Short Thesis for The Degree of
Doctor of Philosophy (PhD)

Extensions of Taylor's theorem and norm estimations of linear functionals

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This thesis contains the primary results from the doctoral dissertation. Our research has yielded several important lemmas, propositions, theorems and corollaries that are thoroughly elaborated in the papers [1, 2] and also within the doctoral dissertation itself.

Taylor's theorem

Taylor's theorem contains a polynomial, which is referred to as a Taylor polynomial of a corresponding degree, is used to estimate a differentiable function about a given point up to a certain order. A polynomial truncates the Taylor series of a function in a specified order and can provide linear or quadratic approximations based on its degree. There are variations of this method that provide clear estimates of error. Brooke Taylor [22] developed one version in 1715 after it had been predicted by James Gregory in 1671 [12]. This concept is an important tool for introductory calculus and mathematical analysis courses, as it provides exact formulas for many transcendental functions such as exponential, trigonometric, and hyperbolic functions. In addition, it can be used in various fields such as numerical analysis and mathematical physics, and can be extended to multivariate or vector-valued functions.

Given a function $f : I \to \mathbb{R}$, which is n times differentiable at $a \in I$ (where I is a non-degenerate real interval), the polynomial $T_{n;a}(f)$ defined by

$$T_{n;a}(f)(x) := \sum_{j=0}^{n} f^{(j)}(a) \cdot \frac{(x-a)^j}{j!}$$

is called the *nth-order Taylor polynomial of the function f at the base point a.*

The form with integral remainder term can be formulated as follows.

THEOREM. *Let I be a real interval and let $f : I \to \mathbb{R}$ be $(n+1)$ times continuously differentiable. Then, for all $a, x \in I$,*

$$f(x) = T_{n;a}(f)(x) + \int_a^x f^{(n+1)}(t) \cdot \frac{(x-t)^n}{n!} dt.$$

The variant as an intermediate value theorem is the following assertion.

THEOREM. *Let I be a real interval and let $f : I \to \mathbb{R}$ be $(n+1)$ times differentiable. Then, for all $a, x \in I$, there exists a point ξ between a and x such that*

$$f(x) = T_{n;a}(f)(x) + f^{(n+1)}(\xi) \cdot \frac{(x-a)^{n+1}}{(n+1)!}. \tag{1}$$

The formula for the remainder term in (1) is called Lagrange's form of the remainder term.

The previous two theorems are contained in most of the textbooks on basic analysis (see, e.g., [4], [8], [19], [20]). In addition to the two remainders in (1) and (2), there are many other types of remainders such as:

- The *Cauchy form* of the remainder, which is given by

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n)!}(x-\xi)^n(x-a). \tag{2}$$

- The *Schlömilch form* of the remainder, which is given by

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^{n+1-p}\frac{(x-a)^p}{p},$$

  which is also known as the *Schlömilch-Roché form* [7], and it is the general case of (1) and (2), where selecting $p = n + 1$ represents the Lagrange form, while choosing $p = 1$ corresponds to the Cauchy form.

Moreover, there have been several papers where extensions, generalizations and applications of these fundamental results can be found, cf. [3], [14], [16], [15], [17], [18], [25]. The main content of these results is that they give a high order approximation of the function f near the point $a \in I$ in terms of the polynomial $T_{n;a}(f)$, which is of degree at most n and therefore it is in the kernel of the differential operator D given by

$$D(f) = f^{(n+1)}.$$

The aim of this section is to derive a general form of the Taylor theorem related to a linear differential operator with constant coefficients.

Given an interval $I \subseteq \mathbb{R}$, let $\mathcal{C}_{\mathbb{K}}(I)$ stand for the space of continuous $\mathbb{K}$-valued functions defined on I, where $\mathbb{K}$ denotes either the field of real or complex numbers. If additionally $n \in \mathbb{N}$, then let $\mathcal{C}^n_{\mathbb{K}}(I)$ denote the space of n-times continuously differentiable $\mathbb{K}$-valued functions defined on I.

For $c = (c_0, \dots, c_n) \in \mathbb{K}^{n+1}$ with $c_n = 1$, let the nth-order linear differential operator $D_c \colon \mathcal{C}^n_{\mathbb{K}}(I) \to \mathcal{C}_{\mathbb{K}}(I)$ be defined by the formula

$$D_c(f) := c_n f^{(n)} + \dots + c_1 f' + c_0 f \qquad (f \in \mathcal{C}^n_{\mathbb{K}}(I)).$$

Let $\omega_c \in \mathcal{C}^n_{\mathbb{C}}(\mathbb{R})$ denote the unique solution of the initial value problem

$$D_c(\omega_c) = 0, \qquad \omega_c^{(\ell)}(0) = \delta_{\ell,n-1} \quad (\ell \in \{0, \dots, n-1\}). \tag{3}$$

The function ω_c will be called the *characteristic solution* of the differential equation

$$D_c(\omega) = 0.$$

In order to provide a more or less explicit formula for P_c, we define the $(n-1)$*st order divided difference* $f(\lambda_1, \dots, \lambda_n)$ by

$$f(\lambda_1, \dots, \lambda_n) := \sum_{i=1}^{n} \frac{f(\lambda_i)}{\prod_{j \in \{1,\dots,n\} \setminus \{i\}} (\lambda_i - \lambda_j)},$$

see [9] for more details and alternative definitions. Moreover, in the following lemma, we compute divided differences of f with repeated arguments under natural regularity assumptions.

LEMMA. *Let $D \subseteq \mathbb{K}$ be open, let $n, k, m_1, \dots, m_k \in \mathbb{N}$ with $m_1 + \dots + m_k = n$, let $(\lambda_1, \dots, \lambda_k) \in \sigma_k(D)$ and define the polynomials $P_1, \dots, P_k, P : \mathbb{C} \to \mathbb{C}$ by*

$$\begin{aligned} P_i(\lambda) &:= \prod_{j \in \{1,\dots,k\} \setminus \{i\}} (\lambda - \lambda_j)^{m_j} \qquad (i \in \{1, \dots, k\}), \\ P(\lambda) &:= \prod_{j=1}^{k} (\lambda - \lambda_j)^{m_j}. \end{aligned} \tag{4}$$

If $f : D \to \mathbb{C}$ is $(m_i - 1)$ times continuously differentiable at λ_i for all $i \in \{1, \dots, k\}$, then

$$f((\lambda_1)^{m_1}, \dots, (\lambda_k)^{m_k}) = \sum_{i=1}^{k} \sum_{\ell=0}^{m_i-1} \frac{(P_i^{-1})^{(m_i-1-\ell)}(\lambda_i)}{(m_i-1-\ell)!} \cdot \frac{f^{(\ell)}(\lambda_i)}{\ell!}.$$

Furthermore,

$$\begin{aligned} &f((\lambda_1)^{m_1}, \dots, (\lambda_k)^{m_k}) \\ &\quad = \sum_{i=1}^{k} \sum_{\ell=0}^{m_i-1} \frac{f^{(\ell)}(\lambda_i)}{\ell!} \Bigg(\sum_{j=0}^{m_i-1-\ell} (-1)^j \\ &\qquad \times \frac{j! B_{m_i-1-\ell,j}\big(\frac{1!}{(m_i+1)!} P^{(m_i+1)}(\lambda_i), \dots, \frac{(m_i-\ell-j)!}{(2m_i-\ell-j)!} P^{(2m_i-\ell-j)}(\lambda_i)\big)}{(m_i-1-\ell)! \big(\frac{0!}{m_i!} P^{(m_i)}(\lambda_i)\big)^{j+1}} \Bigg). \end{aligned}$$

In fact, the ith term of the first (equivalently, of the second) formula of the lemma becomes very simple in the particular cases when $1 \le m_i \le 3$. Indeed,

$$\begin{aligned}
&\sum_{\ell=0}^{m_i-1} \frac{(P_i^{-1})^{(m_i-1-\ell)}(\lambda_i)}{(m_i-1-\ell)!} \cdot \frac{f^{(\ell)}(\lambda_i)}{\ell!} \\
&\quad = \begin{cases} \dfrac{1}{P'(\lambda_i)} f(\lambda_i) & \text{if } m_i = 1, \\[2ex] \dfrac{2}{P''(\lambda_i)} f'(\lambda_i) - \dfrac{2P'''(\lambda_i)}{3P''(\lambda_i)^2} f(\lambda_i) & \text{if } m_i = 2, \\[2ex] \dfrac{3}{P'''(\lambda_i)} f''(\lambda_i) - \dfrac{3P^{(4)}(\lambda_i)}{2P'''(\lambda_i)^2} f'(\lambda_i) & \\ \qquad + \left(\dfrac{3P^{(4)}(\lambda_i)^2}{8P'''(\lambda_i)^3} - \dfrac{3P^{(5)}(\lambda_i)}{10P'''(\lambda_i)^2} \right) f(\lambda_i) & \text{if } m_i = 3. \end{cases}
\end{aligned}$$

LEMMA. *Let* $n \in \mathbb{N}$, $c = (c_0, \dots, c_n) \in \mathbb{K}^{n+1}$ *with* $c_n = 1$, *and let* $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ *be pairwise distinct roots of the characteristic polynomial* P_c *with multiplicities* $m_1, \dots, m_k \in \mathbb{N}$, *respectively. Then*

$$\omega_c(t) = \sum_{i=1}^{k} \sum_{\ell=0}^{m_i-1} \frac{(P_i^{-1})^{(m_i-1-\ell)}(\lambda_i)}{(m_i-1-\ell)!} \cdot \frac{t^\ell \exp(\lambda_i t)}{\ell!},$$

where P_i *is defined by* (4).

For the formulation of some consequences of our main results, for $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $k < n$ and $\gamma \in \mathbb{C}$, we define the function $\zeta_{n,k,\gamma} : \mathbb{R} \to \mathbb{R}$ by

$$\zeta_{n,k,\gamma}(t) := \sum_{i=0}^{\infty} \frac{\gamma^i t^{i(n-k)+n}}{(i(n-k)+n)!}. \tag{5}$$

By applying the ratio test, it follows that the series is convergent for all $t \in \mathbb{R}$. For further properties, we have the following statement.

LEMMA. *Let* $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ *with* $k < n$ *and* $\gamma \in \mathbb{C}$. *Then, the function* $\zeta_{n,k,\gamma}$ *is the (unique) solution of the initial value problem*

$$\zeta^{(n+1)} = \gamma \zeta^{(k+1)}, \qquad \zeta^{(i)}(0) = \delta_{i,n} \quad (i \in \{0, \dots, n\}).$$

In addition, if $j \in \{0, \dots, k\}$, *then*

$$\zeta_{n,k,\gamma}^{(j)} = \zeta_{n-j,k-j,\gamma}.$$

If $\gamma \neq 0$, *then, for all* $t \in \mathbb{R}$,

$$\zeta_{n,k,\gamma^{n-k}}(t) = \gamma^{-n} \zeta_{n,k,1}(\gamma t).$$

Furthermore, for all $t \in \mathbb{R}$*,*

$$\zeta_{n,k,0}(t) = \frac{t^n}{n!},$$
$$\zeta_{n,0,1}(t) = -1 + \frac{1}{n}\sum_{j=0}^{n-1} \exp\left(\cos\left(\frac{2\pi j}{n}\right)t\right) \cdot \cos\left(\sin\left(\frac{2\pi j}{n}\right)t\right).$$

We note that the series involved in the right hand side of (5) has closed forms, more precisely, it is the linear combinations of the hyperbolic or trigonometric functions and a polynomial of degree at most $n-3$. Indeed, if $\gamma = 1$, then we get

$$\zeta_{n,n-2,1}(t) = \sum_{i=0}^{\infty} \frac{t^{2i+n}}{(2i+n)!} = \begin{cases} \cosh(t) - \displaystyle\sum_{i=0}^{\frac{n-2}{2}} \frac{t^{2i}}{(2i)!} & \text{if } n \text{ is even,} \\ \sinh(t) - \displaystyle\sum_{i=0}^{\frac{n-3}{2}} \frac{t^{2i+1}}{(2i+1)!} & \text{if } n \text{ is odd.} \end{cases}$$

While for $\gamma = -1$, we can obtain

$$\begin{aligned} \zeta_{n,n-2,-1}(t) &= \sum_{i=0}^{\infty} \frac{(-1)^i t^{2i+n}}{(2i+n)!} \\ &= \begin{cases} (-1)^{\frac{n-2}{2}}\left(\cos(t) - \displaystyle\sum_{i=0}^{\frac{n-2}{2}} \frac{(-1)^i t^{2i}}{(2i)!}\right) & \text{if } n \text{ is even,} \\ (-1)^{\frac{n-3}{2}}\left(\sin(t) - \displaystyle\sum_{i=0}^{\frac{n-3}{2}} \frac{(-1)^i t^{2i+1}}{(2i+1)!}\right) & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Now, our main results can be stated as follows.

THEOREM. *Let* $n \in \mathbb{N}$, $c = (c_0, \dots, c_n) \in \mathbb{K}^{n+1}$ *with* $c_n = 1$, *and assume that* $f : I \to \mathbb{K}$ *is* $(n-1)$ *times differentiable at* $a \in I$. *Define* $T_{a,c}f : \mathbb{R} \to \mathbb{K}$ *by*

$$(T_{a,c}f)(x) := \sum_{j=0}^{n-1}\left(f^{(j)}(a)\sum_{i=0}^{n-1-j} c_{i+j+1}\omega_c^{(i)}(x-a)\right),$$

where ω_c *is defined by* (3). *Then,* $T_{a,c}f$ *belongs to the kernel of* D_c *and*

$$f^{(\ell)}(a) = (T_{a,c}f)^{(\ell)}(a) \qquad (\ell \in \{0, \dots, n-1\}).$$

The function $T_{a,c}f$ is termed the *generalized Taylor polynomial at the point* a *with respect to the differential operator* D_c.

THEOREM. *Let $n \in \mathbb{N}$, $c = (c_0, \ldots, c_n) \in \mathbb{K}^{n+1}$ with $c_n = 1$. Then, for all $f \in \mathcal{C}^n_{\mathbb{K}}(I)$ and $x, a \in I$, we have*

$$f(x) = (T_{a,c}f)(x) + \int_a^x D_c(f)(t) \cdot \omega_c(x-t)dt.$$

The following result is a consequence of the previous theorem in which the main part contains the Taylor expansion of order k and the rest is in terms of the function $\zeta_{n,k,\gamma}$.

THEOREM. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $k < n$, and $\gamma \in \mathbb{K}$. Then, for all $f \in \mathcal{C}^n_{\mathbb{K}}(I)$ and $x, a \in I$,*

$$\begin{aligned} f(x) = \sum_{j=0}^{k-1} f^{(j)}(a)\frac{(x-a)^j}{j!} + \sum_{j=k}^{n-1} f^{(j)}(a)\zeta_{n,k,\gamma}^{(n-j)}(x-a) \\ + \int_a^x (f^{(n)}(t) - \gamma f^{(k)}(t))\zeta'_{n,k,\gamma}(x-t)dt. \end{aligned}$$

The subsequent results will be corollaries of the previous theorem. First we note that the classical Taylor theorem with an integral remainder term follows from the previous theorem by taking $k = 0$ and $\gamma = 0$.

COROLLARY. *For all $f \in \mathcal{C}^2_{\mathbb{K}}(I)$ and $a, x \in I$, we have*

$$\begin{aligned} f(x) = f(a)\cos(x-a) + f'(a)\sin(x-a) \\ + \int_a^x (f''(t) + f(t))\sin(x-t)dt. \end{aligned}$$

COROLLARY. *For all $f \in \mathcal{C}^2_{\mathbb{K}}(I)$ and $a, x \in I$, we have*

$$\begin{aligned} f(x) = f(a)\cosh(x-a) + f'(a)\sinh(x-a) \\ + \int_a^x (f''(t) - f(t))\sinh(x-t)dt. \end{aligned}$$

COROLLARY. *For all $f \in \mathcal{C}^4_{\mathbb{K}}(I)$ and $a, x \in I$, we have*

$$\begin{aligned} f(x) = f(a)&\frac{\cosh(x-a)+\cos(x-a)}{2} \\ &+ f'(a)\frac{\sinh(x-a)+\sin(x-a)}{2} \\ &+ f''(a)\frac{\cosh(x-a)-\cos(x-a)}{2} \\ &+ f'''(a)\frac{\sinh(x-a)-\sin(x-a)}{2} \\ &+ \int_a^x (f''''(t)-f(t))\frac{\sinh(x-t)-\sin(x-t)}{2}dt. \end{aligned}$$

COROLLARY. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha^2-\beta^2) \neq 0$. Then, for all $f \in \mathcal{C}^4_{\mathbb{K}}(I)$ and $a, x \in I$, we have*

$$\begin{aligned} f(x) = f(a)&\frac{\beta^2\cos(\alpha(x-a))-\alpha^2\cos(\beta(x-a))}{\beta^2-\alpha^2} \\ &+ f'(a)\frac{\beta^3\sin(\alpha(x-a))-\alpha^3\sin(\beta(x-a))}{\alpha\beta(\beta^2-\alpha^2)} \\ &+ f''(a)\frac{\cos(\alpha(x-a))-\cos(\beta(x-a))}{\beta^2-\alpha^2} \\ &+ f'''(a)\frac{\beta\sin(\alpha(x-a))-\alpha\sin(\beta(x-a))}{\alpha\beta(\beta^2-\alpha^2)} \\ &+ \int_a^x (f''''(t)+(\alpha^2+\beta^2)f''(t)+\alpha^2\beta^2 f(t)) \\ &\qquad \times \frac{\beta\sin(\alpha(x-t))-\alpha\sin(\beta(x-t))}{\alpha\beta(\beta^2-\alpha^2)}dt. \end{aligned}$$

The limiting case of the above corollary (i.e., when $\alpha^2 = \beta^2 \neq 0$) is formulated as follows.

COROLLARY. *Let* $\alpha \in \mathbb{R}$ *with* $\alpha \neq 0$. *Then, for all* $f \in \mathcal{C}^4_{\mathbb{K}}(I)$ *and* $a, x \in I$, *we have*

$$\begin{aligned}
f(x) = f(a)&\frac{2\cos(\alpha(x-a)) + \alpha(x-a)\sin(\alpha(x-a))}{2} \\
&+ f'(a)\frac{3\sin(\alpha(x-a)) - \alpha(x-a)\cos(\alpha(x-a))}{2\alpha} \\
&+ f''(a)\frac{(x-a)\sin(\alpha(x-a))}{2\alpha} \\
&+ f'''(a)\frac{\sin(\alpha(x-a)) - \alpha(x-a)\cos(\alpha(x-a))}{2\alpha^3} \\
&+ \int_a^x (f''''(t) + 2\alpha^2 f''(t) + \alpha^4 f(t)) \\
&\qquad \times \frac{\sin(\alpha(x-t)) - \alpha(x-t)\cos(\alpha(x-t))}{2\alpha^3} dt.
\end{aligned}$$

COROLLARY. *Let* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha\beta(\alpha^2 - \beta^2) \neq 0$. *Then, for all* $f \in \mathcal{C}^4_{\mathbb{K}}(I)$ *and* $a, x \in I$, *we have*

$$\begin{aligned}
f(x) = f(a)&\frac{\beta^2\cosh(\alpha(x-a)) - \alpha^2\cosh(\beta(x-a))}{\beta^2 - \alpha^2} \\
&+ f'(a)\frac{\beta^3\sinh(\alpha(x-a)) - \alpha^3\sinh(\beta(x-a))}{\alpha\beta(\beta^2 - \alpha^2)} \\
&+ f''(a)\frac{\cosh(\beta(x-a)) - \cosh(\alpha(x-a))}{\beta^2 - \alpha^2} \\
&+ f'''(a)\frac{\alpha\sinh(\beta(x-a)) - \beta\sinh(\alpha(x-a))}{\alpha\beta(\beta^2 - \alpha^2)} \\
&+ \int_a^x (f''''(t) - (\alpha^2 + \beta^2) f''(t) + \alpha^2\beta^2 f(t)) \\
&\qquad \times \frac{\alpha\sinh(\beta(x-t)) - \beta\sinh(\alpha(x-t))}{\alpha\beta(\beta^2 - \alpha^2)} dt.
\end{aligned}$$

COROLLARY. *Let $\alpha \in \mathbb{R}$ with $\alpha \neq 0$. Then, for all $f \in \mathcal{C}^4(I)$ and $a, x \in I$, we have*

$$\begin{aligned} f(x) &= f(a)\frac{2\cosh(\alpha(x-a)) - \alpha(x-a)\sinh(\alpha(x-a))}{2} \\ &+ f'(a)\frac{3\sinh(\alpha(x-a)) - \alpha(x-a)\cosh(\alpha(x-a))}{2\alpha} \\ &+ f''(a)\frac{(x-a)\sinh(\alpha(x-a))}{2\alpha} \\ &+ f'''(a)\frac{\alpha(x-a)\cosh(\alpha(x-a)) - \sinh(\alpha(x-a))}{2\alpha^3} \\ &+ \int_a^x (f''''(t) - 2\alpha^2 f''(t) + \alpha^4 f(t)) \\ &\qquad \times \frac{\alpha(x-t)\cosh(\alpha(x-t)) - \sinh(\alpha(x-t))}{2\alpha^3} dt. \end{aligned}$$

To formulate the results of Taylor mean value theorem, we recall the extended mean value theorem for integrals.

LEMMA. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function and $g : [a,b] \to \mathbb{R}$ a nonnegative (or nonpositive) integrable function. Then there exists $\xi \in [a,b]$ such that*

$$\int_a^b fg = f(\xi)\int_a^b g.$$

Also, for any continuous function $h : \mathbb{R} \to \mathbb{R}$, let $\rho^+(h) \in [0, +\infty]$ (resp. $\rho^-(h) \in [-\infty, 0]$) denote the infimum of the positive roots (resp. the supremum of the negative roots) of h.

LEMMA. *Let $n \in \mathbb{N}$ and $c = (c_0, \ldots, c_n) \in \mathbb{R}^{n+1}$ with $c_n = 1$. Then, for $k \in \{1, \ldots, n-1\}$,*

$$\big[\rho^-(\omega_c^{(k)}), \rho^+(\omega_c^{(k)})\big] \subseteq \big[\rho^-(\omega_c^{(k-1)}), \rho^+(\omega_c^{(k-1)})\big].$$

Furthermore, $\big[\rho^-(\omega_c^{(n-1)}), \rho^+(\omega_c^{(n-1)})\big]$ is a neighborhood of 0.

The generalization of the Taylor mean value theorem is given as follows.

THEOREM. *Let $n \in \mathbb{N}$, $c = (c_0, \ldots, c_n) \in \mathbb{R}^{n+1}$ with $c_n = 1$. Then, for all $f \in \mathcal{C}^n_{\mathbb{R}}(I)$ and $a, x \in I$ with $\rho^-(\omega_c) \leq x - a \leq \rho^+(\omega_c)$, there exists a point ξ between a and x such that*

$$f(x) = (T_{a,c}f)(x) + D_c(f)(\xi) \cdot \int_0^{x-a} \omega_c(t)dt.$$

THEOREM. *Let* $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ *with* $k < n$ *and* $\gamma \in \mathbb{R}$ *and define* $\zeta_{n,k,\gamma} : \mathbb{R} \to \mathbb{R}$ *by* (5). *Then, for all* $f \in \mathcal{C}^n_{\mathbb{R}}(I)$ *and* $x, a \in I$ *with* $\rho^-(\zeta'_{n,k,\gamma}) \le x - a \le \rho^+(\zeta'_{n,k,\gamma})$*, there exists a point* ξ *between* a *and* x *such that*

$$\begin{aligned} f(x) = \sum_{j=0}^{k-1} f^{(j)}(a) \frac{(x-a)^j}{j!} + \sum_{j=k}^{n-1} f^{(j)}(a) \zeta^{(n-j)}_{n,k,\gamma}(x-a) \\ + (f^{(n)}(\xi) - \gamma f^{(k)}(\xi)) \zeta_{n,k,\gamma}(x-a). \end{aligned}$$

The subsequent results will be corollaries of the previous theorem. Moreover, the classical Taylor Mean Value Theorem is the particular case of the previous theorem when $k = 0$ and $\gamma = 0$. In this setting, we have that

$$\zeta_{n,0,0}(t) = \frac{t^n}{n!}$$

and hence

$$\rho^{\pm}(\zeta'_{n,0,0}) = \pm\infty.$$

COROLLARY. *For all* $f \in \mathcal{C}^2_{\mathbb{R}}(I)$ *and* $a, x \in I$ *with* $|a - x| \le \pi$*, there exists a point* ξ *between* a *and* x *such that*

$$\begin{aligned} f(x) = f(a)\cos(x-a) + f'(a)\sin(x-a) \\ + (f''(\xi) + f(\xi))(1 - \cos(x-a)). \end{aligned}$$

COROLLARY. *For all* $f \in \mathcal{C}^2_{\mathbb{R}}(I)$ *and* $a, x \in I$*, there exists a point* ξ *between* a *and* x *such that*

$$\begin{aligned} f(x) = f(a)\cosh(x-a) + f'(a)\sinh(x-a) \\ + (f''(\xi) - f(\xi))(\cosh(x-a) - 1). \end{aligned}$$

COROLLARY. *For all* $f \in \mathcal{C}^4_{\mathbb{R}}(I)$ *and* $a, x \in I$*, there exists a point* ξ *between* a *and* x *such that*

$$\begin{aligned} f(x) = f(a)\frac{\cosh(x-a) + \cos(x-a)}{2} \\ + f'(a)\frac{\sinh(x-a) + \sin(x-a)}{2} \\ + f''(a)\frac{\cosh(x-a) - \cos(x-a)}{2} \\ + f'''(a)\frac{\sinh(x-a) - \sin(x-a)}{2} \\ + (f''''(t) - f(t))\frac{\cosh(x-a) + \cos(x-a) - 2}{2}. \end{aligned}$$

COROLLARY. *Let* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha\beta(\alpha^2 - \beta^2) \neq 0$ *and let* t_0 *be the smallest positive root of the equation*

$$\beta \sin(\alpha t) = \alpha \sin(\beta t). \tag{6}$$

Then, for all $f \in \mathcal{C}^4_{\mathbb{R}}(I)$ *and* $a, x \in I$ *with* $|x - a| \leq t_0$*, there exists a point* ξ *between* a *and* x *such that*

$$\begin{aligned}
f(x) = f(a)&\frac{\beta^2 \cos(\alpha(x-a)) - \alpha^2 \cos(\beta(x-a))}{\beta^2 - \alpha^2}\\
&+ f'(a)\frac{\beta^3 \sin(\alpha(x-a)) - \alpha^3 \sin(\beta(x-a))}{\alpha\beta(\beta^2 - \alpha^2)}\\
&+ f''(a)\frac{\cos(\alpha(x-a)) - \cos(\beta(x-a))}{\beta^2 - \alpha^2}\\
&+ f'''(a)\frac{\beta \sin(\alpha(x-a)) - \alpha \sin(\beta(x-a))}{\alpha\beta(\beta^2 - \alpha^2)}\\
&+ (f''''(\xi) + (\alpha^2 + \beta^2) f''(\xi) + \alpha^2\beta^2 f(\xi))\\
&\qquad \times \frac{\alpha^2(\cos(\beta(x-a)) - 1) - \beta^2(\cos(\alpha(x-a)) - 1)}{\alpha^2\beta^2(\beta^2 - \alpha^2)}.
\end{aligned}$$

For the applicability of the previous corollary, it is essential to find the zeroes of the equation (6). In general, beyond the trivial solution $t = 0$, the other solutions cannot be established algebraically. On the other hand, if $\frac{\alpha}{\beta}$ is rational, say $|\frac{\alpha}{\beta}| = \frac{n}{m}$, where n, m are coprime natural numbers, let $s := \frac{|\alpha|}{n} = \frac{|\beta|}{m} \neq 0$. Then $\alpha = \pm ns$ and $\beta = \pm ms$ and (6) is now equivalent to

$$m \sin(nst) = n \sin(mst).$$

In the case when $t = \frac{k}{s}\pi$ for some $k \in \mathbb{N}$, then both sides are equal to zero. If t is not of this form, then $\sin(st) \neq 0$, thus this equation can be rewritten as

$$mU_{n-1}(\cos(st)) = m\frac{\sin(nst)}{\sin(st)} = n\frac{\sin(mst)}{\sin(st)} = nU_{m-1}(\cos(st)),$$

where U_k denotes the kth degree Chebyshev polynomial of the second kind. Therefore, the last equation is an algebraic equation for $\cos(st)$. Solving this equation for $\cos(st)$, the smallest positive solution t_0 can easily be computed.

The limiting case of the previous corollary (i.e., when $\alpha^2 = \beta^2 \neq 0$) is formulated as follows.

COROLLARY. *Let* $\alpha \in \mathbb{R}$ *with* $\alpha \neq 0$ *and let* t_0 *be the smallest positive root of the equation*

$$\sin(\alpha t) = \alpha t \cos(\alpha t).$$

Then, for all $f \in \mathcal{C}^4_{\mathbb{R}}(I)$ *and* $a, x \in I$ *with* $|x-a| \le t_0$*, there exists a point* ξ *between* a *and* x *such that*

$$\begin{aligned}
f(x) = f(a)&\frac{2\cos(\alpha(x-a)) + \alpha(x-a)\sin(\alpha(x-a))}{2} \\
&+ f'(a)\frac{3\sin(\alpha(x-a)) - \alpha(x-a)\cos(\alpha(x-a))}{2\alpha} \\
&+ f''(a)\frac{(x-a)\sin(\alpha(x-a))}{2\alpha} \\
&+ f'''(a)\frac{\sin(\alpha(x-a)) - \alpha(x-a)\cos(\alpha(x-a))}{2\alpha^3} \\
&+ (f''''(\xi) + 2\alpha^2 f''(\xi) + \alpha^4 f(\xi)) \\
&\qquad \times \frac{2 - 2\cos(\alpha(x-a)) - \alpha(x-a)\sin(\alpha(x-a))}{2\alpha^4}.
\end{aligned}$$

COROLLARY. *Let* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha\beta(\alpha^2 - \beta^2) \neq 0$*. Then, for all* $f \in \mathcal{C}^4_{\mathbb{R}}(I)$ *and* $a, x \in I$*, there exists a point* ξ *between* a *and* x *such that*

$$\begin{aligned}
f(x) = f(a)&\frac{\beta^2\cosh(\alpha(x-a)) - \alpha^2\cosh(\beta(x-a))}{\beta^2 - \alpha^2} \\
&+ f'(a)\frac{\beta^3\sinh(\alpha(x-a)) - \alpha^3\sinh(\beta(x-a))}{\alpha\beta(\beta^2 - \alpha^2)} \\
&+ f''(a)\frac{\cosh(\beta(x-a)) - \cosh(\alpha(x-a))}{\beta^2 - \alpha^2} \\
&+ f'''(a)\frac{\alpha\sinh(\beta(x-a)) - \beta\sinh(\alpha(x-a))}{\alpha\beta(\beta^2 - \alpha^2)} \\
&+ (f''''(\xi) - (\alpha^2 + \beta^2) f''(\xi) + \alpha^2\beta^2 f(\xi)) \\
&\qquad \times \frac{\alpha^2(\cosh(\beta(x-a)) - 1) - \beta^2(\cosh(\alpha(x-a)) - 1)}{\alpha^2\beta^2(\beta^2 - \alpha^2)}.
\end{aligned}$$

COROLLARY. *Let* $\alpha \in \mathbb{R}$ *with* $\alpha \neq 0$. *Then, for all* $f \in \mathcal{C}_{\mathbb{R}}^4(I)$ *and* $a, x \in I$, *there exists a point* ξ *between* a *and* x *such that*

$$\begin{aligned} f(x) = f(a)&\frac{2\cosh(\alpha(x-a)) - \alpha(x-a)\sinh(\alpha(x-a))}{2} \\ &+ f'(a)\frac{3\sinh(\alpha(x-a)) - \alpha(x-a)\cosh(\alpha(x-a))}{2\alpha} \\ &+ f''(a)\frac{(x-a)\sinh(\alpha(x-a))}{2\alpha} \\ &+ f'''(a)\frac{\alpha(x-a)\cosh(\alpha(x-a)) - \sinh(\alpha(x-a))}{2\alpha^3} \\ &+ (f''''(\xi) - 2\alpha^2 f''(\xi) + \alpha^4 f(\xi)) \\ &\qquad \times \frac{2 - 2\cosh(\alpha(x-a)) + \alpha(x-a)\sinh(\alpha(x-a))}{2\alpha^4}. \end{aligned}$$

Taylor's theorem plays an important role in the field of the numerical analysis, especially when it comes to understanding errors that arise when approximating functions. This includes analyzing errors related to numerical integration methods such as Simpson's rule and the trapezoidal rule. In the following section, we establish various factorization results and then derive estimates for linear functionals through the use of a generalized Taylor theorem. Moreover, several error bounds are established including applications to the trapezoidal rule as well as to a Simpson formula rule.

Factorization results and estimates

We start with the following theorem which establishes a sufficient condition for such a factorization.

THEOREM. *Let* X *and* Z *be Banach spaces and* Y *be a normed space over* $\mathbb{K}$. *Assume that* $A : X \to Y$ *and* $B : X \to Z$ *are bounded linear maps such that* $\ker B \subseteq \ker A$ *and* $B(X) = Z$. *Then there exists a unique bounded linear map* $C : Z \to Y$ *such that* $A = C \circ B$.

The main goal here is to obtain various estimates for the linear functional $\mathcal{A}_\mu : \mathcal{C}_{\mathbb{K}}(I) \to \mathbb{K}$ defined by

$$\mathcal{A}_\mu(f) := \int_{[a,b]} f(x)d\mu(x).$$

In order to construct $n \in \mathbb{N}$ and a linear map $B : \mathcal{C}_{\mathbb{K}}^n(I) \to \mathcal{C}_{\mathbb{K}}(I)$ such that $\ker B \subseteq \ker \mathcal{A}_\mu$, we search for exponential polynomials in the kernel of $\mathcal{A}_\mu$.

For this aim, let us define the function $\mathcal{S}_\mu : \mathbb{C} \to \mathbb{C}$ by

$$\mathcal{S}_\mu(\lambda) := \int_{[a,b]} e^{\lambda x} d\mu(x).$$

The function $\mathcal{S}_\mu$ will be termed the *spectral function related to the measure* μ.

For fixed elements $\lambda_1, \dots, \lambda_k \in \Lambda_\mu$, we consider the polynomial $P : \mathbb{C} \to \mathbb{C}$ given by

$$P(\lambda) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_k)^{m_k} = c_n \lambda^n + \cdots + c_1 \lambda + c_0, \tag{7}$$

where $1 \le m_i \le m(\mathcal{S}_\mu, \lambda_i)$ for all i and $n := m_1 + \cdots + m_k$. Clearly, $c_n = 1$. Then we define the linear differential operator $D_c : \mathcal{C}^n_{\mathbb{K}}(I) \to \mathcal{C}_{\mathbb{K}}(I)$ by the formula

$$D_c(f) := c_n f^{(n)} + \cdots + c_1 f' + c_0 f \qquad (f \in \mathcal{C}^n(I)). \tag{8}$$

It follows from the theory of ordinary differential equations (see [24]) that D_c is a bounded and surjective linear operator and the exponential polynomials

$$x \mapsto x^j e^{\lambda_i x} \qquad (i \in \{1, \dots, k\},\ j \in \{0, \dots, m_i - 1\})$$

form a fundamental system of solutions of the differential equation $D_c(f) = 0$. In other words, the above exponential polynomials span the kernel of D_c and hence $\ker D_c \subseteq \ker \mathcal{A}_\mu$. In view of the previous theorem, that is mentioned in the beginning of this section, there exists a unique bounded linear map $C : \mathcal{C}_{\mathbb{K}}(I) \to \mathbb{K}$ such that $\mathcal{A}_\mu|_{\mathcal{C}^n_{\mathbb{K}}(I)} = C \circ D_c$. In what follows, we explicitly construct C and then we present several applications to obtain sharp upper bounds for the error terms of quadrature rules. For standard references about error bounds for quadrature rules, we refer to the monographs [5] by Atkinson and [11] by Faires and Burden and to the recent papers [6] by Barnett *et al.*, [10] by Cruz-Uribe and Neugebauer, [13] by Masjed-Jamei *et al.*, [21] by Talman and [23] by Ujević.

Our basic factorization results are stated as follows.

THEOREM. *Let μ be a nonzero bounded $\mathbb{C}$-valued Borel measure on $[a,b]$, let $\lambda_1, \dots, \lambda_k \in \Lambda_\mu$ and $m_1, \dots, m_k \in \mathbb{N}$ with $m_i \le m(\mathcal{S}_\mu, \lambda_i)$ for $i \in \{1, \dots, k\}$. Define $c = (c_0, \dots, c_n) \in \mathbb{C}^{n+1}$ by (7) (where $n := m_1 + \cdots + m_k$) and the differential operator $D_c : \mathcal{C}^n_{\mathbb{C}}([a,b]) \to \mathcal{C}_{\mathbb{C}}([a,b])$ by (8). Let $\omega_c \in \mathcal{C}^n_{\mathbb{C}}(\mathbb{R})$ be the characteristic solution of $D_c(\omega) = 0$. Finally, define $g : [a,b] \to \mathbb{C}$ by*

$$g(t) := \int_{[t,b]} \omega_c(x - t) d\mu(x).$$

Then, for all $f \in \mathcal{C}^n_{\mathbb{C}}([a,b])$,

$$\mathcal{A}_\mu(f) := \int_{[a,b]} f(x)d\mu(x) = \int_{[a,b]} D_c(f)(t)\cdot g(t)dt.$$

In other words, $\mathcal{A}_\mu|_{\mathcal{C}^n_{\mathbb{C}}(I)} = C_g \circ D_c$, *where* $C_g : \mathcal{C}_{\mathbb{C}}(I) \to \mathbb{C}$ *is given by*

$$C_g(h) = \int_{[a,b]} h(t)g(t)dt.$$

THEOREM. *Let* μ *be a nonzero bounded* $\mathbb{C}$*-valued Borel measure on* $[a,b]$, *let* $0 \le k \le n$ *and* $\gamma \in \mathbb{C}$. *Assume that*

$$\begin{aligned} &\int_{[a,b]} x^i d\mu(x) = 0, \quad (i \in \{0,\dots,k-1\}),\\ &\int_{[a,b]} \exp\big(\sqrt[n-k]{\gamma}\exp\big(\tfrac{2j\pi}{n-k}\boldsymbol{i}\big)x\big)d\mu(x) = 0, \quad (j \in \{0,\dots,n-k-1\}), \end{aligned} \tag{9}$$

where $\sqrt[n-k]{\gamma}$ *denotes the root of order* $(n-k)$ *of* γ *with the smallest nonnegative argument in the interval* $[0,2\pi)$. *Define* $g : [a,b] \to \mathbb{C}$ *by*

$$g(t) := \int_{[t,b]} \zeta'_{n,k,\gamma}(x-t)d\mu(x). \tag{10}$$

Then, for all $f \in \mathcal{C}^n_{\mathbb{C}}([a,b])$,

$$\mathcal{A}_\mu(f) = \int_{[a,b]} (f^{(n)}(t) - \gamma f^{(k)}(t))\cdot g(t)dt.$$

Before formulating the next result, we recall the definition of the pth norm of a Lebesgue measurable function $f : I \to \mathbb{C}$ for $p \in [1,\infty]$:

$$\|f\|_p := \begin{cases} \Big(\int_{[a,b]} |f(t)|^p dt\Big)^{\frac{1}{p}} & \text{if } p \in [1,\infty),\\ \inf\{s \ge 0\colon |f(t)| \le s \text{ for a.e. } t \in I\} & \text{if } p = \infty. \end{cases}$$

COROLLARY. *Under the notation and assumptions of the previous theorems, for all* $f \in \mathcal{C}^n_{\mathbb{C}}(I)$ *and for all* $p, q \in [1,\infty]$ *with* $\frac{1}{p} + \frac{1}{q} = 1$,

$$|\mathcal{A}_\mu(f)| \le \|D_c(f)\|_p \cdot \|g\|_q.$$

If, in addition, $c = (c_0,\dots,c_n) \in \mathbb{R}^{n+1}$, μ *is a real-valued measure,* $f \in \mathcal{C}^n_{\mathbb{R}}(I)$, *and both* g *and* $D_c(f)$ *are nonnegative (or nonpositive) on* $[a,b]$, *then*

$$\mathcal{A}_\mu(f) \ge 0. \tag{11}$$

If g *and* $D_c(f)$ *have opposite signs over* $[a,b]$, *then this inequality reverses.*

COROLLARY. *Let μ be a nonzero bounded $\mathbb{C}$-valued Borel measure on $[a,b]$, let $0 \leq k \leq n$ and $\gamma \in \mathbb{C}$. Assume that the equalities in (9) hold. Define $g : [a,b] \to \mathbb{C}$ by (10). Then, for all $f \in \mathcal{C}^n_{\mathbb{C}}([a,b])$ and $p, q \in [1,\infty]$ with $\frac{1}{p}+\frac{1}{q}=1$,*

$$\left| \int_{[a,b]} f(x)d\mu(x) \right| \leq \|f^{(n)} - \gamma f^{(k)}\|_p \cdot \|g\|_q.$$

If, in addition, $\gamma \in \mathbb{R}$, μ is a real-valued measure and both g and $f^{(n)} - \gamma f^{(k)}$ are nonnegative (or nonpositive) on $[a,b]$, then the inequality (11) holds. If g and $f^{(n)} - \gamma f^{(k)}$ have opposite signs over $[a,b]$, then this inequality reverses.

To apply the results to the trapezoidal rule, we begin with the following lemma.

LEMMA. *For all $t \in \mathbb{R}_+$*

$$\frac{t}{\sinh(t)} < 1 < t\coth(t)$$

and, for all $t \in (0,\pi)$,

$$t\cot(t) < 1 < \frac{t}{\sin(t)}.$$

The aim is to establish various further estimates for $R_T(f)$, which is defined by

$$R_T(f) := \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f.$$

Observe that, with $\mu := \frac{1}{2}(\delta_a + \delta_b) - \nu$ (where δ_t denotes the Dirac measure concentrated at t and ν stands for the normalized Lebesgue measure on $[a,b]$), we can obtain that

$$R_T(f) = \mathcal{A}_\mu(f).$$

The corresponding spectral function is given by

$$\mathcal{S}_\mu(\lambda) = \frac{e^{\lambda a}+e^{\lambda b}}{2} - \frac{1}{b-a}\int_a^b e^{\lambda x}dx \qquad (\lambda \in \mathbb{C}). \tag{12}$$

Given a compact interval $[a,b]$, the classical trapezoidal rule asserts that, for a twice differentiable function $f : [a,b] \to \mathbb{R}$,

$$\frac{1}{b-a}\int_a^b f = \frac{f(a)+f(b)}{2} - R_T(f),$$

where the remainder term $R_T(f)$ has various estimates in terms of the norms of the second derivative of f and the length of the interval $[a,b]$. For instance (see [5, pp. 252–253]),

$$|R_T(f)| \leq \frac{(b-a)^2}{12}\|f''\|_\infty.$$

LEMMA. *Let $\lambda \in \mathbb{C}$. Then λ is a root of the spectral function $\mathcal{S}_\mu$ given by* (12) *if and only if $u := \lambda\frac{b-a}{2}$ is a fixed point of the tangent hyperbolic function. The multiplicity of λ equals 1 if $\lambda \neq 0$ and equals 2 if $\lambda = 0$.*

In order to apply our main theorems to the trapezoidal rule, we shall need to describe the fixed points of the tangent hyperbolic function.

LEMMA. *A number $u \in \mathbb{C}$ is a fixed point of the tangent hyperbolic function, i.e.,*

$$\tanh(u) = u$$

holds if and only if $u = v\boldsymbol{i}$, where $v \in \mathbb{R}$ is a fixed point of the tangent function. Furthermore, for all $k \in \mathbb{Z}$, the open interval $\big((k-\frac{1}{2})\pi, (k+\frac{1}{2})\pi\big)$ contains exactly one fixed point of the tangent function.

The unique fixed point of the tangent function in the open interval $\big((k-\frac{1}{2})\pi, (k+\frac{1}{2})\pi\big)$ will be denoted by τ_k in the sequel.

THEOREM. *Let $k \in \mathbb{N}$, $0 \leq n_1 < \cdots < n_k$ be integers, let $a, b \in \mathbb{R}$ with $a < b$ and let $\lambda_j := \frac{2}{(b-a)}\tau_{n_j}$ for $j \in \{1,\dots,k\}$. Define $(c_0, c_1, \dots, c_{2k}) \in \mathbb{R}^{2k+1}$ by the equality*

$$\big(z^2+\lambda_1^2\big)\cdots\big(z^2+\lambda_k^2\big) = c_{2k}z^{2k} + \cdots + c_1 z^1 + c_0 =: P_c(z) \quad (z \in \mathbb{C}).$$

Then, for all $f \in \mathcal{C}^{2k}_{\mathbb{K}}([a,b])$,

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f = \int_a^b D_c(f)(t)\cdot g(t)dt,$$

where

$$g(t) := \begin{cases} \displaystyle\sum_{j=1}^{k} \frac{\sin(\lambda_j(b-t)/2)\sin(\lambda_j(t-a)/2)}{\lambda_j Q_j(\lambda_j)\sin(\lambda_j(b-a)/2)} & \text{if } n_1 > 0, \\ \displaystyle\sum_{j=2}^{k} \frac{\sin(\lambda_j(b-t)/2)\sin(\lambda_j(t-a)/2)}{\lambda_j Q_j(\lambda_j)\sin(\lambda_j(b-a)/2)} & \\ \displaystyle\qquad + \frac{(b-t)(t-a)}{2Q_1(0)(b-a)} & \text{if } n_1 = 0, \end{cases}$$

and $Q_j(z) := \prod_{\ell\in\{1,\dots,k\}\setminus\{j\}}(\lambda_\ell^2 - z^2)$ for $j \in \{1,\dots,k\}$.

In the particular case when $k = 1$, the above theorem simplifies to the following result.

COROLLARY. *Let $n \in \mathbb{N} \cup \{0\}$, let $a, b \in \mathbb{R}$ with $a < b$ and let $\lambda_n := \frac{2\tau_n}{b-a}$. Then, for all $f \in \mathcal{C}^2_{\mathbb{K}}([a, b])$,*

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f = \int_a^b (f'' + \lambda_n^2 f)(t) \cdot g(t) dt,$$

where

$$g(t) := \begin{cases} \dfrac{\sin(\lambda_n(b-t)/2) \sin(\lambda_n(t-a)/2)}{\lambda_n \sin(\lambda_n(b-a)/2)} & \textit{if } n > 0, \\ \dfrac{(b-t)(t-a)}{2(b-a)} & \textit{if } n = 0. \end{cases}.$$

Consequently, the following statement is a new error estimate for the trapezoidal rule.

THEOREM. *Let $n \in \mathbb{N} \cup \{0\}$, let $a, b \in \mathbb{R}$ with $a < b$ and let $\lambda_n := \frac{2\tau_n}{b-a}$. Then, for all $f \in \mathcal{C}^2_{\mathbb{K}}([a, b])$,*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f \right|$$
$$\leq \begin{cases} \frac{1}{12}(b-a)^2 \cdot \|f''\|_\infty & \textit{if } n = 0, \\ \frac{(n+1)n\pi}{2\tau_n^3}(b-a)^2 \cdot \|f'' + \lambda_n^2 f\|_\infty & \textit{if } n > 0, \\ \frac{1}{8}(b-a) \cdot \|f''\|_1 & \textit{if } n = 0, \\ \frac{1+|\cos(\tau_n)|}{4\tau_n|\sin(\tau_n)|}(b-a) \cdot \|f'' + \lambda_n^2 f\|_1 & \textit{if } n > 0. \end{cases}$$

Our final result is an extension of the Simpson formula.

THEOREM. *Let $a, b \in \mathbb{R}$ with $a < b$, $u = w + \boldsymbol{i}v$, where $w \in \mathbb{R}_+$, $v \in (0, \pi)$ and define α_u, β_u by*

$$\alpha_u := \frac{\overline{u} \sinh(u) - u \sinh(\overline{u})}{2u\overline{u}(\cosh(u) - \cosh(\overline{u}))},$$
$$\beta_u := \frac{u \cosh(u) \sinh(\overline{u}) - \overline{u} \cosh(\overline{u}) \sinh(u)}{u\overline{u}(\cosh(u) - \cosh(\overline{u}))}.$$

Then, for all $f \in \mathcal{C}^4_{\mathbb{K}}([a,b])$,

$$\left|\alpha_u f(a) + \beta_u f\Big(\frac{a+b}{2}\Big) + \alpha_u f(b) - \frac{1}{b-a}\int_a^b f\right|$$
$$\leq \begin{cases} \dfrac{(b-a)^3\big(v\sinh(w) - w\sin(v)\big)^2}{32(w^2+v^2)^2 wv\sinh(w)\sin(v)} \\ \qquad\qquad \times\left\| f'''' + \dfrac{8(v^2-w^2)}{(b-a)^2} f'' + \dfrac{16(w^2+v^2)^2}{(b-a)^4} f\right\|_1, \\ \dfrac{(b-a)^4(2\alpha_u + \beta_u - 1)}{16(w^2+v^2)^2} \\ \qquad\qquad \times\left\| f'''' + \dfrac{8(v^2-w^2)}{(b-a)^2} f'' + \dfrac{16(w^2+v^2)^2}{(b-a)^4} f\right\|_\infty. \end{cases}$$

Finally, we deduce the Simpson formula with two error terms by taking the limit $u \to 0$ in the previous theorem.

COROLLARY. *Let* $a, b \in \mathbb{R}$. *Then, for all* $f \in \mathcal{C}^4_{\mathbb{K}}([a,b])$,

$$\left|\frac{1}{6} f(a) + \frac{2}{3} f\Big(\frac{a+b}{2}\Big) + \frac{1}{6} f(b) - \frac{1}{b-a}\int_a^b f\right|$$
$$\leq \begin{cases} \dfrac{(b-a)^3}{1152}\cdot \left\|f''''\right\|_1, \\ \dfrac{(b-a)^4}{2880}\cdot \left\|f''''\right\|_\infty. \end{cases}$$

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List of talks

(1) *A new extension of the Taylor theorem with an application of estimating linear functionals*, The First Sharjah International Conference on Mathematical Sciences, University of Sharjah, Sharjah, UAE, November 6–November 8, 2023.

(2) *Estimating linear functionals via factorization: Theory and applications*, The 59th International Symposium on Functional Equations, Hajdúszoboszló, Hungary, June 18–June 25, 2023.

(3) *Estimates of linear expressions through factorization*, Qualification at the End of the Third Year, Institute of Mathematics, University of Debrecen, June 6, 2023.

(4) *Generalizations of the Taylor theorem with factorization results*, 22nd Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities, Hajdúszoboszló, Hungary, February 1–February 4, 2023.

(5) *Taylor-type expansions in terms of exponential polynomials*, Complex Exam Seminar, Institute of Mathematics, University of Debrecen, June 16, 2022.

(6) *Taylor-type expansions in terms of exponential polynomials*, Analysis Research Seminar, Institute of Mathematics, University of Debrecen, May 18, 2022.

(7) *A generalization of the Taylor theorem*, Qualification at the End of the First Year, Institute of Mathematics, University of Debrecen, June 18, 2021.

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Subject: PhD Publication List

Candidate: Ali Hasan Ali
Doctoral School: Doctoral School of Mathematical and Computational Sciences
MTMT ID: 10077583

List of publications related to the dissertation

Foreign language scientific articles in international journals (2)

1. **Ali, A. H.**, Páles, Z.: Estimates of linear expressions through factorization.
   J. Approx. Theory. 299, 1-23, 2024. ISSN: 0021-9045.
   DOI: http://dx.doi.org/10.1016/j.jat.2024.106019
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2. **Ali, A. H.**, Páles, Z.: Taylor-type expansions in terms of exponential polynomials.
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   IF: 1

List of other publications

Foreign language scientific articles in international journals (43)

3. Amir, M., Ali, Q., Raza, A., Almusawa, M., Hamali, W., **Ali, A. H.**: Computational results of convective heat transfer for fractionalized Brinkman type tri-hybrid nanofluid with ramped temperature and non-local kernel.
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Total IF of journals (all publications): 125,1
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The Candidate's publication data submitted to the iDEa Tudóstér have been validated by DEENK on the basis of the Journal Citation Report (Impact Factor) database.

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