since $0<\mathcal{R}(\gamma)<1$, the integral representation (2.14) would be obtained from the integral equation (1.1) that provided an integral representation for the Fox-Wright function ${}_1\Psi_1^*(z)$ contained within the integer and of (2.14). The integral representations (2.15) and (2.16) proofs are similar; therefore, we will skip the details. □

The following known identity will be generalized in the following theorem, for more details, see [20], [41], and [45]:

$$\int_0^x t^{\lambda-1}(x-t)^{\varepsilon-1}M_{\mathrm{a},\lambda}^{\mathrm{b},1}(\omega t^{\mathrm{a}})\cdot M_{\mathrm{a},\varepsilon}^{\phi,1}(\omega(x-t)^{\mathrm{a}})\,dt=x^{\lambda+\varepsilon-1}M_{\mathrm{a},\lambda+\varepsilon}^{\mathrm{b}+\phi,1}(\omega x^{\mathrm{a}}),\tag{2.19}$$

given that every member of (2.19) exists.

Theorem 5. *The following integral representations are valid:*

$$\int_0^x t^{\lambda-1}(x-t)^{\varepsilon-1}M_{\mathrm{a},\lambda}^{\mathrm{b},\gamma}(\omega t^{\mathrm{a}})\cdot M_{\mathrm{a},\varepsilon}^{\phi,\gamma}(\omega(x-t)^{\mathrm{a}})\,dt$$
$$=\frac{x^{\lambda+\varepsilon-\mathrm{b}-\phi}}{\mathfrak{B}(\mathrm{b},\phi)}\int_0^x t^{\lambda-1}(x-t)^{\varepsilon-1}M_{\mathrm{a},\lambda+\varepsilon}^{\mathrm{b}+\phi,\gamma}\left(\omega t^{\mathrm{a}-\gamma}\left([t^{\gamma}+(x-t)^{\gamma}]\right)\right)dt\tag{2.20}$$
$$(\omega\in C;\ \max\{0,\mathcal{R}(\gamma)-1\}<\mathcal{R}(\mathrm{a});\ \min\{\mathcal{R}(\mathrm{b}),\mathcal{R}(\lambda),\mathcal{R}(\phi),\mathcal{R}(\gamma),\mathcal{R}(\varepsilon)\}>0).$$

Proof. By appropriately using a well-established Laplace transform result

$$\pounds\,(f(x))(\mu)=\int_0^{\infty}f(x)\,e^{-\mu x}dx,\tag{2.21}$$

and based on Srivastava et al. [46] analysis of the Fox-Wright generalized hypergeometric function ${}_s\Psi_t(z)$ under the Laplace transform, we obtain

$$\pounds\left[x^{\rho-1}M_{\mathrm{a},\lambda}^{\mathrm{b},\gamma}(\omega x^{\psi})\right]=s^{-\rho}\frac{\Gamma(\lambda)\Gamma(\phi)}{(\Gamma(\gamma))^2}{}_2\Psi_1^*(z)\left({}^{(\rho,\psi)(\gamma,\mathrm{b})}_{(\lambda,\mathrm{a})};\frac{\omega}{s^{\psi}}\right)\tag{2.22}$$

$$(\omega \in C;\ \max\{0, \mathcal{R}(\gamma)-1\} < \mathcal{R}(\mathrm{a});\ \min\{\mathcal{R}(\mathrm{b}), \mathcal{R}(\gamma), \mathcal{R}(\rho), \mathcal{R}(\psi)(\varepsilon)\} > 0)$$

thus, for the special case where

$$\rho = \lambda \quad \text{and} \quad \psi = \mathrm{a}$$

gives

$$\pounds\left[x^{\lambda-1} M_{\mathrm{a},\lambda}^{\mathrm{b},\gamma}(\omega x^{\mathrm{a}})\right] = s^{-\lambda}\frac{(\Gamma(\lambda))^2\Gamma(\phi)}{(\Gamma(\gamma))^2}\,{}_1\Psi_0^*(z)\left(\frac{(\gamma,\mathrm{b})}{};\frac{\omega}{s^{\mathrm{a}}}\right). \tag{2.23}$$

For the Laplace transform (2.21), by using the convolution theorem in light of results like (2.23), we can now deduce from Theorem 4 that

$$\pounds\left[\int_0^x t^{\lambda-1}(x-t)^{\varepsilon-1} M_{\mathrm{a},\lambda}^{\mathrm{b},\gamma}(\omega t^{\mathrm{a}})\cdot M_{\mathrm{a},\varepsilon}^{\phi,\gamma}(\omega(x-t)^{\mathrm{a}})\,dt\right](s)$$

$$= s^{-\lambda-\phi}\frac{(\Gamma(\lambda))^2\Gamma(\phi)}{(\Gamma(\gamma))^2}\,{}_1\Psi_0^*(z)\left(\frac{(\gamma,\mathrm{b})}{};\frac{\omega}{s^{\mathrm{a}}}\right)\cdot {}_1\Psi_0^*(z)\left(\frac{(\phi,\mathrm{b})}{};\frac{\omega}{s^{\mathrm{a}}}\right)$$

$$= \frac{(\Gamma(\lambda))^2\Gamma(\phi)}{(\Gamma(\gamma))^2}\sum_{k=0}^{\infty}\frac{\Gamma(\mathrm{b}+\phi+\gamma k)\,z^k}{(s^{-\mathrm{a}k-\lambda-\varepsilon})\,k!}\left(\int_0^1 t^{\mathrm{b}-1}(1-t)^{\phi-1}\left[t^\gamma+(1-t)^\gamma\right]dt\right). \tag{2.24}$$

This final result (2.24) provides the following using the inverse Laplace transform:

$$\int_0^x t^{\lambda-1}(x-t)^{\varepsilon-1} M_{\mathrm{a},\lambda}^{\mathrm{b},\gamma}(\omega t^{\mathrm{a}})\cdot M_{\mathrm{a},\varepsilon}^{\phi,\gamma}(\omega(x-t)^{\mathrm{a}})\,dt$$

$$= x^{\lambda+\varepsilon-1}\frac{\Gamma(\lambda)\Gamma(\phi)}{(\Gamma(\gamma))^2}\sum_{k=0}^{\infty}\frac{\Gamma(\mathrm{b}+\phi+\gamma k)\,z^k}{(s^{-\mathrm{a}k-\lambda-\varepsilon})\,k!}\frac{\omega^k x^{\mathrm{a}k}}{\Gamma(\mathrm{a}k+\lambda+\varepsilon)}$$

$$\times\left(\int_0^1 t^{\mathrm{b}-1}(1-t)^{\phi-1}[t^\gamma+(1-t)^\gamma]^k dt\right)$$

$$= \frac{x^{\lambda+\varepsilon-1}}{\mathfrak{B}(\mathrm{b},\phi)}\int_0^1 t^{\mathrm{b}-1}(1-t)^{\phi-1} M_{\mathrm{a},\lambda+\varepsilon}^{\mathrm{b}+\phi,\gamma}(\omega x^{\mathrm{a}}\left[t^\gamma+(1-t)^\gamma\right])\,dt$$

$$= \frac{x^{\lambda+\varepsilon-\mathrm{b}-\phi}}{\mathfrak{B}(\mathrm{b},\phi)}\int_0^x t^{\mathrm{b}-1}(x-t)^{\phi-1} M_{\mathrm{a},\lambda+\varepsilon}^{\mathrm{b}+\phi,\gamma}\left(\omega x^{\mathrm{a}-\gamma}\left[t^\gamma+(x-t)^\gamma\right]\right)dt.$$

This is exactly the second part of the statement (2.20). □

3. Numerical method

In this section, a numerical technique is being developed to solve fractional differential equations of both kinds, linear and nonlinear. MLCHF collocation method basically depends on the basic function defined in Eq. (1.18) as the Mittag-Leffler-Confluent hypergeometric function.

3.1. Methodology for linear fractional differential equations

Consider a linear fractional differential equation with coefficients $q_m(z)$ and the function $g(z)$ where $z \in [0,1]$ as

$$\sum_{m=1}^{p} q_m(z) D_z^{\alpha_m} y(z) = g(z), \tag{3.1}$$

where $q_1(z) = 1$, $\alpha_1 > \alpha_2 > \alpha_3 \cdots > \alpha_{p-1} > \alpha_p = 0$ and the initial condition as

$$y^{(n)}(0) = \psi_n, \qquad \text{where} \quad n = 0, 1, \cdots, \lceil \alpha_1 \rceil. \tag{3.2}$$

In this work, α_m will be considered only. The solution of $y(z)$ is assumed to be represented in the form of a series as

$$y_i(z) = \sum_{k=0}^{i} b_k M_{\mathrm{a},\lambda,i}^{\mathrm{b},\gamma}(z), \tag{3.3}$$

where $M_{\mathrm{a},\lambda,i}^{\mathrm{b},\gamma}(z)$ are known and the coefficients $\{b_k\}_{k=0}^{i}$ need to be obtained from the series solution. Now, to approximate the solution for linear fractional differential equation, let us consider some points, i.e. z_q, for $q = 0, 1, \cdots, n$, then Eq. (3.1) becomes $\sum_{m=1}^{p} q_m(z) \sum_{k=0}^{i} b_k D_z^{\alpha_m} M_{\mathrm{a},\lambda,i}^{\mathrm{b},\gamma}(z) = g(z)$. Now using the term $D_z^{\alpha_m} M_{\mathrm{a},\lambda,i}^{\mathrm{b},\gamma}(z)$ defined in Theorem 3, the initial conditions $\sum_{k=0}^{i} b_k D_z^{n} M_{\mathrm{a},\lambda,i}^{\mathrm{b},\gamma}(z_0) = \psi_n$, where $n = 0, 1, \cdots, \alpha_1$ for $q = 0, 1, \cdots, n$, it becomes

$$\sum_{k=0}^{i} b_k \sum_{m=1}^{p} q_m(z_q) D_z^{\alpha_m} M_{\mathrm{a},\lambda,i}^{\mathrm{b},\gamma}(z_q) = g(z_q). \tag{3.4}$$

Now, to evaluate the values of coefficients $\{b_k\}_{k=0}^{i}$, we solve Eq. (3.4) along with its initial conditions. Since it is not an easy process, let us write it in the form of an optimization problem as an unconstrained least-cost squares function,

$$J = \sum_{q=1}^{n} \left[\sum_{m=1}^{p} q_m(z_q) D_z^{\alpha_m} M_{\mathrm{a},\lambda,i}^{\mathrm{b},\gamma}(z_q) - g(z_q) \right]^2 + \sum_{n=0}^{\lceil \alpha_1 \rceil} \left[\sum_{k=0}^{i} b_k D_z^{n} M_{\mathrm{a},\lambda,i}^{\mathrm{b},\gamma}(z_0) - \psi_n \right]^2. \tag{3.5}$$

To solve this, the optimization technique known as Leap Frog mentioned in [16] can be utilized.

3.2. Methodology for nonlinear fractional differential equations

We consider nonlinear fractional differential equations that can be written in general as

$$D_z^{\alpha_1} y(z) = g(D_z^{\alpha_2} y(z), D_z^{\alpha_3} y(z), \cdots, D_z^{\alpha_n} y(z), y(z), z) \tag{3.6}$$

where $q_1(z) = 1$, $\alpha_1 > \alpha_2 > \alpha_3 \cdots > \alpha_{p-1} > \alpha_p = 0$ and with the initial condition as

$$y^{(n)}(0) = \psi_n, \qquad \text{where} \quad n = 0, 1, \cdots, \lceil \alpha_1 \rceil. \tag{3.7}$$

To approximate this solution, let us define a new nonlinear optimization problem as

$$J = \Bigg[\sum_{k=0}^{i} b_k D_z^{\alpha_1} M_{\mathrm{a},\lambda,i}^{\mathrm{b},\gamma}(z) - g\Big(\sum_{k=0}^{i} b_k D_z^{\alpha_2} M_{\mathrm{a},\lambda,i}^{\mathrm{b},\gamma}(z), \sum_{k=0}^{i} b_k D_z^{\alpha_3} M_{\mathrm{a},\lambda,i}^{\mathrm{b},\gamma}(z), \cdots, \sum_{k=0}^{i} b_k M_{\mathrm{a},\lambda,i}^{\mathrm{b},\gamma}(z), z\Big) \Bigg]^2 + \Bigg[\sum_{n=0}^{\lceil \alpha_1 \rceil} \sum_{k=0}^{i} b_k D_z^{n} M_{\mathrm{a},\lambda,i}^{\mathrm{b},\gamma}(z_0) - \psi_n \Bigg]^2. \tag{3.8}$$

To solve this, the optimization technique known as Leap Frog mentioned in [9] can be utilized.

3.3. Example 1

Consider the first example as a linear fractional differential equation

$$D^{\alpha} y - 2y = g(z), \quad g(z) = \frac{\Gamma[3]}{\Gamma[\frac{5}{2}]} z^{\frac{3}{2}} - 2z^2, \quad z \in [0,1] \tag{3.9}$$

with the initial condition $y(0) = 0$. To solve this problem with MLCHF, assume $i = 2$ for the sake of easier calculations, then Eq. (3.9) becomes

$$y_2(z) = \sum_{i=0}^{2} c_i M_{\mathrm{a},\lambda,i}^{\mathrm{b},\gamma}(z) = c_0 M_{\mathrm{a},\lambda,0}^{\mathrm{b},\gamma}(z) + c_1 M_{\mathrm{a},\lambda,1}^{\mathrm{b},\gamma}(z) + c_2 M_{\mathrm{a},\lambda,2}^{\mathrm{b},\gamma}(z) \tag{3.10}$$

then

$$D^{\alpha} y_2(z) = c_1 D^{\alpha} M_{\mathrm{a},\lambda,1}^{\mathrm{b},\gamma}(z) + c_2 D^{\alpha} M_{\mathrm{a},\lambda,2}^{\mathrm{b},\gamma}(z). \tag{3.11}$$

By using the above equation, Eq. (3.9) can be written as

$$c_1 D^{\alpha} M_{\mathrm{a},\lambda,1}^{\mathrm{b},\gamma}(z) + c_2 D^{\alpha} M_{\mathrm{a},\lambda,2}^{\mathrm{b},\gamma}(z) - 2\big(c_0 M_{\mathrm{a},\lambda,0}^{\mathrm{b},\gamma}(z) + c_1 M_{\mathrm{a},\lambda,1}^{\mathrm{b},\gamma}(z) + c_2 M_{\mathrm{a},\lambda,2}^{\mathrm{b},\gamma}(z)\big) = g(z). \tag{3.12}$$

By collecting the coefficients c_2, c_1, c_0

$$\begin{aligned} & c_2 \Bigg(\frac{z^{3/2}\Gamma(3)\Gamma(\lambda)\Gamma(2b+\gamma)}{2\Gamma\left(\frac{5}{2}\right)\Gamma(\gamma)\Gamma(2a+\lambda)} - \frac{z^2\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)} \\ & \qquad + \frac{\sqrt{z}\Gamma(2)\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma\left(\frac{3}{2}\right)\Gamma(\gamma)\Gamma(a+\lambda)} - \frac{2(z\Gamma(\lambda)\Gamma(b+\gamma))}{\Gamma(\gamma)\Gamma(a+\lambda)} - 2 \Bigg) \\ & + c_1 \left(\frac{\sqrt{z}\Gamma(2)\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma\left(\frac{3}{2}\right)\Gamma(\gamma)\Gamma(a+\lambda)} - \frac{2(z\Gamma(\lambda)\Gamma(b+\gamma))}{\Gamma(\gamma)\Gamma(a+\lambda)} - 2 \right) \\ & - 2c_0 - \frac{z^{3/2}\Gamma(3)}{\Gamma\left(\frac{5}{2}\right)} + 2z^2 = 0 \end{aligned} \tag{3.13}$$

and utilizing them for any three points of z_i i.e. $z_0 = 0, z_1 = 0.5, z_2 = 1$ and $\alpha = \frac{1}{2}$, we obtain three equations

$$-2c_0 - 2c_1 - 2c_2 = 0 \tag{3.14}$$

$$
\begin{aligned}
& c_2\left(\frac{z^{3/2}\Gamma(3)\Gamma(\lambda)\Gamma(2b+\gamma)}{2\Gamma\left(\frac{5}{2}\right)\Gamma(\gamma)\Gamma(2a+\lambda)}-\frac{z^2\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)}\right.\\
&\qquad\qquad\left.+\frac{\sqrt{z}\Gamma(2)\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma\left(\frac{3}{2}\right)\Gamma(\gamma)\Gamma(a+\lambda)}-\frac{2(z\Gamma(\lambda)\Gamma(b+\gamma))}{\Gamma(\gamma)\Gamma(a+\lambda)}-2\right)\\
&+c_1\left(\frac{\sqrt{z}\Gamma(2)\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma\left(\frac{3}{2}\right)\Gamma(\gamma)\Gamma(a+\lambda)}-\frac{2(z\Gamma(\lambda)\Gamma(b+\gamma))}{\Gamma(\gamma)\Gamma(a+\lambda)}-2\right)\\
&-2c_0-\frac{z^{3/2}\Gamma(3)}{\Gamma\left(\frac{5}{2}\right)}+2z^2=0
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
& c_2\left(\frac{z^{3/2}\Gamma(3)\Gamma(\lambda)\Gamma(2b+\gamma)}{2\Gamma\left(\frac{5}{2}\right)\Gamma(\gamma)\Gamma(2a+\lambda)}-\frac{z^2\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)}\right.\\
&\qquad\qquad\left.+\frac{\sqrt{z}\Gamma(2)\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma\left(\frac{3}{2}\right)\Gamma(\gamma)\Gamma(a+\lambda)}-\frac{2(z\Gamma(\lambda)\Gamma(b+\gamma))}{\Gamma(\gamma)\Gamma(a+\lambda)}-2\right)\\
&+c_1\left(\frac{\sqrt{z}\Gamma(2)\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma\left(\frac{3}{2}\right)\Gamma(\gamma)\Gamma(a+\lambda)}-\frac{2(z\Gamma(\lambda)\Gamma(b+\gamma))}{\Gamma(\gamma)\Gamma(a+\lambda)}-2\right)\\
&-2c_0-\frac{z^{3/2}\Gamma(3)}{\Gamma\left(\frac{5}{2}\right)}+2z^2=0.
\end{aligned}
\tag{3.16}
$$

In solving this linear system of equations to get

$$
C_0=0, C_1=-\frac{2\Gamma(\gamma)\Gamma(2a+\lambda)}{\Gamma(\lambda)\Gamma(2b+\gamma)}, C_2=\frac{2\Gamma(\gamma)\Gamma(2a+\lambda)}{\Gamma(\lambda)\Gamma(2b+\gamma)},
$$

the solution becomes

$$
\begin{aligned}
y_2(z) &= C_0\sum_{k=0}^{0}\frac{z^k\Gamma(\lambda)\Gamma(bk+\gamma)}{k!\Gamma(\gamma)\Gamma(ak+\lambda)}+C_1\sum_{k=0}^{1}\frac{z^k\Gamma(\lambda)\Gamma(bk+\gamma)}{k!\Gamma(\gamma)\Gamma(ak+\lambda)}+C_2\sum_{k=0}^{2}\frac{z^k\Gamma(\lambda)\Gamma(bk+\gamma)}{k!\Gamma(\gamma)\Gamma(ak+\lambda)}\\
&= \frac{2\Gamma(\gamma)\Gamma(2a+\lambda)\left(\frac{z^2\Gamma(\lambda)\Gamma(2b+\gamma)}{2\Gamma(\gamma)\Gamma(2a+\lambda)}+\frac{z\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)}+1\right)}{\Gamma(\lambda)\Gamma(2b+\gamma)}\\
&\quad-\frac{2\Gamma(\gamma)\Gamma(2a+\lambda)\left(\frac{z\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)}+1\right)}{\Gamma(\lambda)\Gamma(2b+\gamma)},
\end{aligned}
\tag{3.17}
$$

which is the numerical solution for this problem. For $b=1, \lambda=1, a=0, \gamma=1$, this solution becomes $y_2(z)=z^2$ which is its exact solution.

3.4. Example 2

Consider the second example as a fractional differential equation

$$
D_z^{\alpha}y+y=g(z), \quad g(z)=\frac{\Gamma[q+1]}{\Gamma[q+1-\alpha]}z^{q-\alpha}, \quad z\in[0,1]
\tag{3.18}
$$

with the initial condition $y(0)=0$ and $\alpha\in(0,1]$. To solve this problem with MLCHF, assume $i=2$ for the sake of easier calculations, then Eq. (3.18) becomes

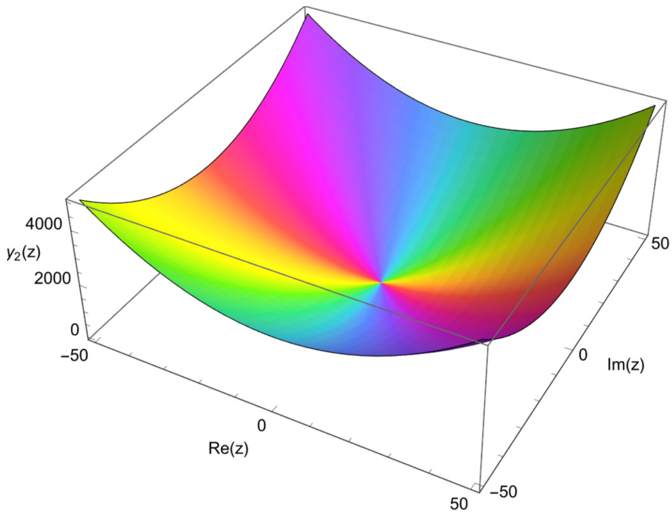

Fig. 1. Visualization of numerical solution obtained by solving Ex. 1 (3.9) at $\alpha = \frac{1}{2}, b = 0, \lambda = 1, a = 0, \gamma = 1$ where $z \in [-50 - 50i, 50 + 50i]$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

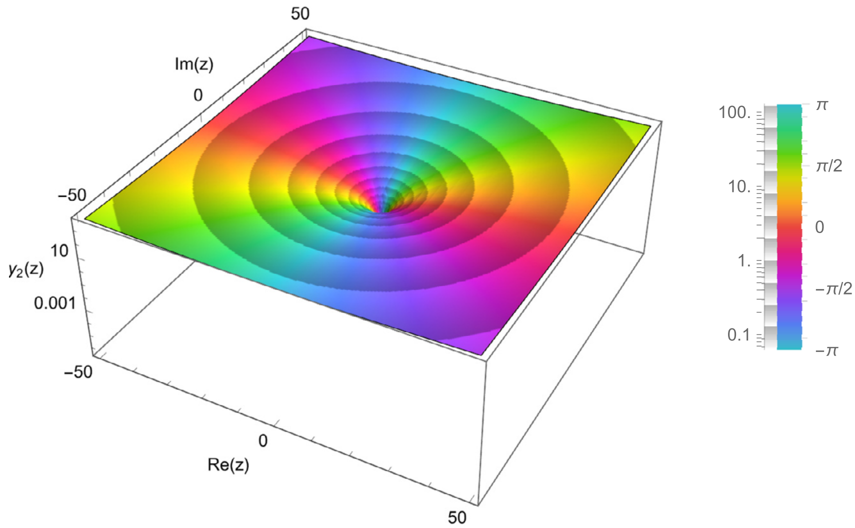

Fig. 2. Visualization of numerical solution obtained by solving Ex. 1 (3.9) at $\alpha = \frac{1}{2}, b = 0, \lambda = 5i, a = -1.5i, \gamma = -0.5i$ where $z \in [-50 - 50i, 50 + 50i]$.

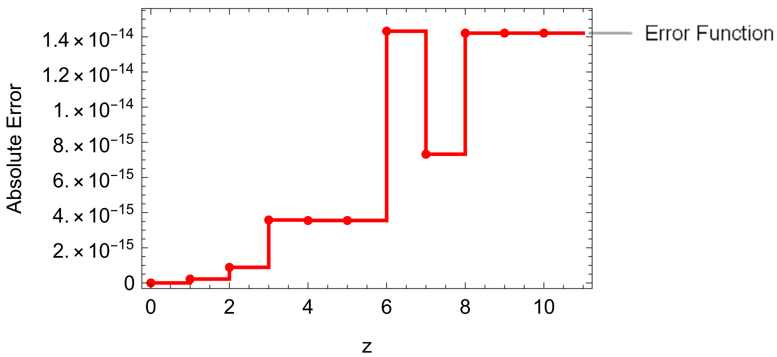

Fig. 3. Visualization of absolute error at $\alpha = \frac{1}{2}, b = 0, \lambda = 1, a = 0, \gamma = 1$ where $z \in [-50 - 50i, 50 + 50i]$.

$$y_2(z) = \sum_{i=0}^{2} c_i M_{\mathrm{a},\lambda,i}^{\mathrm{b},\gamma}(z) = c_0 M_{\mathrm{a},\lambda,0}^{\mathrm{b},\gamma}(z) + c_1 M_{\mathrm{a},\lambda,1}^{\mathrm{b},\gamma}(z) + c_2 M_{\mathrm{a},\lambda,2}^{\mathrm{b},\gamma}(z) \tag{3.19}$$

then

$$D^\alpha y_2(z) = c_1 D^\alpha M_{\mathrm{a},\lambda,1}^{\mathrm{b},\gamma}(z) + c_2 D^\alpha M_{\mathrm{a},\lambda,2}^{\mathrm{b},\gamma}(z). \tag{3.20}$$

By using the above equation, Eq. (3.18) can be written as

$$\begin{aligned} &c_1 D^\alpha M_{\mathrm{a},\lambda,1}^{\mathrm{b},\gamma}(z) + c_2 D^\alpha M_{\mathrm{a},\lambda,2}^{\mathrm{b},\gamma}(z) \\ &\quad + \left(c_0 M_{\mathrm{a},\lambda,0}^{\mathrm{b},\gamma}(z) + c_1 M_{\mathrm{a},\lambda,1}^{\mathrm{b},\gamma}(z) + c_2 M_{\mathrm{a},\lambda,2}^{\mathrm{b},\gamma}(z) \right) = g(z) \end{aligned} \tag{3.21}$$

$$\begin{aligned} &c_1 \left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)\Gamma(2) z^{1-\alpha}}{\Gamma(\gamma)\Gamma(a+\lambda)\Gamma(2-\alpha)} \right) \\ &c_2 \left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)\Gamma(2) z^{1-\alpha}}{\Gamma(\gamma)\Gamma(a+\lambda)\Gamma(2-\alpha)} + \frac{\Gamma(\lambda)\Gamma(2b+\gamma)\Gamma(3) z^{2-\alpha}}{2!\Gamma(\gamma)\Gamma(2a+\lambda)\Gamma(3-\alpha)} \right) \\ &+ \left(c_0 + c_1 + c_1 \frac{\Gamma(\lambda)\Gamma(b+\gamma) z}{\Gamma(\gamma)\Gamma(a+\lambda)} + c_2 + c_2 \frac{\Gamma(\lambda)\Gamma(b+\gamma) z}{\Gamma(\gamma)\Gamma(a+\lambda)} + c_2 \frac{\Gamma(\lambda)\Gamma(2b+\gamma) z^2}{\Gamma(\gamma)\Gamma(2a+\lambda) 2!} \right) \\ &= g(z). \end{aligned} \tag{3.22}$$

Now collect the coefficients c_2, c_1, c_0 at $\alpha = 1$

$$\begin{aligned} &c_2 \left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + \frac{z^2\Gamma(\lambda)\Gamma(2b+\gamma)}{2\Gamma(\gamma)\Gamma(2a+\lambda)} + \frac{z\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + \frac{z\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)} + 1 \right) \\ &+ c_1 \left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + \frac{z\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + 1 \right) + c_0 - 2z = 0 \end{aligned} \tag{3.23}$$

and utilize it for any three points of z_i i.e. $z_0 = 0, z_1 = 0.5, z_2 = 1$ and $\alpha = 1$, we obtain three equations

$$c_1 \left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + 1 \right) + c_2 \left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + 1 \right) + c_0 = 0 \tag{3.24}$$

$$\begin{aligned} &c_1 \left(\frac{1.5\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + 1 \right) \\ &+ c_2 \left(\frac{1.5\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + \frac{0.625\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)} + 1 \right) + c_0 - 1 = 0 \end{aligned} \tag{3.25}$$

$$c_1 \left(\frac{2\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + 1 \right) + c_2 \left(\frac{2\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + \frac{3\Gamma(\lambda)\Gamma(2b+\gamma)}{2\Gamma(\gamma)\Gamma(2a+\lambda)} + 1 \right) + c_0 - 2 = 0. \tag{3.26}$$

In solving this linear system of equations to get $C_0 = 0, C_1 = -\frac{2\Gamma(\gamma)\Gamma(2a+\lambda)}{\Gamma(\lambda)\Gamma(2b+\gamma)}, C_2 = \frac{2\Gamma(\gamma)\Gamma(2a+\lambda)}{\Gamma(\lambda)\Gamma(2b+\gamma)}$, the solution becomes

$$\begin{aligned} y_2(z) =& C_0 \sum_{k=0}^{0} \frac{z^k\Gamma(\lambda)\Gamma(bk+\gamma)}{k!\Gamma(\gamma)\Gamma(ak+\lambda)} + C_1 \sum_{k=0}^{1} \frac{z^k\Gamma(\lambda)\Gamma(bk+\gamma)}{k!\Gamma(\gamma)\Gamma(ak+\lambda)} + C_2 \sum_{k=0}^{2} \frac{z^k\Gamma(\lambda)\Gamma(bk+\gamma)}{k!\Gamma(\gamma)\Gamma(ak+\lambda)} \\ =& \frac{2\Gamma(\gamma)\Gamma(2a+\lambda) \left(\frac{z^2\Gamma(\lambda)\Gamma(2b+\gamma)}{2\Gamma(\gamma)\Gamma(2a+\lambda)} + \frac{z\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + 1 \right)}{\Gamma(\lambda)\Gamma(2b+\gamma)} \\ &- \frac{2\Gamma(\gamma)\Gamma(2a+\lambda) \left(\frac{z\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + 1 \right)}{\Gamma(\lambda)\Gamma(2b+\gamma)} \end{aligned} \tag{3.27}$$

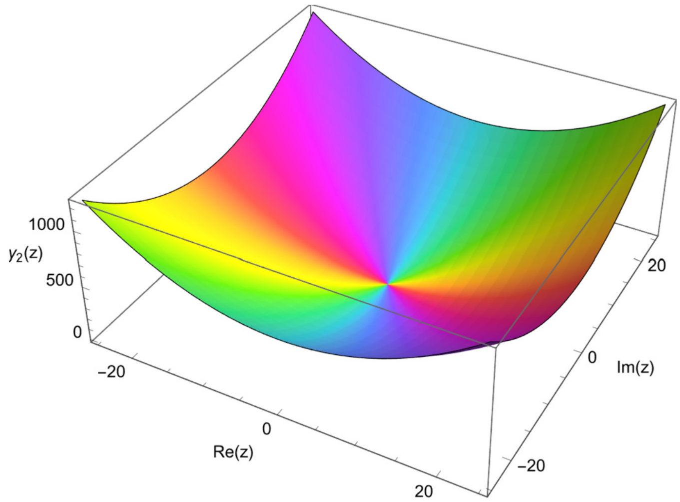

Fig. 4. Visualization of numerical solution obtained by solving Ex. 2 (3.18) at $\alpha = 1, b = 0, \lambda = 1.5i, a = 0, \gamma = 1.5i$ where $z \in [-1 - 1i, 1 + 1i]$.

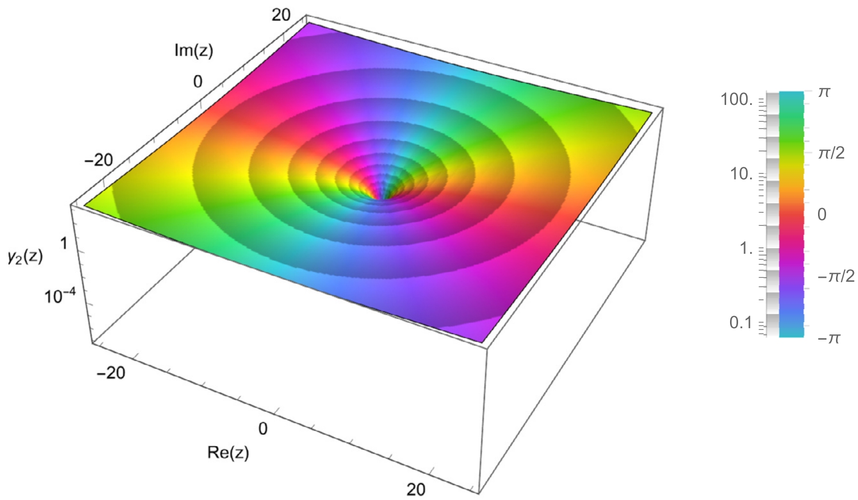

Fig. 5. Visualization of numerical solution obtained by solving Ex. 2 (3.18) at $\alpha = 1, b = 0, \lambda = 2.5i, a = 0, \gamma = 2.5i$ where $z \in [-1 - 1i, 1 + 1i]$ and $z \in [-25 - 25i, 25 + 25i]$.

which is the numerical solution for this problem. For $b = 0, \lambda = 1, a = 0, \gamma = 1, q = 2$, this solution becomes $y_2(z) = z^2$ which is the exact solution for $q = 2$. Next two examples are two cases of fractional Riccati differential equation

$$y' + p(z)y = g(z)y^2 + h(z).$$

Since this equation cannot be solved analytically, and numerical techniques are used to obtain its solution, we will use the MLCHF collocation method to solve the two cases of fractional Riccati differential equation where $g(z) = \pm 1, h(z) = z^0, p(z) = 0$.

3.5. Example 3

Consider another example of Riccati fractional differential equation [57]

$$D^\alpha y - 1 + y^2 = 0, \quad z \in [0, 1] \quad y(0) = 0. \tag{3.28}$$

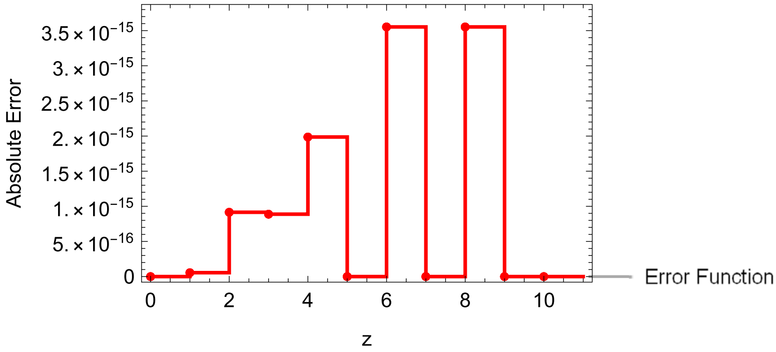

Fig. 6. Visualization of absolute error at $\alpha = 1, b = 1, \lambda = 1.5i, a = 1, \gamma = 1.5i$ where $z \in [0, 10]$.

To solve this problem with MLCHF, assume $i = 3$ for the sake of easier calculations, then Eq. (3.28) becomes

$$y_2(z) = \sum_{i=0}^{2} c_i M_{\mathrm{a},\lambda,i}^{\mathrm{b},\gamma}(z) = c_0 M_{\mathrm{a},\lambda,0}^{\mathrm{b},\gamma}(z) + c_1 M_{\mathrm{a},\lambda,1}^{\mathrm{b},\gamma}(z) + c_2 M_{\mathrm{a},\lambda,2}^{\mathrm{b},\gamma}(z) + c_3 M_{\mathrm{a},\lambda,3}^{\mathrm{b},\gamma}(z) \tag{3.29}$$

then

$$D^{\alpha} y_2(z) = c_1 D^{\alpha} M_{\mathrm{a},\lambda,1}^{\mathrm{b},\gamma}(z) + c_2 D^{\alpha} M_{\mathrm{a},\lambda,2}^{\mathrm{b},\gamma}(z) + c_3 D^{\alpha} M_{\mathrm{a},\lambda,3}^{\mathrm{b},\gamma}(z). \tag{3.30}$$

By using the above equation, Eq. (3.28) can be written as

$$c_1 D^{\alpha} M_{\mathrm{a},\lambda,1}^{\mathrm{b},\gamma}(z) + c_2 D^{\alpha} M_{\mathrm{a},\lambda,2}^{\mathrm{b},\gamma}(z) + c_3 M_{\mathrm{a},\lambda,3}^{\mathrm{b},\gamma}(z) - 1 \tag{3.31}$$

$$+\left(c_1 D^{\alpha} M_{\mathrm{a},\lambda,1}^{\mathrm{b},\gamma}(z) + c_2 D^{\alpha} M_{\mathrm{a},\lambda,2}^{\mathrm{b},\gamma}(z) + c_3 D^{\alpha} M_{\mathrm{a},\lambda,3}^{\mathrm{b},\gamma}(z)\right)^2 = 0. \tag{3.32}$$

By collecting the coefficients c_3, c_2, c_1, c_0 and utilizing it for any four points of z_i i.e. $z_0 = 0, z_1 = 0.5, z_2 = 0.75, z_3 = 1$ and $\alpha = 1$, we obtain the following four equations

$$\frac{c_1 \Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)}$$
$$+c_2\left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)}\right) + c_3\left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)}\right) + (c_0 + c_1 + c_2 + c_3)^2 - 1 = 0.$$

For $z = 1$

$$\Bigg(c_1\left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + 1\right) + c_2\left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + \frac{\Gamma(\lambda)\Gamma(2b+\gamma)}{2\Gamma(\gamma)\Gamma(2a+\lambda)} + 1\right) + c_3\Bigg(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)}$$
$$+\frac{\Gamma(\lambda)\Gamma(2b+\gamma)}{2\Gamma(\gamma)\Gamma(2a+\lambda)} + \frac{\Gamma(\lambda)\Gamma(3b+\gamma)}{6\Gamma(\gamma)\Gamma(3a+\lambda)} + 1\Bigg) + c_0\Bigg)^2 + \frac{1.c_1\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + c_2\Bigg(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)}$$
$$+\frac{\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)}\Bigg) + c_3\left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + \frac{\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)} + \frac{0.5\Gamma(\lambda)\Gamma(3b+\gamma)}{\Gamma(\gamma)\Gamma(3a+\lambda)}\right) - 1 = 0.$$

For $z = 0.5$

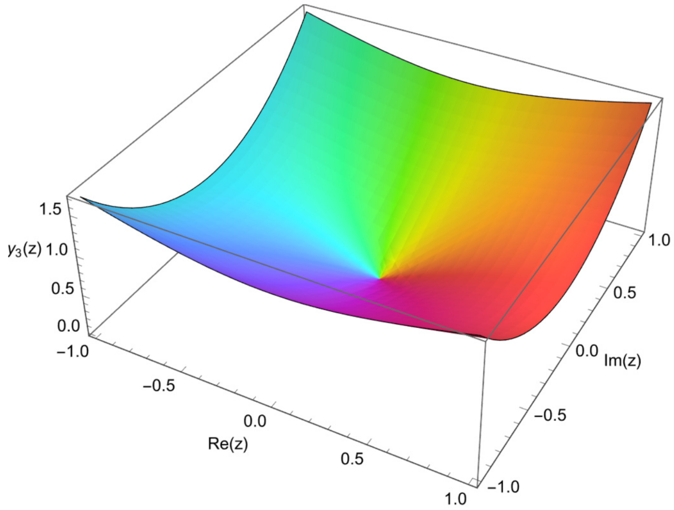

Fig. 7. Visualization of numerical solution obtained by solving Ex. 3 (3.28) at $\alpha = 1, b = 0, \lambda = 1, a = 0, \gamma = 1$ where $z \in [-40 - 40i, 40 + 40i]$.

$$\left(c_1\left(\frac{0.5\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)}+1\right)+c_2\left(\frac{0.5\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)}+\frac{0.125\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)}+1\right)\right.$$
$$\left.+c_3\left(\frac{0.5\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)}+\frac{0.125\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)}+\frac{0.0208333\Gamma(\lambda)\Gamma(3b+\gamma)}{\Gamma(\gamma)\Gamma(3a+\lambda)}+1\right)+c_0\right)^2$$
$$+\frac{c_1\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)}+c_2\left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)}+\frac{0.5\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)}\right)+c_3\left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)}\right.$$
$$\left.+\frac{0.5\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)}+\frac{0.125\Gamma(\lambda)\Gamma(3b+\gamma)}{\Gamma(\gamma)\Gamma(3a+\lambda)}\right)-1=0.$$

For $z = 0.75$

$$\left(c_1\left(\frac{0.75\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)}+1\right)+c_2\left(\frac{0.75\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)}+\frac{0.28125\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)}+1\right)\right.$$
$$\left.+c_3\left(\frac{0.75\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)}+\frac{0.28125\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)}+\frac{0.0703125\Gamma(\lambda)\Gamma(3b+\gamma)}{\Gamma(\gamma)\Gamma(3a+\lambda)}+1\right)+c_0\right)^2$$
$$+\frac{c_1\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)}+c_2\left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)}+\frac{0.75\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)}\right)+c_3\left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)}\right.$$
$$\left.+\frac{0.75\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)}+\frac{0.28125\Gamma(\lambda)\Gamma(3b+\gamma)}{\Gamma(\gamma)\Gamma(3a+\lambda)}\right)-1=0.$$

In solving this linear system of equations to get $C_0 = -1, C_1 = 1, C_2 = 2, C_3 = -2$ at $\alpha = 1$, the solution becomes

$$y_3(z) = \frac{z\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} - \frac{z^3\Gamma(\lambda)\Gamma(3b+\gamma)}{3\Gamma(\gamma)\Gamma(3a+\lambda)} \tag{3.33}$$

which is the numerical solution for this problem. For $b = 0, \lambda = 1, a = 0, \gamma = 1$, this solution becomes $y_3(z) = z - \frac{z^3}{3}$ which is the series form of its exact solution i.e. $\tanh(z)$.

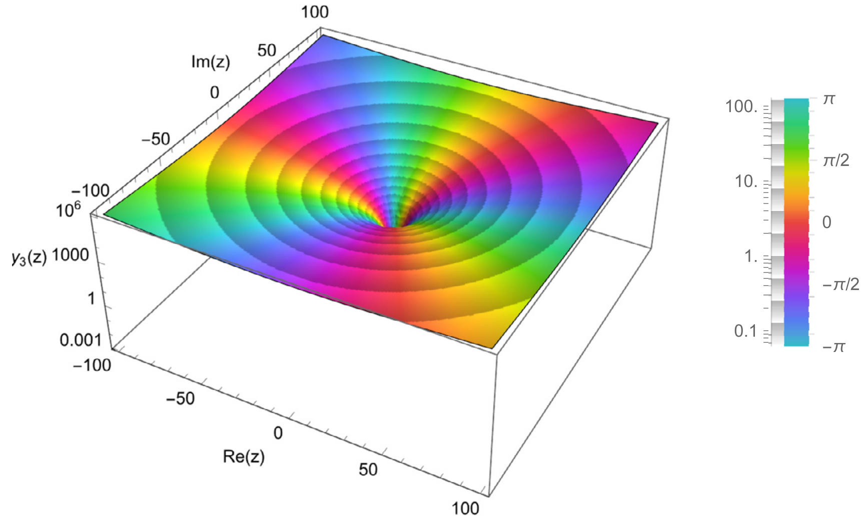

Fig. 8. Visualization of numerical solution obtained by solving Ex. 3 (3.28) at $\alpha = \frac{1}{2}, b = 2, \lambda = 1.5i, a = 2, \gamma = 1.5i$ where $z \in [-100 - 100i, 100 + 100i]$.

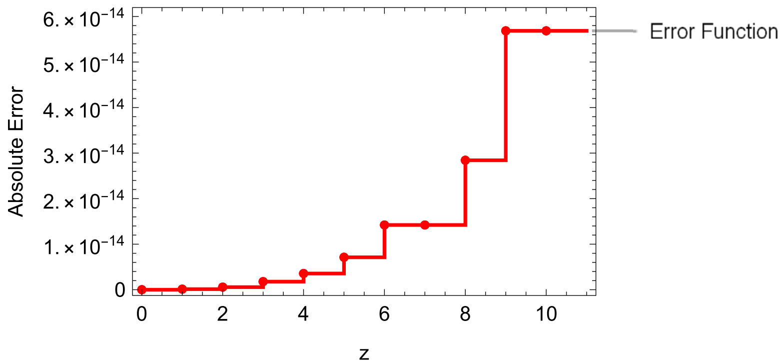

Fig. 9. Visualization of absolute error at $\alpha = 1, b = 1, \lambda = 1.5i, a = 1, \gamma = 1.5i$ where $z \in [0, 10]$.

3.6. Example 4

Consider another example of Riccati fractional differential equation [57]

$$D^{\alpha}y - 1 - y^2 = 0, \quad z \in [0,1] \quad y(0) = 0. \tag{3.34}$$

To solve this problem with MLCHF, assume $i = 3$ for the sake of easier calculations, then Eq. (3.34) becomes

$$y_2(z) = \sum_{i=0}^{2} c_i M^{\mathrm{b},\gamma}_{\mathrm{a},\lambda,i}(z) = c_0 M^{\mathrm{b},\gamma}_{\mathrm{a},\lambda,0}(z) + c_1 M^{\mathrm{b},\gamma}_{\mathrm{a},\lambda,1}(z) + c_2 M^{\mathrm{b},\gamma}_{\mathrm{a},\lambda,2}(z) + c_3 M^{\mathrm{b},\gamma}_{\mathrm{a},\lambda,3}(z) \tag{3.35}$$

then

$$D^{\alpha}y_2(z) = c_1 D^{\alpha} M^{\mathrm{b},\gamma}_{\mathrm{a},\lambda,1}(z) + c_2 D^{\alpha} M^{\mathrm{b},\gamma}_{\mathrm{a},\lambda,2}(z) + c_3 D^{\alpha} M^{\mathrm{b},\gamma}_{\mathrm{a},\lambda,3}(z). \tag{3.36}$$

By using the above equation, Eq. (3.34) can be written as

$$c_1 D^{\alpha} M^{\mathrm{b},\gamma}_{\mathrm{a},\lambda,1}(z) + c_2 D^{\alpha} M^{\mathrm{b},\gamma}_{\mathrm{a},\lambda,2}(z) + c_3 M^{\mathrm{b},\gamma}_{\mathrm{a},\lambda,3}(z) - 1$$

$$-\left(c_1 D^\alpha M^{\mathrm{b},\gamma}_{\mathrm{a},\lambda,1}(z) + c_2 D^\alpha M^{\mathrm{b},\gamma}_{\mathrm{a},\lambda,2}(z) + c_3 D^\alpha M^{\mathrm{b},\gamma}_{\mathrm{a},\lambda,3}(z)\right)^2 = 0. \tag{3.37}$$

By collecting the coefficients c_3, c_2, c_1, c_0 and utilizing it for any four points of z_i i.e. $z_0 = 0, z_1 = 0.5, z_2 = 0.75, z_3 = 1$ and $\alpha = 1$, we obtain the following four equations

$$\frac{c_1\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + c_2\left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + 0.\right) + c_3\left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + 0.\right) - (c_0 + c_1 + c_2 + c_3)^2 - 1 = 0. \tag{3.38}$$

For $z = 1$

$$\begin{aligned} &-\left(c_1\left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + 1\right) + c_2\left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + \frac{\Gamma(\lambda)\Gamma(2b+\gamma)}{2\Gamma(\gamma)\Gamma(2a+\lambda)} + 1\right)\right. \\ &\left. + c_3\left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + \frac{\Gamma(\lambda)\Gamma(2b+\gamma)}{2\Gamma(\gamma)\Gamma(2a+\lambda)} + \frac{\Gamma(\lambda)\Gamma(3b+\gamma)}{6\Gamma(\gamma)\Gamma(3a+\lambda)} + 1\right) + c_0\right)^2 \\ &+ \frac{c_1\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + c_2\left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + \frac{\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)}\right) \\ &+ c_3\left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + \frac{\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)} + \frac{0.5\Gamma(\lambda)\Gamma(3b+\gamma)}{\Gamma(\gamma)\Gamma(3a+\lambda)}\right) - 1 = 0. \end{aligned}$$

For $z = 0.5$

$$\begin{aligned} &-\left(c_1\left(\frac{0.5\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + 1\right) + c_2\left(\frac{0.5\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + \frac{0.125\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)} + 1\right)\right. \\ &+ c_3\left(\frac{0.5\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + \frac{0.125\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)} + \frac{0.0208333\Gamma(\lambda)\Gamma(3b+\gamma)}{\Gamma(\gamma)\Gamma(3a+\lambda)} + 1\right) \\ &\left. + c_0\right)^2 + \frac{c_1\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + c_2\left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + \frac{0.5\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)}\right) \\ &+ c_3\left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + \frac{0.5\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)} + \frac{0.125\Gamma(\lambda)\Gamma(3b+\gamma)}{\Gamma(\gamma)\Gamma(3a+\lambda)}\right) - 1 = 0. \end{aligned}$$

For $z = 0.75$

$$\begin{aligned} &-\left(c_1\left(\frac{0.75\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + 1\right) + c_2\left(\frac{0.75\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + \frac{0.28125\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)} + 1\right)\right. \\ &\left. + c_3\left(\frac{0.75\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + \frac{0.28125\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)} + \frac{0.0703125\Gamma(\lambda)\Gamma(3b+\gamma)}{\Gamma(\gamma)\Gamma(3a+\lambda)} + 1\right) + c_0\right)^2 \\ &+ \frac{c_1\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + c_2\left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + \frac{0.75\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)}\right) \\ &+ c_3\left(\frac{\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} + \frac{0.75\Gamma(\lambda)\Gamma(2b+\gamma)}{\Gamma(\gamma)\Gamma(2a+\lambda)} + \frac{0.28125\Gamma(\lambda)\Gamma(3b+\gamma)}{\Gamma(\gamma)\Gamma(3a+\lambda)}\right) - 1 = 0. \end{aligned}$$

In solving this linear system of equations to get $C_0 = -1, C_1 = 1, C_2 = -2, C_3 = 2$ at $\alpha = 1$, the solution becomes

$$y_3(z) = \frac{z^3\Gamma(\lambda)\Gamma(3b+\gamma)}{3\Gamma(\gamma)\Gamma(3a+\lambda)} + \frac{z\Gamma(\lambda)\Gamma(b+\gamma)}{\Gamma(\gamma)\Gamma(a+\lambda)} \tag{3.39}$$

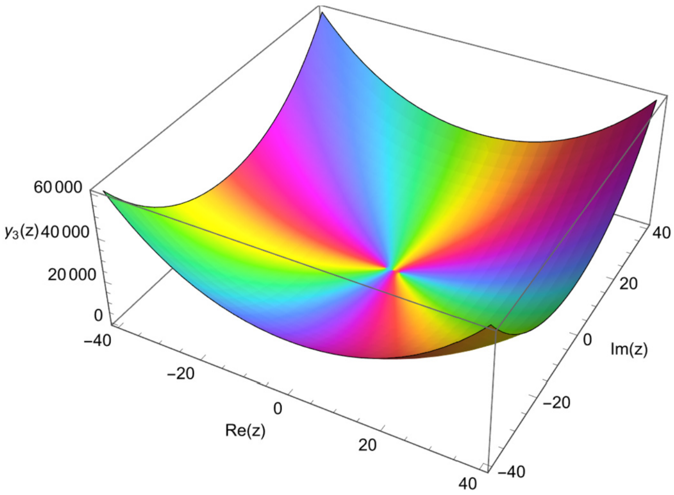

Fig. 10. Visualization of numerical solution obtained by solving Ex. 4 (3.34) at $\alpha = 1, b = 0, \lambda = 1, a = 0, \gamma = 1$ where $z \in [-40 - 40i, 40 + 40i]$.

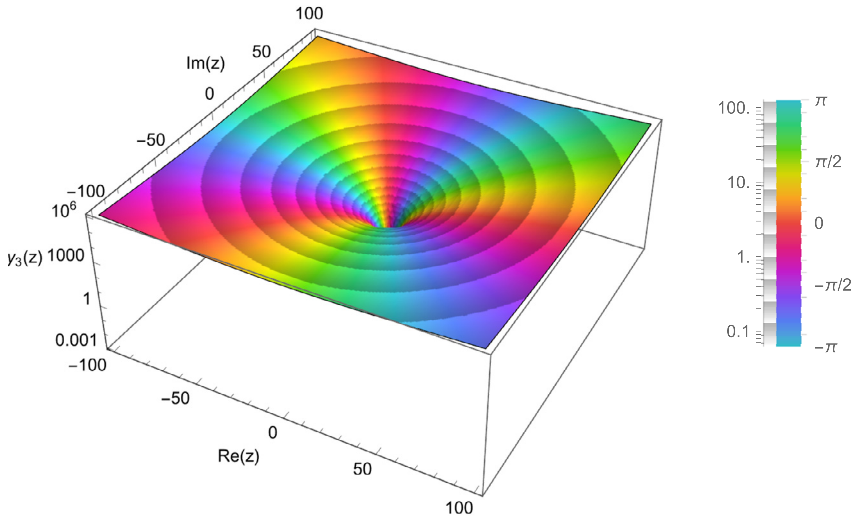

Fig. 11. Visualization of numerical solution obtained by solving Ex. 4 (3.34) at $\alpha = 1, b = 2, \lambda = 1.5i, a = 2, \gamma = 1.5i$ where $z \in [-100 - 100i, 100 + 100i]$.

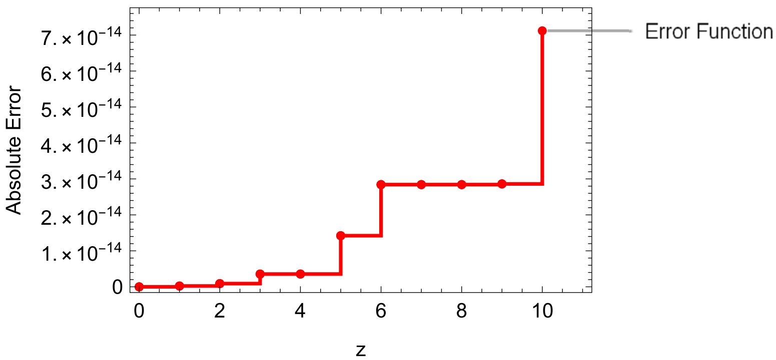

Fig. 12. Visualization of absolute error at $\alpha = 1, b = 2, \lambda = 1.5i, a = 2, \gamma = 1.5i$ where $z \in [0, 10]$.

which is the numerical solution for this problem. For $b = 0, \lambda = 1, a = 0, \gamma = 1$, this solution becomes $y_3(z) = z + \frac{z^3}{3}$ which is the series form of its exact solution i.e. $\tan(z)$.

Graphical results of the examples solved above show that this method is very effective in solving fractional linear and nonlinear differential equations. In Fig. 1, numerical solution for $\alpha = \frac{1}{2}$ is shown. For $b = 0, a = 0, \gamma = 1, \lambda = 1$, it becomes the Mittag-Leffler-Confluent hypergeometric function, and one can obtain an analytical solution. In Fig. 1, it can be observed that the graph is counterclockwise, which shows that there is a zero present in the solution. The graph is repeating its colors and pointing towards a simple multiple zero at $z = 0 + 0i$. In Fig. 2, the multiple zero can be noticed. Also, the legend describes the direction of the graph i.e. anticlockwise. Anticlockwise graphs contain zeros, whereas clockwise graphs describe poles. Fig. 3 shows the graph of absolute error obtained from the exact solution of Eq. (3.9) and the numerical solution obtained by the MLCHF collocation method. Hence, the absolute error shows the accuracy of this method as it goes up to 1.4×10^{-14} in Fig. 3.

Upon solving Eq. (3.18), the results can be seen graphically in Fig. 4. At $\alpha = 1$, the numerical solution is plotted for $a = 1, b = 0, \lambda = 1.5i, \gamma = 1.5i$. This is an anticlockwise graph as after green comes the blue color, and as we increase the plotting range, it adds more colors to it, representing the multiple zeros in this solution. In Fig. 5, the plot range is $-25 - 25i, 25 + 25i$ and we can clearly see that the zero lies at $z = 0 + 0i$. To check the accuracy of the numerical solution of Eq. (3.18) obtained from MLCHF, the absolute error was obtained, and Fig. 6 shows that its absolute error fluctuates between 5×10^{-16} and 3.5×10^{-15}.

Eq. (3.28) and Eq. (3.34) describe the two cases of fractional Riccati differential equation where $g(z) = \pm 1, h(z) = z^0, p(z) = 0$. Fig. 9 and Fig. 12 show the absolute error of both cases at $\alpha = 1$ and the error reaches 6×10^{-14} and 7×10^{-14} respectively. Hence, these values show the accuracy of the MLCHF collocation method. Fig. 7 and Fig. 10 show the numerical solution graphically, and by looking at the legend and graph, it is clear that Fig. 7 and Fig. 10 are anticlockwise showing multiple zeros in the graph. Fig. 8 and Fig. 11 point at the multiple zero in graphs at $z = 0 + 0i$. Since the numerical solution obtained was a polynomial when $z = 0 + 0i$, then the solution becomes zero. Hence, the z-intercept is shown in Fig. 3, Fig. 6, Fig. 8 and Fig. 11.

4. Conclusion

To solve fractional differential and integral equations effectively, the utilization of Mittag-Leffler and confluent hypergeometric functions is indispensable. Consequently, the initial and most critical step in their numerical computation involves thorough investigation and development of reliable techniques. Results found in this paper suggest that many other studies on fractional differential and integral models can be expressed using conventional terms. Integral representations of Mittag-Leffler and confluent hypergeometric functions of various types play a pivotal role in the study of complete functions. Effective graphic representations for solving fractional linear and nonlinear differential equations have been made possible by the numerical method for solving fractional differential equations that was developed in this paper. In essence, this research serves as a continuation of the authors' ongoing work and future investigations into intriguing applications based on generalized multi-parameter Mittag-Leffler functions, their related forms, and numerical techniques.

Acknowledgment

The authors are thankful for the valuable comments and suggestions of the anonymous referee.

References

[1] W.J. Archibald, The complete solution of the differential equation for the confluent hypergeometric function, Lond. Edinb. Dublin Philos. Mag. J. Sci. 26 (1938) 415–419.

[2] R.A. Askey, A.B. Olde Daalhuis, NIST Handbook of Mathematical Functions, Chapters 13, 15 and 16. Olver, Lozier, Boisvert, Clark (eds.), Cambridge University Press, Cambridge, 2010.
[3] M.K. Bansal, N. Jolly, R. Jain, D. Kumar, An integral operator involving generalized Mittag-Leffler function and associated fractional calculus results, J. Anal. 27 (2019) 727–740.
[4] S. Bhatter, A. Mathur, D. Kumar, K.S. Nisar, J. Singh, Fractional modified Kawahara equation with Mittag-Leffler law, Chaos Solitons Fractals 131 (2020) 109508.
[5] R.G. Buschman, Convolution equations with generalized Laguerre polynomial kernels, SIAM Rev. 6 (2) (1964) 166–167.
[6] J. Choi, R.K. Parmar, P. Chopra, Extended Mittag-Leffler function and associated fractional calculus operators, Georgian Math. J. 27 (2) (2020) 199–209.
[7] H.T. Davis, Tables of the Higher Mathematical Functions, vol. I, The Principia Press, Bloomington, Ind., 1933.
[8] J.D. Djida, G. Mophou, I. Area, Optimal control of diffusion equation with fractional time derivative with nonlocal and nonsingular Mittag-Leffler kernel, J. Optim. Theory Appl. 182 (2019) 540–557.
[9] M.A. El-Khateb, H.S. Hussien, An optimization method for solving some differential algebraic equations, Commun. Nonlinear Sci. Numer. Simul. 14 (5) (2009) 1970–1977.
[10] A. Erdélyi, Magnus Wilhelm, Oberhettinger Fritz, G. Tricomi Francesco, Higher Transcendental Functions, vol. I-III, McGraw-Hill, 1953.
[11] C. Fox, The asymptotic expansion of generalized hypergeometric functions, Proc. Lond. Math. Soc. 2 (1) (1928) 389–400.
[12] F. Ghanim, S. Bendak, A. Al Hawarneh, Certain implementations in fractional calculus operators involving Mittag Leffler-confluent hypergeometric functions, Proc. R. Soc. A 478 (2022) 20210839.
[13] B. Gonzalez, J. Matera, H.M. Srivastava, Some q-generating functions and associated generalized hypergeometric polynomials, Math. Comput. Model. 34 (1/2) (2001) 133–175.
[14] R. Gorenflo, A.A. Kilbas, F. Mainardi, S.V. Rogosin, Mittag-Leffler Functions, Related Topics and Applications, 2nd ed., Springer, Berlin, Germany, 2020.
[15] H.J. Haubold, A.M. Mathai, R.K. Saxena, Mittag-Leffler functions and their applications, J. Appl. Math. 2011 (2011) 298628.
[16] S. Kazem, S. Abbasbandy, Sunil Kumar, Fractional-order Legendre functions for solving fractional-order differential equations, Appl. Math. Model. 37 (7) (2013) 5498–5510.
[17] P.R. Khalendekar, On a convolution trans form involving generalized Laguerre polynomial as its kernel, J. Math. Pures Appl. 44 (2) (1965) 195–197.
[18] A.A. Kilbas, Fractional calculus of the generalized Wright function, Fract. Calc. Appl. Anal. 8 (2) (2005) 113–126.
[19] A.A. Kilbas, M. Saigo, J.J. Trujillo, On the generalized Wright function, Fract. Calc. Appl. Anal. 5 (4) (2002) 437–460.
[20] A.A. Kilbas, M. Saigo, R.K. Saxena, Generalized Mittag-Leffler function and generalized fractional calculus operators, Integral Transforms Spec. Funct. 15 (2004) 31–49.
[21] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, the Netherlands, 2006.
[22] V. Kiryakova, Generalized Fractional Calculus and Applications, CRC Press, 1993.
[23] V. Kiryakova, The special functions of fractional calculus as generalized fractional calculus operators of some basic functions, Comput. Math. Appl. 59 (3) (2010) 1128–1141, https://doi.org/10.1016/j.camwa.2009.05.014.
[24] V. Kiryakova, A guide to special functions in fractional calculus, Mathematics 9 (2021) 106, https://doi.org/10.3390/math9010106.
[25] V.S. Kiryakova, The multi-index Mittag-Leffler functions as important class of special function of fractional calculus, Comput. Math. Appl. 59 (5) (2010) 1885–1895.
[26] V.S. Kiryakova, The role of special functions and generalized fractional calculus in studying classes of univalent functions, Adv. Math. Sci. J. 4 (2) (2015) 161–174.
[27] D. Kumar, J. Singh, D. Baleanu, A new analysis of the Fornberg-Whitham equation pertaining to a fractional derivative with Mittag-Leffler-type kernel, Eur. Phys. J. Plus 133 (2018) 1–10, https://doi.org/10.1140/epjp/i2018-11934-y.
[28] C. Lavault, Fractional calculus and generalized Mittag-Leffler type functions, Preprint, arXiv:1703.01912, 2017.
[29] S.-D. Lin, H.M. Srivastava, P.-Y. Wang, Some families of hypergeometric transformations and generating relations, Math. Comput. Model. 36 (2002) 445–459.
[30] S.-D. Lin, Y.-S. Chao, H.M. Srivastava, Some families of hypergeometric polynomials and associated integral representations, J. Math. Anal. Appl. 294 (2) (2004) 399–411.
[31] A.D. Macdonald, Properties of the confluent hypergeometric function, J. Math. Phys. 28 (1949) 183–191.
[32] A.M. Mathai, H.J. Haubold, Mittag-Leffler functions and fractional calculus, in: Special Functions for Applied Scientists, Springer, New York, NY, USA, 2008, pp. 79–134.
[33] K.B. Oldham, J. Spanier, The Fractional Calculus; Theory and Applications of Differentiation and Integration to Arbitrary Order, Academic Press, New York and London, 1974.
[34] J. Paneva-Konovska, V. Kiryakova, The generalized Fox-Wright function: the Laplace transform, the Erdélyi-Kober fractional integral and its role in fractional calculus, Mathematics 12 (2024) 1918, https://doi.org/10.3390/math12121918.
[35] R.N. Pillai, On Mittag-Leffler functions and related distributions, Ann. Inst. Stat. Math. 42 (1990) 157–161.
[36] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1998.
[37] T.R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J. 19 (1971) 7–15.
[38] G. Rahman, D. Baleanu, M.A. Qurashi, S.D. Purohit, S. Mubeen, M. Arshad, The extended Mittag-Leffler function via fractional calculus, J. Nonlinear Sci. Appl. 10 (8) (2017) 4244–4253.
[39] K.C. Rusia, An integral equation involving generalized Laguerre polynomial, Math. Jpn. 11 (1966) 15–18.
[40] S. Samko, A. Kilbas, O. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Yverdon, 1993.

[41] A.K. Shukla, J.C. Prajapati, On a generalization of Mittag-Leffler function and its properties, J. Math. Anal. Appl. 336 (2) (2007) 797–811.

[42] H. Singh, H.M. Srivastava, R.K. Pandey, Special Functions in Fractional Calculus and Engineering, 1st ed., CRC Press, Boca Raton, 2023.

[43] H.M. Srivastava, On an extension of the Mittag-Leffler function, Yokohama Math. J. 16 (2) (1968) 77–88.

[44] H.M. Srivastava, A contour integral involving Fox's H-function, Indian J. Math. 14 (1972) 1–6.

[45] H.M. Srivastava, Ž Tomovski, Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel, J. Appl. Math. Comput. 211 (2009) 198–210.

[46] H.M. Srivastava, R.K. Saxena, C. Ram, A unified presentation of the Gamma-type functions occurring in diffraction theory and associated probability distributions, Appl. Math. Comput. 162 (2) (2005) 931–947.

[47] K.N. Srivastava, A class of integral equations involving Laguerre polynomials as kernel, Proc. Edinb. Math. Soc. 15 (1966) 33–36.

[48] K.N. Srivastava, On integral equations involving Whittaker's function, Proc. Glasg. Math., Assoc. 7 (3) (1966) 125–127.

[49] J.A. Tenreiro Machado, V. Kiryakova, The chronicles of fractional calculus, Fract. Calc. Appl. Anal. 20 (2) (2017) 307–336, https://doi.org/10.1515/fca-2017-0017.

[50] E.T. Whittaker, G.N. Watson, A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; with an Account of the Principal Transcendental Functions, fourth ed., Cambridge University Press, Cambridge, London and New York, 1927.

[51] D.V. Widder, The inversion of a convolution transform whose kernel is a Laguerre polynomial, Am. Math. 70 (3) (1963) 291–293.

[52] A. Wiman, Uber die nullstellen der funktionen $E_\alpha(x)$, Acta Math. 29 (1905) 217–234.

[53] A. Wiman, Uber den fundamentalsatz in der theorie der funcktionen $E_\alpha(x)$, Acta Math. 29 (1905) 191–201.

[54] J. Wimp, Two integral transform pairs involving hypergeometric functions, Proc. Glasg. Math. Assoc. 7 (1965) 42–44.

[55] E.M. Wright, The asymptotic expansion of the generalized Bessel function, J. Lond. Math. Soc. 2 (1) (1934) 257–270.

[56] E.M. Wright, The asymptotic expansion of the generalized hypergeometric function, J. Lond. Math. Soc. 1 (4) (1935) 286–293.

[57] Ş. Yüzbasi, Numerical solutions of fractional Riccati type differential equations by means of the Bernstein polynomials, Appl. Math. Comput. 219 (11) (2013) 6328–6343.