

Functional renormalization group approach to the sine-Gordon model

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The renormalization group flow is presented for the two-dimensional sine-Gordon model within the framework of the functional renormalization group method by including the wave-function renormalization constant. The Kosterlitz-Thouless-Berezinski type phase structure is recovered as the interpolating scaling law between two competing IR attractive area of the global renormalization group flow.

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I. Introduction.— The two-dimensional (2D) sine-Gordon (SG) model, defined by the bare action

$$S = \int_x \left[\frac{1}{2} (\partial_\mu \varphi)^2 + u \cos(\beta \varphi) \right] \quad (1)$$

in Euclidean spacetime, has already received a considerable amount of attention [1–7] since it is the simplest non-trivial quantum field theory with compact variables. This feature is common with non-Abelian gauge theories and is supposed to be the key to their confinement mechanism. In two dimensions this is the driving force to form a non-trivial phase structure. The SG model is known to belong to the universality class of the 2D Coulomb gas and the 2D-XY spin model which have received important applications in condensed matter systems, e.g. describing the Kosterlitz-Thouless-Berezinski (KTB) [8] phase transition of vortices in a thin superfluid film. There is a continuous interest in the literature in constructing SG type models to understand better the vortex dynamics of condensed matter systems [9].

A more detailed relation between the SG model and the XY model in the Villain-approximation is obtained by using lattice regularization [10]. The kinetic energy is periodic with the same period length as the potential energy. Therefore the model supports vortices and has a third adjustable parameter, the vortex fugacity z . For $z \rightarrow 0$ the vortices are suppressed and the SG model of Eq. (1) is recovered in the continuum. The duality transformation, $(\beta, u, z) \rightarrow (2\pi/\beta, 2z, u/2)$ maps the continuum SG model ($z = 0$) into the XY model without external field ($u = 0$). The Coleman point [1], separating the renormalizable, asymptotically free phase ($\beta^2 < 8\pi$) and the non-renormalizable phase ($\beta^2 > 8\pi$) of the SG model is mapped into the KTB point of the XY model.

The perturbative RG results beyond the local potential approximation (LPA) [3] can account for the KTB phase transition and provide $\beta^2 \rightarrow 0$ for $\beta^2 < 8\pi$ in the infrared (IR) limit. Recently, by using the flow equation approach [4] a different IR limit is obtained for the frequency, i.e. $\beta^2 \rightarrow 4\pi$. However, the latter method is not able to recover the leading order perturbative UV results for $\beta^2 < 4\pi$, due to the wrong sign of the evolution

equation derived for the frequency. Functional RG approaches have also been used to map the phase structure of the SG model but their description is not complete since on the one hand the LPA is used [6, 7] and on the other hand, the SG model is mapped onto other models belonging to the same universality class [5]. Therefore, the analysis of the SG model is still incomplete.

Our aim with this work is to determine the complete phase structure of the original SG model by extending the functional RG analysis beyond the LPA, by including the field-independent wave-function renormalization, as well. We use the functional RG method for the effective average action [11–13] which enables us to treat the wave-function renormalization. The evolution arises as the result of the gradual turning on of the field fluctuations according to their increasing amplitude by decreasing the control parameter k from the initial value $\Lambda \ll k_0$ (with k_0 the UV cut-off which goes to infinity) to zero.

The phase structure is found to be the global result of a competition between an IR fixed line and an IR fixed point. The traditional KTB scaling law is actually an interpolation between these two effects.

II. The sine-Gordon model.— The functional renormalization group equation for the effective action of an Euclidean field theory is [11]

$$k \partial_k \Gamma_k = \frac{1}{2} \text{Tr} \frac{k \partial_k R_k}{R_k + \Gamma_k''} \quad (2)$$

where the notation $' = \partial/\partial\varphi$ is used and the trace Tr stands for the integration over all momenta. We use a power-law type regulator function

$$R_k = p^2 \left(\frac{k^2}{p^2} \right)^b \quad (3)$$

with the parameter $b \geq 1$. Eq. (2) has been solved over the functional subspace defined by the ansatz

$$\Gamma_k = \int_x \left[\frac{z}{2} (\partial_\mu \varphi_x)^2 + V_k(\varphi_x) \right], \quad (4)$$

with the local potential $V_k(\varphi) = \sum_{n=1}^{\infty} u_n(k) \cos(n\varphi)$ and the field-independent wave-function renormalization

$z(k)$. Eq. (2) leads to the evolution equations [14]

$$\partial_k V_k = \frac{1}{2} \int_p \mathcal{D}_k k \partial_k R_k, \quad (5)$$

$$k \partial_k z = \mathcal{P}_0 V_k''^2 \int_p \mathcal{D}_k^2 k \partial_k R_k \left(\frac{\partial^2 \mathcal{D}_k}{\partial p^2 \partial p^2} p^2 + \frac{\partial \mathcal{D}_k}{\partial p^2} \right) \quad (6)$$

with $\mathcal{D}_k = 1/(zp^2 + R_k + V_k'')$ and $\mathcal{P}_0 = (2\pi)^{-1} \int_0^{2\pi} d\varphi$ being the projection onto the field-independent subspace.

III. Linearized scaling at the Coleman point.— We assume $\Lambda^2 \gg k^2 \gg |V_k''|$, keep the leading order terms in V_k'' in the Taylor-expansion of the r.h.s of Eqs. (5) and (6) and retain a single Fourier mode in the potential V_k for simplicity. The two-dimensional momentum integrals can easily be performed, giving

$$(2 + k \partial_k) \tilde{u}_1 = \frac{1}{4\pi z} \tilde{u}_1, \quad (7)$$

$$k \partial_k z = -\frac{\tilde{u}_1^2}{z^{2-2/b}} c_b, \quad (8)$$

where the dimensionless couplings $\tilde{u}_n = k^{-2} u_n$ are introduced, and

$$c_b = \frac{b}{48\pi} \Gamma\left(3 - \frac{2}{b}\right) \Gamma\left(1 + \frac{1}{b}\right). \quad (9)$$

The UV evolution Eqs. (7) and (8) clearly show that the critical value $z^* = 1/8\pi$ at the Coleman point is independent of the blocking parameter b . Furthermore the sharp cutoff limit $b \rightarrow \infty$ gives infinite value in the r.h.s. of Eq. (8) signalling the impossibility of introducing the wave-function renormalization in that case. The RG trajectories obtained by integrating Eqs. (7) and (8),

$$\tilde{u}_1^2(z) = \frac{2}{(8\pi)^{1-2/b} c_b} (z - z^*)^2 + \tilde{u}_1^{*2}, \quad (10)$$

indicate turning points in the vicinity of the fixed point, at $(\tilde{u}_1^* = \tilde{u}_1(z^*), z^*)$. Such a flow exhibits the well-known features of the KTB type phase transition. Actually we see the dual of that transition as explained in the Introduction.

Thus Eqs. (7) and (8) provide similar evolution around the KTB fixed point as the one already obtained by a perturbative RG analysis [3] and the flow equation approach [4] for the SG model, and also by the real-space RG for the two-dimensional Coulomb gas [15]. The KTB phase transition is characterized by the exponential dependence of the correlation length on the inverse of the square-root of the reduced temperature $t \propto \tilde{u}_1^{*2}$. The correlation length ξ can be read off from the scale $k^* \sim 1/\xi$ where the RG trajectories show up their turning points. Inserting back the solution (10) into Eq. (8) one obtains

$$\xi \sim e^{\sqrt{\pi}/(\tilde{u}_1^* 8\sqrt{c_b}) + \tilde{u}_1^*(b-1)(2b-1)\sqrt{c_b} 2^{1-6/b} \pi^{5/2-2/b} b^{-2} + \mathcal{O}(\tilde{u}_1^{*2})} \quad (11)$$

which is the typical scaling law for KTB type phase transitions, modified by analytic corrections vanishing for $\tilde{u}_1^* \rightarrow 0$. It is worthwhile mentioning that only the quantitative details depend on the choice of the parameter b in the formula (11). Using Eqs. (7) and (8) the anomalous dimension η can also be calculated via the vortex-vortex correlation function [15] and it is proved to take the value $\eta = 1/4$ independently of the parameter b .

IV. Coleman point, revisited as the dual KTB point.— Let us now take into account the higher-order terms of the Taylor expansion in V_k'' , as well as the higher harmonics of the local potential [7]. We choose $b = 1$, corresponding to the Callan-Symanzik RG scheme [13] which is free of UV divergences for $d = 2$ and ultralocal. The evolution equations assume a simpler form rendering easier the handling of the higher Fourier modes.

The Fourier transform of Eqs. (5) and (6) produces a set of coupled equations for \tilde{u}_n [6] and z . We refer to the solution of these equations with 10 Fourier modes as the *full* solution. According to our experience the retaining of more Fourier harmonics modifies the flow in a negligible manner. By restricting the solution to a single Fourier mode one obtains the evolution equations

$$(2 + k \partial_k) \tilde{u}_1 = \frac{1}{2\pi \tilde{u}_1 z} \left[1 - \sqrt{1 - \tilde{u}_1^2} \right], \quad (12)$$

$$k \partial_k z = -\frac{1}{24\pi} \frac{\tilde{u}_1^2}{(1 - \tilde{u}_1^2)^{3/2}}, \quad (13)$$

whose solution will be referred as *exact in \tilde{u}_1* . The first two terms in the Taylor expansion of Eq. (12) can be identified with the approximation used in [3] by a proper transformation of the parameters in Callan-Symanzik RG scheme. The higher-order terms make negligible effect on the evolution even in the neighbourhood of the turning point (\tilde{u}_1^*, z^*) of the RG trajectory. We refer to the solution of Eqs. (7) and (8) as the *linearized* solution. The RG

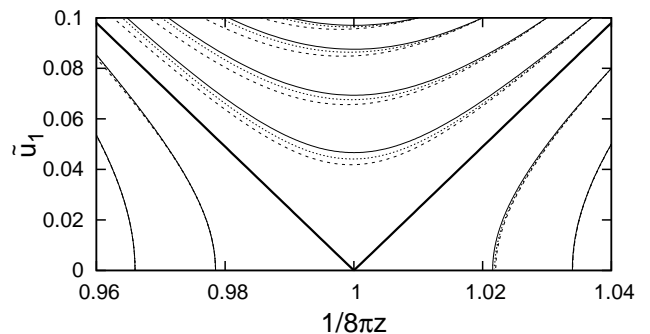


FIG. 1: Phase diagram of the SG model. The solid/dashed/dotted lines show the RG trajectories for the linearized/exact in \tilde{u}_1 /full solutions, respectively. The wide solid line depicts the separatrix.

trajectories are plotted in Fig. 1, they move to the left as k is decreased. This picture is reminiscent of the usual

KTB phase structure. What we see here is actually the vicinity of the dual of the KTB point of the XY model [10]. One can see that the higher harmonics modify the RG trajectories rather slightly as compared to both the linearized and the exact in \tilde{u}_1 solutions. Although the Coleman point lies at a crossover scale between the UV and IR scaling regions for the trajectories above the separatrix, the UV scaling remains valid in its vicinity.

Note **that** the smallness of the z -interval covered. The z -dependence of Eq. (5) is weak and free of singularities under the two solid lines of the separatrix, bordering three different phases, where the left and right sides correspond to the renormalizable and the non-renormalizable, asymptotically free phases of the SG model, respectively, seen already in the LPA approximation. It was verified that the IR values of the Fourier amplitudes, $\tilde{u}_n(0)$ are determined by the initial, UV value of the fundamental mode $\tilde{u}_1(\Lambda)$ in the renormalizable phase.

V. Non-perturbative scaling at the crossover.— Let us now turn our attention to the new phase which is opened up by the evolution of the wave function renormalization constant in the middle of the figure, above the separatrix. The z -dependence is crucial here, it prevents the system to come to standstill where the renormalized trajectory is stationary in \tilde{u}_1 , just above the Coleman point. It has been established that all Fourier modes are irrelevant (decrease with k) before the crossover and they turn to relevant (starts to increase as k is further decreased) at k_n^* , showing very weak n -dependence at the location of the turning point, $k_n^* \approx k^*$. We find $z(k^*) = z^* = 1/(8\pi)$, in a manner similar to the case $z = 1$ [7].

The vertical line $z = z^*$ appears to be a single IR stable fixed point as far as the evolution of the potential is considered only. In fact, the values of the coupling constants, $\tilde{u}_n^* = \tilde{u}_n(k^*)$, determined by the Fourier transform of the evolution equation Eq. (5) satisfy at this line the condition that the ratios

$$c_n = \frac{\tilde{u}_n^*}{\tilde{u}_1^{*2n}}, \quad (14)$$

are universal constants, $c_2 = 1/12$, $c_3 = 1/96$, $c_4 = 13/8640$, $c_5 = 97/414720$, etc. [6]. We recover renormalizability and asymptotical freedom in this phase because the dynamics is characterized by a single coupling strength, $\tilde{u}_1 > 0$ at and below the crossover scale. This is a non-perturbative phenomenon because the crossover "fixed point" is not Gaussian.

We see furthermore the subtle meaning of the "KTB fixed point". As soon as one goes beyond the LPA the Coleman point ceases to be a fixed point and is separating different phases only under the separatrix of Fig. 1, where the beta functions have a common analytic structure [7] and something irregularity shows up in the deep IR region of the symmetry broken phase only.

A more detailed and explicit similarity with the KTB

scaling of the XY model is found by introducing the correlation length ξ by identifying it with the inverse cut-off at the crossover. The numerical results are shown in Fig. 2. The various approximations, i.e. the full solution, the solution exact in \tilde{u}_1 , and the linearized one give the same critical behaviour as Eq. (11), showing that neither the inclusion of higher-order terms in Eqs. (7) and (8) nor that of the higher-order harmonics affect the type of the phase transition, their effects are negligible. The reduced temperature is given formally by the wave-function renormalization constant at the UV cutoff as $t = [z(\Lambda) - z_s(\Lambda)]/z_s(\Lambda)$, where $(1/8\pi z_s(\Lambda), \tilde{u}_1^*(\Lambda))$ is a point of the separatrix. The turning point \tilde{u}_1^* is shown in the inset of Fig. 2 as the function of the reduced temperature t for the linearized solution, giving

$$\tilde{u}_1^{*2} = qt + \mathcal{O}(t^2) \quad (15)$$

as in the XY model [8], and in the Coulomb gas [15]. The same relation is recovered for the exact solution in \tilde{u}_1 and for the full solution, as well. The critical scaling

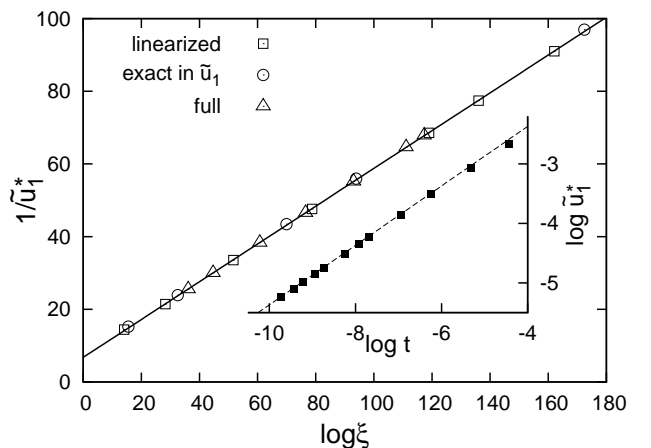


FIG. 2: The inverse of the turning value \tilde{u}_1^* of the fundamental amplitude is plotted versus the correlation length $\xi \sim 1/k^*$ for the linearized, exact in \tilde{u}_1 and full solutions, denoted by different point types. The slope of the solid line is $\sqrt{8/3\pi^2} \approx 0.52$ according to Eq. (11). In the inset the turning value \tilde{u}_1^* is plotted against the reduced temperature t . The slope of the fitted dashed line is 0.5.

relations (11) and (15) signal directly that there is a KTB type phase structure in the SG model.

VI. The IR scaling regime.— The trajectories end in a line of Gaussian IR fixed point in the non-renormalizable phase. All coupling strengths of the potential are irrelevant and this implies that the evolution of the wave-function renormalization z is extremely weak. The LPA can be used and the well-known IR scaling is recovered, including the unusual feature of the non-availability of the concept of relevant or irrelevant operators [7]. There is a line of Gaussian fixed points in the asymptotically free phase, too, but these fixed points are UV and their

scaling laws are linearizable. The IR scaling is difficult to establish numerically because of the instability of the Fourier expansion [16] in any RG scheme, used so far. Nevertheless it is unambiguous from numerics that z tends to be big as the scale k is decreased, while \tilde{u}_1 remains finite. One can get a clear picture of the phase

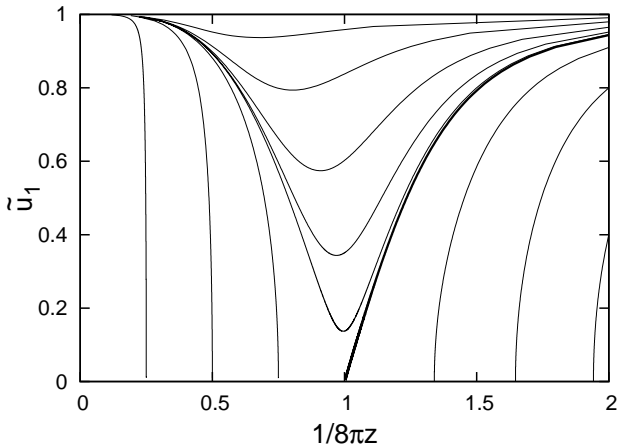


FIG. 3: The phase structure of the SG model given by the numerical solution of Eqs. (12) and (13).

structure of the IR scaling regime by omitting the effect of the higher harmonics, a frequently used approximation [3, 4]. After introducing $\omega = \sqrt{1 - \tilde{u}^2}$, $\chi = 1/z\omega$ and $\partial_t = \omega^2 k \partial_k$ we arrive at the evolution equations

$$\begin{aligned}\partial_t \omega &= 2\omega(1 - \omega^2) - \frac{\omega^2 \chi}{2\pi}(1 - \omega), \\ \partial_t \chi &= \chi^2 \frac{1 - \omega^2}{24\pi} - 2\chi(1 - \omega^2) + \frac{\omega \chi^2}{2\pi}(1 - \omega),\end{aligned}\quad (16)$$

possessing two lines of Gaussian fixed points separated by the well-known Coleman (alias KTB) fixed point, $(1/8\pi z, \tilde{u}_1) = (1, 0)$, and an additional (IR) fixed point $(1/8\pi z, \tilde{u}_1) = (0, 1)$ (see Fig. 3 for the RG trajectories).

Such a modification of the scaling laws which is believed to preserve the qualitative features of the RG flow makes the IR fixed points explicit in the complete phase diagram. It also demonstrates that the hyperbolic nature of the flow in the vicinity of the KTB-Coleman point stems from global effects, the competition between two regions of the phase diagram. This is the attraction of the line of Gaussian IR fixed points of the symmetrical phase, dominated by the kinetic energy on the one hand and of the non-Gaussian, IR fixed point which is dominated by the potential energy on the other.

The feature lost in this approximation is that the effective potential, built up in the IR region makes the evolution equation singular, which is typical for phases with spontaneously broken symmetry [17], automatically

guaranteeing the superuniversality for the potentials. In fact, the precise treatment with no expansion should give the superuniversal potential $\tilde{V}_{k \rightarrow 0} = -\frac{1}{2}\phi^2$ due to the Maxwell cut [7, 18, 19] and $1/z(k \rightarrow 0) = 0$.

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