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Article

Harmonic Synthesis on Group Extensions

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Abstract: Harmonic synthesis describes translation invariant linear spaces of continuous complex valued functions on locally compact abelian groups. The basic result due to L. Schwartz states that such spaces on the reals are topologically generated by the exponential monomials in the space; in other words, the locally compact abelian group of the reals is synthesizable. This result does not hold for continuous functions in several real variables, as was shown by D.I. Gurevich's counterexamples. On the other hand, if two discrete abelian groups have this synthesizability property, then so does their direct sum, as well. In this paper, we show that if two locally compact abelian groups have this synthesizability property and at least one of them is discrete, then their direct sum is synthesizable. In fact, more generally, we show that any extension of a synthesizable locally compact abelian group by a synthesizable discrete abelian group is synthesizable. This is an important step toward the complete characterization of synthesizable locally compact abelian groups.

Keywords: variety; spectral synthesis

MSC: 43A45; 22D99

1. Introduction

Harmonic (or spectral) synthesis deals with uniformly closed translation invariant linear spaces of continuous complex valued functions on a locally compact abelian group. Such a function space is called a *variety*. The first classical result on this field is due to Laurent Schwartz [1], which states that given any complex valued continuous function on the reals, it can be uniformly approximated on compact sets by exponential polynomials, which belong to the smallest variety including the given function. This can be reformulated in the following manner: if a continuous complex valued function on the reals satisfies a system of homogeneous convolution equations corresponding to compactly supported complex Borel measures, then this function is the uniform limit on compact sets of exponential polynomials, which are also solutions of the same system of convolution equations. In particular, every such system has exponential solutions. It is obvious that this problem makes sense on locally compact abelian groups using appropriate definition of exponential polynomials. One possible definition is the following: we call a continuous complex valued function on a locally compact abelian group an exponential polynomial if the smallest variety including the function is finite dimensional. We say that *spectral analysis* holds for a variety if every nonzero subvariety contains a one-dimensional subvariety. We say that a variety is *synthesizable* if its finite-dimensional subvarieties span a dense subspace in the variety. Finally, we say that *spectral synthesis* holds for a variety if every subvariety is synthesizable. In particular, if every variety on the group is synthesizable, then we say that *spectral synthesis holds on the group*, or *the group is synthesizable*. For more about spectral analysis and synthesis on groups, see [2,3].

In [4], the authors characterized synthesizable discrete abelian groups: spectral synthesis holds on the discrete abelian group G if and only if G has finite torsion free rank. In particular, from this result, it follows that if G and H are synthesizable, then so is $G \times H$. Unfortunately, such a result does not hold in the non-discrete case. Namely, by the fundamental result of L. Schwartz [1], $\mathbb{R}$ is synthesizable, but D. I. Gurevich showed in [5]

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that spectral synthesis fails to hold on $\mathbb{R} \times \mathbb{R}$. In other words, a direct product does not necessarily preserve synthesizability in non-discrete cases. However, in this paper, we show that if spectral synthesis holds onto two locally compact abelian groups, then it holds onto their direct product, assuming at least one of them is synthesizable.

2. Preliminaries

On commutative topological groups, finite-dimensional varieties of continuous functions are completely characterized: they consist of the elements of the complex algebra of continuous complex valued functions generated by all continuous homomorphisms into the multiplicative group of nonzero complex numbers (*exponentials*), as well as all continuous homomorphisms into the additive group of all complex numbers (*additive functions*). In particular, an *exponential monomial* (or *m-exponential monomial*) is a function of the form

$$x \mapsto P\big(a_1(x), a_2(x), \dots, a_n(x)\big) m(x),$$

where P is a complex polynomial in n variables, the a_is are additive functions, and m is an exponential. Every exponential polynomial is a linear combination of exponential monomials. The m-exponential monomials with $m = 1$ are called *polynomials*.

Let G be a locally compact abelian group. It is known that the dual space of $\mathcal{C}(G)$ can be identified with the space $\mathcal{M}_c(G)$ of all compactly supported complex Borel measures on G. This space is called the *measure algebra* of G—it is a topological algebra with the linear operations, with the convolution of measures and with the weak* topology arising from $\mathcal{C}(G)$. On the other hand, the space $\mathcal{C}(G)$ is a topological vector module over the measure algebra under the action realized by the convolution of measures and functions. The annihilators of subsets in $\mathcal{C}(G)$ and the annihilators of subsets in $\mathcal{M}_c(G)$ play an important role in our investigation. For each closed ideal I in $\mathcal{M}_c(G)$ and for every variety V in $\mathcal{C}(G)$, $\operatorname{Ann} I$ is a variety in $\mathcal{C}(G)$, and $\operatorname{Ann} V$ is a closed ideal in $\mathcal{M}_c(G)$; further, we have

$$\operatorname{Ann}\operatorname{Ann} I = I \quad \text{and} \quad \operatorname{Ann}\operatorname{Ann} V = V.$$

The Fourier–Laplace transformation (Fourier transformation, in short) on the measure algebra is defined as follows: for every exponential m on G and for each measure μ in $\mathcal{M}_c(G)$, its *Fourier transform* is

$$\widehat{\mu}(m) = \int \check{m}\, d\mu,$$

where $\check{m}(x) = m(-x)$ for each x in G. The Fourier transform $\widehat{\mu}$ is a complex valued function defined on the set of all exponentials on G. As the mapping $\mu \mapsto \widehat{\mu}$ is linear and $(\mu * \nu)\widehat{} = \widehat{\mu} \cdot \widehat{\nu}$, all Fourier transforms form a function algebra. By the injectivity of the Fourier transform, this algebra is isomorphic to $\mathcal{M}_c(G)$. If we equip the algebra of Fourier transforms by the topology arising from the topology of $\mathcal{M}_c(G)$, then we get the *Fourier algebra* of G, denoted by $\mathcal{A}(G)$. In fact, $\mathcal{A}(G)$ can be identified with $\mathcal{M}_c(G)$. We utilize this identification as follosw: for instance, every ideal in $\mathcal{A}(G)$ is of the form $\widehat{I}$, where I is an ideal in $\mathcal{M}_c(G)$. Based on this fact, we say that *the ideal $\widehat{I}$ in $\mathcal{A}(G)$ is synthesizable*, or *spectral synthesis holds for the ideal $\widehat{I}$ in $\mathcal{A}(G)$* if the variety $\operatorname{Ann} I$ in $\mathcal{M}_c(G)$ is synthesizable, or spectral synthesis holds for it.

We shall use the polynomial derivations on the Fourier algebra. A *derivation* on $\mathcal{A}(G)$ is a linear operator $D : \mathcal{A}(G) \to \mathcal{A}(G)$ such that

$$D(\widehat{\mu} \cdot \widehat{\nu}) = D(\widehat{\mu}) \cdot \widehat{\nu} + \widehat{\mu} \cdot D(\widehat{\nu})$$

holds for each $\widehat{\mu}, \widehat{\nu}$. We say that D is a *first-order derivation*. Higher-order derivations are defined inductively: for a positive integer n, we say that the linear operator D on $\mathcal{A}(G)$ is a *derivation of order* $n + 1$, if the two-variable operator

$$(\widehat{\mu}, \widehat{\nu}) \mapsto D(\widehat{\mu} \cdot \widehat{\nu}) - D(\widehat{\mu}) \cdot \widehat{\nu} - \widehat{\mu} \cdot D(\widehat{\nu})$$

is a derivation of order n in both variables. The identity operator id is considered a derivation of order 0. All derivations form an algebra of operators, and the derivations in subalgebra generated by all first-order derivations are called *polynomial derivations*. They have the form $P(D_1, D_2, \ldots, D_k)$, where $D_1, D_2, \ldots, D_k$ are first-order derivations, and P is a complex polynomial in k variables. In [6], we proved the following result:

Theorem 1. *The linear operator D on $\mathcal{A}(G)$ is a polynomial derivation if and only if there exists a unique polynomial f_D such that*

$$D\widehat{\mu}(m) = \int f_D(x)m(-x)\,d\mu(x)$$

holds for each $\widehat{\mu}$ in $\mathcal{A}(G)$ and for every exponential m on G.

In [6], we introduced the following concepts. Given an ideal $\widehat{I}$ in $\mathcal{A}(G)$ and an exponential m, we denoted by $\mathcal{P}_{\widehat{I},m}$ the family of all polynomial derivations $P(D_1, D_2, \ldots, D_k)$ which annihilate $\widehat{I}$ at m. This means that

$$\partial^\alpha P(D_1, D_2, \ldots, D_k)\widehat{\mu}(m) = 0$$

for each multi-index α in $\mathbb{N}^k$, for every exponential m, and for every $\widehat{\mu}$ in $\widehat{I}$. The dual concept is the following: given a family $\mathcal{P}$ of polynomial derivations and an exponential m we denote by $\widehat{I}_{\mathcal{P},m}$ the set of all functions $\widehat{\mu}$ which are annihilated by every derivation in the family $\mathcal{P}$ at m. Then, $\widehat{I}_{\mathcal{P},m}$ is a closed ideal. Obviously,

$$\widehat{I} \subseteq \bigcap_m \widehat{I}_{\mathcal{P}_{\widehat{I},m},m}$$

holds for every ideal $\widehat{I}$. We call $\widehat{I}$ *localizable*, if we have equality in this inclusion. In other words, the ideal $\widehat{I}$ in $\mathcal{A}(G)$ is localizable if and only if it has the following property: if $\widehat{\mu}$ is annihilated by all polynomial derivations, which annihilate $\widehat{I}$ at each m, then $\widehat{\mu}$ is in $\widehat{I}$. The main result in [6] is the following:

Theorem 2. *Let G be a locally compact abelian group. The ideal $\widehat{I}$ in the Fourier algebra is localizable if and only if* Ann *I is synthesizable.*

Corollary 1. *Spectral synthesis holds on a discrete abelian group if and only if its torsion-free rank is finite.*

Proof. Although a quite long and complicated proof for this result was given in [4], we note that a simple application of Theorem 2 above gives Theorem 1 and Corollary 3 in [7], yielding our statement. □

3. Main Result

We have seen above that if G, H are discrete abelian groups and spectral synthesis holds on G and on H, then spectral synthesis holds on $G \times H$. On the other hand, by the cited results of D. I. Gurevich, the corresponding result may not hold if G and H are non-discrete. Our main result in this paper is the following: if G or H is discrete, then spectral synthesis holds on $G \times H$. In fact, we shall prove the following stronger result:

Theorem 3. *Let G be a locally compact abelian group and H a closed subgroup of G. If G/H is discrete, then spectral synthesis holds on G if and only if spectral synthesis holds on H and on G/H.*

In this situation, some authors (see [8,9]) call G an *extension of H by the group $D = G/H$*. To prove this theorem, we need a couple of preliminary results. The first one is interesting on its own. In fact, the corresponding result on discrete abelian groups was proved in ([4], Lemma 6), but we verified it on general locally compact abelian groups.

Lemma 1. *Every closed subgroup of a locally compact synthesizable abelian group is synthesizable.*

Proof. The proof is based on our localization result Theorem 2. Namely, we show that if G is a locally compact synthesizable abelian group and H is a closed subgroup of G, then every ideal in the Fourier algebra of H is localizable.

Let $\widehat{I}$ be an ideal in $\mathcal{A}(H)$, where $I = \operatorname{Ann} V$ is the annihilator of a variety V on H. Clearly, every measure in $\mathcal{M}_c(H)$ can be considered as a measure in $\mathcal{M}_c(G)$: in fact, $\mathcal{M}_c(H)$ is a closed subalgebra of $\mathcal{M}_c(G)$. It follows that $\mathcal{A}(H)$ can be considered as a closed subalgebra of $\mathcal{A}(G)$. We remark that some confusion may arise from the fact that the functions in $\mathcal{A}(G)$ are defined on the set of exponentials on G and the functions in $\mathcal{A}(H)$ are defined only on the set of exponentials on H; however, clearly, this minor inconvenience can be overcome by the fact that every exponential on H can be extended to an exponential on G, and the values of the Fourier transforms of the measures in $\mathcal{A}(H)$ are independent of the particular extension of the measure, as its support is included in H.

It follows that $\widehat{I}$ is a closed subset in $\mathcal{A}(G)$. Let $\widehat{I}_G$ denote the closure of the ideal generated by the set $\widehat{I}$ in $\mathcal{A}(G)$. It follows that the functions of the form $\widehat{\xi} \cdot \widehat{\mu}$ with $\widehat{\xi}$ in $\mathcal{A}(G)$ and $\widehat{\mu}$ in $\widehat{I}$ span a dense subset in $\widehat{I}_G$. We show that if a polynomial derivation $P(D)$ annihilates $\widehat{I}$ at the exponential m, then for any extension, it annihilates $\widehat{I}_G$ at m. Indeed, let P be a complex polynomial in n variables and $D = (D_1, D_2, \ldots, D_n)$, where D_i is a first-order derivation on $\mathcal{A}(H)$. By Theorem 1, every polynomial derivation on $\mathcal{A}(H)$ can be extended to $\mathcal{A}(G)$. On the other hand, we have the following identity:

$$P(D)(\widehat{\xi} \cdot \widehat{\mu}) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} D^\alpha \widehat{\xi} \cdot [\partial^\alpha P(D)]\widehat{\mu},$$

which can be verified easily with Leibniz's Rule. It follows that if $P(D)$ annihilates $\widehat{I}$ at the exponential m, then

$$[\partial^\alpha P(D)]\widehat{\mu}(m) = 0$$

for each α; hence, $P(D)$ vanishes on each product $\widehat{\xi} \cdot \widehat{\mu}$ at m, which implies that $P(D)$ annihilates $\widehat{I}_G$ at m. As G is synthesizable, we have that if $\widehat{\nu}$ is annihilated by each $P(D)$ which annihilates $\widehat{I}_G$ at each m, then $\widehat{\nu}$ is in $\widehat{I}_G$. Now assume that $\widehat{\nu}$ is in $\mathcal{A}(H)$ and it is annihilated by every polynomial derivation which annihilates $\widehat{I}$ at m in the Fourier algebra $\mathcal{A}(H)$. Then, considering $\widehat{\nu}$ as a function in $\mathcal{A}(G)$, it is annihilated by every polynomial derivation which annihilates $\widehat{I}_G$ at m, where m denotes any extension of m to an exponential on G. By the above argument, $\widehat{\nu}$ is in $\widehat{I}_G$. However, this obviously means that $\widehat{\nu}$ is in $\widehat{I}$, as the support of ν is in H. This proves that $\widehat{I}$ is localizable; hence, $I = \operatorname{Ann} V$ is synthesizable, and we conclude that spectral synthesis holds on H and that our proof is complete. □

We recall the following result as well (see [10]).

Lemma 2. *Every continuous homomorphic image of a synthesizable locally compact abelian group is synthesizable.*

We also need the following simple known result.

Lemma 3. *Let G be a locally compact abelian group and let H be a closed subgroup of G. Then, G is topologically be embedded in $H \times G/H$.*

Proof. We define the continuous homomorphism $\varphi : H \times G \to G$ by

$$\varphi(h, g) = h + g$$

for each h in H and g in G. Clearly, φ is surjective, and the First, Homomorphism Theorem says that G is topologically isomorphic to $(H \times G)/\operatorname{Ker} \varphi$. We note that $\operatorname{Ker} \varphi$ is isomorphic to H. Now we define the following continuous homomorphism $\psi : (H \times G)/\mathrm{Ker}\,\varphi \to H \times G/H$ by

$$\psi\big((h,g) + \mathrm{Ker}\,\varphi\big) = (h, g + \mathrm{Ker}\,\varphi) = (0, g + H)$$

for each h in H and g in G. Clearly, ψ is open, and the kernel of ψ is $\{(0, \mathrm{Ker}\,\varphi)\}$; hence, ψ is a topological isomorphism. As $(H \times G)/\mathrm{Ker}\,\varphi$ is topologically isomorphic to G, our lemma is proved. □

Our next result is the last step toward the proof of our main result.

Lemma 4. *If G is a synthesizable locally compact abelian group, then so is $G \times \mathbb{Z}^k$ for each natural number k.*

Proof. It is enough to show that if G is a synthesizable locally compact abelian group, then so is $G \times \mathbb{Z}$. We shall prove this statement next.

It is known that every exponential $e : \mathbb{Z} \to \mathbb{C}$ has the form

$$e(k) = \lambda^k$$

for k in $\mathbb{Z}$, where λ is a nonzero complex number, which is uniquely determined by e. For this exponential, we use the notation e_λ. It follows that for every commutative topological group G, the exponentials on $G \times \mathbb{Z}$ have the form $m \otimes e_\lambda : (g,k) \mapsto m(g)e_\lambda(k)$, where m is an exponential on G, and λ is a nonzero complex number. Hence, the Fourier transforms in $\mathcal{A}(G \times \mathbb{Z})$ can be thought as two-variable functions defined on the pairs (m, λ), where m is an exponential on G, and λ is a nonzero complex number.

For each measure μ in $\mathcal{M}_c(G \times \mathbb{Z})$ and for every k in $\mathbb{Z}$, we let

$$S_k(\mu) = \{g : \ g \in G \ \text{ and } \ (g,k) \in \mathrm{supp}\,\mu\}.$$

This is the k-projection of the support of μ onto G. As μ is compactly supported, there are only finitely many ks in $\mathbb{Z}$, such that $S_k(\mu)$ is nonempty. We have

$$\mathrm{supp}\,\mu = \bigcup_{k \in \mathbb{Z}} (S_k(\mu) \times \{k\}),$$

and

$$S_k(\mu) \times \{k\} = (G \times \{k\}) \cap \mathrm{supp}\,\mu.$$

It follows that the sets $S_k(\mu) \times \{k\}$ are pairwise disjoint compact sets in $G \times \mathbb{Z}$, and they are nonempty for finitely many ks only. The restriction of μ to $S_k(\mu) \times \{k\}$ is denoted by μ_k. Then, by definition,

$$\langle \mu_k, f \rangle = \int f \cdot \chi_k \, d\mu$$

for each f in $\mathcal{C}(G \times \mathbb{Z})$, where χ_k denotes the characteristic function of the set $S_k(\mu) \times \{k\}$. In other words,

$$\int f \, d\mu_k = \int f(g,k) \, d\mu(g,l)$$

holds for each k in $\mathbb{Z}$ and for every f in $\mathcal{C}(G \times \mathbb{Z})$. Clearly, $\mu = \sum_{k \in \mathbb{Z}} \mu_k$, and this sum is finite.

Let $\delta_{(0,k)}$ denote the Dirac measure at the point $(0,k)$ in $G \times \mathbb{Z}$. For each f in $\mathcal{C}(G \times \mathbb{Z})$, we have

$$\langle \mu_0 * \delta_{(0,k)}, f \rangle = \int\int f(g+h, l+n) \, d\mu_0(g,l) \, d\delta_{(0,k)}(h,n) =$$

$$\int f(g, l+k) \, d\mu_0(g,l) = \int f(g,k) \, d\mu(g,l) = \langle \mu_k, f \rangle.$$

Given a measure μ in $\mathcal{M}_c(G \times \mathbb{Z})$, we define μ_G in $\mathcal{M}_c(G)$ by

$$\langle \mu_G, \varphi \rangle = \int \varphi(g)\, d\mu(g,l)$$

whenever φ is in $\mathcal{C}(G)$. Clearly, every φ in $\mathcal{C}(G)$ can be considered as a function in $\mathcal{C}(G \times \mathbb{Z})$; hence, this definition makes sense. Further, we have

$$\langle \mu_G, \varphi \rangle = \int \varphi(g)\, d\mu_0(g,l).$$

If I is a closed ideal in $\mathcal{M}_c(G \times \mathbb{Z})$, then the set I_G of all measures μ_G with μ in I is a closed ideal in $\mathcal{M}_c(G)$. Indeed, $\mu_G + \nu_G = (\mu + \nu)_G$ and $\lambda \cdot \mu_G = (\lambda \cdot \mu)_G$. Further, if μ_G is in I and ξ is in $\mathcal{M}_c(G)$, then we have the following for each φ in $\mathcal{C}(G)$:

$$\langle \xi * \mu_G, \varphi \rangle = \int\int \varphi(g+h)\, d\xi(g)\, d\mu_G(h) = \int\int \varphi(g+h)\, d\xi(g)\, d\mu(h,l).$$

On the other hand, we extend ξ from $\mathcal{M}_c(G)$ to $\mathcal{M}_c(G \times \mathbb{Z})$ by the definition

$$\langle \tilde{\xi}, f \rangle = \int f(g,0)\, d\xi(g)$$

whenever f is in $\mathcal{C}(G \times \mathbb{Z})$. Then,

$$\langle \tilde{\xi}_G, \varphi \rangle = \int \varphi(g)\, d\tilde{\xi}_0(g,l) = \int \varphi(g)\, d\xi(g) = \langle \xi, \varphi \rangle,$$

that is $\tilde{\xi}_G = \xi$. Finally, a simple calculation shows that

$$\langle \xi * \mu_G, \varphi \rangle = \langle (\tilde{\xi} * \mu)_G, \varphi \rangle;$$

hence, $\xi * \mu_G = (\tilde{\xi} * \mu)_G$ is in I_G, as $\tilde{\xi} * \mu$ is in I. This proves that I_G is an ideal in $\mathcal{M}_c(G)$.

To show that I_G is closed, we assume that (μ_α) is a generalized sequence in I such that the generalized sequence $(\mu_{\alpha,G})$ converges to ξ in $\mathcal{M}_c(G)$. This means that

$$\lim_\alpha \int \varphi(g)\, d\mu_{\alpha,G}(g) = \int \varphi(g)\, d\xi(g)$$

holds for each φ in $\mathcal{C}(G)$. In particular, for each exponential m on G, we have

$$\lim_\alpha \int \check{m}(g)\, d\mu_{\alpha,0}(g,l) = \lim_\alpha \int \check{m}(g)\, d\mu_{\alpha,G}(g) = \int \check{m}(g)\, d\xi(g) = \int \check{m}(g)\, d\tilde{\xi}_0(g,l).$$

In other words, $\lim_\alpha \widehat{\mu}_{\alpha,0} = \widehat{\tilde{\xi}}_0$ holds, which implies $\lim_\alpha \mu_{\alpha,0} = \tilde{\xi}_0$. Consequently,

$$\tilde{\xi}_k = \tilde{\xi}_0 * \delta_{(0,k)} = \lim_\alpha \mu_{\alpha,0} * \delta_{(0,k)} = \lim_\alpha \mu_{\alpha,k}.$$

Hence, we infer

$$\tilde{\xi} = \sum_k \tilde{\xi}_k = \sum_k \lim_\alpha \mu_{\alpha,k} = \lim_\alpha \sum_k \mu_{\alpha,k} = \lim_\alpha \mu_\alpha,$$

as each sum is finite. Since I is closed, $\tilde{\xi}$ is in I, which proves that $\tilde{\xi} = \tilde{\xi}_G$ is in I_G, that is, I_G is closed. □

Now we are ready to prove our main result of Theorem 3.

Proof. If spectral synthesis holds on the locally compact abelian group G, then it holds on every closed subgroup, by Lemma 1. Hence, it holds on D, as every discrete subgroup is closed. On the other hand, if spectral synthesis holds on the locally compact abelian group

G, then it holds on every continuous homomorphic image of G, by Lemma 2. Hence, it holds on G/D. This proves the necessity part of the theorem.

Now suppose that the conditions on G and H are satisfied: H is a synthesizable closed subgroup of G and $D = G/H$ is a synthesizable discrete abelian group. By Corollary 1, D has finite torsion-free rank; hence, it is the continuous homomorphic image of $\mathbb{Z}^k$ with some natural number k. By Lemma 1 and Lemma 3, it is enough to show that $H \times D$ is synthesizable. As $H \times D$ is a continuous image of $H \times \mathbb{Z}^k$, by Lemma 2, it is enough to show that this latter group is synthesizable. However, this follows from Lemma 4. □

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