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Cauchy–Schwarz-type inequalities for solutions of Levi-Civita-type functional equations

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Abstract

The main goal of this paper is to show that if a real-valued function defined on a groupoid satisfies a certain Levi-Civita-type functional equation, then it also fulfills a Cauchy–Schwarz-type functional inequality. In particular, if the groupoid is the multiplicative structure of a commutative ring, then we can establish the existence of nontrivial additive functions satisfying inequalities connected to the multiplicative structure.

Keywords Levi-Civita-type functional equation · Cauchy–Schwarz-type functional inequality · Additive function

Mathematics Subject Classification Primary 39B22 · 39B52 · 39B62 · 39B72

1 Introduction

In the theory of real and additive functions (see the monograph [8] of Kuczma) there are several results that establish the existence of a discontinuous additive function that satisfies further algebraic conditions. One of the first problems of this kind was posed by Szabó [10] motivated by a question of Benz [2] and solved by Kominek, Reich, and Schwaiger [7]. They proved that if $A\colon \mathbb{R} \to \mathbb{R}$ is an additive function that satisfies the equality $A(x)A(y) = 0$ for all $(x, y) \in C$, where C is the unit circle, a hyperbola, or an algebraic curve given by polynomials, then A has to be equal to zero identically. Boros and Fechner [4] and Boros, Fechner and Kutas [5] extended these results to sets defined via generalized polynomials and to quadratic functions instead of additive ones, respectively, and they also examined the stability versions of such problems.

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In [5], the case when C is the graph of the hyperbola $xy = 1$ was left open. Kannappan [6, Chapter 1] proved that if, for some positive constant a, an additive function A satisfies the condition $A(x)A(1/x) = a$ for all $x \neq 0$, then A has to be continuous. On the other hand, according to the remarks in [1] and [3], there exist discontinuous additive functions that fulfill the inequality $A(x)A(1/x) > 0$ for all $x \neq 0$. Moreover, using the theory of valuations of fields, Kutas [9, Theorem 24] proved that there exists a nonzero (henceforth discontinuous) additive function that satisfies the equality $A(x)A(1/x) = 0$ for all $x \neq 0$.

The above results motivated us to construct discontinuous additive real functions that enjoy properties that are connected to the multiplicative structure. It turned out that such properties could be possessed if the additive function satisfies Levi-Civita-type functional equations with respect to the multiplicative structure.

More generally, let $(G, *)$ be a groupoid. (Recall that a pair $(G, *)$ is said to be a groupoid if $*$ is a binary operation on G, i.e., $*\colon G \times G \to G$.) Let $A\colon G \to \mathbb{R}$ be a function such that there exist functions $f_1, \ldots, f_n, g_1, \ldots, g_n\colon G \to \mathbb{R}$ such that the functional equation

$$A(x * y) = f_1(x)g_1(y) + \cdots + f_n(x)g_n(y) \qquad (x, y \in G)$$

is fulfilled. Under certain assumptions on n and on the functions $f_1, \ldots, f_n, g_1, \ldots, g_n$, we are going to prove that A will satisfy either the inequality $A(x * y)^2 \leq A(x * x)A(y * y)$ or the reversed one $A(x * x)A(y * y) \leq A(x * y)^2$. In the important particular case when the groupoid is the multiplicative structure of a commutative ring and A is additive, we will establish the existence of nontrivial additive functions that satisfy one of the above-mentioned inequalities.

2 The inequality $A(x * y)^2 \leq A(x * x)A(y * y)$

In our first result we assume that the function A satisfies a Levi-Civita-type functional equation over a groupoid.

Proposition 2.1 *Let $(G, *)$ be a groupoid and let $A\colon G \to \mathbb{R}$ be a function. Assume that there exist $n \in \mathbb{N}$ and functions $f_1, \ldots, f_n\colon G \to \mathbb{R}$ such that A satisfies the Levi-Civita-type functional equation*

$$A(x * y) = f_1(x)f_1(y) + \cdots + f_n(x)f_n(y) \tag{2.1}$$

for all $x, y \in G$. Then A fulfills the functional inequality

$$A(x * y)^2 \leq A(x * x)A(y * y) \tag{2.2}$$

for all $x, y \in G$.

Proof Let $x, y \in G$. In view of the functional equation (2.1), inequality (2.2) can be rewritten as

$$\big(f_1(x)f_1(y) + \cdots + f_n(x)f_n(y)\big)^2 \leq \big(f_1(x)^2 + \cdots + f_n(x)^2\big)\big(f_1(y)^2 + \cdots + f_n(y)^2\big),$$

which follows from the Cauchy–Schwarz inequality when we apply it to the n-dimensional vectors $(f_1(x), \ldots, f_n(x))$ and $(f_1(y), \ldots, f_n(y))$. □

If the groupoid is the multiplicative semigroup of a commutative ring $(R, +, \cdot)$ and A is additive, then we can establish a characterization of the corresponding inequality. Recall that in a ring the product $x \cdot y$ of the elements $x, y \in R$ is simply denoted by xy, and x^2 is defined to be the product $x \cdot x$.

Theorem 2.2 *If $(R,+,\cdot)$ is a commutative ring and $A\colon R\to\mathbb{R}$ is an additive function, then A satisfies the inequality*

$$A(xy)^2 \le A(x^2)A(y^2) \tag{2.3}$$

for all $x,y\in R$ if and only if at least one of the following conditions hold:

(i) $A(x^2)\ge 0$ *for all* $x\in R$,
(ii) $A(x^2)\le 0$ *for all* $x\in R$.

Proof Assume first that A satisfies inequality (2.3) for all $x,y\in R$, but none of the conditions (i) and (ii) is valid. Then there exist $x,y\in R$ such that

$$A(x^2)<0<A(y^2).$$

These inequalities imply

$$A(x^2)A(y^2)<0\le A(xy)^2,$$

which contradicts (2.3).

To prove the reverse implication, assume that A satisfies condition (i) and let $x,y\in R$ be fixed. Then, for all $n\in\mathbb{N}$ and $k\in\mathbb{Z}$, we get

$$0\le A((nx+ky)^2)=A(n^2x^2+2nkxy+k^2y^2)=n^2A(x^2)+2nkA(xy)+k^2A(y^2).$$

Dividing this inequality by n^2, we infer

$$0\le A(x^2)+2\frac{k}{n}A(xy)+\frac{k^2}{n^2}A(y^2).$$

Since $n\in\mathbb{N}$ and $k\in\mathbb{Z}$ were arbitrary, we obtain that

$$0\le A(x^2)+2rA(xy)+r^2A(y^2)$$

is valid for all rational numbers r. By the density of the rational numbers, it follows that the above inequality is true for all real numbers r. The polynomial on the right side cannot have two distinct real roots, so its discriminant has to be nonpositive, i.e.,

$$(2A(xy))^2-4A(x^2)A(y^2)\le 0.$$

This inequality reduces to (2.3).

In the case where (ii) holds, the additive function $(-A)$ satisfies (i) and hence (2.3) holds with $(-A)$ instead of A, which again shows that (2.3) is valid. □

3 The inequality $A(x*x)A(y*y)\le A(x*y)^2$

In the subsequent two propositions, we present Levi-Civita-type functional equations that imply the inequality in the title of this section.

Proposition 3.1 *Let $(G,*)$ be a groupoid and $A\colon G\to\mathbb{R}$ be a function. Assume that there exist functions $f,g\colon G\to\mathbb{R}$ such that the Levi-Civita-type functional equation*

$$A(x*y)=f(x)f(y)-g(x)g(y) \tag{3.1}$$

holds for all $x,y\in G$. Then A satisfies the functional inequality

$$A(x*x)A(y*y)\le A(x*y)^2 \tag{3.2}$$

for all $x,y\in G$.

Proof Let $x, y \in G$. According to (3.1), inequality (3.2) can be rewritten as

$$(f(x)^2 - g(x)^2)(f(y)^2 - g(y)^2) \le (f(x)f(y) - g(x)g(y))^2.$$

Observe that this is equivalent to

$$0 \le (g(x)f(y) - f(x)g(y))^2,$$

which is obviously valid. □

Proposition 3.2 *Let $(G, *)$ be a groupoid. Let $A\colon R \to \mathbb{R}$ be a function. Assume that there exist $f, g\colon R \to \mathbb{R}$ such that the Levi-Civita-type functional equation*

$$A(x * y) = f(x)g(y) + g(x)f(y) \tag{3.3}$$

holds for all $x, y \in G$. Then, for all $x, y \in G$, A satisfies the functional inequality (3.2).

Proof Let $x, y \in G$. According to (3.3), inequality (3.2) can be rewritten as

$$4f(x)g(x)f(y)g(y) \le (f(x)g(y) + g(x)f(y))^2.$$

Observe that this is equivalent to

$$0 \le (g(x)f(y) - f(x)g(y))^2,$$

which is obviously valid. □

Corollary 3.3 *Assume that $A\colon \mathbb{R} \to \mathbb{R}$ satisfies the Leibniz Rule with respect to multiplication, i.e.,*

$$A(xy) = xA(y) + A(x)y \qquad (x, y \in \mathbb{R}).$$

Then, for all $x, y \in \mathbb{R}$, the inequality

$$A(x^2)A(y^2) \le A(xy)^2 \tag{3.4}$$

holds.

Proof Observe that with the groupoid $(G, *) := (\mathbb{R}, \cdot)$ and with the notations $g := A$ and $f(x) := x$, $(x \in G)$, equality (3.3) of Proposition 3.2 holds. Therefore, A satisfies (3.2) for all $x, y \in \mathbb{R}$, hence (3.4) is also satisfied. □

In particular, if $A\colon \mathbb{R} \to \mathbb{R}$ is a derivation (i.e., A is additive and satisfies the Leibniz Rule with respect to multiplication), then the above corollary implies that it fulfills inequality (3.4).

If the groupoid is the multiplicative semigroup of a commutative ring $(R, +, \cdot)$ and A is additive, then we can establish a characterization of (3.2) over a particular subset of the ring.

Theorem 3.4 *Let $(R, +, \cdot)$ be a commutative ring with a multiplicative unit element e and $A\colon R \to \mathbb{R}$ be an additive function with $A(e) \ne 0$. Let the subset $R_A \subseteq R$ be defined by*

$$R_A := \{x \in R \mid 0 \le A(x^2)A(e)\}.$$

Then $e \in R_A$ and A satisfies the functional inequality

$$A(x^2)A(y^2) \le A(xy)^2 \tag{3.5}$$

for all $x, y \in R_A$ if and only if

$$A(x^2)A(e) \le A(x)^2 \tag{3.6}$$

for all $x \in R_A$.

Proof The inclusion $e \in R_A$ is obvious. Now, putting $y := e$, we can see that (3.5) implies (3.6).

To prove the converse, assume that (3.6) is valid for all $x \in R_A$. Then it is also valid for all $x \in R$, since, for $x \in R \setminus R_A$, the left side of the inequality is negative, while the right side is nonnegative. Introduce the function $A_0 := A/A(e)$. Then A_0 is additive, $A_0(e) = 1$ and, dividing (3.6) by $A(e)^2 > 0$ side by side, for all $x \in R$, we get

$$A_0(x^2) \le A_0(x)^2. \tag{3.7}$$

Let $x, y \in R_A$ be fixed and $n \in \mathbb{N}$, $k \in \mathbb{Z}$ be arbitrary. Then (3.7) yields

$$A_0((nx + ky)^2) \le A_0(nx + ky)^2.$$

Using the additivity of A_0, we get

$$n^2 A_0(x^2) + 2nk A_0(xy) + k^2 A_0(y^2) \le n^2 A_0(x)^2 + 2nk A_0(x)A_0(y) + k^2 A_0(y)^2.$$

Dividing this inequality by n^2, we obtain

$$A_0(x^2) + 2\tfrac{k}{n} A_0(xy) + \left(\tfrac{k}{n}\right)^2 A_0(y^2) \le A_0(x)^2 + 2\tfrac{k}{n} A_0(x)A_0(y) + \left(\tfrac{k}{n}\right)^2 A_0(y)^2.$$

Therefore, for any rational number $r \in \mathbb{Q}$,

$$0 \le (A_0(x)^2 - A_0(x^2)) + 2r(A_0(x)A_0(y) - A_0(xy)) + r^2(A_0(y)^2 - A_0(y^2)).$$

Using the continuity of both sides as a function of r, it follows that the same inequality is valid for all $r \in \mathbb{R}$. Thus, the discriminant of this quadratic polynomial has to be nonpositive, i.e.,

$$(A_0(x)A_0(y) - A_0(xy))^2 \le (A_0(x)^2 - A_0(x^2))(A_0(y)^2 - A_0(y^2)) \tag{3.8}$$

so

$$|A_0(x)A_0(y) - A_0(xy)| \le \sqrt{(A_0(x)^2 - A_0(x^2))(A_0(y)^2 - A_0(y^2))} = Q(x)Q(y),$$

thus,

$$\big||A_0(x)A_0(y)| - |A_0(xy)|\big| \le |A_0(x)A_0(y) - A_0(xy)| \le Q(x)Q(y),$$

where $Q(u) := \sqrt{A_0(u)^2 - A_0(u^2)} \ge 0$ ($u \in R$). Then, for all $u \in R$,

$$A_0(u)^2 = Q(u)^2 + A_0(u^2). \tag{3.9}$$

Therefore $|A_0(xy)|$ satisfies the inequality

$$|A_0(x)A_0(y)| - Q(x)Q(y) \le |A_0(xy)| \le |A_0(x)A_0(y)| + Q(x)Q(y). \tag{3.10}$$

We are going to show that

$$A_0(x^2)A_0(y^2) \le A_0(xy)^2. \tag{3.11}$$

To verify this inequality, we will prove

$$Q(x)^2 Q(y)^2 \le A_0(x)^2 A_0(y)^2 \tag{3.12}$$

and

$$A_0(x^2)A_0(y^2) \le \left(|A_0(x)A_0(y)| - Q(x)Q(y)\right)^2. \tag{3.13}$$

Since x and y belong to R_A, we have $A_0(x^2) \geq 0$ and $A_0(y^2) \geq 0$; then, with $u \in \{x, y\}$, equality (3.9) implies

$$Q(x)^2 \leq A_0(x)^2 \quad \text{and} \quad Q(y)^2 \leq A_0(y)^2.$$

Multiplying these inequalities side by side, we get that (3.12) holds. Therefore,

$$Q(x)Q(y) \leq |A_0(x)A_0(y)| \tag{3.14}$$

which is equivalent to (3.12).

By the obvious inequality

$$\big(|A_0(x)|Q(y) - |A_0(y)|Q(x)\big)^2 \geq 0,$$

we have

$$2|A_0(x)A_0(y)|Q(x)Q(y) \leq A_0(x)^2Q(y)^2 + A_0(y)^2Q(x)^2. \tag{3.15}$$

Therefore, using (3.9) with $u \in \{x, y\}$ and (3.15), we obtain

$$\begin{aligned} A_0(x^2)A_0(y^2) &= A_0(x)^2A_0(y)^2 - A_0(x)^2Q(y)^2 - Q(x)^2A_0(y)^2 + Q(x)^2Q(y)^2 \\ &\leq A_0(x)^2A_0(y)^2 - 2|A_0(x)A_0(y)|Q(x)Q(y) + Q(x)^2Q(y)^2 \\ &= (|A_0(x)A_0(y)| - Q(x)Q(y))^2. \end{aligned}$$

This shows that (3.13) holds.

In view of (3.14), the first inequality in (3.10) implies

$$(|A_0(x)A_0(y)| - Q(x)Q(y))^2 \leq A_0(xy)^2.$$

This, combined with (3.13) yields that (3.11) is valid, indeed. Therefore,

$$A(x^2)A(y^2) = A_0(x^2)A_0(y^2)A(e)^2 \leq A_0^2(xy)A(e)^2 = A(xy)^2,$$

which completes the proof of (3.5) for $x, y \in R_A$. □

In the following example we show that the additivity of the function A in Theorem 3.4 is necessary.

Example 3.5 Let $q \in (0, 1)$ and let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x & \text{if } x \neq 1, \\ q & \text{if } x = 1. \end{cases}$$

Clearly, f is not additive. Therefore, for $x \notin \{1, -1\}$,

$$f(x^2)f(1) = qx^2 \leq x^2 = f(x)^2;$$

for $x = \pm 1$,

$$f(1^2)f(1) = q^2 = f(1)^2, \qquad f((-1)^2)f(1) = q^2 < 1 = f(-1)^2,$$

which shows that (3.6) is satisfied for all $x \in \mathbb{R}$. On the other hand, for $x, y \in \mathbb{R} \setminus \{1, -1\}$ with $xy = 1$,

$$f(x^2)f(y^2) = x^2y^2 = 1 > q^2 = f(xy)^2,$$

which shows that (3.5) is not satisfied.

The next example shows that if the function A in Theorem 3.4 is not additive, continuous and satisfies $A(e) = 0$, then the conclusion of Theorem 3.4 may not be valid.

Example 3.6 Let $A: \mathbb{R} \to \mathbb{R}$ be defined as $A(x) = |x - 1|$. Note that A is continuous and not additive. Since $A(1) = 0$, this implies

$$A(x^2)A(1) = 0 \leq (x-1)^2 = A(x)^2.$$

Thus, (3.6) holds for all $x \in \mathbb{R}$. On the other hand, we have

$$A(x^2)A(y^2) = |x^2 - 1||y^2 - 1| \qquad \text{and} \qquad A(xy)^2 = (xy-1)^2.$$

Hence for $x = 2$ and $y = \frac{1}{2}$ we have $A(x^2)A(y^2) = \frac{9}{4}$ but $A(xy)^2 = 0$, so (3.5) does not hold.

4 Consequences of systems of Levi-Civita-type functional equations

Theorem 4.1 *Let $(G, *)$ be a groupoid and $A, B: G \to \mathbb{R}$ be functions. Assume that there exist $f, g: G \to \mathbb{R}$ such that A and B satisfy the system of Levi-Civita-type functional equations*

$$\begin{aligned} A(x * y) &= f(x)f(y) - g(x)g(y), \\ B(x * y) &= f(x)g(y) + g(x)f(y) \end{aligned} \tag{4.1}$$

for all $x, y \in G$. Then the inequalities

$$-B(x*y)^2 \leq A(x*x)A(y*y) \leq A(x*y)^2 \tag{4.2}$$

and

$$-A(x*y)^2 \leq B(x*x)B(y*y) \leq B(x*y)^2 \tag{4.3}$$

hold for all $x, y \in G$.

Proof In view of (4.1), for all $x, y \in G$, we have

$$\begin{aligned} &B(x*y)^2 + A(x*x)A(y*y) \\ &= f(x)^2 g(y)^2 + 2f(x)g(y)g(x)f(y) + g(x)^2 f(y)^2 + (f(x)^2 - g(x)^2)(f(y)^2 - g(y)^2) \\ &= (f(x)f(y) + g(x)g(y))^2 \geq 0, \end{aligned}$$

which proves the left inequality in (4.2). The right inequality in (4.2) is a direct consequence of Proposition 3.1.

Again, in view of (4.1), for all $x, y \in G$, we have

$$\begin{aligned} &A(x*y)^2 + B(x*x)B(y*y) \\ &= f(x)^2 f(y)^2 - 2f(x)f(y)g(x)g(y) + g(x)^2 g(y)^2 + 4f(x)g(x)f(y)g(y) \\ &= (f(x)f(y) + g(x)g(y))^2 \geq 0. \end{aligned}$$

This implies the first inequality in (4.3). On the other hand, applying Proposition 3.2 to the function B instead of A, we obtain

$$B(x*x)B(y*y) \leq B(x*y)^2.$$

This shows that the second inequality of (4.3) holds for all $x, y \in G$. $\square$

An interesting consequence of (4.1) is that A and B satisfy the following identity:

$$B(x*y)^2 + A(x*x)A(y*y) = A(x*y)^2 + B(x*x)B(y*y) \qquad (x, y \in G).$$

Therefore, (4.2) and (4.3) can be expressed as the following chain of inequalities:

$$\begin{aligned} 0 &\le A(x*x)A(y*y) + (B(x*y))^2 \\ &= B(x*x)B(y*y) + A(x*y)^2 \le A(x*y)^2 + B(x*y)^2 \qquad (x, y \in G). \end{aligned}$$

The following result is probably well known, but we could not find an exact reference for it.

Corollary 4.2 *For all $x, y \in \mathbb{R}$, we have*

$$\begin{aligned} -\sin(x+y)^2 \le \cos(2x)\cos(2y) \le \cos(x+y)^2 \qquad \textit{and} \\ -\cos(x+y)^2 \le \sin(2x)\sin(2y) \le \sin(x+y)^2. \end{aligned}$$

Proof Observe that the trigonometric functions $\cos\colon \mathbb{R} \to \mathbb{R}$ and $\sin\colon \mathbb{R} \to \mathbb{R}$ satisfy the functional equations

$$\begin{aligned} \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \qquad \text{and} \\ \sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \end{aligned}$$

for all $x, y \in \mathbb{R}$. Therefore, (4.1) holds with $A := f := \cos$ and $B := g := \sin$ over the groupoid $(\mathbb{R}, +)$. Consequently, (4.2) and (4.3) are satisfied for all $x, y \in \mathbb{R}$, which imply the assertion. □

Corollary 4.3 *Let $(G, *)$ be a groupoid and let $\varphi\colon G \to \mathbb{C}$ be a homomorphism into the multiplicative semigroup of complex numbers. Define $A := \Re\varphi$ and $B := \Im\varphi$. Then, for all $x, y \in G$, inequalities (4.2) and (4.3) hold.*

Proof Using the multiplicativity of φ, for all $x, y \in G$, we get

$$\begin{aligned} A(x*y) &= \Re(\varphi(x*y)) = \Re(\varphi(x)\varphi(y)) \\ &= \Re((A(x) + iB(x))(A(y) + iB(y))) = A(x)A(y) - B(x)B(y), \\ B(x*y) &= \Im(\varphi(x*y)) = \Im(\varphi(x)\varphi(y)) \\ &= \Im((A(x) + iB(x))(A(y) + iB(y))) = A(x)B(y) + B(x)A(y). \end{aligned}$$

Therefore, the functional equations in (4.1) are satisfied with $f := A$ and $g := B$. Thus, according to Theorem 4.1, we obtain that (4.2) and (4.3) hold for all $x, y \in G$, which was to be shown. □

Corollary 4.4 *Let $\varphi\colon \mathbb{C} \to \mathbb{C}$ be an automorphism of the field $\mathbb{C}$. Define $A := \Re\varphi$ and $B := \Im\varphi$. Then $A\colon \mathbb{C} \to \mathbb{R}$ and $B\colon \mathbb{C} \to \mathbb{R}$ are additive mappings, furthermore, for all $x, y \in \mathbb{C}$,*

$$-B(xy)^2 \le A(x^2)A(y^2) \le A(xy)^2 \quad \textit{and} \quad -A(xy)^2 \le B(x^2)B(y^2) \le B(xy)^2. \tag{4.4}$$

Proof Using the additivity of φ, for all $x, y \in \mathbb{C}$, we obtain

$$\begin{aligned} A(x+y) &= \Re(\varphi(x+y)) = \Re(\varphi(x) + \varphi(y)) \\ &= \Re((A(x) + iB(x)) + (A(y) + iB(y))) = A(x) + A(y), \\ B(x+y) &= \Im(\varphi(x+y)) = \Im(\varphi(x) + \varphi(y)) \end{aligned}$$

$$= \Im((A(x) + iB(x)) + (A(y) + iB(y))) = B(x) + B(y).$$

These equalities show that A and B are additive mappings.

By the multiplicativity of φ, it maps the groupoid $(G, *) := (\mathbb{C}, \cdot)$ into itself. Thus, according to Corollary 4.3, we obtain that (4.2) and (4.3) hold for all $x, y \in \mathbb{C}$. This yields the assertion. □

The following result is a counterpart of Theorem 4.1.

Theorem 4.5 *Let $(G, *)$ be a groupoid and $A, B\colon G \to \mathbb{R}$ be functions. Assume that there exist $f, g\colon G \to \mathbb{R}$ such that A and B satisfy the Levi-Civita-type functional equations*

$$\begin{aligned} A(x * y) &= f(x)f(y) + g(x)g(y) \qquad \textit{and} \\ B(x * y) &= f(x)g(y) + g(x)f(y) \end{aligned} \tag{4.5}$$

for all $x, y \in G$. Then the inequalities

$$B(x * x)B(y * y) \le A(x * y)^2 \le A(x * x)A(y * y) \tag{4.6}$$

and

$$B(x * x)B(y * y) \le B(x * y)^2 \le A(x * x)A(y * y). \tag{4.7}$$

hold for all $x, y \in G$.

Proof In view of (4.5), for all $x, y \in G$, we have

$$\begin{aligned} &A(x * y)^2 - B(x * x)B(y * y) \\ &= f(x)^2 f(y)^2 + 2f(x)f(y)g(x)g(y) + g(x)^2 g(y)^2 - 4f(x)g(x)f(y)g(y) \\ &= (f(x)f(y) - g(x)g(y))^2 \ge 0. \end{aligned}$$

This implies the first inequality in (4.6). On the other hand, applying Proposition 2.1 to the function A and $n = 2$, $f_1 := f$, $f_2 := g$, we obtain that the second inequality of (4.6) holds for all $x, y \in G$.

Again, in view of (4.5), for all $x, y \in G$, we have

$$\begin{aligned} &A(x * x)A(y * y) - B(x * y)^2 \\ &= (f(x)^2 + g(x)^2)(f(y)^2 + g(y)^2) - f(x)^2 g(y)^2 - 2f(x)g(y)g(x)f(y) - g(x)^2 f(y)^2 \\ &= (f(x)f(y) - g(x)g(y))^2 \ge 0, \end{aligned}$$

which proves the second inequality in (4.7). The first inequality in (4.7) is a direct consequence of Proposition 3.2 (applied to B instead of A). □

An interesting consequence of (4.1) is that A and B satisfy the following identity:

$$B(x * x)B(y * y) + A(x * x)A(y * y) = A(x * y)^2 + B(x * y)^2 \qquad (x, y \in G).$$

The following result is probably also well known, but we could not find a reference for it.

Corollary 4.6 *For all $x, y \in \mathbb{R}$, we have*

$$\sinh(2x)\sinh(2y) \le \sinh(x + y)^2 < \cosh(x + y)^2 \le \cosh(2x)\cosh(2y). \tag{4.8}$$

Proof Observe that the hyperbolic functions $\cosh\colon \mathbb{R} \to \mathbb{R}$ and $\sinh\colon \mathbb{R} \to \mathbb{R}$ satisfy the functional equations

$$\begin{aligned}\cosh(x+y) &= \cosh(x)\cosh(y) + \sinh(x)\sinh(y) \qquad \text{and}\\ \sinh(x+y) &= \sinh(x)\cosh(y) + \cosh(x)\sinh(y)\end{aligned}$$

for all $x, y \in \mathbb{R}$. Therefore, (4.5) holds with $A := f := \cosh$ and $B := g := \sinh$ over the groupoid $(\mathbb{R}, +)$. Consequently, (4.6) and (4.7) are satisfied for all $x, y \in \mathbb{R}$, which imply the first and last inequalities in (4.8). The central inequality follows from the identity $\cosh^2 - \sinh^2 = 1$. ◻

To formulate the next result, let p be a square-free positive integer and let $\mathbb{Q}(\sqrt{p})$ denote the subfield of $\mathbb{R}$ generated by $\sqrt{p}$. Clearly, $\mathbb{Q}(\sqrt{p}) = \{a + b\sqrt{p} : a, b \in \mathbb{Q}\}$. In what follows, we equip $\mathbb{Q}(\sqrt{p})$ with the topology inherited from $\mathbb{R}$.

Theorem 4.7 *If p is a square-free positive integer, then there exist discontinuous additive functions $A\colon \mathbb{Q}(\sqrt{p}) \to \mathbb{R}$ and $B\colon \mathbb{Q}(\sqrt{p}) \to \mathbb{R}$ such that*

$$\begin{aligned}B(x^2)B(y^2) &\le A(xy)^2 \le A(x^2)A(y^2),\\ B(x^2)B(y^2) &\le B(xy)^2 \le A(x^2)A(y^2)\end{aligned} \tag{4.9}$$

hold for all $x, y \in \mathbb{Q}(\sqrt{p})$.

Proof Define the functions $A\colon \mathbb{Q}(\sqrt{p}) \to \mathbb{R}$ and $B\colon \mathbb{Q}(\sqrt{p}) \to \mathbb{R}$ by

$$A(a + b\sqrt{p}) := a \qquad \text{and} \qquad B(a + b\sqrt{p}) := b\sqrt{p} \qquad (a, b \in \mathbb{Q}).$$

We show that A and B are additive. Indeed, let $x = a_1 + b_1\sqrt{p}$ and $y = a_2 + b_2\sqrt{p}$ be arbitrary points of $\mathbb{Q}(\sqrt{p})$, where $a_1, a_2, b_1, b_2 \in \mathbb{Q}$. According to the definition of A and B,

$$\begin{aligned}A(x+y) &= A(a_1 + a_2 + (b_1 + b_2)\sqrt{p})\\ &= a_1 + a_2 = A(x) + A(y)\end{aligned}$$

and

$$\begin{aligned}B(x+y) &= B(a_1 + a_2 + (b_1 + b_2)\sqrt{p})\\ &= (b_1 + b_2)\sqrt{p} = B(x) + B(y).\end{aligned}$$

This proves that A and B are additive, indeed.

Next we prove that A and B satisfy (4.5) with $f := A$, and $g := B$, where the groupoid $(G, *)$ is equal to $(\mathbb{Q}(\sqrt{p}), \cdot)$. Indeed, let $x = a_1 + b_1\sqrt{p}$ and $y = a_2 + b_2\sqrt{p}$ be arbitrary points of $\mathbb{Q}(\sqrt{p})$, where $a_1, a_2, b_1, b_2 \in \mathbb{Q}$. Then

$$A(xy) = A((a_1a_2 + b_1b_2p) + (a_1b_2 + a_2b_1)\sqrt{p}) = a_1a_2 + b_1b_2p = A(x)A(y) + B(x)B(y)$$

and, similarly,

$$\begin{aligned}B(xy) &= B((a_1a_2 + pb_1b_2) + (a_1b_2 + a_2b_1)\sqrt{p})\\ &= (a_1b_2 + a_2b_1)\sqrt{p} = A(x)B(y) + B(x)A(y).\end{aligned}$$

Therefore, according to Theorem 4.5, (4.6) and (4.7) hold, so the inequalities in (4.9) are also valid.

Finally, we show that A and B are discontinuous at the point $u := 1 + \sqrt{p}$. By the density of the set $\mathbb{Q}$ in $\mathbb{R}$, there exists a sequence (x_n) of rational numbers converging to $u/\sqrt{p}$. Then the sequence $(x_n\sqrt{p})$ converges to u. Now $A(u) = 1$ but, for all $n \in \mathbb{N}$, $A(x_n\sqrt{p}) = 0$, so A is discontinuous at u. Furthermore, there exists a sequence (y_n) of rational numbers that converges to u. Since $B(u) = \sqrt{p}$ and $B(y_n) = 0$ for all $n \in \mathbb{N}$, the function B is discontinuous at u. □

We note that there does not exist a discontinuous additive function $A\colon \mathbb{R} \to \mathbb{R}$ such that the inequality $A(xy)^2 \le A(x^2)A(y^2)$ is valid for all real numbers x, y. Indeed, if A is a discontinuous additive function satisfying this inequality, then $A(u)$ is not zero for some $u > 0$. With the substitution $y := \sqrt{u}$, the inequality shows that A is either nonnegative (if $A(u) < 0$) or nonpositive (if $A(u) < 0$) on the set of positive numbers. This, by classical results on additive functions (see [8]), implies that $A(x) = ax$ for some $a \in \mathbb{R}$, and hence A is continuous.

Motivated by the above remark, we could formulate the following open problem: Find a description or characterization of the maximal subrings (or subfields) of $\mathbb{R}$ such that system of inequalities in (4.9) holds for a discontinuous pair (A, B) of additive functions which are defined on this subring (or subfield).

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