



Dilative stability

Outline of Ph.D. Thesis

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1. Introduction

In this thesis we summarize the most important results given in our dissertation “*Dilative stability*”.

In the dissertation we define and study a scaling property of stochastic processes, which we call dilative stability. This property is comparable with the well-known self-similarity and for those processes for which both property are defined, dilative stability is a generalization of self-similarity. The main difference between them is that while dilative stability refelects the “branching or confluent” structure of the distribution of the process, self-similarity has nothing to do with this.

The dissertation consists of five parts. The first chapter is the introduction. Chapter 2 draws a parallel between self-similarity and dilative stability. Chapters 3 and 4 deal with the superposition of stationary Ornstein–Uhlenbeck (OU) type processes and the superposition of stationary continuous state branching processes with immigration (CBI processes), respectively and with the dilatively stable renormalization functional limit theorem giving the so-called LISOU and LISCBI (particularly LISDLG) limit process, respectively. These dilatively stable processes with stationary increments provide a new opportunity to model non-Gaussian long memory processes as well. In Chapter 5 we present a dilatively stable renormalization functional limit theorem for Cox processes.

In what follows we use some special notations:

$\sim, \overset{1d}{\sim}, \overset{fd}{\sim}$: equivalence of the distributions, 1-dimensional distributions, finite-dimensional distributions

$\otimes, \otimes^c, \otimes^f$: convolution, c -th and function f -th convolution power

$\xrightarrow{w}$: weak convergence of distributions (probability measures)

$\xrightarrow{fd}$: weak convergence of the finite-dimensional distributions

cum, cum_n : (joint) cumulant, cumulant of order n

$(t_1^*, \dots, t_m^*)$: the components of $(t_1, \dots, t_m)$ listed in nondecreasing order

$C[0, \infty)$: the space of continuous functions on $[0, \infty)$ with the local uniform topology and Borel (cylinder) σ -algebra
 $D[0, \infty)$: the set of càdlàg functions on $[0, \infty)$ with the extended Skorokhod topology and Borel (cylinder) σ -algebra
 Γ : the Gamma function

2. Self-similarity and dilative stability

Let $\mathcal{S} \doteq \{\text{stochastic process } \{X(t), t \geq 0\} : X(0) = 0, \{X(t), t \geq 0\} \not\equiv 0 \text{ and right-continuous in distribution}\},$
 $\mathcal{I} \doteq \{\{X(t), t \geq 0\} \in \mathcal{S} : X(1) \text{ is non-Gaussian, the finite-dimensional distributions are infinitely divisible and functions } c_n(t) \doteq \text{cum}_n(X(t)), t \geq 0, n \geq 2, \text{ exist and are right-continuous}\}$

and $f, g : (0, \infty) \rightarrow (0, \infty)$. We call a stochastic process $\{X(t), t \geq 0\} \in \mathcal{I}$ (f, g) -dilatively stable if

$$\forall T > 0 : X(Tt) \stackrel{\text{fd}}{\sim} \frac{f(T)}{\sqrt{g(T)}} X^{\otimes g(T)}(t).$$

It was Lamperti's paper [7], which had attracted attention to the significance of the property of self-similarity. We present the dilatively stable analogues of Lamperti's theorems and show that a lot of results can be transferred from the self-similar to the dilatively stable case. Our most important results in this context are the following.

Theorem. *If a process $\{X(t), t \geq 0\}$ is (f, g) -dilatively stable, then there exist a unique (α, δ) such that $\alpha > 0$, $\delta \leq 2\alpha$ (which we call an admissible (α, δ)) and $f(t) = t^\alpha$, $g(t) = t^\delta$. Therefore instead of the expression “ (t^α, t^δ) -dilatively stable” we simply use “ (α, δ) -dilatively stable”. Moreover, for all admissible (α, δ) there exists an (α, δ) -dilatively stable process.*

Introducing the notations

$$\mathcal{S}_{\alpha\text{ss}} \doteq \{\alpha\text{-self-similar processes}\},$$

$$\mathcal{I}_{(\alpha,\delta)\text{ds}} \doteq \{(\alpha,\delta)\text{-dilatively stable processes}\},$$

we have $\mathcal{I}_{(\alpha,0)\text{ds}} = \mathcal{S}_{\alpha\text{ss}} \cap \mathcal{I}$, $\alpha > 0$, so in $\mathcal{I}$ dilative stability is a generalization of self-similarity. By the following theorem the processes in $\mathcal{I}$ which are the limit processes in renormalization limit theorems, are exactly the dilatively stable processes.

Theorem. *If $\{X(t), t \geq 0\} \in \mathcal{I}$ is a process such that there exist an infinitely divisible process $\{Y(t), t \geq 0\}$ with finite moments of all orders and functions $f, g: (0, \infty) \rightarrow (0, \infty)$ for which*

$$\frac{\sqrt{g(T)}}{f(T)} Y^{\otimes \frac{1}{g(T)}}(Tt) \xrightarrow[T \rightarrow \infty]{fd} X(t) \quad (1)$$

and the convergence corresponding to (1) holds also for every cumulants of the one-dimensional distributions, then there exists an admissible (α, δ) such that process $\{X(t), t \geq 0\}$ is (α, δ) -dilatively stable and f, g are regularly varying functions of orders α, δ , respectively. Conversely, every dilatively stable process $\{X(t), t \geq 0\}$ is a limit process in the previous sense.

The above two theorems are proved by the help of the cumulant functions. The keys are the Cauchy functional equation in the first case and a theorem of Bingham et al. [1] about the existence of a regularly varying limit function in the second case.

The next topic is the description of the relation of self-similarity and dilative stability. We give the connection between self-similar and dilatively stable processes by the help of the so-called function-th convolution power operation. Using this operation, established by a lemma, the self-similar–dilatively stable connection is the following.

Theorem. *Let parameter (α, δ) be admissible and let*

$$T_S : \mathcal{S}_{(\alpha-\frac{\delta}{2})\text{ss}} \cap \mathcal{I} \longrightarrow \mathcal{I}_{(\alpha,\delta)\text{ds}}, \quad T_S(\{Y(t), t \geq 0\}) \doteq \{Y(t), t \geq 0\}^{\otimes t^\delta},$$

$$T_D : \mathcal{I}_{(\alpha,\delta)\text{ds}} \longrightarrow \mathcal{S}_{(\alpha-\frac{\delta}{2})\text{ss}} \cap \mathcal{I}, \quad T_D(\{X(t), t \geq 0\}) \doteq \{X(t), t \geq 0\}^{\otimes t^{-\delta}},$$

where on the right-hand sides there are the t^δ function-th and $t^{-\delta}$ function-th convolution powers. Then the mapping T_S is well-defined (i.e. its values are in $\mathcal{I}_{(\alpha,\delta)\text{ds}}$) and it is an injection. It is a bijection if and only if $\delta = 0$. The analogous statements hold for the mapping T_D .

The most difficult part of the proof is to show that the mappings are not surjective in general. Now, the lack of surjectivity depends on that both mappings reduces the modulus of the correlation coefficient. As a corollary of the above theorem we obtain that dilative stability is not simply the combination of self-similarity and a power function-th convolution power, so dilative stability is not a redundant concept.

In the next section we define the notion of translative stability, a generalization of stationarity: let

$$\begin{aligned} \tilde{\mathcal{I}} \doteq \{ & \text{stochastic processes } \{S(t), t \in \mathbb{R}\} : S(0) \text{ is non-Gaussian, the} \\ & \text{finite-dimensional distributions are infinitely divisible, and functions} \\ & c_n(t) \doteq \text{cum}_n(S(t)), t \in \mathbb{R}, n \geq 2, \text{ exist and are right-continuous} \} \end{aligned}$$

and let $g : \mathbb{R} \rightarrow (0, \infty)$. Process $\{S(t), t \in \mathbb{R}\} \in \tilde{\mathcal{I}}$ is called g -translatively stable if

$$\forall T \in \mathbb{R} : S(t+T) \stackrel{\text{fd}}{\sim} S^{\otimes g(T)}(t).$$

Proposition. *If process $\{S(t), t \in \mathbb{R}\}$ is g -translatively stable, then there exists a unique $\delta \in \mathbb{R}$ such that $g(t) = e^{\delta t}$. Therefore instead of the expression “ $e^{\delta t}$ -translatively stable” we simply use “ δ -translatively stable”.*

Introducing the notations

$$\begin{aligned} \mathcal{S}_{\text{st}} &\doteq \{\text{strictly stationary processes over } (-\infty, \infty)\}, \\ \mathcal{I}_{\delta\text{ts}} &\doteq \{\delta\text{-translatively stable processes}\}, \\ \mathcal{I}_{\text{wsOU}\lambda} &\doteq \{\text{wide sense OU type processes with parameter } \lambda\}, \\ \mathcal{I}_{\text{add}} &\doteq \{\text{the set of additive processes on } [0, \infty)\} \end{aligned}$$

we have $\mathcal{I}_{0\text{ts}} = \mathcal{S}_{\text{st}} \cap \tilde{\mathcal{I}}$. By the help of translative stability we generalize the Lamperti transform, introduced in [7], which gives, for each $\alpha > 0$, a one-to-one correspondence between α -self-similar and stationary processes.

So, our next result is the analogous statement: for each admissible (α, δ) the Lamperti transform gives a one-to-one correspondence between (α, δ) -dilatively stable and δ -translatively stable processes. Furthermore, by Jeanblanc et al. [5] the Lamperti transform links stationary OU type processes with parameter $-\alpha$ and α -self-similar additive processes in a one-to-one manner. We prove the dilatively stable analogue: the Lamperti transform links δ -translatively stable, so-called wide sense OU type processes with parameter $-(\alpha-\delta/2)$ and (α, δ) -dilatively stable additive processes in a one-to-one manner:

Theorem. (*Lamperti transforms*) *Let parameter (α, δ) be admissible. Then the mappings*

$$L_{TS} : \mathcal{I}_{\delta\text{ts}} \longrightarrow \mathcal{I}_{(\alpha, \delta)\text{ds}}, \quad X(t) \doteq (L_{TS}(S))(t) \doteq t^{\alpha - \frac{\delta}{2}} S(\log t), \quad t > 0,$$

$$X(0) \doteq (L_{TS}(S))(0) \doteq 0,$$

$$L_{DS} : \mathcal{I}_{(\alpha, \delta)\text{ds}} \longrightarrow \mathcal{I}_{\delta\text{ts}}, \quad S(t) \doteq (L_{DS}(X))(t) \doteq e^{-(\alpha - \frac{\delta}{2})t} X(e^t), \quad t \in \mathbb{R},$$

are well-defined, i.e. L_{TS} maps into $\mathcal{I}_{(\alpha, \delta)\text{ds}}$, and L_{DS} maps into $\mathcal{I}_{\delta\text{ts}}$ and they are the inverse of each other. Moreover, assuming that all processes have zero mean if $\delta < 0$, the restrictions

$$L_{TS}| : \mathcal{I}_{\delta\text{ts}} \cap \mathcal{I}_{\text{wsOU}-(\alpha - \frac{\delta}{2})} \longrightarrow \mathcal{I}_{(\alpha, \delta)\text{ds}} \cap \mathcal{I}_{\text{add}},$$

$$L_{DS}| : \mathcal{I}_{(\alpha, \delta)\text{ds}} \cap \mathcal{I}_{\text{add}} \longrightarrow \mathcal{I}_{\delta\text{ts}} \cap \mathcal{I}_{\text{wsOU}-(\alpha - \frac{\delta}{2})}$$

are well-defined, $L_{TS}|$ maps onto $\mathcal{I}_{(\alpha, \delta)\text{ds}} \cap \mathcal{I}_{\text{add}}$ and $L_{DS}|$ maps onto $\mathcal{I}_{\delta\text{ts}} \cap \mathcal{I}_{\text{wsOU}-(\alpha - \frac{\delta}{2})}$ and (consequently) they are the inverse of each other.

There is a section about self-similar and dilatively stable processes with stationary increments. The importance of that processes lies in the fact that the autocovariance function of a non-degenerate dilatively stable process with stationary increments is the same as that of the corresponding self-similar process with L^2 -stationary increments, i.e. as the autocovariance function of a fractional Brownian motion. Having stated this we have a new possibility for long memory modelling.

The next section treats dilatively stable renormalization operators. We show that the analogy with the self-similar case holds true in this regard as well. That is, for any fixed admissible (α, δ) the set $\{A_T^{(\alpha, \delta)} : T > 0\}$ of the dilatively stable renormalization operators

$$A_T^{(\alpha, \delta)} : \mathcal{I} \longrightarrow \mathcal{I}, \quad (A_T^{(\alpha, \delta)} X)(t) \doteq T^{-(\alpha - \frac{\delta}{2})} X^{\otimes T^{-\delta}}(Tt), \quad t \geq 0,$$

endowed with the composition operation constitutes a semigroup. Hence, dilatively stable renormalization operators can be iterated. Furthermore, for any admissible (α, δ) , a process in $\mathcal{I}$ is (α, δ) -dilatively stable if and only if it is a fixed point of $A_T^{(\alpha, \delta)}$ for every $T > 0$. This explains why just the renormalized processes happen to converge in dilatively stable (functional) limit theorems.

3. The LISOU process

In this chapter we study the superposition of stationary OU type processes and state a dilatively stable renormalization functional limit theorem for the integrated superposition process. The properties called regularity and affinity of Markov processes were defined and studied in Duffie et al. [2]. Regular affine processes are exactly those regular, homogeneous Markov processes, which are infinitely decomposable (roughly, infinitely divisible, in the terminology of Markov processes). On the other hand, infinitely decomposable processes are just those processes which are suitable for being superposed. The two types of regular affine processes: OU type processes and continuous state branching processes with immigration (CBI processes). This is the reason why in Chapter 3 and 4 we happen to superpose OU type processes and CBI processes, respectively.

The construction called superposition is the following one.

Assumption SPL. (superpositional law) Let $1/2 < H < 1$ be a parameter (serving later as the Hurst parameter), $0 < d_1, d_2, \dots$ be a sequence of constants (time dilation constants), $0 < p_1, p_2, \dots$ be a discrete probability distribution (superposition weight sequence) and let ξ be a discrete random variable with the law $P(\xi = d_i) = p_i$, $i \in \mathbb{N}$, called the superposition law. Let

the left tail (i.e. around zero) of the distribution function of ξ be regularly varying of order $2-2H$, i.e.

$$F_\xi(x) = P(\xi < x) = x^{2-2H} \ell\left(\frac{1}{x}\right),$$

where ℓ is a slowly varying function (i.e. slowly varying at ∞).

Furthermore, let $\{L(t), t \in \mathbb{R}\}$ be a two-sided Lévy process with the generating triplet (σ^2, γ, ν) , where $\gamma = \mathbf{E} L(1)$ and Lévy measure $\nu \not\equiv 0$, for which $\int_{-1}^1 x^2 \nu(dx) + \int_{\mathbb{R} \setminus (-1,1)} e^{u|x|} \nu(dx) < \infty$, for some $u > 0$. Let $\{X(t), t \geq 0\}$ be a stationary OU type process with parameter $\alpha = -1$ and driving process $\{L(t), t \in \mathbb{R}\}$ and let $\{X_j(t), t \geq 0\}$, $j \in \mathbb{N}$, be independent processes such that

$$\{X_j(t), t \geq 0\} \stackrel{\text{fd}}{\sim} \{X(d_j t), t \geq 0\}^{*p_j}, \quad j \in \mathbb{N}. \quad (2)$$

Theorem. For each $t \geq 0$ the series $\sum_j X_j(t)$ converges in L^2 (uniformly in t) and also almost surely.

The limit process $Y(t) \doteq \sum_{j=1}^\infty X_j(t)$, $t \geq 0$, is called the (infinite) superposition of OU type processes (SOU process) with parameters (H, ℓ) and (σ^2, γ, ν) and process

$$J_{\sigma^2, \gamma, \nu}^{(H, \ell)}(t) \doteq \int_0^t Y(s) ds, \quad t \geq 0,$$

defined almost surely, is called the ISOU (integrated SOU) process with parameters (H, ℓ) and (σ^2, γ, ν) . The following theorem states that in the construction of the ISOU process the order of the superposition and the integration can be interchanged and a functional limit theorem also holds.

Theorem. We also have

$$\forall t \geq 0 : \sum_{j=1}^n \int_0^t X_j(s) ds \xrightarrow[n \rightarrow \infty]{} J_{\sigma^2, \gamma, \nu}^{(H, \ell)}(t) \quad \text{in } L^2 \text{ and a.s.}$$

and

$$\left\{ \sum_{j=1}^n \int_0^t X_j(s) ds, t \geq 0 \right\} \xrightarrow[n \rightarrow \infty]{w} \left\{ J_{\sigma^2, \gamma, \nu}^{(H, \ell)}(t), t \geq 0 \right\} \quad \text{on } C[0, \infty).$$

The main theorem in this chapter is the following.

Theorem. *The family of distributions of the renormalized centered ISOU processes converges weakly on $C[0, \infty)$ to a limit distribution. We denote the process with this distribution by $\{J_{\sigma^2, \nu}^{(H)}(t), t \geq 0\}$ and we call it the LISOU process (limit of [renormalized centered] ISOU processes) (with parameters H and (σ^2, ν)):*

$$\frac{1}{T} \left\{ J_{\sigma^2, \gamma, \nu}^{(H, \ell)}(Tt) - \mathbb{E} J_{\sigma^2, \gamma, \nu}^{(H, \ell)}(Tt), t \geq 0 \right\} \stackrel{\circledast \frac{T^2-2H}{\ell(T)}}{\xrightarrow{T \rightarrow \infty}} \left\{ J_{\sigma^2, \nu}^{(H)}(t), t \geq 0 \right\} \quad \text{on } C[0, \infty).$$

The distribution on $C[0, \infty)$ of the LISOU process with parameters H and (σ^2, ν) is uniquely determined by its zero mean and its joint cumulants, which are:

$$\begin{aligned} \text{cum} \left(J_{\sigma^2, \nu}^{(H)}(t_1), \dots, J_{\sigma^2, \nu}^{(H)}(t_m) \right) &= \frac{\text{cum}_m(L(1))}{m} \Gamma(3 - 2H) \\ &\times \int_0^{t_1} \cdots \int_0^{t_m} (s_2^* - s_1^* + \cdots + s_m^* - s_1^*)^{2H-2} d\underline{s}, \quad (3) \end{aligned}$$

$0 \leq t_1, \dots, t_m, m \geq 2$. Moreover, $\{J_{\sigma^2, \nu}^{(H)}(t), t \geq 0\}$ has stationary increments and it is $(H, 2H - 2)$ -dilatively stable.

4. The LISCBI process

The main results in this chapter are parallel with those in the previous chapter, only the starting process is the continuous state branching process with immigration (CBI process) instead of the OU type process. The CBI process is defined in Kawazu–Watanabe [6] as a Feller process $\{X(t), t \geq 0\}$ with state space $[0, \infty)$ with a transition Laplace transform of the form

$$\varphi_{|x}(t, u) = \mathbb{E}(e^{-uX(t)} \mid X(0) = x) = e^{-\phi(t, u) - \psi(t, u)x}, \quad u, t, x \geq 0.$$

The functions ϕ and ψ , called the Laplace exponent functions are given by Kawazu–Watanabe [6, Thm. 1.1–1.1'] and Filipović [3, Thm. 4.3] as the unique solutions of the differential equations

$$\partial_1 \psi(t, u) = R(\psi(t, u)), \quad \psi(0, u) = u, \quad (4)$$

$$\partial_1 \phi(t, u) = S(\psi(t, u)), \quad \phi(0, u) = 0, \quad (5)$$

where

$$R(u) = -\frac{\sigma^2}{2} u^2 + \alpha u + \int_0^\infty (1 - e^{-uy} - u(1 \wedge y)) m_1(dy), \quad u \geq 0, \quad (6)$$

$$S(u) = \vartheta u + \int_0^\infty (1 - e^{-uy}) m_2(dy), \quad u \geq 0, \quad (7)$$

$\sigma^2 \geq 0$, $\vartheta \geq 0$, $-\infty < \alpha < \infty$ and m_1, m_2 are Borel measures on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge y^2) m_1(dy) + \int_0^\infty (1 \wedge y) m_2(dy) < \infty$ and function R fulfills the condition of conservativity. Conversely, given the parameters $\sigma^2, \vartheta, \alpha$, Borel measures m_1, m_2 on $(0, \infty)$ and functions R and S as above, there exists a unique CBI process with the Laplace exponent functions ϕ and ψ , satisfying the differential equations (4–5). The distributions corresponding to the Laplace exponents (6–7) are called the branching and the immigration distributions, respectively. The most important among our own results with respect to stationary CBI processes is the following one.

Proposition. *If $\{X(t), t \geq 0\}$ is a stationary CBI process with branching and immigration distributions of finite moments of all orders, then for each $m \geq 2$ its m -fold joint cumulant taken at $(t_1, \dots, t_m)$ is some multivariate polynomial of $e^{R'(0)(t_2^* - t_1^*)}, \dots, e^{R'(0)(t_m^* - t_{m-1}^*)}$:*

$$\text{cum}(X(t_1), \dots, X(t_m)) = P_m(e^{R'(0)(t_2^* - t_1^*)}, \dots, e^{R'(0)(t_m^* - t_{m-1}^*)}), \quad (8)$$

$t_1, \dots, t_m \geq 0$, where the polynomial

$$P_m(\underline{x}) = P_m(x_1, \dots, x_{m-1}) = \sum_{k_1, \dots, k_{m-1}} c_{k_1, \dots, k_{m-1}} x_1^{k_1} \cdots x_{m-1}^{k_{m-1}} \quad (9)$$

has the following properties:

- i) $c_{k_1, \dots, k_{m-1}} = \sum_{j=1}^m \lambda_{m,j}(k_1, \dots, k_{m-1}) S^{(j)}(0)$ where the coefficients $\lambda_{m,j}(k_1, \dots, k_{m-1})$ do not contain the parameter (ϑ, m_2) ,
- ii) all the coefficients $c_{k_1, \dots, k_{m-1}}$ are positive,
- iii) the transform $(\sigma^2, \alpha, m_1) \mapsto (c\sigma^2, c\alpha, cm_1)$, i.e. $R(t) \mapsto cR(t)$, $t \geq 0$, implies the change of polynomial $P_m(\underline{x})$ as $P_m(\underline{x}) \mapsto \frac{1}{c} P_m(\underline{x})$, $c > 0$,
- iv) the transform $(\vartheta, m_2) \mapsto (c\vartheta, cm_2)$, i.e. $S(t) \mapsto cS(t)$, $t \geq 0$, implies the change of polynomial $P_m(\underline{x})$ as $P_m(\underline{x}) \mapsto cP_m(\underline{x})$, $c > 0$.

The assumptions on the CBI processes which we superpose are the following. Let R given by (6) be the Laplace exponent function of a branching distribution with parameter $(\sigma^2, \alpha = \int_1^\infty (1-y)m_1(dy) - 1, m_1)$ and let S given by (7) be the Laplace exponent function of an immigration distribution with parameter (ϑ, m_2) . Assume that $\int_1^\infty e^{ux}(m_1 + m_2)(dx) < \infty$ for some $u > 0$. Let $\{X(t), t \geq 0\}$ be a stationary CBI process with branching and immigration Laplace exponent functions R and S , respectively. Let $\{X_j(t), t \geq 0\}$, $j \in \mathbb{N}$, be independent processes such that

$$\{X_j(t), t \geq 0\} \stackrel{\text{fd}}{\sim} \{X(d_j t), t \geq 0\}^{\otimes p_j}, \quad j \in \mathbb{N},$$

where the d_j s and p_j s are given by Assumption SPL.

Theorem. *Beginning with the theorem under (2), everything in Chapter 3 (on pp. 7–8) holds true if we replace the initial word “OU” by “CBI” and the parameter vectors (σ^2, γ, ν) and (σ^2, ν) by $(\sigma^2, m_1, \vartheta, m_2)$. Moreover, the cumulants of the LISCBI process with parameter $(\sigma^2, m_1, \vartheta, m_2)$ are, instead of (3):*

$$\begin{aligned} & \text{cum} \left(J_{\sigma^2, m_1, \vartheta, m_2}^{(H)}(t_1), \dots, J_{\sigma^2, m_1, \vartheta, m_2}^{(H)}(t_m) \right) \\ &= \Gamma(3 - 2H) \int_0^{t_1} \dots \int_0^{t_m} \varphi_H(P_m)(s_2^* - s_1^*, \dots, s_m^* - s_{m-1}^*) d\underline{s}, \quad (10) \end{aligned}$$

$0 \leq t_1, \dots, t_m$, $m \geq 2$, where P_m , $m \geq 2$, are the polynomials (9), with the restriction $R'(0) = -1$ and φ_H is a mapping defined on the set of

multivariate polynomials with positive variables as

$$P(x_1, \dots, x_n) = \sum_{k_1, \dots, k_n} c_{k_1, \dots, k_n} x_1^{k_1} \cdots x_n^{k_n} \\ \longmapsto \varphi_H(P)(x_1, \dots, x_n) \doteq \sum_{k_1, \dots, k_n} c_{k_1, \dots, k_n} (k_1 x_1 + \cdots + k_n x_n)^{2H-2},$$

where the variables of the images $\varphi_H(P)$ are also positive.

The most difficult part of the proof is to show that the cumulants (10) (and the zero mean) uniquely determine the distribution. To prove it, we use the proposition on p. 9.

The particular case when the CBI process is the so-called diffusion process with linear generator (DLG process), given by the stochastic differential equation

$$dX(t) = (\vartheta + \alpha X(t)) dt + \sigma \sqrt{X(t)} dB(t),$$

($\{B(t), t \geq 0\}$ is the standard Brownian motion), is enlarged upon in a separate section. In this case we give an explicit expression for the cumulants (10). So, the LISDLG process can be defined also by (its zero mean and) the cumulants

$$\text{cum} \left(J_{\sigma^2, \vartheta}^{(H)}(t_1), \dots, J_{\sigma^2, \vartheta}^{(H)}(t_m) \right) = 2^{3-2H-m} \Gamma(3-2H) \vartheta \sigma^{2m-2} (m-1)! \\ \times \text{sym}_{\underline{t}} \int_0^{t_1} \cdots \int_0^{t_m} \left(|s_2 - s_1| + |s_3 - s_2| + \cdots + |s_m - s_{m-1}| + |s_1 - s_m| \right)^{2H-2} d\underline{s},$$

$$0 \leq t_1, \dots, t_m, \quad m \geq 2.$$

5. Cox processes and dilative stability

If $\{Y(t), t \geq 0\}$ is a random measure, independent of the standard Poisson process $\{N(t), t \geq 0\}$, then process $\{N \triangleleft Y(t), t \geq 0\} \doteq \{N(Y(t)), t \geq 0\}$ is called a Cox process with intensity process $\{Y(t), t \geq 0\}$. According to Grandell [4, Thm. 4.2.2], if a self-similar renormalization functional limit

theorem holds for the intensity process of a Cox process, then a similar self-similar renormalization functional limit theorem holds also for the Cox process itself. In this last part of the dissertation we present the analogous statement for dilatively stable renormalization functional limit theorems.

If $\{Y(t), t \geq 0\}$ is an infinitely divisible random measure, then we say that a dilatively stable renormalization functional limit theorem holds for $\{Y(t), t \geq 0\}$, if there exists a càdlàg function a , an admissible parameter (α, δ) , an α -varying function $f : (0, \infty) \rightarrow (0, \infty)$, a δ -varying function $g : (0, \infty) \rightarrow (0, \infty)$, for which $\lim_{t \rightarrow \infty} (f(t)/\sqrt{g(t)}) = \infty$ and an (α, δ) -dilatively stable càdlàg process $\{X(t), t \geq 0\}$, such that

$$\left\{ \frac{\sqrt{g(T)}}{f(T)} (Y(Tt) - a(Tt)), t \geq 0 \right\}^{\otimes \frac{1}{g(T)}} \xrightarrow[T \rightarrow \infty]{w} \{X(t), t \geq 0\} \quad \text{on } D[0, \infty). \quad (11)$$

Our theorem is the dilatively stable analogue of [4, Thm. 4.2.2]:

Theorem. *Let $\{Y(t), t \geq 0\}$ be an infinitely divisible random measure for which the dilatively stable renormalization functional limit theorem (11) holds, where $a \in D_0[0, \infty)$. If there exists the limit $\lim_{T \rightarrow \infty} \frac{a(T)}{f^2(T)} = \kappa \in [0, \infty)$, then for the Cox process $\{N \triangleleft Y(t), t \geq 0\}$ the following dilatively stable renormalization functional limit theorem holds:*

$$\left\{ \frac{\sqrt{g(T)}}{f(T)} (N \triangleleft Y(Tt) - a(Tt)), t \geq 0 \right\}^{\otimes \frac{1}{g(T)}} \xrightarrow[T \rightarrow \infty]{w} \{X(t) + B(\kappa t^{2\alpha}), t \geq 0\} \quad \text{on } D[0, \infty),$$

where $\{B(t), t \geq 0\}$ is a standard Brownian motion, independent of $\{X(t), t \geq 0\}$.

Letting the intensity process to be the ISDLG process $\{J_{\sigma^2, \vartheta}^{(H, \ell)}(t), t \geq 0\}$, we obtain as a corollary of the previous theorem, that for Cox process $\{U(t), t \geq 0\} \doteq \{N \triangleleft J_{\sigma^2, \vartheta}^{(H, \ell)}(t), t \geq 0\}$ we have

$$\left\{ \frac{1}{T} (U(Tt) - Tt\vartheta), t \geq 0 \right\}^{\otimes \frac{T^{2-2H}}{\ell(T)}} \xrightarrow[T \rightarrow \infty]{w} \{J_{\sigma^2, \vartheta}^{(H)}(t), t \geq 0\} \quad \text{on } D[0, \infty), \quad (12)$$

and the limit process is the LISDLG process. Next it is proven that process $\{U(t), t \geq 0\}$ is just the superposition of death counting processes of birth and death processes with immigration (SDC process). The superposition construction is similar to the ones taken place so far. So, (12) can be restated as follows, and this is the end of the dissertation:

Theorem. *The family of distributions of the renormalized centered versions of an SDC process $\{U(t), t \geq 0\}$ with parameters (H, ℓ) and $\nu, \lambda > 0$ (the rates of immigration and birth) converges weakly on $D[0, \infty)$ to the LISDLG process with parameters H and $(2\lambda(\lambda + 1), \nu(\lambda + 1))$:*

$$\frac{1}{T} \left\{ U(Tt) - \mathbb{E}U(Tt), t \geq 0 \right\}^{\otimes \frac{T^2 - 2H}{\ell(T)}} \xrightarrow[T \rightarrow \infty]{w} \left\{ J_{2\lambda(\lambda+1), \nu(\lambda+1)}^{(H)}(t), t \geq 0 \right\} \text{ on } D[0, \infty).$$

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