A Symmetric Algorithm for Hyperharmonic and Fibonacci Numbers

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Abstract

In this work, we introduce a symmetric algorithm obtained by the recurrence relation $a_n^k = a_{n-1}^k + a_n^{k-1}$. We point out that this algorithm can be applied to hyperharmonic-, ordinary and incomplete Fibonacci- and Lucas numbers. An explicit formula for hyperharmonic numbers, general generating functions of the Fibonacci- and Lucas numbers are obtained.

Besides we define "hyper-Fibonacci numbers", "hyper-Lucas numbers". Using these new concepts, some relations between ordinary and incomplete Fibonacci- and Lucas numbers are investigated.

1 Introduction

The algorithm introduced below is an analog of the Euler-Seidel algorithm [4]. These kind of algorithms are useful to investigate some recurrence relations and identities for some numbers and polynomials.

Having this concept, we give some applications for hyperharmonic numbers, ordinary and incomplete Fibonacci and Lucas numbers.

First of all, two real initial sequences, denoted by (a_n) and (a^n), be given. Then an infinite matrix which we call "symmetric infinite matrix" with entries a_n^k corresponding to these sequences is determined recursively by the formulae

$$
\begin{aligned}
a_n^0 &= a_n,\ a_0^n = a^n,\ (n \geq 0), \\
a_n^k &= a_{n-1}^k + a_n^{k-1},\ (n \geq 1,\ k \geq 1),
\end{aligned}
\tag{1}
$$

i.e.,

$$
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & & \begin{matrix} a_n^{k-1} \\ \downarrow \end{matrix} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & a_{n-1}^k & \longrightarrow & a_n^k & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
$$

With induction, we get following symmetric relation which gives us any entries a_n^k (k denotes the row, n is the column) in terms of the first row's and the first column's elements:

$$
a_n^k = \sum_{i=1}^{k} \binom{n+k-i-1}{n-1} a_0^i + \sum_{j=1}^{n} \binom{k+n-j-1}{k-1} a_j^0. \tag{2}
$$

By the relation (2) we get the generating function of any row and column for the symmetric infinite matrix (see Theorem 3). It is proved that the relation (2) is useful to investigate the structures of familiar sequences.

There are some papers related with this work. Among which Dumont [4] used another recurrence relation which was given in [5], [11] and he gave many applications for Bernoulli, Euler, Genocchi etc. numbers. In [3], there is a generalization of Euler-Seidel matrices for Bernoulli, Euler and Genocchi polynomials. Present authors used Dumont's method for hyperharmonic numbers, r-Stirling numbers and for classification of second order recurrence sequences in [9].

2 Definitions and notation

2.1 Euler-Seidel infinite matrices

Let a sequence (a_n) be given. Then the Euler-Seidel infinite matrix corresponding to this sequence is determined recursively by the formulae;

$$
\begin{aligned}
a_n^0 &= a_{n,} \quad (n \geq 0); \\
a_n^k &= a_n^{k-1} + a_{n+1}^{k-1}, \quad (n \geq 0,\ k \geq 1).
\end{aligned}
\tag{3}
$$

i.e.,

$$\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \begin{matrix} a_n^{k-1} \\ \downarrow \end{matrix} & \begin{matrix} a_{n+1}^{k-1} \\ \swarrow \end{matrix} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & a_n^k & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}$$

The first row and column can be transformed into each other via Dumont's identities [4]

$$a_0^n = \sum_{k=0}^{n} \binom{n}{k} a_k^0, \tag{4}$$

$$a_n^0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a_0^k.$$

There is a connection between the generating functions of the initial sequence $(a_n) = (a_n^0)$ and the generating functions of the first column (a_0^n). Namely,

Proposition 1 (Euler [5]) *Let*

$$a(t) = \sum_{n=0}^{\infty} a_n^0 t^n \tag{5}$$

be the generating function of the initial sequence (a_n^0)*. Then the generating function of the sequence* (a_0^n) *is*

$$\overline{a}(t) = \sum_{n=0}^{\infty} a_0^n t^n = \frac{1}{1-t} a\left(\frac{t}{1-t}\right). \tag{6}$$

In the sequel, the generating functions for the columns of the Euler-Seidel infinite matrix will be denoted by overline.

2.2 Hyperharmonic numbers

The n-th harmonic number is the n-th partial sum of the harmonic series:

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Let $H_n^{(1)} := H_n$, and for all $r > 1$

$$H_n^{(r)} = \sum_{k=1}^{n} H_k^{(r-1)} \tag{7}$$

be the n-th hyperharmonic number of order r. By agreement, $H_0^{(r)} = 0$ for all $r \geq 1$. These numbers can be expressed by binomial coefficients and ordinary harmonic numbers as

$$H_n^{(r)} = \binom{n+r-1}{r-1}(H_{n+r-1} - H_{r-1}).$$

It turned out that the hyperharmonic numbers have many combinatorial connections. To present these facts, we refer to [1] and [2]. Present authors gave a new closed form for these numbers in [9].

2.3 Fibonacci and Lucas numbers

The sequence of the Fibonacci numbers is given by the recursion formula

$$F_n = F_{n-1} + F_{n-2}, \quad (n \geq 2)$$

with initial values $F_0 = 0,\ F_1 = 1$. The Lucas sequence L_n has the same recursion formula, but $L_0 = 2,\ L_1 = 1$. The numbers L_n and F_n are connected with the formula

$$L_n = F_{n-1} + F_{n+1}, \quad (n \geq 1). \tag{8}$$

One can read more on these numbers in [2], [8], [12] and [13]. Now we recall a general generating function of Fibonacci numbers from [8] (page 230) which we need later, namely,

$$\sum_{n=0}^{\infty} F_{kn+r} t^n = \frac{F_r + (-1)^r F_{k-r} t}{1 - L_k t + (-1)^k t^2}. \tag{9}$$

We can derive similar generating function for Lucas numbers easily by using formula (8) and the generating function (9) as follows

$$\sum_{n=0}^{\infty} L_{kn+r} t^n = \frac{L_r + (-1)^{r-1} L_{k-r} t}{1 - L_k t + (-1)^k t^2}. \tag{10}$$

2.4 Incomplete Fibonacci and Incomplete Lucas numbers

The incomplete Fibonacci and incomplete Lucas numbers are defined in [7] by:

$$F_n(k) = \sum_{j=0}^{k} \binom{n-1-j}{j}, \quad \left(n = 1, 2, 3, ...;\ 0 \leq k \leq \left[\frac{n-1}{2}\right]\right); \qquad (11)$$

$$L_n(k) = \sum_{j=0}^{k} \frac{n}{n-j} \binom{n-j}{j}, \left(n = 1, 2, 3, ...;\ 0 \leq k \leq \left[\frac{n}{2}\right]\right),$$

where $[n]$ denotes the integer part of n.

The connection between ordinary and incomplete Fibonacci and Lucas numbers are also given in [7] as

$$F_n(k) = 0 \qquad 0 \leq n \leq 2k+1, \quad F_{2k+1}(k) = F_{2k+1,} \quad F_{2k+2}(k) = F_{2k+2}; \ (12)$$

We also need the following properties of incomplete Fibonacci and Lucas numbers which are given in [7] as

$$\sum_{j=0}^{h} \binom{h}{j} F_{n+j}(k+j) = F_{n+2h}(k+h), \ \left(0 \leq k \leq \frac{n-h-1}{2}\right), \qquad (13)$$

$$\sum_{j=0}^{h} \binom{h}{j} L_{n+j}(k+j) = L_{n+2h}(k+h), \ \left(0 \leq k \leq \frac{n-h}{2}\right). \qquad (14)$$

Generating functions of these numbers are given in [10] as

$$R_k(t) = \sum_{j=0}^{\infty} F_j(k)\, t^j = t^{2k+1} \frac{(F_{2k+1} + F_{2k}t)(1-t)^{k+1} - t^2}{(1-t)^{k+1}(1-t-t^2)}, \qquad (15)$$

$$S_k(t) = \sum_{j=0}^{\infty} L_j(k)\, t^j = t^{2k} \frac{(L_{2k} + L_{2k-1}t)(1-t)^{k+1} - t^2(2-t)}{(1-t)^{k+1}(1-t-t^2)}. \qquad (16)$$

3 Generating Function of any Row and Column of the Symmetric Infinite Matrix

After these introductory steps we are ready to formulate our results.

First we give general terms and generating functions of any row and column of the symmetric infinite matrix by using the symmetric algorithm.

Proposition 2 *If the relation (1) holds then any entries of the symmetric infinite matrix is*

$$a_n^k = \sum_{i=1}^{k} \binom{n+k-i-1}{n-1} a_0^i + \sum_{j=1}^{n} \binom{k+n-j-1}{k-1} a_j^0. \tag{17}$$

Proof. It follows from induction on $n+k$ by using relation (1). ■

Theorem 3 *Let (a_n^0) and (a_0^n) be two initial sequences. Then the generating functions of the kth row and nth column of the symmetric infinite matrix are*

$$^k a(t) = \sum_{n=1}^{\infty} a_n^k t^n = \frac{1}{(1-t)^k} \left\{ {}^0 a(t) + \frac{t}{1-t} \sum_{r=1}^{k} a_0^r (1-t)^r \right\}, \tag{18}$$

and

$$^n \overline{a(t)} = \sum_{k=1}^{\infty} a_n^k t^k = \frac{1}{(1-t)^n} \left\{ {}^0 \overline{a(t)} + \frac{t}{1-t} \sum_{j=1}^{n} a_j^0 (1-t)^j \right\}. \tag{19}$$

Proof. We will prove just the first equation, the proof of the second equation is similar and it is omitted here. From equation (17),

$$\sum_{n=0}^{\infty} a_{n+1}^{k+1} t^n = \sum_{n=0}^{\infty} \left\{ \sum_{r=1}^{k+1} \binom{n+k+1-r}{n} a_0^r + \sum_{j=1}^{n+1} \binom{k+n+1-j}{k} a_j^0 \right\} t^n$$

$$= a_0^1 \sum_{n=0}^{\infty} \binom{n+k}{k} t^n + \sum_{r=1}^{k} a_0^{r+1} \sum_{n=0}^{\infty} \binom{n+k-r}{n} t^n + \sum_{n=0}^{\infty} a_{n+1}^0 t^n \sum_{n=0}^{\infty} \binom{k+n}{k} t^n$$

$$= \sum_{n=0}^{\infty} \binom{n+k}{k} t^n \left\{ a_0^1 + \sum_{n=0}^{\infty} a_{n+1}^0 t^n \right\} + \sum_{r=1}^{k} a_0^{r+1} \sum_{n=0}^{\infty} \binom{n+k-r}{n} t^n.$$

Then

$$\sum_{n=1}^{\infty} a_n^{k+1} t^n = \sum_{n=0}^{\infty} \binom{n+k}{k} t^n \left\{ a_0^1 t + {}^0 a(t) \right\} + \sum_{r=1}^{k} a_0^{r+1} t \sum_{n=0}^{\infty} \binom{n+k-r}{k-r} t^n.$$

If we write related series in terms of Newton's binomial series we get

$$\sum_{n=1}^{\infty} a_n^{k+1} t^n = \frac{1}{(1-t)^{k+1}} \left\{ {}^0 a(t) + \sum_{r=0}^{k} a_0^{r+1} t (1-t)^r \right\}.$$

The last equation gives the statement. ■

4 Applications

In this section we obtain some results on hyperharmonic-, ordinary Fibonacci- and Lucas numbers using the symmetric algorithm have introduced.

4.1 Application for Hyperharmonic Numbers

We start with two suitable initial sequences for hyperharmonic numbers.

Let $a_n^0 = \frac{1}{n+1}$ and $a_0^n = 1$, $n \geq 1$ be given. If we calculate the elements of the corresponding infinite matrix by using the recursive formula (1), it turns out that it equals to

$$\begin{pmatrix} H_1^{(0)} & H_2^{(0)} & H_3^{(0)} & H_4^{(0)} & \cdots \\ H_1^{(1)} & H_2^{(1)} & H_3^{(1)} & H_4^{(1)} & \cdots \\ H_1^{(2)} & H_2^{(2)} & H_3^{(2)} & H_4^{(2)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{20}$$

where $H_n^{(0)} = \frac{1}{n}, n \geq 1$.

Now we are ready to obtain the well-known generating function of hyperharmonic numbers with our method as a corollary of Theorem 3.

Corollary 4 *We have*

$$\sum_{n=1}^{\infty} H_n^{(k)} t^n = -\frac{\ln(1-t)}{(1-t)^k}.$$

Proof. In Theorem 3 by taking $a_n^0 = \frac{1}{n+1}$ and $a_0^n = 1$, $(n \geq 1)$ we obtain

$$^k a(t) = \sum_{n=2}^{\infty} H_n^{(k)} t^n = \frac{t}{(1-t)^k} \left\{ {^0 a(t)} + \frac{t}{1-t} \sum_{r=1}^{k} (1-t)^r \right\}.$$

From the identities

$$^0 a(t) = -\frac{\ln(1-t)}{t} - 1,$$

and

$$\sum_{r=1}^{k} (1-t)^r = \frac{(1-t)}{t} \left\{ 1 - (1-t)^k \right\},$$

we can write,

$$\sum_{n=2}^{\infty} H_n^{(k)} t^n = \frac{t}{(1-t)^k} \left\{ -\frac{\ln(1-t)}{t} - (1-t)^k \right\} = -\frac{\ln(1-t)}{(1-t)^k} - t.$$

It completes the proof. ■

Next theorem indicates the relation between binomial coefficients and hyperharmonic numbers. In [1], authors gave combinatorial proof of this statement whereas here it will be proven by the symmetric algorithm.

Theorem 5 *If* $n \geq 1$, $k \geq 1$ *then*

$$H_n^{(k)} = \sum_{j=1}^{n} \binom{n+k-j-1}{k-1} \frac{1}{j}.$$

Proof. Let us take $a_n^0 = \frac{1}{n+1}$ and $a_0^n = 1$, $(n \geq 1)$. From the formula (17),

$$\begin{aligned} a_{n+1}^{k+1} &= \sum_{i=1}^{k+1} \binom{n+k-i+1}{n} + \sum_{j=1}^{n+1} \binom{k+n-j+1}{k} \frac{1}{j+1} \\ &= \sum_{i=0}^{k} \binom{n+k-i}{n} + \sum_{j=0}^{n} \binom{k+n-j}{k} \frac{1}{j+2} \\ &= \sum_{r=0}^{k} \binom{n+r}{n} + \sum_{s=0}^{n} \binom{k+s}{k} \frac{1}{n-s+2}, \end{aligned}$$

where $k - i = r$ and $n - j = s$. From [6, page 160] we have

$$\sum_{t=a}^{b} \binom{t}{a} = \binom{b+1}{a+1}.$$

Hence

$$a_{n+1}^{k+1} = \binom{k+n+1}{n+1} + \sum_{s=0}^{n} \binom{k+s}{k} \frac{1}{n-s+2} = \sum_{s=0}^{n+1} \binom{k+s}{k} \frac{1}{n-s+2}.$$

Then the matrix (20) yields

$$a_{n-1}^{k} = H_n^{(k)} = \sum_{s=0}^{n-1} \binom{k+s-1}{k-1} \frac{1}{n-s},$$

which completes the proof. ■

4.2 Applications for the Ordinary Fibonacci and Lucas Numbers

In the sequel we point out that the symmetric algorithm is quite applicable for ordinary Fibonacci and Lucas numbers. By starting with two different initial sequences we get an application which gives us new identities.

Let us consider the initial sequences $a_n^0 = F_{n-1}$ and $a_0^n = F_{2n-1}$, $n \geq 1$. This special case gives the following infinite matrix:

$$\begin{pmatrix} 0 & F_0 & F_1 & F_2 & \cdots \\ F_1 & F_2 & F_3 & F_4 & \cdots \\ F_3 & F_4 & F_5 & F_6 & \cdots \\ F_5 & F_6 & F_7 & F_8 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{21}$$

One can consider the similar infinite matrix for the Lucas numbers just by substitution F_n with L_n.

We will prove some famous relations (cf. [8]) for Fibonacci and Lucas numbers with our method.

Proposition 6 *The following equalities hold*

$$F_{2n} = \sum_{i=1}^{n} F_{2i-1} \text{ and } \sum_{i=1}^{n} F_i = F_{n+2} - 1 \tag{22}$$

and

$$L_{2n} - 2 = \sum_{i=1}^{n} L_{2i-1} \text{ and } \sum_{i=0}^{n} L_i = L_{n+2} - 1. \tag{23}$$

Proof. Here, we consider only the equation (22). Equation (23) can be proven similarly.

For $a_n^0 = F_{n-1}$ and $a_0^n = F_{2n-1}$, $n \geq 1$ we can write $a_1^1 = F_2$, $a_1^2 = F_4$, and by induction $a_1^n = F_{2n}$. From equation (2) we see that

$$a_1^n = F_0 + \sum_{i=1}^{n} F_{2i-1}.$$

and complete the proof of (22). ■

The generating function of the first row or the first column in the matrix (21) is well-known. In the following proposition we obtain generating function of any row or column of this matrix and later we observe some results of them.

For the sake of simplicity hereafter we will denote

$$\sum_{i=0}^{k-1} \binom{n+k-i-2}{n-1} F_{2i+1} \quad \text{and} \quad \sum_{i=0}^{n-1} \binom{n+k-i-2}{k-1} F_i \tag{24}$$

by $A_{n,k}$ and $B_{n,k}$, respectively.

Proposition 7 *For the values $a_n^0 = F_{n-1}$ and $a_0^n = F_{2n-1}$, $(n \geq 1)$, we have*

$$^{k}a\left(t\right) = \sum_{n=1}^{\infty} \left(A_{n,k} + B_{n,k}\right) t^{n} = \frac{t\left\{F_{2k} + tF_{2k-1}\right\}}{1-t-t^{2}} \tag{25}$$

and

$$^{n}\overline{a\left(t\right)} = \sum_{k=1}^{\infty} \left(A_{n,k} + B_{n,k}\right) t^{k} = \frac{t\left(F_{n+1} - tF_{n-1}\right)}{t^{2} - 3t + 1}. \tag{26}$$

Proof. From the relation (18),

$$^{k}a\left(t\right) = \frac{1}{\left(1-t\right)^{k}} \left\{ ^{0}a\left(t\right) + \frac{t}{1-t} \sum_{r=1}^{k} F_{2r-1} \left(1-t\right)^{r} \right\}.$$

Considering the generating function (9),

$$^{0}a\left(t\right) = \sum_{n=1}^{\infty} F_{n-1} t^{n} = \frac{t^{2}}{1-t-t^{2}},$$

and

$$\begin{aligned} \sum_{r=1}^{k} F_{2r-1} \left(1-t\right)^{r} &= \sum_{r=1}^{\infty} F_{2r-1} \left(1-t\right)^{r} - \sum_{r=k+1}^{\infty} F_{2r-1} \left(1-t\right)^{r} \\ &= \frac{\left(1-t\right) t - \left(1-t\right)^{k+1} \left\{F_{2k+1} - \left(1-t\right) F_{1-2k}\right\}}{t^{2} + t - 1}. \end{aligned}$$

By definition, $F_{-n} = \left(-1\right)^{n+1} F_{n}$, thus

$$\sum_{r=1}^{k} F_{2r-1} \left(1-t\right)^{r} = \frac{\left(1-t\right) \left\{ t - \left(1-t\right)^{k} \left(F_{2k} + tF_{2k-1}\right) \right\}}{t^{2} + t - 1}.$$

Then

$$^{k}a\left(t\right) = \frac{1}{\left(1-t\right)^{k}} \left\{ \frac{t\left(1-t\right)^{k} \left\{F_{2k} + tF_{2k-1}\right\}}{1-t-t^{2}} \right\} = \frac{t\left\{F_{2k} + tF_{2k-1}\right\}}{1-t-t^{2}}.$$

The identity (26) can be proven by the same approach. ■

Let us consider a similar proposition for even and odd Fibonacci numbers.

Proposition 8 *With initial sequences* $a_n^0 = F_{2n-1}$ *and* $a_0^n = F_{2n}$*,* $n \geq 1$*, we have*

$$\sum_{n=1}^{\infty} (C_{n,k} + A_{k,n})\, t^n = \frac{-t}{t^2+t-1} \left\{ \frac{t\,(t^2-t+1)}{(1-t)^k\,(t^2-3t+1)} + F_{2k+1} + tF_{2k} \right\}$$

and

$$\sum_{k=1}^{\infty} (C_{n,k} + A_{k,n})\, t^k = \frac{t}{t^2+t-1} \left\{ \frac{2t\,(t^2-t+1)}{(1-t)^n\,(t^2-3t+1)} - F_{2n} - tF_{2n-1} \right\},$$

where

$$C_{n,k} := \sum_{i=0}^{k-1} \binom{n+k-i-2}{n-1} F_{2i}.$$

Remark 9 *By considering Proposition 6, Proposition 7 and Proposition 8 we have the generating functions for tails of the Fibonacci sequence.*

Remark 10 *We obtain similar propositions to Proposition 7 and Proposition 8 for the Lucas numbers just by substituting* F_n *with* L_n*.*

4.3 Applications with the Incomplete Fibonacci and Incomplete Lucas Numbers

In this subsection we will investigate the results obtained when the entries of the Euler-Seidel and symmetric infinite matrices are particularly choosen to be the incomplete Fibonacci and incomplete Lucas numbers.

4.3.1 Euler-Seidel Algorithm

We will give some applications on incomplete Fibonacci numbers with Euler-Seidel method.

Let us take the incomplete Fibonacci numbers $F_{r+n}(s+n)$ as a_n^0. From formulae (4) we have

$$a_0^n = \sum_{k=0}^{n} \binom{n}{k} F_{r+k}(s+k).$$

The identity (13) implies

$$a_0^n = F_{r+2n}(s+n).$$

Because of the choice of a_n^0 and the last equation of a_0^n, we obtain the dual formula of (13):

$$F_{r+n}\left(s+n\right)=\sum_{k=0}^{n}\binom{n}{k}\left(-1\right)^{n-k}F_{r+2k}\left(s+k\right),\quad 0\leq s\leq\frac{r-n-1}{2}. \tag{27}$$

Similarly,

$$L_{r+2n}\left(s+n\right)=\sum_{k=0}^{n}\binom{n}{k}L_{r+k}\left(s+k\right),\quad 0\leq s\leq\frac{r-n}{2},$$

and its dual is

$$L_{r+n}\left(s+n\right)=\sum_{k=0}^{n}\binom{n}{k}\left(-1\right)^{n-k}L_{r+2k}\left(s+k\right),\quad 0\leq s\leq\frac{r-n}{2}.$$

Secondly, let $a_n^0=F_n\left(k\right)$. Then from (4),

$$a_0^n=\sum_{l=0}^{n}\binom{n}{l}F_l\left(k\right).$$

In the followig theorem we present a new formula to this quantity.

Theorem 11

$$\sum_{l=0}^{n}\binom{n}{l}F_l\left(k\right)=\begin{cases}0, & \textit{if } n<2k+1;\\ F_{2k+1}, & \textit{if } n=2k+1;\\ F, & \textit{if } n\geq 2k+2.\end{cases}$$

where,

$$\begin{aligned}F\ :=&\sum_{r=2k+1}^{n}\left[F_{2k}\binom{r}{2k}+F_{2k-1}\binom{r-1}{2k-1}\right]F_{2n-2r}\\&-\sum_{r=0}^{n}\sum_{m=0}^{r}F_{2n-2r-4k-2}\binom{r+k-m-1}{k}\binom{m+k}{k}2^m.\end{aligned}$$

Proof. The relation (15) gives that

$$a\left(t\right)=\sum_{j=0}^{\infty}F_j\left(k\right)t^j=t^{2k+1}\frac{\left(F_{2k+1}+tF_{2k}\right)\left(1-t\right)^{k+1}-t^2}{\left(1-t\right)^{k+1}\left(1-t-t^2\right)}.$$

From equation (6) we obtain

$$
\begin{aligned}
\overline{a}(t) &= \sum_{n=0}^{\infty}\left[\sum_{l=0}^{n}\binom{n}{l}F_l\left(k\right)\right]t^n = \frac{t^{2k+1}}{\left(1-2t\right)^{k+1}\left(t^2-3t+1\right)\left(1-t\right)^{k-1}} \\
&\quad \times\left\{\left(F_{2k+1}-tF_{2k}\right)\frac{\left(1-2t\right)^{k+1}}{\left(1-t\right)^{k+2}}-\frac{t^2}{\left(1-t\right)^2}\right\}.
\end{aligned}
$$

By substracting the generating function of even Fibonacci numbers outside the equation above we have

$$
\begin{aligned}
\overline{a}(t) &= \frac{t^{2k+1}}{\left(t^2-3t+1\right)}\left\{\left(F_{2k+1}-tF_{2k}\right)\sum_{n=0}^{\infty}\binom{n+2k}{n}t^n\right. \\
&\quad \left.-t^2\sum_{n=0}^{\infty}\binom{n+k}{n}2^n t^n\sum_{n=0}^{\infty}\binom{n+k}{n}t^n\right\} \\
&= t^{2k}\sum_{n=0}^{\infty}\sum_{r=0}^{n}F_{2n-2r}\left\{F_{2k+1}\binom{r+2k}{r}-tF_{2k}\binom{r+2k}{r}\right. \\
&\quad \left.-t^2\left[\sum_{m=0}^{r}\binom{r-m+k}{r-m}\binom{m+k}{m}2^m\right]\right\}t^n \\
&= F_2F_{2k+1}t^{2k+1}+\sum_{n=2k+2}^{\infty}\left\{\sum_{r=0}^{n-2k}F_{2n-4k-2r}F_{2k+1}\binom{r+2k}{r}\right. \\
&\quad -\sum_{r=1}^{n-2k}F_{2n-4k-2r}F_{2k}\binom{r+2k-1}{r-1} \\
&\quad \left.-\sum_{r=2}^{n-2k}\sum_{m=0}^{r-2}F_{2n-4k-2r}\binom{r-2-m+k}{r-2-m}\binom{m+k}{m}2^m\right\}t^n.
\end{aligned}
$$

After some rearrangement,

$$
\begin{aligned}
\overline{a}(t) &= F_2F_{2k+1}t^{2k+1}+\sum_{n=2k+2}^{\infty}\left\{\sum_{r=2}^{n-2k}F_{2n-4k-2r}\left[F_{2k+1}\binom{r+2k}{r}\right.\right. \\
&\quad \left.-F_{2k}\binom{r+2k-1}{r-1}-\sum_{m=0}^{r-2}\binom{r-2-m+k}{r-2-m}\binom{m+k}{m}2^m\right] \\
&\quad \left.+F_{2n-4k}F_{2k+1}+\left(2k+1\right)F_{2n-4k-2}F_{2k+1}-F_{2n-4k-2}F_{2k}\right\}t^n \\
&= F_2F_{2k+1}t^{2k+1}+\sum_{n=2k+2}^{\infty}\left\{\sum_{r=2}^{n-2k}F_{2n-4k-2r}\left[\binom{r+2k-1}{r-1}\frac{rF_{2k}+2kF_{2k+1}}{r}\right.\right.
\end{aligned}
$$

$$-\sum_{m=0}^{r-2}\binom{r-2-m+k}{r-2-m}\binom{m+k}{m}2^{m}\Bigg]$$
$$+F_{2n-4k}F_{2k+1}+(2k+1)\,F_{2n-4k-2}F_{2k+1}-F_{2n-4k-2}F_{2k}\}\,t^{n}$$

and thereby the theorem is proved. ■

A same approach proves a parallel result for incomplete Lucas numbers

Theorem 12

$$\sum_{l=0}^{n}\binom{n}{l}L_{l}\,(k)=\begin{cases}0, & \text{if } n<2k;\\ L_{2k}, & \text{if } n=2k;\\ (2k+1)\,L_{2k}+L_{2k+2}, & \text{if } n=2k+1;\\ L, & \text{if } n\geq 2k+2.\end{cases}$$

where,

$$\begin{aligned}L\ :&=\sum_{r=0}^{n-2k-2}\Bigg(\{F_{2n-4k-2r+2}L_{2k}-F_{2n-4k-2r}L_{2k-2}\}\binom{r+2k-1}{r}\\ &\quad-\{F_{2n-4k-2r-5}+F_{2n-4k-2r-3}\}\sum_{m=0}^{r}\binom{r-m+k}{r-m}\binom{m+k}{m}2^{m}\\ &\quad+L_{2k}\binom{n-1}{n-2k}+L_{2k+2}\binom{n-2}{n-2k-1}\Bigg).\end{aligned}$$

4.3.2 Symmetric Algorithm

Here we introduce new concepts as "hyper-Fibonacci numbers" and "hyper-Lucas numbers" similar to the concept of "hyperharmonic numbers". They will be useful for us.

Definition 13 *Let the hyper-Fibonacci numbers* $F_n^{(r)}$ *and the hyper-Lucas numbers* $L_n^{(r)}$ *be defined respectively as*

$$F_n^{(r)}=\sum_{k=0}^{n}F_k^{(r-1)},\ \text{with } F_n^{(0)}=F_n,\ F_0^{(r)}=0,\ \text{and } F_1^{(r)}=1; \qquad (28)$$
$$L_n^{(r)}=\sum_{k=0}^{n}L_k^{(r-1)},\ \text{with } L_n^{(0)}=L_n,\ L_0^{(r)}=0,\ \text{and } L_1^{(r)}=1.$$

Proposition 14 *The generating functions of the hyper-Fibonacci numbers and the hyper-Lucas numbers as follows, respectively,*

$$\sum_{n=0}^{\infty}F_n^{(r)}t^{n}=\frac{t}{(1-t-t^{2})\,(1-t)^{r}},$$

$$\sum_{n=0}^{\infty} L_n^{(r)} t^n = \frac{2-t}{(1-t-t^2)(1-t)^r}.$$

Proof. Proof is obtained immediately by using Cauchy product and induction r. ■

Now we are ready for the application. Let us recall the equtions (24). By generating function (15) we have

$$\sum_{j=0}^{\infty} F_j(k) t^j = t^{2k} \sum_{n=1}^{\infty} (A_{n,k} + B_{n,k}) t^n - \frac{t^{2k+2}}{(1-t)^{k+1}} \sum_{n=0}^{\infty} F_n t^n.$$

Applying the concept of hyper-Fibonacci numbers, we can rewrite the right-hand side of the equality as

$$\sum_{j=0}^{\infty} F_j(k) t^j = \sum_{n=2k+1}^{\infty} (A_{n-2k,k} + B_{n-2k,k}) t^n - \sum_{n=2k+2}^{\infty} F_{n-2k-2}^{(k+1)} t^n.$$

Here with help of the Proposition 6 we have the following theorem.

Theorem 15 *We have*

$$F_n(k) = \begin{cases} 0, & \text{if } 0 \le n < 2k+1; \\ F_{2k+1}, & \text{if } n = 2k+1; \\ A_{n-2k,k} + B_{n-2k,k} - F_{n-2k-2}^{(k+1)}, & \text{if } n > 2k+1. \end{cases} \tag{29}$$

Theorem 15 provides an interesting corollary.

Corollary 16 *For the ordinary Fibonacci numbers, the following equalities are valid:*

$$F_{2k+1} - 1 = \sum_{i=0}^{k-1} (k-i) F_{2i+1}, \tag{30}$$

$$F_{2k+2} - k - 1 = \sum_{i=0}^{k-1} \binom{k+1-i}{2} F_{2i+1}. \tag{31}$$

Proof. Proof of the equations (30) and (31) seen by taking $n = 2k+2$ and $n = 2k+3$ in the formula (29), respectively. ■

The similar result for incomplete Lucas numbers is:

Theorem 17 *We have*

$$L_n(k) = \begin{cases} 0, & \text{if } 0 \le n < 2k; \\ L_{2k}, & \text{if } n = 2k; \\ A_{2,k} + B_{2,k}, & \text{if } n = 2k+1; \\ A_{n-2k+1,k} + B_{n-2k+1,k} - L_{n-2k-2}^{(k+1)}, & \text{if } n \ge 2k+2. \end{cases} \tag{32}$$

Proof. Proof is similar to the proof of the Theorem 15. ■

Corollary 18 *We have*

$$L_{2k+1} = \sum_{i=0}^{k-1} (k-i)\, L_{2i+1} + 2k + 1.$$

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