# ON GEOMETRIC VECTOR FIELDS OF MINKOWSKI SPACES AND THEIR APPLICATIONS

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Abstract. As it is well-known, a Minkowski space is a finite dimensional real vector space equipped with a Minkowski functional $F$. By the help of its second order partial derivatives we can introduce a Riemannian metric on the vector space and the indicatrix hypersurface $S := F^{-1}(1)$ can be investigated as a Riemannian submanifold in the usual sense.

Our aim is to study affine vector fields on the vector space which are, at the same time, affine with respect to the Funk metric associated with the indicatrix hypersurface. We give an upper bound for the dimension of their (real) Lie algebra and it is proved that equality holds if and only if the Minkowski space is Euclidean. Criteria of the existence is also given in lower dimensional cases. Note that in case of a Euclidean vector space the Funk metric reduces to the standard Cayley-Klein metric perturbed with a nonzero 1-form.

As an application of our results we present the general solution of Matsumoto's problem on conformal equivalent Berwald and locally Minkowski manifolds. The reasoning is based on the theory of harmonic vector fields on the tangent spaces as Riemannian manifolds or, in an equivalent way, as Minkowski spaces. Our main result states that the conformal equivalence between two Berwald manifolds must be trivial unless the manifolds are Riemannian.

## 1. Preliminaries

1.1. **Minkowski functionals.** [1], [15]. Let $V$ be an $n$-dimensional ($n \geq 2$) real vector space. The elements of $V$ will be interpreted both as points $p, q, ...$ and vectors $v, w, ...$ as usual. A *Minkowski functional* on $V$ is a function $F\colon V \to \mathbb{R}$ with the following properties:

(F0) $\forall p \in V \setminus \{0\} : F(p) > 0$ and $F(0) = 0$.
(F1) $F$ is positive homogeneous of degree 1, i.e. $\forall t \in \mathbb{R}^+ : F(tp) = tF(p)$.
(F2) $F$ is continuous on $V$ and smooth over the set $V \setminus \{0\}$.
(F3) $\forall p \in V \setminus \{0\}$ :

$$g_p := E''(p)\colon V \times V \to \mathbb{R}$$

is an inner product on $V$, where $E := \frac{1}{2}F^2$ is the *energy function.*

The condition (F1) implies the energy function $E$ to be homogeneous of degree 2 and we have

$$g_p(p, v) = E'(p)(v),\ g_p(p, p) = 2E(p). \tag{1}$$

1991 *Mathematics Subject Classification.* 53C60, 58B20.

*Key words and phrases.* Minkowski spaces, Affine vector fields, Finsler manifolds, Conformal equivalence, Berwald manifolds.

Supported by FKFP (0184/2001), Hungary.

1.2. **Cartan tensors.** Let $(V,F)$ be a Minkowski space and consider the mappings

$$\mathcal{C}_\flat(p) := E'''(p)\colon V\times V\times V\to\mathbb{R},\ \mathcal{C}_p\colon V\times V\to V \tag{2}$$

defined by the formula

$$g_p(\mathcal{C}_p(v,w),z)=\mathcal{C}_\flat(p)(v,w,z); \tag{3}$$

$\mathcal{C}$ is called the *first Cartan tensor*. The first Cartan tensor, as well as its lowered tensor $\mathcal{C}_\flat$ is totally symmetric and, of course, multilinear. This means that the mapping

$$\mathcal{C}_p(v,\cdot)\colon V\to V,\quad \mathcal{C}_p(v,\cdot)(w):=\mathcal{C}_p(v,w)$$

is a self-adjoint linear operator with respect to the inner product $g_p$. Since the energy function is homogeneous of degree 2 it follows that

$$\mathcal{C}_p(p,\cdot)=0. \tag{4}$$

It is well-known that the vanishing of the first Cartan tensor implies the Minkowski space to be Euclidean. The contracted Cartan tensor is defined by the formula

$$\tilde{\mathcal{C}_p}(v):=\operatorname{tr}\,\mathcal{C}_p(v,\cdot); \tag{5}$$

Deicke's classical theorem states that *the contracted Cartan tensor vanishes if and only if the space is Euclidean*; see e.g. [2], [3] and [1].

1.3. **The associated Funk metric.** [12], [15]. Let $(V,F)$ be a Minkowski space and consider the set

$$B^\circ := \{p\in V | F(p)<1\}; \tag{6}$$

the associated Funk metric

$$L\colon TB^\circ\to\mathbb{R} \tag{7}$$

is defined by the property

$$F\big(p+\frac{v}{L(v_p)}\big)=1,$$

where $v_p\in T_pB^\circ$ is an arbitrary nonzero tangent vector at the point $p\in B^\circ$. Then, of course, the pair $(B^\circ,L)$ is a Finsler manifold in the usual sense, i.e. for any point $p\in B^\circ$ the restriction

$$L_p := L|_{T_pB^\circ} \tag{8}$$

is a Minkowski functional. Let $e_1,\dots,e_n$ be an arbitrary basis of the vector space V with the dual basis $u^1,\dots,u^n$ and consider the standard induced coordinate system $(x^i,y^i)_{i=1}^n$ on the tangent manifold $TV$. Okada's theorem states that for any indeces $i\in\{1,\dots,n\}$:

$$\frac{\partial}{\partial x^i}L=L\frac{\partial}{\partial y^i}L;$$

for a proof see e.g. [15], Lemma 2.3.1. In terms of differential geometric structures we can write the previous formula in the form

$$d_hL=Ld_JL, \tag{9}$$

where $h$ is the horizontal distribution determined by the first $n$ coordinate vector fields

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$$

and $J$ is the canonical almost tangent structure on the tangent manifold $TB^\circ$. We have that

$$d_J d_h L = d_J L \wedge d_J L + L d_J^2 L = 0, \tag{10}$$

which is just the coordinate-free expression for the classical Rapcsák equation of projective equivalence; see [16]. This means that $V$ as affine vector space and the associated Funk manifold $(B^\circ, L)$ are projectively equivalent; simply put the Funk manifold is projectively flat. Note that the geodesics of $V$ as affine vector space are the usual parametrized lines

$$c\colon \mathbb{R} \to V, \quad t \longrightarrow c(t) := p + tv. \tag{11}$$

Using the fundamental relation we can write the formula of projective equivalence between the canonical spray $\xi$ of the Funk manifold and $\xi_V$ in the form

$$\xi = \xi_V - LC, \tag{12}$$

where $C$ is the so-called *Liouville vector field*; [16], 3.8. Proposition, for the details of spray geometry see also [4], [5] and [15].

**Definition.** *Let $(M, \xi)$ be an arbitrary spray manifold; the vector field $X \in \mathfrak{X}(M)$ is called an* affine vector field *if its local 1-parameter group consists of geodesic-preserving maps. The vector field is* projective *if the local 1-parameter group consists of maps preserving the geodesics up to a strictly increasing reparametrization.*

For lots of equivalent characterizations see e.g. [9] and [13].

1.4. **Example.** Suppose that $X$ is an affine vector field on the vector space $V$ and consider a point $p$ together with its open neighbourhood $U \subset V$ such that any integral curve starting from a point $q \in U$ is defined on the open intervall $(-\epsilon, \epsilon)$; the mapping

$$\varphi\colon t \in (-\epsilon, \epsilon) \to \varphi_t$$

denotes the local 1-parameter group of the vector field $X$. We set

$$c(s) := q + sv,$$

where the parameter $s$ is small enough satisfying the condition $Im\ c \subset U$. Since $X$ is affine, the curve $\tilde{c} := \varphi_t \circ c$ is a geodesic, i.e. its second order derivative vanishes; especially

$$\tilde{c}''(0) = 0 \Rightarrow (\varphi_t)''(q)(v, v) = 0 \tag{13}$$

and, consequently, $\varphi_t'' = 0$. In other words, $\varphi_t'$ is independent of the point $q$ which implies the vectorial part

$$q \in U \to \lim_{t \to 0} \frac{(\varphi_{-t})'(q)(e_i) - e_i}{t} \tag{14}$$

of the Lie bracket $[X, \frac{\partial}{\partial u^i}]$ to be constant. Taking the Lie bracket again it follows that

$$[[X, \frac{\partial}{\partial u^i}], \frac{\partial}{\partial u^j}] = 0 \tag{15}$$

and we have the following simple differential equation

$$\frac{\partial^2}{\partial u^i \partial u^j} X^k = 0$$

for the coefficients of the vector field $X$. Therefore

$$X = (\alpha^i_j u^j + \beta^i)\frac{\partial}{\partial u^i}, \tag{16}$$

where $A := (\alpha^j_i)_{1\le i,j\le n}$ is a matrix of real numbers and $\beta^1, \ldots, \beta^n \in \mathbb{R}$. As a routine calculation shows,

$$\varphi_t(q) = e^{tA}q + w_t, \tag{17}$$

where the part of translation is independent of $q$.

1.5. **Riemannian quantities.** Let $(V, F)$ be a Minkowski space; according to the regularity property (F3), the vector space can be considered as a Riemannian manifold in the usual sense. After identifying the tangent spaces with $V$, consider the following special vector fields:

$$\begin{aligned} X &\colon V \to V, \quad p \longrightarrow X_p := x, \\ Y &\colon V \to V, \quad p \longrightarrow Y_p := y, \\ Z &\colon V \to V, \quad p \longrightarrow Z_p := z, \end{aligned}$$

where $x, y$ and $z \in V$ are arbitrarily fixed vectors. It can be easily seen that the Lévi-Civita connection $\nabla$ associated with $g$ acts as follows:

$$\nabla_{X_p} Y = \mathcal{C}_p(x, y) \tag{18}$$

and, consequently, the curvature tensor has the following simple form:

$$\mathbb{Q}_p(x, y)z = \mathcal{C}_p(\mathcal{C}_p(x, z), y) - \mathcal{C}_p(x, \mathcal{C}_p(y, z)). \tag{19}$$

We set

$$R_p(x, y) := \sum_{i=1}^n g_p(\mathbb{Q}_p(e_i, x)y, e_i), \tag{20}$$

where $e_1, \ldots, e_n \in V$ is a $g_p$-orthonormal system; as usual $R$ is called the *Ricci tensor* of the Riemannian manifold $V \setminus \{0\}$.

## 2. Affine vector fields of the associated Funk metric

In what follows $V$ denotes a Minkowski vector space equipped with the Minkowski functional $F$. As we have seen above $V$ as affine vector space and the Funk manifold $(B^\circ, L)$ are projectively equivalent and, consequently, the restriction of projective vector fields on the vector space are projective with respect to the Funk metric and vice versa. In what follows we are going to study *affine* vector fields on the vector space which are, at the same time, affine with respect to the Funk metric. Suppose that $X$ is one of them; if

$$c\colon t \in \mathbb{R} \to c(t) := q + tv,$$

then, by the projective equivalence, there is a strictly increasing reparametrization $\theta$ such that the curve $\tilde{c} := c \circ \theta$ is a geodesic of the Funk manifold. According to the formula (12), the reparametrization is just the solution of the differential equation

$$\theta'' = -(\theta')^2 L(v_q); \tag{21}$$

see e.g. [7]. Under the initial conditions $\theta(0) = 0$ and $\theta'(0) = 1$ we have that

$$\theta(s) = \frac{1}{L(v_q)} \ln(1 + sL(v_q)), \tag{22}$$

i.e.

$$\tilde{c}(s) = q + \frac{1}{L(v_q)} \ln(1 + sL(v_q))v$$

is a geodesic of the Funk manifold. Let $\varphi\colon t \in \mathbb{R} \to \varphi_t$ be the 1-parameter group of the vector field $X$; using the formula of reparametrization it follows that $X$ is affine with respect to the Funk metric if and only if

$$\varphi_t(q + \theta(s)v) = \varphi_t(q) + \frac{1}{L \circ T\varphi_t(v_q)} \ln(1 + sL \circ T\varphi_t(v_q)) f(v), \tag{23}$$

where $f := (\varphi_t)'(q)$ which is actually independent of the point $q \in B^\circ$ as we have seen above. On the other hand

$$\varphi_t(q + \theta(s)v) \overset{(17)}{=} \varphi_t(q) + \theta(s) f(v), \tag{24}$$

i.e.

$$\theta(s) = \frac{1}{L \circ T\varphi_t(v_q)} \ln(1 + sL \circ T\varphi_t(v_q)).$$

Differentiating by $s$, it can be easily seen that

$$L \circ T\varphi_t(v_q) = L(v_q) \tag{25}$$

provided, of course, that the parameter $t$ is small enough satisfying the condition $\varphi_t(q) \in B^\circ$. Since the mapping

$$T\varphi\colon t \in \mathbb{R} \to T\varphi_t \tag{26}$$

is just the 1-parameter group of the complete lift $X^c$, the relation

$$X^c_{v_q} L = \lim_{t \to 0} \frac{L \circ T\varphi_t(v_q) - L(v_q)}{t} = 0 \tag{27}$$

follows immediately.

**Proposition 1.** *Suppose that $X$ is an affine vector field on the vector space $V$; then the following conditions are equivalent* [1]*:*

(i) *$X$ is an affine vector field with respect to the Funk metric.*
(ii) *$X^c L = 0$.*

[1]For the implication $(i) \Rightarrow (ii)$ of Proposition 1 and 2 we should refer to the lecture *Affine and projective vector fields on spray manifolds* presented by L. R. Lovas; Workshop on Finsler Geometry and its Applications, August 11-15, 2003, Debrecen, Hungary. See also [9].

In terms of coordinates we have the expression

$$X^c = (\alpha^i_j x^j + \beta^i)\frac{\partial}{\partial x^i} + \alpha^i_j y^j \frac{\partial}{\partial y^i}. \tag{28}$$

Consider now the projection

$$\rho\colon TB^\circ \to S, \quad v_q \to \rho(v_q) := q + \frac{v}{L(v_q)}; \tag{29}$$

it is clear that $F \circ \rho = 1$ and, consequently, $TF \circ T\rho = 0$. On the other hand, as a staightforward calculation shows

$$TF \circ T\rho(X^c)(v_q) = -\frac{1}{L^2(v_q)} F'(\rho(v_q))(v) X^c_{v_q} L + X_{\rho(v_q)} F. \tag{30}$$

The strictly convexity of the indicatrix hypersurface implies that

$$F'(\rho(v_q))(v) \neq 0; \tag{31}$$

in a geometrical interpretation this means that $v$ couldn't be tangential to the indicatrix hypersurface at the point $\rho(v_q) \in S$. Using the previous result we have the following proposition immediately.

**Proposition 2.** *Suppose that $X$ is an affine vector field on the vector space $V$; then the following conditions are equivalent:*

(i) *$X$ is an affine vector field with respect to the Funk metric.*
(ii) *$XF \circ \rho = 0$.*

Since $\rho$ is surjective, (ii) means that the restriction $X|_S$ must be tangential to the indicatrix hypersurface. In other words, if $c$ is an integral curve of the vector field $X$ starting from a point $p \in S$, then $Im\ c \subset S$.

**Proposition 3.** *Suppose that $X$ is an affine vector field on the vector space $V$ which is, at the same time, affine with respect to the Funk metric. Then*

$$\mathrm{tr}\ A := \sum_{i=1}^n \alpha^i_i = 0. \tag{32}$$

*If the Minkowski functional is reversibile, then $X$ is a linear vector field, i.e. its 1-parameter group consists of linear transformations and $X$ can be written in the form*

$$X = \alpha^i_j u^j \frac{\partial}{\partial u^i}. \tag{33}$$

*i.e. the 1-parameter group consists of special linear transformations.*

Proof. Since for any $t \in \mathbb{R}$ the indicatrix hypersurface is invariant under the transformation $\varphi_t$ preserving the affine (especially convex) combination it follows that $B^\circ$ is also invariant. Therefore

$$\int_{B^\circ} du^1 \dots du^n = \int_{\varphi_{-t}(B^\circ)} \det \varphi'_t\, du^1 \dots du^n = \int_{B^\circ} e^{t\ \mathrm{tr}A}\, du^1 \dots du^n;$$

differentiating by $t$, we have

$$0 = \mathrm{tr}\ A \int_{B^\circ} e^{t\ \mathrm{tr}A}\, du^1 \dots du^n \Rightarrow\ \mathrm{tr}\ A = 0 \tag{34}$$

as was to be stated.

Suppose that $F$ is reversible, i.e.

$$F(v) = F(-v); \tag{35}$$

then □

**Theorem 1.** *Suppose that $X$ is an affine vector field on the vector space $V$ which is, at the same time, affine with respect to the Funk metric. Then $X$ is a Killing vector field on the vector space $V$ as Riemannian manifold and, at the same time, it is a Killing vector field on the indicatrix hypersurface with respect to the induced Riemannian structure.*

Proof. As an easy calculation shows, for any indeces $i, j \in \{1, \ldots, n\}$

$$(\mathcal{L}_X g)(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}) = \frac{\partial^2}{\partial u^i \partial u^j}(XE) - \big[[X, \frac{\partial}{\partial u^i}], \frac{\partial}{\partial u^j}\big](E).$$

By the relation (ii) of Proposition 2, $XF = 0$ on the indicatrix hypersurface; since X is actually a linear or, in an equivalent way, it is a homogeneous vector field, the relation $XF = 0$ holds on the *whole* vector space $V$. This means that $XE = 0$ and the first term vanishes. The vanishing of the Lie bracket follows immediately from the formula (15). □

**Proposition 4.** *Suppose that $X$ is an affine vector field on the vector space $V$ which is, at the same time, affine with respect to the Funk metric; then $\tilde{\mathcal{C}}(X) = 0$.*

Proof. Since $\mathcal{L}_X g = 0$, it follows that the divergence of the vector field $X$ vanishes; indeed, for any vector fields $Y$ and $Z$

$$\begin{aligned} 0 &= Xg(Y, Z) - g([X, Y], Z) - g(Y, [X, Z]) = g(\nabla_X Y - [X, Y], Z) + \\ &+ g(\nabla_X Z - [X, Z], Y) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y), \end{aligned}$$

i.e. the Hesse form $(\nabla X)_\flat$ is antisymmetric and, of course, *div* $X = 0$. Let now $e_1, \ldots, e_n$ be a basis of the vector space; the relation (18) shows that the parameters of the Lévi-Civita connection with respect to the dual basis $u^1, \ldots, u^n$ are just the components of the Cartan tensor. Thus we have

$$div\ X = \sum_{i=1}^{n} \frac{\partial}{\partial u^i} X^i + \tilde{\mathcal{C}}(X) = \sum_{i=1}^{n} \alpha_i^i + \tilde{\mathcal{C}}(X) \overset{(35)}{=} \tilde{\mathcal{C}}(X)$$

and the vanishing of $\tilde{\mathcal{C}}(X)$ follows immediately. □

**Theorem 2.** *Suppose that $V$ is of dimension $n \geq 3$ and let $\mathcal{A}_\circ(V)$ be the (real) Lie algebra of affine vector fields on the vector space which are, at the same time, affine with respect to the Funk metric; then*

$$\dim \mathcal{A}_\circ(V) \leq \frac{n(n-1)}{2} \tag{36}$$

*and equality holds if and only if the Minkowski space is Euclidean.*

Proof. Since the elements of $\mathcal{A}_\circ(V)$ are tangential to the indicatrix hypersurface and $\dim S = n - 1$, the estimation is a direct consequence of Theorem 1; see [13], section 3.53. Suppose that $\dim \mathcal{A}_\circ(V) = \frac{n(n-1)}{2}$ and consider a basis

$$X_1, \ldots, X_k$$

where, for the sake of simplicity, $k = \frac{n(n-1)}{2}$. Let $p \in S$ be an arbitrarily fixed point; since $n \geq 3$ the vector fields $X_1, \ldots, X_k$ is linearly *dependent* at the point $p$, i.e. there exist real numbers $r_1, \ldots, r_k$ such that $r_1 \neq 0$ and

$$r_1 X_1(p) + \ldots + r_k X_k(p) = 0. \tag{37}$$

The vector field

$$Y_1 := r_1 X_1 + \ldots + r_k X_k \tag{38}$$

is, of course, nontrivial. On the other hand, $Y_1$ vanishes at the point $p$ which means that its 1-parameter group consisting of isometries with respect to the Riemannian metric $g$ has a fixpoint and, consequently,

$$\begin{aligned} g_p(v,w) = (\varphi_t^* g)_p(v,w) = g_{\varphi_t(p)}(\varphi_t'(v), \varphi_t'(w)) = g_p(\varphi_t'(v), \varphi_t'(w)) = \\ = g_p(\varphi_t(v), \varphi_t(w)), \end{aligned} \tag{39}$$

i.e. the group consists of orthogonal transformations with respect to the inner product $g_p$. It also follows that $Y_1$ can be interpreted as a nontrivial element of $\mathcal{A}_\circ(H)$, where the subspace $H$ is orthogonal to the point $p$ with respect to $g_p$. Indeed, the invariance of $H$ under the transformations of the 1-parameter group implies the vector field $Y_1$ to be tangential to the subspace $H$. On the other hand, if the restriction $Y_1|_H$ vanishes then the transformations of the 1-parameter group have further fixpoints; by setting a basis of them we can see that the group is trivial and, consequently, $Y_1 = 0$ which is a contradiction.

Consider now the basis $Y_1, X_2, \ldots, X_k$; if the vector fields $X_2, \ldots, X_k$ are linearly *dependent* at the point $p$, then there exist real numbers $r_2, \ldots, r_k$ such that $r_2 \neq 0$ and

$$r_2 X_2(p) + \ldots + r_k X_k(p) = 0. \tag{40}$$

In a similar way as above we define the nontrivial vector field

$$Y_2 := r_2 X_2 + \ldots + r_k X_k. \tag{41}$$

Since $Y_2$ vanishes at the point $p$, its 1-parameter group consists of orthogonal transformations with respect to the inner product $g_p$. It also follows that $Y_2$ can be interpreted as a nontrivial element of $\mathcal{A}_\circ(H)$. Using this proccess as far as possible we can construct the vector fields $Y_1, Y_2, \ldots, Y_l$; in what follows it is proved that their restrictions to the subspace $H$ are linearly *independent*. Suppose, in contrary, that

$$\big(s_1 Y_1 + \ldots + s_l Y_l\big)|_H = 0 \tag{42}$$

is a nontrivial combination; if $s_1 \neq 0$, then $Y_1|_H \in \mathcal{L}(Y_2, \ldots, Y_l)$ and, by the constructing proccess, the relation

$$X_1|_H \in \mathcal{L}(X_2, \ldots, X_k) \tag{43}$$

follows immediately. Let us introduce the vector field

$$X := X_1 - \eta_2 X_2 + \ldots - \eta_k X_k \tag{44}$$

where $(\eta_2)^2+\ldots+(\eta_k)^2\neq 0$ and $X|_H=0$. Then $X$ is a linear vector field and its 1-parameter group can be represented in the form

$$
\left(\begin{array}{ccccccc}
1 & 0 & . & . & . & 0 & 0\\
0 & 1 & . & . & . & 0 & 0\\
. & & . & & & & .\\
. & & & . & & & .\\
. & & & & . & & .\\
0 & 0 & . & . & . & 1 & 0\\
0 & 0 & . & . & . & 0 & \alpha(t)
\end{array}\right)_{n\times n} \tag{45}
$$

where the condition $\det\varphi_t=1$ should be also satisfied. This means that $\alpha\equiv 1$ and, consequently, the 1-parameter group of the vector field $X$ is trivial, i.e. $X=0$ which is a contradiction.

In case of $s_1=0$, the reasoning is similar for the first nontrivial coefficient; the contradiction shows that $Y_1,\ldots,Y_l$ are linearly independent as the elements of $\mathcal{A}_\circ(H)$. Since the proccess ends at the step

$$
l=\dim\mathcal{A}_\circ(H)=\frac{(n-1)(n-2)}{2}, \tag{46}
$$

we have that the X's block of the basis $Y_1,\ldots,Y_l,X_{l+1},\ldots,X_k$ must be *linearly independent at the point* $p$. Here

$$
k-l=\frac{n(n-1)}{2}-\frac{(n-1)(n-2)}{2}=n-1 \tag{47}
$$

and, by Proposition 4,

$$
\tilde{\mathcal{C}}(X_{l+1})=\ldots=\tilde{\mathcal{C}}(X_k)=0.
$$

This means that $\tilde{\mathcal{C}}_p$ vanishes on a basis of the tangent space $T_pS$ and, consequently, $\tilde{\mathcal{C}}_p=0$; Deicke's theorem implies the space to be Euclidean. □

We have a more transparent picture in case of lower dimensional spaces as the following theorem shows.

**Theorem 3.** *Suppose that* $\dim V=2$*; then the Lie algebra* $\mathcal{A}_\circ(V)$ *is trivial unless the space is Euclidean.*

*If* $V$ *is of dimension* 3*, then we have the following cases:*

(i) $\dim\mathcal{A}_\circ(V)=0$.
(ii) $\dim\mathcal{A}_\circ(V)=1$ *and the indicatrix is a rotation surface with respect to the inner product* $g_p$*, where* $p\in S$ *is a zero of any vector field* $X\in\mathcal{A}_\circ(V)$.
(iii) $\dim\mathcal{A}_\circ(V)=3$ *and the space is Euclidean.*

Proof. Let $X\in\mathcal{A}_\circ(V)$ be a nontrivial vector field and $\dim V=2$; if $X$ has no zero except the origin then, by Proposition 4, $\tilde{\mathcal{C}}=0$ and Deicke's theorem implies the space to be Euclidean.

If $X(p)=0$, then its 1-parameter group consists of orthogonal transformations with respect to the inner product $g_p$:

$$
\begin{aligned}
g_p(v,w)&=(\varphi_t^*g)_p(v,w)=g_{\varphi_t(p)}(\varphi_t'(v),\varphi_t'(w))=g_p(\varphi_t'(v),\varphi_t'(w))=\\
&=g_p(\varphi_t(v),\varphi_t(w)).
\end{aligned} \tag{48}
$$

Since there is a fixpoint, the group is trivial, i.e. $X = 0$ which is a contradiction.

Suppose that $\dim V = 3$; the key observation is that the existence of a zero is guaranteed on the indicatrix surface and (49) shows the rotation property (ii). Indeed, if $X \in \mathcal{A}_\circ(V)$ is a nontrivial vector field with the 1-parameter group

$$\varphi \colon t \in \mathbb{R} \to \varphi_t$$

and $X(p) = 0$, then the group can be represented in the form

$$\begin{pmatrix} \cos\alpha(t) & \sin\alpha(t) & 0 \\ -\sin\alpha(t) & \cos\alpha(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{49}$$

where the function $\alpha$ is additive with the "initial condition" $\alpha(0) = 0$. Since for any $t \in \mathbb{R}$

$$\alpha'(t) = \lim_{s\to 0} \frac{\alpha(t+s) - \alpha(t)}{s} = \lim_{s\to 0} \frac{\alpha(s)}{s} = \lim_{s\to 0} \frac{\alpha(s) - \alpha(0)}{s} = \alpha'(0), \tag{50}$$

it follows that $\alpha$ is linear and the representation reduces to the following simple form

$$\begin{pmatrix} \cos Kt & \sin Kt & 0 \\ -\sin Kt & \cos Kt & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{51}$$

where $K$ is a constant. The basis is, of course, a $g_p$-orthonormal system $(e_1, e_2, p)$ with vectors $e_1$ and $e_2$ spanning the invariant subspace $H$ of the "rotation" group. As we have seen above, $H$ is orthogonal to $p \in S$ with respect to $g_p$ which means that the invariant subspace is tangential to the indicatrix surface at the point $p$. If $(u^1, u^2, u^3)$ is the dual basis then we have the expression

$$X = K\big(u^2 \frac{\partial}{\partial u^1} - u^1 \frac{\partial}{\partial u^2}\big). \tag{52}$$

It remains only to prove that there couldn't be two different rotation axes; indeed, by the strictly convexity of the indicatrix surface, the existence of two different axes implies different invariant subspaces for the rotation groups. This means that the velocity vector fields $X$ and $Y$ are linearly independent "almost anywhere". The critical points on the indicatrix surface belong to one of the following types:

(i) the points of the axes,
(ii) the points where the common line of the translated invariant subspaces is tangential to the indicatrix surface.

As it can be easily seen, the points of the axes are of type (ii), too. Therefore, it is enough to investigate the second case. Suppose that the vector fields $X$ and $Y$ are linearly *dependent* at the point $q \in S$ and the intersection of the invariant subspaces is generated by the vector $v \in V$. Then we have that $E'(q)(v) = 0$. Let us form the mapping

$$\eta \colon q \in S \to \eta(q) := E'(q)(v)$$

and consider the set $\Omega := \eta^{-1}(0)$. It can be easily seen that for any point $q \in \Omega$ and $w \in T_q S$

$$\eta'(q)(w) = g_q(v, w)$$

and, consequently, $\eta'(q) \neq 0$ because of the relation $g_q(v, v) = 2E(v)$. Note that by the construction of $\eta$ the vector $v$ is always tangential to S at the point $q \in \Omega$. This means that $\Omega$ is a regular curve on the indicatrix surface, i.e. the set of critical points belonging to (ii) forms a set of measure zero. Therefore, the contracted Cartan tensor vanishes and Deicke's theorem implies the space to be Euclidean; then, by Theorem 2, the Lie algebra $\mathcal{A}_\circ(V)$ is maximal. □

## 3. An application: The Matsumoto's problem

In what follows we are going to solve the problem of conformally equivalent Berwald and locally Minkowski manifolds. As it is well-known, a Finsler manifold is a *Berwald manifold* if and only if the canonical connection is linear; a Berwald manifold is called a *locally Minkowski manifold* if the linear connection has zero curvature. The problem given by M. Matsumoto in his paper [11] is that wheter there exist conformally equivalent Berwald or locally Minkowski manifolds. In our previous paper [20] we used a further condition to prove that the conformal equivalence between two Berwald manifolds must be trivial unless the manifolds are Riemannian: it was supposed that one and therefore all indicatrices have positive curvature. This condition will be omitted in the following argumentation.

3.1. **Finsler manifolds.** [1], [15] and [18]. Let $M$ be a differentiable manifold equipped with a function $F\colon TM \to \mathbb{R}$ such that

- (F0) $\forall v \in TM \setminus \{0\} : F(v) > 0$ and $F(0) = 0$.
- (F1) $F$ is homogeneous of degree 1, i.e. $\forall t \in \mathbb{R}^+ : F(tv) = tF(v)$.
- (F2) $F$ is continuous on the tangent manifold $TM$ and smooth except the zero section.
- (F3) The *fundamental form* $\omega := dd_J E$ is nondegenerate, where $E := \frac{1}{2}F^2$ is the so-called *energy function*.

The *Riemann-Finsler metric* of the Finsler manifold $(M, E)$ is defined by the formula

$$g(JX, JY) := \omega(JX, Y),$$

where $X$, $Y$ are vector fields on $TM$ and $J$ is the canonical almost tangent structure or, in an equivalent terminology, the vertical endomorphism on the tangent bundle $\pi\colon TM \to M$; for the details see [8]. In what follows we suppose that *the Riemann-Finsler metric is positive definite.*

As it can be easily seen, for any point $p \in M$ the restriction

$$F_p := F|_{T_pM}$$

is a Minkowski functional. On the other hand, for any tangent vector $v \in T_pM$ the vertical subspace can be identified with the tangent space of the "manifold" $T_pM$ at the "point" $v$. This means that the Riemann-Finsler metric works as a usual Riemannian metric on the vector space $T_pM$.

**Remark 1.** *Note that if the energy function $E$ is smooth on the* whole *tangent manifold, then we have a Riemannian manifold in the usual sense; indeed, the property (ii) implies $E$ to be homogeneous of degree 2 and, consequently, it is a quadratic function. For this reason, in case of a non-Riemannian Finsler manifold, differentiability is required only on the splitting tangent manifold $\mathcal{T}M := TM \setminus \{0\}$.*

**Definition.** *Consider the Finsler manifolds $(M, E)$ and $(M, \tilde{E})$ with Riemann-Finsler metrics $g$ and $\tilde{g}$, respectively; $g$ and $\tilde{g}$ are said to be* conformally equivalent *if there exists a positive smooth function $\varphi \colon \mathcal{T}M \to \mathbb{R}$ such that $\tilde{g} = \varphi g$. The function $\varphi$ is called the* scale function *or the* proportionality function. *If the scale function is constant, then we say that the conformal change is* homothetic

**Remark 2.** *If $\tilde{g} = \varphi g$ then*

$$\tilde{E} = \frac{1}{2}\tilde{g}(C, C) = \frac{1}{2}\varphi g(C, C) = \varphi E. \tag{53}$$

*It is also well-known due to M.S. Knebelman, that the scale function between conformally equivalent Finsler manifolds is a vertical lift, i.e. $\varphi$ can always be written in the form*

$$\varphi = exp \circ \alpha^v := exp \circ \alpha \circ \pi; \tag{54}$$

*see e.g. [14]. Moreover, if a Finsler manifold $(M, E)$ with Riemann-Finsler metric $g$ and a function $\alpha \in C^\infty(M)$ are given, then*

$$g_\alpha := \varphi g \quad (\varphi = exp \circ \alpha^v) \tag{55}$$

*is the Riemann-Finsler metric of the Finsler manifold $(M, E_\alpha)$, where the energy function $E_\alpha$ is defined by the formula $E_\alpha := \varphi E$. According to these elementary facts we also speak of a* conformal change $g_\alpha = \varphi g$ *of the metric $g$; for the details see [6], [17] and [19].*

3.2. **Further formulas (a practical summary).** [4], [5] and [18]. Let $(M, E)$ be a Finsler manifold. The covariant derivatives with respect to the *Cartan connection* can be explicitly calculated by the following formulas:

(C1) $D_{JX}JY = J[JX, Y] + \mathcal{C}(X, Y) = \overset{\circ}{D}_{JX} JY + \mathcal{C}(X, Y)$,
(C2) $D_{hX}JY = \nu[hX, JY] + \mathcal{C}'(X, Y) = \overset{\circ}{D}_{hX} JY + \mathcal{C}'(X, Y)$,
(C3) $D_{JX}hY = h[JX, Y] + \mathrm{F}\mathcal{C}(X, Y) = \overset{\circ}{D}_{JX} hY + \mathrm{F}\mathcal{C}(X, Y)$,
(C3) $D_{hX}hY = h\mathrm{F}[hX, JY] + \mathrm{F}\mathcal{C}'(X, Y) = \overset{\circ}{D}_{hX} hY + \mathrm{F}\mathcal{C}'(X, Y)$,

where $\overset{\circ}{D}$ denotes the *Berwald connection* and $h$ is the canonical horizontal endomorphism (nonlinear connection) or, in an equivalent terminology, the Barthel endomorphism of the Finsler manifold; $\nu := 1 - h$ is the so-called *vertical projector* and F denotes the almost complex structure associated with the Barthel endomorphism:

$$\mathrm{F} \circ J = h, \ \mathrm{F} \circ h = -J.$$

The first and second Cartan tensors $\mathcal{C}$ and $\mathcal{C}'$ are defined by the formulas

$$\omega(\mathcal{C}(X,Y),Z) = \frac{1}{2}\big(\mathcal{L}_{JX}J^*g_h\big)(Y,Z),$$
$$\omega(\mathcal{C}'(X,Y),Z) = \frac{1}{2}\big(\mathcal{L}_{hX}\ g_h\big)(JY,JZ), \tag{56}$$

where

$$g_h(X,Y) := g(JX,JY) + g(\nu X, \nu Y)$$

is the prolongation of the Riemann-Finsler metric along $h$ and

$$(J^*g_h)(X,Y) := g_h(JX,JY).$$

It is well-known that for any Finsler connection $(D,h)$ its curvature tensor field $\mathbb{K}$ is uniquely determined by the following three mappings

$$\mathbb{R}(X,Y)Z := \mathbb{K}(hX,hY)JZ \ - \ \text{h-curvature},$$
$$\mathbb{P}(X,Y)Z := \mathbb{K}(hX,JY)JZ \ - \ \text{hv-curvature}, \tag{57}$$
$$\mathbb{Q}(X,Y)Z := \mathbb{K}(JX,JY)JZ \ - \ \text{v-curvature}.$$

The $v$-curvature tensor of the Cartan connection can be calculated by the formula

$$\mathbb{Q}(X,Y)Z = \mathcal{C}(\mathrm{F}\mathcal{C}(X,Z),Y) - \mathcal{C}(X,\mathrm{F}\mathcal{C}(Y,Z)). \tag{58}$$

**Remark 3.** *Note that the vertical covariant differentiation with respect to the Cartan connection is essentially the same as that with respect to the Lévi-Civita connection $\nabla$ on the "manifold" $T_pM$ as a vector space equipped with the Minkowski functional $F_p$; see the formula (19) for the curvature of the Lévi-Civita connection. Here we give a short list of coordinate expressions as the simpliest way to clarify how the different interpretations are related.*

3.3. **Local characterizations.** [10], [14]. Consider a coordinate system $(U,(u^i)_{i=1}^n)$ on the underlying manifold $M$ together with the induced coordinate system $(\pi^{-1}(U),(x^i,y^i)_{i=1}^n)$ on the tangent manifold. As it is well-known, the first $n$ coordinate fixes a point and the second $n$ coordinate gives the vectorial part of the tangent vectors. In what follows we briefly summarize the basic geometrical objects in terms of local coordinates.

Let $(M,E)$ be a Finsler manifolds. The functions

$$g_{ij} := \frac{\partial^2}{\partial y^i \partial y^j}(E) \quad (1 \le i,j \le n)$$

are the components of the *Riemann-Finsler metric.* The *lowered first Cartan tensor* $\mathcal{C}_\flat$ is just the vertical Lie-derivative of the Riemann-Finsler metric multiplied by $\frac{1}{2}$:

$$\mathcal{C}_{ijk} := \mathcal{C}_\flat(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k}) = \frac{1}{2}\frac{\partial}{\partial y^i}g_{jk}, \quad \mathcal{C}_{ij}^l := g^{lk}\mathcal{C}_{ijk},$$

where the functions $\mathcal{C}_{ij}^l$ $(1 \le i,j,l \le n)$ are the components of the *first Cartan tensor* $\mathcal{C}$. At the same time, they are the coefficients of the vertical covariant differentiation with respect to the *Cartan connection* $D$. In terms of local coordinates its *v-curvature* tensor $\mathbb{Q}$ has the following simple form:

$$\mathbb{Q}_{ijk}^l := \mathbb{Q}(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k}) = \mathcal{C}_{ik}^r\mathcal{C}_{rj}^l - \mathcal{C}_{jk}^r\mathcal{C}_{ri}^l.$$

As it can be easily seen from the local characterizations all of previous quantities can be interpreted on the vertical subbundle or, in an equivalent way, on the tangent spaces as "differentiable manifolds". The following group of geometrical objects are more closely related to the underlying manifold via the changing of the based point $p \in M$.

The *canonical spray* is given by the formula

$$S = y^k \frac{\partial}{\partial x^k} - 2G^k \frac{\partial}{\partial y^k},$$

where

$$G^k := \frac{1}{2} g^{kj} \left( y^i \frac{\partial^2}{\partial x^i \partial y^j} E - \frac{\partial}{\partial x^j} E \right).$$

The functions

$$\Gamma^k_i := \frac{\partial}{\partial y^i} G^k \quad (1 \leq i, k \leq n)$$

are the coefficients of the *canonical horizontal endomorphism* $h$, i.e.

$$h(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} - \Gamma^k_i \frac{\partial}{\partial y^k}.$$

The *lowered second Cartan tensor* $\mathcal{C}'_\flat$ is just the horizontal Lie-derivative of the Riemann-Finsler metric multiplied by $\frac{1}{2}$:

$$\begin{aligned} \mathcal{C}'_{ijk} \quad &:= \quad \mathcal{C}'_\flat(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^k}) = \frac{1}{2} \Bigg( \frac{\delta}{\delta x^i} g(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}) - \\ &- \quad g([\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}], \frac{\partial}{\partial y^k}) - g(\frac{\partial}{\partial y^j}, [\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^k}]) \Bigg), \end{aligned}$$

where

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - \Gamma^k_i \frac{\partial}{\partial y^k}.$$

The *Berwald connection* of a Finsler manifold is completely determined by the functions

$$\Gamma^k_{ij} := \frac{\partial}{\partial y^j} \Gamma^k_i = \frac{\partial^2}{\partial y^i \partial y^j} G^k;$$

they are the coefficients of the horizontal covariant differentiation with respect to the Berwald connection (coefficients of the vertical covariant differentiation are identically zero). Traditionally *Berwald manifolds* are defined as follows: *the coefficients of the Berwald connection depend only on the position*, i.e.

$$\Gamma^k_{ijl} := \frac{\partial}{\partial y^l} \Gamma^k_{ij} = 0,$$

where the functions $-\Gamma^k_{ijl}$ $(1 \leq i, j, k, l \leq n)$ are just the components of the *hv-curvature* $\overset{\circ}{\mathbb{P}}$ of the Berwald connection. As it can be easily seen, the vanishing of this curvature implies the canonical spray to be quadratic and, consequently, the canonical horizontal endomorphism arises from a linear connection on the underlying manifold $M$.

3.4. **The gradient operator.** [17], [19]. Let a smooth function

$$\varphi\colon TM \to R \text{ (or } \varphi\colon \mathcal{T}M \to R)$$

be given. Since the fundamental form $\omega$ is nondegenerate, there exists a unique vector field $\operatorname{grad}\varphi \in \mathfrak{X}(\mathcal{T}M)$ such that

$$\iota_{\operatorname{grad}\varphi}\omega = d\varphi; \tag{59}$$

this vector field is called the *gradient* of $\varphi$. Consider the vertical lift $\alpha^v := \alpha \circ \pi$ of a function $\alpha \in C^\infty(M)$; then $\operatorname{grad}\alpha^v$ is a vertical vector field with the following properties:

(i) $[C, \operatorname{grad}\alpha^v] = -\operatorname{grad}\alpha^v$,
(ii) $\operatorname{grad}\alpha^v(E) = \alpha^c$, where $\alpha^c := S\alpha^v$ is the complete lift of $\alpha$,
(iii) $D\operatorname{grad}\alpha^v = -\iota_{\mathrm{F}\operatorname{grad}\alpha^v}\mathcal{C}$.

In terms of local coordinates:

$$\operatorname{grad}\alpha^v = g^{ij}\frac{\partial\alpha}{\partial u^j}\circ\pi\frac{\partial}{\partial y^i}, \quad \mathrm{F}\operatorname{grad}\alpha^v = g^{ij}\frac{\partial\alpha}{\partial u^j}\circ\pi\frac{\delta}{\delta x^i}.$$

**Lemma 1.** *If* $\operatorname{grad}\alpha^v = \mu C$, *where* $\mu \in C^\infty(\mathcal{T}M)$, *then* $\mu = 0$ *and, consequently, the function* $\alpha$ *is constant.*

For a proof see [17].

3.5. **The generalized Matsumoto's problem.** The generalized problem is that whether there exists a nontrivial conformal change of a Riemann-Finsler metric such that the *hv*-curvature tensor of the Berwald connection is invariant. In what follows we are going to solve this generalized problem using the standard technical tools of tangent bundle differential geometry such as vertical, complete and horizontal lifts of a vector field $X \in \mathfrak{X}(M)$:

$$\begin{aligned}
X^v &= X^i\circ\pi\frac{\partial}{\partial y^i},\\
X^c &= X^i\circ\pi\frac{\partial}{\partial x^i} + y^j\frac{\partial X^i}{\partial u^j}\circ\pi\frac{\partial}{\partial y^i},\\
X^h &= X^i\circ\pi\left(\frac{\partial}{\partial x^i} - \Gamma_i^k\frac{\partial}{\partial y^k}\right);
\end{aligned}$$

see [8], [18] and [22]. In terms of complete lifts the *hv*-curvature tensor of the Berwald connection can be calculated as follows:

$$\overset{\circ}{\mathbb{P}}(X^c, Y^c)Z^c = [[X^h, Y^v], Z^v];$$

recall that the vanishing of this curvaure characterizes the so-called *Berwald manifolds.*

The *vertical* and *complete* lifts of a function $\alpha \in C^\infty(M)$ are given by the formulas $\alpha^v := \alpha\circ\pi$ and

$$\alpha^c = y^i\frac{\partial\alpha}{\partial u^i}\circ\pi,$$

respectively.

**Theorem 4.** *Let* $(M, E)$ *be a Finsler manifold and suppose that the hv-curvature tensor* $\overset{\circ}{\mathbb{P}}$ *is invariant under the conformal change* $g_\alpha = \varphi g$. *If the function* $\alpha$ *is regular at the point* $p \in M$, *then the manifold is locally*

*Riemannian, i.e. there is a neighbourhood $U$ of the point $p$ such that the restricted energy function $E|_{TU}$ is quadratic.*

Proof. Since the $hv$-curvature tensor of the Berwald connection is invariant, for any vector field $Y, Z$ and $W \in \mathfrak{X}(M)$ it follows that

$$\begin{aligned} 0 =& \overset{\circ}{\mathbb{P}}_\alpha (Y^c, Z^c)W^c - \overset{\circ}{\mathbb{P}} (Y^c, Z^c)W^c = \\ &= [[Y^{h_\alpha}, Z^v], W^v] - [[Y^h, Z^v], W^v] = [[Y^{h_\alpha}, Z^v] - [Y^h, Z^v], W^v]. \end{aligned}$$

This means that the vector field

$$[Y^{h_\alpha}, Z^v] - [Y^h, Z^v] = [Y^{h_\alpha} - Y^h, Z^v]$$

is a vertical lift and, consequently, the difference vector field $Y^{h_\alpha} - Y^h$ is linear on *any* tangent space $T_pM$. As an easy calculation shows

$$\big(Y^{h_\alpha} - Y^h\big)E = -(Y\alpha)^v E.$$

Consider the vector field

$$X := \big(Y^{h_\alpha} - Y^h\big) + \frac{1}{2}(Y\alpha)^v C;$$

since it is tangential to the indicatrix hypersurface, the restriction $X|_{T_pM}$ is an element of the Lie algebra $\mathcal{A}_\circ(T_pM)$. This follows immediately from Proposition 2. Therefore, by Proposition 4, $\tilde{\mathcal{C}}(\mathrm{F}X) = 0$, where $\tilde{\mathcal{C}}$ is the semibasic trace of the first Cartan tensor. Of course, we have a well-known transformation formula for changing of the Barthel endomorphism under a conformal change, namely,

$$Y^{h_\alpha} = Y^h - \frac{1}{2}\alpha^c Y^v - \frac{1}{2}(Y\alpha)^v C - E\mathcal{C}(\mathrm{F}\,\mathrm{grad}\,\alpha^v, Y^c) + \frac{1}{2}Y^v E\,\mathrm{grad}\,\alpha^v;$$

[6], see also [17], [19] and [20] for the coordinate-free expression. It follows that

$$X = \frac{1}{2}Y^v E\,\mathrm{grad}\,\alpha^v - \frac{1}{2}\alpha^c Y^v - E\mathcal{C}(\mathrm{F}\,\mathrm{grad}\,\alpha^v, Y^c)$$

and, consequently,

$$0 = \frac{1}{2}Y^v E\tilde{\mathcal{C}}(\mathrm{F}\,\mathrm{grad}\,\alpha^v) - \frac{1}{2}\alpha^c\tilde{\mathcal{C}}(Y^c) - E\tilde{\mathcal{C}}(\mathrm{F}\mathcal{C}(\mathrm{F}\,\mathrm{grad}\,\alpha^v, Y^c)).$$

Since it is a tensorial relation, the substitution of the canonical spray $S$ instead of $Y^c$ shows that

$$\tilde{\mathcal{C}}(\mathrm{F}\,\mathrm{grad}\,\alpha^v) = 0 \Rightarrow -\frac{1}{2}\alpha^c\tilde{\mathcal{C}}(Y^c) - E\tilde{\mathcal{C}}(\mathrm{F}\mathcal{C}(\mathrm{F}\,\mathrm{grad}\,\alpha^v, Y^c)) = 0. \tag{60}$$

By substituting the vector field $\mathrm{F}\,\mathrm{grad}\,\alpha^v$ instead of $Y^c$ we have that

$$\tilde{\mathcal{C}}(\mathrm{F}\mathcal{C}(\mathrm{F}\,\mathrm{grad}\,\alpha^v, \mathrm{F}\,\mathrm{grad}\,\alpha^v)) = 0. \tag{61}$$

Let now $v \in T_pM$ be a nonzero tangent vector; since the lowered first Cartan tensor is totally symmetric, the mapping

$$\mathcal{C}(\mathrm{F}\,\mathrm{grad}\,\alpha^v, \cdot)(v)\colon T_vTM \to T_vTM$$

is "self-adjoint" with respect to the metric $g_v$ in the following sense: we can consider a $g_v$-orthonormal system $Y_1^v, \dots, Y_n^v$ at the point $v$ such that

$$\mathcal{C}(\mathrm{F}\,\mathrm{grad}\,\alpha^v, Y_i^c)(v) = \lambda_i Y_i^v(v).$$

Then, by the formula (62), it follows that

$$
\begin{aligned}
0 &= \sum_{i=1}^{n} g(\mathcal{C}(\mathrm{F}\mathcal{C}(\mathrm{F}\,\mathrm{grad}\,\alpha^v, \mathrm{F}\,\mathrm{grad}\,\alpha^v), Y_i^c), Y_i^v)(v) = \\
&= \sum_{i=1}^{n} g(\mathbb{Q}(\mathrm{F}\,\mathrm{grad}\,\alpha^v, Y_i^c)\mathrm{F}\,\mathrm{grad}\,\alpha^v, Y_i^v)(v) + \\
&+ \sum_{i=1}^{n} g(\mathcal{C}(\mathrm{F}\,\mathrm{grad}\,\alpha^v, \mathrm{F}\mathcal{C}(Y_i^c, \mathrm{F}\,\mathrm{grad}\,\alpha^v)), Y_i^v)(v) = \\
&= -R(\mathrm{F}\,\mathrm{grad}\,\alpha^v, \mathrm{F}\,\mathrm{grad}\,\alpha^v) + \lambda_1^2 + \ldots + \lambda_n^2,
\end{aligned} \tag{62}
$$

where $R$ is the vertical Ricci tensor of the Cartan connection. Therefore

$$R(\mathrm{F}\,\mathrm{grad}\,\alpha^v, \mathrm{F}\,\mathrm{grad}\,\alpha^v) \geq 0.$$

Since the vertical covariant differentiation with respect to the Cartan connection is just the same as that with respect to the Lévi-Civita connection $\nabla$ on the "manifold" $T_pM$, the formula 3.4 (iii) shows that $\mathrm{div}\,\mathrm{grad}\,\alpha^v = 0$ where the divergence operator, of course, is taken with respect to the connection $\nabla$. On the other hand, the Hesse form $\nabla\,\mathrm{grad}\,\alpha^v$ is automatically self-adjoint. This means, by a theorem due to G. de Rham (see [13], section 5.4) that $\mathrm{grad}\,\alpha^v$ is a harmonic vector field. Moreover, de Rham's theorem states that

$$g(\mathrm{tr}\ \nabla^2\,\mathrm{grad}\,\alpha^v, \cdot) = R(\mathrm{F}\,\mathrm{grad}\,\alpha^v, \cdot)$$

and, by a theorem due to S. Bochner (see [13], section 4.18) we have that

$$2g(\mathrm{tr}\ \nabla^2\,\mathrm{grad}\,\alpha^v, \mathrm{grad}\,\alpha^v) + 2\|\nabla\,\mathrm{grad}\,\alpha^v\|^2 + \Delta\|\,\mathrm{grad}\,\alpha^v\|^2 = 0, \tag{63}$$

where the norm, of course, is taken with respect to the metric $g$. Therefore

$$\Delta\|\,\mathrm{grad}\,\alpha^v\|^2 \leq 0. \tag{64}$$

Since the function $\|\,\mathrm{grad}\,\alpha^v\|^2$ is homogeneous of degree 0 it attains both its maximum and minimum on the vector space $T_pM$. In this case a subharmonic function must be constant as the Hopf's maximum principle states; see [21], Theorem 2.1. This means that we can write the function $\|\,\mathrm{grad}\,\alpha^v\|^2$ in the form

$$\|\,\mathrm{grad}\,\alpha^v\|^2 = \beta \circ \pi \tag{65}$$

and the proof can be finished as follows. The hyphotesis on the $hv$-curvature tensor of the Berwald connection implies that the difference of the canonical sprays is a quadratic vector fields. Of course, we have a well-known transformation formula for changing of the canonical spray under a conformal change, namely,

$$S_\alpha = S - \alpha^c C + E\,\mathrm{grad}\,\alpha^v;$$

[6]; see also [17], [19] and [20] for the coordinate-free expression. It follows that the function

$$E\|\,\mathrm{grad}\,\alpha^v\|^2 = \big(S_\alpha - S\big)\alpha^c + (\alpha^c)^2$$

is quadratic. Since $d_p\alpha \neq 0$, the left hand side is nontrivial on a neighbourhood $U$ of the point $p$. Therefore, by the formula (66), the restriction $E|_{TU}$ must be the energy function of a Riemannian manifold. □

**Theorem 5.** *The conformal equivalence between two Berwald manifolds must be trivial unless the manifolds are Riemannian.*

Proof. It remains only to show that if a Berwald manifold is locally Riemannian, then it is a Riemannian manifold; but this is trivial. The local property can be easily extended by the help of the (linear) parallel transport provided, of course, that $M$ is a connected manifold. $\square$

**Exercise.** Using the same technic on the "manifold" $T_pM$ as in the proof of theorem 4 prove Deicke's classical theorem for Finsler manifolds. (Hint: Substitute an arbitrary vertical lifted vector field into the formulas (61)-(66) instead of $\operatorname{grad} \alpha^v$.)

**Exercise.** Find a short proof of theorem 4 in case of dimension 2. (Hint: Suppose that $\mathcal{A}_\circ(T_pM)$ is trivial; then $X = 0$, i.e. $\operatorname{grad} \alpha^v$ and the Liouville vector field $C$ are linearly *dependent* on the "manifold" $T_pM$. By the help of Lemma 1 we get a contradiction immediately.)

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