

Exact and fractional solution of MHD generalized Couette hybrid nanofluid flow with Mittag–Leffler and power law kernel

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ABSTRACT

This study investigates the complex behavior of Jeffrey nanofluid flow in a porous oscillating microchannel under the influence of magnetohydrodynamic (MHD) effects. The research explores how magnetic field distortions lead to diverse accumulation patterns of nanofluid particles, a phenomenon attributed to homogeneous magnetization in fluid dynamics. Nanoparticles ranging from 0.25 % to 0.5 % (low and inexpensive concentrations) are remarkably consistent for best results in most machining procedures. Different concentrations of various nanomaterials is utilized ($\varphi_1 = \varphi_2 = 0.01, 0.02, 0.03, 0.04$) to make the simple nanofluid and hybrid nanofluid suspensions. By employing fractal-fractional derivatives governed by power law, a mathematical model developed to describe the time-varying, compressible MHD flow of Jeffrey nanofluid. The model incorporates the effects of heat transfer, pressure, and magnetic fields on the fluid dynamics. A novel fractional approach utilizing the Laplace transform is applied to solve the fractal MHD hybrid-fluid model integrated into a porous medium. The study reveals velocity flow decrease with increasing Reynolds numbers but increase with channel inclination. Additionally, both the Darcy number and magnetic field orientation enhance heat transfer rates. In addition, the velocity profile enhanced by the hybrid nanofluid suspension as compared to simple nanofluid flow. The research validates its findings by demonstrating the convergence of fractional and numerical solution methods. Furthermore, the study compares the performance of different hybrid nanofluids, concluding that water-based ($H_2O + GO + MoS_2$) hybrid fluids exhibit slightly superior characteristics compared to (CMC + GO + MoS_2) hybrid nanofluids.

1. Introduction

Nanofluids are a recent development in fluid dynamics and materials research. They are made up of a dispersed combination of nanoparticles, either metallic or non-metallic, in a base fluid. The size of the nanoparticles used is generally between 1 and 100 nanometers. In 1995, Choi was among the first to demonstrate nanofluids with nanoparticles. Thermal conductivity is major of the many advantages that nanofluids may offer. The majority of nanofluid suspensions were developed by

Choi and Eastman [1] for use in mechanical design, fluid dynamics, and biomedical engineering applications. The authors of [2] investigated the feasibility of using nanofluids in various solar thermal systems. According to the researchers, nanofluids may be utilized to generate solar energy in thermal engineering, as well as solar energy devices in industries. This work investigates a viscous, incompressible nanofluid made up of various nanoparticles suspended in carboxymethyl cellulose (CMC), including graphene oxide and molybdenum disulfide. A "hybrid nanofluid" is created by combining several types of nanoparticles or adding additional chemicals to an existing nanofluid. Velasco et al. [3]

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Nomenclatures

Symbol Quantity Unit

 T_d Ambient temperature (K) U Fluid velocity (m/s) θ_1 The angle of magnetic inclination (–) U_0 Characteristic velocity (ms^{-1}) μ_{nf} Dynamic viscosity (Kg/ms) q Laplace transform variable (–) σ Electrical Conductivity (–) C_f Skin friction (–) α, β Fractional parameters (–) β_T Thermal expansion coefficient (1/k) Gr Heat Grashof number (–) B_o Strength of Magnetic field (Kg/s^2) Gm Mass Grashof number (–) M Magnetic field (–) ρ_{nf} Nano-fluid density (Kg/m^3) θ_2 Inclination of plates (–) g Gravity acceleration (m/s^2) Sh Sherwood number (–) C_p Specific heat at the constant pressure (J/kgK) t Times (s) k_{nf} Thermal Conductivity of the Nanofluid (W/mk) T_w Wall temperature (K)

discuss the hybrid nanofluid characteristics with various fractional and numerical solutions. Because hybrid nanofluids are a relatively new class of nanofluids, they require considerably more research before they can be employed in industrial applications. Their synergistic effect, which imparts favorable properties on all of its elements, suggests that they might be used in almost any heat transfer application. It combines several drugs to form a hybrid system with enhanced or tailored characteristics. For example, a hybrid nanofluid might contain both nonmetallic nanoparticles, such as carbon nanotubes or graphene oxide, and metallic nanoparticles, such as copper or silver. Domairry et al. [4] discovered a free-convective, non-Newtonian fluid with parallel vertical flat surfaces with hybrid nanofluid suspension. The differential modeling approach was utilized to turn the leading model into an ODE. Ikram et al. [5] used a CPC-fractional operator to create a fractional model of a Brinkman-type fluid transporting hybrid nanoparticles in a confined micro-channel.

Fractional calculus (FC) is the study of non-integer-order integral and derivative operations and their applications in realistic or complex domains. Fractional calculus (FC) extends integration and differentiation to non-integer orders. It broadens the scope of traditional calculus, focusing on differentiation and integration at integer levels. Kulish and Lage [6] explain using fractional calculus, often known as the calculus of arbitrary (noninteger) differentiation, to solve issues involving viscous diffusion in time-dependent fluid mechanics. Nadeem et al. [7] investigated how often uses of fractional calculus might emerge in several settings. Sunthrayuth et al. [8] employ the fractional derivative to investigate the outcome of fractional Casson fluid movement via a permeable channel. Using the CF-fractional derivative, they explored the effect of porosity on the limits of the oscillating channel. Yilmaz et al. [9] carried out an experiment comparing non-Newtonian (distilled water) and Newtonian flows through porous material. Amir et al. [10] studied the entropy generation and convection of water based hybrid nanofluid flowing through a poured channel. Ramesh et al. [11] investigated the MHD fluid using many numerical approaches. Srayyih et al. [12] investigated the fluid oscillatory motion flowing through an elliptic surface with Cu and Al_2O_3 based hybrid nanofluid and water as a base fluid. Ahmed et al. [13] deliberated the natural convection of non-Newtonian fluid employing the finite difference method. Dulaimi et al. [14] examined the Ag – MgO based hybrid nanofluid natural convection of the flowing fluid through poured plates. They have utilized the residual finite element for the solution of the governed model.

The analytical answers for the CFD-described fractional fluid computations. The variations of the fractional mobility issue with the reaction term and fractional heat solution are investigated using the Caputo derivative sequence. Sene in [15] investigated the fractional free convection flow of a fluid of the Brinkman type, which is a particular fluid model used in this work. The present research are primarily concerned with providing exact analytical answers using the Laplace transformation and its inverse approach. Kausel et al. [16] employed an

updated Bessel equation and the Laplace transformation to examine the shear stress and velocity field connected to the oscillating movement of a fractional Burgers' fluid model in a continuously circular cylinder. The fluid first remains stationary before rapidly starts to travel down the cylinder's axis owing to shear. The derived results meet the beginning and boundary conditions and are presented as modified Bessel fields. Chillingo et al. [17] investigated the unstable and incompressible fluid flow using CPC and Caputo type fractional derivatives. Porosity and MHD are also considered as additional repercussions. Dimensional evaluation of the signal's leading equations into a non-dimensional form is subsequently extended using a new fractional operator with a single kernel to generate series explanations. Rehman et al. [18] looked at the unsteady (MHD) flows of a Prabhakar-like noninteger Maxwell fluid near an endlessly perpendicular plate with constant concentration. The effects of Newtonian heat transfer and the wall sliding condition on temperature are carefully examined. Rehman et al. [19] explored how thermal energy affected Prabhakar-like non-integer second-grade fluid disturbances MHD motions along a constantly increased vertical surface with exponentially varying velocity diffusion via porous medium. To account for generalized memory effects, a novel computational fractional model is developed using the recently found Prabhakar fractional operator and generalized Fourier's and Fick's laws. Gul et al. [20] investigated a viscous, unstable Williamson fluid that moved on an oscillating infinite inclined plate.

The use of newly designed fractal fractional operators with power-law kernels in water dynamics is demonstrated in [21]. MHD viscous fluid passes across two plates, with the upper plate continually moving while the lower plate remains fixed. In [22], the authors observed that non-Newtonian fluid flow is appropriate in commercial and manufacturing activities. The basic goal of this study is to create and improve a Jeffery flow model in a channel with two parallel and perpendicular plates. Unlike prior fractional derivative techniques, the Prabhakar operators used the three elements of the Mittag-Leffler projection as its kernel, making it more broad and sophisticated. Rehman et al. [23] investigated second-grade MHD fluid's irregular and insoluble flow across a vertical vibrating plate while passing under an inclined magnetic field. The Prabhakar fractional derivative is a sophisticated and practical fractional formulation frequently based on enhanced Fourier's and Fick's laws. It is used to transform non-dimensional controlled solutions into fractional models. Rasool et al. [24] looked at a Casson-type fluid that is volatile and incompressible, traveling on a sloping, rotating poured plane with an increasing temperature profile. Turq et al. [25] evaluated whether this fractional approach takes into account blood base fluid while formulating quantitative predictions for the hybrid nanofluid problem with various nanoparticles. When assessing the thermal growth of the hybrid nanofluid model, aluminum oxide (Al_2O_3), titanium oxide (TiO_2), and copper (Cu) are all considered. While Chu et al. [26] examined the use of Newtonian heating, our research will focus on heat conduction in hybrid

nanofluids across an angled pours plate. Akgul et al. [27] investigated the fluid dynamics applications of newly discovered nonlocal differential equations with fractional order and fractal character. Murtaza et al. [28] discovered fractal fractional, which was just introduced. Abro and Atangana [29] sought to regulate the analysis and coexisting attractants. The differential equations regulating memristors are highly nonlinear regarding mathematical connections. Imran mentioned in [30] that he has recently created planned fractal fractional controllers with power law. He examined MHD fluid flow between two plates with a constant speed upper plate movement and a stationary lower plate. Raza et al. [31] investigated how the Fractal Fractional derivative was employed in this study to control the flow of non-Newtonian fluids. Abro et al. [32] proposed a fractal-fractional model of convective fluid motion in a revolving hollow with non-uniform external heating. Arif et al. [33] investigated a unique fractional derivative type with a power Law kernel known as the fractal-fractional derivative, which has several practical applications.

To the best of the author's knowledge, no prior research has been conducted to examine the unsteady distribution of hybrid nanofluids in relation to pressure gradient under the influence of a magnetic field and orientation. This study explores the fundamental and modified Couette flows of a MHD dynamic Jeffrey-type nanofluid through inclined narrow channels. Two different types of nanomaterial's (GO , MoS_2) with two different base fluids (H_2O, CMC) for the hybrid nanofluid suspensions. The analysis employs the fractal non-integer order derivative, which represents the most recent and updated definition of fractional derivatives. The study also addresses the nondimensional partial differential equation (PDE) problem using an integral transformation, including the Laplace transform. Numerical methods such as the Stehfest and Tzou algorithms are utilized to obtain solutions for the velocity profiles. For data analysis in perpendicular channels, the angle is set to $\theta_2 = \pi/2$, while for straight channels, it is set to 0. Finally, graphical illustrations are examined for various parameters included in the study. The validity of this study can be accomplished by comparing the attained results with the previous published work.

2. Formulation of governed model

This paper introduces a Jeffery-type fluid through free convection that travels between two heated side surfaces arranged in the xy -plane with unique equidistant $h = 1$ separated in a rectangular coordinate structure. The circulating hybrid nanofluid consists of ($MOS_2 + GO$) nanoparticles combined with base fluids (H_2O, CMC). This flow moves past an inclined medium of consistent density while a gradient magnetic field is established. At constant temperature T_0 , the whole system as a whole fluid and its restrictions—remain in its innovative steadiness. When a period-dependent shear strength is exerted, the top plate ($y = h$) starts to interchange in its direction at time $t > 0$. The pressure incline, heating field enhancement, and vibration cause the stationary hybrid nanofluid to move within the inclined micro-channel. The subsequent factors are proposed to aid in the formulation of controlled PDEs.

- The microchannel is assumed to have an infinite length. It has a constant width denoted as h , allowing for a uniform cross-sectional area throughout the channel.
- The microchannel undergoes oscillatory motion along the x -axis, perpendicular to the y -axis. This oscillation introduces time-dependent effects on the fluid flow and heat transfer processes within the channel.
- At the initial time ($t = 0$), the entire system is at a uniform temperature T_0 , representing the baseline thermal condition before the flow begins.
- With the passage of time ($t > 0$), a heat source increases the temperature from T_0 to T_w . This temperature change the heat transfer process and influences the fluid movement within the microchannel.

- The combination of temperature gradients and pressure differences induces fluid flow with an increased velocity in the x -direction. This flow is a result of both thermal and pressure-driven effects.
- An inclined magnetic field with strength B_0 is applied at an angle θ_2 to the flow direction. This magnetic field acts as a resistive force against the fluid motion, influencing the flow characteristics and heat transfer properties.

This study have also some limitations as stated below.

- Assuming an infinitely long microchannel may not adequately reflect practical uses with finite channel widths.
- The model only takes into account one-dimensional fluctuation, which may fail to describe more complicated multi-directional oscillations in actual circumstances.
- The abrupt temperature rise from T_0 to T_w may not accurately depict gradual or non-uniform overheating in real-world systems.
- The investigation may have overlooked issues such as rough exteriors, flow-structure conversations, or non-Newtonian fluid movement.
- The model presupposes an uninterrupted magnetic field, which may not be applicable in scenarios involving time-varying electromagnetic energies.
- The investigation may not have taken into account temperature-dependent fluid characteristics, which might impair the preciseness of results at bigger variations in temperature.

The mathematical formulation of the physical model can be execute as [11]

$$\nabla \cdot \overline{\mathcal{F}} = 0 \quad (1)$$

$$\rho \left(\frac{\partial \overline{\mathcal{F}}}{\partial \mathfrak{T}} + \overline{\mathcal{F}} \cdot \nabla \overline{\mathcal{F}} \right) = -\nabla p + \nabla \cdot \overline{\mathcal{E}} + \overline{\mathfrak{R}} + \overline{\mathcal{F}} \times \overline{\mathcal{B}} \quad (2)$$

Where the extra stress tensor can be identified as for the Jeffery type hybrid fluid.

$$\overline{\mathcal{E}} = \frac{\mu}{1 + \lambda} (\dot{\mathfrak{r}} + \lambda_0 \ddot{\mathfrak{r}})$$

Where μ , λ_0 , λ , $\ddot{\mathfrak{r}}$, and $\dot{\mathfrak{r}}$ are the viscosity, retardation, relaxation time, material derivative, and deformation tensor, respectively with their mathematical representations as

$$\dot{\mathfrak{r}} = \nabla \cdot \overline{\mathcal{F}} + (\nabla \cdot \overline{\mathcal{F}})^T, \quad \ddot{\mathfrak{r}} = \frac{d\dot{\mathfrak{r}}}{dt} = \frac{\partial \dot{\mathfrak{r}}}{\partial t} + (\overline{\mathcal{F}} \cdot \nabla) \dot{\mathfrak{r}}$$

Further the Darcy resistance and current vector for poured mediocre can be distinct as

$$\overline{\mathfrak{R}} = -\frac{\mu}{k} \overline{\mathcal{F}}, \quad \overline{\mathcal{F}} = \sigma (\overline{E} + \overline{q} \times \overline{\mathcal{B}})$$

As due to low magnetic Reynolds number, the induced magnetic strength is ignored. So the magnetic vector will become as $\overline{\mathcal{B}} = (0, B_0, 0)$. So Darcy's law will become as

$$\overline{\mathcal{F}} \times \overline{\mathcal{B}} = (-\sigma B_0^2 U, 0, 0)$$

So the governed PDE with the above supposition can be [34,35]

$$\begin{aligned} \rho_{hmf} \frac{\partial U(\mathcal{Y}, t)}{\partial t} = & -\frac{\partial \mathcal{P}}{\partial x} + \frac{\mu_{hmf}}{1 + \Phi_1} \frac{\partial^2 U(\mathcal{Y}, t)}{\partial \mathcal{Y}^2} + g \rho_{hmf} \sin(\theta_2) \\ & - \sigma_{hmf} B_0^2 \sin(\theta_1) U(\mathcal{Y}, t) - \frac{\mu_{hmf}}{k} U(\mathcal{Y}, t) \end{aligned} \quad (3)$$

Where $\frac{\partial \mathcal{P}}{\partial x}$ signifies the pressure ascent.

Tables 1 and 2 provide a list of the base material, the proposed nanofluid model, and the thermal properties of solid nanoparticles.

Exploring the combined impacts of MHD and Couette hybrid

Table 1
The thermos-physical properties of basic measures.

Thermal features	Hybrid nanofluid
Density	$\rho_f = \frac{\rho_{hnf}}{(1 - \varphi_2) \left((1 - \varphi_1) + \varphi_1 \frac{\rho_{s1}}{\rho_f} \right) + \varphi_2 \rho_{s2}}$
Dynamic viscosity	$\mu_f = \mu_{hnf} (1 - \varphi_1)^{2.5} (1 - \varphi_2)^{2.5}$
Electrical conductivity	$\sigma_{bf} = \frac{\sigma_{hnf}}{\left(1 + \frac{3\varphi(\varphi_1\sigma_1 + \varphi_2\sigma_2 - \sigma_{bf}(\varphi_1 + \varphi_2))}{(\varphi_1\sigma_1 + \varphi_2\sigma_2 + 2\varphi\sigma_{bf} - \varphi\sigma_{bf}(\varphi_1\sigma_1 + \varphi_2\sigma_2 - \sigma_{bf}(\varphi_1 + \varphi_2)))} \right)}$
Thermal conductivity	$k_{bf} = \frac{k_{hnf}}{\left(\frac{k_{s2} + (n-1)k_{bf} - (n-1)(k_{bf} - k_{s2})\varphi_2}{k_{s2} + (n-1)k_{bf} + (k_{bf} - k_{s2})\varphi_2} \right)}$ and $k_f = \frac{k_{bf}}{\left(\frac{k_{s1} + (n-1)k_f - (n-1)(k_f - k_{s1})\varphi_1}{k_{s1} + (n-1)k_f + (k_f - k_{s1})\varphi_1} \right)}$
Heat capacitance	$(\rho C_p)_s = \frac{(\rho C_p)_{hnf}}{(1 - \varphi_2) \left((1 - \varphi_1) + \varphi_1 \frac{(\rho C_p)_{s1}}{(\rho C_p)_f} \right) + \varphi_2 (\rho C_p)_{s2}}$
Thermal expansion coefficient	$(\rho\beta)_f = \frac{(\rho\beta)_{hnf}}{(1 - \varphi_2) \left((1 - \varphi_1) + \varphi_1 \frac{(\rho\beta)_{s1}}{(\rho\beta)_{f1}} \right) + \varphi_2 (\rho\beta)_{s2}}$

Table 2
Considered nanofluids with their thermal characteristics [36].

Material	CMC	H ₂ O	MOS ₂	GO
$\rho(\text{kg/m}^3)$	997	997.1	5060	1800
$C_p(\text{J/kgK})$	4179	4179	397.20	717
$k(\text{W/mK})$	0.613	0.613	904.4	5000
$\beta_T(\text{K}^{-1})$	20×10^{-5}	21×10^{-5}	2.842×10^{-5}	2.84×10^{-4}

nanofluid flow can help improve heat transfer effectiveness and regulate fluid dynamics in technologically sophisticated fields. The existence of a magnetic field in MHD substantially impacts the flow pattern and thermal characteristics of the hybrid nanofluid, which consists of nanoparticles from several materials floating in a base fluid. The combination of these factors can increase thermal conductivity and viscosity, resulting in more efficient cooling or heating operations in a variety of industrial systems, including cooling in electronic devices, medicinal purposes, and electrical systems. Furthermore, the hybrid nanofluid's capacity to change flow properties in response to a magnetic field enables more accurate flow regulation in limited patterns, such as microfluidic equipment. In general, this field of study has the possibility to improve the development and execution of systems requiring controlling heat and fluid oversight, opening the door to more sustainable and regulated operations.

Fractal Integral: The integral definition of the Fractal fractional derivative in the sense of RL-definition for a continuous function $y(x)$ can be defined as [37]:

$${}^{FFP}\mathcal{D}_x^{\alpha, \beta} y(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx^\beta} \int_0^x y(t)(x-t)^{-\alpha} dt, \quad 0 < \alpha, \beta \leq 1$$

$$\frac{dy(t)}{dx^\beta} = \lim_{x \rightarrow t} \frac{y(x) - y(t)}{x^\beta - t^\beta}$$

The FFD is an application of mathematics that integrates fractional calculus with the geometry of fractals to describe the dynamics of phenomena having both fractional-order characteristics and fractal-like features. This derivative is especially useful for simulating operations that take place in fractal media or exhibit memory and hereditary features.

Laplace Transformation: Consider $y(x)$ for instance where $x > 0$. The Laplace integral offers a well-defined continuous function called the LT of $y(x)$ as $Y(s)$ or $\mathcal{L}\{y(x)\}$

$$\mathcal{L}\{y(x)\} = Y(x) = \int_0^\infty \exp(-sx)f(x)dx$$

Fractal Laplace Transformation: The fractal-Laplace of a continuous function $g(x)$ can be defined as [38]

$${}^F\mathcal{L}_p^\alpha\{g(x)\} = G(x) = \int_0^\infty \exp(-sx)x^{\alpha-1}g(x)dx$$

3. Solution of the fractional PDEs

In this portion, the well-ordered PDE solution is concluded with the presentation of a recent fractional derivative divination, specifically the Fractal fractional derivatives. Additionally, the Laplace transformation integral transform methodology is utilized for all simple, Poiseuille and generalized Couette flow solutions subsequent the non-dimensionalization procedures.

The mathematical implications and the flowchart illustration of fractional derivatives and the Laplace transform are presented in Fig. 1.

3.1. Solution of fundamental flow

In an inclined micro-channel, we studied the unstable laminar distribution of an inflexible MHD fractional Jeffrey fluid passing through two infinite comparable plates separated by h . The flow scenario is expected to be via a homogeneous porous medium. The lower plate is also assumed to be immobile, while the top plate travels at a constant velocity U_o with no pressure gradient. The motion of the top plate induces fluid flow in the optimistic horizontal plane (see Fig. 2).

With such presumptions, the formula that governs may be represented as without pressure.

$$\begin{aligned} \frac{\partial U(\mathcal{Y}, t)}{\partial t} &= \frac{\mu_{hnf}}{\rho_{hnf}} \frac{1}{1 + \Phi_1} \left(\frac{\partial^2 U(\mathcal{Y}, t)}{\partial \mathcal{Y}^2} \right) + g \sin(\theta_2) - \frac{\sigma_{hnf}}{\rho_{hnf}} B_0^2 \sin(\theta_1) U(\mathcal{Y}, t) \\ &\quad - \frac{\mu_{hnf}}{\rho_{hnf}} \frac{1}{k} U(\mathcal{Y}, t) \end{aligned} \quad (4)$$

With corresponding conditions

$$U(\mathcal{Y}, 0) = 0; \quad U(h, t) = U_o R(t); \quad U(0, t) = 0$$

Presenting the non-dimensional quantities

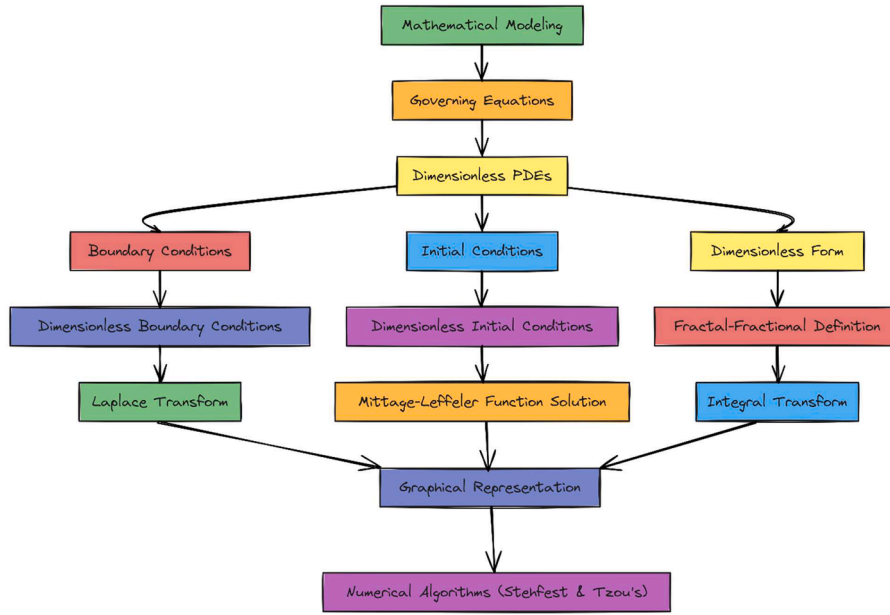


Fig. 1. Flow chart of the solution.

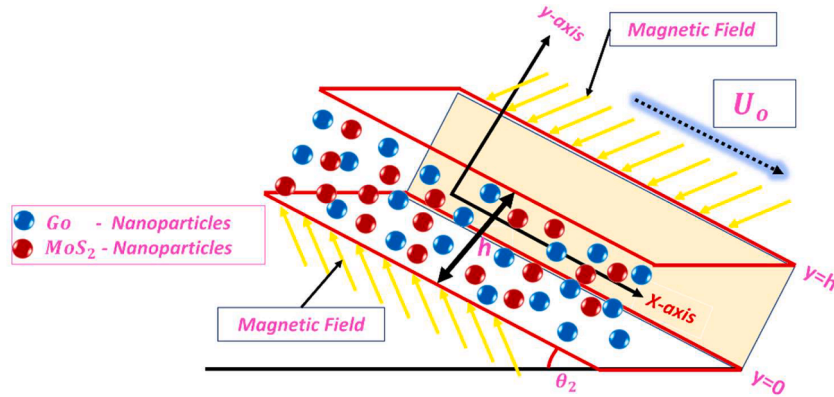


Fig. 2. Geometry of simple Couette flow.

$$\overline{y} = \frac{y}{l}, \quad \overline{U} = \frac{U}{U_0}, \quad D_a = \frac{k}{l^2}, \quad R_e = \frac{\rho U_0 l}{\mu}, \quad \bar{t} = \frac{U_0 t}{l}, \quad \bar{h} = \frac{h}{l},$$

$$Fr = \frac{U^2}{gl}, \quad \mathcal{M} = \sqrt{\frac{\sigma}{\mu}} \beta_0 l$$

Using above non-dimensional quantities

$$Re \frac{\partial U(\mathcal{Y}, t)}{\partial t} = \frac{1}{1 + \Phi_1} \frac{\partial^2 U(\mathcal{Y}, t)}{\partial \mathcal{Y}^2} - \frac{1}{D_a} U(\mathcal{Y}, t) - \mathcal{M}^2 \sin(\theta_1) U(\mathcal{Y}, t) + \frac{Re}{Fr} \sin(\theta_2) \quad (5)$$

With it corresponding dimensionless conditions

$$U(0, t) = 0, \quad \forall t; \quad U(h, t) = R(t), \quad t > 0; \quad U(\mathcal{Y}, 0) = 0; \quad 0 \leq \mathcal{Y} < h$$

Under the given non-dimensional governing conditions, the system initially remains at rest with no fluid flow at time $t = 0$. As time progresses, the upper plate begins to oscillate at a constant velocity, denoted by $R(t)$, while the lower plate remains stationary. These oscillations, combined with temperature variations, induce movement in the initially static fluid, causing it to flow at the same velocity as the oscillating plate. Employing the fractal definition on the assessed PDE and conditions, we yield

$$R_e^{FFD} \mathfrak{D}_t^{\alpha, \beta} U(\mathcal{Y}, t) = \beta t^{\beta-1} \left[\frac{1}{1 + \Phi_1} \frac{\partial^2 U(\mathcal{Y}, t)}{\partial \mathcal{Y}^2} - \frac{1}{D_a} U(\mathcal{Y}, t) - \mathcal{M}^2 \sin(\theta_1) U(\mathcal{Y}, t) + \frac{Re}{Fr} \sin(\theta_2) \right] - \frac{U(\mathcal{Y}, 0)}{\Gamma(1 - \alpha)} t^{-\alpha} \quad (6)$$

It is a more facilitating way to utilize the LT on the above-governed PDE of simple Couette flow. By retaining the LT on the PDEs with the consistent Laplace characterization of FFD and its consistent conditions, we get the solution of a simple Couette flow momentum field such that

$$\begin{aligned} \overline{U}(\mathcal{Y}, q) = & \frac{a_3}{a_5 + a_6 q^{\alpha+\beta}} \left(1 - \frac{\sinh(\mathcal{Y} \sqrt{a_8 + a_9 q^{\alpha+\beta}})}{\sinh(\sqrt{a_8 + a_9 q^{\alpha+\beta}})} \right) \\ & + r(q) \frac{\sinh(\mathcal{Y} \sqrt{a_7(a_5 + a_6 q^{\alpha+\beta})})}{\sinh(\sqrt{a_7(a_5 + a_6 q^{\alpha+\beta})})} \end{aligned} \quad (7)$$

where:

$$a_1 = \frac{Re}{Fr}, \quad a_2 = \sin(\theta_2), \quad a_3 = a_1 a_2, \quad a_4 = \mathfrak{M}^2 \sin(\theta_1), \quad a_5 = a_4 + \frac{1}{\mathcal{D}_a},$$

$$a_6 = \frac{Re}{\beta\Gamma(\beta)}, \quad a_7 = \frac{1}{1 + \Phi_1}, \quad a_8 = a_7 a_5, \quad a_9 = a_7 a_6$$

Eq. (7) further can be simplified in its relevant exponential form as follows

$$\begin{aligned} \bar{U}(\mathcal{Y}, q) = & \frac{a_3}{a_5 + a_6 q^{\alpha+\beta}} \sum_{m=0}^{\infty} \left(1 - e^{-(2m+1-\mathcal{Y})(a_8 + a_9 q^{\alpha+\beta})^{\frac{1}{2}}} \right. \\ & + e^{-(2m+1+\mathcal{Y})(a_8 + a_9 q^{\alpha+\beta})^{\frac{1}{2}}} \Big) + r(q) \sum_{m=0}^{\infty} \left(e^{-(2m+1-\mathcal{Y})\sqrt{a_7(a_5 + a_6 q^{\alpha+\beta})}} \right. \\ & \left. - e^{-(2m+1+\mathcal{Y})\sqrt{a_7(a_5 + a_6 q^{\alpha+\beta})}} \right) \end{aligned}$$

Writing the above expression in its relevant series form and simplifying, we get

$$\begin{aligned} \frac{\partial U(\mathcal{Y}, t)}{\partial t} = & \mathfrak{S} + \frac{1}{1 + \Phi_1} \frac{\partial^2 U(\mathcal{Y}, t)}{\partial \mathcal{Y}^2} - \frac{1}{D_a} U(\mathcal{Y}, t) - \mathcal{M}^2 \sin(\theta_1) U(\mathcal{Y}, t) \\ & + \frac{Re}{Fr} \sin(\theta_2) \end{aligned} \quad (9)$$

With

$$U(0, t) = 0, \quad U(h, t) = K(t), \quad U(\mathcal{Y}, 0) = 0$$

According to above non-dimensional governing conditions, the system initially remains at rest with no fluid flow at time $t = 0$. As time progresses, the upper plate begins to oscillate at a constant velocity, denoted by $R(t)$, while the lower plate remains stationary. These oscillations, combined with temperature variations, and due to the pressure

$$\begin{aligned} \bar{U}(\mathcal{Y}, q) = & \sum_{n=0}^{\infty} \frac{a_3}{a_5} (1 - a_{10} q^{\alpha+\beta}) + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \left(1 - \frac{(\mathcal{Y} - 2m - 1)^n \Gamma(\frac{n}{2} + p) q^{\frac{n}{2}(\alpha+\beta)}}{n! \Gamma(\frac{n}{2}) (a_9)^{\frac{n}{2}}} + \frac{(-2m - 1 - \mathcal{Y})^n (a_8)^{\frac{n}{2}} \Gamma(\frac{n}{2} + p) q^{\frac{n}{2}(\alpha+\beta)}}{n! \Gamma(\frac{n}{2}) (a_9)^{\frac{n}{2}}} \right) + r(q) + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \\ & \times \sum_{p=0}^{\infty} \left(\frac{(\mathcal{Y} - 1 - 2m)^n (a_5 a_7)^{\frac{n}{2}} \Gamma(\frac{n}{2} + p) q^{\frac{n}{2}(\alpha+\beta)}}{n! \Gamma(\frac{n}{2}) (a_9)^{\frac{n}{2}}} - \frac{(-\mathcal{Y} - 2m - 1)^n (a_8 a_5 a_7)^{\frac{n}{2}} \Gamma(\frac{n}{2} + p) q^{\frac{n}{2}(\alpha+\beta)}}{(a_9)^{\frac{n}{2}} n! \Gamma(\frac{n}{2})} \right) \end{aligned}$$

Now by employing the Laplace inverse

gradient the static fluid starts to flow with the same velocity of the oscillations. Again after employing the Fractal mathematical description,

$$\begin{aligned} U(\mathcal{Y}, t) = & R(t) + \sum_{n=0}^{\infty} \frac{a_3}{a_5} \left(\delta(1) - \frac{a_{10}}{\Gamma(\alpha+\beta)} \frac{1}{t^{1+\alpha+\beta}} \right) + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \left(\delta(1) - \frac{(\mathcal{Y} - 2m - 1)^n \Gamma(\frac{n}{2} + p)}{n! \Gamma(\frac{n}{2}) (a_9)^{\frac{n}{2}} \Gamma(\frac{n}{2}(\alpha+\beta))} t^{1+\frac{n}{2}(\alpha+\beta)} + \frac{(-2m - 1 - \mathcal{Y})^n (a_8)^{\frac{n}{2}} \Gamma(\frac{n}{2} + p)}{n! \Gamma(\frac{n}{2}) (a_9)^{\frac{n}{2}} \Gamma(\frac{n}{2}(\alpha+\beta))} t^{1+\frac{n}{2}(\alpha+\beta)} \right) \\ & + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \left(\frac{(\mathcal{Y} - 1 - 2m)^n (a_5 a_7)^{\frac{n}{2}} \Gamma(\frac{n}{2} + p)}{n! \Gamma(\frac{n}{2}) (a_9)^{\frac{n}{2}} \Gamma(\frac{n}{2}(\alpha+\beta))} t^{1+\frac{n}{2}(\alpha+\beta)} - \frac{(-\mathcal{Y} - 2m - 1)^n (a_8 a_5 a_7)^{\frac{n}{2}} \Gamma(\frac{n}{2} + p) q^{\frac{n}{2}(\alpha+\beta)}}{(a_9)^{\frac{n}{2}} n! \Gamma(\frac{n}{2}) \Gamma(\frac{n}{2}(\alpha+\beta))} t^{1+\frac{n}{2}(\alpha+\beta)} \right) \end{aligned}$$

3.2. Plane poiseuille flow

We address the unsteady laminar motion of an inflexible MHD Jeffrey fluid through two infinitely persuaded plates separated by a porous material of length h . Assuming both plates are at equilibrium, the flow is caused by the continuous pressure gradient $\mathfrak{S} = -\frac{\partial \mathcal{P}}{\partial x}$ in the x -direction (see Fig. 3). Assuming these conditions, the equation that governs it is presented by

$$\begin{aligned} \rho_{hmf} \frac{\partial U(\mathcal{Y}, t)}{\partial t} = & \mathfrak{S} + \frac{\mu_{hmf}}{1 + \Phi_1} \frac{\partial^2 U(\mathcal{Y}, t)}{\partial \mathcal{Y}^2} + g \rho_{hmf} \sin(\theta_2) \\ & - \sigma_{hmf} B_0^2 \sin(\theta_1) U(\mathcal{Y}, t) - \frac{\mu_{hmf}}{k} U(\mathcal{Y}, t) \end{aligned} \quad (8)$$

With corresponding conditions

$$U(\mathcal{Y}, 0) = 0; \quad U(h, t) = U_o K(t); \quad U(0, t) = 0$$

Again utilizing the above non-dimensional constraints of simple Couette flow, we gain

we get

$$\begin{aligned} {}^{FFD} \mathfrak{S}_t^{\alpha, \beta} U(\mathcal{Y}, t) = & \beta t^{\beta-1} \left[\mathfrak{S} + \frac{1}{1 + \Phi_1} \frac{\partial^2 U(\mathcal{Y}, t)}{\partial \mathcal{Y}^2} - \frac{1}{D_a} U(\mathcal{Y}, t) \right. \\ & \left. - \mathcal{M}^2 \sin(\theta_1) U(\mathcal{Y}, t) + \frac{Re}{Fr} \sin(\theta_2) \right] - \frac{U(\mathcal{Y}, 0)}{\Gamma(1-\alpha)} t^{-\alpha} \end{aligned} \quad (10)$$

Employing the LT on the Eq. (10) and using prospective conditions we get the solution

$$\begin{aligned} \bar{U}(\mathcal{Y}, q) = & \frac{(\mathfrak{S} + 1)}{a_5 + \frac{1}{\beta\Gamma(\beta)} q^{\alpha+\beta}} \left(1 - \frac{\sinh\left(\mathcal{Y} \sqrt{a_7 \left(a_5 + \frac{1}{\beta\Gamma(\beta)} q^{\alpha+\beta}\right)}\right)}{\sinh\sqrt{a_7 \left(a_5 + \frac{1}{\beta\Gamma(\beta)} q^{\alpha+\beta}\right)}} \right) \\ & + k(q) \frac{\sinh\left(\mathcal{Y} \sqrt{a_8 + \frac{a_7}{\beta\Gamma(\beta)} q^{\alpha+\beta}}\right)}{\sinh\left(\sqrt{a_8 + \frac{a_7}{\beta\Gamma(\beta)} q^{\alpha+\beta}}\right)} \end{aligned} \quad (11)$$

where:

$$a_1 = \frac{Re}{Fr}, \quad a_2 = \sin(\theta_2), \quad a_3 = a_1 a_2, \quad a_4 = \mathcal{M}^2 \sin(\theta_1), \quad a_5 = a_4 + \frac{1}{\mathcal{J}_a}$$

$$a_6 = \frac{1}{\beta \Gamma(\beta)}, \quad a_7 = \frac{1}{1 + \Phi_1}, \quad a_8 = a_7 a_5, \quad a_9 = a_7 a_6$$

Eq. (11) further can be simplified in its relevant exponential form as follows

$$\bar{U}(\mathcal{Y}, q) = (\mathfrak{S} + 1) \bar{Q}_1(\mathcal{Y}, q) (1 - \bar{Q}_2(\mathcal{Y}, q) + \bar{Q}_3(\mathcal{Y}, q)) + \mathfrak{F}(q) (\bar{Q}_4(\mathcal{Y}, q) - \bar{Q}_5(\mathcal{Y}, q))$$

with:

$$\bar{Q}_1(\mathcal{Y}, q) = \frac{\beta \Gamma(\beta)}{\beta \Gamma(\beta) a_5 + q^{\alpha+\beta}}$$

$$\bar{Q}_2(\mathcal{Y}, q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{(\mathcal{Y} - 2m - 1)^n (a_7)^{\frac{n}{2}} + m (a_5)^{\frac{n}{2}} \Gamma\left(\frac{n}{2} + p\right) q^{\frac{p}{2}(\alpha+\beta)}}{n! (\beta)^{\frac{m}{2}} \Gamma(\beta) \Gamma\left(\frac{n}{2}\right)} \right)$$

$$\bar{Q}_3(\mathcal{Y}, q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \left(\frac{(-\mathcal{Y} - 2m - 1)^n (a_7)^{\frac{n}{2}} + m \Gamma\left(\frac{n}{2} + p\right) q^{\frac{\alpha+\beta}{2}}}{n! (\beta)^{\frac{m}{2}} \Gamma(\beta) \Gamma\left(\frac{n}{2}\right) (a_9)^{\frac{n}{2}}} \right)$$

$$\bar{Q}_4(\mathcal{Y}, q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \left(\frac{(\mathcal{Y} - 1 - 2m)^n (a_7)^{\frac{m}{2}} + p \Gamma\left(\frac{n}{2} + p\right) q^{\frac{p}{2}(\alpha+\beta)}}{n! \Gamma(\beta) \Gamma\left(\frac{n}{2}\right) (a_8)^{\frac{m}{2}}} \right)$$

$$\bar{Q}_5(\mathcal{Y}, q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \left(\frac{(-\mathcal{Y} - 2m - 1)^n (a_8 a_7)^{\frac{n}{2}} \Gamma\left(\frac{n}{2} + p\right) q^{\frac{\alpha+\beta}{2}}}{n! \Gamma(\beta) \Gamma\left(\frac{n}{2}\right)} \right)$$

Now by employing the Laplace inverse

$$\mathfrak{U}(\mathcal{Y}, t) = (\mathfrak{S} + 1) Q_1(\mathcal{Y}, t) * (1 - Q_2(\mathcal{Y}, t) + Q_3(\mathcal{Y}, t)) + k(q) * (Q_4(\mathcal{Y}, t) - Q_5(\mathcal{Y}, t))$$

Where the symbol " * " signifying the convolution of Laplace inverse with

$$Q_1(\mathcal{Y}, t) = \sum_{n=0}^{\infty} \beta \Gamma(\beta) a_5 \left(\frac{1}{\Gamma(\alpha + \beta)} \frac{1}{t^{1+\alpha+\beta}} \right)$$

$$Q_2(\mathcal{Y}, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \left(\frac{(\mathcal{Y} - 2m - 1)^n (a_7)^{\frac{n}{2}} + m (a_5)^{\frac{n}{2}} \Gamma\left(\frac{n}{2} + p\right)}{n! (\beta)^{\frac{m}{2}} \Gamma(\beta) \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{p}{2}(\alpha + \beta)\right) t^{1+\frac{p}{2}(\alpha + \beta)}} \right)$$

$$Q_3(\mathcal{Y}, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{(-\mathcal{Y} - 2m - 1)^n (a_7)^{\frac{n}{2}} + m \Gamma\left(\frac{n}{2} + p\right)}{n! (\beta)^{\frac{m}{2}} \Gamma(\beta) \Gamma\left(\frac{n}{2}\right) (a_9)^{\frac{n}{2}} \Gamma\left(\frac{1}{2}(\alpha + \beta)\right) t^{1+\frac{1}{2}(\alpha + \beta)}} \right)$$

$$Q_4(\mathcal{Y}, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \left(\frac{(\mathcal{Y} - 1 - 2m)^n (a_7)^{\frac{m}{2}} + p \Gamma\left(\frac{n}{2} + p\right)}{n! \Gamma(\beta) \Gamma\left(\frac{n}{2}\right) (a_8)^{\frac{m}{2}} \Gamma\left(\frac{p}{2}(\alpha + \beta)\right) t^{1+\frac{p}{2}(\alpha + \beta)}} \right)$$

$$Q_5(\mathcal{Y}, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \left(\frac{(-\mathcal{Y} - 2m - 1)^n (a_8 a_7)^{\frac{n}{2}} \Gamma\left(\frac{n}{2} + p\right)}{n! \Gamma(\beta) \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{1}{2}(\alpha + \beta)\right) t^{1+\frac{1}{2}(\alpha + \beta)}} \right)$$

3.3. Generalized couette flow

In this scenario, we look at the unstable laminar motion of an inflexible MHD Jeffrey fluid through two unbounded straight comparable surfaces spaced h in a disposed micro-channel. The flow condition is analyzed across a homogenous porous media. The condition is analogous to the Couette flow. It is supposed that the bottom platter is at rest while the top plate moves at a continuous speed U_0 . So due to oscillation and pressure gradient $\mathfrak{S} = -\frac{\partial \mathcal{P}}{\partial x}$ the fluid flows in the x-direction as shown Fig. 4.

The motion of the fluid is in the constructive x-direction, which is related to

$$\begin{aligned} \frac{\partial U(\mathcal{Y}, t)}{\partial t} = & \mathfrak{S} + \frac{\mu_{hnf}}{\rho_{hnf}} \frac{1}{1 + \Phi_1} \left(\frac{\partial^2 U(\mathcal{Y}, t)}{\partial \mathcal{Y}^2} \right) + g \sin(\theta_2) \\ & - \frac{\sigma_{hnf}}{\rho_{hnf}} B_0^2 \sin(\theta_1) U(\mathcal{Y}, t) - \frac{\mu_{hnf}}{\rho_{hnf} k} U(\mathcal{Y}, t) \end{aligned} \quad (12)$$

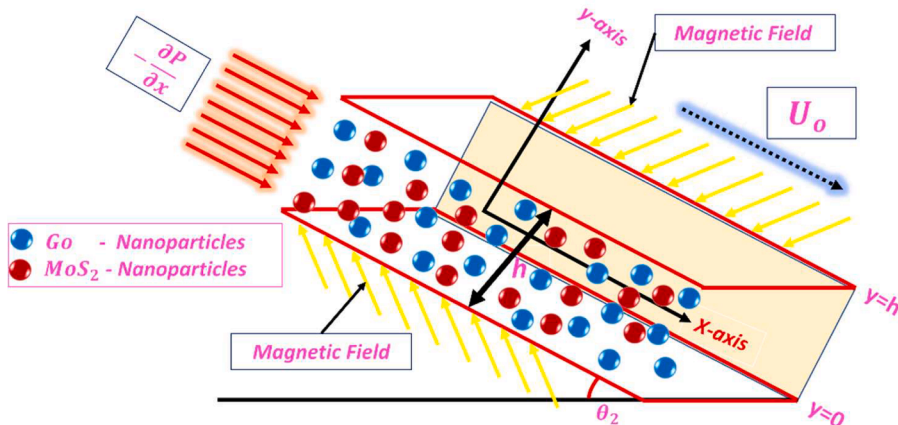


Fig. 3. Flow geometry of Poiseuille flow.

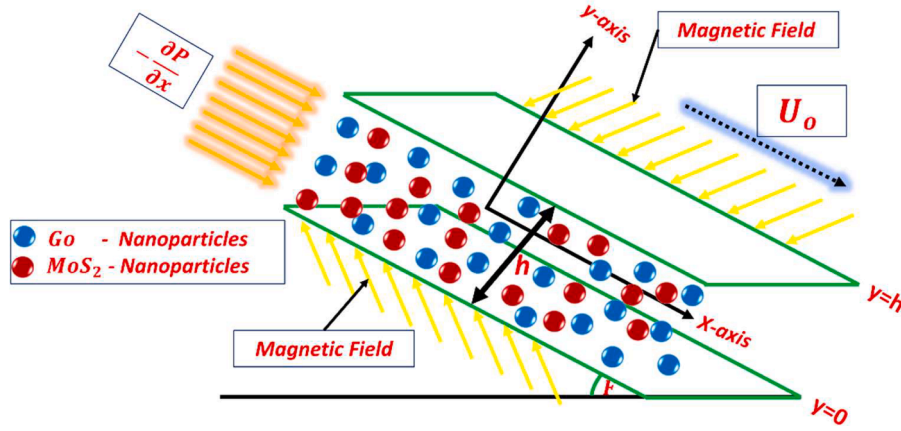


Fig. 4. Flow geometry of generalized Couette flow.

$$U(0, t) = 0; \quad U(h, t) = U_0 R(t); \quad U(\mathcal{Y}, 0) = 0$$

Acquainting the non-dimensional quantities

$$\overline{\mathcal{Y}} = \frac{\mathcal{Y}}{l}, \quad \overline{U} = \frac{U}{U_0}, \quad D_a = \frac{k'}{l^2}, \quad Re = \frac{\rho U_0 l}{\mu}, \quad \bar{t} = \frac{U_0 t}{l}, \quad \bar{h} = \frac{h}{l}, \quad \mathcal{M} = \sqrt{\frac{\sigma}{\mu}} \beta_0 l, \quad \overline{\mathcal{S}} = \frac{l^2 \mathcal{S}}{\mu U_0}$$

Using the above parameters

$$Re \frac{\partial U(\mathcal{Y}, t)}{\partial t} = \mathcal{S} + \frac{1}{1 + \Phi_1} \left(\frac{\partial^2 U(\mathcal{Y}, t)}{\partial \mathcal{Y}^2} \right) + \frac{Re}{Fr} \sin(\theta_2) - M^2 \sin(\theta_1) U(\mathcal{Y}, t) - \frac{1}{D_a} U(\mathcal{Y}, t) \quad (13)$$

$$U(0, t) = 0; \quad U(h, t) = R(t); \quad U(\mathcal{Y}, 0) = 0$$

With its respective fractional model

$${}^{FFD} \mathfrak{D}_t^{\alpha, \beta} U(\mathcal{Y}, t) = \beta t^{\beta-1} \left[\frac{\mathcal{S}}{Re} + \frac{\Lambda_1}{Re} \frac{\partial^2 U(\mathcal{Y}, t)}{\partial \mathcal{Y}^2} - \frac{1}{Re D_a} U(\mathcal{Y}, t) - \frac{M^2}{Re} \sin(\theta_1) U(\mathcal{Y}, t) + \frac{1}{Fr} \sin(\theta_2) \right] - \frac{U(\mathcal{Y}, 0)}{\Gamma(1-\alpha)} t^{-\alpha}$$

Engaging the Laplace definition

$$\overline{U}(0, q) = 0; \quad \overline{U}(h, q) = r(q); \quad \overline{U}(\mathcal{Y}, 0) = 0$$

And

$$\begin{aligned} \overline{U}(\mathcal{Y}, q) = & \frac{\mathcal{S}}{a_5 + a_6 q^{\alpha+\beta}} \left(1 - \frac{\sinh(\mathcal{Y} \sqrt{a_7(a_5 + a_6 q^{\alpha+\beta})})}{\sinh \sqrt{a_7(a_5 + a_6 q^{\alpha+\beta})}} \right) \\ & + \frac{a_3}{a_5 + a_6 q^{\alpha+\beta}} \left(1 - \frac{\sinh(\mathcal{Y} \sqrt{a_8 + a_9 q^{\alpha+\beta}})}{\sinh \sqrt{a_8 + a_9 q^{\alpha+\beta}}} \right) \\ & + r(q) \frac{\sinh(\mathcal{Y} \sqrt{a_7(a_5 + a_6 q^{\alpha+\beta})})}{\sinh \sqrt{a_7(a_5 + a_6 q^{\alpha+\beta})}} \end{aligned} \quad (14)$$

Where:

$$a_1 = \frac{Re}{Fr}, \quad a_2 = \sin(\theta_2), \quad a_3 = a_1 a_2, \quad a_4 = M^2 \sin(\theta_1), \quad a_5 = a_4 + \frac{1}{\mathcal{D}_a},$$

$$a_6 = \frac{Re}{\beta \Gamma(\beta)}, \quad a_7 = \frac{1}{1 + \Phi_1}, \quad a_8 = a_7 a_5, \quad a_9 = a_7 a_6$$

Eq. (14) further can be simplified in its relevant exponential form as follows

$$\begin{aligned} \overline{U}(\mathcal{Y}, q) = & \overline{B}_1(\mathcal{Y}, q)(1 - \overline{B}_2(\mathcal{Y}, q) + \overline{B}_3(\mathcal{Y}, q)) \\ & + \overline{B}_4(\mathcal{Y}, q)(1 - \overline{B}_2(\mathcal{Y}, q) + \overline{B}_3(\mathcal{Y}, q)) \\ & + r(q)(\overline{B}_5(\mathcal{Y}, q) + \overline{B}_6(\mathcal{Y}, q)) \end{aligned}$$

where:

$$\overline{B}_1(\mathcal{Y}, q) = \frac{\mathcal{S}}{a_5 + a_6 q^{\alpha+\beta}}$$

$$\overline{B}_2(\mathcal{Y}, q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \left(\frac{(\mathcal{Y} - 2m - 1)^n \Gamma\left(\frac{n}{2} + p\right) q^{\frac{n}{2}(\alpha + \beta)}}{n! \Gamma\left(\frac{n}{2}\right) (\mathfrak{E}_9)^{\frac{n}{2}}} \right)$$

$$\overline{B}_3(\mathcal{Y}, q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \left(\frac{(-2m - 1 - \mathcal{Y})^n (a_8)^{\frac{n}{2}} \Gamma\left(\frac{n}{2} + p\right) q^{\frac{n}{2}(\alpha + \beta)}}{n! \Gamma\left(\frac{n}{2}\right) (a_9)^{\frac{n}{2}}} \right)$$

$$\overline{B}_4(\mathcal{Y}, q) = \frac{a_3}{a_5 + a_6 q^{\alpha+\beta}}$$

$$\overline{B}_5(\mathcal{Y}, q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \left(\frac{(\mathcal{Y} - 1 - 2m)^n (a_5 a_7)^{\frac{n}{2}} \Gamma\left(\frac{n}{2} + p\right) q^{\frac{n}{2}(\alpha + \beta)}}{n! \Gamma\left(\frac{n}{2}\right) (a_9)^{\frac{n}{2}}} \right)$$

$$\overline{B}_6(\mathcal{Y}, q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \left(\frac{(-\mathcal{Y} - 2m - 1)^n (a_8 a_5 a_7)^{\frac{n}{2}} \Gamma\left(\frac{n}{2} + p\right) q^{\frac{n}{2}(\alpha + \beta)}}{(a_9)^{\frac{n}{2}} n! \Gamma\left(\frac{n}{2}\right)} \right)$$

Now by employing the Laplace inverse

$$\begin{aligned} U(\mathcal{Y}, t) = & B_1(\mathcal{Y}, t) * (\delta(1) - B_2(\mathcal{Y}, t) + B_3(\mathcal{Y}, t)) + B_4(\mathcal{Y}, t) \\ & * (\delta(1) - \mathfrak{B}_2(\mathcal{Y}, t) + \mathfrak{B}_3(\mathcal{Y}, t)) + R(t) * (B_5(\mathcal{Y}, t) + B_6(\mathcal{Y}, t)) \end{aligned}$$

Where the symbol “*” signifying the convolution of Laplace inverse with

$$B_1(\mathcal{Y}, t) = \frac{\mathcal{S}}{a_5} \sum_{n=0}^{\infty} \left(\frac{1}{\Gamma(\alpha + \beta)} \frac{1}{t^{1+\alpha+\beta}} \right)$$

$$B_2(\mathcal{Y}, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \left(\frac{(\mathcal{Y} - 2m - 1)^n \Gamma\left(\frac{n}{2} + p\right)}{n! \Gamma\left(\frac{n}{2}\right) (a_9)^{\frac{n}{2}} \Gamma\left(\frac{n}{2}(\alpha + \beta)\right) t^{1+\frac{n}{2}(\alpha + \beta)}} \right)$$

$$B_3(\mathcal{Y}, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \left(\frac{(-2m-1-\mathcal{Y})(a_8)^{\frac{n}{2}} \Gamma(\frac{n}{2}+p)}{n! \Gamma(\frac{n}{2})(a_9)^{\frac{n}{2}} \Gamma(\frac{1}{2}(\alpha+\beta)) t^{1+\frac{n}{2}(\alpha+\beta)}} \right)$$

$$B_4(\mathcal{Y}, t) = \sum_{n=0}^{\infty} \frac{a_3}{a_5} \left(\delta(1) - \frac{1}{\Gamma(\alpha+\beta)} \frac{1}{t^{1+\alpha+\beta}} \right)$$

$$B_5(\mathcal{Y}, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \left(\frac{(\mathcal{Y}-1-2m)^n (a_5 a_7)^{\frac{n}{2}} \Gamma(\frac{n}{2}+p)}{n! \Gamma(\frac{n}{2})(a_9)^{\frac{n}{2}} \Gamma(\frac{1}{2}(\alpha+\beta)) t^{1+\frac{n}{2}(\alpha+\beta)}} \right)$$

$$B_6(\mathcal{Y}, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \left(\frac{(-\mathcal{Y}-2m-1)^n (a_8 a_5 a_7)^{\frac{n}{2}}}{(a_9)^{\frac{n}{2}} n! \Gamma(\frac{n}{2}) \Gamma(\frac{1}{2}(\alpha+\beta)) t^{1+\frac{n}{2}(\alpha+\beta)}} \right)$$

Furthermore, many scholars [39–41] have used numerous numerical approaches to get the Laplace converse of many governed equation solutions. Stehfest's approach is a numerical and inverted approach used to perform the inverse Laplace transform. It is especially useful for Laplace structures that exist in closed form but whose inverse transform is challenging or hard to get analytically. The approach corresponds to the inverse Laplace using a weighted sum of functionality assessments at certain places. Stehfest's approach offers a close fit to the inverse Laplace $f(t)$ utilizing the following formula:

$$U(\mathcal{Y}, t) = \frac{\ln(2)}{t} \sum_{k=1}^{\infty} C_k \bar{U}\left(\mathcal{Y}, k \frac{\ln(2)}{t}\right)$$

$$C_k = (-1)^{k+p} \sum_{j=\frac{k+1}{2}}^{\min(k,p)} \frac{j^p (2j)!}{(p-j)! j! (i-1)! (k-1)! (2i-k)!}$$

Where C_k is the Stehfest coefficient. Tzou's approach is an alternative numerical inversion methodology commonly used to solve heat transfer issues and analyze instantaneous responses. It corresponds to the answer to an inverse Laplace by extending the function into a numerically

invertible series. Tzou suggested evaluating this integral numerically, such as using a finite sum across particular spots on a complicated plane or utilizing series expansion procedures. Tzou's method commonly employs the residue theorem or other complicated analytical tools to estimate the integral by adding components from each of the poles of $f(s)$. The mathematical form of the above numerical scheme can be express as:

$$U(\mathcal{Y}, t) = \frac{e^{4.7}}{t} \left[0.5 \bar{U}\left(\mathcal{Y}, \frac{4.7}{t}\right) + \text{Re} \left\{ \sum_{l=1}^M (-1)^l \bar{U}\left(\mathcal{Y}, \frac{4.7 + l\pi i}{t}\right) \right\} \right]$$

Anyhow both numerical approaches have also some limitations like numerical instability, limited accuracy, fixed number of terms, sensitivity of function, and computational cost issue.

4. Discussion of results

This paper investigates the MHD influence on the fractional Jeffrey fluid by considering nanoparticles. Different nanoparticles (MoS_2, GO) with (H_2O, CMC) as base fluids. A new definition of fractional integrals, known as the Fractal fractional derivative, is used in conjunction with the perspective Laplace transformation. This section examines the effects of various emergent factors, including variable values and pressure gradients, on trajectory profiles in all three cases using visual evidence.

Fig. 5 investigates the physical impact of the fractal fractional limitation on the three Jeffery-type hybrid nanofluid instances. As fractional restrictions increase, the momentum profile decreases in all circumstances due to the fractional and generalized manifestations of the Fractal integral kernel. Fractional derivatives provide a form of memory into the entire structure, which means that previous states impact the present state of circulation. This often leads to a slower reaction of the fluid speed to boundary motion, which delays the formation of the velocity spectrum. In Couette flow, fractional factors can create unusual transport behavior, resulting in sub-diffusion, in which particles or substances travel more slowly than classical diffusion. Furthermore, with the augmentation of fractional variables, the momentum profile for water-based ($H_2O + MoS_2 + GO$) hybrid nanofluid is significantly higher than CMC-based ($CMC + MoS_2 + GO$) hybrid nanofluid due to the physiological implication of measured nanomaterials. Fig. 6 demonstrates correspondingly D_a impact on simple, Poiseuille, and enlarged Couette flows. In fluid mechanics, the Darcy component defines the

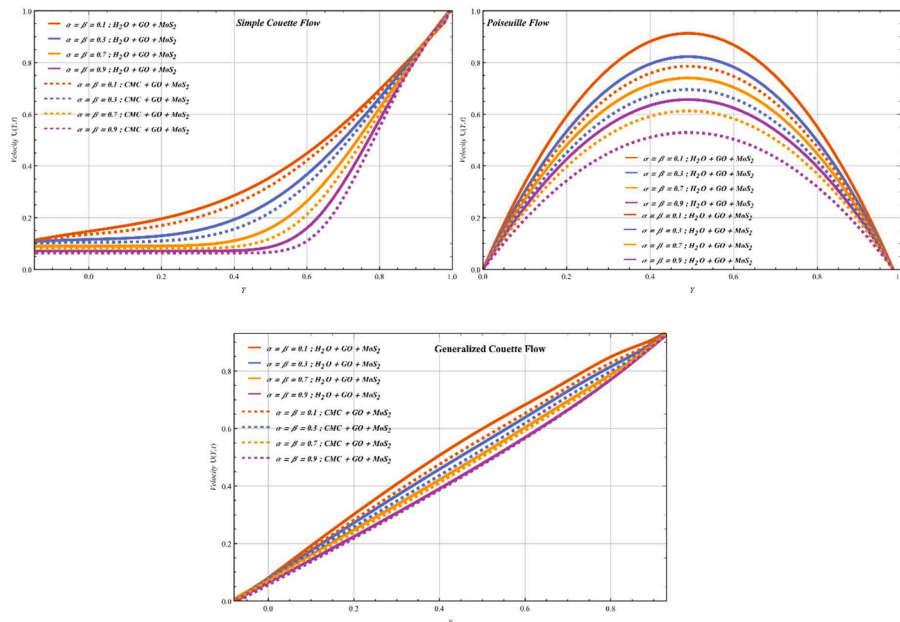


Fig. 5. Impact of α, β on $U(\mathcal{Y}, t)$ with $\theta_1 = \theta_2 = \frac{\pi}{3}$, $M = 2.0$, $Re = 0.3$, $Fr = 3.6$, $t = 1.4$, $D_a = 0.5$, $\Phi_1 = 2.0$.

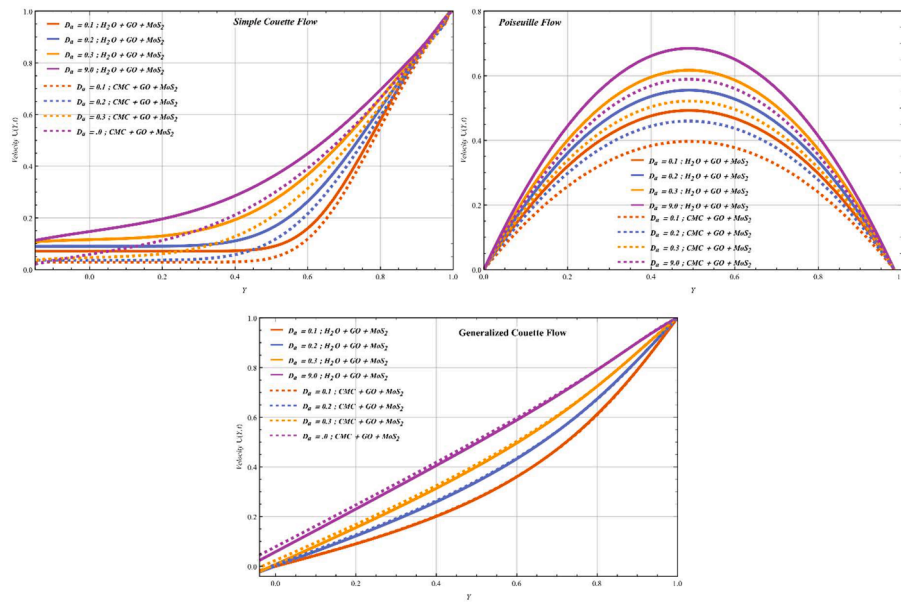


Fig. 6. Impact of D_a on $U(\mathcal{Y}, t)$ with $\alpha = \beta = 0.5$, $\theta_1 = \theta_2 = \frac{\pi}{3}$, $M = 2.0$, $Re = 0.3$, $Fr = 3.6$, $t = 1.4$, $\Phi_1 = 2.0$.

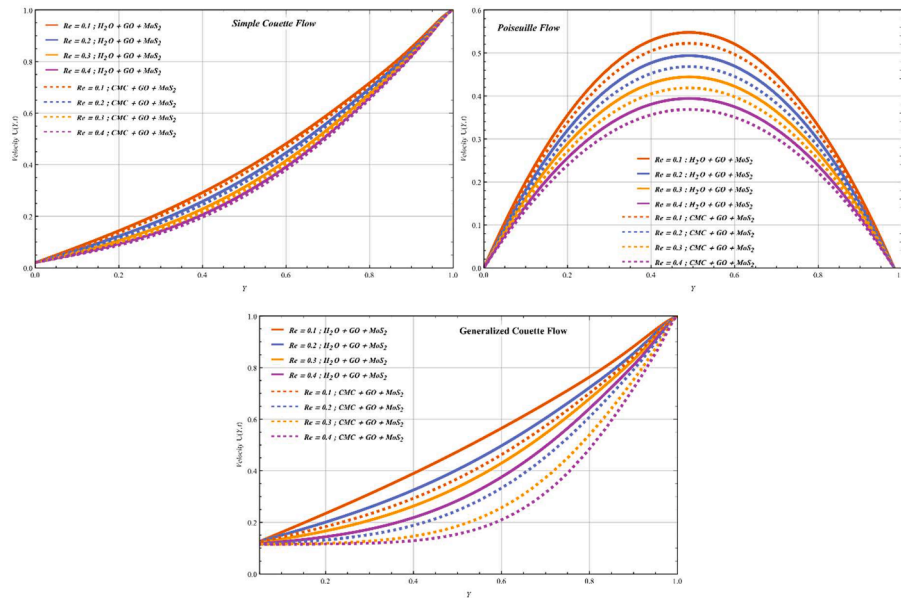


Fig. 7. Impact of Re on $U(\mathcal{Y}, t)$ with $\alpha = \beta = 0.5$, $\theta_1 = \theta_2 = \frac{\pi}{3}$, $M = 2.0$, $Fr = 3.6$, $t = 1.4$, $D_a = 0.5$, $\Phi_1 = 2.0$.

relationship between channel restriction and flow speed through the base material. The D_a factor is connected to the Coulomb-Weisbach equation and represents how friction reduces an object's pressure. As D_a grows, the velocity exhibits shear-weakened or thickening behavior, depending on the fluid's unique properties. Furthermore, due to the physical effects of the examined nanoparticles, the $(H_2O + GO + MoS_2)$ hybrid nanofluid has a somewhat stronger effect than the $(CMC + GO + MoS_2)$ combination. Fig. 7 shows a full description of Reynolds number (Re). The (Re) is a dimensionless measure that fluids research uses to assess an ingredient's flow. In classical Couette flow, when fluid becomes sheared through two identical plates with one oscillating compared to the other, a low Re implies that viscous forces are dominant, resulting in a smooth, linear velocity distribution throughout the separation. As the Re rises, inertial influences become more evident, potentially causing departures from the linear description, increased flow turbulence, and the initial stage of turbulence, especially in

generalized Couette flows with extra stresses or geometrical complexity. It is defined as the ratio of the effects of adherence to viscous forces. At a given Re value, the flow characteristic changes from laminar to turbulent.

Figs. 8 and 9 show how the magnetic field (M) and rotation (θ_1) affect the flow. The existence of a magnetic field has a considerable influence on flow velocity owing to MHD effects. The magnetic field generates a Lorentz force on the conductive fluid, which typically acts to impede fluid motion, lowering flow velocity. This impact is more noticeable when the field is particularly strong, or the fluid conductance is high. Furthermore, the angle of the magnetic field compared to the flow direction changes the occurrence of the Lorentz force, which can cause anisotropy impacts on the flow profile. For instance, placing a magnetic field at an inclination to the flow might create secondary flows or change the velocity gradient, resulting in a more complicated flow pattern. Overall, the intensity and direction of the magnetic field are important

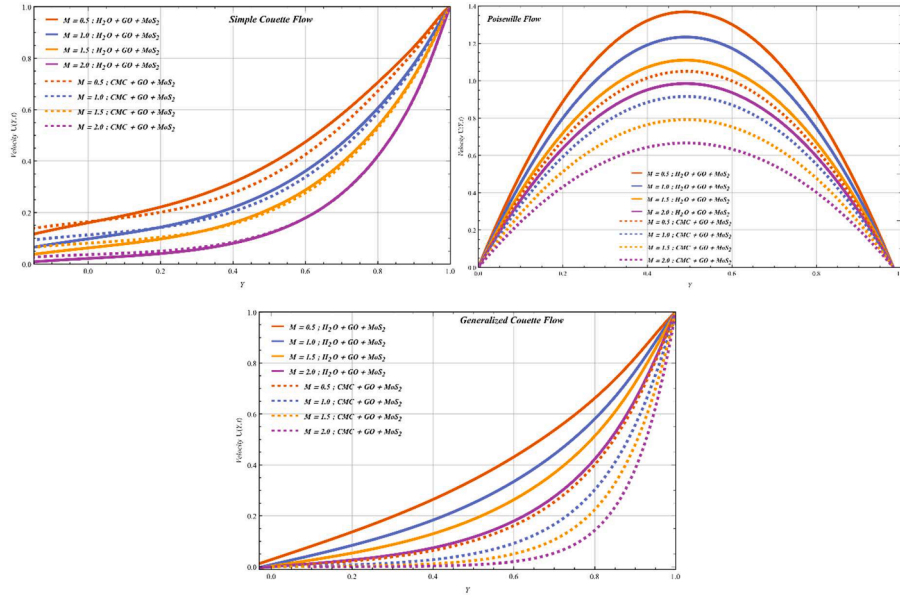


Fig. 8. Impact of M on $U(\mathcal{Y}, t)$ with $\alpha = \beta = 0.5$, $\theta_1 = \theta_2 = \frac{\pi}{3}$, $Re = 0.3$, $Fr = 3.6$, $t = 1.4$, $D_a = 0.5$, $\Phi_1 = 2.0$.

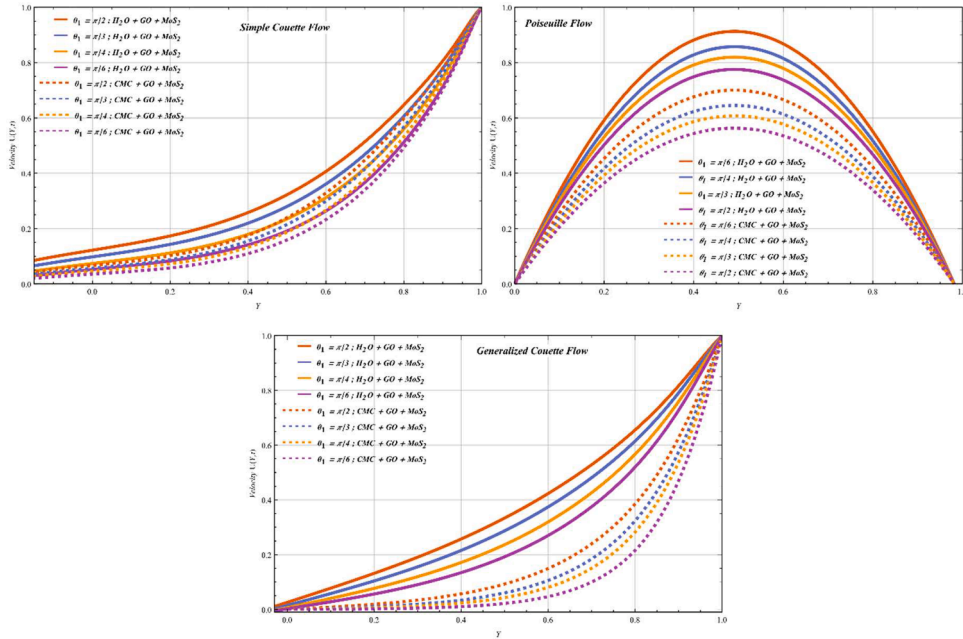


Fig. 9. Impact of θ_1 on $U(\mathcal{Y}, t)$ with $\alpha = \beta = 0.5$, $\theta_2 = \frac{\pi}{3}$, $M = 2.0$, $Re = 0.3$, $Fr = 3.6$, $t = 1.4$, $D_a = 0.5$, $\Phi_1 = 2.0$.

in regulating and altering the velocity patterns. The inspiration of ($H_2O + GO + MoS_2$) hybrid nanofluid is somewhat higher than that of ($CMC + GO + MoS_2$) hybrid fluid with respect to both factors (M) and (θ_1) in all circulation situations. Fig. 10 depicts the influence of Froude number (Fr) on the enhancement of ($H_2O + GO + MoS_2$) and ($CMC + GO + MoS_2$) based hybrid nanofluids in flow. A larger Fr implies that inertial forces take precedence over forces of gravity, resulting in a more consistent spread of velocity throughout the flow channel because fluid velocity is more probable to be preserved along the lines of the flowing surface. A lower Fr , on the other hand, indicates that gravitational pulls are more significant, which might lead the fluid velocity to be more variable and categorized, with possible changes in the velocity gradient caused by gravity operating transversely to the flow path. This can result in a slower average flow velocity and a more noticeable velocity profile due to gravity setting or stratified processess.

Fig. 11 shows demonstrations of the effects of the inclination of poured microchannels on the velocity profiles of all types of flows. The slope of the surfaces in Couette and generalized Couette flow substantially influence the velocity profile because it introduces a gravitational component along the flow path, which assists the flow according to the inclination angle. When the plates are tilted, gravitational force affects the fluid's movement, possibly increasing speed towards the inclined sides. If gravity aids the flow speed, this might result in an asymmetrical velocity profile, with fluid velocity being greater closer the lower plate. Hence due to the inclination the fluid speed will enhance due to the gravitational impact. Fig. 12 shows how the Jeffery-fluid limitation (Φ_1) affects velocity fields. The Φ_1 limitation, which defines viscoelastic fluids as having both fluid-like and solid-like conduct, has a substantial influence on the speed profile in Couette and modified Couette processes. The extra elasticity enables the fluid to endure distortion more

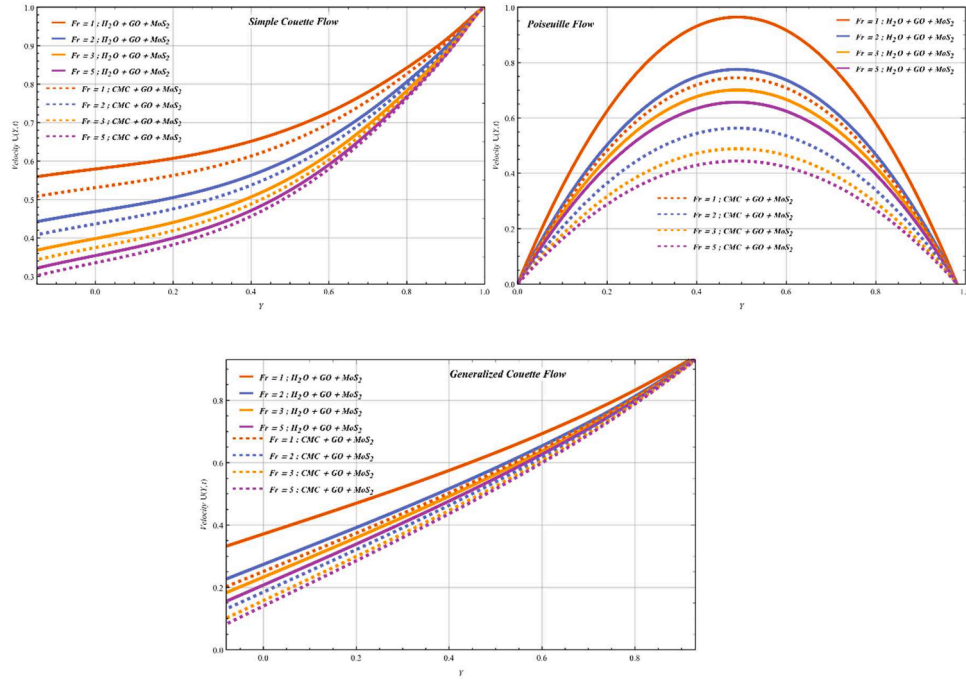


Fig. 10. Impact of Fr on $U(\mathcal{Y}, t)$ with $\alpha = \beta = 0.5$, $\theta_1 = \theta_2 = \frac{\pi}{3}$, $M = 2.0$, $Re = 0.3$, $t = 1.4$, $D_a = 0.5$, $\Phi_1 = 2.0$.

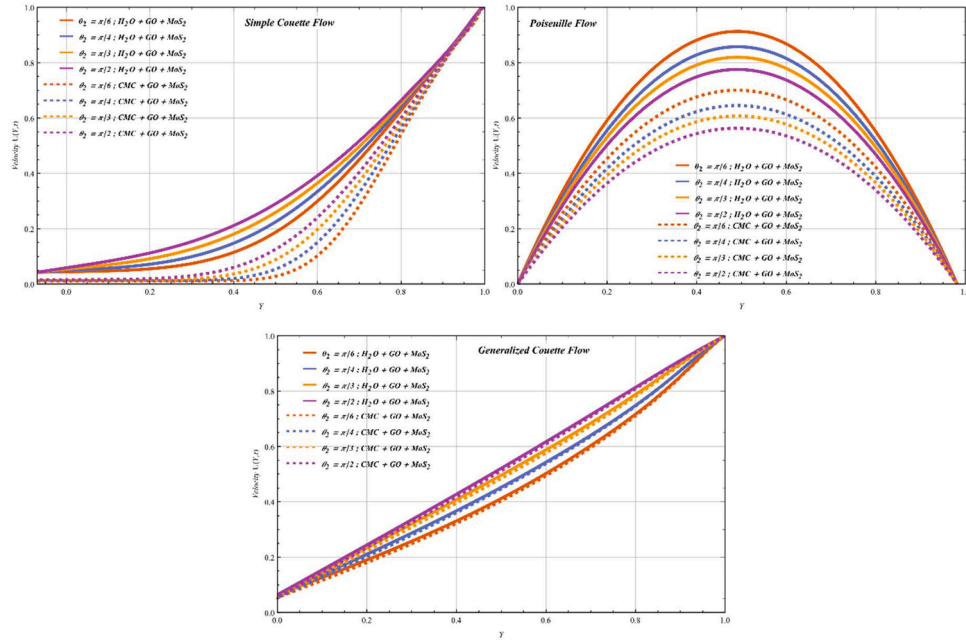


Fig. 11. Impact of θ_2 on $U(\mathcal{Y}, t)$ with $\alpha = \beta = 0.5$, $\theta_1 = \frac{\pi}{3}$, $M = 2.0$, $Re = 0.3$, $Fr = 3.6$, $t = 1.4$, $D_a = 0.5$, $\Phi_1 = 2.0$.

than a basic viscous fluid, resulting in velocity gradients that are proportional to both the shear rate and the fluid's relaxation period. In extended Couette flows, the Φ_1 limitation can result in more complicated flow characteristics, such as possible velocity exceeds or underflows along border regions. As a result of the viscoelastic behavior, fluid flow accelerates in all flow circumstances. Fig. 13 show that time reduction (t) corresponds with the accelerating field throughout the two simple nanofluids and hybrid nanofluid suspensions. This shows that across all flows, the flow velocity field increases in the case of hybrid nanofluid suspension as compared to simple nanofluids suspension.

Fig. 14 is used to evaluate numerical methods, notably Stehfest and Tzou's procedures. Once again, the underlying context of the individual

curves in both cases demonstrates that the obtained results and numerical approaches are credible. To support the conclusions of this investigation, Fig. 15 shows a graphical comparison of the accomplished velocity graph with the implications estimated by Ramesh et al. [11]. The integrated impact of graphs verifies this research. The comparable nature of graphs gives direct evidence of the work's reliability. This validation supports the theoretical foundation upon which the study is based. Finally, Tables 3–5 compare our study's numerical results to those obtained from Ramesh et al. [11] for conducting fluid. The outcomes of this analysis for all scenarios are relatively comparable to those of Ramesh et al. [11], indicating their validity. At the end for the validity of numerical schemes, Table 6 is depicted. Again very close of results

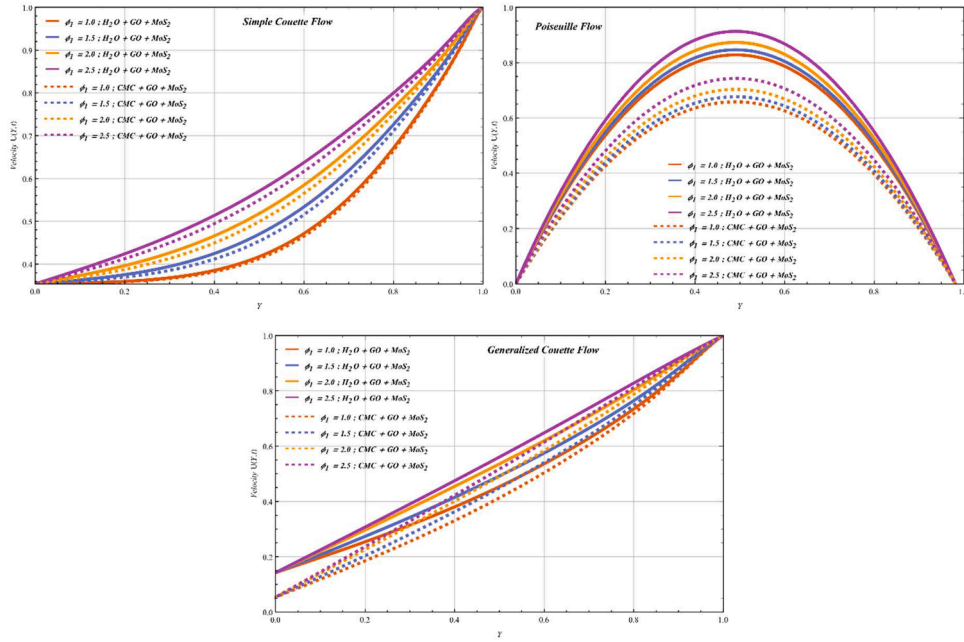


Fig. 12. Impact of Φ_1 on $U(\mathcal{Y}, t)$ with $\alpha = \beta = 0.5$, $\theta_1 = \theta_2 = \frac{\pi}{3}$, $M = 2.0$, $Re = 0.3$, $Fr = 3.6$, $t = 1.4$, $Da = 0.5$.

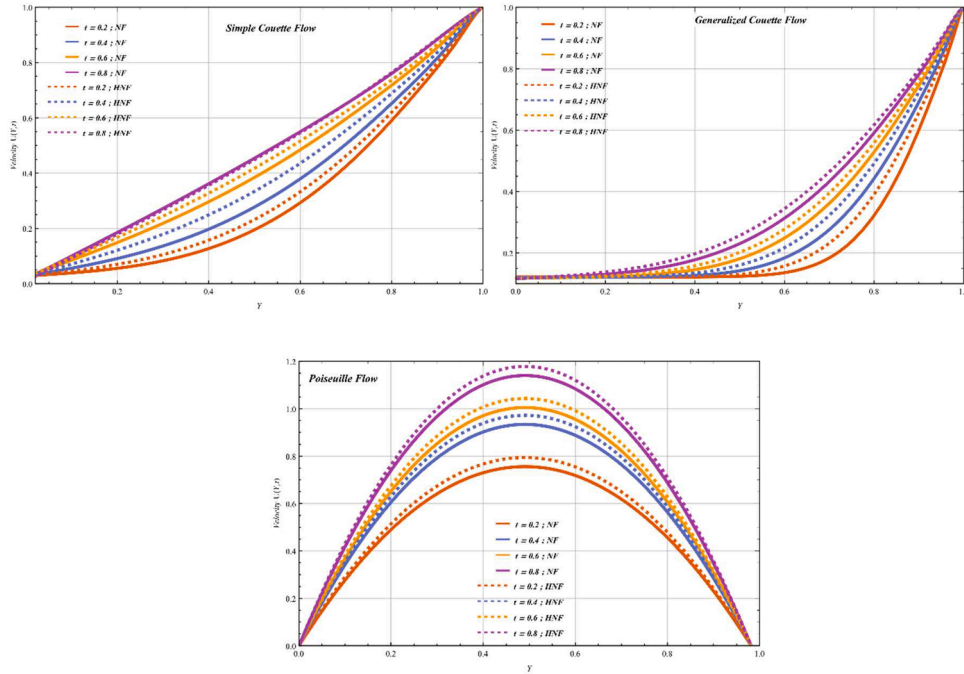


Fig. 13. Impact of t on $U(\mathcal{Y}, t)$ and comparison of simple nanofluid and hybrid nanofluid with $\alpha = \beta = 0.5$, $\theta_1 = \theta_2 = \frac{\pi}{3}$, $M = 2.0$, $Re = 0.3$, $Fr = 3.6$, $Da = 0.5$, $\Phi_1 = 2.0$.

validates the implication of numerical schemes.

5. Conclusions

The influence of an electromagnetic field on the unsteady movements of a Jeffrey fluid in a poured material between angled two parallel plates is investigated. The fractional solutions of all non-dimensional PDEs have been approximated using a new definition of fractional derivatives, known as Fractal Fractional derivatives. The Laplace transformation is also used for the fractional solution. Fractional Jefferey-type hybrid nanofluids are investigated in planar, poiseuille, and

generalized Couette flow characteristics. The influence of the aforementioned parameters on flow characteristics and flow incidences produced by wall degradation is carefully examined. The foremost outcomes of this work are outlined below:

- In-simple and generalized streams experience continuous pressure, causing velocity to increase.
- Lorentz forces cause the speed of the fluid to drop as the M and θ_1 parameters increase. However, when the Da , which represents the medium's accessibility, rises, and the obstacle from the porous material falls, inverting the trend and increasing the velocity.

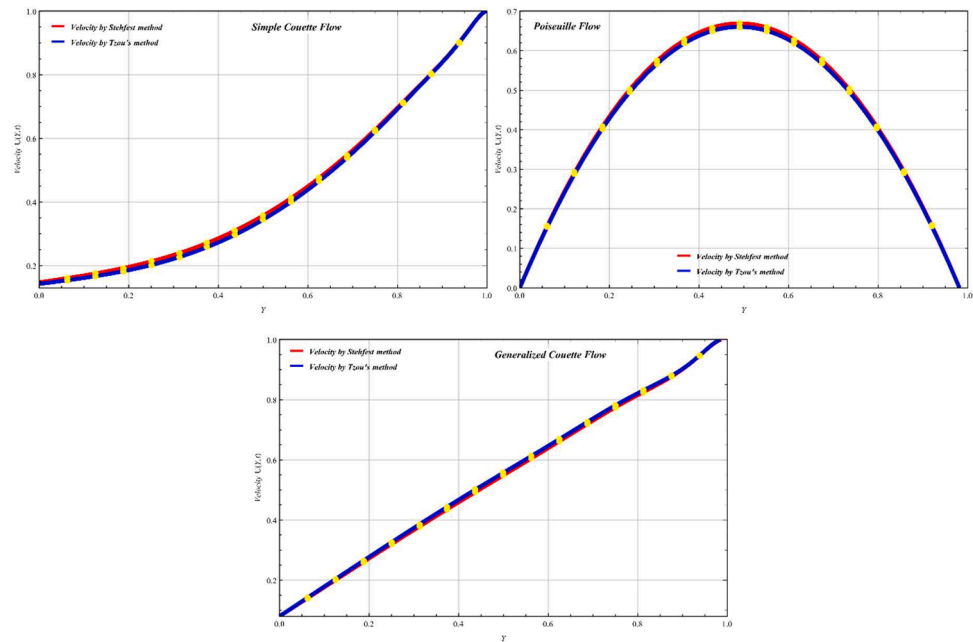


Fig. 14. Comparison of numerical schemes.

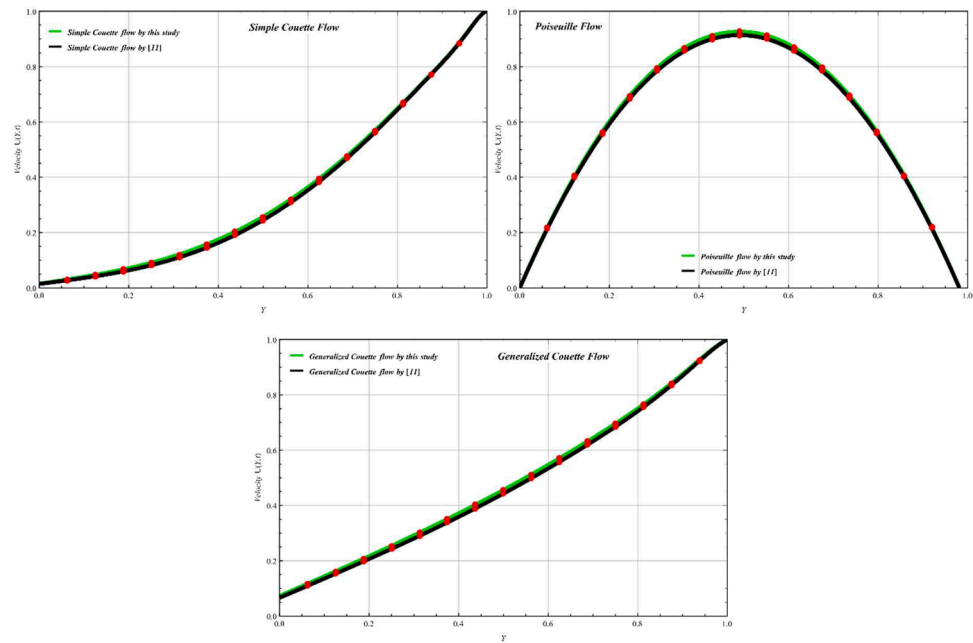


Fig. 15. Comparison with Ramesh et al. [11].

Table 3
Comparison of *Couetteflow* attained by this study with the outcomes of [11].

α, β	<i>Couetteflow</i> by this study	<i>Couetteflow</i> by [11]
0.1	0.086222	0.086159
0.2	0.17252	0.172513
0.3	0.260899	0.260947
0.4	0.352408	0.352504
0.5	0.448057	0.448185
0.6	0.548731	0.548864
0.7	0.655007	0.655105
0.8	0.766825	0.766851
0.9	0.882935	0.88289

Table 4
Assessment of *Poiseuilleflow* outcomes with the results obtained by Ramesh et al. [11].

	<i>Poiseuilleflow</i> by this study	<i>Poiseuilleflow</i> by [11]
0.1	1.04015	1.02979
0.2	1.05606	1.04541
0.3	1.07727	1.06644
0.4	1.10681	1.09583
0.5	1.15126	1.13797
0.6	1.22487	1.21254
0.7	1.37091	1.35720
0.8	1.80040	1.78244
0.9	1.93390	1.93018

Table 5

Numerical comparison of *GeneralizedCouetteflow* results attained by this study and [11].

α, β	<i>GeneralizedCouetteflow</i> by this study	<i>GeneralizedCouetteflow</i> by [11]
0.1	0.26633	0.274039
0.2	0.528969	0.544501
0.3	0.784137	0.807722
0.4	1.027888	1.059887
0.5	1.256061	1.296987
0.6	1.464288	1.514829
0.7	1.648019	1.709064
0.8	1.802445	1.875087
0.9	1.922037	2.007577

Table 6

Numerical comparison of utilized numerical schemes.

α, β	<i>SimpleCouetteflow</i>		<i>GeneralizedCouetteflow</i>	
	<i>StehfestMethod</i>	<i>Tzou'sMethod</i>	<i>StehfestMethod</i>	<i>Tzou'sMethod</i>
0.1	0.049394	0.049396	0.177566	0.17762
0.2	0.101296	0.101307	0.355961	0.356072
0.3	0.158378	0.158413	0.536093	0.536264
0.4	0.223641	0.223716	0.719034	0.719274
0.5	0.300552	0.300674	0.906164	0.906481
0.6	0.393116	0.393273	1.099357	1.099766
0.7	0.505776	0.505927	1.301245	1.301763
0.8	0.642965	0.643043	1.515435	1.516084
0.9	0.807903	0.807866	1.746509	1.747316

The motion of the fluid improves as the time variable increases, showing that it is accelerating over time. This might be owing to a steady decrease in friction as the flow expands, or to other forces operating on the mixture over time.

- The fluid's movement may be described as a Newtonian fluid through adjusting variable $\lambda_1 = 0$.
- A vertical channel has a greater fluid velocity than a straight channel. This discrepancy is most likely caused by gravity's impact on the vertical structure.8
- A rise in the pressure gradient causes an upsurge in the flow rate in both simple and generalized Couette processes. The pressure gradient functions as a propellant that accelerates the fluid, hence a higher gradient causes quicker fluid flow.
- To change the acceleration field, modify Re, Fr, M, θ_1 .
- The initial impression of (H_2O)-based mixed nanofluid is slightly superior to (CMC)-based combination because of the physical characteristics of the studied nanomaterial's(GO, MoS_2).
- The coinciding of the outcomes of this research with the implications of Ramesh et al. [11] indicates that the study's findings are correct and precise.
- The approaches offered can aid in proper information interpretation and, if necessary, can be utilized to assess likely response estimates.
- The novelty of this study can be accomplished by the overlapping of the curves with previous published studies and numerical algorithms.

Future study might look at several potential extensions of this work. One strategy is to investigate the effects of changing the shape and density of various nanoparticles on fluid movement and heat transfer properties, resulting in a more thorough understanding of hybrid nanofluids in several industrial applications. Furthermore, applying the existing computational framework to different dimensions and flow configurations, such as porous solids or non-Newtonian fluids, may result in a new understanding of fluid dynamics in various scenarios. Further study might include empirical testing of the conceptual frameworks to confirm their feasibility. Investigating the durability and efficacy of the proposed hybrid nanofluids under continuous operation and changing ambient circumstances might provide valuable insights for

practical applications.

CRediT authorship contribution statement

Ali Hasan Ali: Writing – review & editing, Software, Project administration, Methodology, Conceptualization. **Ali Raza:** Writing – review & editing, Supervision, Conceptualization. **Belal Batiha:** Writing – original draft, Investigation, Data curation. **Ahmed M. Abed:** Writing – original draft, Validation, Supervision, Data curation. **Zaid Ameen Abduljabbar:** Writing – original draft, Visualization, Validation, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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