

On Statistical Problems of Discrete and Continuous Time Autoregressive Processes

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2003

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Ezen értekezést a Debreceni Egyetem Matematika doktori program Valószínűségelmélet és matematikai statisztika alprogramja keretében készítettem 1995–2003 között és ezúton benyújtom a Debreceni Egyetem doktori Ph.D. fokozatának elnyerése céljából.

Debrecen, 2003. július 5.

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jelölt

Tanúsítom, hogy Varga Katalin doktorjelölt 1995–2003 között a fent megnevezett doktori alprogram keretében irányításommal végezte munkáját. Az értekezésben foglaltak a jelölt önálló munkáján alapulnak, az eredményekhez önálló alkotó tevékenységével meghatározóan hozzájárult. Az értekezés elfogadását javaslom.

Debrecen, 2003. július 5.

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Dr. Pap Gyula
témavezető

Acknowledgement

Here I would like to thank all the people who have contributed to my dissertation.

To my supervisor, Dr. Gyula Pap from whom I have learned and still learn a lot, and who has helped me with his instructions and advice in writing the dissertation.

To my professors, Dr. Mátyás Arató and Dr. Gábor Tusnányi for the valuable discussions, which widened my sphere of vision.

Köszönetnyilvánítás

Itt szeretnék köszönetet mondani azoknak, akik hozzájárultak disszertációm elkészítéséhez.

Témavezetőmnek, Dr. Pap Gyulának, akitől sokat tanultam és tanulok ma is, aki tanácsaival és útmutatásával segített abban, hogy ez a dolgozat létrejöhessen.

Tanáraimnak, Dr. Arató Mátyásnak és Dr. Tusnádý Gábornak a gondolatébresztő beszélgetésekért, amelyek szélesítették látókörömet.

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List of Notations

$\mathbb{N}$	set of natural numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^d$	d -dimensional Euclidean space
$\mathbb{C}$	set of complex numbers
$ x $	the absolute value of x
$\ x\ $	the norm of x in a linear space
$\bar{x}$	complex conjugate of x , $x \in \mathbb{C}^d$
$\operatorname{Re} x$	the real part of the complex number x
$\operatorname{Im} x$	the imaginary part of the complex number x
$\langle \cdot, \cdot \rangle$	scalar product
$\mathbb{E}$	expectation
$\mathbb{E}(X Y)$	conditional expectation of X given Y
$A^\top$	transpose of matrix A
$\operatorname{Tr} A$	trace of matrix A
$\operatorname{Det} A$	determinant of A
$\varrho(A)$	spectral radius of matrix A
$A^\ominus$	pseudo inverse of matrix A
I_d	$d \times d$ identity matrix
$\Psi_T(\alpha)$	Laplace transform
$\hat{Q}$	estimator for Q
$\xrightarrow{\mathcal{D}}$	weak convergence
MLE	maximum likelihood estimate
LSE	least squares estimate
$\mathbb{P}_X$	the measure generated by process X
$\mathcal{N}(m, D)$	normal distribution with mean (vector) m and covariancia (matrix) D
$\frac{dP_X}{dP_W}(x)$	Radon-Nikodym derivative
$\stackrel{\mathcal{D}}{=}$	equality in distribution
SDE	stochastic differential equation

Chapter 1

Preface

Why linear stochastic models?

The idea of using mathematical models to describe the behaviour of physical, economical phenomenon is well established. In particular, it is sometimes possible to derive a model based on physical laws, which enables us to calculate the value of some time-dependent quantity nearly exactly any instant of time. If such exact calculations were possible the model would be entirely deterministic. In many problems we have to consider a time-dependent phenomenon, in which there are many unknown factors and for which it is not possible to write a deterministic model that allows exact calculations of the future behaviour of the phenomenon. Nevertheless, it may be possible to derive a model that can be used to calculate the probability of the future values lying between two specified limits. Such a model is called stochastic model.

Among stochastic models the class of linear models is wide enough to describe many important empirical time series and can be handled well mathematically. In a linear stochastic model the time series is supposed to be linear aggregation of random shocks. For practical representation it is desirable to employ models which use parameters parsimoniously. Parsimony can be achieved by representation of the linear process in terms of a small number of autoregressive and moving average terms. In these models the current value of the process is expressed as a finite, linear aggregate of previous values of the process and random noise. Since 1970, when Box and Jenkins worked out the theory of ARMA modelling, the importance of these models have been growing continuously.

An important class of stochastic models for describing time series is the so called stationary models, which assume that the process remains in equilibrium about a constant mean level. However, in industry, business and economics there are many time

series, which are often better represented as nonstationary. There are an unlimited number of ways in which a process can be nonstationary. Nevertheless, many important types of economic and financial time series frequently exhibit a particular kind of homogeneous nonstationary behaviour, that can be represented by a stochastic model, which is modified form of the autoregressive model.

The thesis deals with some statistical questions of linear, particularly autoregressive models in discrete and continuous time. Stationary and nonstationary time series are discussed as well. We study the limit distribution of the estimators of the model-parameters.

In Chapter (2) there is a summary of the results which we consider as a starting point of the dissertation. In Chapter 3, in Chapter 4 and Chapter 5 the new results of the author are presented. These chapters are based on one paper of the author and three papers of the author with coauthors.

In Chapter (3) we treat the generalization of the result of Arató, Kolmogorov, Sinay [6] on the exact distribution of the maximum likelihood estimate (MLE) of the period of the complex-valued stationary first order autoregressive process.

The complex-valued stationary autoregressive process $\xi(t) = \xi_1(t) + i\xi_2(t)$, $t \geq 0$, can be given by the stochastic differential equation (SDE)

$$d\xi(t) = -\gamma\xi(t)dt + dw(t),$$

where $w(t) = w_1(t) + iw_2(t)$, $t \geq 0$ is a standard complex Wiener process (ie. $w_1(t)$ and $w_2(t)$ are independent standard real-valued Wiener processes) and $\gamma = \lambda - i\omega$ with $\lambda > 0$, $\omega \in \mathbb{R}$. This equation describes the rotation of the instantaneous axis of rotation of the earth with respect to the minor axis of the terrestrial ellipsoid, after the elimination of the one year periodic component, and it is called Chandler wobble. Arató in [4] gave a new proof that the suitably normalized MLE of the period ω exactly normally distributed.

If we rewrite the SDE as a two-dimensional real-valued SDE, the following natural question arises. Consider the d -dimensional process $X(t)$, $t \geq 0$, given by the SDE

$$dX(t) = AX(t)dt + dW(t), \quad t \geq 0,$$

where $W(t)$, $t \geq 0$ is a standard d -dimensional Wiener process and A is a $d \times d$ matrix. Which conditions should be imposed on the matrix A and on the distribution of the initial value $X(0)$ in order to get similar results on the suitably normalized MLE of some entries of the matrix A ? Pap and van Zuijlen gave a multidimensional generalization. In [1] we could weaken the conditions of Pap and van Zuijlen by showing that a part of the conditions is superfluous in [27]. Consider the following

multidimensional process:

$$dX(t) = \left(-\lambda I_d + \sum_{i=1}^m \omega_i C_i \right) X(t) dt + dW(t), \quad X(0) = 0 \quad (1.1)$$

where I_d is the $d \times d$ unit matrix, $\lambda, \omega_1, \dots, \omega_m \in \mathbb{R}$ are unknown parameters and $C_1, \dots, C_m$ are fixed skew-symmetric matrices. The MLE of $\omega = (\omega_1, \dots, \omega_m)^\top$ is given by

$$\hat{\omega}_X(t) = \sigma_X^{-1}(t) r_X(t),$$

where $\sigma_X(t)$ is the $m \times m$ matrix

$$\sigma_X(t) = \left(\int_0^t \langle C_i X(s), C_j X(s) \rangle ds \right)_{1 \leq i, j \leq m},$$

and $r_X(t)$ is the m -dimensional column vector

$$r_X(t) = \left(\int_0^t \langle C_i X(s), dX(s) \rangle \right)_{1 \leq i \leq m}^\top.$$

In [1] we proved that under the weekend conditions, the suitably normalized MLE of the periodic parameter ω is exactly normally distributed.

In Chapter (4) nearly unstable, nearly stable and explosive AR models are studied. A multidimensional AR model can be characterized by the help of its coefficient matrix. Consider the d -dimensional autoregressive model

$$\begin{cases} X_k = QX_{k-1} + \varepsilon_k, & k = 1, 2, \dots, \\ X_0 = 0, \end{cases}$$

where the d -dimensional real-valued random vector ε_k contains the random noise at time k , and the $d \times d$ matrix Q is the unknown parameter of the model. Let $\varrho(Q)$ denote the spectral radius of the matrix Q , i.e., the maximum of the absolute values of the eigenvalues of the matrix Q . When $\varrho(Q) < 1$, the model is said to be stable or asymptotically stationary. It is well-known that under the assumption that the ε_k 's are i.i.d. with $E\varepsilon_k = 0$, $E\varepsilon_k \varepsilon_k^\top = \Sigma$, the least squares estimate (LSE) of Q is asymptotically normal (see [1], [22]). When $\varrho(Q) = 1$, the model is said to be unstable or unstationary, when $\varrho(Q) > 1$ the model is explosive. The different types of models behaves diversely. The condition $\varrho(Q) < 1$, is needed for the process to have a stationary solution. In this case the process is damped. The explosive model "breaks loose" after a short induction period, with the generating noise playing almost no further part. The unstable model is also called unit root case and have been the subject of much recent attention in the econometrics literature. In part, this is

because the unit root hypothesis is of considerable interest in applications, not only with data from financial and commodity markets where it has a long history but also with aggregate time series. As we will see later, the estimators of parameters in these models behaves differently.

In the first part of Chapter (4), which is based on [3], we study a sequence of nearly unstable AR models:

$$\begin{cases} X_k^{(n)} = \alpha^{(n)} X_{k-1}^{(n)} + \varepsilon_k^{(n)}, & k = 1, 2, \dots, \\ X_0^{(n)} = 0, \end{cases} \quad (1.2)$$

where

$$\alpha^{(n)} = 1 - \frac{\gamma^{(n)}}{n}, \quad \gamma^{(n)} \rightarrow \gamma \in \mathbb{R}.$$

We discuss the questions concerning the LSE of the shift-parameter $m^{(n)}$ of the model

$$Z_k^{(n)} := X_k^{(n)} + m^{(n)} h_k^{(n)}, \quad k = 1, 2, \dots, \quad n = 1, 2, \dots,$$

where $m^{(n)} \in \mathbb{R}$ is an unknown parameter and $h_k^{(n)}$, $k = 1, \dots, n$, $n = 1, 2, \dots$ are known constraints.

Our purpose was to investigate the limit behaviour of $\widehat{m}_{T_1, T_2}^{(n)}$ as $n \rightarrow \infty$ and the connection between $\widehat{m}_{T_1, T_2}^{(n)}$ and the MLE $\widehat{m}_{T_1, T_2}$ of the shift-parameter of the related continuous time model. We discuss these questions in the nearly stable, nearly unstable and nearly explosive cases.

It turns out that $\widehat{m}_{T_1, T_2}^{(n)}$ is always asymptotically normal, but the speed of convergence is much less in the nearly unstable case than in the unstable case. The stationary case is also treated, when $\widehat{m}_{T_1, T_2}^{(n)}$ might not be asymptotically normal if $\gamma^{(n)} \rightarrow 0$, where $\gamma^{(n)}$ is the damping parameter of the model.

In the second part of Chapter (4), which is based on [2], nearly unstable multidimensional AR processes are treated. We discuss models where the coefficient matrices are in Jordan normal form. Consider the d -dimensional complex-valued autoregressive model

$$\begin{cases} X_k = J(\lambda, d) X_{k-1} + \varepsilon_k, & k=1, 2, \dots, \\ X_0 = 0, \end{cases}$$

where $\lambda \in \mathbb{C}$ and $d \in \mathbb{N}$, $J(\lambda, d)$ is a $d \times d$ matrix in Jordan normal form, with

eigenvalue λ :

$$J(\lambda, d) := \begin{pmatrix} \lambda & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & \lambda & 0 & & & \cdot \\ 0 & 1 & \lambda & 0 & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \lambda & 0 \\ 0 & \cdot & \cdot & 0 & 1 & \lambda \end{pmatrix}$$

and λ is the unknown parameter of the model.

If the random innovations ε_k satisfy a usual, rather general condition, then we found that the suitably normalised LSE of the eigenvalue $\hat{\lambda}_n$ converges weakly to a functional of a special d -dimensional Ornstein-Uhlenbeck process:

$$n^d(\hat{\lambda}_n - \lambda_n) \xrightarrow{\mathcal{D}} e^{(d-1)\theta i} \frac{\int_0^1 \overline{Y_d(t)} dW_d(t)}{\int_0^1 |Y_d(t)|^2 dt},$$

where the process $Y(t)$, $t \in [0, 1]$ is given by

$$dY(t) = AY(t)dt + \Sigma dW(t) \quad Y(0) = 0,$$

with a certain matrix A , and $W(t)$, $t \in [0, 1]$ is a standard complex d -dimensional Wiener process.

We also prove that in the one-dimensional model the LSE of the discrete time models converges to the MLE of the coefficient of the corresponding continuous time model.

If $d \geq 2$ then such relationship does not hold.

In Chapter (5), which is based on [4], we treat the estimation of the mean of multivariate AR processes. In Arató [3] the estimation of the unknown mean of real- and complex-valued AR(1) processes is discussed. Our aim was to investigate the problem of estimation of the mean for stationary and zero start multidimensional autoregressive processes. We show that for autoregressive processes the estimators of the mean are consistent if the component of the process is 'periodical', and it is not the case if the component is a 'damping' one. In the one-dimensional AR(1) case, the mean cannot be estimated well. In the complex AR(1), where the process behaves periodically, the mean can be estimated well. For an AR(2) process, the mean can be estimated well, if the roots of the characteristic equation are complex.

It is clear that the present results can be used fruitfully in a statistical context such as in problems of testing hypotheses, estimating parameters, and also in constructing confidence regions for the unknown parameters.

Chapter 2

Preliminaries

In this chapter we introduce discrete and continuous time multidimensional AR models and show how one can get estimates of the most important parameters e.g. the coefficient matrix or the shift-parameter.

2.1 Processes in Discrete Time

A d -dimensional stochastic vector process $X^\top(t) = (X_1(t), \dots, X_d(t))$ is called elementary Gaussian if it is stationary, Markov and Gaussian. In continuous time it is assumed that the process is of diffusion type. It is supposed that the process $X(t)$ is nondegenerate and it is linearly regular (or purely non-deterministic). Nondegeneracy means that the components of $X(t)$ are pointwise linearly independent.

$X(t)$ is called linearly regular if it has no deterministic component. In this case the Wold's expansion holds in discrete time, (see eg. Arató [3], Shirayev [30])

$$X(t) = \sum_{k=0}^{\infty} A_k \varepsilon(t-k),$$

where $\varepsilon(t)$ is d -dimensional process, $\mathbb{E}X(t) = 0$, $\text{Cov}(\varepsilon(t), \varepsilon(t))$ exists, $\text{Cov}(\varepsilon(t), \varepsilon(t-k)) = 0, k \geq 0$. Such an $\varepsilon(t)$ process will be called a white noise process.

The connection between stochastic difference equations and elementary processes can be characterized.

In the following we denote the time by n in the discrete time case.

Let ε_n be a d -dimensional Gaussian white noise process with parameters

$$\mathbb{E}\varepsilon_n = 0, \quad \text{Cov}(\varepsilon_n, \varepsilon_n) = B_\varepsilon,$$

where $\text{rank } B_\varepsilon \geq 1$. Let Q be a non-singular $d \times d$ matrix. Let us assume that the equation

$$B(0) = QB(0)Q^\top + B_\varepsilon \quad (2.1)$$

has a non-singular, symmetric, positive definite solution $B(0)$. The connection is the following, see Arató [3]:

Theorem 2.1.1. *The process X_n is a d -dimensional Gaussian process if and only if it is the solution of the following stochastic difference equation*

$$X_n = QX_{n-1} + \varepsilon_n, \quad n = 0, 1, 2, \dots, \quad (2.2)$$

in the following sense:

- (i) *Let ε_n be a d -dimensional Gaussian white noise (independent, identically distributed sequence of random vectors) with covariance matrix B_ε ($B_\varepsilon \neq 0, \mathbb{E}\varepsilon_n = 0$) and let Q be a non-singular $d \times d$ matrix with eigenvalues in the open unit disc. Then equation (2.2) has a unique regular stationary solution which is a Gaussian Markov process where $\text{Cov}(X_n, X_n) = B(0)$ is the solution of (2.1).*
- (ii) *Let X_n ($n = 0, \pm 1, \pm 2, \dots$) be a nondegenerate, linearly regular d -dimensional elementary Gaussian process with $\mathbb{E}X_n = 0$, $\text{Cov}(X_n, X_n) = B(0)$, $B(0) \neq 0$. Then there exists a nonsingular $d \times d$ matrix Q , with eigenvalues in the open unit disc and a sequence of independent, identically distributed Gaussian vectors ε_n such that equation (2.2) holds. $\mathbb{E}\varepsilon_n = 0$ and B_ε is uniquely determined by (2.1).*

2.1.1 Estimators for Different Models

Often the practitioner must determine the degree of the autoregressive process as well as estimate the parameters. If it is possible to specify a maximum for the degree of the process, a process of that degree can first be estimated and higher order terms discarded using standard regression statistics. Anderson (1962) gave a procedure for this decision problem. Various other model building methods based on regression theory can be used. In the dissertation we don't treat the question of model building. We are dealing with estimating the parameters of some specified model. The following model is one of the simplest and most heavily used models in time series analysis.

The vector AR(1) process is given by:

$$X_n = QX_{n-1} + \varepsilon_n, \quad n = 0, 1, 2, \dots \quad (2.3)$$

The d -dimensional random column vector ε_n contains the unobservable random innovation at time n . We study the case when the $d \times d$ matrix Q is the unknown parameter of the model.

The least squares estimator (LSE) of Q based on the observations $X_1, \dots, X_n$ is given by

$$\hat{Q}_n = \left(\sum_{k=1}^n X_k X_{k-1}^\top \right) \left(\sum_{k=1}^n X_{k-1} X_{k-1}^\top \right)^{-1}.$$

Let $\varrho(Q)$ denote the spectral radius of the matrix Q , i.e., the maximum of the absolute values of the eigenvalues of the matrix Q .

Asymptotically stationary models, when $\varrho(Q) < 1$, were studied by Mann and Wald [22] and Anderson [1].

Under the assumption that the ε_k 's are i.i.d. with $\mathbb{E}\varepsilon_k = 0$, $\mathbb{E}\varepsilon_k \varepsilon_k^\top = \Sigma$, the LSE of Q is asymptotically normal:

$$(\hat{Q}_n - Q) \left(\sum_{k=1}^n X_{k-1} X_{k-1}^\top \right)^{1/2} \xrightarrow{\mathcal{D}} \mathcal{N}_{d \times d}(0, I), \quad \text{as } n \rightarrow \infty,$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution and I is the unit matrix.

When $\varrho(Q) = 1$, the model is said to be unstable. The one-dimensional unstable AR(1) model $X_k = QX_{k-1} + \varepsilon_k$, $k \geq 1$, with $Q = 1$ was studied by White [33] and it was shown that the variables $n(\hat{Q} - Q)$ converge in law to a random variable:

$$n(\hat{Q} - Q) \xrightarrow{\mathcal{D}} \frac{\int_0^1 W(t) dW(t)}{\int_0^1 W^2(t) dt},$$

where $\{W(t), t \geq 0\}$ is a standard Wiener process. Multidimensional unstable models are studied in Sims, Stock and Watson [31], Tsay and Tiao [29], and in Arató [4].

In the explosive case:

$$\begin{cases} X_k = \alpha X_{k-1} + \varepsilon_k, & k = 1, 2, \dots, \\ X_0 = 0.2 \end{cases} \quad (2.4)$$

when $|\alpha| > 1$, the sequence $(\hat{\alpha}_n)_{n \geq 1}$ is again not asymptotically normal. If, for example, $\varepsilon_1 \sim \mathcal{N}(0, 1)$, then

$$\alpha^n (\hat{\alpha}_n - \alpha) \xrightarrow{\mathcal{D}} \text{Cauchy}(0, \alpha^2 - 1).$$

In general, the limit distribution depends on the distribution of ε_1 (White [33], Anderson [1]).

By another normalization,

$$\left(\sum_{j=1}^n X_{j-1}^2 \right)^{1/2} (\hat{\alpha}_n - \alpha) \xrightarrow{\mathcal{D}} \begin{cases} \mathcal{N}(0, 1), & \text{if } |\alpha| \neq 1, \\ \frac{\int_0^1 W(t) dW(t)}{\left(\int_0^1 W^2(t) dt \right)^{1/2}}, & \text{if } |\alpha| = 1, \end{cases}$$

which holds in case $|\alpha| > 1$ if $\varepsilon_1 \sim \mathcal{N}(0, 1)$.

These results led to the study of the following so-called nearly unstable models. Nearly unstable or nearly nonstationary multidimensional AR processes are generated according to the scheme

$$\begin{cases} X_k^{(n)} = Q_n X_{k-1}^{(n)} + \varepsilon_k^{(n)}, & k = 1, 2, \dots, n, \\ X_0^{(n)} = 0, \end{cases} \quad (2.5)$$

where $\{\varepsilon_k^{(n)}\}$ is an array of d -dimensional random vectors and Q_n , $n \geq 1$, is a sequence of $d \times d$ matrices such that $Q_n \rightarrow Q$, where Q is a matrix with $\rho(Q) = 1$.

The case when $Q_n = e^{A/n}$, $n \geq 1$, where A is a fixed $d \times d$ matrix was studied by Phillips [7]. Kormos and Pap [8] treated the case when $Q_n = e^{(\gamma I + A)/n}$, $n \geq 1$, where $\gamma \in \mathbb{R}$ and A is a skew-symmetric matrix under the assumption that ε_k -s are i.i.d. variables.

Stockmarr and Jacobsen [32] investigated the case when $Q_n = I + A/n$. Pap and Zuijlen [26] studied the case when $Q_n = e^{A_n/n} \cdot e^B$, $n \geq 1$, where $A_n \rightarrow A$, B is a known skew-symmetric $d \times d$ matrix, and $A_n B = B A_n$, $n \geq 1$. Pap and van Zuijlen [11] also studied the case when the model is complex-valued and the coefficient matrices are in Jordan normal form.

In the nearly unstable zero start AR(1) model

$$\begin{cases} X_k^{(n)} = \alpha^{(n)} X_{k-1}^{(n)} + \varepsilon_k^{(n)}, & k = 1, 2, \dots, \\ X_0^{(n)} = 0, \end{cases} \quad (2.6)$$

where

$$\alpha^{(n)} = 1 - \frac{\gamma^{(n)}}{n}, \quad \gamma^{(n)} \rightarrow \gamma \in \mathbb{R},$$

the LSE $\hat{\alpha}_n^{(n)}$ of $\alpha^{(n)}$ based on the observations $\{X_k^{(n)} : k = 1, 2, \dots, n\}$ has the asymptotic behaviour

$$n(\hat{\alpha}_n^{(n)} - \alpha^{(n)}) \xrightarrow{\mathcal{D}} \frac{\int_0^1 Y(t) dW(t)}{\int_0^1 Y^2(t) dt}, \quad (2.7)$$

where $\{Y(t) : t \in [0, 1]\}$ is a continuous time AR(1) process, i.e., an Ornstein-Uhlenbeck process, defined as the solution of the stochastic differential equation

$$\begin{cases} dY(t) = -\gamma Y(t) dt + dW(t), & t \geq 0, \\ Y(0) = 0, \end{cases} \quad (2.8)$$

(Bobkoski [9], Phillips [28], Chan and Wei [11]). Arató, Kolmogorov and Sinay [6] and Arató [2], [4] has drawn the attention to the connection between discrete and continuous time models. The result (2.7) can also be formulated as

$$\hat{\gamma}_n^{(n)} \xrightarrow{\mathcal{D}} \hat{\gamma},$$

where

$$\hat{\gamma}_n^{(n)} = -\frac{n \sum_{j=1}^n X_{j-1}^{(n)} (X_j^{(n)} - X_j^{(n)})}{\sum_{j=1}^n \left(X_{j-1}^{(n)}\right)^2}$$

is the LSE of $\gamma^{(n)}$ in the discrete time model (2.6) and

$$\hat{\gamma} = -\frac{\int_0^1 Y(t) dY(t)}{\int_0^1 Y^2(t) dt}$$

is the LSE of γ based on the observations $\{Y(t) : t \in [0, 1]\}$ in the continuous time model (2.8), which is also the maximum likelihood estimator (MLE) of γ (see [3], Meer, Pap, Zuijlen [23]). The distribution of $\hat{\gamma}$ is tabulated in Arató [3].

In the dissertation the nearly unstable shifted AR(1) model is also treated. The nearly unstable AR(1) model:

$$\begin{cases} X_k^{(n)} = \alpha^{(n)} X_{k-1}^{(n)} + \varepsilon_k^{(n)}, & k = 1, 2, \dots, \\ X_0^{(n)} = 0, \end{cases} \quad (2.9)$$

where

$$\alpha^{(n)} = 1 - \frac{\gamma^{(n)}}{n}, \quad \gamma^{(n)} \rightarrow \gamma \in \mathbb{R},$$

Now, consider the shifted model

$$Z_k^{(n)} := X_k^{(n)} + m^{(n)} h_k^{(n)}, \quad k = 1, 2, \dots, \quad n = 1, 2, \dots,$$

where $m^{(n)} \in \mathbb{R}$ is an unknown parameter and $h_k^{(n)}$, $k = 1, \dots, n$, $n = 1, 2, \dots$ are known constraints.

2.2 Processes in Continuous Time

The real observation are realized by discrete time. Nevertheless, the time-continuous model has its advantages. In many cases it is much better to work with the time-continuous model. Some phenomena can be described more adequately in that way, in other cases the result have more simpler form. The statistical questions of continuous time AR processes are studied in Arató [3].

Let $(W(t), F_t)$ be a d -dimensional standard Wiener process with local parameters $\mathbb{E}W(t) = 0$ (the drift is 0), $\mathbb{E}(W(t)W(t)^\top) = tI$ (with unit diffusion parameter). Let us consider the linear stochastic differential equation with the non-singular $d \times d$ matrix A and, may be singular, positive semidefinite matrix $B_W \neq 0$

$$dX(t) = AX(t)dt + B_W^{\frac{1}{2}}dW(t), \quad (2.10)$$

or in integral form

$$X(t) = X(t_0) + A \int_{t_0}^t X(s)ds + B_W^{\frac{1}{2}}(W(t) - W(t_0)),$$

where $X(t_0)$ is normally distributed and independent of $F_{[t_0, t]}^W$, $t \geq t_0$. In one dimension we can interpret this SDE as a the differential equation which is describing the motion of a particle under the effect of random collisions in the presence of friction, which is proportional to the speed of the particle. This is the so-called Ornstein-Uhlenbeck process. The following statement is true.

Theorem 2.2.1. *The continuous d -dimensional random process $X(t)$ is an elementary (i.e. stationary Markov) Gaussian process if and only if it is the solution of stochastic differential equation (2.10) in the following sense.*

- (i) *If $X(t)$ is a continuous, $\mathbb{E}X(t) = 0$, elementary Gaussian process then there exists a unique $d \times d$ matrix A with eigenvalues in the left halfplane and a Wiener process $(W(t), F_t^X) = 0$, $\mathbb{E}(W(t)W^\top(t)) = B_W t$, such that (2.10) holds and*

$$B(t) = \mathbb{E}X(s+t)X^\top(s) = e^{At}B(0), \quad t \geq 0, \quad (2.11)$$

where

$$AB(0) + B(0)A^\top = -B_W \quad (2.12)$$

with the solution

$$B(0) = \int_0^\infty e^{As}B_W e^{A^\top s}ds.$$

- (ii) Let $d \times d$ matrix A be a non-singular matrix with eigenvalues in the left halfplane, and B_W non-negative definite, then the only stationary regular solution of (2.10) with continuous $X(t)$, $-\infty < t < \infty$, is an elementary Gaussian process. Its covariance matrix function has the form 2.11 with $B(0)$ satisfying (2.12). In the case when $X(t)$ is defined for $t > 0$, $X(0)$ and F_t^W are independent, $X(0)$ is normally distributed with parameters 0, $B(0)$.

We can conclude from the above theorem that a Gaussian process is elementary if and only if its covariance matrix has the form

$$B(t, s) = \mathbb{E}X(t)X^\top(s) = \begin{cases} e^{A(t-s)}B_s, & t \geq s \\ B_t e^{A^\top(s-t)}, & t \leq s \end{cases}.$$

2.2.1 Radon-Nikodym Derivatives and MLE's

In the statistical investigation of elementary Gaussian processes with continuous time parameter, similarly to the statistics of independent observations, the maximum-likelihood principle plays an important role. For this purpose it is important to determine the Radon-Nikodym derivative of the measure generated by the process with respect to some standard measure. Elementary Gaussian processes with common diffusion matrix generate equivalent measures, and these measures are equivalent to the Wiener-measure with the same local matrix of variance, see in Liptser, Shirayev [21].

Let $C^d[0, T]$ be the metric space of d -dimensional vector-valued continuous functions on the interval $[0, T]$ with the uniform metric. It will be convenient to assume $C^d[0, T]$ as direct product of the space $C_z^d[0, T]$ of d -dimensional continuous functions $x = \{x(t), 0 \leq t \leq T\}$ with the initial condition $x(0) = z$ and the d -dimensional Euclidean space R^d .

We consider a Gaussian Markov process $X(t)$ satisfying the stochastic differential equation (2.10) and having $f(0)$ as initial probability density function. Let P_X be the probability measure on $C^d[0, T]$ generated by the above process $X(t)$ and P_W be the "conditional" product of the d -dimensional Lebesgue-measure and the measure generated by the Wiener process on the right hand side of (2.10).

Definition 2.2.2. For a $d \times d$ matrix A , we denote by $A^\ominus$ the **pseudo inverse** of A , which is the unique $d \times d$ matrix satisfying the following properties.

- (i) $AA^\ominus A = A$,
- (ii) There exist matrices U and V such that $A^\ominus = UA^\top = A^\top V$.

On contrary to the inverse of A , the pseudo inverse of A always exists.

Theorem 2.2.3. *If X satisfies the SDE (2.10) then the measures P_X and P_W are equivalent and their Radon-Nikodym derivative has the form*

$$\frac{dP_X}{dP_W}(x(t)) = f(x(0))e^{\int_0^T \langle Cx(t), dx(t) \rangle - \frac{1}{2} \int_0^T \langle Ax(t), Cx(t) \rangle dt}, \quad (2.13)$$

where $C = B_W^\ominus A$, and f is the density function of the initial value $X(0)$, that is the distribution of $X(0)$ is absolutely continuous.

The value of stochastic integral $\int_0^T \langle x(t), dx(t) \rangle$ can be determined for almost all realizations $x(t)$ with respect to P_W .

Formula (2.13) may also be written in the following form

$$\frac{dP_X}{dP_W}(x(t)) = f(x(0))e^{\text{Tr}[C \int_0^T x(t) dx^\top(t)] - \frac{1}{2} \text{Tr}[A^\top C \int_0^T x(t) x^\top(t)] dt},$$

where $\text{Tr}A$ means the trace of A .

In the time-continuous case we use the Radon-Nikodym derivative of X in the likelihood equation to determine the maximum likelihood estimate of some parameters and we can get the sufficient statistics as well (if they exist).

In the dissertation we are dealing with the estimation of the mean in multivariate continuous time AR processes. Arató in [3] studies the estimation of the mean in the one-dimensional real and complex models. He also studied the case in which both the damping parameter and the mean are unknown in the one-dimensional model. Here we summarize his results as a starting point of our study.

Consider a stationary AR(1) process $\{\tilde{X}(t) : t \in \mathbb{R}\}$ which is the weakly stationary solution of

$$d\tilde{X}(t) = -\alpha\tilde{X}(t)dt + dW(t), \quad t \in \mathbb{R}, \quad (2.14)$$

where $\{W(t) : t \in \mathbb{R}\}$ is a standard Wiener process (hence $\mathbb{E}W(t) = 0$, $\mathbb{E}W(t)^2 = t$), and $\alpha > 0$ is the damping parameter.

Let $\tilde{Z}(t) := \tilde{X}(t) + m$, $t \in \mathbb{R}$, where $m \in \mathbb{R}$ is an unknown parameter. Then, the maximum likelihood estimator (MLE) of m based on the observation of $\{\tilde{Z}(t) : t \in [T_1, T_2]\}$ is given by

$$\tilde{m} = \frac{\tilde{Z}(T_1) + \tilde{Z}(T_2) + \alpha \int_{T_1}^{T_2} \tilde{Z}(t) dt}{2 + (T_2 - T_1)\alpha} \sim \mathcal{N}\left(m, \frac{1}{2\alpha + (T_2 - T_1)\alpha^2}\right), \quad (2.15)$$

see, e.g., Arató [3].

Since the Radon-Nikodym derivative of the measure $\mathbb{P}_{\alpha,m}$ generated by the process $\tilde{Z}(t)$ with respect to the measure $\mathbb{P}_{\alpha,0}$ generated by $\tilde{X}(t)$ is the following:

$$\frac{d\mathbb{P}_{\alpha,m}}{d\mathbb{P}_{\alpha,0}} = \exp\{-\alpha(m(\tilde{Z}(T_1) + \tilde{Z}(T_2) + \alpha \int_{T_1}^{T_2} \tilde{Z}(t)dt) + m^2(1 + \frac{\alpha(T_2 - T_1)}{2}))\}.$$

We can calculate the Radon-Nikodym derivative of the shifted process with respect to original one by the help of the Ito's formula from (2.13).

We have the following asymptotic behaviour of the variance of $\tilde{m}$

$$\lim_{T_2-T_1 \rightarrow \infty} (T_2 - T_1) \text{Var } \tilde{m} = \frac{1}{\alpha^2}, \quad \lim_{\alpha \downarrow 0} \alpha \text{Var } \tilde{m} = \frac{1}{2}.$$

Especially,

$$\lim_{T_2-T_1 \rightarrow \infty} \text{Var } \tilde{m} = 0, \quad \lim_{\alpha \downarrow 0} \text{Var } \tilde{m} = \infty,$$

hence $\tilde{m}$ is asymptotically consistent as $T_2 - T_1 \rightarrow \infty$, although not uniformly in α . Furthermore, $\text{Var } \tilde{m}$ is unbounded as $\alpha \downarrow 0$.

The number of parameters can be reduced in the following way. We may suppose $T_1 = 0$ and $T_2 = T$ because of the stationarity. Moreover, let us consider the process $\{\tilde{\tilde{Z}}(t) := \tilde{Z}(Tt)/\sqrt{T} : t \in [0, 1]\}$. Then $\tilde{\tilde{Z}}(t) = \tilde{\tilde{X}}(t) + m/\sqrt{T}$, where $\tilde{\tilde{X}}(t) := \tilde{X}(Tt)/\sqrt{T}$, hence

$$d\tilde{\tilde{X}}(t) = -\kappa \tilde{\tilde{X}}(t) dt + dW(t), \quad t \in \mathbb{R},$$

where $\kappa := \alpha T$ is a new parameter. Considering $a := m/\sqrt{T}$ as a new parameter, we obtain that the MLE of a based on the observation of $\{\tilde{\tilde{Z}}(t) : t \in [0, 1]\}$ is given by

$$\tilde{a} = \frac{\tilde{\tilde{Z}}(0) + \tilde{\tilde{Z}}(1) + \kappa \int_0^1 \tilde{\tilde{Z}}(t) dt}{2 + \kappa} \sim \mathcal{N}\left(a, \frac{1}{2\kappa + \kappa^2}\right), \quad (2.16)$$

see Arató [3]. For the variance of $\tilde{a}$ we have

$$\lim_{\kappa \rightarrow \infty} \kappa^2 \text{Var } \tilde{a} = 1, \quad \lim_{\kappa \downarrow 0} \kappa \text{Var } \tilde{a} = \frac{1}{2}.$$

Especially,

$$\lim_{\kappa \rightarrow \infty} \text{Var } \tilde{a} = 0, \quad \lim_{\kappa \downarrow 0} \text{Var } \tilde{a} = \infty,$$

hence $\tilde{a}$ is asymptotically consistent as $\kappa \rightarrow \infty$, but $\text{Var} \tilde{a}$ is unbounded as $\kappa \downarrow 0$.

Arató in [3] also showed that if the damping parameter and the mean both are unknown in the one-dimensional model, they cannot be estimated well. In statistical investigations of independent random variables (observations) it is well known that if $X_1, X_2, \dots, X_n$ are normally distributed with parameters (m, σ^2) , when both of them are unknown, then with an arbitrary degree of confidence, a finite confidence interval can be constructed, e.g., by the help of t-statistics. This is not the situation in the case of a stationary Gaussian Markov process. In Arató [3] the following is stated. When the parameters m and κ of a stationary Gaussian Markov processes are unknown, it is impossible to construct finite confidence intervals for m using continuous functionals.

Chapter 3

Parameter Estimation with Exact Distribution for Multivariate AR Processes

3.1 MLE of the Multidimensional Case

The complex-valued stationary autoregressive process $\xi(t) = \xi_1(t) + i\xi_2(t)$, $t \geq 0$, can be given by the stochastic differential equation (SDE)

$$d\xi(t) = -\gamma\xi(t)dt + dw(t),$$

where $w(t) = w_1(t) + iw_2(t)$, $t \geq 0$ is a standard complex Wiener process (ie. $w_1(t)$ and $w_2(t)$ are independent standard real-valued Wiener processes) and $\gamma = \lambda - i\omega$ with $\lambda > 0$, $\omega \in \mathbb{R}$.

Consider the statistics

$$s_\xi^2(t) = \int_0^t |\xi(u)|^2 du, \quad r_\xi(t) = \int_0^t |\xi(u)|^2 d\theta(u),$$

where $\theta(t)$, $t \geq 0$ is defined by

$$\xi(t) = |\xi(t)| e^{i\theta(t)}.$$

The process

$$r_\xi(t) = \int_0^t (\xi_1(u)d\xi_2(u) - \xi_2(u)d\xi_1(u)), \quad t \geq 0, \quad (3.1)$$

is called Levy's stochastic area process. It is known that the maximum likelihood estimate (MLE) of the period ω is

$$\hat{\omega}_\xi(t) = \frac{r_\xi(t)}{s_\xi^2(t)},$$

and

$$s_\xi(t) (\hat{\omega}_\xi(t) - \omega) \stackrel{\mathcal{D}}{=} \mathcal{N}(0, 1) \quad \text{for all } t \geq 0,$$

where $\stackrel{\mathcal{D}}{=}$ denotes equality in distribution. Surprisingly, we have an exact distribution, not only an asymptotic property. This result was first formulated and applied in astronomy in Arató, Kolmogorov, Sinay [6]. Complicated proofs can be found in Novikov [24], Liptser and Shiryaev [21]. Recently Arató [4] gave an elegant new proof using Novikov's method.

The statement can be reformulated also in the following way. Let us consider the two-dimensional real-valued stationary autoregressive process $X(t)$, $t \geq 0$, given by the SDE

$$\begin{pmatrix} dX_1(t) \\ dX_2(t) \end{pmatrix} = \begin{pmatrix} -\lambda & -\omega \\ \omega & -\lambda \end{pmatrix} \begin{pmatrix} X_1(t)dt \\ X_2(t)dt \end{pmatrix} + \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix}, \quad (3.2)$$

where $W(t)$, $t \geq 0$, is a standard two-dimensional Wiener process, and $\lambda \geq 0$, $\omega \in \mathbb{R}$. Consider the statistics

$$s_X^2(t) = \int_0^t (X_1^2(u) + X_2^2(u)) du, \quad r_X(t) = \int_0^t (X_1(u) dX_2(u) - X_2(u) dX_1(u)).$$

The maximum likelihood estimate of the period ω is

$$\hat{\omega}_X(t) = \frac{r_X(t)}{s_X^2(t)},$$

and

$$s_X(t) (\hat{\omega}_X(t) - \omega) \stackrel{\mathcal{D}}{=} \mathcal{N}(0, 1) \quad \text{for all } t > 0.$$

The following natural question can be formulated. Consider the q -dimensional process $X(t)$, $t \geq 0$, given by the SDE

$$dX(t) = AX(t)dt + dW(t), \quad t \geq 0,$$

where $W(t)$, $t \geq 0$ is a standard q -dimensional Wiener process and A is a $q \times q$ matrix. Which conditions should be imposed on the matrix A and on the distribution of the initial value $X(0)$ in order that the suitably normalised MLE of some of its entries will

have exactly a normal distribution? This process is the so-called Ornstein-Uhlenbeck process and considered as a generalization of Wiener processes.

Pap [25] and Fazekas [14] found some examples for stationary multidimensional Ornstein-Uhlenbeck processes which have the above property. Pap and van Zuijlen gave a multidimensional generalization. We could weaken the condition of Pap and van Zuijlen by showing that a part of the conditions is superfluous in [27].

Consider the following multidimensional process:

$$dX(t) = \left(-\lambda I_d + \sum_{i=1}^m \omega_i C_i \right) X(t) dt + dW(t), \quad X(0) = 0 \quad (3.3)$$

where I_d is the $d \times d$ unit matrix, $\lambda, \omega_1, \dots, \omega_m \in \mathbb{R}$ are unknown parameters and $C_1, \dots, C_m$ are fixed skew-symmetric matrices.

The MLE of $\omega = (\omega_1, \dots, \omega_m)^\top$ is given by

$$\hat{\omega}_X(t) = \sigma_X^{-1}(t) r_X(t),$$

where $\sigma_X(t)$ is the $m \times m$ matrix

$$\sigma_X(t) = \left(\int_0^t \langle C_i X(s), C_j X(s) \rangle ds \right)_{(1 \leq i, j \leq m)},$$

and $r_X(t)$ is the m -dimensional column vector

$$r_X(t) = \left(\int_0^t \langle C_i X(s), dX(s) \rangle \right)_{(1 \leq i \leq m)}^\top.$$

In [27] it is proved that

$$\sigma_X^{1/2}(t)(\hat{\omega}_X(t) - \omega) \stackrel{\mathcal{D}}{=} \mathcal{N}(0, I_m) \quad \text{for all } t > 0, \quad (3.4)$$

if conditions (C1) – (C3) are satisfied, where

$$(C1) \quad C_i^\top = -C_i, \quad i = 1, \dots, m$$

$$(C2) \quad (C_i C_j + C_j C_i) C_k = C_k (C_i C_j + C_j C_i), \quad i, j, k = 1, \dots, m$$

$$(C3) \quad (C_i C_j + C_j C_i)(C_k C_l + C_l C_k) \in \mathcal{L}(C_u C_v, 1 \leq u, v \leq m), \quad i, j, k, l = 1, \dots, m$$

where $\mathcal{L}(C_u C_v, 1 \leq u, v \leq m)$ denotes the linear hull of the matrixes

$C_u C_v, 1 \leq u, v \leq m$.

We showed that condition (C3) is superfluous.

Theorem 3.1.1. *Let $X(t)$, $t \geq 0$, be the process given by (3.3). Let us suppose that the conditions (C1) and (C2) are satisfied. Then (3.4) holds. We need the following lemmas to prove the theorem.*

We shall make use of the following explicite formula which is a special case of Lemma 11.6 in [21].

Lemma 3.1.2. *Consider a standard d -dimensional Wiener process $W(t)$, $t \geq 0$. For all $t \geq 0$ let $B(t)$ and $Q(t)$ be $d \times d$ matrices such that $Q(t)$ is symmetric, positive semidefinite and*

$$\text{Tr} \int_0^T (B(t)B^\top(t) + Q(t)) dt < \infty. \quad (3.5)$$

Then

$$\begin{aligned} & \mathbb{E} \exp \left\{ - \int_0^T \left(\int_0^t B(s) dW(s) \right)^\top Q(t) \left(\int_0^t B(s) dW(s) \right) dt \right\} \\ &= \exp \left\{ \frac{1}{2} \text{Tr} \int_0^T B(t)B'(t)\Gamma(t) dt \right\}, \end{aligned}$$

where $\Gamma(t)$, $t \geq 0$ are negative semidefinite matrices determined by the Riccati differential equation

$$\dot{\Gamma}(t) = 2Q(t) - \Gamma(t)B(t)B(t)^\top\Gamma(t), \quad \Gamma(T) = 0.$$

Let us denote the cone of the symmetric, positive semidefinite $d \times d$ matrices by C_d . We shall also use that the distribution of a symmetric, positive semidefinite $d \times d$ random matrix is uniquely determined by the value of its Laplace transform on the cone C_d .

Lemma 3.1.3. *If σ is a random matrix with $\sigma^\top = \sigma$ and $\sigma \geq 0$ then the distribution of σ is uniquely determined by the Laplace transform $\Psi : C_d \rightarrow (0, \infty)$ given by*

$$\Psi(\alpha) := \mathbb{E} \exp \{ -\text{Tr}(\alpha^\top \sigma) \} = \mathbb{E} \exp \left\{ - \sum_{i=1}^d \sum_{j=1}^d \alpha_{ij} \sigma_{ij} \right\}, \quad \alpha \in C_d$$

Proof. First we prove that for $\alpha \in C_d$ we have $\text{Tr}(\alpha^\top \sigma) \geq 0$. It is well known that there is a symmetric, positive semidefinite matrix $\beta \in C_d$ such that $\alpha = \beta^2 = \beta^\top \beta$. The matrix $\beta\sigma\beta^\top$ is again symmetric and positive definite since

$$\langle \beta\sigma\beta^\top x, x \rangle = \langle \sigma(\beta^\top x), (\beta^\top x) \rangle \geq 0, \quad x \in \mathbb{R}^d.$$

Hence, indeed

$$\text{Tr}(\alpha^\top \sigma) = \text{Tr}(\beta^\top \beta \sigma) = \text{Tr}(\beta \sigma \beta^\top) \geq 0$$

For fixed $k \in \{1, \dots, d\}$ let us consider the matrix $\alpha^{(k)} \in C_d$ with entries

$$\alpha_{ij}^{(k)} = \begin{cases} 1 & \text{if } i = j = k, \\ 0 & \text{else.} \end{cases}$$

Then $\text{Tr}((\alpha^{(k\ell)})^\top \sigma) = \sigma_{kk} + 2\sigma_{k\ell} + \sigma_{\ell\ell}$.

Using the classical result on the Laplace transform of a random vector with non-negative coordinates we know that the joint distribution of the random variables

$$\sigma_{kk} : 1 \leq k \leq d \cup \{\sigma_{kk} + 2\sigma_{k\ell} + \sigma_{\ell\ell} : 1 \leq k \leq \ell \leq d\}, \quad (3.6)$$

is uniquely determined by the Laplace transform

$$\begin{aligned} & \varphi(s_k, 1 \leq k \leq d, s_{k\ell}, 1 \leq k \leq \ell \leq d) \\ &:= \mathbb{E} \exp \left\{ - \sum_{k=1}^d s_k \sigma_{kk} - \sum_{1 \leq k \leq \ell \leq d} s_{k\ell} \sigma_{k\ell} \right\}, \quad s_k, s_{k\ell} \geq 0. \end{aligned}$$

Clearly

$$\begin{aligned} & \varphi(s_k, 1 \leq k \leq d, s_{k\ell}, 1 \leq k \leq \ell \leq d) \\ &= \mathbb{E} \exp \left\{ \sum_{k=1}^d s_k \text{Tr}((\alpha^{(k)})^\top \sigma) - \sum_{1 \leq k \leq \ell \leq d} \text{Tr}((\alpha^{(k\ell)})^\top \sigma) \right\} \\ &= \mathbb{E} \exp \{-\text{Tr}(\alpha^\top \Sigma)\} = \Psi(\alpha), \end{aligned}$$

where

$$\alpha = \sum_{k=1}^d s_k \alpha^{(k)} + \sum_{1 \leq k \leq \ell \leq d} s_{k\ell} \alpha^{k\ell} \in C_d.$$

Consequently the joint distribution of the random variables in (3.6) is uniquely determined by the Laplace transform $\Psi : C_d \rightarrow (0, \infty)$ of the random matrix σ , hence, the joint distribution of the entries of the matrix σ is also uniquely determined by $\Psi : C_d \rightarrow (0, \infty)$ since there is a one-to-one correspondence between the entries of σ and the random variables in (3.6).

3.2 Proof of the Theorem

The proof can be carried out as in [27]. We have to show only that for all $T > 0$ the distribution of the symmetric, positive semidefinite random matrix $\sigma_X(T)$ does not

depend on the parameter $\omega = (\omega, \dots, \omega_m)^\top$. Using Lemma (3.1.3.) it is sufficient to show that the Laplace transform

$$\Psi_T(\alpha) = \mathbb{E} \exp \left\{ - \sum_{i,j=1}^m \alpha_{i,j} \int_0^\top \langle C_i X(t), C_j X(t) \rangle dt \right\}, \quad \alpha \in C_d,$$

does not depend on the parameter ω . Using the notation

$$C := \sum_{i=1}^m \sum_{j=1}^m \alpha_{ij} C_i^\top C_j,$$

we have

$$\Psi_T(\alpha) = \mathbb{E} \exp \left\{ - \int_0^T X(t)^\top C X(t) dt \right\}.$$

Next we show that C is a symmetric, positive semidefinite matrix. We use again that there exists a matrix $\beta \in C_d$ such that $\alpha = \beta^2 = \beta^\top \beta$, hence $\alpha_{ij} = \sum_{k=1}^d \beta_{ki} \beta_{kj}$. We have

$$\langle Cx, x \rangle = \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^d \beta_{ki} \beta_{kj} \langle C_i^\top C_j x, x \rangle = \sum_{k=1}^d \left| \sum_{i=1}^m \beta_{ki} C_i x \right|^2 \geq 0,$$

thus $C \in C_d$, indeed.

Let

$$A = -\lambda I_d + \sum_{i=1}^m \omega_i C_i.$$

It is known that the solution $X(t)$, $t \geq 0$, of the SDE(1) can be represented in the form

$$X(t) = \int_0^t e^{(t-s)A} dW(s),$$

Consequently,

$$\int_0^T X(t)^\top C X(t) dt = \int_0^T \left(\int_0^t e^{-sA} dW(s) \right)^\top e^{(tA^\top)} C e^{(tA)} \left(\int_0^t e^{-sA} dW(s) \right) dt.$$

We will show that Lemma (3.1.2.) can be applied with $B(t) = e^{-tA}$ and $Q(t) = e^{tA^\top} C e^{tA}$. Clearly the condition (C1) and (C2) imply

$$B(t)B(t)^\top = e^{2\lambda t} I_d$$

and $AC = CA$, hence

$$Q(t) = Ce^{ta^\top} e^{ta} = e^{2\lambda t} C,$$

and we concluded the validity of the condition (3.5). Applying Lemma 3.1.2. and using the above formulae we obtain

$$\Psi_T(\alpha) = \exp \left\{ \frac{1}{2} \text{Tr} \int_0^T e^{2\lambda t} \Gamma(t) dt \right\}, \quad \alpha \in C_d,$$

where $\Gamma(t)$, $t \geq 0$, is defined by

$$\dot{\Gamma}(t) = 2e^{-2\lambda t} C - e^{2\lambda t} \Gamma^2(t), \quad \Gamma(T) = 0.$$

Consequently the Laplace transform Ψ_T does not depend on the parameter ω and the proof is completed.

3.3 Special Cases

We give some application of the Theorem.

Corollary 1. *Consider the d -dimensional Ornstein–Uhlenbeck process $X(t)$, $t \geq 0$, given by*

$$dX(t) = \left(-\lambda I + \sum_{i=1}^m \omega_i C_i \right) X(t) dt + dW(t), \quad X(0) = 0,$$

where

$$C_i^\top = -C_i, \quad i = 1, \dots, m,$$

$$C_i C_j = -C_j C_i, \quad 1 \leq i < j \leq m.$$

Then the maximum likelihood estimators of the parameters $\omega_1, \dots, \omega_m$ are given by

$$\hat{\omega}_X^{(i)}(t) = \frac{r_X^{(i)}(t)}{(s_X^{(i)}(t))^2},$$

where

$$r_X^{(i)}(t) = \int_0^t \langle C_i X(s), dX(s) \rangle, \quad s_X^{(i)}(t) = \int_0^t \langle C_i X(s), C_i X(s) \rangle ds,$$

and

$$\left(\sqrt{s_X^{(1)}(t)} \left(\hat{\omega}_X^{(1)} - \omega_1 \right), \dots, \sqrt{s_X^{(m)}(t)} \left(\hat{\omega}_X^{(m)} - \omega_m \right) \right) \stackrel{\mathcal{D}}{=} \mathcal{N}_m(0, I) \quad \text{for all } t > 0.$$

Corollary 2. *Consider the d -dimensional process $X(t)$, $t \geq 0$, given by*

$$dX(t) = (-\lambda I + \omega C)X(t)dt + dW(t), \quad X(0) = 0,$$

where $C^\top = -C$.

Then the maximum likelihood estimator of the parameter ω is

$$\hat{\omega}_X(t) = \frac{r_X(t)}{s_X^2(t)},$$

where

$$r_X(t) = \int_0^t \langle CX(s), dX(s) \rangle, \quad s_X^2(t) = \int_0^t |CX(s)|^2 ds,$$

and

$$s_X(t) (\hat{\omega}_X(t) - \omega) \stackrel{\mathcal{D}}{=} \mathcal{N}(0, 1) \quad \text{for all } t > 0.$$

Chapter 4

Nearly Unstable AR Models

4.1 The Shifted AR(1) Process

Let $\hat{m}_{T_1, T_2}^{(n)}$ be the LSE of $m^{(n)}$ based on the observations $\{Z_k^{(n)} : [T_1 n] \leq k \leq [T_2 n]\}$, where $0 \leq T_1 < T_2$. Our purpose is to investigate the limit behaviour of $\hat{m}_{T_1, T_2}^{(n)}$ as $n \rightarrow \infty$, and the connection between $\hat{m}_{T_1, T_2}^{(n)}$ and the MLE $\hat{m}_{T_1, T_2}$ of m based on the observations $\{Z(t) : t \in [T_1, T_2]\}$, where $Z(t) := Y(t) + mh(t)$ is the shifted Ornstein-Uhlenbeck process with some appropriate function h . We discuss these questions concerning nearly stable, nearly unstable and nearly explosive models, as well. It turns out that $\hat{m}_{T_1, T_2}^{(n)}$ is always asymptotically normal, but the speed of convergence is much less in the nearly unstable case. The stationary case will be also treated, when $\hat{m}_{T_1, T_2}^{(n)}$ might not be asymptotically normal if $\gamma^{(n)} \rightarrow 0$.

4.1.1 Case of Zero Start AR Model

First, consider a shifted zero start Ornstein-Uhlenbeck process $\{Z(t) : t \geq 0\}$ given by

$$\begin{cases} dY(t) = -\gamma Y(t) dt + dW(t), & t \geq 0, \\ Y(0) = 0, \\ Z(t) = Y(t) + mh(t), & t \geq 0, \end{cases}$$

where $\gamma \in \mathbb{R}$ is a known parameter, $h : [0, \infty) \rightarrow \mathbb{R}$ is a known function and $m \in \mathbb{R}$ is an unknown parameter.

Let $0 \leq T_1 < T_2$. Baran, Pap and Zuijlen [7] proved the following result on the MLE of m . Denote by $\mathbb{P}_Z$ and $\mathbb{P}_Y$ the measures generated on $C([T_1, T_2] \rightarrow \mathbb{R})$ by the processes Z and Y , respectively.

If h is twice continuously differentiable with $h(0) = 0$, then the measures $\mathbb{P}_Z$ and $\mathbb{P}_Y$ are equivalent and the MLE of m based on the observations $\{Z(t) : t \in [T_1, T_2]\}$ has the form

$$\hat{m}_{T_1, T_2} = \frac{\hat{\zeta}_{T_1, T_2}}{\hat{A}_{T_1, T_2}},$$

where

$$\hat{A}_{T_1, T_2} = \begin{cases} \gamma \coth(\gamma T_1) h(T_1)^2 + \gamma h(T_2)^2 + \int_{T_1}^{T_2} (\gamma^2 h(t)^2 + h'(t)^2) dt, \\ \text{if } \gamma \neq 0, T_1 \neq 0, \\ \frac{h(T_1)^2}{T_1} + \int_{T_1}^{T_2} h'(t)^2 dt, & \text{if } \gamma = 0, T_1 \neq 0, \\ \gamma h(T_2)^2 + \int_0^{T_2} (\gamma^2 h(t)^2 + h'(t)^2) dt, & \text{if } T_1 = 0, \end{cases}$$

$$\hat{\zeta}_{T_1, T_2} = \begin{cases} (\gamma \coth(\gamma T_1) h(T_1) - h'(T_1)) Z(T_1) + (\gamma h(T_2) + h'(T_2)) Z(T_2) \\ + \int_{T_1}^{T_2} (\gamma^2 h(t) - h''(t)) Z(t) dt, & \text{if } \gamma \neq 0, T_1 \neq 0, \\ \frac{h(T_1) Z(T_1)}{T_1} + h'(T_2) Z(T_2) - h'(T_1) Z(T_1) \\ - \int_{T_1}^{T_2} h''(t) Z(t) dt, & \text{if } \gamma = 0, T_1 \neq 0, \\ (\gamma h(T_2) + h'(T_2)) Z(T_2) + \int_0^{T_2} (\gamma^2 h(t) - h''(t)) Z(t) dt, & \text{if } T_1 = 0. \end{cases}$$

Moreover, $\hat{m}_{T_1, T_2}$ is normally distributed with mean m and variance $1/\hat{A}_{T_1, T_2}$. We note that the assumption $h(0) = 0$ is needed because if $h(0) \neq 0$ and $m \neq 0$, then the measures $\mathbb{P}_Z$ and $\mathbb{P}_Y$ are singular with respect to each other, since $Y(0) = 0$ and $Z(0) = mh(0) \neq 0$.

Next, consider a sequence of shifted zero start AR(1) models

$$\begin{cases} X_k^{(n)} = \alpha^{(n)} X_{k-1}^{(n)} + \varepsilon_k^{(n)}, & k = 1, 2, \dots, \\ X_0^{(n)} = 0, \\ Z_k^{(n)} = X_k^{(n)} + m^{(n)} h_k^{(n)}, & k = 0, 1, 2, \dots, \end{cases}$$

for $n = 1, 2, \dots$, where $\{\alpha^{(n)} : n = 1, 2, \dots\}$ and $\{h_k^{(n)} : n = 1, 2, \dots; k = 0, 1, 2, \dots\}$ are known and $\{m^{(n)} : n = 1, 2, \dots\}$ are unknown parameters.

The LSE $\widehat{m}_{T_1, T_2}^{(n)}$ of the parameter $m^{(n)}$ based on the observations $\{Z_k^{(n)} : [T_1 n] \leq k \leq [T_2 n]\}$, where $0 \leq T_1 < T_2$, can be obtained by minimizing the sum of squares

$$\frac{\left(Z_{[T_1 n]}^{(n)} - m^{(n)} h_{[T_1 n]}^{(n)}\right)^2}{\text{Var } Z_{[T_1 n]}^{(n)}} + \sum_{j=[T_1 n]+1}^{[T_2 n]} \left(Z_j^{(n)} - m^{(n)} h_j^{(n)} - \alpha^{(n)} (Z_{j-1}^{(n)} - m^{(n)} h_{j-1}^{(n)})\right)^2 \quad (4.1)$$

if $T_1 \neq 0$, and by minimizing the sum of squares

$$\sum_{j=1}^{[T_2 n]} \left(Z_j^{(n)} - m^{(n)} h_j^{(n)} - \alpha^{(n)} (Z_{j-1}^{(n)} - m^{(n)} h_{j-1}^{(n)})\right)^2 \quad (4.2)$$

if $T_1 = 0$.

We can observe that

$$X_k^{(n)} = \sum_{j=1}^k \left(\alpha^{(n)}\right)^{k-j} \varepsilon_j^{(n)},$$

hence,

$$\text{Var } Z_k^{(n)} = \text{Var } X_k^{(n)} = \sum_{j=1}^k \left(\alpha^{(n)}\right)^{2(k-j)} = \begin{cases} \frac{1 - (\alpha^{(n)})^{2k}}{1 - (\alpha^{(n)})^2} & \text{if } |\alpha^{(n)}| \neq 1, \\ k & \text{if } |\alpha^{(n)}| = 1. \end{cases}$$

Consequently,

$$\widehat{m}_{T_1, T_2}^{(n)} = \frac{\widehat{\zeta}_{T_1, T_2}^{(n)}}{\widehat{A}_{T_1, T_2}^{(n)}},$$

where

$$\widehat{A}_{T_1, T_2}^{(n)} = \begin{cases} \frac{\left(1 - (\alpha^{(n)})^2\right) \left(h_{[T_1 n]}^{(n)}\right)^2}{1 - (\alpha^{(n)})^{2[T_1 n]}} + \sum_{j=[T_1 n]+1}^{[T_2 n]} (\overline{\Delta} h_j^{(n)})^2, \\ \text{if } |\alpha^{(n)}| \neq 1, \quad T_1 \neq 0, \\ \\ \frac{\left(h_{[T_1 n]}^{(n)}\right)^2}{[T_1 n]} + \sum_{j=[T_1 n]+1}^{[T_2 n]} (\overline{\Delta} h_j^{(n)})^2 & \text{if } |\alpha^{(n)}| = 1, \quad T_1 \neq 0, \\ \\ \sum_{j=1}^{[T_2 n]} (\overline{\Delta} h_j^{(n)})^2 & \text{if } T_1 = 0, \end{cases}$$

$$\widehat{\zeta}_{T_1, T_2}^{(n)} = \begin{cases} \frac{\left(1 - (\alpha^{(n)})^2\right) h_{[T_1 n]}^{(n)} Z_{[T_1 n]}^{(n)}}{1 - (\alpha^{(n)})^{2[T_1 n]}} + \sum_{j=[T_1 n]+1}^{[T_2 n]} \overline{\Delta} h_j^{(n)} \overline{\Delta} Z_j^{(n)}, \\ \text{if } |\alpha^{(n)}| \neq 1, \quad T_1 \neq 0, \\ \\ \frac{h_{[T_1 n]}^{(n)} Z_{[T_1 n]}^{(n)}}{[T_1 n]} + \sum_{j=[T_1 n]+1}^{[T_2 n]} \overline{\Delta} h_j^{(n)} \overline{\Delta} Z_j^{(n)}, & \text{if } |\alpha^{(n)}| = 1, \quad T_1 \neq 0, \\ \\ \sum_{j=1}^{[T_2 n]} \overline{\Delta} h_j^{(n)} \overline{\Delta} Z_j^{(n)}, & \text{if } T_1 = 0, \end{cases}$$

with the notation

$$\overline{\Delta} h_j^{(n)} := h_j^{(n)} - \alpha^{(n)} h_{j-1}^{(n)}, \quad \overline{\Delta} Z_j^{(n)} := Z_j^{(n)} - \alpha^{(n)} Z_{j-1}^{(n)}.$$

We remark that $\widehat{m}_{T_1, T_2}^{(n)}$ has mean $m^{(n)}$ and variance $1/\widehat{A}_{T_1, T_2}^{(n)}$.

The next statement describes the asymptotic behaviour of $m^{(n)}$ and the connection between discrete and continuous time models.

Theorem 4.1.1. *Suppose that $h_k^{(n)} = h\left(\frac{k}{n}\right)$ for $n = 1, 2, \dots$, $k = 0, 1, 2, \dots$ with some twice continuously differentiable function $h : [0, \infty) \rightarrow \mathbb{R}$ with $h(0) = 0$. Suppose that for all n , $\{\varepsilon_j^{(n)} : j = 1, \dots, n\}$ are i.i.d., mean zero and variance 1.*

(i) If $n(1 - \alpha^{(n)}) \rightarrow \gamma \in \mathbb{R}$, then

$$\frac{\widehat{m}_{T_1, T_2}^{(n)} - m^{(n)}}{\sqrt{n}} \xrightarrow{\mathcal{D}} \widehat{m}_{T_1, T_2} - m \stackrel{\mathcal{D}}{=} \mathcal{N}\left(0, \frac{1}{\widehat{A}_{T_1, T_2}}\right).$$

(ii) If $n(1 - \alpha^{(n)}) \rightarrow \pm\infty$, then

$$\sqrt{n}(1 - \alpha^{(n)})(\widehat{m}_{T_1, T_2}^{(n)} - m^{(n)}) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{1}{\int_{T_1}^{T_2} h(t)^2 dt}\right).$$

Proof. (i). First, suppose that $T_1 \neq 0$, and for all sufficiently large $n \in \mathbb{N}$, $\alpha^{(n)} \neq 1$. We investigate the asymptotic behaviour of $\widehat{A}_{T_1, T_2}^{(n)}$ which can be written in the form

$$\begin{aligned} \widehat{A}_{T_1, T_2}^{(n)} = & \left(\frac{1 - (\alpha^{(n)})^2}{1 - (\alpha^{(n)})^{2[T_1 n]}} - \alpha^{(n)}(1 - \alpha^{(n)}) \right) \left(h_{[T_1 n]}^{(n)} \right)^2 + (1 - \alpha^{(n)}) \left(h_{[T_2 n]}^{(n)} \right)^2 \\ & + (1 - \alpha^{(n)})^2 \sum_{j=[T_1 n]+1}^{[T_2 n]} \left(h_j^{(n)} \right)^2 + \alpha^{(n)} \sum_{j=[T_1 n]+1}^{[T_2 n]} (h_j^{(n)} - h_{j-1}^{(n)})^2. \end{aligned} \quad (4.3)$$

We will prove that $n\widehat{A}_{T_1, T_2}^{(n)} \rightarrow \widehat{A}_{T_1, T_2}$ as $n \rightarrow \infty$. We have

$$n(1 - \alpha^{(n)}) \rightarrow \gamma, \quad \alpha^{(n)} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Moreover,

$$(\alpha^{(n)})^{2[T_1 n]} = \left(\left(1 - \frac{n(1 - \alpha^{(n)})}{n} \right)^n \right)^{2[T_1 n]/n} \rightarrow e^{-2\gamma T_1}.$$

If $\gamma \neq 0$ then

$$n \cdot \frac{1 - (\alpha^{(n)})^2}{1 - (\alpha^{(n)})^{2[T_1 n]}} = \frac{n(1 - \alpha^{(n)})(1 + \alpha^{(n)})}{1 - (\alpha^{(n)})^{2[T_1 n]}} \rightarrow \frac{2\gamma}{1 - e^{-2\gamma T_1}}.$$

If $\gamma = 0$ then

$$\frac{1 - (\alpha^{(n)})^{2[T_1 n]}}{n(1 - (\alpha^{(n)})^2)} = \frac{1}{n} \sum_{j=0}^{[T_1 n]-1} (\alpha^{(n)})^{2j} \rightarrow T_1.$$

Consequently,

$$n \cdot \frac{1 - (\alpha^{(n)})^2}{1 - (\alpha^{(n)})^{2[T_1 n]}} \rightarrow \begin{cases} \frac{2\gamma}{1 - e^{-2\gamma T_1}} & \text{if } \gamma \neq 0, \\ \frac{1}{T_1} & \text{if } \gamma = 0. \end{cases}$$

Clearly, we have for all $T \in [T_1, T_2]$,

$$h_{[Tn]}^{(n)} = h\left(\frac{[Tn]}{n}\right) \rightarrow h(T).$$

Moreover,

$$\begin{aligned} \frac{1}{n} \sum_{j=[T_1n]+1}^{[T_2n]} \left(h_j^{(n)}\right)^2 &\rightarrow \int_{T_1}^{T_2} h(t)^2 dt, \\ n \sum_{j=[T_1n]+1}^{[T_2n]} (h_j^{(n)} - h_{j-1}^{(n)})^2 &\rightarrow \int_{T_1}^{T_2} h'(t)^2 dt, \end{aligned}$$

hence we obtain $n\hat{A}_{T_1, T_2}^{(n)} \rightarrow \hat{A}_{T_1, T_2}$.

Now, we investigate the asymptotic behaviour of $\hat{\zeta}_{T_1, T_2}^{(n)}$ which can be written in the form

$$\begin{aligned} \hat{\zeta}_{T_1, T_2}^{(n)} &= m^{(n)} \hat{A}_{T_1, T_2}^{(n)} + \left(\frac{1 - (\alpha^{(n)})^2}{1 - (\alpha^{(n)})^{2[T_1n]}} - \alpha^{(n)}(1 - \alpha^{(n)}) \right) h_{[T_1n]}^{(n)} X_{[T_1n]}^{(n)} \\ &\quad + (1 - \alpha^{(n)}) h_{[T_2n]}^{(n)} X_{[T_2n]}^{(n)} + (1 - \alpha^{(n)})^2 \sum_{j=[T_1n]+1}^{[T_2n]-1} h_j^{(n)} X_j^{(n)} \\ &\quad - \alpha^{(n)} \sum_{j=[T_1n]+1}^{[T_2n]-1} (h_{j+1}^{(n)} - 2h_j^{(n)} + h_{j-1}^{(n)}) X_j^{(n)} \\ &\quad + \alpha^{(n)} \left(h_{[T_2n]}^{(n)} - h_{[T_2n]-1}^{(n)} \right) X_{[T_2n]}^{(n)} - \alpha^{(n)} \left(h_{[T_1n]+1}^{(n)} - h_{[T_1n]}^{(n)} \right) X_{[T_1n]}^{(n)}. \end{aligned}$$

The random step functions

$$Y^{(n)}(t) := \frac{1}{\sqrt{n}} X_{[nt]}^{(n)}, \quad t \in [T_1, T_2],$$

can be considered as random elements in the Skorokhod space $D([T_1, T_2] \rightarrow \mathbb{R})$. It is known (see, e.g., Pap and Zuijlen [26]) that

$$Y^{(n)} \rightarrow Y, \quad \text{in } D([T_1, T_2] \rightarrow \mathbb{R}).$$

Particularly, for all $T \in [T_1, T_2]$,

$$\frac{1}{\sqrt{n}} X_{[Tn]}^{(n)} \xrightarrow{\mathcal{D}} Y(T).$$

By the Continuous Mapping Theorem we obtain

$$\begin{aligned} \frac{1}{n^{3/2}} \sum_{j=[T_1 n]+1}^{[T_2 n]-1} h_j^{(n)} X_j^{(n)} &\xrightarrow{\mathcal{D}} \int_{T_1}^{T_2} h(t) Y(t) dt, \\ \sqrt{n} \sum_{j=[T_1 n]+1}^{[T_2 n]-1} (h_{j+1}^{(n)} - 2h_j^{(n)} + h_{j-1}^{(n)}) X_j^{(n)} &\xrightarrow{\mathcal{D}} \int_{T_1}^{T_2} h''(t) Y(t) dt, \end{aligned}$$

hence, we get

$$\begin{aligned} \sqrt{n} \left(\widehat{\zeta}_{T_1, T_2}^{(n)} - m^{(n)} \widehat{A}_{T_1, T_2}^{(n)} \right) &\xrightarrow{\mathcal{D}} \begin{cases} \gamma \coth(\gamma T_1) h(T_1) Y(T_1) + \gamma h(T_2) Y(T_2) \\ + h'(T_2) Y(T_2) - h'(T_1) Y(T_1) \\ + \int_{T_1}^{T_2} (\gamma^2 h(t) - h''(t)) Y(t) dt, & \text{if } \gamma \neq 0, \\ \frac{h(T_1) Y(T_1)}{T_1} + h'(T_2) Y(T_2) \\ - h'(T_1) Y(T_1) - \int_{T_1}^{T_2} h''(t) Y(t) dt, & \text{if } \gamma = 0, \end{cases} \\ &= \widehat{\zeta}_{T_1, T_2} - m \widehat{A}_{T_1, T_2}. \end{aligned}$$

Finally, we conclude

$$\begin{aligned} \frac{\widehat{m}_{T_1, T_2}^{(n)} - m^{(n)}}{\sqrt{n}} &= \frac{\sqrt{n} \left(\widehat{\zeta}_{T_1, T_2}^{(n)} - m^{(n)} \widehat{A}_{T_1, T_2}^{(n)} \right)}{n \widehat{A}_{T_1, T_2}^{(n)}} \\ &\xrightarrow{\mathcal{D}} \frac{\zeta_{T_1, T_2} - m \widehat{A}_{T_1, T_2}}{\widehat{A}_{T_1, T_2}} = \widehat{m}_{T_1, T_2} - m \stackrel{\mathcal{D}}{=} \mathcal{N} \left(0, \frac{1}{\widehat{A}_{T_1, T_2}} \right). \end{aligned}$$

Next, we suppose that $T_1 \neq 0$, and for all sufficiently large $n \in \mathbb{N}$, $\alpha^{(n)} = 1$. Then, we obviously have

$$n \widehat{A}_{T_1, T_2}^{(n)} = \frac{\left(h_{[T_1 n]}^{(n)} \right)^2}{[T_1 n]/n} + n \sum_{j=[T_1 n]+1}^{[T_2 n]} (h_j^{(n)} - h_{j-1}^{(n)})^2 \rightarrow \frac{h(T_1)^2}{T_1} + \int_{T_1}^{T_2} h'(t)^2 dt = \widehat{A}_{T_1, T_2}.$$

Moreover,

$$\widehat{\zeta}_{T_1, T_2}^{(n)} = m^{(n)} \widehat{A}_{T_1, T_2}^{(n)} + \frac{h_{[T_1 n]}^{(n)} X_{[T_1 n]}^{(n)}}{[T_1 n]} + \sum_{j=[T_1 n]+1}^{[T_2 n]} (h_j^{(n)} - h_{j-1}^{(n)}) (X_j^{(n)} - X_{j-1}^{(n)}),$$

hence,

$$\begin{aligned} \sqrt{n} \left(\zeta_{T_1, T_2}^{(n)} - m^{(n)} \hat{A}_{T_1, T_2}^{(n)} \right) &\xrightarrow{\mathcal{D}} \frac{h(T_1)Y(T_1)}{T_1} + h'(T_2)Y(T_2) - h'(T_1)Y(T_1) \\ &= - \int_{T_1}^{T_2} h''(t)Y(t) dt \hat{\zeta}_{T_1, T_2} - m \hat{A}_{T_1, T_2}, \end{aligned}$$

and we can conclude again that

$$\frac{\hat{m}_{T_1, T_2}^{(n)} - m^{(n)}}{\sqrt{n}} \xrightarrow{\mathcal{D}} \hat{m}_{T_1, T_2} - m \stackrel{\mathcal{D}}{=} \mathcal{N} \left(0, \frac{1}{\hat{A}_{T_1, T_2}} \right).$$

The case $T_1 = 0$ can be handled in a similar way.

(ii). Let us start with the case $T_1 \neq 0$. Now, we have

$$\frac{\hat{A}_{T_1, T_2}^{(n)}}{n(1 - \alpha^{(n)})^2} \rightarrow \int_{T_1}^{T_2} h(t)^2 dt \quad \text{as } n \rightarrow \infty,$$

since we can use again the representation (4.3) of $\hat{A}_{T_1, T_2}^{(n)}$ and

$$(\alpha^{(n)})^{2[T_1 n]} = \left(\left(1 - \frac{n(1 - \alpha^{(n)})}{n} \right)^n \right)^{2[T_1 n]/n} \rightarrow \begin{cases} +\infty, & \text{if } n(1 - \alpha^{(n)}) \rightarrow -\infty, \\ 0, & \text{if } n(1 - \alpha^{(n)}) \rightarrow +\infty. \end{cases}$$

Next, we write $\hat{\zeta}_{T_1, T_2}^{(n)}$ in the form

$$\hat{\zeta}_{T_1, T_2}^{(n)} = m^{(n)} \hat{A}_{T_1, T_2}^{(n)} + \frac{(1 - (\alpha^{(n)})^2) h_{[T_1 n]}^{(n)} X_{[T_1 n]}^{(n)}}{1 - (\alpha^{(n)})^{2[T_1 n]}} + \sum_{j=[T_1 n]+1}^{[T_2 n]} (h_j^{(n)} - \alpha^{(n)} h_{j-1}^{(n)}) \varepsilon_j^{(n)}.$$

We will prove that

$$\frac{\hat{\zeta}_{T_1, T_2}^{(n)} - m^{(n)} \hat{A}_{T_1, T_2}^{(n)}}{\sqrt{n}(1 - \alpha^{(n)})} \xrightarrow{\mathcal{D}} \mathcal{N} \left(0, \int_{T_1}^{T_2} h(t)^2 dt \right). \quad (4.4)$$

Clearly

$$U_{T_1, T_2}^{(n)} := \frac{(1 - (\alpha^{(n)})^2) h_{[T_1 n]}^{(n)} X_{[T_1 n]}^{(n)}}{\sqrt{n}(1 - \alpha^{(n)}) (1 - (\alpha^{(n)})^{2[T_1 n]})} \xrightarrow{\mathbb{L}^2} 0,$$

since $\mathbb{E} U_{T_1, T_2}^{(n)} = 0$ and

$$\text{Var } U_{T_1, T_2}^{(n)} = \frac{(1 + \alpha^{(n)}) (h_{[T_1 n]})^2}{n(1 - \alpha^{(n)}) (1 - (\alpha^{(n)})^{2[T_1 n]})} \rightarrow 0.$$

By the Central Limit Theorem,

$$\frac{1}{\sqrt{n}(1 - \alpha^{(n)})} \sum_{j=[T_1 n]+1}^{[T_2 n]} (h_j^{(n)} - \alpha^{(n)} h_{j-1}^{(n)}) \varepsilon_j^{(n)} \xrightarrow{\mathcal{D}} \mathcal{N} \left(0, \int_{T_1}^{T_2} h(t)^2 dt \right),$$

since

$$\begin{aligned} & \text{Var} \left(\frac{1}{\sqrt{n}(1 - \alpha^{(n)})} \sum_{j=[T_1 n]+1}^{[T_2 n]} (h_j^{(n)} - \alpha^{(n)} h_{j-1}^{(n)}) \varepsilon_j^{(n)} \right) \\ &= \frac{1}{n(1 - \alpha^{(n)})^2} \sum_{j=[T_1 n]+1}^{[T_2 n]} (h_j^{(n)} - \alpha^{(n)} h_{j-1}^{(n)})^2 \\ &\rightarrow \int_{T_1}^{T_2} h(t)^2 dt \end{aligned}$$

and

$$\begin{aligned} & \max_{[T_1 n] \leq j \leq [T_2 n]} \text{Var} \left(\frac{(h_j^{(n)} - \alpha^{(n)} h_{j-1}^{(n)}) \varepsilon_j^{(n)}}{\sqrt{n}(1 - \alpha^{(n)})} \right) \\ &= \max_{[T_1 n] \leq j \leq [T_2 n]} \frac{(h_j^{(n)} - \alpha^{(n)} h_{j-1}^{(n)})^2}{n(1 - \alpha^{(n)})^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Consequently, we obtain (4.4).

Hence, finally we obtain

$$\begin{aligned} & \sqrt{n}(1 - \alpha^{(n)}) (\widehat{m}_{T_1, T_2}^{(n)} - m^{(n)}) \\ &= \frac{\frac{1}{\sqrt{n}(1 - \alpha^{(n)})} (\widehat{\zeta}_{T_1, T_2}^{(n)} - m^{(n)} \widehat{A}_{T_1, T_2}^{(n)})}{\frac{1}{n(1 - \alpha^{(n)})^2} \widehat{A}_{T_1, T_2}^{(n)}} \xrightarrow{\mathcal{D}} \mathcal{N} \left(0, \frac{1}{\int_{T_1}^{T_2} h(t)^2 dt} \right). \end{aligned}$$

The case $T_1 = 0$ can be handled in a similar way. □

Corollary 4.1.2. (i) If $\alpha^{(n)} = 1 - \frac{\gamma^{(n)}}{n}$ with $\gamma^{(n)} \rightarrow \gamma \in \mathbb{R}$, then

$$\frac{\widehat{m}_{T_1, T_2}^{(n)} - m^{(n)}}{\sqrt{n}} \xrightarrow{\mathcal{D}} \widehat{m}_{T_1, T_2} - m \stackrel{\mathcal{D}}{=} \mathcal{N}\left(0, \frac{1}{\widehat{A}_{T_1, T_2}}\right).$$

(ii) If $\alpha^{(n)} = 1 - \frac{\gamma^{(n)}}{n}$ with $\gamma^{(n)} \rightarrow \pm\infty$, then

$$\sqrt{n}(1 - \alpha^{(n)})(\widehat{m}_{T_1, T_2}^{(n)} - m^{(n)}) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{1}{\int_{T_1}^{T_2} h(t)^2 dt}\right).$$

(iii) If $\alpha^{(n)} \rightarrow \alpha \neq 1$, then

$$\sqrt{n}(\widehat{m}_{T_1, T_2}^{(n)} - m^{(n)}) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{1}{(1 - \alpha)^2 \int_{T_1}^{T_2} h(t)^2 dt}\right).$$

Example 4.1.3. Let $\alpha^{(n)} = 1 - \frac{\beta}{n^q}$, $q \in \mathbb{R}$, $\beta \neq 0$, $T_1 > 0$ and $h(t) = 1$ for all $t \geq T_1$.

(i) If $q > 1$, then

$$\frac{\widehat{m}_{T_1, T_2}^{(n)} - m^{(n)}}{\sqrt{n}} \xrightarrow{\mathcal{D}} \widehat{m}_{T_1, T_2} - m \stackrel{\mathcal{D}}{=} \mathcal{N}(0, T_1),$$

hence the rate of convergence does not depend on q , and the limit law does not depend on β , T_2 .

(ii) If $q = 1$, then

$$\frac{\widehat{m}_{T_1, T_2}^{(n)} - m^{(n)}}{\sqrt{n}} \xrightarrow{\mathcal{D}} \widehat{m}_{T_1, T_2} - m \stackrel{\mathcal{D}}{=} \mathcal{N}\left(0, \frac{1}{\beta \coth(\beta T_1) + \beta + \beta^2(T_2 - T_1)}\right),$$

hence the limit law depends on β , T_1 and T_2 .

(iii) If $q < 1$, then

$$\frac{\widehat{m}_{T_1, T_2}^{(n)} - m^{(n)}}{n^{q-1/2}} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{1}{\beta^2(T_2 - T_1)}\right),$$

hence the rate of convergence depends on q , and the limit law depends on β , T_1 and T_2 .

Remark 4.1.4. We note that no assumption is needed concerning the sequence $m^{(n)}$, $n = 1, 2, \dots$, unlike in case of the LSE of the coefficient $\alpha^{(n)}$.

4.1.2 Case of Stationary Nearly Unstable AR Model

A stationary AR(1) time series is the weakly stationary solution of

$$\tilde{X}_k = \alpha \tilde{X}_{k-1} + \varepsilon_k, \quad k \in \mathbb{Z},$$

where necessarily $|\alpha| < 1$.

Consider a sequence of shifted stationary AR(1) models

$$\begin{cases} \tilde{X}_k^{(n)} = \alpha^{(n)} \tilde{X}_{k-1}^{(n)} + \varepsilon_k^{(n)}, & k \in \mathbb{Z}, \\ \tilde{Z}_k^{(n)} = \tilde{X}_k^{(n)} + m^{(n)} h_k^{(n)}, & k \in \mathbb{Z}, \end{cases}$$

for $n = 1, 2, \dots$, where $|\alpha^{(n)}| < 1$, $n = 1, 2, \dots$ and $h_k^{(n)}$, $n = 1, 2, \dots$, $k \in \mathbb{Z}$ are known and $m^{(n)}$, $n = 1, 2, \dots$ are unknown parameters. Suppose again that for all n , $\{\varepsilon_j^{(n)} : j \in \mathbb{Z}\}$ are i.i.d., mean zero and variance 1.

Then,

$$\mathbb{E} \tilde{X}_k^{(n)} = 0, \quad \mathbb{D}^2 \tilde{X}_k^{(n)} = \frac{1}{1 - (\alpha^{(n)})^2}, \quad \text{for } k \in \mathbb{Z}.$$

The LSE $\tilde{m}^{(n)}$ of the parameter $m^{(n)}$ based on the observations $\{\tilde{Z}_k^{(n)} : [T_1 n] \leq k \leq [T_2 n]\}$, where $T_1 < T_2$, can be obtained by minimizing the sum of squares (4.1), hence

$$\tilde{m}_{T_1, T_2}^{(n)} = \frac{\tilde{\zeta}_{T_1, T_2}^{(n)}}{\tilde{A}_{T_1, T_2}^{(n)}},$$

where

$$\begin{aligned} \tilde{A}_{T_1, T_2}^{(n)} &= \left(1 - (\alpha^{(n)})^2\right) \left(h_{[T_1 n]}^{(n)}\right)^2 + \sum_{j=[T_1 n]+1}^{[T_2 n]} (h_j^{(n)} - \alpha^{(n)} h_{j-1}^{(n)})^2, \\ \tilde{\zeta}_{T_1, T_2}^{(n)} &= \left(1 - (\alpha^{(n)})^2\right) h_{[T_1 n]}^{(n)} Z_{[T_1 n]}^{(n)} + \sum_{j=[T_1 n]+1}^{[T_2 n]} (h_j^{(n)} - \alpha^{(n)} h_{j-1}^{(n)}) (\tilde{Z}_j^{(n)} - \alpha^{(n)} \tilde{Z}_{j-1}^{(n)}). \end{aligned}$$

Next, consider a stationary Ornstein-Uhlenbeck process $\{\tilde{Y}(t) : t \in \mathbb{R}\}$, which is the stationary solution of

$$d\tilde{Y}(t) = -\gamma \tilde{Y}(t) dt + dW(t), \quad t \in \mathbb{R},$$

where $\gamma > 0$ necessarily.

It is a zero mean Gaussian process with

$$\mathbb{E}\tilde{Y}(t_1)\tilde{Y}(t_2) = \frac{e^{-\gamma|t_2-t_1|}}{2\gamma}.$$

Consider a shifted stationary Ornstein-Uhlenbeck process $\{\tilde{Z}(t) : t \in \mathbb{R}\}$, given by

$$\begin{cases} d\tilde{Y}(t) = -\gamma\tilde{Y}(t)dt + dW(t), & t \in \mathbb{R}, \\ \tilde{Z}(t) = \tilde{Y}(t) + mh(t), & t \in \mathbb{R}, \end{cases}$$

where $\gamma > 0$ is a known parameter, $h : \mathbb{R} \rightarrow \mathbb{R}$ is a known function and $m \in \mathbb{R}$ is an unknown parameter.

In [7] the following result is proved on the MLE of m . Denote by $\mathbb{P}_{\tilde{Z}}$ and $\mathbb{P}_{\tilde{Y}}$ the measures generated on $C([T_1, T_2] \rightarrow \mathbb{R})$ by the processes $\tilde{Z}$ and $\tilde{Y}$, respectively. If h is twice continuously differentiable, then the measures $\mathbb{P}_{\tilde{Z}}$ and $\mathbb{P}_{\tilde{Y}}$ are equivalent and the MLE of m based on the observations $\{\tilde{Z}(t) : t \in [T_1, T_2]\}$ has the form

$$\tilde{m}_{T_1, T_2} = \frac{\tilde{\zeta}_{T_1, T_2}}{\tilde{A}_{T_1, T_2}},$$

where

$$\begin{aligned} \tilde{A}_{T_1, T_2} &= \gamma h(T_1)^2 + \gamma h(T_2)^2 + \int_{T_1}^{T_2} (\gamma^2 h(t)^2 + h'(t)^2) dt, \\ \tilde{\zeta}_{T_1, T_2} &= \gamma h(T_1)\tilde{Z}(T_1) + \gamma h(T_2)\tilde{Z}(T_2) + h'(T_2)\tilde{Z}(T_2) - h'(T_1)\tilde{Z}(T_1) \\ &\quad + \int_{T_1}^{T_2} (\gamma^2 h(t) - h''(t)) \tilde{Z}(t) dt. \end{aligned}$$

Moreover, $\tilde{m}_{T_1, T_2}$ is normally distributed with mean m and variance $1/\tilde{A}_{T_1, T_2}$.

Theorem 4.1.5. *Suppose that $h_k^{(n)} = h(\frac{k}{n})$ for $n = 1, 2, \dots$, $k \in \mathbb{Z}$ with some twice continuously differentiable function $h : \mathbb{R} \rightarrow \mathbb{R}$. Suppose that for all n , $\{\varepsilon_j^{(n)} : j \in \mathbb{Z}\}$ are i.i.d., mean zero and variance 1.*

(i) *If $n(1 - \alpha^{(n)}) \rightarrow \gamma \in (0, \infty)$ then*

$$\frac{\tilde{m}_{T_1, T_2}^{(n)} - m^{(n)}}{\sqrt{n}} \xrightarrow{\mathcal{D}} \tilde{m}_{T_1, T_2} - m \stackrel{\mathcal{D}}{=} \mathcal{N}\left(0, \frac{1}{\tilde{A}_{T_1, T_2}}\right).$$

(ii) If $n(1 - \alpha^{(n)}) \rightarrow \infty$ then

$$\sqrt{n}(1 - \alpha^{(n)})(\tilde{m}_{T_1, T_2}^{(n)} - m^{(n)}) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{1}{\int_{T_1}^{T_2} h(t)^2 dt}\right).$$

(iii) Let $n(1 - \alpha^{(n)}) \rightarrow 0$ and $\varepsilon_k^{(n)} \sim \mathcal{N}(0, 1)$.

(a) If $h(t) = c$ for all $t \in \mathbb{R}$ with some $c \in \mathbb{R}$, then

$$\sqrt{1 - \alpha^{(n)}}(\tilde{m}_{T_1, T_2}^{(n)} - m^{(n)}) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{1}{2c^2}\right).$$

(b) Otherwise,

$$\frac{\tilde{m}_{T_1, T_2}^{(n)} - m^{(n)}}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{1}{\int_{T_1}^{T_2} h'(t)^2 dt}\right).$$

Proof. The proof is similar to the proof of Theorem (4.1.1.). $\square$

Example 4.1.6. Let $\alpha^{(n)} = 1 - \frac{\beta}{n^q}$, $q \geq 0$, $\beta > 0$, and $h(t) = 1$ for all $t \in \mathbb{R}$.

(i) If $q = 1$, then

$$\frac{\tilde{m}_{T_1, T_2}^{(n)} - m^{(n)}}{\sqrt{n}} \xrightarrow{\mathcal{D}} \tilde{m}_{T_1, T_2} - m \stackrel{\mathcal{D}}{=} \mathcal{N}\left(0, \frac{1}{2\beta + \beta^2(T_2 - T_1)}\right).$$

(ii) If $0 \leq q < 1$, then

$$\frac{\tilde{m}_{T_1, T_2}^{(n)} - m^{(n)}}{n^{q-1/2}} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{1}{\beta^2(T_2 - T_1)}\right),$$

hence the rate of convergence depends on q .

(iii) If $q > 1$ and $\varepsilon_k^{(n)} \sim \mathcal{N}(0, 1)$, then

$$\frac{\tilde{m}_{T_1, T_2}^{(n)} - m^{(n)}}{n^{q/2}} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{1}{2\beta}\right),$$

hence the rate of convergence depends on q , but the limit distribution does not depend on T_1, T_2 .

4.2 Nearly Unstable Multidimensional AR Processes

Our aim was to investigate nearly unstable complex-valued models, where the coefficient matrices are in Jordan normal form and to study the limit behaviour of the suitable normalised LSE of the eigenvalue. It will turn out that the limit distribution depends only on the last, d -th component of the process.

This type of models is rather special. However from a practical point of view it is often useful to investigate the behaviour of the greatest eigenvalue.

We also compared it with the maximum likelihood estimator (MLE) of the eigenvalue of the coefficient matrix of the related continuous time model. It is interesting to note that the MLE of the eigenvalue in the related continuous time model depends only on the first coordinate of the process. Matrices consisting of two or more Jordan blocks were also studied.

4.2.1 Nearly Unstable AR Models with Coefficient Matrices in Jordan Normal Form

Let $\mathbb{C}^d$ be the space of the d -dimensional complex column vectors. Let us introduce the widely used notation

$$\langle x, y \rangle := x^\top \bar{y} \quad \text{for } x, y \in \mathbb{C}^d$$

for the scalar product in $\mathbb{C}^d$, where $\bar{y}$ denotes the complex conjugate of y . We introduce the norm of $x \in \mathbb{C}^d$ by

$$\|x\| := \sqrt{\langle x, x \rangle}.$$

For $\lambda \in \mathbb{C}$ and $d \in \mathbb{N}$ we introduce the notation

$$J(\lambda, d) := \begin{pmatrix} \lambda & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & \lambda & 0 & & & \cdot \\ 0 & 1 & \lambda & 0 & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \lambda & 0 \\ 0 & \cdot & \cdot & 0 & 1 & \lambda \end{pmatrix}$$

for a $d \times d$ matrix in Jordan normal form with eigenvalue λ . We shall use the short notation $J(\lambda)$ if it does not cause misunderstanding.

Consider the d -dimensional complex-valued autoregressive model

$$\begin{cases} X_k = J(\lambda)X_{k-1} + \varepsilon_k, & k=1,2,\dots, \\ X_0 = 0, \end{cases}$$

where $\lambda \in \mathbb{C}$ is the unknown parameter of the model. If we take into consideration the special form of the coefficient matrix then we can calculate the LSE of λ as follows.

Lemma 4.2.1. The LSE of λ , based on the observations $X_1, \dots, X_n$ is given by

$$\hat{\lambda} := \frac{\sum_{k=1}^n \sum_{j=1}^d \langle X_{k,j} - X_{k-1,j-1}, X_{k-1,j} \rangle}{\sum_{k=1}^n \sum_{j=1}^d |X_{k-1,j}|^2},$$

where $X_k = (X_{k,1}, \dots, X_{k,d})^\top$ and $X_{k,0} := 0$.

Proof. Let us consider the sum of squares:

$$\begin{aligned} \Lambda_n(\lambda) &:= \sum_{k=1}^n \|X_k - J(\lambda)X_{k-1}\|^2 \\ &= \sum_{k=1}^n \left\| \begin{pmatrix} X_{k,1} \\ \vdots \\ X_{k,d} \end{pmatrix} - \begin{pmatrix} \lambda & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & \lambda & 0 & & & \cdot \\ 0 & 1 & \lambda & 0 & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \lambda & 0 \\ 0 & \cdot & \cdot & 0 & 1 & \lambda \end{pmatrix} \begin{pmatrix} X_{k-1,1} \\ \vdots \\ X_{k-1,d} \end{pmatrix} \right\|^2 \\ &= \sum_{k=1}^n \left\| \begin{pmatrix} X_{k,1} - X_{k-1,0} - \lambda X_{k-1,1} \\ X_{k,2} - X_{k-1,1} - \lambda X_{k-1,2} \\ \vdots \\ X_{k,d} - X_{k-1,d-1} - \lambda X_{k-1,d} \end{pmatrix} \right\|^2. \end{aligned}$$

Obviously

$$\begin{aligned} \Lambda_n(\lambda) &= \sum_{k=1}^n \sum_{j=1}^d |X_{k,j} - X_{k-1,j-1}|^2 - \lambda \sum_{k=1}^n \sum_{j=1}^d X_{k-1,j} \overline{(X_{k,j} - X_{k-1,j-1})} \\ &\quad - \bar{\lambda} \sum_{k=1}^n \sum_{j=1}^d \overline{X_{k-1,j}} (X_{k,j} - X_{k-1,j-1}) + |\lambda|^2 \sum_{k=1}^n \sum_{j=1}^d |X_{k-1,j}|^2. \end{aligned}$$

Clearly $\hat{\lambda}$ is a least squares estimate of λ if $\Lambda_n(\hat{\lambda})$ is minimal. We write the complex number λ in the form $\lambda = \alpha + i\beta$, where $\alpha, \beta \in \mathbb{R}$. Rewriting $\Lambda_n(\hat{\lambda})$ as

a function of α and β we get the following:

$$\begin{aligned}\Lambda_n(\alpha, \beta) &= \sum_{k=1}^n \sum_{j=1}^d |X_{k,j} - X_{k-1,j-1}|^2 - \\ &\quad 2\alpha \cdot \operatorname{Re} \left(\sum_{k=1}^n \sum_{j=1}^d X_{k-1,j} \overline{(X_{k,j} - X_{k-1,j-1})} \right) + \\ &\quad 2\beta \cdot \operatorname{Im} \left(\sum_{k=1}^n \sum_{j=1}^d X_{k-1,j} \overline{(X_{k,j} - X_{k-1,j-1})} \right) + \\ &\quad (\alpha^2 + \beta^2) \sum_{k=1}^n \sum_{j=1}^d |X_{k-1,j}|^2.\end{aligned}$$

This expression takes minimum if the following equations hold,

$$\begin{aligned}-2 \sum_{k=1}^n \sum_{j=1}^d \operatorname{Re} \left[X_{k-1,j} \overline{(X_{k,j} - X_{k-1,j-1})} \right] + 2\alpha \sum_{k=1}^n \sum_{j=1}^d |X_{k-1,j}|^2 &= 0, \\ 2 \sum_{k=1}^n \sum_{j=1}^d \operatorname{Im} \left[X_{k-1,j} \overline{(X_{k,j} - X_{k-1,j-1})} \right] + 2\beta \sum_{k=1}^n \sum_{j=1}^d |X_{k-1,j}|^2 &= 0.\end{aligned}$$

Hence the LSE of α and β are given by

$$\begin{aligned}\hat{\alpha}_n &= \frac{\sum_{k=1}^n \sum_{j=1}^d \operatorname{Re} \left[X_{k-1,j} \overline{(X_{k,j} - X_{k-1,j-1})} \right]}{\sum_{k=1}^n \sum_{j=1}^d |X_{k-1,j}|^2}, \\ \hat{\beta}_n &= -\frac{\sum_{k=1}^n \sum_{j=1}^d \operatorname{Im} \left[X_{k-1,j} \overline{(X_{k,j} - X_{k-1,j-1})} \right]}{\sum_{k=1}^n \sum_{j=1}^d |X_{k-1,j}|^2}.\end{aligned}$$

Thus we obtain the assertion of the lemma. $\square$

4.2.2 Convergence of a Related Step Process

For $n = 1, 2, \dots$ consider the d -dimensional complex-valued AR model:

$$\begin{cases} X_k^{(n)} = J(\lambda_n)X_{k-1}^{(n)} + \varepsilon_k^{(n)}, & k = 1, 2, \dots, n, \\ X_0^{(n)} = 0, \end{cases} \quad (4.5)$$

where $\{\varepsilon_k^{(n)}\}$ is an array of random vectors in $\mathbb{C}^d$ and $\lambda_n = e^{h_n/n+i\theta}$ with $h_n \in \mathbb{C}$, $h_n \rightarrow h \in \mathbb{C}$ and $\theta \in (-\pi, \pi]$. Clearly $J(\lambda_n) \rightarrow J(e^{i\theta})$ since $\lambda_n \rightarrow e^{i\theta}$. It is easy to see that the model is nearly unstable, since $\varrho(J(e^{i\theta})) = 1$.

We consider the following transformation on the elements of $\mathbb{C}^d$:

$$z \mapsto e^{ik\theta} z.$$

This transformation can be considered as a rotation in $\mathbb{C}^d$, since

$$\langle e^{ik\theta} u, e^{ik\theta} v \rangle = \langle u, v \rangle \quad \text{for } u, v \in \mathbb{C}^d.$$

We define the following rotated elements (the construction is the same as in Pap, Zuijlen [26]):

$$\begin{aligned} Z_k^{(n)} &:= e^{-ik\theta} X_k^{(n)}, \\ \zeta_k^{(n)} &:= e^{-ik\theta} \varepsilon_k^{(n)}. \end{aligned}$$

Let

$$\tilde{J}(\lambda_n) := e^{-i\theta} J(\lambda_n) = \begin{pmatrix} e^{h_n/n} & 0 & \cdot & 0 \\ e^{-i\theta} & \cdot & & 0 \\ \cdot & & \cdot & \cdot \\ 0 & \cdot & e^{-i\theta} & e^{h_n/n} \end{pmatrix}.$$

The following model is clearly equivalent with (4.5):

$$\begin{cases} Z_k^{(n)} = \tilde{J}(\lambda_n) Z_{k-1}^{(n)} + \zeta_k^{(n)}, & k = 1, 2, \dots, n, \\ Z_0^{(n)} = 0. \end{cases}$$

Now we consider the normalization matrices

$$L_n := \text{diag} \left\{ n^{-1/2}, n^{-3/2} e^{i\theta}, \dots, n^{-d+1/2} e^{(d-1)i\theta} \right\},$$

and random step functions

$$Y^{(n)}(t) := e^{-i[nt]\theta} L_n X_{[nt]}^{(n)} = L_n Z_{[nt]}^{(n)},$$

$$M^{(n)}(t) := \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} e^{-ik\theta} \varepsilon_k^{(n)} = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \zeta_k^{(n)}.$$

Let us put the following condition on the random disturbances $\{\varepsilon_k^{(n)}\}$.

(C) $\varepsilon_k^{(n)}$, $k = 1, \dots, n$, $n \geq 1$, is a triangular array of square integrable martingale differences in $\mathbb{C}^d$ with respect to the filtrations $(\mathcal{F}_k^{(n)})_{k=0,1,\dots,n;n \geq 1}$ such that for all $t \in [0, 1]$

$$\frac{1}{nt} \sum_{k=1}^{[nt]} \mathbb{E} \left(\eta_k^{(n)} \left(\eta_k^{(n)} \right)^\top \middle| \mathcal{F}_{k-1}^{(n)} \right) \xrightarrow{\mathbb{P}} \frac{1}{2} I_{2d} \quad \text{as } n \rightarrow \infty,$$

where $\eta_k^{(n)} := (\operatorname{Re}(\zeta_k^{(n)}), \operatorname{Im}(\zeta_k^{(n)}))^\top$ and

$$\forall \alpha > 0 \quad \frac{1}{n} \sum_{k=1}^{[nt]} \mathbb{E}(\|\varepsilon_k^{(n)}\|^2 \chi_{\{\|\varepsilon_k^{(n)}\| > \alpha \sqrt{n}\}} | \mathcal{F}_{k-1}^{(n)}) \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \rightarrow \infty.$$

From Pap, Zuijlen [26] we know that if the condition (C) is satisfied then the following statements are true for $Y^{(n)}$, $M^{(n)}$:

$$\begin{aligned} M^{(n)} &\xrightarrow{\mathcal{D}} W \quad \text{in } D([0, 1] \rightarrow \mathbb{C}^d), \\ Y^{(n)} &\xrightarrow{\mathcal{D}} Y \quad \text{in } D([0, 1] \rightarrow \mathbb{C}^d), \\ (M^{(n)}, Y^{(n)}) &\xrightarrow{\mathcal{D}} (W, Y) \quad \text{in } D([0, 1] \rightarrow \mathbb{C}^{2d}), \end{aligned}$$

where $D([0, 1] \rightarrow \mathbb{C}^d)$ denotes the Skorokhod space, $\{W(t), t \in [0, 1]\}$ is a complex, standard Wiener process with values in $\mathbb{C}^d$ and $\{Y(t), t \in [0, 1]\}$ is a complex, continuous time autoregressive process, defined as the solution of the stochastic differential equation

$$dY(t) = AY(t) dt + \Sigma dW(t), \quad Y(0) = 0,$$

where $\Sigma := \operatorname{diag}(1, 0, 0, \dots, 0)$, and A is the Jordan matrix with eigenvalue h :

$$A := \begin{pmatrix} h & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & h & 0 & & & \cdot \\ 0 & 1 & h & 0 & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & h & 0 \\ 0 & \cdot & \cdot & 0 & 1 & h \end{pmatrix}.$$

4.2.3 Convergence of the LSE

Consider the d -dimensional complex-valued nearly unstable AR process generated according to the scheme (4.5). We are interested in the limit behaviour of the LSE of

λ_n which has the following form according to Lemma (4.2.1):

$$\hat{\lambda}_n = \frac{\sum_{k=1}^n \sum_{j=1}^d \langle X_{k,j}^{(n)} - X_{k-1,j-1}^{(n)}, X_{k-1,j}^{(n)} \rangle}{\sum_{k=1}^n \sum_{j=1}^d |X_{k-1,j}^{(n)}|^2},$$

where $X_{k,0}^{(n)} := 0$.

Theorem 4.2.2. Suppose that array $\varepsilon_k^{(n)}$ $k = 1, \dots, n$, $n \geq 1$ satisfies the condition (C). Then

$$n^d(\hat{\lambda}_n - \lambda_n) \xrightarrow{\mathcal{D}} e^{(d-1)\theta i} \frac{\int_0^1 \overline{Y_d(t)} dW_d(t)}{\int_0^1 |Y_d(t)|^2 dt}, \quad (4.6)$$

where the process $Y(t)$, $t \in [0, 1]$ is given by

$$dY(t) = AY(t)dt + \Sigma dW(t) \quad Y(0) = 0,$$

and $W(t)$, $t \in [0, 1]$ is a standard complex d -dimensional Wiener process.

Proof. If we take into consideration that

$$X_{k,j}^{(n)} = X_{k-1,j-1}^{(n)} + \lambda_n X_{k-1,j}^{(n)} + \varepsilon_{k,j}^{(n)} \quad \text{for } 1 \leq k \leq n, \quad 1 \leq j \leq d,$$

we get the following:

$$\begin{aligned} \hat{\lambda}_n &= \frac{\sum_{j=1}^d \sum_{k=1}^n [\lambda_n |X_{k-1,j}^{(n)}|^2 + \langle \varepsilon_{k,j}^{(n)}, X_{k-1,j}^{(n)} \rangle]}{\sum_{k=1}^n \sum_{j=1}^d |X_{k-1,j}^{(n)}|^2} \\ &= \lambda_n + \frac{\sum_{j=1}^d \sum_{k=1}^n \langle \varepsilon_{k,j}^{(n)}, X_{k-1,j}^{(n)} \rangle}{\sum_{k=1}^n \sum_{j=1}^d |X_{k-1,j}^{(n)}|^2}, \\ \hat{\lambda}_n - \lambda_n &= \frac{\sum_{j=1}^d \sum_{k=1}^n \langle \varepsilon_{k,j}^{(n)}, X_{k-1,j}^{(n)} \rangle}{\sum_{k=1}^n \sum_{j=1}^d |X_{k-1,j}^{(n)}|^2}. \end{aligned}$$

The orthogonality of the transformation $z \mapsto e^{-ik\theta}$ implies that

$$\hat{\lambda}_n - \lambda_n = \frac{\sum_{j=1}^d \sum_{k=1}^n \langle \zeta_{k,j}, Z_{k-1,j} \rangle}{\sum_{k=1}^n \sum_{j=1}^d |Z_{k-1,j}|^2}.$$

Moreover the following is valid:

$$Z_{k,j}^{(n)} = e^{-ik\theta} X_{k,j}^{(n)},$$

$$\begin{aligned}
& \int_0^1 Y_j^{(n)}(t) \overline{Y_j^{(n)}(t)} dt = \\
& \sum_{k=1}^n \int_{k-1/n}^{k/n} n^{-j+1/2} e^{(j-1)\theta i} Z_{k-1,j}^{(n)}(t) n^{-j+1/2} e^{-(j-1)\theta i} \overline{Z_{k-1,j}^{(n)}(t)} dt = \\
& n^{-2j+1} \cdot \frac{1}{n} \sum_{k=1}^n |Z_{k-1,j}^{(n)}|^2,
\end{aligned}$$

hence

$$\sum_{k=1}^n |X_{k-1,j}^{(n)}|^2 = \sum_{k=1}^n |Z_{k-1,j}^{(n)}|^2 = n^{2j} \int_0^1 Y_j^{(n)}(t) \overline{Y_j^{(n)}(t)} dt.$$

Similarly

$$\int_0^1 \overline{Y_j^{(n)}(t)} dM_j^{(n)}(t) = \frac{1}{\sqrt{n}} e^{-(j-1)\theta i} n^{-j+1/2} \sum_{k=1}^n \zeta_{k-1,j}^{(n)} \overline{Z_{k-1,j}^{(n)}},$$

thus

$$\sum_{k=1}^n \varepsilon_{k-1,j}^{(n)} \overline{X_{k-1,j}^{(n)}} = \sum_{k=1}^n \zeta_{k-1,j}^{(n)} \overline{Z_{k-1,j}^{(n)}} = n^j e^{(j-1)\theta i} \int_0^1 \overline{Y_j^{(n)}(t)} dM_j^{(n)}(t).$$

Consequently

$$\begin{aligned}
n^d (\hat{\lambda}_n - \lambda_n) &= n^d \frac{\sum_{j=1}^d e^{(j-1)\theta i} n^j \int_0^1 \overline{Y_j^{(n)}(t)} dM_j^{(n)}(t)}{\sum_{j=1}^d n^{2j} \int_0^1 Y_j^{(n)}(t) \overline{Y_j^{(n)}(t)} dt} \\
&= \frac{\sum_{j=1}^d e^{(j-1)\theta i} \int_0^1 \overline{Y_j^{(n)}(t)} dM_j^{(n)}(t)}{\sum_{j=1}^d n^{j-d} \int_0^1 Y_j^{(n)}(t) \overline{Y_j^{(n)}(t)} dt}.
\end{aligned}$$

Taking into account that $(M^{(n)}, Y^{(n)}) \xrightarrow{\mathcal{D}} (W, Y)$ in $D([0, 1] \rightarrow \mathbb{C}^{2d})$ and using, for example, Proposition 6 in Jeganathan [17] we obtain the assertion. $\square$

4.2.4 MLE of the Coefficient of the Related Continuous Time Model

Consider the d -dimensional complex-valued Ornstein–Uhlenbeck process defined as the solution of the stochastic differential equation

$$dY(t) = AY(t) dt + \Sigma dW(t), \quad Y(0) = 0,$$

where $\{W(t), t \in [0, 1]\}$ is a standard Wiener process with values in $\mathbb{C}^d$. Let us identify the complex process $\{W(t), t \in [0, 1]\}$ with the standard Wiener process $\{\widetilde{W}(t), t \in [0, 1]\}$ in $\mathbb{R}^{2d}$, where

$$\widetilde{W}(t) := (\operatorname{Re} W_1(t), \operatorname{Im} W_1(t), \operatorname{Re} W_2(t), \operatorname{Im} W_2(t), \dots, \operatorname{Re} W_d(t), \operatorname{Im} W_d(t)).$$

Let us identify the complex process $\{Y(t), t \in [0, 1]\}$ with the process

$$\widetilde{Y}(t) := (\operatorname{Re} Y_1(t), \operatorname{Im} Y_1(t), \operatorname{Re} Y_2(t), \operatorname{Im} Y_2(t), \dots, \operatorname{Re} Y_d(t), \operatorname{Im} Y_d(t))$$

with values in $\mathbb{R}^{2d}$.

Then the above complex stochastic differential equation is equivalent with the following real stochastic differential equation:

$$\widetilde{Y}(t) = \widetilde{A}\widetilde{Y}(t) dt + \widetilde{\Sigma} d\widetilde{W}(t), \quad \widetilde{Y}(0) = 0,$$

where $\widetilde{A}$ and $\widetilde{\Sigma}$ are defined as the following blocks of matrices:

$$\begin{aligned} \widetilde{A} &:= \begin{pmatrix} K & 0 & \cdot & 0 \\ I_2 & K & 0 & \cdot \\ 0 & I_2 & K & \\ \cdot & & \cdot & \cdot \\ 0 & & 0 & I_2 & K \end{pmatrix}, \\ \widetilde{\Sigma} &:= \begin{pmatrix} I_2 & 0 & \cdot & 0 \\ 0 & 0 & \cdot & \\ \cdot & & \cdot & \\ 0 & & & 0 \end{pmatrix} = \operatorname{diag}(1, 1, 0, 0, \dots), \end{aligned}$$

where

$$K := \begin{pmatrix} \operatorname{Re} h & -\operatorname{Im} h \\ \operatorname{Im} h & \operatorname{Re} h \end{pmatrix}, \quad I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

($\widetilde{A}$ and $\widetilde{\Sigma}$ are real $2d \times 2d$ matrices.)

Consider the measures $\mathbb{P}_{\widetilde{W}}$ and $\mathbb{P}_{\widetilde{Y}}$ on $C([0, 1] \rightarrow \mathbb{R}^{2d})$ generated by the real processes $\widetilde{W}$ and $\widetilde{Y}$, respectively. It is known that $\mathbb{P}_{\widetilde{Y}}$ is absolutely continuous with respect to the measure $\mathbb{P}_{\widetilde{W}}$ and the Radon-Nikodym derivative has the form

$$\frac{d\mathbb{P}_{\widetilde{Y}}}{d\mathbb{P}_{\widetilde{W}}}(\widetilde{Y}) = \exp \left\{ \int_0^1 \langle \widetilde{\Sigma}^\ominus \widetilde{A}\widetilde{Y}(u), d\widetilde{Y}(u) \rangle - \frac{1}{2} \int_0^1 \langle \widetilde{A}\widetilde{Y}(u), \widetilde{\Sigma}^\ominus \widetilde{A}\widetilde{Y}(u) \rangle du \right\}, \quad (4.7)$$

where $\widetilde{\Sigma}^\ominus$ denotes the pseudo inverse (or generalized inverse) of the matrix $\widetilde{\Sigma}$ (see Arató [3]). It can be easily verified that $\widetilde{\Sigma}^\ominus = \widetilde{\Sigma}$.

Obviously if we maximize the right hand side of the equation (4.7) in $\operatorname{Re} h$ and $\operatorname{Im} h$, we can obtain the MLE of $\operatorname{Re} h$ and $\operatorname{Im} h$, ie., the MLE of h . We have

$$\begin{aligned} \frac{d\mathbb{P}_{\tilde{Y}}}{d\mathbb{P}_{\tilde{W}}}(\tilde{Y}) = & \operatorname{Re} h \cdot \left\{ \int_0^1 \operatorname{Re} Y_1(u) d\operatorname{Re} Y_1(u) + \int_0^1 \operatorname{Im} Y_1(u) d\operatorname{Im} Y_1(u) \right\} \\ & + \operatorname{Im} h \cdot \left\{ \int_0^1 \operatorname{Re} Y_1(u) d\operatorname{Im} Y_1(u) - \int_0^1 \operatorname{Im} Y_1(u) d\operatorname{Re} Y_1(u) \right\} \\ & - \frac{1}{2} ((\operatorname{Re} h)^2 + (\operatorname{Im} h)^2) \int_0^1 \{(\operatorname{Re} Y_1(u))^2 + (\operatorname{Im} Y_1(u))^2\} du, \end{aligned}$$

where Y_1 is the first component of the complex process Y .

Finally we obtain

$$\hat{h} = \frac{\int_0^1 \overline{Y_1(t)} dY_1(t)}{\int_0^1 |Y_1(t)|^2 dt}.$$

So the MLE of h depends only on the first component of Y .

We remark that in the case $d = 1$ Theorem (4.2.2.) says that

$$n(\hat{\lambda}_n - \lambda_n) \xrightarrow{\mathcal{D}} \frac{\int_0^1 \overline{Y_1(t)} dW_1(t)}{\int_0^1 |Y_1(t)|^2 dt},$$

and the Itô's formula gives

$$\hat{h} = \frac{\int_0^1 \overline{Y_1(t)} dY_1(t)}{\int_0^1 |Y_1(t)|^2 dt} = h + \frac{\int_0^1 \overline{Y_1(t)} dW_1(t)}{\int_0^1 |Y_1(t)|^2 dt},$$

hence $n(\hat{\lambda}_n - \lambda) \rightarrow h$ implies

$$n(\hat{\lambda}_n - \lambda_n) \xrightarrow{\mathcal{D}} \hat{h},$$

that is, the LSE of the coefficient of the discrete time models converges to the MLE of the coefficient of the corresponding continuous time model. (See also [20].)

If $d \geq 2$ then such a relationship does not hold, because the limit distribution of the LSE of the coefficient of the discrete time models depends on the last component of the complex process Y .

It is also interesting to mention that the complex process Y is driven, in fact, only by the first component W_1 of the Wiener process W , and the limit distribution of the LSE contains the last component W_d , which is, by definition, independent of the first component W_1 (if $d \geq 2$).

4.2.5 The Case of Several Jordan Blocks

The Case of Different Eigenvalues.

Let $\lambda_n^{(j)}$, $j = 1, \dots, r$, be different complex numbers and let $d_1, \dots, d_r \in \mathbb{N}$. Let the coefficient matrix Q_n of the model (2.5) consists of the corresponding Jordan blocks:

$$Q_n := \text{diag} \{J(\lambda_n^{(1)}, d_1), \dots, J(\lambda_n^{(r)}, d_r)\},$$

where $\lambda_n^{(j)} = e^{h_n^{(j)}/n + i\theta^{(j)}}$ with $h_n^{(j)} \in \mathbb{C}$, $h_n^{(j)} \rightarrow h^{(j)} \in \mathbb{C}$ and $\theta^{(j)} \in (-\pi, \pi]$ for $j = 1, \dots, r$.

Now the sum of squares has the following form:

$$\begin{aligned} \Lambda_n(\lambda_n^{(1)}, \dots, \lambda_n^{(r)}) = \\ = \sum_{j=1}^r \sum_{k=1}^n \left\| \begin{pmatrix} X_{k, d_1 + \dots + d_{j-1} + 1}^{(n)} \\ \vdots \\ X_{k, d_1 + \dots + d_{j-1} + d_j}^{(n)} \end{pmatrix} - J(\lambda_n^{(j)}, d_1) \begin{pmatrix} X_{k-1, d_1 + \dots + d_{j-1} + 1}^{(n)} \\ \vdots \\ X_{k-1, d_1 + \dots + d_{j-1} + d_j}^{(n)} \end{pmatrix} \right\|^2. \end{aligned}$$

Obviously $\lambda_n^{(j)}$ is estimated from the j -th block of components of X_k in the same way as in the case of a single Jordan block.

Hence, under the Condition (C), the LSE of $\lambda_n^{(j)}$, $j = 1, \dots, r$ are asymptotically independent, and the limit behaviour can be obtained as we have seen it in the case of a single Jordan block.

The Case of Two Jordan Blocks with the Same Eigenvalue

Let $\lambda \in \mathbb{C}$ and let $d_1, \dots, d_r \in \mathbb{N}$. Let the coefficient matrix Q_n of the model (2.5) be

$$Q_n := \text{diag} \{J(\lambda_n, d_1), \dots, J(\lambda_n, d_r)\},$$

where $\lambda_n = e^{h_n/n + i\theta}$ with $h_n \in \mathbb{C}$, $h_n \rightarrow h \in \mathbb{C}$ and $\theta \in (-\pi, \pi]$.

Now the LSE of λ_n will be

$$\hat{\lambda}_n = \frac{\sum_{k=1}^n \sum_{\ell=1}^{d_1 + \dots + d_r} \langle X_{k, \ell}^{(n)} - \tilde{X}_{k-1, \ell-1}^{(n)}, X_{k-1, j}^{(n)} \rangle}{\sum_{k=1}^n \sum_{\ell=1}^{d_1 + \dots + d_r} |X_{k-1, \ell}^{(n)}|^2},$$

where

$$\tilde{X}_{k,\ell-1}^{(n)} := \begin{cases} 0 & \text{for } \ell = d_1, \dots, d_r, \\ X_{k,\ell-1}^{(n)} & \text{otherwise.} \end{cases}$$

Now the normalization matrices are

$$L_n := \text{diag} \left\{ n^{-1/2}, \dots, n^{-d_1+1/2} e^{(d_1-1)i\theta}, \dots, n^{-1/2}, \dots, n^{-d_r+1/2} e^{(d_r-1)i\theta} \right\},$$

and under the Condition **(C)** we have the convergence of the random step functions

$$(M^{(n)}, Y^n) \xrightarrow{\mathcal{D}} (W, Y) \quad \text{in } D([0, 1] \rightarrow \mathbb{C}^{2(d_1+\dots+d_r)}),$$

where the process $Y(t)$, $t \in [0, 1]$ is defined by

$$dY(t) = AY(t) dt + \Sigma dW(t), \quad Y(0) = 0,$$

with

$$A := \text{diag} \{J(h, d_1), \dots, J(h, d_r)\}, \quad \Sigma := \text{diag} \{\Sigma^{(1)}, \dots, \Sigma^{(r)}\},$$

where $\Sigma^{(j)} := \text{diag}(1, 0, \dots, 0)$, $\Sigma^{(j)}$ is a $d_j \times d_j$ matrix, $j = 1, \dots, r$.

Let $d := \max\{d_1, \dots, d_r\}$. As in Theorem 1, we obtain that

$$n^d (\hat{\lambda}_n - \lambda_n) \xrightarrow{\mathcal{D}} e^{(d-1)\theta i} \frac{\sum_{j:d_j=d} \int_0^1 \overline{Y_{d_1+\dots+d_j}(t)} dW_{d_1+\dots+d_j}(t)}{\sum_{j:d_j=d} \int_0^1 |Y_{d_1+\dots+d_j}(t)|^2 dt},$$

thus the submodels of the greatest dimension cooperate with each other and dominate the rest.

It is interesting to note that for $d \geq 2$ the processes $W_{d_1+\dots+d_j}$, and $Y_{d_1+\dots+d_j}$ for $\{j : d_j = d\}$ are independent since the processes $Y_{d_1+\dots+d_j}$, $\{j : d_j = d\}$ are driven by $W_{d_1+\dots+d_{j-1}+1}$, $\{j : d_j = d\}$.

Chapter 5

Estimation of the Mean of Multivariate AR Processes

Our aim was to investigate the problem of estimation of the mean for stationary and zero start multidimensional autoregressive processes.

5.1 Case of Stationary AR Process

Consider a d -dimensional stationary AR process $\{\tilde{Y}(t) : t \in \mathbb{R}\}$, which is the weakly stationary solution of

$$d\tilde{Y}(t) = -A\tilde{Y}(t) dt + \Sigma dW(t), \quad t \in \mathbb{R}, \quad (5.1)$$

where A and Σ are $d \times d$ matrices such that all eigenvalues of A have necessarily positive real part, and $\{W(t) : t \in \mathbb{R}\}$ is a d -dimensional standard Wiener process, i.e., $\{W(t) : t \geq 0\}$ and $\{W(-t) : t \geq 0\}$ are independent d -dimensional standard Wiener processes. In fact, $\{W(t) : t \in \mathbb{R}\}$ is a continuous zero mean Gaussian process with

$$\text{Cov}(W(s), W(t)) = \mathbb{E}W(s)W(t)^\top = \begin{cases} (|s| \wedge |t|) \cdot I, & \text{if } st \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

or, equivalently, $\{W(t) : t \in \mathbb{R}\}$ is a continuous process with stationary independent increments such that for the increments we have

$$W(t) - W(s) \sim \mathcal{N}_d(0, |t - s| \cdot I).$$

The process $\{\tilde{Y}(t) : t \in \mathbb{R}\}$ is a zero mean Gaussian process with

$$\text{Cov}(\tilde{Y}(t), \tilde{Y}(s)) = \mathbb{E}\tilde{Y}(t)\tilde{Y}(s)^\top = e^{-(t-s)A}B(0), \quad \text{for } s \leq t,$$

where the $d \times d$ matrix $B(0)$ is the only solution of the matrix equation

$$AB(0) + B(0)A^\top = \Sigma\Sigma^\top. \quad (5.2)$$

We remark that $B(0)$ is a symmetric, positive definite matrix and

$$B(0) = \int_0^\infty e^{-uA}\Sigma\Sigma^\top e^{-uA^\top} du. \quad (5.3)$$

Moreover,

$$\tilde{Y}(t) = e^{-tA} \left(\tilde{Y}(0) + \int_0^t e^{uA}\Sigma dW(u) \right) = \int_{-\infty}^t e^{-(t-u)A}\Sigma dW(u). \quad (5.4)$$

Now, consider a shifted stationary AR process $\{\tilde{Z}(t) : t \in \mathbb{R}\}$, given by

$$\tilde{Z}(t) = \tilde{Y}(t) + Hm, \quad t \in \mathbb{R},$$

where H is a known $d \times d$ constraint matrix and $m \in \mathbb{R}^d$ is an unknown parameter vector.

Theorem 5.1.1. *Let $[T_1, T_2] \subset \mathbb{R}$. If H^{-1} exists then the maximum likelihood estimator of m based on the observation of $\{\tilde{Z}(t) : t \in [T_1, T_2]\}$ is given by*

$$\tilde{m} = U^{-1}V_{\tilde{Z}}, \quad (5.5)$$

where

$$U = H^\top B(0)^{-1}H + (T_2 - T_1)H^\top A^\top (\Sigma\Sigma^\top)^\ominus AH, \quad (5.6)$$

$$V_{\tilde{Z}} = H^\top B(0)^{-1}\tilde{Z}(T_1) + H^\top A^\top (\Sigma\Sigma^\top)^\ominus \left(\tilde{Z}(T_2) - \tilde{Z}(T_1) + A \int_{T_1}^{T_2} \tilde{Z}(t) dt \right). \quad (5.7)$$

Moreover,

$$\tilde{m} \sim \mathcal{N}_d(m, U^{-1}). \quad (5.8)$$

We need the following lemma.

Lemma 5.1.2. *Denote by $\mathbb{P}_{\tilde{Y}}$ and $\mathbb{P}_{\tilde{Z}}$ the measures generated on $C([T_1, T_2] \rightarrow \mathbb{R}^d)$ by the processes $\tilde{Y}$ and $\tilde{Z}$, respectively. Then, the measures $\mathbb{P}_{\tilde{Y}}$ and $\mathbb{P}_{\tilde{Z}}$ are equivalent and the Radon–Nikodym derivative has the form*

$$\frac{d\mathbb{P}_{\tilde{Z}}}{d\mathbb{P}_{\tilde{Y}}}(X) = \exp \left\{ -\frac{1}{2} \langle Um, m \rangle + \langle V_X, m \rangle \right\}. \quad (5.9)$$

Proof. Denote by $\mathbb{P}_W$ the measure generated on $C([T_1, T_2] \rightarrow \mathbb{R}^d)$ by the process W . Then, the measures $\mathbb{P}_{\tilde{Y}}$ and $\mathbb{P}_W$ are equivalent and the Radon–Nikodym derivative has the form

$$\begin{aligned} \frac{d\mathbb{P}_{\tilde{Y}}}{d\mathbb{P}_W}(X) &= \frac{1}{\sqrt{2\pi \text{Det} B(0)}} \exp \left\{ -\frac{1}{2} \langle B(0)^{-1} X(T_1), X(T_1) \rangle \right. \\ &\quad \left. - \frac{1}{2} \int_{T_1}^{T_2} \langle (\Sigma \Sigma^\top)^\ominus AX(t), AX(t) \rangle dt \right. \\ &\quad \left. - \int_{T_1}^{T_2} \langle (\Sigma \Sigma^\top)^\ominus AX(t), dX(t) \rangle \right\}, \end{aligned} \quad (5.10)$$

see, e.g., Arató [3, § 2.3.3, Theorem 2].

Consequently, the measures $\mathbb{P}_{\tilde{Z}}$ and $\mathbb{P}_W$ are equivalent and the Radon–Nikodym derivative has the form

$$\begin{aligned} \frac{d\mathbb{P}_{\tilde{Z}}}{d\mathbb{P}_W}(X) &= \frac{d\mathbb{P}_{\tilde{Y}}}{d\mathbb{P}_W}(X - Hm) \\ &= \frac{1}{\sqrt{2\pi \text{Det} B(0)}} \exp \left\{ -\frac{1}{2} \langle B(0)^{-1} (X(T_1) - Hm), X(T_1) - Hm \rangle \right. \\ &\quad \left. - \frac{1}{2} \int_{T_1}^{T_2} \langle (\Sigma \Sigma^\top)^\ominus A(X(t) - Hm), A(X(t) - Hm) \rangle dt \right. \\ &\quad \left. - \int_{T_1}^{T_2} \langle (\Sigma \Sigma^\top)^\ominus A(X(t) - Hm), dX(t) \rangle \right\}. \end{aligned} \quad (5.11)$$

We have

$$\begin{aligned} &\langle B(0)^{-1} (X(T_1) - Hm), X(T_1) - Hm \rangle \\ &= \langle B(0)^{-1} X(T_1), X(T_1) \rangle - 2 \langle H^\top B(0)^{-1} X(T_1), m \rangle + \langle H^\top B(0)^{-1} Hm, m \rangle, \end{aligned} \quad (5.12)$$

$$\begin{aligned}
& \int_{T_1}^{T_2} \langle (\Sigma \Sigma^\top)^\ominus A(X(t) - Hm), A(X(t) - Hm) \rangle dt \\
&= \int_{T_1}^{T_2} \langle (\Sigma \Sigma^\top)^\ominus AX(t), AX(t) \rangle dt - 2 \int_{T_1}^{T_2} \langle H^\top A^\top (\Sigma \Sigma^\top)^\ominus AX(t), m \rangle dt \quad (5.13) \\
&+ \int_{T_1}^{T_2} \langle H^\top A^\top (\Sigma \Sigma^\top)^\ominus AHm, m \rangle dt,
\end{aligned}$$

$$\begin{aligned}
& \int_{T_1}^{T_2} \langle (\Sigma \Sigma^\top)^\ominus A(X(t) - Hm), dX(t) \rangle \\
&= \int_{T_1}^{T_2} \langle (\Sigma \Sigma^\top)^\ominus AX(t), dX(t) \rangle - \int_{T_1}^{T_2} \langle H^\top A^\top (\Sigma \Sigma^\top)^\ominus dX(t), m \rangle, \quad (5.14)
\end{aligned}$$

hence

$$\frac{d\mathbb{P}_{\tilde{Z}}}{d\mathbb{P}_{\tilde{Y}}}(X) = \frac{d\mathbb{P}_{\tilde{Z}}}{d\mathbb{P}_W}(X) \frac{d\mathbb{P}_W}{d\mathbb{P}_{\tilde{Y}}}(X) = \exp \left\{ -\frac{1}{2} \langle Um, m \rangle + \langle V_X, m \rangle \right\}.$$

□

Proof of Theorem (5.1.1.) Obviously, U is a symmetric, positive definite matrix. Denoting by $\sqrt{U}$ and $\sqrt{U^{-1}}$ the only symmetric, positive definite square root of U and U^{-1} , respectively, we obtain

$$\begin{aligned}
-\frac{1}{2} \langle Um, m \rangle + \langle V_{\tilde{Z}}, m \rangle &= -\frac{1}{2} \left(\langle \sqrt{U}m, \sqrt{U}m \rangle - 2 \langle \sqrt{U^{-1}}V_{\tilde{Z}}, \sqrt{U}m \rangle \right) \\
&= -\frac{1}{2} \left(\|\sqrt{U}m - \sqrt{U^{-1}}V_{\tilde{Z}}\|^2 - \langle \sqrt{U^{-1}}V_{\tilde{Z}}, \sqrt{U^{-1}}V_{\tilde{Z}} \rangle \right) \\
&= -\frac{1}{2} \left(\|\sqrt{U}(m - U^{-1}V_{\tilde{Z}})\|^2 - \langle U^{-1}V_{\tilde{Z}}, V_{\tilde{Z}} \rangle \right),
\end{aligned}$$

which implies that $\tilde{m} = U^{-1}V_{\tilde{Z}}$.

Clearly, $\tilde{m}$ is normally distributed with mean m . The aim of the following discussion is to show that $\text{Var } \tilde{m} = U^{-1}$. We have $V_{\tilde{Z}} = V_{\tilde{Y}} + Um$. Consequently, $\tilde{m} = U^{-1}V_{\tilde{Y}} + m$, hence

$$\text{Var } \tilde{m} = \mathbb{E}(\tilde{m} - m)(\tilde{m} - m)^\top = U^{-1} \mathbb{E} \left(V_{\tilde{Y}} V_{\tilde{Y}}^\top \right) U^{-1}.$$

Thus, it is sufficient to prove that $\mathbb{E}V_{\tilde{Y}}V_{\tilde{Y}}^\top = U$. Let us write $V_{\tilde{Y}}$ in the form

$$\begin{aligned} V_{\tilde{Y}} &= H^\top B(0)^{-1}\tilde{Y}(T_1) + H^\top A^\top (\Sigma\Sigma^\top)^\ominus \int_{T_1}^{T_2} (A\tilde{Y}(t) dt + d\tilde{Y}(t)) \\ &= H^\top B(0)^{-1}\tilde{Y}(T_1) + H^\top A^\top (\Sigma\Sigma^\top)^\ominus \int_{T_1}^{T_2} \Sigma dW(t) \\ &= H^\top B(0)^{-1}\tilde{Y}(T_1) + H^\top A^\top (\Sigma\Sigma^\top)^\ominus \Sigma(W(T_2) - W(T_1)), \end{aligned}$$

where $\tilde{Y}(T_1) = \int_{-\infty}^{T_1} e^{-(T_1-u)A} \Sigma dW(u)$ and $W(T_2) - W(T_1)$ are independent, hence

$$\begin{aligned} \text{Var } V_{\tilde{Y}} &= \text{Var} \left(H^\top B(0)^{-1}\tilde{Y}(T_1) \right) + \text{Var} \left(H^\top A^\top (\Sigma\Sigma^\top)^\ominus \Sigma(W(T_2) - W(T_1)) \right) \\ &= H^\top B(0)^{-1}B(0)B(0)^{-1}H + (T_2 - T_1)H^\top A^\top (\Sigma\Sigma^\top)^\ominus \Sigma\Sigma^\top (\Sigma\Sigma^\top)^\ominus AH \\ &= H^\top B(0)^{-1}H + (T_2 - T_1)H^\top A^\top (\Sigma\Sigma^\top)^\ominus AH = U. \end{aligned}$$

□

Remark 5.1.3. Lemma (5.1.2.) can be proved directly by the help of finite dimensional approximation as well. For $T_1 = t_1 < t_2 < \dots < t_n = T_2$, the random vectors

$$\tilde{Y}(t_1), \quad \tilde{Y}(t_2) - e^{-(\Delta t_2)A}\tilde{Y}(t_1), \quad \dots, \quad \tilde{Y}(t_n) - e^{-(\Delta t_n)A}\tilde{Y}(t_{n-1})$$

are uncorrelated and have a joint normal distribution, hence they are independent with zero mean, $\text{Var } \tilde{Y}(t_1) = B(0)$ and

$$\text{Var} \left(\tilde{Y}(t_j) - e^{-(\Delta t_j)A}\tilde{Y}(t_{j-1}) \right) = B(0) - e^{-(\Delta t_j)A}B(0)e^{-(\Delta t_j)A^\top},$$

where $\Delta t_j := t_j - t_{j-1}$. Thus, the joint density of the random vectors $(\tilde{Y}(t_1), \dots, \tilde{Y}(t_n))$ has the form

$$\begin{aligned} f_{\tilde{Y}(t_1), \dots, \tilde{Y}(t_n)}(x_1, \dots, x_n) &= c \exp \left\{ -\frac{1}{2} \langle B(0)^{-1}x_1, x_1 \rangle - \right. \\ &\quad \left. -\frac{1}{2} \sum_{j=2}^n \left\langle \left(B(0) - e^{-(\Delta t_j)A}B(0)e^{-(\Delta t_j)A^\top} \right)^{-1} \right. \right. \\ &\quad \left. \left. \left(x_j - e^{-(\Delta t_j)A}x_{j-1} \right), x_j - e^{-(\Delta t_j)A}x_{j-1} \right\rangle \right\}, \end{aligned}$$

with a suitable normalizing constant c . Clearly, the joint density of the random vectors $(\tilde{Z}(t_1), \dots, \tilde{Z}(t_n))$ has the form

$$f_{\tilde{Z}(t_1), \dots, \tilde{Z}(t_n)}(x_1, \dots, x_n) = f_{\tilde{Y}(t_1), \dots, \tilde{Y}(t_n)}(x_1 - Hm, \dots, x_n - Hm).$$

If $\mathbb{P}_{\tilde{Y}(t_1), \dots, \tilde{Y}(t_n)}$ and $\mathbb{P}_{\tilde{Z}(t_1), \dots, \tilde{Z}(t_n)}$ denote the distribution on $\mathbb{R}^n$ of the random vectors $(\tilde{Y}(t_1), \dots, \tilde{Y}(t_n))$ and $(\tilde{Z}(t_1), \dots, \tilde{Z}(t_n))$, respectively, then $\mathbb{P}_{\tilde{Y}(t_1), \dots, \tilde{Y}(t_n)}$ and $\mathbb{P}_{\tilde{Z}(t_1), \dots, \tilde{Z}(t_n)}$ are equivalent and

$$\frac{d\mathbb{P}_{\tilde{Z}(t_1), \dots, \tilde{Z}(t_n)}}{d\mathbb{P}_{\tilde{Y}(t_1), \dots, \tilde{Y}(t_n)}}(x_1, \dots, x_n) = \frac{f_{\tilde{Z}(t_1), \dots, \tilde{Z}(t_n)}(x_1, \dots, x_n)}{f_{\tilde{Y}(t_1), \dots, \tilde{Y}(t_n)}(x_1, \dots, x_n)}.$$

We have $e^{-(\Delta t_j)A} \approx I - (\Delta t_j)A$, hence

$$B(0) - e^{-(\Delta t_j)A}B(0)e^{-(\Delta t_j)A^\top} \approx (\Delta t_j)(AB(0) + B(0)A^\top) = (\Delta t_j)\Sigma\Sigma^\top.$$

Moreover,

$$x_j - Hm - e^{-(\Delta t_j)A}(x_{j-1} - Hm) \approx \Delta x_j + (\Delta t_j)A(x_{j-1} - Hm),$$

where $\Delta x_j := x_j - x_{j-1}$. Hence,

$$\begin{aligned} & \frac{d\mathbb{P}_{\tilde{Z}(t_1), \dots, \tilde{Z}(t_n)}}{d\mathbb{P}_{\tilde{Y}(t_1), \dots, \tilde{Y}(t_n)}}(x_1, \dots, x_n) \\ & \approx \exp \left\{ -\frac{1}{2} \left\langle \left(B(0)^{-1} + \sum_{j=2}^n (\Delta t_j)A^\top (\Sigma\Sigma^\top)^\ominus A \right) Hm, Hm \right\rangle \right. \\ & \quad \left. + \left\langle B(0)^{-1}x_1 + \sum_{j=2}^n A^\top (\Sigma\Sigma^\top)^\ominus (\Delta x_j + (\Delta t_j)Ax_{j-1}), Hm \right\rangle \right\}. \end{aligned}$$

It can be easily shown that, as $n \rightarrow \infty$ and $\max_{1 \leq j \leq n} \Delta t_j \rightarrow 0$,

$$\frac{d\mathbb{P}_{\tilde{Z}(t_1), \dots, \tilde{Z}(t_n)}}{d\mathbb{P}_{\tilde{Y}(t_1), \dots, \tilde{Y}(t_n)}}(\tilde{Y}(t_1), \dots, \tilde{Y}(t_n)) \xrightarrow{\mathbb{P}} \exp \left\{ -\frac{1}{2} \langle Um, m \rangle + \langle V_{\tilde{Y}}, m \rangle \right\}.$$

Proving $V_{\tilde{Y}} \sim \mathcal{N}_d(0, U)$, we obtain

$$\mathbb{E} \exp \left\{ -\frac{1}{2} \langle Um, m \rangle + \langle V_{\tilde{Y}}, m \rangle \right\} = 1,$$

and by Theorem VII.6.2 in Shiriyayev [30], we conclude the statement of Lemma (5.1.2.).

5.1.1 Examples of Stationary Processes

Example 5.1.4. One-dimensional complex-valued AR(1). Consider a one-dimensional complex-valued stationary AR(1) process $\{\tilde{X}(t) : t \in \mathbb{R}\}$ which is the weakly stationary solution of

$$d\tilde{X}(t) = -\alpha\tilde{X}(t)dt + dW(t), \quad t \in \mathbb{R},$$

where $\{W(t) : t \in \mathbb{R}\}$ is a complex-valued standard Wiener process (i.e., $W(t) = \widetilde{W}_1(t) + i\widetilde{W}_2(t)$, where $\{\widetilde{W}_1(t), \widetilde{W}_2(t) : t \in \mathbb{R}\}$ is a 2-dimensional standard Wiener process), and $\alpha = \lambda + i\omega$ with the damping parameter $\lambda > 0$ and the periodical parameter $\omega \in \mathbb{R}$. (See Arató, Kolmogorov and Sinay [6], Arató [2].) Let us use the notation $\operatorname{Re}\tilde{X}(t) = \tilde{Y}_1(t)$, $\operatorname{Im}\tilde{X}(t) = \tilde{Y}_2(t)$. Clearly,

$$\begin{pmatrix} d\tilde{Y}_1(t) \\ d\tilde{Y}_2(t) \end{pmatrix} = - \begin{pmatrix} \lambda & \omega \\ -\omega & \lambda \end{pmatrix} \begin{pmatrix} \tilde{Y}_1(t)dt \\ \tilde{Y}_2(t)dt \end{pmatrix} + \begin{pmatrix} d\widetilde{W}_1(t) \\ d\widetilde{W}_2(t) \end{pmatrix},$$

hence $\{(\operatorname{Re}\tilde{X}(t), \operatorname{Im}\tilde{X}(t))^\top : t \in \mathbb{R}\}$ is a 2-dimensional stationary AR process with coefficient matrix

$$A = \begin{pmatrix} \lambda & \omega \\ -\omega & \lambda \end{pmatrix}.$$

The only solution of the matrix equation $AB(0) + B(0)A^\top = I$ is $B(0) = \frac{1}{2\lambda}I$, hence $B(0)^{-1} = 2\lambda I$. Obviously, $A^\top A = (\lambda^2 + \omega^2)I$, thus $U = (2\lambda + (T_2 - T_1)(\lambda^2 + \omega^2))I$. Let $\tilde{Z}(t) := \tilde{Y}(t) + m$, $t \in \mathbb{R}$, where $m = (m_1, m_2) \in \mathbb{R}^2$ is unknown. Then,

$$V_{\tilde{Z}} = 2\lambda\tilde{Z}(T_1) + A^\top(\tilde{Z}(T_2) - \tilde{Z}(T_1)) + (\lambda^2 + \omega^2) \int_{T_1}^{T_2} \tilde{Z}(t) dt,$$

consequently,

$$\tilde{m}_1 = \frac{\lambda(\tilde{Z}_1(T_1) + \tilde{Z}_1(T_2)) - \omega(\tilde{Z}_2(T_2) - \tilde{Z}_2(T_1)) + (\lambda^2 + \omega^2) \int_{T_1}^{T_2} \tilde{Z}_1(t) dt}{2\lambda + (T_2 - T_1)(\lambda^2 + \omega^2)}, \quad (5.15)$$

$$\tilde{m}_2 = \frac{\lambda(\tilde{Z}_2(T_1) + \tilde{Z}_2(T_2)) + \omega(\tilde{Z}_1(T_2) - \tilde{Z}_1(T_1)) + (\lambda^2 + \omega^2) \int_{T_1}^{T_2} \tilde{Z}_2(t) dt}{2\lambda + (T_2 - T_1)(\lambda^2 + \omega^2)}, \quad (5.16)$$

see Arató [3, §4.4.1].

Moreover,

$$(\tilde{m}_1, \tilde{m}_2)^\top \sim \mathcal{N}_2 \left(m, \frac{1}{2\lambda + (T_2 - T_1)(\lambda^2 + \omega^2)} I \right).$$

Particularly, $\tilde{m}_1$ and $\tilde{m}_2$ are independent.

We have the following asymptotic behaviour

$$\lim_{T_2-T_1 \rightarrow \infty} (T_2 - T_1) \text{Var } \tilde{m}_1 = \lim_{T_2-T_1 \rightarrow \infty} (T_2 - T_1) \text{Var } \tilde{m}_2 = \frac{1}{\lambda^2 + \omega^2},$$

hence $\tilde{m}_1$ and $\tilde{m}_2$ are asymptotically consistent as $T_2 - T_1 \rightarrow \infty$. Moreover, if $\omega \neq 0$ is fixed, then we get, in contrast to the one-dimensional real-valued case, that $\text{Var } \tilde{m}_1$ and $\text{Var } \tilde{m}_2$ remain bounded uniformly in λ , since

$$\lim_{\lambda \downarrow 0} \text{Var } \tilde{m}_1 = \lim_{\lambda \downarrow 0} \text{Var } \tilde{m}_2 = \frac{1}{(T_2 - T_1)\omega^2}.$$

Example 5.1.5. One-dimensional real-valued AR(2). Consider a one-dimensional stationary AR(2) process $\{\tilde{X}(t) : t \in \mathbb{R}\}$ which is the weakly stationary solution of

$$d^2 \tilde{X}(t) = -c_1 d\tilde{X}(t) - c_2 \tilde{X}(t) dt + dW(t), \quad t \in \mathbb{R}. \quad (5.17)$$

(We remark that now $\frac{d\tilde{X}(t)}{dt} = \tilde{X}'(t)$ exists.) Let us write the characteristic polynomial of the process in the form

$$\psi(\lambda) = \lambda^2 + c_1 \lambda + c_2 = (\lambda + \lambda_1)(\lambda + \lambda_2).$$

Equation (5.17) has a (unique) weakly stationary solution if and only if for the damping parameters we have $\text{Re} \lambda_1 > 0$, $\text{Re} \lambda_2 > 0$. Let us use the notation $\tilde{X}(t) = \tilde{Y}_1(t)$, $\tilde{X}'(t) = \tilde{Y}_2(t)$. Clearly,

$$\begin{pmatrix} d\tilde{Y}_1(t) \\ d\tilde{Y}_2(t) \end{pmatrix} = - \begin{pmatrix} 0 & -1 \\ c_2 & c_1 \end{pmatrix} \begin{pmatrix} \tilde{Y}_1(t) dt \\ \tilde{Y}_2(t) dt \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ d\tilde{W}(t) \end{pmatrix},$$

hence $\{(\tilde{X}(t), \tilde{X}'(t))^\top : t \in \mathbb{R}\}$ is a 2-dimensional stationary AR process with coefficient matrices

$$A = \begin{pmatrix} 0 & -1 \\ c_2 & c_1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The only solution of the matrix equation $AB(0) + B(0)A^\top = I$ is

$$B(0) = \begin{pmatrix} \frac{1}{2c_1 c_2} & 0 \\ 0 & \frac{1}{2c_1} \end{pmatrix}, \quad \text{hence} \quad B(0)^{-1} = \begin{pmatrix} 2c_1 c_2 & 0 \\ 0 & 2c_1 \end{pmatrix}.$$

Obviously,

$$(\Sigma \Sigma^\top)^\ominus = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

hence

$$A^\top (\Sigma \Sigma^\top)^\ominus = \begin{pmatrix} 0 & c_2 \\ 0 & c_1 \end{pmatrix}, \quad A^\top (\Sigma \Sigma^\top)^\ominus A = \begin{pmatrix} c_2^2 & c_1 c_2 \\ c_1 c_2 & c_1^2 \end{pmatrix}.$$

Let $\tilde{Z}(t) := \tilde{X}(t) + m$, $t \in \mathbb{R}$, where $m \in \mathbb{R}$ is unknown. Then $\tilde{Z}'(t) = \tilde{X}'(t)$, $t \in \mathbb{R}$, hence

$$\begin{pmatrix} d\tilde{Z}(t) \\ d\tilde{Z}'(t) \end{pmatrix} = Y(t) + H \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad t \in \mathbb{R},$$

where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} m \\ 0 \end{pmatrix}.$$

Thus

$$U = \begin{pmatrix} 2c_1 c_2 + (T_2 - T_1)c_2^2 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\begin{aligned} V_{\tilde{Z}} &= \begin{pmatrix} 2c_1 c_2 \tilde{Z}(T_1) + \int_{T_1}^{T_2} (c_2^2 \tilde{Z}(t) + c_1 c_2 \tilde{Z}'(t)) dt + \int_{T_1}^{T_2} c_2 d\tilde{Z}'(t) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} c_1 c_2 (\tilde{Z}(T_1) + \tilde{Z}(T_2)) + c_2 (\tilde{Z}'(T_2) - \tilde{Z}'(T_1)) + c_2^2 \int_{T_1}^{T_2} \tilde{Z}(t) dt \\ 0 \end{pmatrix}. \end{aligned}$$

By Lemma (5.1.2.),

$$\frac{d\mathbb{P}_{\tilde{Z}}}{d\mathbb{P}_{\tilde{Y}}}(\tilde{Z}) = \exp \left\{ -\frac{1}{2} u_{1,1} m^2 + v_1(\tilde{Z}) m \right\},$$

where $\mathbb{P}_{\tilde{Z}}$ denotes the measure generated on $C([T_1, T_2] \rightarrow \mathbb{R}^2)$ by the process $(\tilde{Z}, \tilde{Z}')$, and $u_{1,1}$, $v_1(\tilde{Z})$ denote the first entry of U and $V_{\tilde{Z}}$, respectively. Consequently, the maximum likelihood estimator of m is

$$\begin{aligned} \tilde{m} &= \frac{c_1(\tilde{Z}(T_1) + \tilde{Z}(T_2)) + \tilde{Z}'(T_2) - \tilde{Z}'(T_1) + c_2 \int_{T_1}^{T_2} \tilde{Z}(t) dt}{2c_1 + (T_2 - T_1)c_2} \\ &\sim \mathcal{N}_1 \left(m, \frac{1}{2c_1 c_2 + (T_2 - T_1)c_2^2} \right), \end{aligned} \tag{5.18}$$

see Arató [3, § 2.2.2, § 4.4.5]. We have the following asymptotic behaviour:

$$\lim_{T_2 - T_1 \rightarrow \infty} (T_2 - T_1) \text{Var } \tilde{m} = \frac{1}{c_2^2}, \quad \text{hence} \quad \lim_{T_2 - T_1 \rightarrow \infty} \text{Var } \tilde{m} = 0,$$

thus $\tilde{m}$ is asymptotically consistent as $T_2 - T_1 \rightarrow \infty$. Furthermore,

$$\lim_{\lambda_1 \downarrow 0} \lambda_1 \text{Var } \tilde{m} = \frac{1}{2\lambda_2^2}, \quad \text{hence } \lim_{\lambda_1 \downarrow 0} \text{Var } \tilde{m} = \infty,$$

hence $\text{Var } \tilde{m}$ is unbounded as $\lambda_1 \downarrow 0$.

If $\lambda_1 = \lambda_2 = \lambda > 0$ are double roots, then

$$\lim_{\lambda \downarrow 0} \lambda^3 \text{Var } \tilde{m} = \frac{1}{4},$$

hence $\text{Var } \tilde{m}$ is unbounded as $\lambda \downarrow 0$.

If the characteristic polynomial has conjugated complex roots $\lambda_1 = \lambda + i\omega$, $\lambda_2 = \lambda - i\omega$, where $\lambda, \omega \in \mathbb{R}$, $\lambda > 0$ is the damping parameter and $\omega \neq 0$ is the periodical parameter, then

$$\lim_{\lambda \downarrow 0} \text{Var } \tilde{m} = \frac{1}{(T_2 - T_1)\omega^4},$$

and we get that $\text{Var } \tilde{m}$ remains bounded uniformly in λ .

Example 5.1.6. One-dimensional real-valued $AR(p)$. Consider a one-dimensional stationary $AR(p)$ process $\{\tilde{X}(t) : t \in \mathbb{R}\}$ which is the weakly stationary solution of

$$d^p \tilde{X}(t) = -c_1 d^{p-1} \tilde{X}(t) - \dots - c_{p-1} d \tilde{X}(t) - c_p \tilde{X}(t) dt + dW(t), \quad t \in \mathbb{R}. \quad (5.19)$$

Let us write the characteristic polynomial of the process in the form

$$\psi(\lambda) = \lambda^p + c_1 \lambda^{p-1} + c_2 \lambda^{p-2} + \dots + c_p = \prod_{j=1}^p (\lambda + \lambda_j).$$

Equation (5.19) has a (unique) weakly stationary solution if and only if $\text{Re} \lambda_j > 0$ for all $j = 1, 2, \dots, p$. If we use the notation $\tilde{X}(t) = \tilde{Y}_1(t)$, $\tilde{X}'(t) = \tilde{Y}_2(t)$, $\dots$, $\tilde{X}^{(p-1)}(t) = \tilde{Y}_p(t)$ and $\tilde{Y}(t) = (\tilde{Y}_1(t), \dots, \tilde{Y}_p(t))^T$, then we have

$$d\tilde{Y}(t) = -A\tilde{Y}(t) dt + \Sigma dW(t), \quad t \in \mathbb{R}, \quad (5.20)$$

where

$$A = \begin{pmatrix} 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -1 \\ c_p & c_{p-1} & c_{p-2} & \dots & c_2 & c_1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix},$$

hence $\{(\tilde{X}(t), \tilde{X}'(t), \dots, \tilde{X}^{(p-1)}(t))^\top : t \in \mathbb{R}\}$ is a p -dimensional stationary AR process.

We have $B(0)^{-1} = (d_{j,k})_{1 \leq j,k \leq p}$, where

$$d_{j,k} = \begin{cases} 2 \sum_{\ell=1}^p (-1)^\ell c_{p-k-\ell} c_{p-j+\ell+1} & \text{if } j \equiv k \pmod{2} \text{ and } j \leq k, \\ 2 \sum_{\ell=1}^p (-1)^\ell c_{p-j-\ell} c_{p-k+\ell+1} & \text{if } j \equiv k \pmod{2} \text{ and } j > k, \\ 0 & \text{otherwise,} \end{cases} \quad (5.21)$$

where

$$c_\ell := \begin{cases} 1, & \text{if } \ell = 0, \\ 0, & \text{if } \ell < 0 \text{ or } \ell > p, \end{cases}$$

see Arató [3, § 2.2.2, § 4.4.5].

For example, if $p = 3$, $p = 4$ or $p = 5$, then

$$B(0)^{-1} = 2 \begin{pmatrix} c_2 c_3 & 0 & c_3 \\ 0 & c_1 c_2 - c_3 & 0 \\ c_3 & 0 & c_1 \end{pmatrix},$$

$$B(0)^{-1} = 2 \begin{pmatrix} c_3 c_4 & 0 & c_1 c_4 & 0 \\ 0 & c_2 c_3 - c_1 c_4 & 0 & c_3 \\ c_1 c_4 & 0 & c_1 c_2 - c_3 & 0 \\ 0 & c_3 & 0 & c_1 \end{pmatrix},$$

$$B(0)^{-1} = 2 \begin{pmatrix} c_4 c_5 & 0 & c_2 c_5 & 0 & c_5 \\ 0 & c_3 c_4 - c_2 c_5 & 0 & c_1 c_4 - c_5 & 0 \\ c_2 c_5 & 0 & c_2 c_3 - c_1 c_4 + c_5 & 0 & c_3 \\ 0 & c_1 c_4 - c_5 & 0 & c_1 c_2 - c_3 & 0 \\ c_5 & 0 & c_3 & 0 & c_1 \end{pmatrix}.$$

Clearly $(\Sigma \Sigma^\top)^\ominus = \Sigma$, hence

$$A^\top (\Sigma \Sigma^\top)^\ominus = \begin{pmatrix} 0 & 0 & \cdots & 0 & c_p \\ 0 & 0 & \cdots & 0 & c_{p-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & c_2 \\ 0 & 0 & \cdots & 0 & c_1 \end{pmatrix},$$

$$\begin{aligned}
A^\top (\Sigma \Sigma^\top)^\ominus A &= \begin{pmatrix} c_p^2 & c_{p-1}c_p & \cdots & c_1c_p \\ c_{p-1}c_p & c_{p-1}^2 & \cdots & c_1c_{p-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_1c_p & c_1c_{p-1} & \cdots & c_1^1 \end{pmatrix} \\
&= \begin{pmatrix} c_p \\ c_{p-1} \\ \vdots \\ c_1 \end{pmatrix} (c_p \quad c_{p-1} \quad \cdots \quad c_1).
\end{aligned}$$

Let $\tilde{Z}(t) := \tilde{X}(t) + m$, $t \in \mathbb{R}$, where $m \in \mathbb{R}$ is unknown. Then $\tilde{Z}^{(j)}(t) = \tilde{X}^{(j)}(t)$, $t \in \mathbb{R}$, $j \geq 1$, hence

$$\begin{pmatrix} \tilde{Z}(T_1) \\ \tilde{Z}'(T_1) \\ \vdots \\ \tilde{Z}^{(p-1)}(T_1) \end{pmatrix} = Y(t) + H \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_p \end{pmatrix}, \quad t \in \mathbb{R},$$

where

$$H = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_p \end{pmatrix} = \begin{pmatrix} m \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus

$$U = \begin{pmatrix} 2c_{p-1}c_p + (T_2 - T_1)c_p^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (5.22)$$

$$V_{\tilde{Z}} = \begin{pmatrix} c_p \sum_{j=1}^p c_{p-j} (\tilde{Z}^{(j-1)}(T_2) + (-1)^{j-1} \tilde{Z}^{(j-1)}(T_1)) + c_p^2 \int_{T_1}^{T_2} \tilde{Z}(t) dt \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (5.23)$$

By Lemma (5.1.2.)

$$\frac{d\mathbb{P}_{\tilde{Z}}}{d\mathbb{P}_{\tilde{Y}}}(\tilde{Z}) = \exp \left\{ -\frac{1}{2} u_{1,1} m^2 + v_1(\tilde{Z}) m \right\},$$

where $\mathbb{P}_{\tilde{Z}}$ denotes the measure generated on $C([T_1, T_2] \rightarrow \mathbb{R}^2)$ by the process $(\tilde{Z}, \tilde{Z}', \dots, \tilde{Z}^{(p-1)})$, and $u_{1,1}, v_1(\tilde{Z})$ denote the first entry of U and $V_{\tilde{Z}}$, respectively.

Consequently, the maximum likelihood estimator of m is

$$\begin{aligned} \tilde{m} &= \frac{\sum_{j=1}^p c_{p-j}(\tilde{Z}^{(j-1)}(T_2) + (-1)^{j-1} \tilde{Z}^{(j-1)}(T_1)) + c_p \int_{T_1}^{T_2} \tilde{Z}(t) dt}{2c_{p-1} + (T_2 - T_1)c_p} \\ &\sim \mathcal{N}_1 \left(m_1, \frac{1}{2c_{p-1}c_p + (T_2 - T_1)c_p^2} \right), \end{aligned} \quad (5.24)$$

see Arató [3, § 2.2.2, § 4.4.5].

We have the following asymptotic behaviour:

$$\lim_{T_2 - T_1 \rightarrow \infty} (T_2 - T_1) \text{Var } \tilde{m} = \frac{1}{c_p^2},$$

hence $\hat{m}$ is asymptotically consistent as $T_2 - T_1 \rightarrow \infty$. Furthermore,

$$\lim_{\lambda_1 \downarrow 0} \lambda_1 \text{Var } \tilde{m} = \frac{1}{2\lambda_2^2 \dots \lambda_p^2}, \quad \text{hence } \lim_{\lambda_1 \downarrow 0} \text{Var } \tilde{m} = \infty,$$

thus $\text{Var } \tilde{m}$ is unbounded as $\lambda_1 \downarrow 0$.

If $\lambda_1 = \lambda_2 = \lambda > 0$ then

$$\lim_{\lambda \downarrow 0} \lambda^3 \text{Var } \tilde{m} = \frac{1}{4\lambda_3^2 \dots \lambda_p^2},$$

hence $\text{Var } \tilde{m}$ is unbounded as $\lambda \downarrow 0$.

If $\lambda_1 = \lambda + i\omega$, $\lambda_2 = \lambda - i\omega$, where $\lambda, \omega \in \mathbb{R}$, $\lambda > 0$, $\omega \neq 0$, then

$$\lim_{\lambda \downarrow 0} \text{Var } \tilde{m} = \frac{1}{2\omega^4 \lambda_3^2 \dots \lambda_p^2 \left(\frac{1}{\lambda_3} + \dots + \frac{1}{\lambda_p} + \frac{T_2 - T_1}{2} \right)},$$

and we get that $\text{Var } \tilde{m}$ remains bounded as $\lambda \downarrow 0$. In a similar way, if $\lambda_1 = \lambda' + i\omega'$, $\lambda_2 = \lambda' - i\omega'$, $\lambda_3 = \lambda'' + i\omega''$, $\lambda_4 = \lambda'' - i\omega''$, where $\lambda', \lambda'', \omega', \omega'' \in \mathbb{R}$, $\lambda', \lambda'' > 0$, $\omega', \omega'' \neq 0$, then

$$\lim_{\lambda', \lambda'' \downarrow 0} \text{Var } \tilde{m} = \frac{1}{2(\omega' \omega'')^4 \lambda_5^2 \dots \lambda_p^2 \left(\frac{1}{\lambda_5} + \dots + \frac{1}{\lambda_p} + \frac{T_2 - T_1}{2} \right)},$$

and we get that $\text{Var } \tilde{m}$ remains bounded as $\lambda', \lambda'' \downarrow 0$.

Example 5.1.7. Multidimensional real-valued AR. Consider a d -dimensional ($d \geq 2$) real-valued stationary AR process $\{\tilde{Y}(t) : t \in \mathbb{R}\}$ which is the weakly stationary solution of

$$d\tilde{Y}(t) = -A\tilde{Y}(t) dt + dW(t), \quad t \in \mathbb{R}, \quad (5.25)$$

where $\{W(t) : t \in \mathbb{R}\}$ is a d -dimensional standard Wiener process, and the $d \times d$ coefficient matrix A has the form $A = \lambda I + \Omega$ with the damping parameter $\lambda > 0$ and a skew-symmetric matrix Ω (i.e., $\Omega^\top = -\Omega$), containing parameters of periodical type. The only solution of the matrix equation $AB(0) + B(0)A^\top = I$ is $B(0) = \frac{1}{2\lambda}I$, hence $B(0)^{-1} = 2\lambda I$. Obviously, $A^\top A = \lambda^2 I + \Omega^\top \Omega$, thus $U = 2\lambda I + (T_2 - T_1)(\lambda^2 I + \Omega^\top \Omega)$. Let $\tilde{Z}(t) := \tilde{Y}(t) + m$, $t \in \mathbb{R}$, where $m \in \mathbb{R}^d$ is unknown. Then,

$$V_{\tilde{Z}} = 2\lambda \tilde{Z}(T_1) + A^\top (\tilde{Z}(T_2) - \tilde{Z}(T_1)) + (\lambda^2 I + \Omega^\top \Omega) \int_{T_1}^{T_2} \tilde{Z}(t) dt,$$

and $\tilde{m} = U^{-1}V_{\tilde{Z}} \sim \mathcal{N}_d(m, U^{-1})$.

We have the following asymptotic behaviour

$$\lim_{T_2 - T_1 \rightarrow \infty} (T_2 - T_1) \text{Var } \tilde{m} = (\lambda^2 I + \Omega^\top \Omega)^{-1},$$

hence $\tilde{m}$ is asymptotically consistent as $T_2 - T_1 \rightarrow \infty$. Moreover, if $\Omega \neq 0$ is fixed, then we get, in contrast to the one-dimensional real-valued case, that $\text{Var } \tilde{m}_1$ and $\text{Var } \tilde{m}_2$ remain bounded uniformly in λ , since

$$\lim_{\lambda \downarrow 0} \text{Var } \tilde{m} = \frac{1}{T_2 - T_1} (\Omega^\top \Omega)^{-1}.$$

5.2 Case of Zero Start AR Process

Consider a d -dimensional zero start AR process

$$\begin{cases} dY(t) = -AY(t) dt + \Sigma dW(t), & t \geq 0, \\ Y(0) = 0, \end{cases} \quad (5.26)$$

where A and Σ are arbitrary $d \times d$ matrices. The process $\{Y(t) : t \geq 0\}$ is a zero mean Gaussian process with

$$\text{Cov}(Y(t), Y(s)) = \mathbb{E}Y(t)Y(s)^\top = e^{-(t-s)A}B(0, s), \quad \text{for } 0 \leq s \leq t,$$

where the $d \times d$ matrix $B(0, s)$ is the only solution of the matrix equation

$$AB(0, s) + B(0, s)A^\top = \Sigma\Sigma^\top - e^{-sA}\Sigma\Sigma^\top e^{-sA^\top}.$$

We remark that $B(0, s)$ is a symmetric, positive definite matrix and

$$B(0, s) = \int_0^s e^{-uA}\Sigma\Sigma^\top e^{-uA^\top} du.$$

Moreover,

$$Y(t) = \int_0^t e^{-(t-u)A}\Sigma dW(u).$$

Now, consider a shifted stationary AR process $\{Z(t) : t \geq 0\}$, given by

$$Z(t) = Y(t) + m, \quad t \geq 0,$$

where $m \in \mathbb{R}^d$ is an unknown parameter vector. The following theorem can be proved as in the stationary case.

Theorem 5.2.1. *Let $0 < T_1 < T_2$. The maximum likelihood estimator of m based on the observation of $\{Z(t) : t \in [T_1, T_2]\}$ is given by*

$$\tilde{m} = U^{-1}V_Z,$$

where

$$U = B(0, T_1)^{-1} + (T_2 - T_1)A^\top (\Sigma\Sigma^\top)^\ominus A,$$

$$V_Z = B(0, T_1)^{-1}\tilde{Z}(T_1) + A^\top (\Sigma\Sigma^\top)^\ominus \left(\tilde{Z}(T_2) - \tilde{Z}(T_1) + A \int_{T_1}^{T_2} \tilde{Z}(t) dt \right).$$

Moreover,

$$\tilde{m} \sim \mathcal{N}_d(m, U^{-1}).$$

Remark 5.2.2. *If $T_1 = 0$ then $Z(0) = Y(0) + m = m$ gives the precise value of the parameter m .*

5.2.1 Examples of Zero Start Processes

Example 5.2.3. One-dimensional AR(1). *Consider a one-dimensional zero start AR(1) process*

$$\begin{cases} dX(t) = -\alpha X(t) dt + dW(t), & t \geq 0, \\ X(0) = 0, \end{cases} \quad (5.27)$$

where $\alpha \in \mathbb{R}$. Then,

$$B(0, s) = \frac{1 - e^{-2\alpha s}}{2\alpha}, \quad B(0, s)^{-1} = \frac{2\alpha}{1 - e^{-2\alpha s}} = \alpha(\coth(\alpha s) + 1),$$

consequently,

$$U = \alpha(\coth(\alpha T_1) + 1 + \alpha(T_2 - T_1)).$$

Let $Z(t) := X(t) + m$, $t \geq 0$, where $m \in \mathbb{R}$ is an unknown parameter. Then,

$$V_Z = \alpha \left(\coth(\alpha T_1)Z(T_1) + Z(T_2) + \alpha \int_{T_1}^{T_2} Z(t) dt \right),$$

consequently,

$$\begin{aligned} \tilde{m} &= \frac{\coth(\alpha T_1)Z(T_1) + Z(T_2) + \alpha \int_{T_1}^{T_2} Z(t) dt}{\coth(\alpha T_1) + 1 + (T_2 - T_1)\alpha} \\ &\sim \mathcal{N} \left(m, \frac{1}{\alpha(\coth(\alpha T_1) + 1 + (T_2 - T_1)\alpha)} \right). \end{aligned}$$

We have the following asymptotic behaviour:

$$\lim_{T_1 \downarrow 0} T_1^{-1} \text{Var } \tilde{m} = 1, \quad \lim_{T_2 \rightarrow \infty} (T_2 - T_1) \text{Var } \tilde{m} = \frac{1}{\alpha^2},$$

and

$$\lim_{T_1 \downarrow 0} \text{Var } \tilde{m} = \lim_{T_2 \rightarrow \infty} \text{Var } \tilde{m} = \lim_{T_1 \downarrow 0, T_2 \rightarrow \infty} \text{Var } \tilde{m} = 0,$$

hence $\tilde{m}$ is asymptotically consistent as $T_1 \downarrow 0$, or $T_2 \rightarrow \infty$, or $T_1 \downarrow 0, T_2 \rightarrow \infty$. Moreover,

$$\lim_{\alpha \downarrow 0} \text{Var } \tilde{m} = T_1,$$

and we get, in contrast to the stationary case, that $\text{Var } \tilde{m}_1$ remains bounded as $\alpha \downarrow 0$.

Example 5.2.4. One-dimensional complex-valued AR(1). Consider a one-dimensional complex-valued zero start AR(1) process

$$\begin{cases} dX(t) = -\alpha X(t) dt + dW(t), & t \geq 0, \\ X(0) = 0, \end{cases} \quad (5.28)$$

where $\{W(t) : t \in \mathbb{R}\}$ is a complex-valued standard Wiener process and $\alpha = \lambda + i\omega$ with $\lambda, \omega \in \mathbb{R}$. Then, $(Y_1(t), Y_2(t))^\top := (\text{Re}X(t), \text{Im}X(t))^\top$, $t \geq 0$, is a 2-dimensional zero start AR process with coefficient matrix

$$A = \begin{pmatrix} \lambda & \omega \\ -\omega & \lambda \end{pmatrix}.$$

The only solution of the matrix equation $AB(0, s) + B(0, s)A^\top = I$ is $B(0, s) = \frac{1 - e^{-2\lambda s}}{2\lambda}I$, hence $B(0, s)^{-1} = \lambda(\coth(\lambda s) + 1)I$. Obviously, $A^\top A = (\lambda^2 + \omega^2)I$, thus $U = (\lambda(\coth(\lambda T_1) + 1) + (T_2 - T_1)(\lambda^2 + \omega^2))I$. Let $Z(t) := Y(t) + m$, $t \geq 0$, where $m = (m_1, m_2)^\top \in \mathbb{R}^2$ is unknown. Then,

$$V_Z = \lambda(\coth(\lambda T_1) + 1)Z(T_1) + A^\top(Z(T_2) - (T_1)) + (\lambda^2 + \omega^2) \int_{T_1}^{T_2} Z(t) dt,$$

consequently,

$$\tilde{m}_1 = \frac{\lambda(\coth(\lambda T_1)Z_1(T_1) + Z_1(T_2)) - \omega(Z_2(T_2) - Z_2(T_1)) + (\lambda^2 + \omega^2) \int_{T_1}^{T_2} Z_1(t) dt}{\lambda(\coth(\lambda T_1) + 1) + (T_2 - T_1)(\lambda^2 + \omega^2)}, \quad (5.29)$$

$$\tilde{m}_2 = \frac{\lambda(\coth(\lambda T_1)Z_2(T_1) + Z_2(T_2)) + \omega(Z_1(T_2) - Z_1(T_1)) + (\lambda^2 + \omega^2) \int_{T_1}^{T_2} Z_2(t) dt}{2\lambda(\coth(\lambda T_1) + 1) + (T_2 - T_1)(\lambda^2 + \omega^2)}. \quad (5.30)$$

Moreover,

$$(\tilde{m}_1, \tilde{m}_2)^\top \sim \mathcal{N}_2 \left(m, \frac{1}{\lambda(\coth(\lambda T_1) + 1) + (T_2 - T_1)(\lambda^2 + \omega^2)} I \right). \quad (5.31)$$

Particularly, $\tilde{m}_1$ and $\tilde{m}_2$ are independent.

We have the following asymptotic behaviour:

$$\lim_{T_2 \rightarrow \infty} (T_2 - T_1) \text{Var } \tilde{m}_1 = \lim_{T_2 \rightarrow \infty} (T_2 - T_1) \text{Var } \tilde{m}_2 = \frac{1}{\lambda^2 + \omega^2},$$

$$\lim_{T_1 \downarrow 0} T_1^{-1} \text{Var } \tilde{m}_1 = \lim_{T_1 \downarrow 0} T_1^{-1} \text{Var } \tilde{m}_2 = 1,$$

$$\lim_{T_2 \rightarrow \infty} \text{Var } \tilde{m}_1 = \lim_{T_2 \rightarrow \infty} \text{Var } \tilde{m}_2 = \lim_{T_1 \downarrow 0} \text{Var } \tilde{m}_1 = \lim_{T_1 \downarrow 0} \text{Var } \tilde{m}_2 = 0,$$

hence $\tilde{m}_1$ and $\tilde{m}_2$ are asymptotically consistent as $T_1 \downarrow 0$, or $T_2 \rightarrow \infty$, or $T_1 \downarrow 0, T_2 \rightarrow \infty$. Moreover,

$$\lim_{\lambda \downarrow 0} \text{Var } \tilde{m}_1 = \lim_{\lambda \downarrow 0} \text{Var } \tilde{m}_2 = \frac{1}{\frac{1}{T_1} + (T_2 - T_1)\omega^2},$$

and we get, in contrast to the one-dimensional real-valued case, that $\text{Var } \tilde{m}_1$ and $\text{Var } \tilde{m}_2$ remain bounded as $\lambda \downarrow 0$. It is valid even if $\omega = 0$, in contrast to the stationary case.

Összefoglaló

A disszertáció diszkrét és folytonos idejű autoregressziós modellek paramétereinek becslésével foglalkozik. Célja a becslések asszimptotikus tulajdonságainak vizsgálata, valamint a diszkrét és a megfelelő folytonos idejű modellek paraméter becslései közötti kapcsolat tanulmányozása.

A disszertáció öt fejezetből áll, és egy a szerző által önállóan és három társ-szerzőkkel írt cikkén alapszik, lásd Appendix A. Az 1. fejezet összegzése a disszertáció eredményeinek, áttekintést ad a tanulmányozott témakörökről. A 2. fejezet a disszertáció kiinduló pontjának tekintett modelleket és becsléseket ismerteti, rámutatva a már meglévő és az új eredmények közti kapcsolatra. Ismerteti azokat a segédeszközöket, amelyek szükségesek a további fejezetek megértéséhez. A szerző eredményeit a 3., a 4. és az 5. fejezet tartalmazza.

A 3. fejezet a szerző Dr. Pap Gyulával közösen írt cikkén, lásd [1] alapszik. Arató, Kolmogorov és Sinay klasszikus eredményét általánosítja, amely a komplex-értékű elsőrendű stacionárius AR folyamat periódusának maximum-likelihood becslésének pontos eloszlásáról szól.

A komplex-értékű stacionárius autoregressziós folyamat $\xi(t) = \xi_1(t) + i\xi_2(t)$, $t \geq 0$ esetén a következő sztochasztikus differenciálegyenlet megoldása:

$$d\xi(t) = -\gamma\xi(t)dt + dw(t),$$

ahol $w(t) = w_1(t) + iw_2(t)$, $t \geq 0$ a standard komplex Wiener-folyamat (azaz $w_1(t)$ és $w_2(t)$ független standard valós-értékű Wiener-folyamatok) és $\gamma = \lambda - i\omega$, ahol $\lambda > 0$, $\omega \in \mathbb{R}$. Ez az egyenlet a csillagászatban közismert, a Föld rotációs tengelyének mozgását írja le, az egyéves periódus eltávolítása után. Arató[4] bizonyítja, hogy az ω paraméter (periódus) alkalmasan normált maximum-likelihood becslése pontosan normális eloszlású. Ez az eredmény azért fontos, mert pontos eloszlásról van szó és nem a becslés aszimptotikus viselkedéséről.

Ha a komplex egydimenziós egyenletet kétdimenziós valós egyenletté alakítjuk, a következő természetes kérdés adódik. Tekintve a d -dimenziós $X(t)$, $t \geq 0$ folyamatot,

amelyet a következő sztochasztikus differenciálegyenlet ad meg:

$$dX(t) = AX(t)dt + dW(t), \quad t \geq 0,$$

ahol $W(t)$, $t \geq 0$ esetén standard d -dimenziós Wiener-folyamat és A egy $d \times d$ -s mátrix. Milyen feltevésekkel kell élnünk A mátrix és $X(0)$ kezdeti eloszlás esetén, hogy A mátrix bizonyos elemeinek alkalmasan normált maximum-likelihood becslése pontosan normális eloszlású legyen? Pap és van Zuijlen találtak ilyen A mátrixokat, így sikerült a klasszikus eredményt többdimenzióra általánosítaniuk. Az [1] cikkben gyengítjük Pap és van Zuijlen feltevéseit az A mátrixra, megmutatva azt, hogy a feltételek egy része felesleges [27]-ban.

Tekintsük a következő folyamatot:

$$dX(t) = \left(-\lambda I_d + \sum_{i=1}^m \omega_i C_i \right) X(t)dt + dW(t), \quad X(0) = 0$$

ahol I_d a $d \times d$ -s egységmátrix, $\lambda, \omega_1, \dots, \omega_m \in \mathbb{R}$ az ismeretlen paraméterek és $C_1, \dots, C_m$ rögzített ferdénszimmetrikus mátrixok. Ekkor $\omega = (\omega_1, \dots, \omega_m)^\top$ maximum-likelihood becslése a következő:

$$\hat{\omega}_X(t) = \sigma_X^{-1}(t) r_X(t),$$

ahol $\sigma_X(t)$ $m \times m$ mátrix

$$\sigma_X(t) = \left(\int_0^t \langle C_i X(s), C_j X(s) \rangle ds \right)_{1 \leq i, j \leq m},$$

és $r_X(t)$ m -dimenziós oszlop vektor

$$r_X(t) = \left(\int_0^t \langle C_i X(s), dX(s) \rangle \right)_{1 \leq i \leq m}^\top.$$

Az [1] cikkben bizonyítjuk, hogy a gyengített feltételek mellett az ω , periódus jellegű paraméter alkalmasan normált maximum-likelihood becslése pontosan normális eloszlású.

A 4. fejezetben közel instabil, közel stabil és expozív AR modelleket vizsgálunk. A többdimenziós AR modelleket az együttható mátrixuk sajátértékei szerint osztályozhatjuk. Tekintsük a következő d -dimenziós autoregressziós modellt:

$$\begin{cases} X_k = QX_{k-1} + \varepsilon_k, & k = 1, 2, \dots, \\ X_0 = 0, \end{cases}$$

ahol ε_k d -dimenziós valós-értékű véletlen vektor, amely a k időpontbeli véletlen zajt tartalmazza, és Q $d \times d$ -s mátrix a modell ismeretlen paramétere. Jelölje $\varrho(Q)$ a Q mátrix spektrálsugarát, azaz a sajátértékek abszolútértékei közül a legnagyobbat. Ha $\varrho(Q) \leq 1$, a modellt stabilisnak vagy aszimptotikusan stacionáriusnak mondjuk. Közismert az az eredmény, mely szerint, ha a ε_k sorozat független, azonos eloszlású, nulla várható értékkel és véges szórással, akkor Q legkisebb négyzetes becslése aszimptotikusan normális (lásd [1], [22]). Ha $\varrho(Q) = 1$, akkor a modellt instabilnak, ha $\varrho(Q) \geq 1$ explozívnek nevezik. Az instabil esetet egységyökkel rendelkező folyamatnak is hívják, és a közgazdasági és pénzügyi szakirodalomban népszerű modell. A 4. fejezet első felében közel instabil AR modell sorozatokkal foglalkozunk:

$$\begin{cases} X_k^{(n)} = \alpha^{(n)} X_{k-1}^{(n)} + \varepsilon_k^{(n)}, & k = 1, 2, \dots, \\ X_0^{(n)} = 0, \end{cases}$$

ahol

$$\alpha^{(n)} = 1 - \frac{\gamma^{(n)}}{n}, \quad \gamma^{(n)} \rightarrow \gamma \in \mathbb{R}.$$

A folyamat $m^{(n)}$ eltolás paraméterének legkisebb négyzetes becslését vizsgáljuk:

$$Z_k^{(n)} := X_k^{(n)} + m^{(n)} h_k^{(n)}, \quad k = 1, 2, \dots, \quad n = 1, 2, \dots,$$

ahol $m^{(n)} \in \mathbb{R}$ az ismeretlen paraméter és $h_k^{(n)}$, $k = 1, \dots, n$, $n = 1, 2, \dots$ ismert értékek.

A dolgozatban megvizsgáljuk a $\widehat{m}_{T_1, T_2}^{(n)}$, $n \rightarrow \infty$ határeloszlását, és kapcsolatot keresünk $\widehat{m}_{T_1, T_2}^{(n)}$ és a folytonos idejű modell eltolás paraméterének maximum-likelihood becslésével, $\widehat{m}_{T_1, T_2}$ -vel. Az eltolás paraméter viselkedését mind a stabil, mind a közel instabil, mind a közel explozív esetben tanulmányozzuk.

Belátjuk, hogy $\widehat{m}_{T_1, T_2}^{(n)}$ mindig aszimptotikusan normális, de a konvergencia sebesség sokkal kisebb a közel instabil esetben, mint az instabil modellben. A stacionárius esetet megvizsgálva belátjuk, hogy $\gamma^{(n)} \rightarrow 0$ esetén, ahol $\gamma^{(n)}$ a modell csillapítási paramétere, a $\widehat{m}_{T_1, T_2}^{(n)}$ becslés nem feltétlenül aszimptotikusan normális.

A 4. fejezet második részében többdimenziós közel instabil AR idősorokat vizsgálunk. Olyan speciális modellt tekintünk, ahol az együttható mátrix Jordan-alakban van.

Tekintsük a d -dimenziós komplex-értékű autoregressziós modellt:

$$\begin{cases} X_k = J(\lambda, d) X_{k-1} + \varepsilon_k, & k=1, 2, \dots, \\ X_0 = 0, \end{cases}$$

ahol $\lambda \in \mathbb{C}$ és $d \in \mathbb{N}$, $J(\lambda, d)$ egy $d \times d$ Jordan-alakú mátrix λ sajátértékkel:

$$J(\lambda, d) := \begin{pmatrix} \lambda & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & \lambda & 0 & & & \cdot \\ 0 & 1 & \lambda & 0 & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \lambda & 0 \\ 0 & \cdot & \cdot & 0 & 1 & \lambda \end{pmatrix}$$

és λ a modell ismeretlen paramétere.

Ha ε_k véletlen zajfolyamat egy szokásos, kellőképpen általános feltételnek tesz eleget, akkor belátható, hogy a sajátérték megfelelően normált legkisebb négyzetes becslése, $\hat{\lambda}_n$ gyengén konvergál egy speciális, d -dimenziós Ornstein-Uhlenbeck folyamat funkcionáljához:

$$n^d(\hat{\lambda}_n - \lambda_n) \xrightarrow{\mathcal{D}} e^{(d-1)\theta i} \frac{\int_0^1 \overline{Y_d(t)} dW_d(t)}{\int_0^1 |Y_d(t)|^2 dt},$$

itt $Y(t)$, $t \in [0, 1]$ folyamat a következő sztochasztikus differenciálegyenlet megoldása:

$$dY(t) = AY(t)dt + \Sigma dW(t) \quad Y(0) = 0,$$

ahol A egy speciális együttható mátrix, és $W(t)$, $t \in [0, 1]$ a standard, komplex d -dimenziós Wiener folyamat. Bebizonyítjuk, hogy az egydimenziós diszkrét idejű modell paraméterének legkisebb négyzetes becslése konvergál a megfelelő egydimenziós folytonos idejű modell együtthatójának maximum-likelihood becsléséhez. Ad ≥ 2 esetben ilyen összefüggés nem található.

Az 5. fejezetben többdimenziós autoregressziós folyamatok várható értékének becslésével foglalkozunk. Kiindulópontul Dr. Arató Mátyás, [3]-ben leírt eredményeit tekintjük az egydimenziós valós- és komplex-értékű folyamatok várható érték becsléseiről. Mind a stacionárius, mind a nulla-indítású esetet megvizsgáljuk. Belátjuk, hogy a várható érték becslése a folyamat periódusos jellegű komponensei esetén konzisztens, azonban ez nem teljesül a folyamat csillapítás jellegű komponensei esetén. Így például az egydimenziós AR modellben a várható érték nem becsülhető jól. Azonban komplex-értékű AR(1) modell esetén, ahol a folyamat periódikusan viselkedik, a várható érték jól becsülhető. Az AR(2) folyamatban akkor található jó becslés a várható értékre, ha a karakterisztikus egyenlet gyökei komplexek.

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Appendix A

List of Publications

- [1] G. Pap and K. Varga. Statistical inference for multidimensional ar processes. *Publ. Math. Debrecen*, 49/3-4:211–218, 1996.
- [2] K. Varga. Nearly unstable ar models with coefficient matrices in jordan normal form. *Computers Math. Applic.*, 36/9:1–10, 1998.
- [3] M. Arató, G. Pap, and K. Varga. Asymptotic behaviour of the least squares estimator of the mean of ar(1) models. accepted to *Analysis Mathematica*, 2001.
- [4] M. Arató, G. Pap, and K. Varga. Estimation of the mean of multivariate ar processes. *Computers Math. Applic.*, 43:707–719, 2002.

Appendix B

Conference Talks

- [1] G. Pap and K. Varga. Parameter estimation with exact distribution for multi-dimensional AR processes parameter estimation. In *XVI. Seminar on Stability Problems of Stochastic Models*, Eger, Hungary, 1994.
- [2] K. Varga. Nearly unstable AR models with coefficient matrices in Jordan Normal Form. In *XVIII. Seminar on Stability Problems of Stochastic Models*, Hajdúszoboszló, Hungary, 1997.
- [3] M. Arató, G. Pap, and K. Varga. Asymptotic behaviour of the least squares estimator of the mean of AR models. In *XXI. Seminar on Stability Problems of Stochastic Models*, Eger, Hungary, 2001.
- [4] K. Varga. Asymptotic properties of the LSE of the mean in nearly unstable AR models. In *XIII. European Young Statisticians Meeting*, Ovronnaz, Switzerland, 2003.

On Statistical Problems of Discrete and Continuous Time Autoregressive Processes

Értekezés a doktori (Ph.D.) fokozat megszerzése érdekében
a matematika tudományában.

Írta: Varga Katalin okleveles matematikus és angol-magyar szakfordító

Készült a Debreceni Egyetem Matematika doktori programja (Valószínűségelmélet és
matematikai statisztika alprogramja) keretében.

Témavezető: Dr. Pap Gyula
Elfogadásra javaslom: 2003. július 25.

.....

A jelölt a doktori szigorlatot 2003, ... -n eredményesen letette.
A bizottság elnöke: Dr. Arató Mátyás

.....

Az értekezést bírálóként elfogadásra javaslom:

Dr.
Dr.
Dr.

A jelölt az értekezést 2003, ... -n sikeresen megvédte:
A bírálóbizottság elnöke: Dr.

A bíráló bizottság tagjai:

Dr.
Dr.
Dr.