



Dynamical analysis of a plant–honeybee mutualism model with dust storm effects[☆]

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ABSTRACT

The interaction between blooming plants and honeybees (pollinators) is one of the most frequently observed phenomena in ecosystems. Research typically examines simplified systems that focus on blooming plants and their pollinators, as well as predators that target one or both populations. These studies overlook the influence of ecological phenomena on the relationship. In light of the observation above, we propose a mathematical model to examine the effects of dust storms on the mutualistic interaction between blooming plants and honeybee populations. Basic results on the positivity, boundedness, and persistence of solutions; the feasibility of equilibria; local and global stability; and certain bifurcations and their characteristics are examined. We observe that an increase in the frequency of dust storms leads to a decrease in the density of blooming plants and bees at the coexistence equilibrium point. The numerical findings indicate that dust storms may lead to the extinction of blooming plants and the decline of bee populations. The blooming plant is essential to a sustainable ecosystem, contributing to the sustainability and stability of both the plant and the honeybee populations within ecosystems. These results may guide conservation programs aimed at safeguarding honeybees and enhancing biodiversity.

Introduction

The process of turning issues that occur in the actual world into mathematical formulations, with examples including equations and algorithms, is known as mathematical modeling. It enables the analysis of systems, the forecasting of behaviors, and the making of sound judgments in engineering, ecology [1–3], diseases [4–6], and biology [7,8]. A well-designed model strikes a balance between real-world accuracy and mathematical simplicity [9]. In ecology, the sustainability of global agricultural systems and local ecosystems depends on the symbiotic relationship between honeybee populations and blooming plants [10]. The reproductive success of approximately 300,000 plant species worldwide is contingent upon the pollination services provided by animal pollinators, such as honeybees, through a symbiotic relationship. Pollinators also significantly enhance agricultural productivity through plant cross-pollination, as well as genetic diversity and quality of crop yield in this symbiosis [11]. As a result, this is an important area of research in ecology and conservation biology. The interdependent associations between plants and pollinator systems have been scrutinized in many studies [12–14]. Hadany studied the symbiosis between plants and pollinators. This association is proven to stay

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the same as long as the starting population is big enough [15]. Biswas et al. considered a plant-pollinator system to measure the effect of predation on pollinator populations. The authors concluded that pollinators face extinction risk if predation rates remain uncontrolled. This hypothesis has led to numerous studies exploring the influences of ants and nectar theft on the plant-pollinator model [16]. The decline in biodiversity is a significant issue, with the reduction of pollinating insects being particularly alarming due to its potential impacts on plant reproduction. A recent report on the worldwide decline in bumblebee and honeybee populations highlighted the economic and ecological significance of pollination [17]. The decline in pollinator species is linked to multiple environmental factors, including habitat destruction, disease, climate change, pesticide use, and predation [18–20]. Invasive predators can substantially impact pollinators by reducing their populations, altering plant reproductive outcomes, and disrupting the interactions between plants and pollinators [21].

Dust storms are one factor that adversely impacts the mutualism in plant-pollinator systems, which results from active winds that transport dust particles from the Earth's loose surface crust. The tempest occurs when two conditions are satisfied: high wind speeds and dry, loose soil devoid of vegetation [22]. Dust storms adversely affect plants by obstructing the stomata on their leaves, thereby impeding the sunlight essential for photosynthesis [23]. Dust obstructs critical operations, including transpiration, heightens water stress, and attracts insects and illnesses. It can also transport hazardous chemicals that adversely affect plants, hindering their growth, diminishing output, and occasionally resulting in plant mortality.

On the other hand, intense winds interfere with floral fragrance signals, diminishing their appeal to honeybees. Intense winds swiftly disseminate floral fragrant compounds in various directions, reducing their "scent signal." As aroma dispersal diminishes, honeybees' capacity to locate food sources is impaired, leading to a decrease in their visits to flowers [24]. Moreover, flying in strong winds requires more drive to maintain balance and progress. Honeybees remain inside the hive or restrict their flight activity to conserve energy. Opportunities for pollination decay with the number of flowers visited. Intense winds can damage flower petals or destroy the structures that would keep honeybees from reaching their reproductive organs. This change in time might make pollination less likely during the peak flowering season. Intense gusts can make honeybees crash, land, or alter course, endangering the colony. Worker attrition or chronic stress can negatively impact the colony's ability to provide adequate personnel for visits [25]. A comprehensive model of blooming plants and honeybees that incorporates dust storms has not been extensively investigated. The significance of investigating the influence of dust storms on the relationship between blooming plants and pollinators lies in clarifying how harsh climatic factors affect pollination, a critical environmental process indispensable for agricultural production and biodiversity. Dust storms transfer efficacy, and bee activity can be diminished by dust cyclones, which also impact the health of blooming plants by depositing dust on their various components. Ultimately, this can result in decreased ecosystem stability and sustainability, lower pollination success, and reduced seed and produce production. The biological justification for modeling dust pollutants as a dynamic state and their relationship to vegetation cover was previously provided by Shyam et al. [26]. Our model findings provide additional context for Shyam et al. by examining the effects of dust pollutants on the mutualistic interaction between flowering plants and honeybees in the presence of wind flow. Therefore, the objective of this work is to elucidate the interplay between dust storms and mutualistic interactions and how these dynamics influence the overall ecological equilibrium and sustainability of the blooming plants–honeybee system. Bifurcation and stability analyses are conducted to assess the stability characteristics of equilibrium states

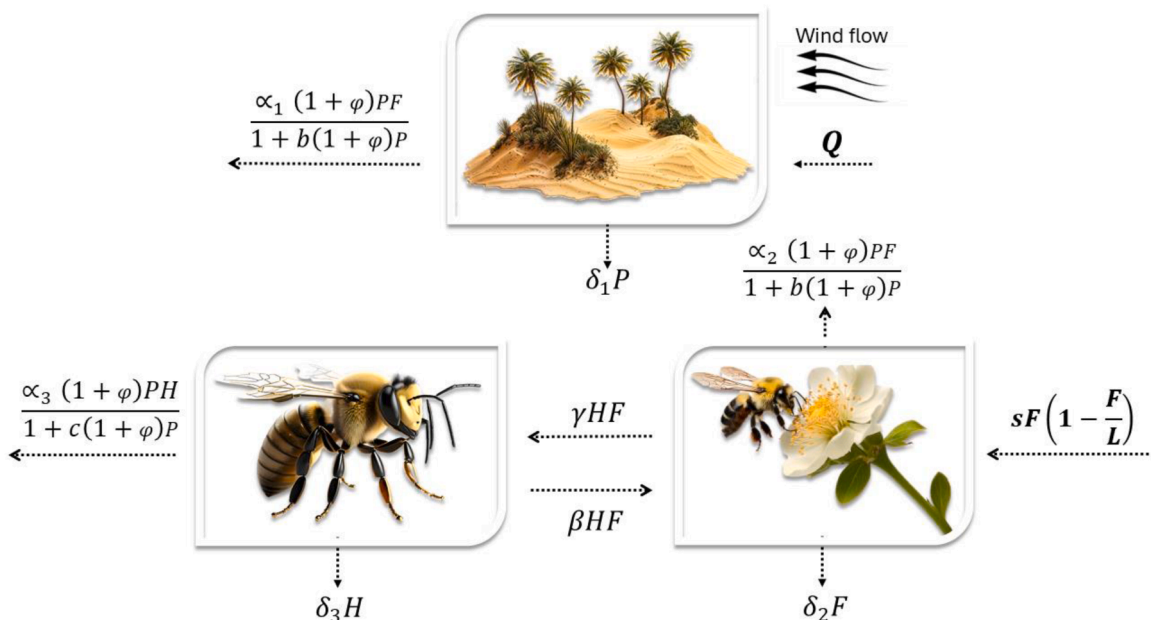


Fig. 1. Schematic diagram of The PFH system.

and the qualitative behavior of the proposed dynamical system. Numerical simulations are performed to ascertain the long-term sustainability of the biological system by studying the impact of each parameter on the proposed system.

Model formulation

This section will involve integrating numerous experimental investigations and mathematical models to develop a model that elucidates the dynamics of the interaction between honeybees and blooming plants [12,16,19,20]. In the interaction, the dust storms and excessive wind flow are formulated. The *PFH* system's dynamics have a substantial impact on the interaction between honeybees and blooming plants, rendering it a considerable risk factor. Additionally, we analyze the interactions among the above factors and the aspects of blooming plants and honeybees. Fig. 1 illustrates the interactions among blooming plants $F(t)$, and honeybees $H(t)$ in the presence of dust storms $P(t)$.

Dust storms are a risk factor that has a detrimental impact on the environment, particularly by negatively affecting the interaction between pollinators and blooming plants. Blooming plants serve as a protective barrier against dust storms, thereby mitigating their impact on the atmosphere. In the first equation of our model, which is for dust storms, it is hypothesized that Q is the emission rate of dust contaminants into the atmosphere, and δ_1 is the natural depletion of dust pollutants [26]. A Holling type II functional response has been modified after adding the effect of dust storms to illustrate the decrease in atmospheric dust pollutant concentration attributable to blooming plants $\left(\frac{\alpha_1(1+\varphi)PF}{1+b(1+\varphi)P}\right)$, which act as dust scavengers. α_1 refers to the depletion rate of dust particles due to the barrier it forms from blooming plants, b is a half-saturation level, and $z(\varphi) = (1+\varphi)$ is a function of wind flow that fulfils the subsequent criteria

- $z(0) = 1$; which indicates that in the absence of wind speed, the depletion of blooming plants-induced dust particle concentrations will be $\left(\frac{\alpha_1 PF}{1+bP}\right)$.
- $z'(\varphi) > 1$; that means the reduction in dust pollutant concentration is consistently heightened by blooming plants [27].

So, the dynamics of dust storms could be presented by

$$\frac{dP}{dt} = Q - \frac{\alpha_1(1+\varphi)PF}{1+b(1+\varphi)P} - \delta_1 P. \quad (1)$$

Blooming plants are honeybees' primary source of nutrition. However, dust storms negatively influence plants by obstructing the pores on the leaf surface, thereby impeding gas exchange and the completion of photosynthesis. So, in the second equation, which is for blooming plants, the term $sF\left(1 - \frac{F}{L}\right)$ represents the logistic growth rate for the blooming plants in the absence of honeybees, where s represents the intrinsic growth rate, L signifies the carrying capacity, and δ_2 denotes the natural depletion rate. $\left(\frac{\alpha_2(1+\varphi)PF}{1+b(1+\varphi)P}\right)$ signifies the depletion of blooming plants as a result of the rising intensity of dust storms, where α_2 is the rate of dust-storm-driven depletion of blooming plants. (βHF) utilizes to illustrate the mutualistic connection, wherein β signifies the beneficial impact of honeybees on blooming plants (as pollinators). Therefore, the dynamics of blooming plants could be explained by

$$\frac{dF}{dt} = sF\left(1 - \frac{F}{L}\right) - \frac{\alpha_2(1+\varphi)PF}{1+b(1+\varphi)P} + \beta HF - \delta_2 F. \quad (2)$$

Honeybees play a crucial role in maintaining crop diversity and quality, providing essential pollination services vital to their production. However, they are adversely affected by environmental factors, such as particulate pollutants. Dust storms directly affect bee movement by impeding their ability to navigate and forage. They disrupt floral scents through complex chemical reactions, obscure navigation paths, and impair bees' ability to remember and navigate. These effects render bees less capable of finding flowers, thereby reducing pollination and honey production. So, in the last equation, which is for honeybees, the term γHF denotes the nutrients that blooming plants supply to honeybees, where γ signifies the corresponding value of honeybee nutrients from blooming plants. The term $\frac{\alpha_3(1+\varphi)PH}{1+c(1+\varphi)P}$ stands for the depletion of honeybees as a result of the increasing dust storms, where α_3 is the depletion rate of honeybees by dust storms. δ_3 represents the natural depletion rate of honeybees. Thus, the dynamics of Honeybees could be obtained by

$$\frac{dH}{dt} = \gamma HF - \frac{\alpha_3(1+\varphi)PH}{1+c(1+\varphi)P} - \delta_3 H. \quad (3)$$

The sum of the three equations above will be the *PFH* system, which consists of dust storms, blooming plants, and honeybees, and could be presented by

$$\begin{aligned}
\frac{dP}{dt} &= Q - \frac{\alpha_1(1+\varphi)PF}{1+b(1+\varphi)P} - \delta_1 P, \\
\frac{dF}{dt} &= sF \left(1 - \frac{F}{L}\right) - \frac{\alpha_2(1+\varphi)PF}{1+b(1+\varphi)P} + \beta HF - \delta_2 F, \\
\frac{dH}{dt} &= \gamma HF - \frac{\alpha_3(1+\varphi)PH}{1+c(1+\varphi)P} - \delta_3 H.
\end{aligned} \tag{4}$$

Every parameter presented in our model, along with its units, ecological meaning, and source, is clearly explained in [Table 1](#).

Remark 1. A modified Holling type II functional response has been used to incorporate the effects of dust storms on the decrease in atmospheric dust pollutant concentration attributable to blooming plants, and to describe the depletion of blooming plants as dust storms intensify. The reason for choosing this functional response to describe the above cases is that the inclusion of the wind factor $(1+\varphi)$ in both the numerator and the denominator, it was intended to reflect the dual role of wind in the dust-scavenging process by plants. Biologically, moderate wind enhances the transport of dust particles toward the plant surface, thereby increasing the effective contact rate between dust particles and foliage. This effect is represented by the increase in the attack rate, given by the term $\alpha_1(1+\varphi)PF$. At the same time, stronger wind conditions may also reduce the efficiency of dust retention due to particle resuspension and shorter residence time of particles near the leaf surface. To capture this saturation-related effect, the same factor was incorporated into the denominator as $b(1+\varphi)P$, effectively increasing the half-saturation constant. Consequently, the functional response assumes that wind simultaneously promotes dust delivery and limits scavenging efficiency at higher dust concentrations. In particular, the chosen formulation allows wind to influence both the encounter process and the saturation dynamics simultaneously, which we consider more biologically realistic for airborne dust–plant interactions. The same rationale has been used to select a functional response to describe the depletion of honeybees due to increasing dust storms. This response was chosen to characterize how winds contribute to the movement of dust particles toward the honeybee population. This, in turn, directly affects bee movement by impeding their ability to navigate and forage.

Model analysis

Uniqueness

Clearly, the equations that are listed in the *PFH* model are $C^1(R_+^3)$. Thus, they fulfill the Lipschitz condition. Consequently, the solution of *PFH* system uniquely exists.

Positivity

Theorem 1. For the positive initial condition $(P(0) = P_0, F(0) = F_0, H(0) = H_0)$, the solution of *PFH* model is positive.

Proof. From the *PFH* system and the initial condition $(P(0) = P_0, F(0) = F_0, H(0) = H_0)$, the following is obtained

$$P(t) \geq P_0 e^{-\int_0^t \left[\delta_1 + \frac{\alpha_1(1+\varphi)F(\tau)}{1+b(1+\varphi)P(\tau)} \right] d\tau} > 0$$

Table 1

clarifies the *PFH* model parameters and their explanations.

Parameter	Explanations	Unit	Value	Source
Q	Dust input.	$\mu\text{gm}^{-3}\text{day}^{-1}$	10	[26]
α_1	Dust scavenging rate by the plant.	$\text{m}^2\text{kg}^{-1}\text{day}^{-1}$	0.25	[26]
φ	Wind speed.	dimensionless	3	Estimated
b	Half saturation level.	$\text{m}^3/\mu\text{g}$	0.2	[27]
δ_1	Dust depletion rate.	day^{-1}	0.03	[28]
s	Plant intrinsic growth.	day^{-1}	0.31	[26]
L	Plant carrying capacity.	kg/m^2	12	[28]
α_2	Dust-induced plant injury.	$\text{m}^2\text{kg}^{-1}\text{day}^{-1}$	0.22	[26]
β	The benefit rate of honeybees to flowers.	day^{-1}	0.35	[29]
δ_2	Plant natural depletion rate.	day^{-1}	0.02	[28]
γ	The benefit rate of flowers to honeybees.	day^{-1}	0.28	[29]
α_3	Dust-induced bee injury.	$\text{m}^2\text{kg}^{-1}\text{day}^{-1}$	0.15	Estimated
c	Half saturation level.	$\text{m}^3/\mu\text{g}$	0.3	Estimated
δ_3	Honeybees' natural depletion rate.	day^{-1}	0.01	[29]

$$F(t) = F_0 e^{\int_0^t \left[s \left(1 - \frac{F(\tau)}{L} \right) - \frac{\alpha_2(1+\varphi)P(\tau)}{1+b(1+\varphi)P(\tau)} + \beta H(\tau) - \delta_2 \right] d\tau} > 0$$

$$H(t) = H_0 e^{\int_0^t \left[\gamma F - \frac{\alpha_3(1+\varphi)P(\tau)}{1+c(1+\varphi)P(\tau)} - \delta_3 \right] d\tau} > 0$$

Thus, all solutions of *PFH* model will stay positive.

Boundedness

Theorem 2. The solutions of *PFH* model with the positive initial condition ($P(0) = P_0, F(0) = F_0, H(0) = H_0$) are uniformly bounded if the following condition is satisfied

$$\gamma L \leq \delta_3 \quad (5)$$

$$s + \beta H_0 > \delta_2 \quad (6)$$

Proof. From the first equation of *PFH* model, we acquire

$$\frac{dP}{dt} = Q - \frac{\alpha_1(1+\varphi)PF}{1+b(1+\varphi)P} - \delta_1 P \leq Q - \delta_1 P$$

Then, multiply both sides by $e^{\delta_1 t}$, we get

$$e^{\delta_1 t} \frac{dP}{dt} + \delta_1 e^{\delta_1 t} P \leq Q e^{\delta_1 t} \Rightarrow \frac{d}{dt} P e^{\delta_1 t} \leq Q e^{\delta_1 t}.$$

By integrating both sides, we obtain $P(t) = \frac{Q}{\delta_1} + C e^{\delta_1 t}$.

Now, by applying the initial condition $P(0) = P_0$, we obtain $C = P_0 - \frac{Q}{\delta_1}$.

Thus, $P(t) \leq \frac{Q}{\delta_1} + \left(P_0 - \frac{Q}{\delta_1} \right) e^{\delta_1 t}$.

That means $P(t) \rightarrow \frac{Q}{\delta_1} = P_{Max}$, as $t \rightarrow \infty$. This result shows $P(t)$ is bounded.

From Equation (3) and the plant carrying capacity, the following is obtained

$$\frac{dH}{dt} = \gamma HF - \frac{\alpha_3(1+\varphi)PH}{1+c(1+\varphi)P} - \delta_3 H < H(\gamma L - \delta_3).$$

Then, by integrating both sides, we attain $H(t) \leq H_0 e^{-(\delta_3 - \gamma L)t}$. Hence, by condition (5), the following is obtained $0 \leq H(t) \leq H_0$, for all $t \geq 0$,

From Equation (2) and the honeybee upper bound, the following results.

$$\frac{dF}{dt} = sF \left(1 - \frac{F}{L} \right) - \frac{\alpha_2(1+\varphi)PF}{1+b(1+\varphi)P} + \beta HF - \delta_2 F \leq F \left((s + \beta H_0 - \delta_2) - \frac{sF}{L} \right)$$

Then, by the Comparison Lemma and condition (6), the following is obtained

$$\lim_{\sup_{t \rightarrow \infty}} F(t) \leq \frac{(s + \beta H_0 - \delta_2)L}{s} = F_{Max}$$

Consequently, all solutions of the *PFH* system will converge towards $\Sigma = \{(P, F, H) \in \mathbb{R}_+^3 : P(t) \leq P_{Max}, F(t) \leq F_{Max}, H(t) \leq H_0\}$.

Remark 2. Condition (5) ensures that the natural depletion rate of honeybees dominates the multiplication of the benefit rate of flowers to honeybees by the plant carrying capacity, thereby preventing unbounded growth and guaranteeing the boundedness of solutions.

Persistence

The following theorem is employed to ensure the survival of all *PFH* system's components in future times.

Theorem 3. The *PFH* system is persistent if the following conditions are satisfied

$$sb > \alpha_2 + b\delta_2 \quad (7)$$

$$\gamma J_1 > \frac{\alpha_3(1+\varphi)P_{Max}}{1+c(1+\varphi)P_{Max}} + \delta_3 \quad (8)$$

$$\alpha_1(1+\varphi)F_{Max} > \delta_1 \quad (9)$$

Proof: From Equation (2) of PFH we have

$$\frac{dF}{dt} = sF\left(1 - \frac{F}{L}\right) - \frac{\alpha_2(1+\varphi)PF}{1+b(1+\varphi)P} + \beta HF - \delta_2 F \geq F\left[\left(s - \frac{\alpha_2}{b} - \delta_2\right) - \left(\frac{s}{L}\right)F\right]$$

By the Comparison Lemma,

$$\liminf_{t \rightarrow \infty} F(t) \geq J_1,$$

where $J_1 = \frac{L(sb - \alpha_2 - b\delta_2)}{bs}$. Clearly, for J_1 to be positive, condition (7) must hold.

From Equation (3) of PFH we have

$$\frac{dH}{dt} = \gamma HF - \frac{\alpha_3(1+\varphi)PH}{1+c(1+\varphi)P} - \delta_3 H \geq H\left(\gamma J_1 - \frac{\alpha_3(1+\varphi)P_{Max}}{1+c(1+\varphi)P_{Max}} - \delta_3\right)$$

Thus $H(t) \geq H_0 e^{\left(\gamma J_1 - \frac{\alpha_3(1+\varphi)P_{Max}}{1+c(1+\varphi)P_{Max}} - \delta_3\right)t}$, $t \geq 0$. So, the honeybee population stays persistent if condition (8) holds.

$$\frac{dP}{dt} = Q - \frac{\alpha_1(1+\varphi)PF}{1+b(1+\varphi)P} - \delta_1 P \geq Q - P(\alpha_1(1+\varphi)F_{Max} - \delta_1)$$

Thus, applying the Comparison Lemma, we have $\liminf_{t \rightarrow \infty} P(t) \geq \frac{Q}{\alpha_1(1+\varphi)F_{Max} - \delta_1}$. So, if condition (9) holds, the blooming plant population stays persistent.

Therefore, PFH system is uniformly persistent.

Equilibrium analysis

The possible equilibria of PFH system are

1) The dust storms' point $E_1 = (P_1, 0, 0)$, where $P_1 = \frac{Q}{\delta_1}$.

2) The honeybees' free point $E_2 = (P_2, F_2, 0)$, where $F_2 = \frac{[(s - \delta_2)(1 + b(1 + \varphi)P_2) - \alpha_1(1 + \varphi)P_2]L}{s[1 + b(1 + \varphi)P_2]}$, and P_2 could be found from the following equation:

$$W_1(P) = C_0 P^3 + C_1 P^2 + C_2 P + C_3 = 0.$$

here

$$C_0 = -\delta_1 b^2 (1 + \varphi)^2 < 0$$

$$C_1 = (1 + \varphi)^2 (s Q b^2 - \alpha_1 L (s - \delta_2) b + \alpha_1 \alpha_2 L) - 2 b \delta_1 (1 + \varphi),$$

$$C_2 = (1 + \varphi) (2 s b Q - \alpha_1 (s - \delta_2) L) - \delta_1$$

$$C_3 = Q S > 0$$

Now, $W_1(P)$ has a unique positive root P_2 , providing one of the following conditions:

$C_1 < 0$, which gives $s Q b^2 + \alpha_1 \alpha_2 L + \alpha_1 b L \delta_2 < s b \alpha_1 L$,

$C_2 > 0$, which gives $(1 + \varphi) (2 s b Q - \alpha_1 (s - \delta_2) L) > \delta_1$.

Further, $F_2 > 0$ if the following condition is held

$$(s - \delta_2)(1 + b(1 + \varphi)P_2) > \alpha_1(1 + \varphi)P_2.$$

3) The coexistence point $E_3 = (P_3, F_3, H_3)$, where $F_3 = \frac{\alpha_3(1+\varphi)P_3 + \delta_3[1+c(1+\varphi)P_3]}{\gamma[1+c(1+\varphi)P_3]}$, $H_3 = \frac{\alpha_3(1+\varphi)P_3 F_3}{(1+c(1+\varphi)P_3)(\gamma F_3 - \delta_3)}$, and P_3 could be found from the following equation:

$$W_2(P) = B_0 P^3 + B_1 P^2 + B_2 P + B_3 = 0,$$

here,

$$B_0 = -\delta_1 \gamma bc(1+\varphi)^2 < 0,$$

$$B_1 = (Q\gamma bc - \alpha_1 \alpha_3 - c\alpha_1 \delta_3)(1+\varphi)^2 - (\gamma c + \delta_1 \gamma b)(1+\varphi)$$

$$B_2 = (Q\gamma c + Q\gamma b + \alpha_1 \delta_3)(1+\varphi) - \delta_1 \gamma,$$

$$B_3 = Q\gamma > 0.$$

Now, $W_2(P)$ has a unique positive root P_3 , providing one of the following conditions:

$$B_1 < 0, \text{ which gives } (Q\gamma bc - \alpha_1 \alpha_3 - c\alpha_1 \delta_3)(1+\varphi) < (\gamma c + \delta_1 \gamma b)$$

$$B_2 > 0, \text{ which gives } (Q\gamma c + Q\gamma b + \alpha_1 \delta_3)(1+\varphi) > \delta_1 \gamma.$$

For H_3 to be positive, the following must be the case

$$\gamma F_3 > \delta_3. \quad (10)$$

Local stability of equilibrium points

The Jacobian matrix at any point is used to investigate the local stability of equilibria. Thus, the Jacobian matrix at (P, F, H) is given by

$$J(P, F, H) = \begin{bmatrix} -\frac{\alpha_1 F(1+\varphi)}{[1+b(1+\varphi)P]^2} - \delta_1 & -\frac{\alpha_1(1+\varphi)P}{1+b(1+\varphi)P} & 0 \\ -\frac{\alpha_2(1+\varphi)F}{[1+b(1+\varphi)P]^2} & c_{22} & \beta F \\ -\frac{\alpha_3(1+\varphi)H}{[1+c(1+\varphi)P]^2} & \gamma H & \gamma F - \frac{\alpha_3(1+\varphi)P}{1+c(1+\varphi)P} - \delta_3 \end{bmatrix},$$

$$\text{where } c_{22} = s - \frac{2\delta F}{L} - \frac{\alpha_2(1+\varphi)P}{1+b(1+\varphi)P} + \beta H - \delta_2.$$

Theorem 4. The dust storms' point $E_1 = (P_1, 0, 0)$ is locally asymptotically stable if

$$s < \frac{\alpha_2(1+\varphi)P_1}{1+b(1+\varphi)P_1} + \delta_2. \quad (11)$$

Proof: The Jacobian matrix at the dust storm's point $E_1 = (P_1, 0, 0)$ is

$$J(E_1) = \begin{bmatrix} -\delta_1 & -\frac{\alpha_1(1+\varphi)P_1}{1+b(1+\varphi)P_1} & 0 \\ 0 & s - \frac{\alpha_2(1+\varphi)P_1}{1+b(1+\varphi)P_1} - \delta_2 & 0 \\ 0 & 0 & -\frac{\alpha_3(1+\varphi)P_1}{1+c(1+\varphi)P_1} - \delta_3 \end{bmatrix}. \quad (12)$$

Then, the eigenvalues of $J(E_1)$ are:

$$\lambda_{11} = -\delta_1 < 0,$$

$$\lambda_{12} = s - \frac{\alpha_2(1+\varphi)P_1}{1+b(1+\varphi)P_1} - \delta_2,$$

$$\lambda_{13} = -\frac{\alpha_3(1+\varphi)P_1}{1+c(1+\varphi)P_1} - \delta_3 < 0.$$

Therefore, if condition 11 is met, then E_1 is asymptotically stable.

Remark 3. The biological explanation of condition 11 indicates that the PFH system reaches the dust-storms point asymptotically when the natural depletion rate of blooming plants and the depletion rate due to dust storms exceed their intrinsic growth rate.

Theorem 5. $E_2 = (P_2, F_2, 0)$ is locally asymptotically stable if

$$\gamma F_2 < \delta_3 + \frac{\alpha_3(1+\varphi)P_2}{1+c(1+\varphi)P_2}, \quad (13)$$

$$\frac{\alpha_1 F_2(1+\varphi)}{[1+b(1+\varphi)P_2]^2} + s < \delta_1 + \delta_2 + \frac{2sF_2}{L} + \frac{\alpha_2(1+\varphi)P_2}{1+b(1+\varphi)P_2}, \quad (14)$$

$$\frac{\alpha_1 F_2(1+\varphi)}{[1+b(1+\varphi)P_2]^2} \left[(s-\delta_2) - \left(\frac{2sF_2}{L} \right) \right] > \delta_1(s-\delta_2) + \frac{2\delta_1 s F_2}{L} + \frac{\alpha_2 \delta_1(1+\varphi)P_2}{1+b(1+\varphi)P_2}. \quad (15)$$

Proof: $J(E_2) = J(P_2, F_2, 0)$ is given by

$$J(E_2) = \begin{bmatrix} \frac{\alpha_1(1+\varphi)F_2}{[1+b(1+\varphi)P_2]^2} - \delta_1 & -\frac{\alpha_1(1+\varphi)P_2}{1+b(1+\varphi)P_2} & 0 \\ \frac{\alpha_2(1+\varphi)F_2}{[1+b(1+\varphi)P_2]^2} & c_{22} & \beta F_2 \\ 0 & 0 & \gamma F_2 - \frac{\alpha_3(1+\varphi)P_2}{1+c(1+\varphi)P_2} - \delta_3 \end{bmatrix}, \quad (16)$$

where $c_{22} = s - \delta_2 - \frac{2sF_2}{L} - \frac{\alpha_2(1+\varphi)P_2}{1+b(1+\varphi)P_2}$.

Then, the characteristic Equation of $J(E_2)$ is given by:

$$\left(\gamma F_2 - \frac{\alpha_3(1+\varphi)P_2}{1+c(1+\varphi)P_2} - \delta_3 - \lambda_{23} \right) [\lambda^2 - Tr\lambda + Det] = 0, \quad (17)$$

where,

$$\lambda_{23} = \gamma F_2 - \frac{\alpha_3(1+\varphi)P_2}{1+c(1+\varphi)P_2} - \delta_3 < 0 \text{ under condition 13.}$$

$$Tr = \frac{\alpha_1(1+\varphi)F_2}{[1+b(1+\varphi)P_2]^2} + s - \delta_1 - \delta_2 - \frac{2sF_2}{L} - \frac{\alpha_2(1+\varphi)P_2}{1+b(1+\varphi)P_2} < 0 \text{ under condition 14.}$$

$$Det = \frac{\alpha_1(1+\varphi)F_2}{[1+b(1+\varphi)P_2]^2} \left[(s-\delta_2) - \left(\frac{2sF_2}{L} \right) \right] - \delta_1(s-\delta_2) + \frac{2\delta_1 s F_2}{L} + \frac{\alpha_2 \delta_1(1+\varphi)P_2}{1+b(1+\varphi)P_2} > 0 \text{ under condition 15.}$$

Thus, E_2 exhibits local stability if conditions (13)- (15) are satisfied.

Theorem 5. The coexistence point $E_3 = (P_3, F_3, H_3)$ is locally asymptotically stable if

$$S_i > 0, \quad i = 1, 3, \quad (18)$$

$$S_1 S_2 > S_3 \quad (19)$$

Proof: $J(E_3) = J(P_3, F_3, H_3)$ is given by

$$J(E_3) = \begin{bmatrix} \frac{\alpha_1(1+\varphi)F_3}{[1+b(1+\varphi)P_3]^2} - \delta_1 & -\frac{\alpha_1(1+\varphi)P_3}{1+b(1+\varphi)P_3} & 0 \\ \frac{\alpha_2(1+\varphi)F_3}{[1+b(1+\varphi)P_3]^2} & d_{22} & \beta F_3 \\ -\frac{\alpha_3(1+\varphi)H_3}{[1+c(1+\varphi)P_3]^2} & \gamma H_3 & \gamma F_3 - \frac{\alpha_3(1+\varphi)P_3}{1+c(1+\varphi)P_3} - \delta_3 \end{bmatrix}, \quad (20)$$

where $d_{22} = s + \beta H_3 - \delta_2 - \frac{2sF_3}{L} - \frac{\alpha_2(1+\varphi)P_3}{1+b(1+\varphi)P_3}$.

Then, the characteristic Equation of $J(E_3)$ is given by:

$$\lambda^3 + S_1 \lambda^2 + S_2 \lambda + S_3 = 0 \quad (21)$$

where,

$$\begin{aligned} S_1 &= -\left(s + \beta H_3 + \gamma F_3 + \frac{\alpha_1(1+\varphi)F_3}{[1+b(1+\varphi)P_3]^2} - \delta_1 - \delta_2 - \delta_3 - \frac{2sF_3}{L} - (\alpha_2 + \alpha_3) \left(\frac{\alpha_2(1+\varphi)P_3}{1+b(1+\varphi)P_3} \right) \right). \\ S_2 &= \left(\frac{\alpha_1 \alpha_2 (1+\varphi)^2 P_3 F_3}{[1+b(1+\varphi)P_3]^3} \right) - \beta \gamma F_3 H_3 + \left(\gamma F_3 - \frac{\alpha_3(1+\varphi)P_3}{1+c(1+\varphi)P_3} - \delta_3 \right) \left(s + \beta H_3 - \delta_2 - \frac{2sF_3}{L} - \frac{\alpha_2(1+\varphi)P_3}{1+b(1+\varphi)P_3} \right) \\ &\quad + \left(\frac{\alpha_1(1+\varphi)F_3}{[1+b(1+\varphi)P_3]^2} - \delta_1 \right) \left(s + \beta H_3 - \delta_2 - \frac{2sF_3}{L} - (\alpha_2 + \alpha_3) \left(\frac{\alpha_2(1+\varphi)P_3}{1+b(1+\varphi)P_3} \right) + \gamma F_3 - \delta_3 \right). \end{aligned}$$

$$S_3 = \left(\delta_1 - \frac{\alpha_1(1+\varphi)F_3}{[1+b(1+\varphi)P_3]^2} \right) \left(\left(\gamma F_3 - \frac{\alpha_3(1+\varphi)P_3}{1+c(1+\varphi)P_3} - \delta_3 \right) \left(s + \beta H_3 - \delta_2 - \frac{2sF_3}{L} - \frac{\alpha_2(1+\varphi)P_3}{1+b(1+\varphi)P_3} \right) - \beta \gamma F_3 H_3 \right) \\ - \left(\frac{\alpha_1(1+\varphi)P_3}{1+b(1+\varphi)P_3} \right) \left(\left(\frac{\alpha_2(1+\varphi)F_3}{[1+b(1+\varphi)P_3]^2} \right) \left(\gamma F_3 - \frac{\alpha_3(1+\varphi)P_3}{1+c(1+\varphi)P_3} - \delta_3 \right) + \frac{\alpha_3\beta(1+\varphi)F_3 H_3}{[1+c(1+\varphi)P_3]^2} \right).$$

$$\Delta = S_1 S_2 - S_3$$

$$= \left(\left(\frac{-\alpha_1\alpha_2(1+\varphi)^2 P_3 F_3}{[1+b(1+\varphi)P_3]^3} \right) - \left(\gamma F_3 - \frac{\alpha_3(1+\varphi)P_3}{1+c(1+\varphi)P_3} - \delta_3 \right)^2 \right) \left(\frac{\alpha_1(1+\varphi)F_3}{[1+b(1+\varphi)P_3]^2} - \delta_1 + s + \beta H_3 - \delta_2 - \frac{2sF_3}{L} - \frac{\alpha_2(1+\varphi)P_3}{1+b(1+\varphi)P_3} \right) \\ - 2 \left(\frac{\alpha_1(1+\varphi)F_3}{[1+b(1+\varphi)P_3]^2} - \delta_1 \right) \left(s + \beta H_3 - \delta_2 - \frac{2sF_3}{L} - \frac{\alpha_2(1+\varphi)P_3}{1+b(1+\varphi)P_3} \right) \left(\gamma F_3 - \frac{\alpha_3(1+\varphi)P_3}{1+c(1+\varphi)P_3} - \delta_3 \right) \\ + \left(\beta \gamma F_3 H_3 - \left(\frac{\alpha_1(1+\varphi)F_3}{[1+b(1+\varphi)P_3]^2} - \delta_1 \right)^2 \right) \left(s + \beta H_3 - \delta_2 - \frac{2sF_3}{L} - \frac{\alpha_2(1+\varphi)P_3}{1+b(1+\varphi)P_3} + \gamma F_3 - \frac{\alpha_3(1+\varphi)P_3}{1+c(1+\varphi)P_3} - \delta_3 \right) \\ + \left(\frac{\alpha_1\beta F_3(1+\varphi)F_3 P_3}{1+b(1+\varphi)P_3} \right) \left(\frac{\alpha_3(1+\varphi)H_3}{[1+c(1+\varphi)P_3]^2} \right) - \left(\gamma F_3 - \frac{\alpha_3(1+\varphi)P_3}{1+c(1+\varphi)P_3} - \delta_3 - \frac{\alpha_1(1+\varphi)F_3}{[1+b(1+\varphi)P_3]^2} + \delta_1 \right).$$

So, by the Routh-Hurwitz criteria, all the eigenvalues of $J(E_3)$ have roots with negative real parts and, therefore E_3 exhibits local stability under the conditions (18) and (19).

Global stability

The global stability of equilibria is analyzed by the Lyapunov approach, as shown in the subsequent theorems.

Theorem 6. The dust storms' point $E_1 = (P_1, 0, 0)$ is GAS if

$$\left(\frac{L}{\delta_1} \right) \left(\frac{\alpha_1(1+\varphi)P}{1+b(1+\varphi)P} \right)^2 \leq s \leq \delta_2 \quad (22)$$

$$(\beta + \gamma)F_{Max} \leq \delta_3 \quad (23)$$

Proof: Let $Y_1 = \left(\frac{P-P_1}{2} \right)^2 + F + H$, where $Y_1 : R_+^3 \rightarrow R$, that satisfies $Y_1(P_1, 0, 0) = 0$ and for all $(P, F, H) \in R_+^3$ with $(P, F, H) \neq (P_1, 0, 0) \Rightarrow Y_1(P, F, H) > 0$, then

$$\frac{dY_1}{dt} = (P - P_1) \frac{dP}{dt} + \frac{dF}{dt} + \frac{dH}{dt} \\ = (P - P_1) \left(-\frac{\alpha_1(1+\varphi)PF}{1+b(1+\varphi)P} - \delta_1(P - P_1) \right) + \left(sF \left(1 - \frac{F}{L} \right) - \frac{\alpha_2(1+\varphi)PF}{1+b(1+\varphi)P} + \beta HF - \delta_2 P \right) + \left(\gamma HF - \frac{\alpha_3(1+\varphi)PH}{1+c(1+\varphi)P} - \delta_3 H \right)$$

After using F_{Max} , the following is obtained

$$\frac{dY_1}{dt} \leq - \left(\delta_1(P - P_1)^2 + \frac{\alpha_1(1+\varphi)P}{1+b(1+\varphi)P} (P - P_1)F + \left(\frac{s}{L} \right) F^2 \right) + (s - \delta_2)F + H((\beta + \gamma)F_{Max} - \delta_3) - \frac{\alpha_2(1+\varphi)PF}{1+b(1+\varphi)P} - \frac{\alpha_3(1+\varphi)PH}{1+c(1+\varphi)P}.$$

Thus,

$$dY_1 dt \leq -\delta_1 P - P_1 + sL F_2 + s - \delta_2 F + H\beta + \gamma F_{Max} - \delta_3 - \alpha_2 1 + \varphi PF_1 + b_1 + \varphi P - \alpha_3 1 + \varphi PH_1 + c_1 + \varphi P.$$

Thus, conditions (22)-(23) guarantee the negative definiteness of dY_1/dt . Therefore, $E_1 = (P_1, 0, 0)$ is GAS.

Remark 4. The biological meaning of the global stability of the dust storms' point E_1 is that the increase in dust storms leads to a loss of persistence in both blooming plants and the honeybee population in long-term behavior. The ecosystem, in turn, loses its coexistence state when conditions (22) and (23) are met.

Theorem 7. If the following condition is satisfied

$$\left(\frac{\alpha_1(1+\varphi)P_2(1+b(1+\varphi)P + \alpha_2)}{[1+b(1+\varphi)P][1+b(1+\varphi)P_2]} \right)^2 \leq \left(\frac{\alpha_1(1+\varphi)F(1+b(1+\varphi)P)}{[1+b(1+\varphi)P][1+b(1+\varphi)P_2]} + \delta_1 \right) \left(\frac{s}{L} \right), \quad (24)$$

along with condition (23), then the honeybees' free point $E_2 = (P_2, F_2, 0)$ is GAS. Proof: - Let $Y_2 = \left(\frac{P-P_2}{2} \right)^2 + F - F_2 - F_2 \ln \left(\frac{F}{F_2} \right) + H$, where $Y_2 : R_+^3 \rightarrow R$, that satisfies $Y_2(P_2, F_2, 0) = 0$ and for all $(P, F, H) \in R_+^3$ with $(P, F, H) \neq (P_2, F_2, 0) \Rightarrow Y_2(P, F, H) > 0$, then

$$\begin{aligned}\frac{dY_2}{dt} &= (P - P_2) \frac{dP}{dt} + \left(\frac{F - F_2}{F} \right) \frac{dF}{dt} + \frac{dH}{dt} \\ &= (P - P_2) \left[Q - \frac{\alpha_1(1+\varphi)PF}{1+b(1+\varphi)P} - \delta_1 P \right] + \left(\frac{F - F_2}{F} \right) \left[sF \left(1 - \frac{P}{L} \right) - \frac{\alpha_2(1+\varphi)PF}{1+b(1+\varphi)P} + \beta HF - \delta_2 F \right] + \gamma HF - \frac{\alpha_3(1+\varphi)PH}{1+c(1+\varphi)P} - \delta_3 H\end{aligned}$$

After using F_{Max} , the following is obtained

$$\begin{aligned}\frac{dY_2}{dt} &\leq - \left[\frac{\alpha_1(1+\varphi)F(1+b(1+\varphi)P)}{[1+b(1+\varphi)P][1+b(1+\varphi)P_2]} + \delta_1 \right] (P - P_2)^2 - \left[\frac{\alpha_1(1+\varphi)P_2(1+b(1+\varphi)P + \alpha_2)}{[1+b(1+\varphi)P][1+b(1+\varphi)P_2]} \right] (P - P_2)(F - F_2) - \frac{s}{L}(F - F_2)^2 \\ &\quad + [(\beta + \gamma)F_{Max} - \delta_3]H - \beta F_2 H - \frac{\alpha_3(1+\varphi)PH}{1+c(1+\varphi)P}\end{aligned}$$

Thus,

$$\frac{dY_2}{dt} \leq - \left(\sqrt{\frac{\alpha_1(1+\varphi)F(1+b(1+\varphi)P)}{[1+b(1+\varphi)P][1+b(1+\varphi)P_2]} + \delta_1} \right) (P - P_2) + \sqrt{\frac{s}{L}} (F - F_2)^2 + [(\beta + \gamma)F_{Max} - \delta_3]H - \beta F_2 H - \frac{\alpha_3(1+\varphi)PH}{1+c(1+\varphi)P}.$$

Thus, conditions (23) and (24) guarantee the negative definiteness of dY_2/dt . Therefore, $E_2 = (P_2, H_2, 0)$ is GAS.

Remark 5. The biological significance of the global stability of the honeybees' free point E_2 is that the PFH system is at risk of extinction of the honeybee population in the long term when conditions (23) and (24) are met.

Theorem 8. $E_3 = (P_3, F_3, H_3)$ is GAS if

$$\left(\frac{(\alpha_1 P_3 + \alpha_1 b(1+\varphi)PP_3 + \alpha_2)(1+\varphi)}{[1+b(1+\varphi)P][1+b(1+\varphi)P_3]} \right)^2 \leq \left(\frac{s}{L} \right) \left(\frac{\alpha_1(1+\varphi)}{[1+b(1+\varphi)P][1+b(1+\varphi)P_3]} + \delta_1 \right), \quad (25)$$

$$(\beta + \gamma)(F - F_3)(H - H_3) \leq \left(\frac{\alpha_3(1+\varphi)}{[1+c(1+\varphi)P][1+c(1+\varphi)P_3]} \right) (H - H_3)(P - P_3). \quad (26)$$

Proof: - Let $E_3 = \left(\frac{P-P_3}{2} \right)^2 + F - F_3 - F_3 \ln \left(\frac{F}{F_3} \right) + H - H_3 - H_3 \ln \left(\frac{H}{H_3} \right)$, where $Y_3 : R_+^3 \rightarrow R$, which satisfies $Y_3(P_3, F_3, H_3) = 0$ and for all $(P, F, H) \in R_+^3$ with $(P, F, H) \neq (P_3, F_3, H_3) \Rightarrow Y_3(P, F, H) > 0$, then

$$\begin{aligned}\frac{dY_3}{dt} &= (P - P_3) \frac{dP}{dt} + \left(\frac{F - F_3}{F} \right) \frac{dF}{dt} + \left(\frac{H - H_3}{H} \right) \frac{dH}{dt} \\ \frac{dY_3}{dt} &= (P - P_3) \left[Q - \frac{\alpha_1(1+\varphi)PF}{1+b(1+\varphi)P} - \delta_1 P \right] + \left(\frac{F - F_3}{F} \right) \left[sF \left(1 - \frac{P}{L} \right) - \frac{\alpha_2(1+\varphi)PF}{1+b(1+\varphi)P} + \beta HF - \delta_2 F \right] \\ &\quad + \left(\frac{H - H_3}{H} \right) \left[\gamma HF - \frac{\alpha_3(1+\varphi)PH}{1+c(1+\varphi)P} - \delta_3 H \right] \\ \frac{dY_3}{dt} &= (P - P_3) \left[Q - \frac{\alpha_1(1+\varphi)PF}{1+b(1+\varphi)P} - \delta_1 P - Q + \frac{\alpha_1(1+\varphi)P_3 F_3}{1+b(1+\varphi)P_3} + \delta_1 P_3 \right] \\ &\quad + (F - F_3) \left[s - \frac{sF}{L} - \frac{\alpha_2(1+\varphi)P}{1+b(1+\varphi)P} + \beta H - \delta_2 - s + \frac{sF_3}{L} + \frac{\alpha_2(1+\varphi)P_3}{1+b(1+\varphi)P_3} - \beta H_3 + \delta_2 \right] \\ &\quad + (H - H_3) \left[\gamma F - \frac{\alpha_3(1+\varphi)P}{1+c(1+\varphi)P} - \delta_3 - \gamma F_3 + \frac{\alpha_3(1+\varphi)P_3}{1+c(1+\varphi)P_3} + \delta_3 \right]\end{aligned}$$

Then,

$$\begin{aligned}\frac{dY_3}{dt} &= - \left(\frac{\alpha_1 F(1+\varphi)}{[1+b(1+\varphi)P][1+b(1+\varphi)P_3]} + \delta_1 \right) (P - P_3)^2 + \left(\frac{(\alpha_1 P_3 + \alpha_1 b(1+\varphi)PP_3 + \alpha_2)(1+\varphi)}{[1+b(1+\varphi)P][1+b(1+\varphi)P_3]} \right) (F - F_3)(P - P_3) \\ &\quad - \left(\frac{s}{L} \right) (F - F_3)^2 + (H - H_3) \left[(\beta + \gamma)(F - F_3) - \left(\frac{\alpha_3(1+\varphi)}{[1+c(1+\varphi)P][1+c(1+\varphi)P_3]} \right) (P - P_3) \right]\end{aligned}$$

Thus,

$$\begin{aligned}\frac{dY_3}{dt} &\leq - \left[\sqrt{\frac{\alpha_1(1+\varphi)}{[1+b(1+\varphi)P][1+b(1+\varphi)P_3]} + \delta_1} \right] (P - P_3) + \sqrt{\frac{s}{L}} (F - F_3)^2 \\ &\quad + (H - H_3) \left[(\beta + \gamma)(F - F_3) - \left(\frac{\alpha_3(1+\varphi)}{[1+c(1+\varphi)P][1+c(1+\varphi)P_3]} \right) (P - P_3) \right]\end{aligned}$$

Therefore, under conditions (25) and (26), dY_3/dt is a negative definite. Consequently, $E_3 = (P_3, F_3, H_3)$ is GAS.

Remark 6. The biological perspective of the global stability of E_3 is that, when conditions (25) and (26) are met, the populations of blooming plants and pollinators converge toward a state of symbiosis despite the presence of dust pollutants, regardless of their initial densities. This, in turn, implies that the ecosystem is resilient to disturbances and that equilibrium is the sole sustainable result of species interaction.

Bifurcation

This section investigates bifurcation types, such as saddle-node (SNB), transcritical (TB), and Hopf (HB), that occur near non-hyperbolic equilibrium points.

Theorem 9. At $\alpha_2^* = \frac{(s-\delta_2)(1+b(1+\varphi)P_1)}{(1+\varphi)P_1}$, where $\alpha_2^* > 0$, the PFH model distinguishes TB at the dust storms' point $E_1 = (P_1, 0, 0)$ if the following condition is satisfied

$$\left(\frac{\alpha_1 \alpha_2^* (1+\varphi)^2 P_1}{\delta_1 [1+b(1+\varphi)P_1]^3} \right) \neq \left(\frac{s}{L} \right) \quad (27)$$

Proof: - The formula of $\alpha_2^* = \frac{(s-\delta_2)(1+b(1+\varphi)P_1)}{(1+\varphi)P_1}$ could be found when we set $\lambda_{12} = 0$, $J(E_1)$ at α_2^* becomes

$$J^*(E_1) = \begin{bmatrix} -\delta_1 & -\frac{\alpha_1(1+\varphi)P_1}{1+b(1+\varphi)P_1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{\alpha_3(1+\varphi)P_1}{1+c(1+\varphi)P_1} - \delta_3 \end{bmatrix}$$

Now, suppose that $\eta^{[1]} = (\eta_1^{[1]}, \eta_2^{[1]}, \eta_3^{[1]})^T$, and $(T^{[1]})^T = (\tau_1^{[1]}, \tau_2^{[1]}, \tau_3^{[1]})^T$ be eigenvectors to $\lambda_{12} = 0$ of $J^*(E_1)$, and $J^{*T}(E_1)$. Then, the calculation gives $\eta^{[1]} = \left(-\frac{\alpha_1(1+\varphi)P_1}{\delta_1 [1+b(1+\varphi)P_1]}, 1, 0 \right)^T$, and $(T^{[1]})^T = (0, 1, 0)^T$.

Further,

$$\frac{\partial F}{\partial \alpha_2} = F_{\alpha_2}(E, \alpha_2) = \left(\frac{\partial f_1}{\partial \alpha_2}, \frac{\partial f_2}{\partial \alpha_2}, \frac{\partial f_3}{\partial \alpha_2} \right)^T = \left(0, -\frac{(1+\varphi)PF}{1+b(1+\varphi)P}, 0 \right)^T \Rightarrow F_{\alpha_2}(E_1, \alpha_2^*) = (0, 0, 0)^T$$

$$(T^{[1]})^T F_{\alpha_2}(E_1, \alpha_2^*) = (0, 1, 0)^T (0, 0, 0) = 0$$

$$(T^{[1]})^T DF_{\alpha_2}(E_1, \alpha_2^*)(\eta^{[1]}) = (0, 1, 0) \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{(1+\varphi)P_1}{1+b(1+\varphi)P_1} & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(\frac{-\alpha_1(1+\varphi)P_1}{\delta_1 [1+b(1+\varphi)P_1]}, 1, 0 \right)^T = \frac{-(1+\varphi)P_1}{1+b(1+\varphi)P_1} \neq 0$$

$$(T^{[1]})^T D^2 F_{\alpha_2}(E_1, \alpha_2^*)(\eta^{[1]}, \eta^{[1]}) = (0, 1, 0) \begin{bmatrix} -2\eta_1^{[1]} \left(\frac{\alpha_1(1+\varphi)}{[1+b(1+\varphi)P_1]^2} \right) \\ -2 \left[\eta_1^{[1]} \left(\frac{\alpha_2^*(1+\varphi)}{[1+b(1+\varphi)P_1]^2} \right) + \left(\frac{s}{L} \right) \right] \\ 0 \end{bmatrix} = 2 \left[\left(\frac{\alpha_1 \alpha_2^* (1+\varphi)^2 P_1}{\delta_1 [1+b(1+\varphi)P_1]^3} \right) - \left(\frac{s}{L} \right) \right]$$

Therefore, if condition (27) is fulfilled, then the PFH model indicates a TB with the parameter α_2^* around E_1 .

Theorem 10. For $\alpha_1^* = \frac{\left[\left[-\delta_1(s-\delta_2) + \frac{2s\delta_1 P_2}{L} \right] [1+b(1+\varphi)P_2] + \alpha_2 \delta_1 (1+\varphi)P_2 \right] [1+b(1+\varphi)P_2]}{F_2(1+\varphi) \left[(s-\delta_2) - \frac{2\delta_1^2}{L} \right]}$, where $\alpha_1^* > 0$, the PFH model faces SNB at the honeybees' free point $E_2 = (P_2, F_2, 0)$ if the following is satisfied

$$(T^{[2]})^T P^2 F_{\alpha_1}(E_2, \alpha_1^*)(\eta^{[2]}, \eta^{[2]}) \neq 0 \quad (28)$$

Proof: - By setting $Det = 0$ in the characteristic equation given by (17), the PFH system faces a zero eigenvalue ($\lambda_{22} = 0$), which gives

$$\alpha_1^* = \frac{\left[\left(-\delta_1(s - \delta_2) + \frac{2s\delta_1 F_2}{L} \right) [1 + b(1 + \varphi)P_2] + \alpha_2 \delta_1 (1 + \varphi)P_2 \right] [1 + b(1 + \varphi)P_2]}{F_2(1 + \varphi) \left[(s - \delta_2) - \frac{2sF_2}{L} \right]}, \text{ and } J(E_2) \text{ at } \alpha_1^* \text{ becomes}$$

$$J^*(E_2) = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & \beta F_2 \\ 0 & 0 & c_{33} \end{bmatrix}$$

where $c_{11} = -\frac{\alpha_1^* F_2(1 + \varphi)}{[1 + b(1 + \varphi)P_2]^2} - \delta_1$, $c_{12} = -\frac{\alpha_1^* (1 + \varphi)P_2}{1 + b(1 + \varphi)P_2}$, $c_{21} = -\frac{\alpha_2 F_2(1 + \varphi)}{[1 + b(1 + \varphi)P_2]^2}$, $c_{22} = (s - \delta_2) - \frac{2sF_2}{L} - \frac{\alpha_2(1 + \varphi)P_2}{1 + b(1 + \varphi)P_2}$, and $c_{33} = \gamma F_2 - \frac{\alpha_3(1 + \varphi)P_2}{1 + c(1 + \varphi)P_2} - \delta_3$.

Now, suppose that $\eta^{[2]} = (\eta_1^{[2]}, \eta_2^{[2]}, \eta_3^{[2]})^T$ and $(T^{[2]})^T = (\tau_1^{[2]}, \tau_2^{[2]}, \tau_3^{[2]})^T$ are the eigenvectors to $\lambda_{22} = 0$ of $J^*(E_2)$ and $J^{*T}(E_2)$. Solving $(J^*(E_2) - \lambda_{22}I)\eta^{[2]} = 0$, and $(J^{*T}(E_2) - \lambda_{22}I)T^{[2]} = 0$ for $\eta^{[2]}$, and $T^{[2]}$ gives $\eta^{[2]} = \left(\frac{-c_{12}}{c_{11}}, 1, \frac{c_{21}c_{12} - c_{11}c_{22}}{\beta F_2 c_{11}} \right)^T$ and $(T^{[2]})^T = \left(\frac{-c_{21}}{c_{11}}, 1, \frac{\beta F_2}{c_{33}} \right)^T$, where $c_{33} \neq 0$.

Further,

$$\frac{\partial F}{\partial \alpha_1} = F_{\alpha_1}(E, \alpha_1) = \left(\frac{\partial f_1}{\partial \alpha_1}, \frac{\partial f_2}{\partial \alpha_1}, \frac{\partial f_3}{\partial \alpha_1} \right)^T F_{\alpha_1}(E, \alpha_1) = \left(-\frac{(1 + \varphi)PF}{1 + b(1 + \varphi)P}, 0, 0 \right)^T \Rightarrow F_{\alpha_1}(E_2, \alpha_1^*) = \left(-\frac{(1 + \varphi)P_2 F_2}{1 + b(1 + \varphi)P_2}, 0, 0 \right)^T$$

$$(T^{[2]})^T F_{\alpha_1}(E_2, \alpha_1^*) = \left(\frac{-c_{21}}{c_{11}}, 1, \frac{\beta F_2}{c_{33}} \right)^T \left(-\frac{(1 + \varphi)P_2 F_2}{1 + b(1 + \varphi)P_2}, 0, 0 \right) = \frac{c_{21}(1 + \varphi)P_2 F_2}{c_{11}[1 + b(1 + \varphi)P_2]} \neq 0$$

$$\begin{aligned} (T^{[2]})^T D^2 F_{\alpha_1}(E_2, \alpha_1^*)(\eta^{[2]}, \eta^{[2]}) &= \begin{bmatrix} \frac{-c_{21}}{c_{11}} \\ 1 \\ \frac{\beta F_2}{c_{33}} \end{bmatrix} \begin{bmatrix} -2\eta_1^{[2]} \left(\frac{\alpha_1^*(1 + \varphi)}{[1 + b(1 + \varphi)P_2]^2} \right) \left(\left(\frac{F_2 b(1 + \varphi)}{1 + \alpha_1 F_2(1 + \varphi)P_2} \right) \eta_1^{[2]} + 1 \right) \\ -2 \left(\frac{\alpha_2(1 + \varphi)\eta_1^{[2]}}{[1 + b(1 + \varphi)P_2]^2} \right) \left(\frac{b p_2(1 + \varphi)\eta_1^{[2]}}{1 + b(1 + \varphi)P_2} + 1 \right) - \left(\frac{2S}{L} \right) + 2\beta\eta_3^{[2]} \\ -2\eta_3^{[2]} \left(\frac{\alpha_3(1 + \varphi)}{[1 + c(1 + \varphi)P_2]^2} \eta_1^{[2]} - \gamma \right) \end{bmatrix} \\ &= \frac{2c_{21}\eta_1^{[2]}}{c_{11}} \left(\frac{\alpha_1^*(1 + \varphi)}{[1 + b(1 + \varphi)P_2]^2} \right) \left(\left(\frac{F_2 b(1 + \varphi)}{1 + \alpha_1 F_2(1 + \varphi)P_2} \right) \eta_1^{[2]} + 1 \right) \\ &\quad - 2 \left(\frac{\alpha_2(1 + \varphi)\eta_1^{[2]}}{[1 + b(1 + \varphi)P_2]^2} \right) \left(\frac{b p_2(1 + \varphi)\eta_1^{[2]}}{1 + b(1 + \varphi)P_2} + 1 \right) - \left(\frac{2S}{L} \right) + 2\beta\eta_3^{[2]} \\ &\quad + \frac{2\beta F_2 \eta_3^{[2]}}{c_{33}} \left(\frac{\alpha_3(1 + \varphi)}{[1 + c(1 + \varphi)P_2]^2} \eta_1^{[2]} - \gamma \right) \end{aligned}$$

Therefore, if condition (28) is satisfied, then there is an SNB around E_2 with the parameter α_1^* .

Theorem 11. For $\alpha_3^* = \frac{[1 + c(1 + \varphi)P_3] \left(S_3 + \left[\frac{\alpha_1(1 + \varphi)F_3}{[1 + b(1 + \varphi)P_3]^2} - \delta_1 + s + \beta H_3 - \delta_2 - \frac{2sF_3}{L} - \frac{\alpha_2(1 + \varphi)P_3}{1 + b(1 + \varphi)P_3} + \gamma F_3 - \delta_3 \right] S_2 \right)}{(1 + \varphi)P_3 S_2}$, where $\alpha_3^* > 0$ and S_i , $i = 2, 3$ are the coefficients of the characteristic equation of $J(E_3)$ which is given in Equation (21), then the PFH model faces SNB at $E_3 = (P_3, F_3, H_3)$ if the following condition is satisfied

$$(T^{[3]})^T P^2 F_{\alpha_3}(E_3, \alpha_3^*)(\eta^{[3]}, \eta^{[3]}) \neq 0 \quad (29)$$

Proof: - At α_3^* , the PFH system has a zero eigenvalue ($\lambda_{33} = 0$) around E_3 . Consequently, $J(E_3)$ at α_3^* becomes

$$J^*(E_3) = [d_{ij}]_{3 \times 3},$$

where $d_{11} = -\frac{\alpha_1 F_3(1 + \varphi)}{[1 + b(1 + \varphi)P_3]^2} - \delta_1$, $d_{12} = -\frac{\alpha_1(1 + \varphi)P_3}{1 + b(1 + \varphi)P_3}$, $d_{13} = 0$, $d_{21} = -\frac{\alpha_2(1 + \varphi)F_3}{[1 + b(1 + \varphi)P_3]^2}$, $d_{22} = s - \frac{2sF_3}{L} - \frac{\alpha_2(1 + \varphi)P_3}{1 + b(1 + \varphi)P_3} + \beta H_3 - \delta_2$, $d_{23} = \beta F_3$, $d_{31} = -\frac{\alpha_3(1 + \varphi)H_3}{[1 + c(1 + \varphi)P_3]^2}$, $d_{32} = \gamma H_3$, $d_{33} = \gamma F_3 - \frac{\alpha_3(1 + \varphi)P_3}{1 + c(1 + \varphi)P_3} - \delta_3$.

Now, suppose that $\eta^{[3]} = (\eta_1^{[3]}, \eta_2^{[3]}, \eta_3^{[3]})^T$ and $(T^{[3]})^T = (\tau_1^{[3]}, \tau_2^{[3]}, \tau_3^{[3]})^T$ be eigenvectors to $\lambda_{33} = 0$ of $J^*(E_3)$ and $J^{*T}(E_3)$. Solving $(J^*(E_3) - \lambda_{33}I)\eta^{[3]} = 0$, and $(J^{*T}(E_3) - \lambda_{33}I)T^{[3]} = 0$ for $\eta^{[3]}$ and $T^{[3]}$ gives $\eta^{[3]} = \left(-\frac{d_{12}}{d_{11}}, 1, \frac{d_{21}d_{12} - d_{11}d_{22}}{\beta F_3 d_{11}} \right)^T$ and $(T^{[3]})^T = \left(\frac{-d_{33}d_{21} + d_{31}\beta F_3}{d_{11}d_{33}}, 1, -\frac{\beta F_3}{d_{33}} \right)^T$, where $d_{33} \neq 0$.

Further

$$\begin{aligned} \frac{\partial F}{\partial \alpha_3} &= F_{\alpha_3}(E, \alpha_3) = \left(\frac{\partial f_1}{\partial \alpha_3}, \frac{\partial f_2}{\partial \alpha_3}, \frac{\partial f_3}{\partial \alpha_3} \right)^T = \left(0, 0, -\frac{(1+\varphi)PH}{1+c(1+\varphi)P} \right)^T \Rightarrow F_{\alpha_3}(E_3, \alpha_3^*) = \left(0, 0, -\frac{(1+\varphi)P_3H_3}{1+c(1+\varphi)P_3} \right)^T \\ (T^{[3]})^T F_{\alpha_3}(E_3, \alpha_3^*) &= (\tau_1^{[3]}, \tau_2^{[3]}, \tau_3^{[3]})^T \left(0, 0, -\frac{(1+\varphi)P_3H_3}{1+c(1+\varphi)P_3} \right) = \frac{(1+\varphi)P_3H_3\tau_3^{[3]}}{(1+c(1+\varphi)P_3)} \neq 0 \\ (T^{[3]})^T D^2 F_{\alpha_3}(E_3, \alpha_3^*)(\eta^{[3]}, \eta^{[3]}) &= \begin{bmatrix} \tau_1^{[3]} \\ 1 \\ \tau_3^{[3]} \end{bmatrix} \begin{bmatrix} \frac{-2\alpha_1(1+\varphi)\eta_1^{[3]}}{[1+b(1+\varphi)P_3]^2} \left(\frac{F_3b(1+\varphi)\eta_1^{[3]}}{1+\alpha_1F_3(1+\varphi)P_3} + 1 \right) \\ \frac{-2\alpha_2(1+\varphi)\eta_1^{[3]}}{[1+b(1+\varphi)P_3]^2} \left(\left(\frac{F_3b(1+\varphi)}{1+b(1+\varphi)P_3} \right) \eta_1^{[3]} + 1 \right) - \left(\frac{2s}{L} \right) \eta_2^{[3]} + 2\beta\eta_3^{[3]} \\ \frac{-\alpha_3(1+\varphi)\eta_1^{[3]}}{[1+c(1+\varphi)P_3]^2} \left(\left(\frac{cH_3(1+\varphi)}{[1+c(1+\varphi)P_3]^2} \right) \eta_1^{[3]} + 2\eta_3^{[3]} \right) + 2\gamma\eta_3^{[3]} \end{bmatrix} \\ &= \frac{-2\alpha_1(1+\varphi)\tau_1^{[3]}\eta_1^{[3]}}{[1+b(1+\varphi)P_3]^2} \left(\frac{F_3b(1+\varphi)\eta_1^{[3]}}{1+\alpha_1F_3(1+\varphi)P_3} + 1 \right) - \frac{2\alpha_2(1+\varphi)\eta_1^{[3]}}{[1+b(1+\varphi)P_3]^2} \left(\left(\frac{F_3b(1+\varphi)}{1+b(1+\varphi)P_3} \right) \eta_1^{[3]} + 1 \right) - \left(\frac{2s}{L} \right) \eta_2^{[3]} + 2\beta\eta_3^{[3]} \\ &\quad - \frac{\alpha_3(1+\varphi)\tau_3^{[3]}\eta_1^{[3]}}{[1+c(1+\varphi)P_3]^2} \left(\left(\frac{cH_3(1+\varphi)}{[1+c(1+\varphi)P_3]^2} \right) \eta_1^{[3]} + 2\eta_3^{[3]} \right) + 2\gamma\tau_3^{[3]}\eta_3^{[3]} \end{aligned}$$

Therefore, if condition (29) is satisfied, then there is an SND around E_3 with the parameter α_3^* .

Theorem 12. At $\delta_2^* = \frac{\alpha_1 F_2(1+\varphi)}{[1+b(1+\varphi)P_2]^2} - \delta_1 + s - \frac{2sF_2}{L} - \frac{\alpha_2(1+\varphi)P_2}{1+b(1+\varphi)P_2}$, where $\delta_2^* > 0$, the PFH system undergoes a Hopf bifurcation around the honeybees' free point $E_2 = (P_2, F_2, 0)$ if the following are satisfied

$$\frac{\alpha_1 F_2(1+\varphi)}{[1+b(1+\varphi)P_2]^2} + s > \delta_1 + \frac{2sF_2}{L} + \frac{\alpha_2(1+\varphi)P_2}{1+b(1+\varphi)P_2} \quad (30)$$

$$\frac{\alpha_1(1+\varphi)F_2}{[1+b(1+\varphi)P_2]^2} \left[(s - \delta_2) - \left(\frac{2sF_2}{L} \right) \right] + \frac{2\delta_1 s F_2}{L} + \frac{\alpha_2 \delta_1 (1+\varphi)P_2}{1+b(1+\varphi)P_2} > \delta_1 (s - \delta_2) \quad (31)$$

Proof: The characteristic equation of $J(E_2)$ is given in Equation (17), and the Hopf bifurcation is contingent upon the following conditions.

- 1) $[Tr]|_{\delta_2=\delta_2^*} = 0$,
- 2) $[Det]|_{\delta_2=\delta_2^*} > 0$,
- 3) $\frac{d}{d\tau}[Re(\lambda_{1,2})]|_{\delta_2=\delta_2^*} \neq 0$.

Conditions (1) and (2) have been fulfilled at $\delta_2^* = \frac{\alpha_1 F_2(1+\varphi)}{[1+b(1+\varphi)P_2]^2} - \delta_1 + s - \frac{2sF_2}{L} - \frac{\alpha_2(1+\varphi)P_2}{1+b(1+\varphi)P_2}$. Clearly $\delta_2^* > 0$ if Condition 30 holds. At $\delta_2 = \delta_2^*$, the equation given in (17) could be rewritten as $\lambda^2 + Det J(E_2) = 0$, which has two roots

$$\lambda_{1,2} = \pm i\sqrt{Det}.$$

Clearly, at $\delta_2 = \delta_2^*$, there are two purely imaginary eigenvalues λ_1 and λ_2 which are complex conjugates under the conditions specified in (31).

The general roots of Eq. (17) in the neighborhood of δ_2^* could be written as $\lambda_{1,2} = \frac{Tr \pm i\sqrt{(Tr)^2 - 4Det}}{2}$, then $\frac{d}{d\tau}[Re(\lambda_{1,2})]|_{\delta_2=\delta_2^*} = \frac{d}{d\tau}\left[\frac{Tr}{2}\right]_{\delta_2=\delta_2^*} = \frac{-1}{2} \neq 0$. That means PFH system undergoes an HB around the honeybees' free point E_2 with $\delta_2 = \delta_2^*$.

Theorem 13. The PFH system undergoes an HB at E_3 to the bifurcation parameter α_1^* if the following are held

$$Y < 0 \quad (32)$$

$$2S_2(\alpha_1^*) \left(S_2'(\alpha_1^*)S_1(\alpha_1^*) - S_3'(\alpha_1^*) - S_1'(\alpha_1^*)S_2(\alpha_1^*) \right) \neq 0, \quad (33)$$

where, S_i , $i = 1, 2, 3$ are given in the local stability of E_3 , and the formula of Y is given in the theorem.

Proof: We obtain $\alpha_1^* = \frac{Y(1+b(1+\varphi)P_3)[1+c(1+\varphi)P_3]^2}{-\beta\alpha_3(1+\varphi)^2P_3F_3H_3}$ when setting $\Delta = 0$ in Eq. (17), where $Y = (d_{12}d_{21} - d_{33}^2)(d_{11} + d_{22}) + (d_{23}d_{32} - d_{11}^2)(d_{22} + d_{33}) - 2d_{11}d_{22}d_{33} - d_{22}^2(d_{33} + d_{11})$ and d_{ij} , $i, j = 1, 2, 3$ are the entries of $J(E_3)$.

Clearly $\alpha_1^* > 0$ if condition 32 holds. At $\alpha_1 = \alpha_1^*$ the equation $\lambda^3 + S_1\lambda^2 + S_2\lambda + S_3 = 0$ can be written as $(\lambda + S_1)(\lambda^2 + S_2) = 0$, which gives $\lambda_1 = -S_1$, $\lambda_{2,3} = \pm i\sqrt{S_2}$. In the neighborhood of α_1^* , $\lambda_{2,3}$ can be written as $\lambda_{2,3} = \psi_1(\alpha_1^*) \pm i\psi_2(\alpha_1^*)$, where $Re(\lambda_{2,3})|_{\alpha_1=\alpha_1^*} = \psi_1(\alpha_1^*) = 0$. That means the first requirement for HB occurrence is met. To satisfy the second condition, i.e., $\phi(\alpha_1^*)\Gamma(\alpha_1^*) + \Theta(\alpha_1^*)\pi(\alpha_1^*) \neq 0$, the following should be calculated

$$\phi(\alpha_1) = \psi_1^2(\alpha_1)\dot{S}_1(\alpha_1) + S_2(\alpha_1)\psi_1(\alpha_1) + \dot{S}_3(\alpha_1) - \dot{S}_1(\alpha_1)\psi_2^2(\alpha_1)$$

$$\Gamma(\alpha_1) = 3\psi_1^2(\alpha_1) + 2S_1(\alpha_1)\psi_1(\alpha_1) + S_2(\alpha_1) - 3\psi_2^2(\alpha_1)$$

$$\Theta(\alpha_1) = 2\psi_1(\alpha_1)\psi_2(\alpha_1)\dot{S}_1(\alpha_1) + \dot{S}_2(\alpha_1)\psi_2(\alpha_1)$$

$$\pi(\alpha_1) = 6\psi_1(\alpha_1)\psi_2(\alpha_1) + 2S_1(\alpha_1)\psi_2(\alpha_1)$$

At $\alpha_1 = \alpha_1^*$, $\psi_1(\alpha_1^*) = 0$, and $\psi_2(\alpha_1^*) = \sqrt{S_2(\alpha_1^*)}$, then

$$\phi(\alpha_1^*) = \dot{S}_3(\alpha_1^*) - \dot{S}_1(\alpha_1^*)S_2(\alpha_1^*)$$

$$\Gamma(\alpha_1^*) = -2S_2(\alpha_1^*)$$

$$\Theta(\alpha_1^*) = \dot{S}_2(\alpha_1^*)\sqrt{S_2(\alpha_1^*)}$$

$$\pi(\alpha_1^*) = 2S_1(\alpha_1^*)\sqrt{S_2(\alpha_1^*)}$$

where,

$$\dot{S}_1(\alpha_1^*) = \frac{(1+\varphi)F_3}{[1+b(1+\varphi)P_3]^2},$$

$$\dot{S}_2(\alpha_1^*) = \frac{\alpha_2(1+\varphi)^2P_3F_3}{[1+b(1+\varphi)P_3]^3} + \frac{(1+\varphi)F_3}{[1+b(1+\varphi)P_3]^2} \left(s + \beta H_3 - \delta_2 - \frac{2sF_3}{L} - (\alpha_2 + \alpha_3) \left(\frac{\alpha_2(1+\varphi)P_3}{1+b(1+\varphi)P_3} \right) + \gamma F_3 - \delta_3 \right)$$

$$\begin{aligned} \dot{S}_3(\alpha_1^*) = & -\frac{(1+\varphi)F_3}{[1+b(1+\varphi)P_3]^2} \left(\left(\gamma F_3 - \frac{\alpha_3(1+\varphi)P_3}{1+c(1+\varphi)P_3} - \delta_3 \right) \left(s + \beta H_3 - \delta_2 - \frac{2sF_3}{L} - \frac{\alpha_2(1+\varphi)P_3}{1+b(1+\varphi)P_3} \right) - \beta \gamma F_3 H_3 \right) \\ & - \left(\frac{(1+\varphi)P_3}{1+b(1+\varphi)P_3} \right) \left(\left(\frac{\alpha_2(1+\varphi)F_3}{[1+b(1+\varphi)P_3]^2} \right) \left(\gamma F_3 - \frac{\alpha_3(1+\varphi)P_3}{1+c(1+\varphi)P_3} - \delta_3 \right) + \frac{\alpha_3\beta(1+\varphi)F_3H_3}{[1+c(1+\varphi)P_3]^2} \right) \end{aligned}$$

Thus,

$$\phi(\alpha_1^*)\Gamma(\alpha_1^*) + \Theta(\alpha_1^*)\pi(\alpha_1^*) = 2S_2(\alpha_1^*) \left(\dot{S}_1(\alpha_1^*)S_2(\alpha_1^*) - \dot{S}_3(\alpha_1^*) \right) + 2S_1(\alpha_1^*)S_2(\alpha_1^*)\dot{S}_2(\alpha_1^*) =$$

$$2S_2(\alpha_1^*) \left(\dot{S}_2(\alpha_1^*)S_1(\alpha_1^*) - \dot{S}_1(\alpha_1^*)S_2(\alpha_1^*) - \dot{S}_3(\alpha_1^*) \right) \neq 0 \text{ under Condition 33.}$$

That means *PFH* system undergoes an HB around E_3 with $\alpha_1 = \alpha_1^*$.

Theorem 14. If

$$\mathcal{B} < 0, \tag{34}$$

where the formula of $\mathcal{B}$ is given in the following proof, then the *PFH* system indicates a stable limit cycle in $R_{(P,F,H)}^3$.

Proof: The following transformations $P = x_1 + P_3$, $F = x_2 + F_3$, $H = x_3 + H_3$ are used to move $E_3 = (P_3, F_3, H_3)$ into $(0, 0, 0)$. Then, the *PFH* system becomes:

$$\frac{dx_1}{dt} = -\frac{\alpha_1(1+\varphi)x_1x_2}{1+b(1+\varphi)(x_1+P_3)}$$

$$\frac{dx_2}{dt} = -\frac{sx_2^2}{L} - \frac{\alpha_2(1+\varphi)x_1x_2}{1+b(1+\varphi)(x_1+P_3)} + \beta x_3x_2$$

$$\frac{dx_3}{dt} = \gamma x_3x_2 - \frac{\alpha_3(1+\varphi)x_1x_3}{1+c(1+\varphi)(x_1+P_3)}$$

The nonlinear part of the above system can be written as

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{bmatrix} \frac{\alpha_1(1+\varphi)x_1x_2}{1+b(1+\varphi)(x_1+P_3)} \\ \frac{sx_2^2}{L} - \frac{\alpha_2(1+\varphi)x_1x_2}{1+b(1+\varphi)(x_1+P_3)} + \beta x_3x_2 \\ \gamma x_3x_2 - \frac{\alpha_3(1+\varphi)x_1x_3}{1+c(1+\varphi)(x_1+P_3)} \end{bmatrix}$$

$$g_{20}^0 = \frac{1}{4} \left\{ \frac{\partial^2 v_1}{\partial x_1^2} - \frac{\partial^2 v_1}{\partial x_2^2} + 2 \frac{\partial^2 v_2}{\partial x_1 \partial x_2} + i \left(\frac{\partial^2 v_2}{\partial x_1^2} - \frac{\partial^2 v_2}{\partial x_2^2} - 2 \frac{\partial^2 v_1}{\partial x_1 \partial x_2} \right) \right\}$$

$$g_{20}^0 = \frac{1}{2} \left\{ \frac{b\alpha_1x_2(1+\varphi)^2[1+b(1+\varphi)P_3]}{[1+b(1+\varphi)(x_1+P_3)]^3} - \frac{\alpha_2(1+\varphi)(1+b(1+\varphi)P_3)}{[1+b(1+\varphi)(x_1+P_3)]^2} \right. \\ \left. + i \left(\frac{s}{L} + \frac{b\alpha_2x_2(1+\varphi)^2(1+b(1+\varphi)P_3)}{[1+b(1+\varphi)(x_1+P_3)]^3} + \frac{\alpha_1(1+\varphi)[1+b(1+\varphi)P_3]}{[1+b(1+\varphi)(x_1+P_3)]^2} \right) \right\}$$

$$g_{11}^0 = \frac{1}{4} \left\{ \frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2} + i \left(\frac{\partial^2 v_2}{\partial x_1^2} + \frac{\partial^2 v_2}{\partial x_2^2} \right) \right\}$$

$$g_{11}^0 = \frac{1}{2} \left\{ \frac{\alpha_1b(1+\varphi)^2x_2[1+b(1+\varphi)P_3]}{[1+b(1+\varphi)(x_1+P_3)]^3} + i \left(\frac{\alpha_2b(1+\varphi)^2x_2(1+b(1+\varphi)P_3)}{[1+b(1+\varphi)(x_1+P_3)]^3} - \frac{s}{L} \right) \right\}$$

$$G_{110}^0 = \frac{1}{2} \left\{ \frac{\partial^2 v_1}{\partial x_1 \partial x_3} + \frac{\partial^2 v_2}{\partial x_2 \partial x_3} + i \left(\frac{\partial^2 v_2}{\partial x_1 \partial x_3} - \frac{\partial^2 v_1}{\partial x_2 \partial x_3} \right) \right\}$$

$$G_{110}^0 = \frac{1}{2}\beta$$

$$G_{101}^0 = \frac{1}{2} \left\{ \frac{\partial^2 v_1}{\partial x_1 \partial x_3} - \frac{\partial^2 v_2}{\partial x_2 \partial x_3} + i \left(\frac{\partial^2 v_2}{\partial x_1 \partial x_3} + \frac{\partial^2 v_1}{\partial x_2 \partial x_3} \right) \right\}$$

$$G_{101}^0 = -\frac{1}{2}\beta$$

$$W_{11}^0 = \frac{-1}{4\lambda_3(S_1(\alpha_1^*))} \left(\frac{\partial^2 v_3}{\partial x_1^2} + \frac{\partial^2 v_3}{\partial x_2^2} \right) = \frac{-1}{2\lambda_3(a_1(k^*))} \left(\frac{\alpha_3c(1+\varphi)^2x_3[1+c(1+\varphi)P_3]}{[1+c(1+\varphi)(x_1+P_3)]^3} \right)$$

$$W_{20}^0 = \frac{-1}{4\lambda_3(S_1(\alpha_1^*))} \left(\frac{\partial^2 v_3}{\partial x_1^2} + \frac{\partial^2 v_3}{\partial x_2^2} - 2i \left(\frac{\partial^2 v_3}{\partial x_1 \partial x_2} \right) \right) = \frac{-1}{2\lambda_3(S_1(\alpha_1^*))} \left(\frac{\alpha_3c(1+\varphi)^2x_3[1+c(1+\varphi)P_3]}{[1+c(1+\varphi)(x_1+P_3)]^3} \right)$$

$$G_{21}^0 = \frac{1}{8} \left\{ \frac{\partial^3 v_1}{\partial x_1^3} + \frac{\partial^3 v_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 v_2}{\partial x_2^3} + \frac{\partial^3 v_2}{\partial x_1^2 \partial x_2} + i \left(\frac{\partial^3 v_2}{\partial x_1^3} + \frac{\partial^3 v_2}{\partial x_1 \partial x_2^2} - \frac{\partial^3 v_1}{\partial x_2^3} - \frac{\partial^3 v_1}{\partial x_1^2 \partial x_2} \right) \right\}$$

$$G_{21}^0 = \frac{1}{4} \left\{ \frac{-3b^2\alpha_1(1+\varphi)^3x_2[1+b(1+\varphi)P_3]}{[1+b(1+\varphi)(x_1+P_3)]^7} + \frac{\alpha_2b(1+\varphi)^2[1+b(1+\varphi)P_3]}{[1+b(1+\varphi)(x_1+P_3)]^3} \right. \\ \left. + i \left(\frac{-3\alpha_2b^2(1+\varphi)^3x_2[1+b(1+\varphi)P_3]}{[1+b(1+\varphi)(x_1+P_3)]^7} - \frac{b\alpha_1(1+\varphi)^2[1+b(1+\varphi)P_3]}{[1+b(1+\varphi)(x_1+P_3)]^3} \right) \right\}$$

Then, the coefficient of the limit cycle's curvature is

$$\rho_1^0 = \text{Re} \left\{ \frac{1}{4} g_{20}^0 g_{11}^0 i + G_{110}^0 W_{11}^0 + \frac{1}{2} (G_{21}^0 + G_{101}^0 W_{20}^0) \right\}$$

$$\rho_1^0 = \text{Re} \left\{ \frac{1}{16} \left(i \left[\frac{(\alpha_1^2 - \alpha_2^2)(bx_2(1+\varphi)V_1)^2}{V_2^6} - \frac{2\alpha_1\alpha_2bx_2(1+\varphi)V_1^2}{V_2^5} + \left(\frac{s}{L} \right)^2 + \frac{s\alpha_1V_1}{LV_2^2} \right] \right. \right. \\ \left. \left. - \frac{2\alpha_1\alpha_2(bx_2(1+\varphi)V_1)^2}{V_2^6} - \frac{(\alpha_1^2 - \alpha_2^2)bx_2(1+\varphi)V_1^2}{V_2^5} - \frac{s\alpha_2V_1}{LV_2^2} \right) \right. \\ \left. + \frac{1}{8} \left\{ \frac{-3b\alpha_1(1+\varphi)x_2V_3}{V_2^7} + \frac{\alpha_2V_3}{V_2^3} + i \left(\frac{-3\alpha_2b(1+\varphi)x_2V_3}{V_2^7} - \frac{\alpha_1V_3}{V_2^3} \right) \right\} \right\}$$

$$\rho_1^0 = \operatorname{Re} \left\{ i \left[\frac{1}{16} \left(\frac{(\alpha_1^2 - \alpha_2^2)(bx_2(1+\varphi)V_1)^2}{V_2^6} - \frac{2\alpha_1\alpha_2bx_2(1+\varphi)V_1^2}{V_2^5} + \left(\frac{s}{L} \right)^2 + \frac{s\alpha_1V_1}{LV_2^2} \right) + \frac{1}{8} \left(\frac{-3\alpha_2b(1+\varphi)x_2V_3}{V_2^7} - \frac{\alpha_1V_3}{V_2^3} \right) \right] + \frac{1}{16} \left(\frac{-2\alpha_1\alpha_2(bx_2(1+\varphi)V_1)^2}{V_2^6} - \frac{(\alpha_1^2 - \alpha_2^2)bx_2(1+\varphi)V_1^2}{V_2^5} - \frac{s\alpha_2V_1}{LV_2^2} \right) + \frac{1}{8} \left(\frac{-3b\alpha_1(1+\varphi)x_2V_3}{V_2^7} + \frac{\alpha_2V_3}{V_2^3} \right) \right\}$$

$$\rho_1^0 = \frac{-V_1}{16V_2^2} \left[\frac{2\alpha_1\alpha_2V_1(bx_2(1+\varphi))^2}{V_2^4} + \frac{(\alpha_1^2 - \alpha_2^2)bx_2(1+\varphi)V_1}{V_2^3} + \frac{s\alpha_2}{L} \right] + \frac{V_3}{8V_2^3} \left[\alpha_2 - \frac{3b\alpha_1(1+\varphi)x_2}{V_2^4} \right] = \mathcal{B},$$

here $V_1 = (1 + \varphi)[1 + b(1 + \varphi)P_3]$, $V_2 = [1 + b(1 + \varphi)(x_1 + P_3)]$, and $V_3 = b(1 + \varphi)^2[1 + b(1 + \varphi)P_3]$.

Thus, $\rho_1^0 < 0$ under condition (34) means the PFH system indicates a stable limit cycle.

Numerical simulations and discussions

In order to verify the analytical results, this section endeavors to provide graphical representations of a variety of dynamical scenarios that result from changes in the numerical values of the environmental parameters. The graphs are constructed using MATLAB with the classical fourth-order Runge–Kutta method, with a fixed step size $h = 0.001$ and with the initial condition $(P_0, F_0, H_0) = (0.4, 0.8, 0.12)$. Simulations are implemented by employing the following parameters:

$$Q = 10, \alpha_1 = 0.25, \varphi = 3, b = 0.2, \delta_1 = 0.03, s = 0.31, \alpha_2 = 0.22, L = 12, \beta = 0.35, \delta_2 = 0.02, \gamma = 0.28, \alpha_3 = 0.15, \delta_3 = 0.01, c = 0.3, . \quad (35)$$

The nullclines of the PFH model are depicted in Fig. 2 for the parameters specified above in Eq. (35). The global behavior of E_3 is illustrated in Fig. 2, while the coexistence point $E_3 = (P_3, F_3, H_3)$ is depicted in Fig. 3. That is, the parameters enumerated in Eq. (35) are consistent with the results of Theorem 8, which demonstrates that all other points, with the exception of E_3 , function as saddle points. The uniform persistence of the PFH system is also confirmed by these findings, which corroborate the yield of Theorem 3.

The importance of the parameter Q within the PFH system is examined in Fig. 4. The figure is generated using the parameter values specified in Eq. (35), with only parameter Q varying. The results show that for each Q belonging to $(0, 1.49)$, the solution of the PFH system stabilized at the coexistence point with a low dust pollutant density, as shown in Fig. 4a. When Q belongs to $[1.49, 9.4]$, the incidence of the Hopf bifurcation is detected. At $Q = 1.49 = Q^{hb}$, the fluctuation is observed at $Q = 1.49 = Q^{hb}$, as shown in Figs. 4b and 5. Further, the PFH system exhibits stability for the coexistence point when $9.4 < Q < 15$, see Fig. 4c. As Q is adjusted to 15, the PFH system loses stability for the coexistence point, and the solution stabilizes at the honeybees' free point E_2 . That means the occurrence of transcritical bifurcation is observed at $Q = 15 = Q^b$, see Fig. 4d Ecologically, the dust storms can induce oscillations in the PFH system. A particular concentration of dust in the environment enables the cohabitation of blooming plants and honey bees. A rise in dust storms is associated with a decline in honeybee populations. Moreover, a defined rate of dust pollutant can cause periodic variations in dust pollutant concentrations, as well as in blooming plants and honeybees. A specific concentration of dust storms may lead to a decline in the honey bee population.

Now, the importance of the depletion rate of dust particles caused by the barrier established by blooming plants, α_1 , within the PFH system is examined. To understand this, the figures are generated using the parameter given in Eq. (35), with different values of α_1 . With the range $\alpha_1 \in (0, 0.014)$, Fig. 6a demonstrates that, in the absence of the barrier provided by blooming plants, dust storms occur at elevated levels, subsequently leading to a decline in both blooming plants and the honeybee population. In this case, the solution of

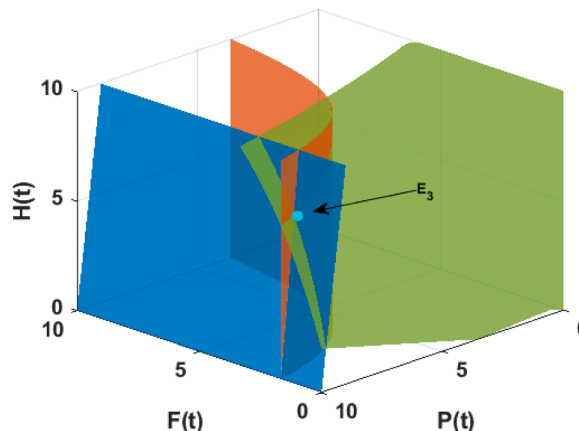


Fig. 2. The nullclines of PFH model.

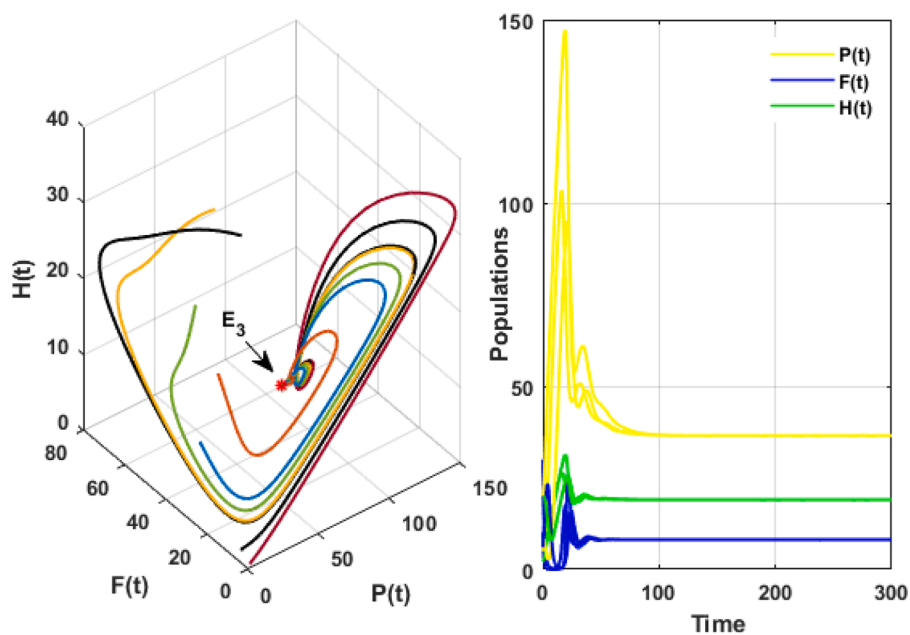
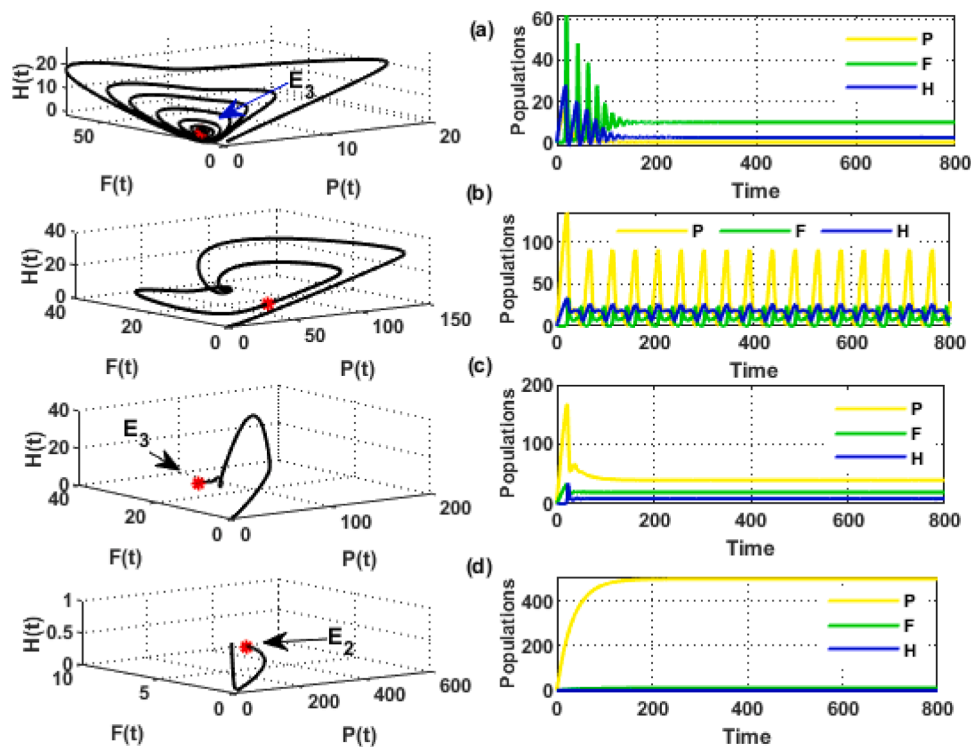
Fig. 3. The global behavior of E_3 .

Fig. 4. The dynamics of the PFH system with varying Q . a) The coexistence point is attained when $Q = 1.42$; b) Emergence of a Hopf bifurcation is reached when $Q = 1.49$; c) The coexistence point is reached when $Q = 9.4$; d) The honeybees' free point E_2 point is achieved when $Q = 15$.

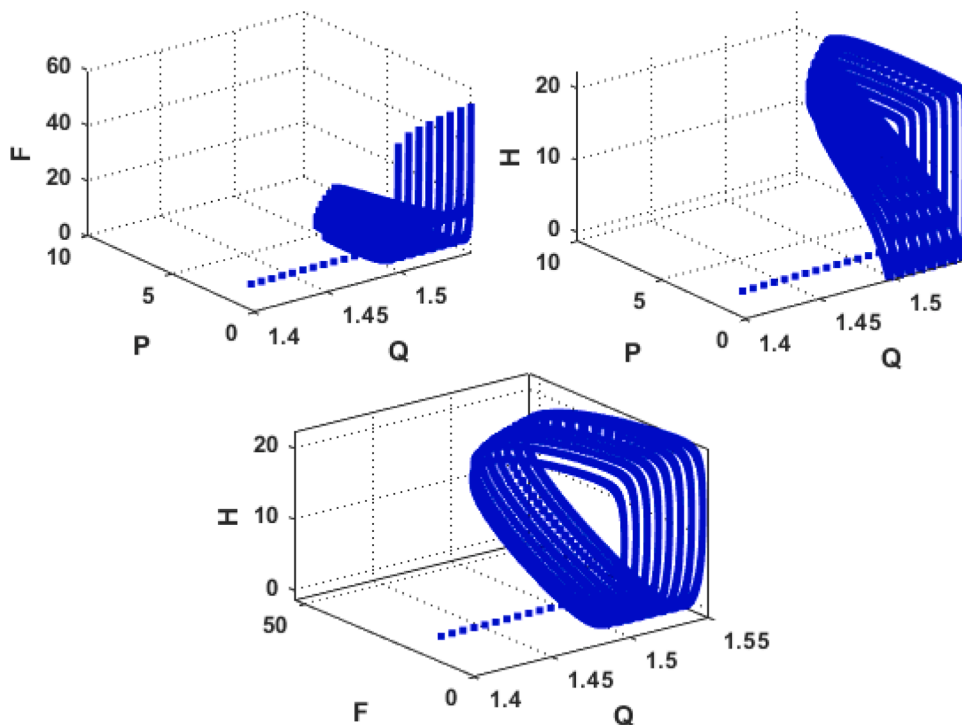


Fig. 5. The incidence of the Hopf bifurcation at $Q = 1.49$.

the *PFH* system stabilized at the dust storm's point $E_1 = (P_1, 0, 0)$. While for the $\alpha_1 \in [0.014, 0.18]$, Fig. 6b establishes the occurrence of the transcritical bifurcation at $\alpha_1 = 0.014 = \alpha_1^{hb}$, and the solution steadied at the honeybees' free point $E_2 = (P_2, F_2, 0)$. Fig. 6c shows that the dust storms, blooming plants, and honeybees can coexist within the interval $[0.18, 0.27]$. Finally, as the parameter α_1 rises to $\alpha_1 = 0.27 = \alpha_1^{hb}$, a bubble of oscillations appears readily in Figs. 6d and 7, confirming the output of 13. That means the solution of the *PFH* system has transitioned from a coexistence equilibrium to continuous oscillatory dynamics upon Hopf bifurcation. In biological terms, this implies that the blooming plants' biomass and the honeybee population in the presence of dust storms no longer converge on stable levels but instead exhibit continuous cyclical fluctuations. This behavior indicates the ecosystem's susceptibility to fluctuations in biological or environmental parameters and may serve as an early indicator of weakening ecological stability. Thus, the parameter α_1 is crucial in modifying the dynamics of the *PFH* system.

An essential parameter to examine is the influence of wind intensity (φ) on the system's continued cohabitation. We deal with two sides of φ : first, how much it affects the species densities, and second, how it might change the stability of the *PFH* system. Fig. 8a illustrates that, in a low wind speed environment, specifically when $0 < \varphi < 15$, the trajectory of the *PFH* model settles down at E_3 (coexistence point) asymptotically. In the case of elevated wind speeds, specifically within the range of $\varphi \geq 15$, the *PFH* system's solution approaches a limit cycle, as illustrated in Fig. 8b. Further, Fig. 9 depicts that the occurrence of the limit cycle starts when the wind speed is set at $\varphi = 15 = \varphi^{hb}$. This signifies that in low-wind conditions, the wind facilitates the pollination and dispersal of seeds from blooming plants, thereby assisting their reproduction. Consequently, the existence of blooming plants enhances the system's sustainability. Conversely, under conditions of elevated wind flow, the populations of blooming plants and bees lose their persistence.

The effects of the parameter associated with the dust storms' natural depletion (δ_1) are depicted in Figs. 10–11. Fig. 10a demonstrates the coexistence of all *PFH* system components within the interval $\delta_1 \in (0, 0.08)$. At the parameter value $\delta_1 = 0.08 = \delta_1^{hb}$, the start of an oscillatory bubble emerges, as shown in Figs. 10b and 11. For higher values of δ_1 , i.e., $\delta_1 \geq 1.43$, the *PFH* system illustrates the coexistence of blooming plants and bees despite the presence of dust storms, as presented in Fig. 10c. Ecologically, this means that when the natural reduction of dust contaminants is enhanced, blooming plants thrive, consequently resulting in a substantial rise in the bee population.

The intrinsic growth rate s of the blooming plants population is a critical parameter for consideration, as it may affect the densities of both dust storms and honeybee populations. Fig. 12a clearly shows that the *PFH* system experiences a Hopf bifurcation for $0 < s \leq 0.045$. Fig. 13 shows that the oscillatory bubble emerges at the parameter value $s = 0.045 = s^{hb}$. On the other hand, the trajectory of the *PFH* system stabilizes at the coexistence point $E_3 = (P_3, F_3, H_3)$ for $s > 0.045$. This finding indicates that s is a critical parameter that influences the system's continued coexistence.

The depletion rate of blooming plants by dust storms α_2 plays a vital role in preserving the blooming plants and the honeybee population within the *PFH* model. Fig. 14a illustrates that, for $0 < \alpha_2 < 0.51$, the solution of the *PFH* system attains stability at the

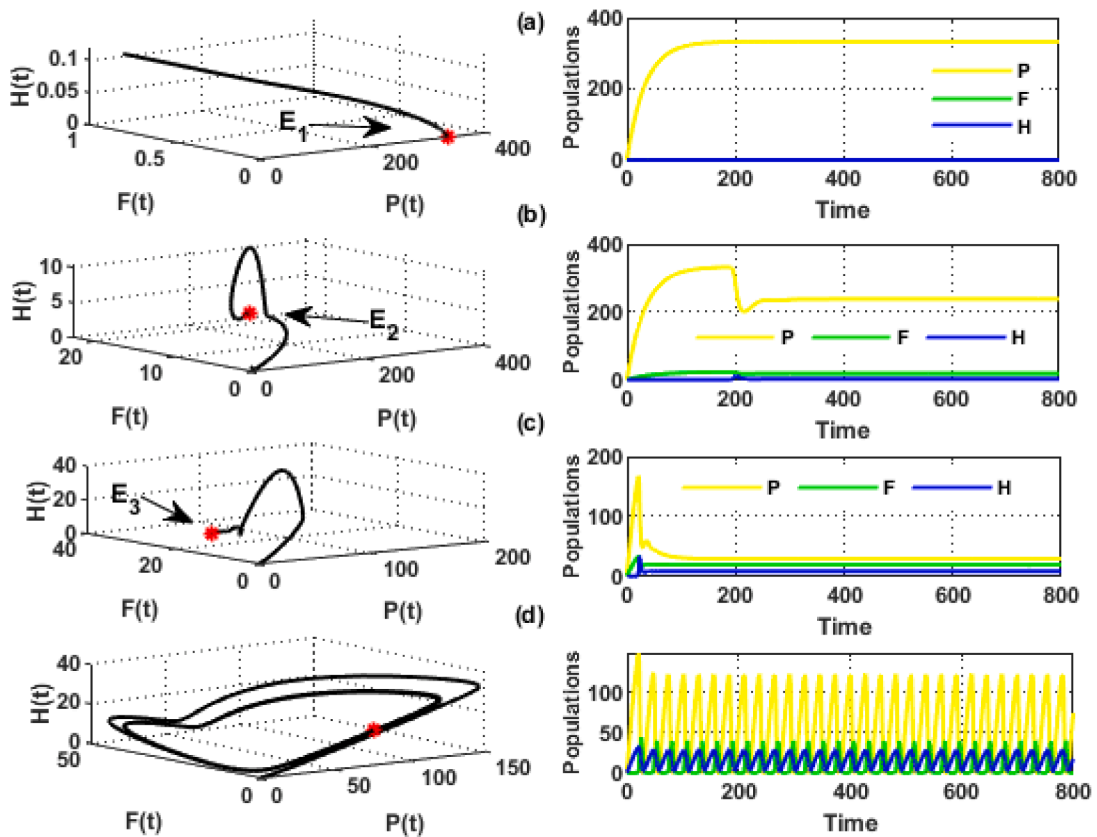


Fig. 6. The dynamics of the PFH system with varying α_1 . a) The dust storm's point E_1 is attained when $\alpha_1 = 0.013$; b) The honeybees' free point E_2 point is achieved when $\alpha_1 = 0.014$; c) The coexistence point E_3 is reached when $\alpha_1 = 0.18$; d) Emergence of a Hopf bifurcation is reached when $\alpha_1 = 0.27$.

coexistence point $E_3 = (P_3, F_3, H_3)$. In contrast, Fig. 14b demonstrates that the honeybee population loses persistence as a result of the elevated depletion rate of blooming plants caused by dust storms for $0.51 \leq \alpha_2 < 0.57$. In this case, the solution steadied at the honeybees' free point $E_2 = (P_2, F_2, 0)$. This result indicates that the PFH system faces a TB at $\alpha_2 = 0.51 = \alpha_2^{th}$. A sustained increase in α_2 , i.e., $\alpha_2 \geq 0.57$ results in a loss of persistence in the blooming plants, and the solution of the PFH system achieves stability at the dust storm's point $E_1 = (P_1, 0, 0)$, as shown in Fig. 14c. This result demonstrates that the PFH system encounters a TB at $\alpha_2 = 0.57 = \alpha_2^{th}$, thereby corroborating the findings presented in Theorem 9. Therefore, the importance of the parameter α_2 to maintain the persistence of the blooming plants and the honeybee population in the presence of dust pollutants, thereby sustaining the ecosystem.

The impact of varying the depletion rate of honeybees by dust storms α_3 on the population of honeybees and blooming plants is explored in Fig. 15. Fig. 15a demonstrates that the solution of the PFH model asymptotically approaches the coexistence point $E_3 = (P_3, F_3, H_3)$ for $0 < \alpha_3 < 0.27$. For $\alpha_3 \geq 0.27$, the PFH system faces an occurrence of TB at $\alpha_3 = 0.27 = \alpha_3^{th}$. So, the PFH system loses its persistence, and the trajectories in this case attain the honeybees' free point $E_2 = (P_2, F_2, 0)$ as displayed in Fig. 15b. It can be inferred that the depletion rate of honeybees by dust storms, denoted as α_3 , is a crucial parameter influencing the survival of bee populations, and it also affects the density of blooming plants, as it is considered one of the important pollinators of blooming plants.

The mutualistic rate between the blooming plants and the honeybee population, β , is examined in Fig. 16. Fig. 16a shows that for $0 < \beta < 0.07$, the solution of the PFH system stabilized at the dust storms' point $E_1 = (P_1, 0, 0)$. While the solution attains the coexistence point $E_3 = (P_3, F_3, H_3)$ for $\beta \geq 0.07$ as shown in Fig. 16b. This result indicates that the TB occurs at $\beta = 0.07 = \beta^{th}$. These findings highlight the essential role that pollinators have in ensuring the persistence of blooming plants and maintaining ecosystem sustainability.

Fig. 17 illustrates the impact of the comparable nutritional value of honeybee nutrients derived from blooming plants γ on the dynamics of PFH system. Fig. 17a illustrates the solutions converging to the honeybees' free point $E_2 = (P_2, F_2, 0)$ for $0 < \gamma < 0.022$. The augmentation of γ , i.e., when $\gamma \geq 0.022$, indicates that the solutions have stabilized at E_3 . The incidence of TB occurs at $\gamma = 0.022 = \gamma^{th}$. The results suggest that the rise in γ significantly contributes to the persistence of blooming plants and honeybees despite the presence of dust storms.

Fig. 18 depicts the substantial influence of the depletion rate of the blooming plants $\delta_2 = 0.02$. Fig. 18a illustrates the solution of

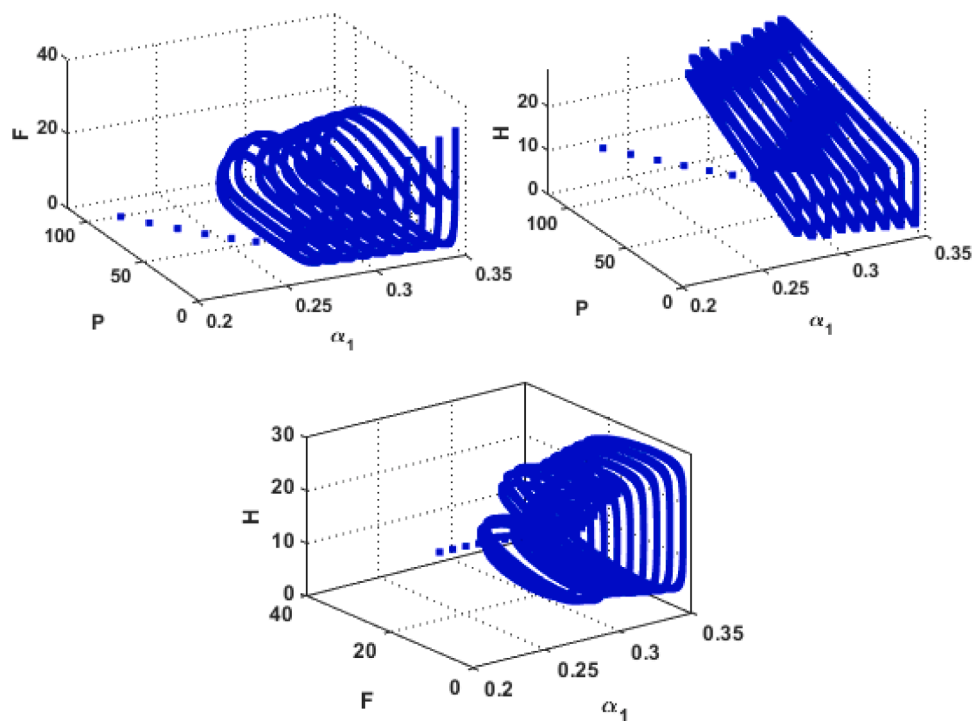


Fig. 7. The occurrence of the Hopf bifurcation at $\alpha_1 = 0.27$.

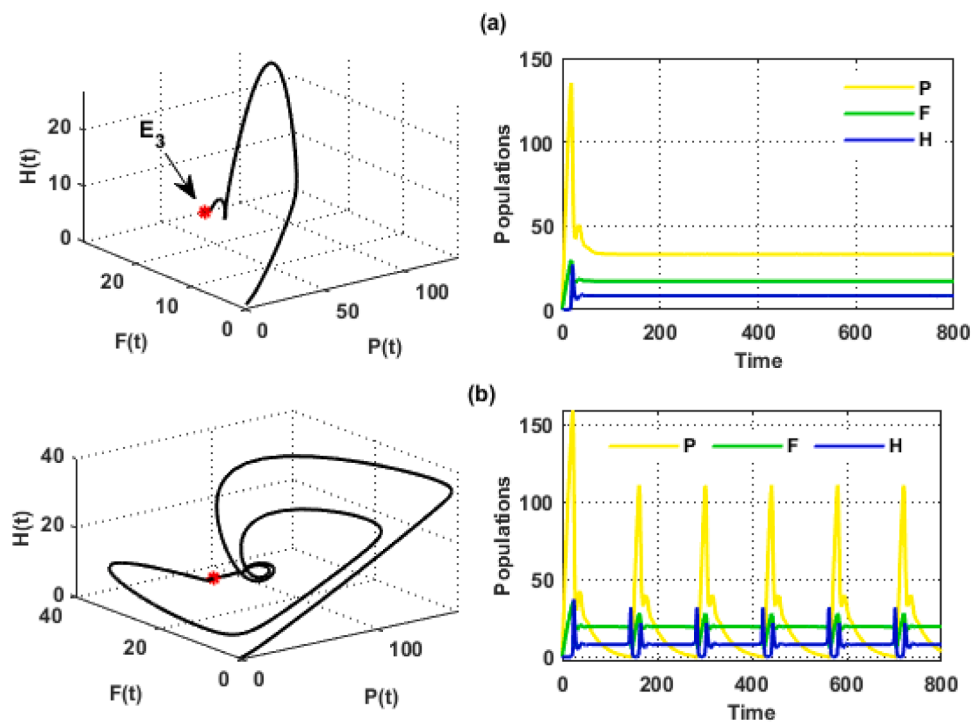


Fig. 8. The dynamics of the PFH system with varying φ . a) The coexistence point E_3 is reached when $\varphi = 14$; d) Emergence of a Hopf bifurcation is reached when $\varphi = 15$.

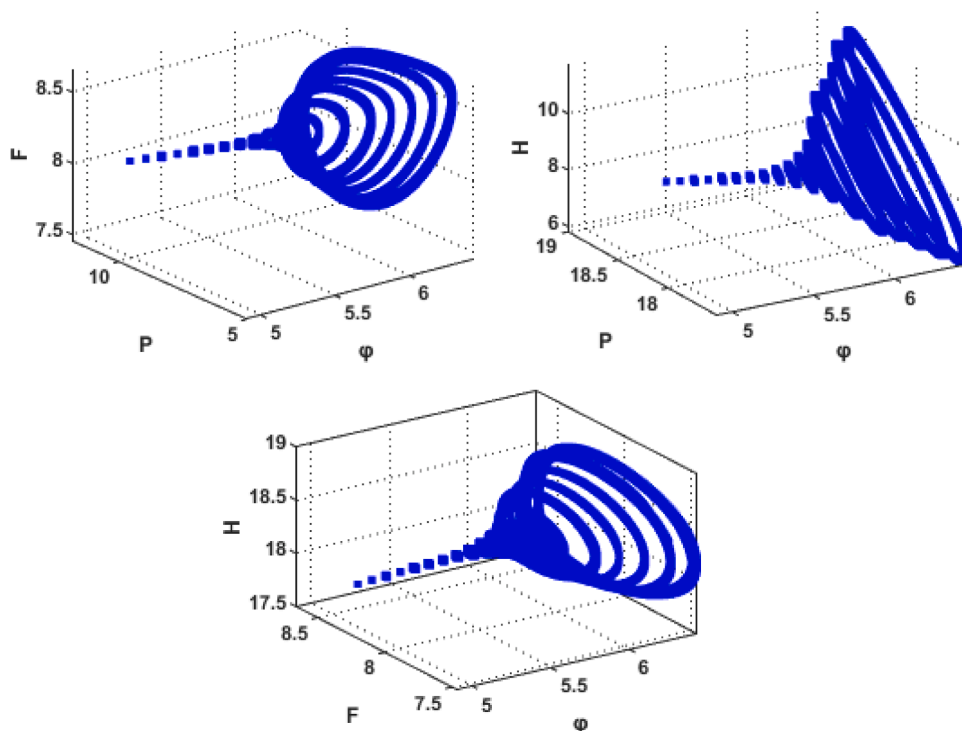


Fig. 9. The incidence of the Hopf bifurcation at $\varphi = 15$.

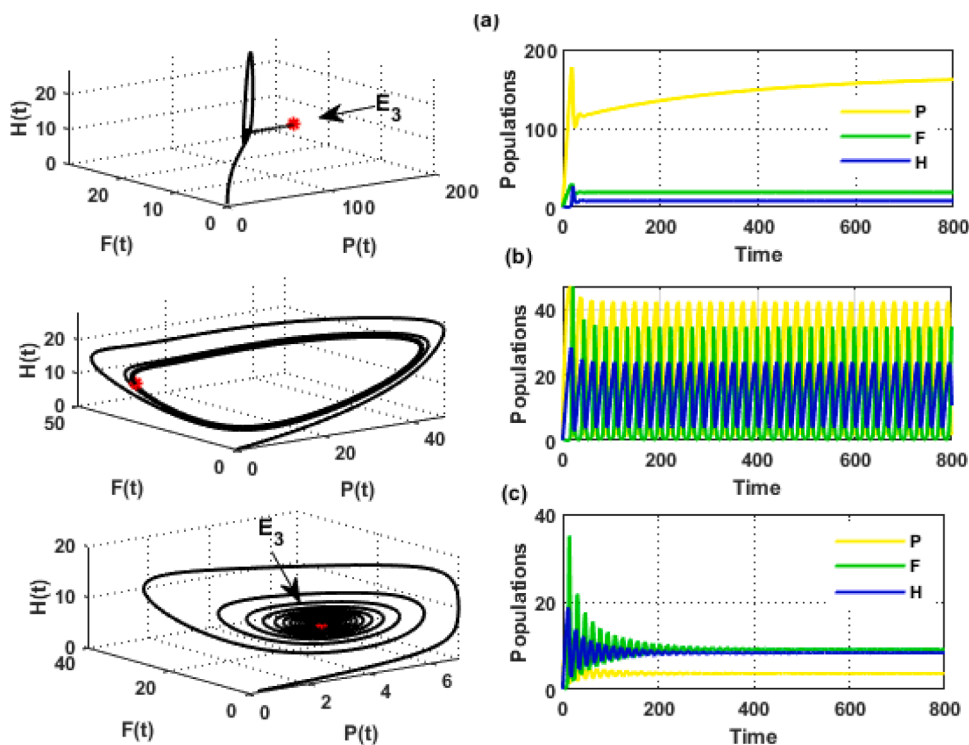


Fig. 10. The dynamics of the PFH system with varying δ_1 . a) The coexistence point E_3 is reached when $\delta_1 = 0.07$; d) Emergence of a Hopf bifurcation is reached when $\delta_1 = 0.08$; c) The coexistence point E_3 is attained when $\delta_1 = 1.4$.

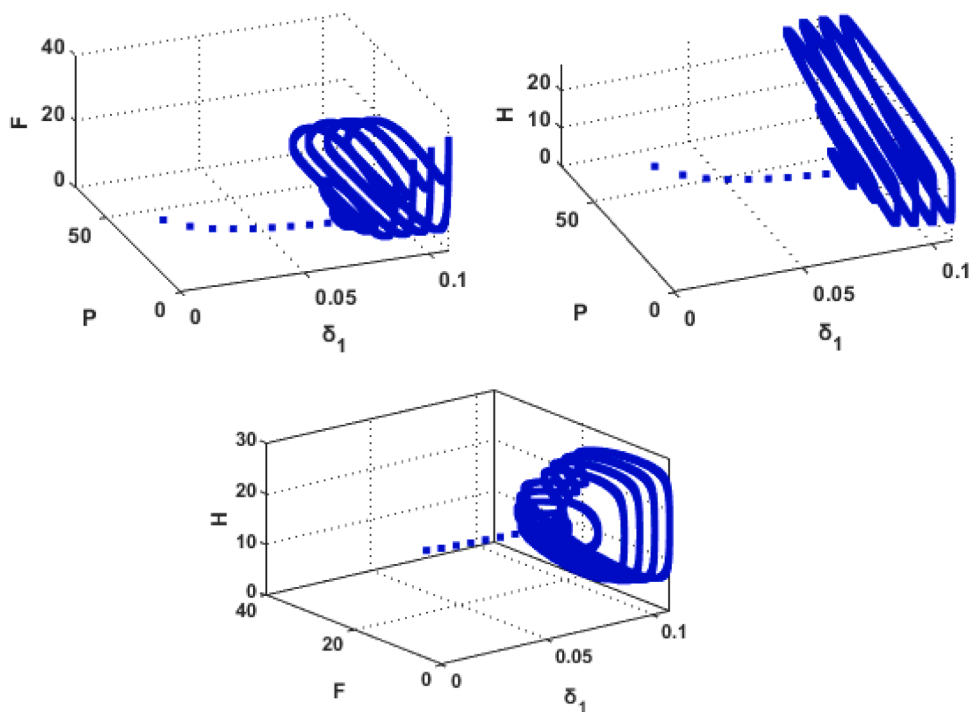


Fig. 11. The incidence of the Hopf bifurcation at $\delta_1 = 0.08$.

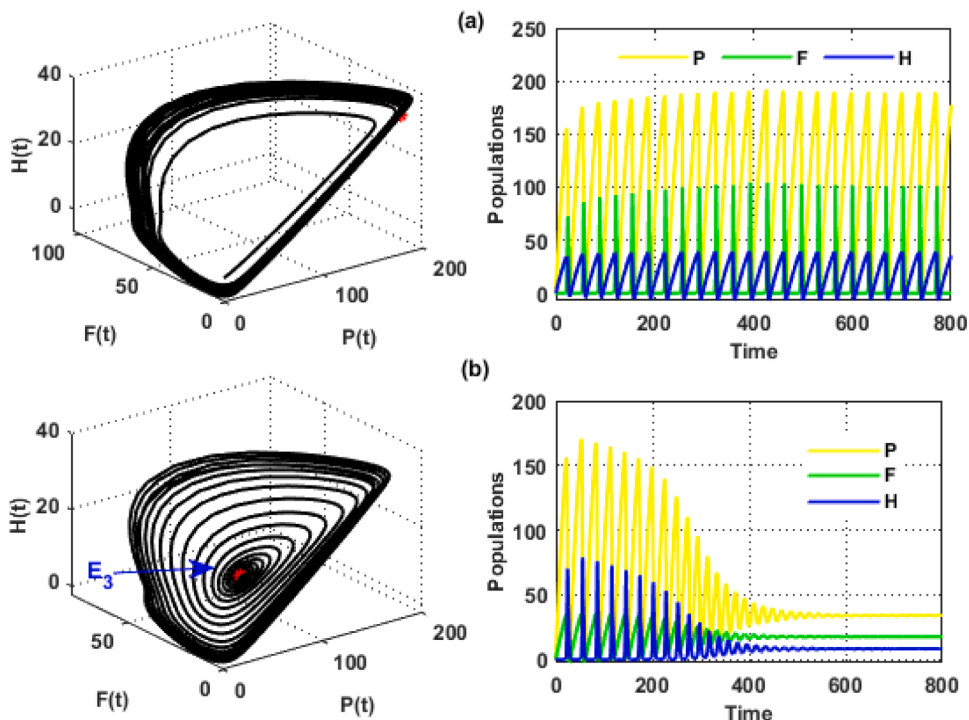


Fig. 12. The dynamics of the PFH system with varying s . a) Emergence of a Hopf bifurcation is reached when $s = 0.045$; b) The coexistence point E_3 is attained when $s = 0.05$.

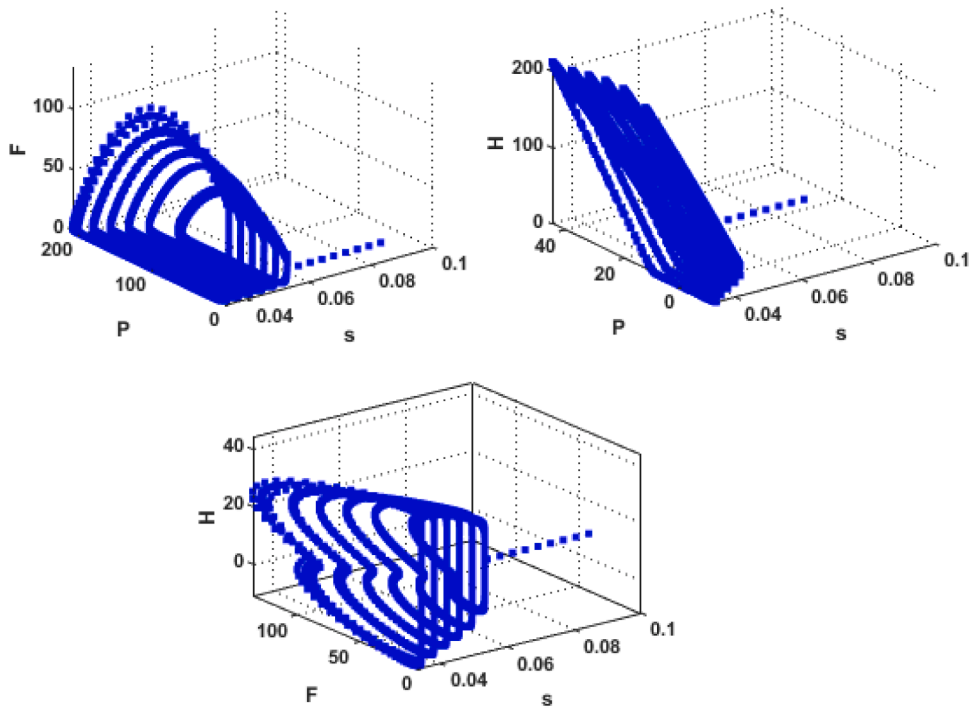


Fig. 13. The existence of the Hopf bifurcation at $s = 0.045$.

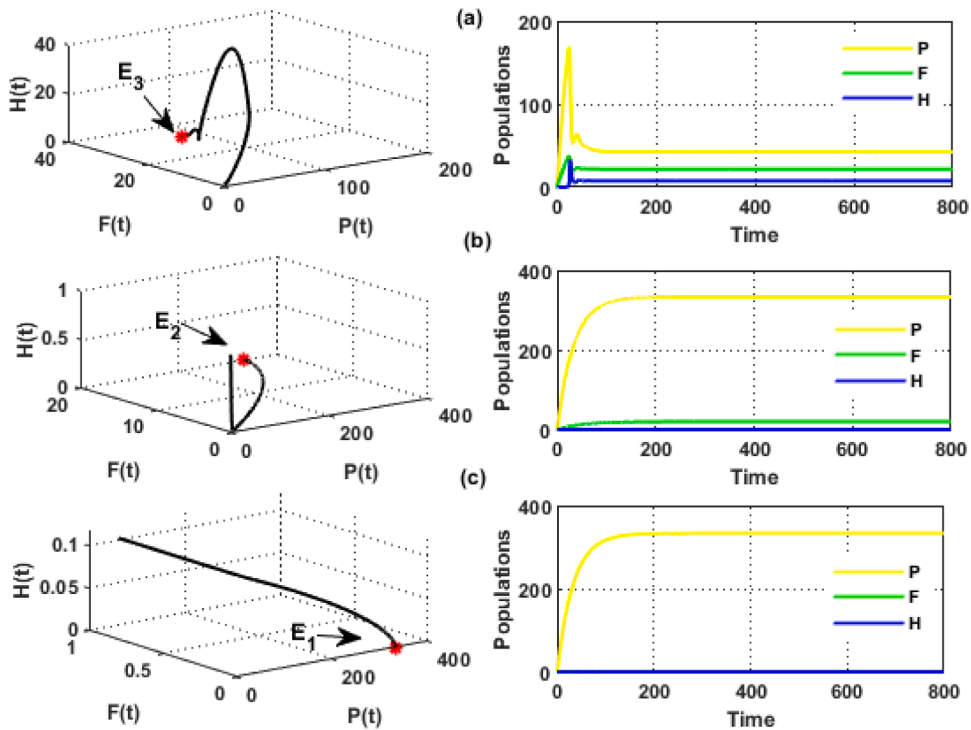


Fig. 14. The dynamics of the PFH system with varying α_2 . a) The coexistence point E_3 is reached when $\alpha_2 = 0.5$; b) The honeybees' free point E_2 point is achieved when $\alpha_2 = 0.51$; c) The dust storm's point E_1 is attained when $\alpha_2 = 0.057$.

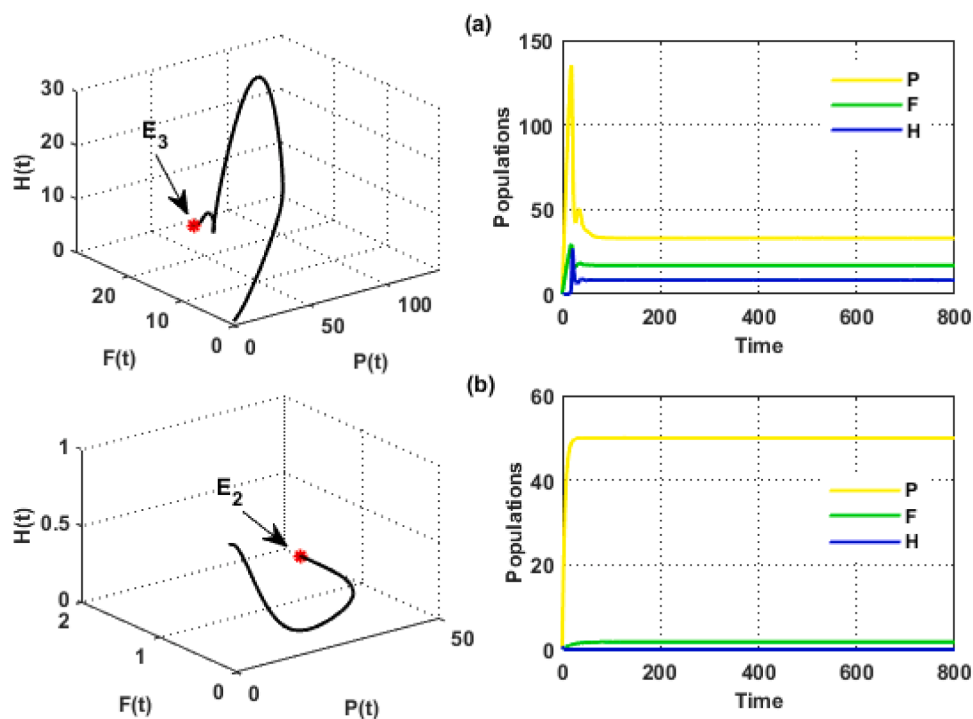


Fig. 15. The dynamics of the *PFH* system with varying α_3 . a) The coexistence point E_3 is reached when $\alpha_3 = 0.26$; b) The honeybees' free point E_2 point is achieved when $\alpha_3 = 0.27$.

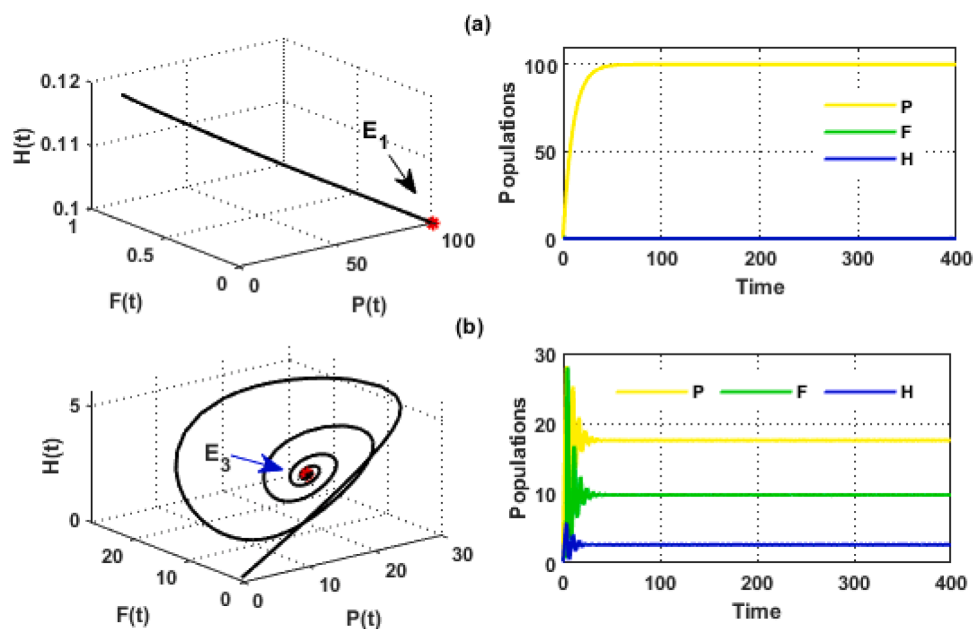


Fig. 16. The dynamics of the *PFH* system with varying β . a) The dust storm's point E_1 is attained when $\beta = 0.06$; b) The coexistence point E_3 is reached when $\beta = 0.07$.

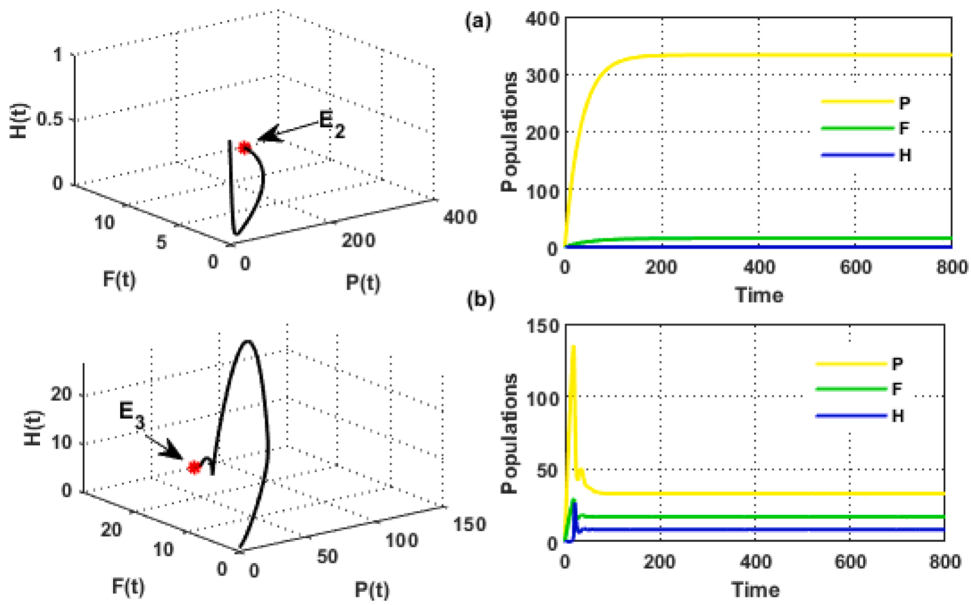


Fig. 17. The dynamics of the PFH system with varying γ . a) The honeybees' free point E_2 point is achieved when $\gamma = 0.021$; b) The coexistence point E_3 is reached when $\gamma = 0.022$.

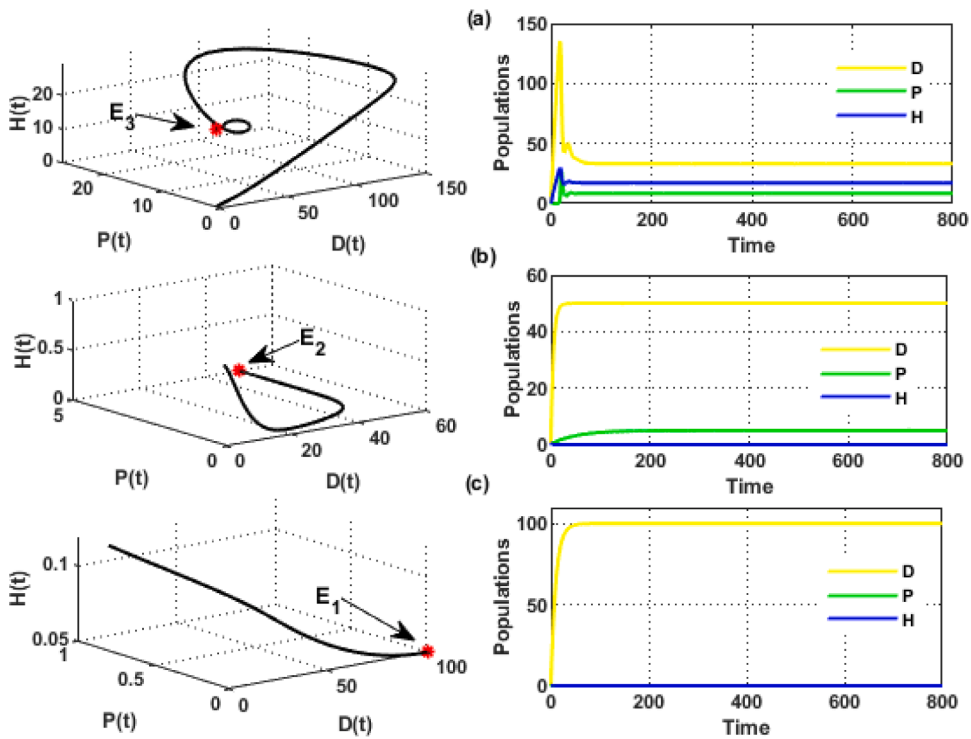


Fig. 18. The dynamics of the PFH system with varying δ_2 . a) The coexistence point E_3 is reached when $\delta_2 = 0.05$; b) The honeybees' free point E_2 point is achieved when $\delta_2 = 0.06$; c) The dust storm's point E_1 is attained when $\delta_2 = 0.14$.

the PFH system approaching the coexistence point E_3 for $0 < \delta_2 < 0.06$. For $0.06 \leq \delta_2 < 0.14$. The results indicate a decline in the bee population within the PFH system, and in this case, the solution converges to the honeybees' equilibrium point $E_2 = (P_2, F_2, 0)$. An increase in δ_2 (where $\delta_2 \geq 0.14$) leads to a reduction in the population of blossoming plants as well, with the solution stabilizing at the dust storms' point $E_1 = (P_1, 0, 0)$. The above result indicates the incidence of TB occurring twice: the first at $\delta_2 = 0.06 = \delta_2^{tb1}$ and the second one at $\delta_2 = 0.14 = \delta_2^{tb2}$. Consequently, the sustained existence of a thriving plant population markedly contributes to mitigating the effects of dust storms and supporting the longevity of the honey bee population, as blooming plants serve as the main source of nourishment for pollinators.

Conclusion

A mathematical model describing the influence of dust storms on the dynamics of blooming plants and honeybee populations has been analyzed through the use of ODEs. The model's solutions have been formulated to preserve key attributes, including uniqueness, non-negativity, and boundedness. The local and global scales of all equilibrium points have been identified. The analytical demonstration of the occurrence of certain types of bifurcations around the equilibria has been conducted. Numerical simulations validate the analytical results. The threshold values for the transcritical, saddle-node, and Hopf bifurcations have been accurately identified.

Furthermore, based on the numerical results, we observe that when the system adopts a unique coexistence equilibrium, the impact of the dust storm can decrease the densities of blooming plants and bees as the dust storm effect intensifies. That escalation in the intensity of dust storm effects has the potential to compromise the stability of the proposed system. The blooming plants is vital to a sustainable ecosystem, supporting the stability and resilience of both plant and honeybee populations within ecological systems. These findings may inform conservation initiatives dedicated to protecting pollinators and promoting biodiversity. In future conversations, we may address the limitations of the current study by examining additional ecological factors, including the influence of external conditions (e.g., moisture, the absence of stochastic dust events, pesticide effects, disease and temperature). By addressing these limitations, we offer a balanced perspective that acknowledges our model's strengths while identifying areas for improvement.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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