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On approximately convex functions

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Makó Judit

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Dr. Páles Zsolt
témavezető

On approximately convex functions

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a matematika tudományágban.

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Contents

Introduction

As usual, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ denote the set of natural, integer, rational and real numbers, respectively, $\mathbb{R}_+$ denotes the set of nonnegative real numbers. Denote by I a nonempty real interval.

The investigation of approximate convexity probably started with the paper by Hyers and Ulam [**35**] who in the year 1952 introduced and investigated ε-convex functions. The Hyers and Ulam decomposition theorem of ε-convex functions was later generalized by Páles in [**58**]. Since then many papers on this subject have been published. Two trends in these papers can be observed. One focuses on investigation of the regularity properties (differentiability, Lipschitz or Hölder property, etc.) of approximately convex functions (cf. [**53**], [**54**], [**64**], [**16**], [**27**], [**62, 63, 65**]). The other concerns, roughly speaking, the connections of approximate Jensen convexity and approximate convexity, see, for example, Házy [**30, 31**], Házy and Páles [**32, 33, 34**], Makó and Páles [**44**], Mureńko, Tabor and Tabor [**51**], Tabor and Tabor [**66, 67**], Tabor, Tabor, and Żołdak [**69, 68**].

Our considerations belong to the second current.

In Chapter 1, we introduce Takagi-like functions, which appear naturally in the investigation of approximate convexity. The main results of this chapter examine the approximate Jensen convexity of these Takagi type functions.

First we recall various notions of approximate Jensen convexity and approximate convexity. Let X be a normed space and D be a nonempty convex subset of X. Denote by D^+ the set $\{\|x-y\| : x, y \in D\}$. Let $\varepsilon, q \in \mathbb{R}_+$. We say that $f : D \to \mathbb{R}$ is (ε, q)-Jensen convex, if

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2} + \varepsilon\|x-y\|^q \qquad (x, y \in D).$$

A more general version of approximate Jensen convexity was examined in [**66**] and [**44**]. Given a nonnegative error function $\varphi : D^+ \to \mathbb{R}$, we say that a function $f : D \to \mathbb{R}$ is φ-Jensen convex if

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2} + \varphi(\|x-y\|) \qquad (x, y \in D).$$

By taking the function $\varphi(t) := \varepsilon t^q$, we can see that (ε, q)-Jensen convexity is a particular case of φ-Jensen convexity.

The fundamental problem of the theory of convexity is to establish a relationship between approximate Jensen convexity and approximate convexity. Let us mention, in chronological order, the most important results in this direction. We should recall first the Bernstein–Doetsch Theorem [**8**], which says that every locally bounded Jensen convex function is convex. In all the results quoted below we assume that f is locally bounded. In the year 1979 Rolewicz [**63**] showed that in the case when $q > 2$ every continuous (ε, q)-Jensen convex function is convex. Ng and Nikodem in [**52**] proved that when $q = 0$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon \sum_{n=0}^{\infty} \frac{\chi_{\mathbb{R}\setminus\mathbb{Z}}(2^n t)}{2^n} \qquad (t \in [0,1],\ x, y \in D),$$

where $\chi_{\mathbb{R}\setminus\mathbb{Z}}$ denotes the characteristic function of $\mathbb{R} \setminus \mathbb{Z}$. They also showed that this error function is optimal. Next, Házy and Páles considered first the case when $q = 1$ in [**32**], and later in [**33**], they showed that, if f is (ε, q)-Jensen convex, then

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon \sum_{n=0}^{\infty} \frac{d_{\mathbb{Z}}^q(2^n t)}{2^n} \|x - y\|^q \quad (t \in [0,1],\ x, y \in D),$$

where

$$d_{\mathbb{Z}}(t) := 2\,\mathrm{dist}(t, \mathbb{Z}) := 2 \min\{|t - z| : z \in \mathbb{Z}\} \qquad (t \in \mathbb{R}).$$

Páles (see [**59**], 7. problem, p. 307) posed the problem whether the above error function is optimal. If $q = 0$, then this reduced the problem of Ng and Nikodem, who showed that their estimation is optimal. In the case when $q = 1$, Boros proved in [**14**] that the previous inequality includes also the best error function. In Chapter 1, we will examine the optimal estimation the case when $q \in]0, 1]$. More generally, we will also prove that the optimal estimation for φ-Jensen convex functions is given by

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \sum_{n=0}^{\infty} \frac{\varphi(d_{\mathbb{Z}}(2^n t)\|x - y\|)}{2^n} \quad (t \in [0,1], x, y \in D),$$

if we have some natural regularity assumptions for φ.

The remaining case $q \in [1, 2]$ was completely solved in [**67**] by Tabor and Tabor, where these authors prove that the optimal estimation for an (ε, q)-Jensen convex function is given by

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon \sum_{n=0}^{\infty} \frac{d_{\mathbb{Z}}(2^n t)}{2^{nq}} \|x - y\|^q \qquad (t \in [0,1],\ x, y \in D).$$

In [**66**], Tabor and Tabor showed that for all φ-Jensen convex functions are approximately convex in the following sense

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)+\sum_{n=0}^{\infty}\varphi\Big(\frac{\|x-y\|}{2^n}\Big)d_{\mathbb{Z}}(2^n t) \quad (t\in[0,1],\ x,y\in D).$$

In Chapter 1, we will also prove that this error function is optimal under certain regularity conditions for the function φ.

The integral average of any standard convex function $f : I \to \mathbb{R}$ can be estimated by the values of the function at the midpoint and at the endpoints of the domain as follows:

$$f\Big(\frac{x+y}{2}\Big) \le \int_0^1 f(tx+(1-t)y)dt \le \frac{f(x)+f(y)}{2} \qquad (x,y\in I). \tag{1}$$

This is the well known Hermite–Hadamard inequality. The above implication was discovered by Hadamard [**28**]. (See also [**49**], [**41**], and [**56**], [**23**], [**55**], [**56**], [**57**] for a historical account.)

More generally, it is easy to see that the ε-convexity of f (cf. [**35**]), i.e., the validity of

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)+\varepsilon \qquad (x,y\in D,\ t\in[0,1]),$$

implies the following ε-Hermite–Hadamard inequalities

$$f\Big(\frac{x+y}{2}\Big) \le \int_0^1 f(tx+(1-t)y)dt+\varepsilon \qquad (x,y\in D). \tag{2}$$

and

$$\int_0^1 f(tx+(1-t)y)dt \le \frac{f(x)+f(y)}{2}+\varepsilon \qquad (x,y\in D). \tag{3}$$

Concerning the reversed implication, Nikodem, Riedel, and Sahoo in [**57**] have recently shown that the ε-Hermite–Hadamard inequalities (2) and (3) do not imply the $c\varepsilon$-convexity of f (with any $c>0$). Thus, in order to obtain results that establish implications between the approximate Hermite–Hadamard inequalities and the approximate convexity type inequality, one has to consider these inequalities with nonconstant error terms.

Let X be a linear space and D be a nonempty convex subset of X and denote by D^* the following set $\{x-y \mid x,y\in D\}$. In Chapter 2, we will investigate the connections between approximate upper Hermite–Hadamard type inequality and Jensen type inequalities. In other words, we consider functions $f : D\to\mathbb{R}$ satisfying

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2}+\alpha(x-y) \qquad (x,y\in D)$$

$$\int_0^1 f(tx+(1-t)y)\rho(t)dt \le \lambda f(x)+(1-\lambda)f(y)+\beta(x-y) \qquad (x,y\in D),$$

where $\alpha, \beta : D^* \to \mathbb{R}$ are given even functions, $\lambda \in [0,1]$, and $\rho : [0,1] \to \mathbb{R}_+$ is an integrable nonnegative function with $\int_0^1 \rho(t)dt = 1$. The heart of our approach is a multiplicative type convolution and its asymptotic properties.

In Chapter 3, we will examine the connections between lower Hermite–Hadamard type inequalities and convexity type inequalities. In other words, we consider f satisfying

$$f((1-\mu_1)x+\mu_1 y) \le \int\limits_{[0,1]} f((1-t)x+ty)d\mu(t) + E(x,y) \quad ((x,y)\in D^2), \tag{4}$$

$$f((1-t)x+ty) \le (1-t)f(x)+tf(y)+e_{x,y}(t) \quad ((x,y)\in D^2,\ t\in[0,1]),$$

where μ is a measure on $[0,1]$, $\mu_1 := \int_{[0,1]} td\mu(t)$, $E : D^2 \to \mathbb{R}$ and, for all $(x,y) \in D^2$, $e_{x,y} : [0,1] \to \mathbb{R}$ is a function. In fact, we consider lower Hermite–Hadamard type inequalities and convexity type inequalities in a more general case using the definition of Chebyshev-system and means. The key for the proof of the main result is a Korovkin type theorem which enables us deduce the approximate convexity property from the approximate lower Hermite–Hadamard type inequality via an iteration process. Consider (4), when μ is a measure on $[0,1]$ defined by $\mu := (1-\tau)\delta_0 + \tau\delta_1$, where $\tau \in]0,1[$ and δ is the Dirac-measure. Then (4) reduces a Jensen type inequality. Thus an important application of our main result is a Bernstein–Doetsch type theorem (cf. [**8**]).

Terminology and notations

In this section, we introduce the necessary notations and terminology. Define the function $d_{\mathbb{Z}} : \mathbb{R} \to \mathbb{R}$ by the following formula:

$$d_{\mathbb{Z}}(x) := 2\,\mathrm{dist}(x,\mathbb{Z}) := 2\min\{|x-z| : z\in\mathbb{Z}\} \qquad (x\in\mathbb{R}).$$

Given a nonempty, convex subset D of the linear space X, denote

$$D^* := (D-D) \qquad \text{and} \qquad D^{2*} := \{(x,y)\in D^2 \mid x\ne y\}.$$

If, in addition, X is a normed space, then we set

$$D^+ := \{\|x-y\| : x,y\in D\}.$$

Let $\alpha : D^* \to \mathbb{R}$ be a given even function (we note that α need not be nonnegative). A function $f : D \to \mathbb{R}$ is called *approximate α-Jensen convex* on D (cf. Makó and Páles [**44**], Tabor and Tabor [**66, 67**]) if, for all $x, y \in D$,

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2} + \alpha(x-y). \tag{5}$$

If X is a normed space and $\alpha(u)$ is a function of $\|u\|$, i.e., $\alpha = \varphi \circ \|\cdot\|$ for some $\varphi : D^+ \to \mathbb{R}$, then (5) reduces to

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2} + \varphi(\|x-y\|) \quad (x, y \in D). \tag{6}$$

In this case, f is termed φ-Jensen convex on D. An important particular case occurs when $\varphi : D^+ \to \mathbb{R}$ is of the form $\varphi(t) := \varepsilon t^q$, where $q \ge 0$ and $\varepsilon \in \mathbb{R}$ are arbitrary constants. Then f is called (ε, q)-*Jensen convex* on D (cf. [**58, 32, 33**]).

We need to introduce the following terminology. For a function $f : D \to \mathbb{R}$, we say that f is *hemi-P* if, for all $x, y \in D$, the mapping

$$t \mapsto f((1-t)x + ty) \qquad (t \in [0,1]) \tag{7}$$

has property P. For example, f is hemi-bounded if for all $x, y \in D$ the mapping defined by (7) is bounded. Analogously, we say that a function $h : D^* \to \mathbb{R}$ is *radially-P* if, for all $u \in D^*$, the mapping

$$t \mapsto h(tu) \qquad (t \in [0,1])$$

has property P on $[0, 1]$.

Approximate convexity of Takagi functions

Let X be a normed space and D be a convex subset of X. For a fixed error function $\varphi : D^+ \to \mathbb{R}_+$, we introduce the Takagi type functions $\mathcal{T}_\varphi : \mathbb{R} \times D^+ \to \mathbb{R}_+$ and $\mathcal{S}_\varphi : \mathbb{R} \times D^+ \to \mathbb{R}_+$ by

$$\mathcal{T}_\varphi(t,u) := \sum_{n=0}^{\infty} \frac{1}{2^n}\varphi(d_{\mathbb{Z}}(2^n t)u) \qquad ((t,u) \in \mathbb{R} \times D^+) \tag{1.1}$$

and

$$\mathcal{S}_\varphi(t,u) := \sum_{n=0}^{\infty} \varphi\Big(\frac{u}{2^n}\Big)d_{\mathbb{Z}}(2^n t) \qquad ((t,u) \in \mathbb{R} \times D^+). \tag{1.2}$$

Note that the first series converges uniformly if φ is bounded, on the other hand, for the uniform convergence of the second series, it is sufficient if $\sum_{n=n_0}^{\infty} \varphi(2^{-n}) < \infty$ for some $n_0 \in \mathbb{N}$.

The importance of the function $\mathcal{T}_\varphi$ introduced above is enlightened by the following result which can be considered as a generalization of the celebrated Bernstein–Doetsch theorem [**8**].

Theorem 1.1. (Makó–Páles [**44**], Tabor–Tabor [**67**])
Let $f : D \to \mathbb{R}$ be locally upper bounded on D. Assume that $\varphi : D^+ \to \mathbb{R}_+$ is a continuous function such that $\varphi(0) = 0$. Then f is φ-Jensen convex on D if and only if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \mathcal{T}_\varphi(t, \|x-y\|) \tag{1.3}$$

for all $x, y \in D$ and $t \in [0,1]$.

The other Takagi type function $\mathcal{S}_\varphi$ was introduced by Jacek Tabor and Józef Tabor. Its role and importance in the theory of approximate convexity is shown by the next theorem.

Theorem 1.2. (Tabor–Tabor [**67**])
Let $f : D \to \mathbb{R}$ be upper semicontinuous on D and let $\varphi : D^+ \to \mathbb{R}_+$ be nondecreasing

such that $\sum_{n=n_0}^{\infty} \varphi(2^{-n}) < \infty$ *for some* $n_0 \in \mathbb{N}$*. Then* f *is* φ*-Jensen convex on* D *if and only if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \mathcal{S}_{\varphi}(t, \|x - y\|) \tag{1.4}$$

for all $x, y \in D$ *and* $t \in [0, 1]$.

Let $\varepsilon, q \geq 0$ be arbitrary constants. When $\varphi(u) := \varepsilon u^q$, $(u \in D^+)$, the two corollaries below (see [**33**] and [**67**]) are immediately consequences of the previous theorems.

For $q > 0$, define the Takagi type functions S_q and T_q by

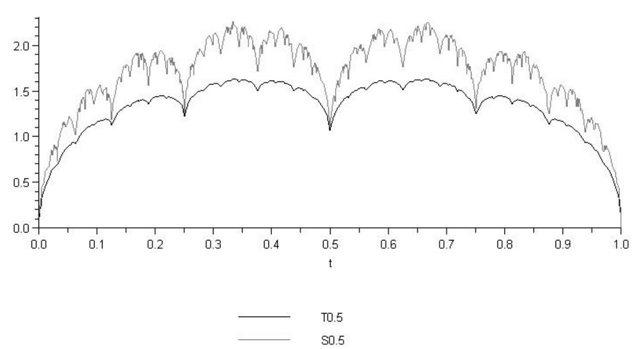
$$T_q(t) := \sum_{n=0}^{\infty} \frac{(d_{\mathbb{Z}}(2^n t))^q}{2^n}, \qquad S_q(t) := \sum_{n=0}^{\infty} \frac{d_{\mathbb{Z}}(2^n t)}{2^{nq}} \qquad (t \in \mathbb{R}). \tag{1.5}$$

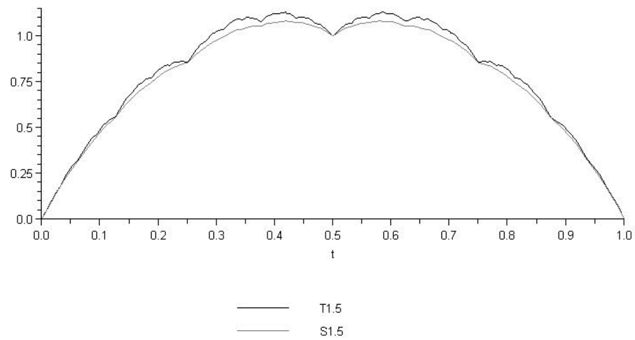
They generalize the classical Takagi function

$$T(t) := \sum_{n=0}^{\infty} \frac{\operatorname{dist}(2^n t, \mathbb{Z})}{2^n} \qquad (t \in \mathbb{R})$$

in two ways, because $T_1 = S_1 = 2T$ holds obviously. This function was introduced by Takagi in [**70**] and it is a well-known example of a continuous but nowhere differentiable real function. For further historical details and remarks, we refer to the papers Billingsley [**13**], Cater [**17**], Kairies [**36**] and Allaart and Kawamura [**6**].

It is less trivial, but it can be proved that $T_2(t) = S_2(t) = 4t(1-t)$ for $t \in [0, 1]$. The following pictures demonstrate the comparison between T_q and S_q for $q = 0.5$ and $q = 1.5$, respectively.





Corollary 1.3. (Házy [**29**])
Let $f : D \to \mathbb{R}$ be locally upper bounded on D, $q > 0$ and $\varepsilon \geq 0$. Then f is (ε, q)-Jensen convex on D, if and only if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon T_q(t)\|x-y\|^q \tag{1.6}$$

for all $x, y \in D$ and $t \in [0,1]$.

Corollary 1.4. (Tabor–Tabor [**67**])
Let $f : D \to \mathbb{R}$ be upper semicontinuous on D, $q > 0$ and $\varepsilon \geq 0$. Then f is (ε, q)-Jensen convex on D if and only if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon S_q(t)\|x-y\|^q \tag{1.7}$$

for all $x, y \in D$ and $t \in [0,1]$.

In [**14**] Boros proved that if $q = 1$ and $t \in [0,1]$ is fixed, then $S_1(t) = T_1(t) = 2T(t)$ is the smallest possible. In [**66**] Tabor and Tabor showed that if $1 \leq q \leq 2$ and $t \in [0,1]$ is fixed, then $S_q(t)$ is the smallest possible value so that (1.7) be valid for all (ε, q)-Jensen convex functions f on D.

It is an important question whether the error terms $\mathcal{T}_\varphi(t, \|x-y\|)$, $\mathcal{S}_\varphi(t, \|x-y\|)$ in (1.3) in (1.4) and $T_q(t)$ in (1.6) are the smallest possible ones. In other words, for all fixed $x, y \in D$, we want to obtain the exact upper bound of the convexity-difference of φ-Jensen convex functions defined by

$$C_\varphi(x, y, t) := \sup_{f \in \mathcal{JC}_\varphi(D)} \{f(tx + (1-t)y) - tf(x) - (1-t)f(y)\}, \tag{1.8}$$

where

$$\mathcal{JC}_\varphi(D) := \{f : D \to \mathbb{R} \mid f \text{ is } \varphi\text{-Jensen convex on } D\}.$$

The statement of Theorem 1.1, Theorem 1.2, Corollary 1.3, and Corollary 1.4 can be stated as

$$C_\varphi(x, y, t) \leq \tau(t, \|x-y\|), \tag{1.9}$$

where $\tau : \mathbb{R} \times D^+ \to \mathbb{R}_+$ is given by

$$\tau := \mathcal{T}_\varphi, \quad \tau := \mathcal{S}_\varphi, \quad \tau(t,u) := \varepsilon T_q(t) u^q, \quad \text{and} \quad \tau(t,u) := \varepsilon S_q(t) u^q,$$

respectively. To obtain also a lower bound for $C_\varphi(x,y,t)$, (and thus to prove the sharpness of the inequality (1.9)), the following important observation was done by Páles in [**59**].

Theorem 1.5. (Páles [**59**])
Let $\varphi : D^+ \to \mathbb{R}_+$ be increasing. Let $\tau : \mathbb{R} \times D^+ \to \mathbb{R}_+$ be continuous and 1-periodic in its first variable, with $\tau(0,u) = 0$ for all $u \in D^+$, which is Jensen convex in the following sense, for all $u \in D^+$ and $s,t \in [0,1]$,

$$\tau\Big(\frac{s+t}{2}, u\Big) \le \frac{\tau(s,u) + \tau(t,u)}{2} + \varphi(|s-t|u). \tag{1.10}$$

Then, for all $x,y \in D$ and $t \in [0,1]$,

$$C_\varphi(x,y,t) \ge \tau(t, \|x-y\|). \tag{1.11}$$

Proof. Let $x,y \in D$ be arbitrary fixed. If $x = y$, the proof is evident. Assume that $x \ne y$. Then by Hahn-Banach theorem, there exists $0 \ne x^* \in X^*$ such that

$$x^*(x-y) = \|x^*\| \cdot \|x-y\|.$$

Define $g_{x,y} : D \to \mathbb{R}$ by the following way:

$$g_{x,y}(u) := \tau\Big(\frac{x^*(u-y)}{\|x^*\| \cdot \|x-y\|}, \|x-y\|\Big) \qquad (u \in D).$$

Now we prove that $g_{x,y} \in \mathcal{JC}_\varphi(D)$. Let $u,v \in D$. Then, using the definition of $g_{x,y}$ and (1.10), we get that

$$\begin{aligned}
g_{x,y}\Big(\frac{u+v}{2}\Big) &- \frac{g_{x,y}(u) + g_{x,y}(v)}{2} \\
&= \tau\Big(\frac{1}{2}\Big(\frac{x^*(u) - x^*(y)}{\|x^*\| \cdot \|x-y\|} + \frac{x^*(v) - x^*(y)}{\|x^*\| \cdot \|x-y\|}\Big), \|x-y\|\Big) \\
&\quad - \frac{1}{2}\Big(\tau\Big(\frac{x^*(u) - x^*(y)}{\|x^*\| \cdot \|x-y\|}, \|x-y\|\Big) + \tau\Big(\frac{x^*(v) - x^*(y)}{\|x^*\| \cdot \|x-y\|}, \|x-y\|\Big)\Big) \\
&\le \varphi\Big(\frac{x^*(u) - x^*(v)}{\|x^*\| \cdot \|x-y\|}\|x-y\|\Big) \\
&\le \varphi\Big(\frac{\|x^*\| \cdot \|u-v\|}{\|x^*\| \cdot \|x-y\|}\|x-y\|\Big) = \varphi(\|u-v\|),
\end{aligned}$$

which proves that $g_{x,y} \in \mathcal{JC}_\varphi(D)$. Then, using $\tau(0, \|x-y\|) = \tau(1, \|x-y\|) = 0$, we get

$$\begin{aligned} C_\varphi(x,y,t) &\geq g_{x,y}(tx + (1-t)y) - tg_{x,y}(x) - (1-t)g_{x,y}(y) \\ &= \tau(t, \|x-y\|) - t\tau(1, \|x-y\|) - (1-t)\tau(0, \|x-y\|) \\ &= \tau(t, \|x-y\|), \end{aligned}$$

which proves (1.11). □

1.1. Approximate convexity of Takagi type functions $\mathcal{T}_\varphi$

Given a bounded function $\Phi : \mathbb{R} \to \mathbb{R}$, we define the Takagi type function $T_\Phi : \mathbb{R} \to \mathbb{R}$ by

$$T_\Phi(x) = \sum_{n=0}^{\infty} \frac{\Phi(2^n x)}{2^n}.$$

Observe that $\Phi \mapsto T_\Phi$ is a bounded linear map on the space of bounded real functions for which $T_C = 2C$ for each constant function C. It is also easy to see that T_Φ satisfies the following identity:

$$T_\Phi(x) = \Phi(x) + \frac{1}{2} T_\Phi(2x) \qquad (x \in \mathbb{R}). \tag{1.12}$$

Furthermore, if Φ is continuous and 1-periodic then T_Φ is also continuous and 1-periodic, respectively. Obviously, with $\Phi := \operatorname{dist}(\cdot, \mathbb{Z})$, we get the classical Takagi function defined. Taking $\Phi := d_{\mathbb{Z}}^q$, the obtain the Takagi type function defined in (1.5).

The main purpose of this section is to prove that a certain class of the Takagi type function T_Φ is also approximately Jensen convex in an appropriate sense. In other words, we are looking for a function $\Psi : \mathbb{R} \to \mathbb{R}$ such that

$$T_\Phi\Big(\frac{x+y}{2}\Big) \leq \frac{T_\Phi(x) + T_\Phi(y)}{2} + \Psi\Big(\frac{x-y}{2}\Big) \qquad (x, y \in \mathbb{R}) \tag{1.13}$$

be satisfied. Putting $y = 0$, replacing x by $2x$, and observing that $T_\Phi(0) = 2\Phi(0)$, (1.13) reduces to

$$T_\Phi(x) \leq \frac{1}{2} T_\Phi(2x) + \Phi(0) + \Psi(x) \qquad (x \in \mathbb{R}),$$

which, by (1.12), is equivalent to

$$\Phi(x) \leq \Phi(0) + \Psi(x) \qquad (x \in \mathbb{R}).$$

This means that $\Psi := \Phi - \Phi(0)$ is the smallest possible choice for Ψ. In Theorem 1.6 below, we will prove that, for a certain class of functions Φ, this choice is indeed appropriate. Then, we apply the results on the approximate φ-Jensen convexity of

Takagi type functions to obtain the sharpness of the error terms for the corresponding approximate convexity property.

Our first main result states that, under certain assumptions on $\Phi : \mathbb{R} \to \mathbb{R}$, the approximate Jensen convexity inequality (1.13) holds with $\Psi := \Phi - \Phi(0)$.

Theorem 1.6. (Makó–Páles [**43**])
Assume that $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous, 1-periodic and, for all $x, y \in \mathbb{R}$,

$$\begin{aligned} 2\Phi(x) - \Phi(x-y) - \Phi(x+y) \\ + \min\Big(0, \Phi(2x) - \Phi(2x+\tfrac{1}{2}) + \Phi(2y+\tfrac{1}{2}) - \Phi(2y)\Big) \\ \le 2\Phi(y) - \Phi(0) - \Phi(2y). \end{aligned} \tag{1.14}$$

Then T_Φ satisfies, for all $x, y \in \mathbb{R}$, the approximate Jensen convexity inequality

$$2T_\Phi(x) \le T_\Phi(x+y) + T_\Phi(x-y) + 2(\Phi(y) - \Phi(0)). \tag{1.15}$$

Proof. Replacing Φ by $\Phi - \Phi(0)$ if necessary, we may assume that $\Phi(0) = 0$.

By the 1-periodicity, continuity of Φ and $\Phi(0) = 0$, the function T_Φ is 1-periodic, continuous, vanishes on $\mathbb{Z}$, and $T_\Phi(x + \frac{1}{2}) = \Phi(\frac{1}{2})$ holds for all $x \in \mathbb{Z}$. Substituting $x = 0$ and $y = 1/2$ into (1.14), it follows that $\Phi(\frac{1}{2}) \ge 0$.

We show, by induction on $n \in \mathbb{N}$, that (1.15) holds for all $x, y \in \mathbb{Z}/2^n$. Then the statement follows from the continuity of T_Φ and the denseness of dyadic rational numbers in $\mathbb{R}$.

To prove the statement in the case $n = 1$, let $x, y \in \mathbb{Z}/2$ be arbitrary.

If $x, y \in \mathbb{Z}$, then $T_\Phi(x) = T_\Phi(x-y) = T_\Phi(x+y) = 0$ and $\Phi(y) = 0$, hence (1.15) holds with equality.

If $x \notin \mathbb{Z}$, $y \in \mathbb{Z}$, then, by the 1-periodicity, we have $T_\Phi(x+y) = T_\Phi(x-y) = T_\Phi(x)$ and $\Phi(y) = 0$, hence (1.15) holds with equality.

If $x \in \mathbb{Z}$, $y \notin \mathbb{Z}$, then $T_\Phi(x) = 0$ and $T_\Phi(x-y) = T_\phi(x+y) = \Phi(y) = \Phi(\frac{1}{2})$, hence (1.15) is equivalent to $0 \le \Phi(\frac{1}{2})$, which was derived from (1.14).

If $x \notin \mathbb{Z}$, $y \notin \mathbb{Z}$, then $T_\Phi(x) = \Phi(y) = \Phi(\frac{1}{2})$ and $T_\Phi(x-y) = T_\Phi(x+y) = 0$ since $x \pm y \in \mathbb{Z}$. Thus (1.15) again holds with equality.

Now assume that the statement holds for all $x, y \in \mathbb{Z}/2^n$ for some $n \in \mathbb{N}$ and let $x, y \in \mathbb{Z}/2^{n+1}$ be arbitrary. We distinguish two cases according to the validity of the inequality

$$0 \le \Phi(2x) - \Phi(2x+\tfrac{1}{2}) + \Phi(2y+\tfrac{1}{2}) - \Phi(2y) \tag{1.16}$$

Case I: When (1.16) *holds.* Then (1.14) reduces to

$$2\Phi(x) - \Phi(x-y) - \Phi(x+y) \le 2\Phi(y) - \Phi(2y). \tag{1.17}$$

Since $2x, 2y \in \mathbb{Z}/2^n$, therefore, by the inductive assumption, we also have

$$T_\Phi(2x) - \tfrac{1}{2}(T_\Phi(2x+2y) + T_\Phi(2x-2y)) \le \Phi(2y). \tag{1.18}$$

Using (1.12), (1.17), and (1.18), we get

$$\begin{aligned}
&2T_\Phi(x) - (T_\Phi(x+y) + T_\Phi(x-y)) \\
&= 2\Phi(x) + T_\Phi(2x) - \Big(\Phi(x+y) + \frac{1}{2}T_\Phi(2x+2y) + \Phi(x-y) + \frac{1}{2}T_\Phi(2x-2y)\Big) \\
&= 2\Phi(x) - \Phi(x-y) - \Phi(x+y) + T_\Phi(2x) - \frac{1}{2}(T_\Phi(2x+2y) + T_\Phi(2x-2y)) \\
&\le 2\Phi(y) - \Phi(2y) + \Phi(2y) = 2\Phi(y).
\end{aligned}$$

Case II: When (1.16) *does not hold.* Then (1.14) reduces to

$$2\Phi(x) - \Phi(x-y) - \Phi(x+y) + \Big(\Phi(2x) - \Phi(2x+\tfrac{1}{2}) + \Phi(2y+\tfrac{1}{2})\Big) \le 2\Phi(y). \tag{1.19}$$

We have that $2x+\frac{1}{2}, 2y+\frac{1}{2} \in \mathbb{Z}/2^n$, whence, by the inductive assumption, we obtain

$$T_\Phi(2x+\tfrac{1}{2}) - \frac{1}{2}(T_\Phi(2x+2y+1) + T_\Phi(2x-2y)) \le \Phi(2y+\tfrac{1}{2}). \tag{1.20}$$

Using (1.12) several times, the 1-periodicity, (1.19), and (1.20), we get

$$\begin{aligned}
&2T_\Phi(x) - (T_\Phi(x+y) + T_\Phi(x-y)) \\
&= 2\Phi(x) + T_\Phi(2x) - \Big(\Phi(x+y) + \frac{1}{2}T_\Phi(2x+2y) + \Phi(x-y) + \frac{1}{2}T_\Phi(2x-2y)\Big) \\
&= 2\Phi(x) - \Phi(x-y) - \Phi(x+y) + T_\Phi(2x) - T_\Phi(2x+\tfrac{1}{2}) \\
&\qquad + T_\Phi(2x+\tfrac{1}{2}) - \tfrac{1}{2}(T_\Phi(2x+2y+1) + T_\Phi(2x-2y)) \\
&\le 2\Phi(x) - 2\Phi(x-y) - \Phi(x+y) \\
&\qquad + \Phi(2x) + \frac{1}{2}T_\Phi(4x) - \Phi(2x+\tfrac{1}{2}) - \frac{1}{2}T_\Phi(4x+1) + \Phi(2y+\tfrac{1}{2}) \\
&\le 2\Phi(y).
\end{aligned}$$

This completes the proof. □

In order to obtain a verifiable form of condition (1.14), we shall need the notion of higher-order monotonicity and convexity. Let $I \subseteq \mathbb{R}$ be a proper interval and $\phi : I \to \mathbb{R}$. Given $h \in \mathbb{R}$, we use the notation $\Delta_h\phi(x) := \phi(x+h) - \phi(x)$ whenever $x \in I \cap (I-h)$. We say that a function ϕ is *n-monotone ($(n-1)$-Wright-convex) on I* if, for all $h_1, \dots, h_n \ge 0$ and for all $x \in I \cap (I - h_1 - \cdots - h_n)$, the inequality

$$\Delta_{h_1} \cdots \Delta_{h_n}\phi(x) \ge 0$$

holds. Observe that 1-monotonicity is equivalent to the nondecreasingness of ϕ. Furthermore, 2-monotonicity can be rephrased as the following property: for all $x, y \in I$ and $t \in [0,1]$,

$$\phi(tx + (1-t)y) + \phi((1-t)x + ty) \le \phi(x) + \phi(y),$$

which is called the Wright-convexity of ϕ (cf. Wright [**72**], Gilányi and Páles [**25**], [**26**]). In the case of continuity, the Wright-convexity is equivalent to convexity of ϕ.

The following lemma characterizes n-monotonicity for n-times differentiable functions.

Lemma 1.7. (Popoviciu [**60**], Roberts–Varberg [**61**], Kuczma [**41**])
Let $\phi : I \to \mathbb{R}$ be a continuous function which is n-times differentiable on the interior I° of I. Then ϕ is n-monotone if and only if its nth derivative $\phi^{(n)}$ is nonnegative on I°.

In the next result, we investigate condition (1.14) for functions Φ of the form $\phi \circ d_{\mathbb{Z}}$, where $\phi : [0,1] \to \mathbb{R}$.

Theorem 1.8. (Makó–Páles [**43**])
Let $\phi : [0,1] \to \mathbb{R}$ be a continuous function such that ϕ is 1- and 3-monotone, and $(-\phi)$ is 2-monotone. Then the function $\Phi := \phi \circ d_{\mathbb{Z}}$ is 1-periodic, even, and fulfills (1.14) *for all $x, y \in \mathbb{R}$.*

Proof. In the proof, we will frequently use the following easy-to-check identity:

$$d_{\mathbb{Z}}(u + \tfrac{1}{2}) = 2 - d_{\mathbb{Z}}(u) \qquad (u \in \mathbb{R}). \tag{1.21}$$

Without loss of generality, we may assume that $\phi(0) = 0$ (otherwise we replace ϕ by $\phi - \phi(0)$). Then, ϕ is nonnegative and hence $\Phi := \phi \circ d_{\mathbb{Z}}$ is 1-periodic, even and nonnegative. In view of the 1-periodicity of Φ, if (1.14) holds for some $(x, y) \in \mathbb{R}^2$, then it also holds for $(x + n, y + m) \in \mathbb{R}^2$ where $n, m \in \mathbb{Z}$ is arbitrary. Therefore, we may assume that $|x|, |y| \le \frac{1}{2}$. Using also the evenness of Φ, it follows that if (1.14) holds for some $(x, y) \in \mathbb{R}^2$ then it also holds for $(\pm x, \pm y) \in \mathbb{R}^2$. Hence, we may also assume that $x, y \ge 0$. In the rest of the proof we distinguish two main cases and two subcases in each case.

Case I: When $d_{\mathbb{Z}}(2y) \le d_{\mathbb{Z}}(2x)$ holds. By (1.21) and the inequality $2 - d_{\mathbb{Z}}(2y) \ge 2 - d_{\mathbb{Z}}(2x)$, we get $d_{\mathbb{Z}}(2y + \frac{1}{2}) \ge d_{\mathbb{Z}}(2x + \frac{1}{2})$. Since ϕ is nondecreasing, we obtain that $\Phi(2y) \le \Phi(2x)$ and $\Phi(2y + \frac{1}{2}) \ge \Phi(2x + \frac{1}{2})$. In this case

$$\Phi(2x) - \Phi(2x + \tfrac{1}{2}) + \Phi(2y + \tfrac{1}{2}) - \Phi(2y) \ge 0,$$

therefore (1.14) reduces to

$$2\Phi(x) - \Phi(x - y) - \Phi(x + y) \le 2\Phi(y) - \Phi(2y). \tag{1.22}$$

Subcase Ia: When $y \in [0, 1/4]$ holds. Then, $d_{\mathbb{Z}}(2y) \le d_{\mathbb{Z}}(2x)$ yields that $0 \le x + y \le \frac{1}{2}$ and $0 \le x - y \le \frac{1}{2}$. By the 3-monotonicity of ϕ, we obtain that $\Delta_{2x-2y}\Delta^2_{2y}\phi(0) \ge 0$, which is equivalent to

$$2\phi(2x) - \phi(2x - 2y) - \phi(2x + 2y) \le 2\phi(2y) - \phi(4y).$$

Therefore,

$$2\Phi(x) - \Phi(x-y) - \Phi(x+y) = 2\phi(2x) - \phi(2x-2y) - \phi(2x+2y)$$
$$\leq 2\phi(2y) - \phi(4y) = 2\Phi(y) - \Phi(2y),$$

which proves (1.22) in this subcase.

Subcase Ib: When $y \in [\frac{1}{4}, \frac{1}{2}]$ *holds.* Then, by $d_{\mathbb{Z}}(2y) \leq d_{\mathbb{Z}}(2x)$, we have that $\frac{1}{2} \leq x + y \leq 1$ and $0 \leq y - x \leq \frac{1}{2}$. Applying the 3-monotonicity of ϕ, we have $\Delta_{2y-2x}\Delta^2_{1-2y}\phi(0) \geq 0$, which yields

$$-\phi(2y-2x) - \phi(2-2x-2y) \leq 2\phi(1-2y) - \phi(2-4y) - 2\phi(1-2x). \tag{1.23}$$

By the 1-monotonicity of ϕ, we also have that

$$\phi(2x) \leq \phi(2y), \qquad \phi(1-2y) \leq \phi(1-2x). \tag{1.24}$$

Thus, using these inequalities, we get

$$\begin{aligned} 2\Phi(x) - \Phi(x-y) - \Phi(x+y) &= 2\phi(2x) - \phi(2y-2x) - \phi(2-2x-2y) \\ &\overset{(1.23)}{\leq} 2\phi(2x) + 2\phi(1-2y) - \phi(2-4y) - 2\phi(1-2x) \\ &\overset{(1.24)}{\leq} 2\phi(2y) - \phi(2-4y) = 2\Phi(y) - \Phi(2y), \end{aligned}$$

which completes the proof of (1.22) in this subcase.

Case II: When $d_{\mathbb{Z}}(2y) \geq d_{\mathbb{Z}}(2x)$ *holds.* Then

$$\Phi(2x) - \Phi(2x + \tfrac{1}{2}) + \Phi(2y + \tfrac{1}{2}) - \Phi(2y) \leq 0,$$

therefore (1.14) reduces to

$$2\Phi(x) - \Phi(x-y) - \Phi(x+y) + \Phi(2x) - \Phi(2x + \tfrac{1}{2}) + \Phi(2y + \tfrac{1}{2}) \leq 2\Phi(y). \tag{1.25}$$

Subcase IIa: When $0 \leq x \leq \frac{1}{4}$ *holds.* Then by $d_{\mathbb{Z}}(2y) \geq d_{\mathbb{Z}}(2x)$, we have $0 \leq y - x \leq \frac{1}{2}$ and $0 \leq x + y \leq \frac{1}{2}$. Applying the 1-monotonicity of ϕ, we have

$$\phi(2x) \leq \phi(2y), \qquad \phi(|1-4y|) \leq \phi(1-4x). \tag{1.26}$$

On the other hand, by the 3-monotonicity of ϕ, we get $\Delta_{2y-2x}\Delta^2_{2x}\phi(0) \geq 0$, which yields

$$2\phi(2y) - \phi(2y-2x) - \phi(2x+2y) + \phi(4x) \leq 2\phi(2x). \tag{1.27}$$

Therefore, by using the above inequalities, we obtain

$$\begin{aligned} &2\Phi(x) - \Phi(x-y) - \Phi(x+y) + \Phi(2x) - \Phi(2x + \tfrac{1}{2}) + \Phi(2y + \tfrac{1}{2}) \\ &\quad = 2\phi(2x) - \phi(2y-2x) - \phi(2x+2y) + \phi(4x) - \phi(1-4x) + \phi(|1-4y|) \\ &\quad \overset{(1.26)}{\leq} 2\phi(2y) - \phi(2y-2x) - \phi(2x+2y) + \phi(4x) \\ &\quad \overset{(1.27)}{\leq} 2\phi(2x) \overset{(1.26)}{\leq} 2\phi(2y) = 2\Phi(y), \end{aligned}$$

which means that (1.25) holds in this subcase.

Subcase IIb: When $\frac{1}{4} \le x \le \frac{1}{2}$ holds.

Then by $d_{\mathbb{Z}}(2y) \ge d_{\mathbb{Z}}(2x)$, we get that $0 \le x - y \le \frac{1}{2}$ and $\frac{1}{2} \le x + y \le 1$. It follows from the 1-monotonicity of ϕ that

$$\phi(|1 - 4y|) \le \phi(4x - 1). \tag{1.28}$$

Using the 3-monotonicity of ϕ, we have that $\Delta_{2x-2y}\Delta^2_{1-2x}\phi(0) \ge 0$, which results

$$-\phi(2x - 2y) - \phi(2 - 2x - 2y) + \phi(2 - 4x) \le 2\phi(1 - 2x) - 2\phi(1 - 2y). \tag{1.29}$$

Finally, by the 2-monotonicity of $-\phi$, we have that $\Delta_{2x-2y}\Delta_{2x+2y-1}\phi(1-2x) \le 0$, which yields that

$$\phi(1 - 2x) + \phi(2x) - \phi(1 - 2y) \le \phi(2y). \tag{1.30}$$

Combining these inequalities, we obtain

$$\begin{aligned}
2\Phi(x) - \Phi(x - y) - \Phi(x + y) + \Phi(2x) - \Phi(2x + \tfrac{1}{2}) + \Phi(2y + \tfrac{1}{2})& \\
= 2\phi(2x) - \phi(2x - 2y) - \phi(2 - 2x - 2y) + \phi(2 - 4x) - \phi(4x - 1) + \phi(|1 - 4y|)& \\
\overset{(1.28)}{\le} 2\phi(2x) - \phi(2x - 2y) - \phi(2 - 2x - 2y) + \phi(2 - 4x)& \\
\overset{(1.29)}{\le} 2\phi(2x) + 2\phi(1 - 2x) - 2\phi(1 - 2y) \overset{(1.30)}{\le} 2\phi(2y) = 2\Phi(y),&
\end{aligned}$$

thus (1.25) has been proved.

The proof of the theorem is complete. □

The next result is an immediate corollary of Theorem 1.6 and Theorem 1.8.

Corollary 1.9. (Makó–Páles [**43**])
Let $\phi : [0, 1] \to \mathbb{R}$ be a continuous function such that ϕ is 1- and 3-monotone, and $(-\phi)$ is 2-monotone. Then, with $\Phi := \phi \circ d_{\mathbb{Z}}$, the Takagi type function T_Φ satisfies (1.15).

In what follows, we consider the particular case when ϕ is a cone combination of power functions.

Lemma 1.10. (Makó–Páles [**43**])
Let ν be a nonnegative bounded Borel measure on $[0, 1]$. *Then the function $\phi_\nu : \mathbb{R}_+ \to \mathbb{R}_+$ defined by*

$$\phi_\nu(t) := \int_{[0,1]} t^q d\nu(q) \qquad (t \in \mathbb{R}_+) \tag{1.31}$$

is 1- and 3-monotone, and $(-\phi)$ *is 2-monotone. Furthermore, with the notation* $\Phi_\nu := \phi_\nu \circ d_{\mathbb{Z}}$,

$$T_{\Phi_\nu}(t) = \int_{[0,1]} T_q(t) d\nu(q) \qquad (t \in \mathbb{R}), \tag{1.32}$$

where, for $q \in [0, 1]$, *the Takagi type function* T_q *was defined in* (1.5) *of the introduction.*

Proof. Applying standard calculus rules, we have

$$\phi_\nu'(t) = \int_{[0,1]} q t^{q-1} d\nu(q), \quad \phi_\nu''(t) = \int_{[0,1]} q(q-1) t^{q-2} d\nu(q)$$

$$\text{and} \quad \phi_\nu'''(t) = \int_{[0,1]} q(q-1)(q-2) t^{q-3} d\nu(q),$$

which, by the nonnegativity of the measure, yield that $\phi_\nu'(t) \ge 0$, $\phi_\nu''(t) \le 0$, and $\phi_\nu'''(t) \ge 0$ hold for $t > 0$, whence, by using Lemma 1.7, the statement follows.

The proof of (1.32) is a consequence of the definitions and the uniform convergence:

$$\begin{aligned} T_{\Phi_\nu}(t) = \sum_{n=0}^{\infty} \frac{1}{2^n} \Phi_\nu(2^n t) &= \sum_{n=0}^{\infty} \frac{1}{2^n} \int_{[0,1]} (d_{\mathbb{Z}}(2^n t))^q d\nu(q) \\ &= \int_{[0,1]} \sum_{n=0}^{\infty} \frac{1}{2^n} (d_{\mathbb{Z}}(2^n t))^q d\nu(q) = \int_{[0,1]} T_q(t) d\nu(q). \end{aligned}$$

□

Thus, Corollary 1.9 and Lemma 1.10 immediately yield the following corollaries:

Corollary 1.11. (Makó–Páles [**43**])
Let ν *be a nonnegative bounded Borel measure on* $[0, 1]$ *and* $\Phi_\nu := \phi_\nu \circ d_{\mathbb{Z}}$. *Then the Takagi type function* T_{Φ_ν} *satisfies* (1.15).

If the measure ν is the Dirac measure δ_q, then $\Phi_\nu = d_Z^q$ and $T_{\Phi_\nu} = T_q$, hence Corollary 1.11 reduces to the following result, which is still more general then that of Boros [**14**].

Corollary 1.12. (Makó–Páles [**43**])
For $0 < q \le 1$, *the Takagi type function* T_q *satisfies, for all* $x, y \in \mathbb{R}$,

$$T_q\Big(\frac{x+y}{2}\Big) \le \frac{T_q(x) + T_q(y)}{2} + d_{\mathbb{Z}}^q\Big(\frac{x-y}{2}\Big).$$

In the subsequent theorem we show that, under certain assumptions on the error function φ, $\mathcal{T}_\varphi$ is sharp.

Theorem 1.13. (Makó–Páles [**43**])
Let $\varphi : D^+ \to \mathbb{R}_+$ be a continuous function with $\varphi(0) = 0$, such that φ is 1- and 3-monotone, and $(-\varphi)$ is 2-monotone on D^+. Then, for all fixed $x, y \in D$,

$$C_\varphi(x, y, t) = \mathcal{T}_\varphi(t, \|x - y\|) \qquad (t \in [0, 1]). \tag{1.33}$$

Proof. Let $x, y \in D$ be fixed. By Theorem 1.1, $C_\varphi(x, y, t) \le \mathcal{T}_\varphi(t, \|x - y\|)$. In order to show the reversed inequality (which then yields (1.33)), by Theorem 1.5, it suffices to prove that $t \mapsto \mathcal{T}_\varphi(t, \|x - y\|)$ is φ-Jensen convex on $[0, 1]$ in the sense of (1.10) and $\mathcal{T}_\varphi(0, \|x - y\|) = 0$. Observe that the assumption $\varphi(0) = 0$ implies that $\mathcal{T}_\varphi(0, \|x - y\|) = 0$.

To verify that $t \mapsto \mathcal{T}_\varphi(t, \|x - y\|)$ is φ-Jensen convex on $[0, 1]$, define the functions $\phi : [0, 1] \to \mathbb{R}_+$ and $\Phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(t) := \varphi(t\|x - y\|) \qquad \text{and} \qquad \Phi := \phi \circ d_{\mathbb{Z}}.$$

Then, by the monotonicity assumptions on φ, we get that ϕ is 1- and 3-monotone, and $(-\phi)$ is 2-monotone. Therefore, by Corollary 1.9, the Takagi type function T_Φ satisfies

$$T_\Phi\Big(\frac{s + t}{2}\Big) \le \frac{T_\Phi(s) + T_\Phi(t)}{2} + \Phi\Big(\frac{s - t}{2}\Big). \tag{1.34}$$

for all $t, s \in \mathbb{R}$. Observe that, for all $t \in \mathbb{R}$,

$$T_\Phi(t) = \sum_{n=0}^{\infty} \frac{\Phi(2^n t)}{2^n} = \sum_{n=0}^{\infty} \frac{\varphi\big(d_{\mathbb{Z}}(2^n t)\|x - y\|\big)}{2^n} = \mathcal{T}_\varphi(t, \|x - y\|).$$

On the other hand, using the increasingness of ϕ, for $s, t \in [0, 1]$, we have

$$\Phi\Big(\frac{s - t}{2}\Big) = \phi\Big(d_{\mathbb{Z}}\Big(\frac{s - t}{2}\Big)\Big) \le \phi(s - t) = \varphi(|s - t| \cdot \|x - y\|).$$

Hence using also (1.34) we get that,

$$\mathcal{T}_\varphi\Big(\frac{s + t}{2}, \|x - y\|\Big) - \frac{\mathcal{T}_\varphi(s, \|x - y\|) + \mathcal{T}_\varphi(t, \|x - y\|)}{2} \le \varphi(|s - t| \cdot \|x - y\|).$$

which completes the proof of the φ-Jensen convexity in the sense of (1.10). □

Corollary 1.14. (Makó–Páles [**43**])
Let ν be a nonnegative bounded Borel measure on $[0, 1]$ with $\nu(\{0\}) = 0$. Define the error function $\varphi_\nu : D^+ \to \mathbb{R}_+$ by

$$\varphi_\nu(u) := \int_{[0,1]} u^q d\nu(q) \qquad (u \in D^+).$$

Then, for all $x, y \in D$ *and* $t \in [0,1]$,

$$C_{\varphi_\nu}(x,y,t) = \int\limits_{[0,1]} T_q(t)\|x-y\|^q d\nu(q),$$

where $T_q : \mathbb{R} \to \mathbb{R}$ *is given by* (1.5).

Proof. By Lemma 1.10, φ_ν is 1- and 3-monotone, and $(-\varphi_\nu)$ is 2-monotone on D^+, and $\nu(\{0\}) = 0$ implies $\varphi_\nu(0) = 0$. Thus, Theorem 1.13 can be applied, and hence, for all $x, y \in D$ and $t \in [0,1]$,

$$C_{\varphi_\nu}(x,y,t) = \mathcal{T}_{\varphi_\nu}(t, \|x-y\|).$$

On the other hand,

$$\begin{aligned}\mathcal{T}_{\varphi_\nu}(t,\|x-y\|) &= \sum_{n=0}^{\infty} \frac{\varphi_\nu(d_{\mathbb{Z}}(2^n t)\|x-y\|)}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^n} \int\limits_{[0,1]} (d_{\mathbb{Z}}(2^n t)\|x-y\|)^q d\nu(q)\\ &= \int\limits_{[0,1]} \sum_{n=0}^{\infty} \frac{(d_{\mathbb{Z}}(2^n t))^q}{2^n} \|x-y\|^q d\nu(q) = \int\limits_{[0,1]} T_q(t)\|x-y\|^q d\nu(q),\end{aligned}$$

which completes the proof. □

When the measure ν is of the form $\varepsilon\delta_q$, where $\varepsilon \geq 0$ is a constant and δ_q is the Dirac measure supported at $q \in]0,1]$, then we get the following consequence of Corollary 1.3, which shows the optimality of $T_q(t)$ in (1.6).

Corollary 1.15. (Makó–Páles [**43**])
Let $0 < q \leq 1$ *and* $\varphi_q : D^+ \to \mathbb{R}_+$ *defined by* $\varphi_q(t) := \varepsilon t^q$. *Then, for all* $x, y \in D$ *and* $t \in [0,1]$,

$$C_{\varphi_q}(x,y,t) = \varepsilon T_q(t)\|x-y\|^q.$$

1.2. Approximate convexity of Takagi type functions $\mathcal{S}_\varphi$

Let introduce the following Takagi type function $S_\phi : \mathbb{R} \to \mathbb{R}$ defined by

$$S_\phi(x) = \sum_{n=0}^{\infty} \phi\left(\tfrac{1}{2^n}\right) d_{\mathbb{Z}}(2^n x), \tag{1.35}$$

where $P := \{1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots,\}$ and $\phi : P \to \mathbb{R}_+$ is a nonnegative function. Taking $\phi(t) := t^q$, $(t \in \mathbb{R}_+)$, we obtain the function S_q defined in (1.5). The main results of this section state that, under certain assumptions on the function $\phi : P \to \mathbb{R}$, S_ϕ is well-defined and approximately Jensen convex in the following sense: For all $x, y \in \mathbb{R}$,

$$S_\phi\Big(\frac{x+y}{2}\Big) \le \frac{S_\phi(x) + S_\phi(y)}{2} + \phi \circ d_{\mathbb{Z}}\Big(\frac{x-y}{2}\Big). \tag{1.36}$$

First we describe the situation when the definition of S_ϕ is correct.

Lemma 1.16. (Makó–Páles [**48**])
Let $\phi : P \to \mathbb{R}_+$ be a nonnegative function. Then S_ϕ is well-defined, i.e., the series on the right hand side of (1.35) *is convergent everywhere if and only if*

$$\sum_{n=0}^{\infty} \phi(\tfrac{1}{2^n}) < \infty. \tag{1.37}$$

Proof. Observe that $d_{\mathbb{Z}}(2^n \frac{1}{3}) = \frac{2}{3}$ for all nonnegative integer n. Hence the convergence of the right hand side of (1.35) at $x = \frac{1}{3}$ yields that (1.37) holds.

Conversely, if (1.37) is satisfied, then, for all $x \in \mathbb{R}$,

$$S_\phi(x) = \sum_{n=0}^{\infty} \phi(\tfrac{1}{2^n}) d_{\mathbb{Z}}(2^n x) \le \sum_{n=0}^{\infty} \phi(\tfrac{1}{2^n}) < \infty,$$

which proves the statement. □

In the sequel, the class of nonnegative functions $\phi : P \to \mathbb{R}_+$ satisfying the condition (1.37) will be denoted by $\mathcal{H}$:

$$\mathcal{H} := \Big\{\phi : P \to \mathbb{R}_+ \mid \sum_{n=0}^{\infty} \phi(\tfrac{1}{2^n}) < \infty\Big\}.$$

The next theorem, which was discovered by Jacek Tabor and Józef Tabor, has an important role in the proof of the main theorem of this section.

Theorem 1.17. (Tabor–Tabor [**66**])
For every $q \in [1, 2]$ and $x, y \in \mathbb{R}$,

$$S_q\Big(\frac{x+y}{2}\Big) \le \frac{S_p(x) + S_q(y)}{2} + d_{\mathbb{Z}}^q\Big(\frac{x-y}{2}\Big).$$

In the next result we give a representation of $S_\phi(x)$ as an infinite linear combination of the values $S_q(2^n x)$, $n = 1, 2, \dots$. The particular case when $q = 2$ and $\phi(x) = x^q$ for some $q \in [1, 2]$ was established in [**66**].

Theorem 1.18. (Makó–Páles [**48**])
Let $\phi \in \mathcal{H}$. Then, for every $q > 0$ and $x \in \mathbb{R}$,

$$S_\phi(x) = \phi(1) S_q(x) + \sum_{n=1}^{\infty} \Big(\phi(\tfrac{1}{2^n}) - \tfrac{1}{2^q}\phi(\tfrac{1}{2^{n-1}})\Big) S_q(2^n x). \tag{1.38}$$

Proof. Since $S_q(x) = \sum_{n=0}^{\infty} \frac{1}{2^{nq}} d_{\mathbb{Z}}(2^n x)$, the right hand side of (1.38) can be written in the form $\sum_{n=0}^{\infty} a_n d_{\mathbb{Z}}(2^n x)$, where

$$\begin{aligned} a_n &= \phi(1)\frac{1}{2^{nq}} + \sum_{k=1}^{n} \frac{1}{2^{(n-k)q}}\Big(\phi(\tfrac{1}{2^k}) - \tfrac{1}{2^q}\phi(\tfrac{1}{2^{k-1}})\Big) \\ &= \phi(1)\frac{1}{2^{nq}} + \sum_{k=1}^{n}\Big(\frac{1}{2^{(n-k)q}}\phi(\tfrac{1}{2^k}) - \frac{1}{2^{(n-k+1)q}}\phi(\tfrac{1}{2^{k-1}})\Big) = \phi(\tfrac{1}{2^n}). \end{aligned}$$

Consequently (1.38) holds. □

An immediate consequence of the previous two theorems is the next result which states the approximate convexity of S_ϕ.

Theorem 1.19. (Makó–Páles [**48**])
Let $\phi \in \mathcal{H}$ such that, for all $u \in \frac{1}{2}P$, $\phi(2u) \le 2^q\phi(u)$, where $q \in [1,2]$ is an arbitrary constant. Then, for all $x, y \in \mathbb{R}$,

$$S_\phi\Big(\frac{x+y}{2}\Big) \le \frac{S_\phi(x) + S_\phi(y)}{2} + \Phi_q\Big(\frac{x-y}{2}\Big), \tag{1.39}$$

where $\Phi_q : \mathbb{R} \to \mathbb{R}$ is defined by

$$\Phi_q(u) := \sum_{n=0}^{\infty} \phi(\tfrac{1}{2^n})\Big(d_{\mathbb{Z}}^q(2^n u) - \frac{1}{2^q} d_{\mathbb{Z}}^q(2^{n+1}u)\Big). \tag{1.40}$$

Proof. For an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ define $Jf : \mathbb{R}^2 \to \mathbb{R}$ by

$$Jf(x,y) := f\Big(\frac{x+y}{2}\Big) - \frac{f(x) + f(y)}{2}.$$

Then, by (1.38), we get

$$JS_\phi(x,y) = \phi(1)JS_q(x,y) + \sum_{n=1}^{\infty}\Big(\phi(\tfrac{1}{2^n}) - \frac{1}{2^q}\phi(\tfrac{1}{2^{n-1}})\Big)JS_q(2^n x, 2^n y).$$

By Theorem 1.17, for all $x, y \in \mathbb{R}$, we have that

$$JS_q(x,y) \le d_{\mathbb{Z}}^q\Big(\frac{x-y}{2}\Big)$$

On the other hand, by the assumption on ϕ, the coefficient $(\phi(\frac{1}{2^n}) - \frac{1}{2^q}\phi(\frac{1}{2^{n-1}}))$ is nonnegative, hence for all $x, y \in \mathbb{R}$, we get

$$\begin{aligned} JS_\phi(x,y) &\le \phi(1)d_{\mathbb{Z}}^q\Big(\frac{x-y}{2}\Big) + \sum_{n=1}^{\infty}\Big(\phi(\tfrac{1}{2^n}) - \frac{1}{2^q}\phi(\tfrac{1}{2^{n-1}})\Big)d_{\mathbb{Z}}^q\Big(\frac{2^n(x-y)}{2}\Big) \\ &= \sum_{n=0}^{\infty}\phi(\tfrac{1}{2^n})\Big(d_{\mathbb{Z}}^q\Big(\frac{2^n(x-y)}{2}\Big) - \frac{1}{2^q}d_{\mathbb{Z}}^q\Big(\frac{2^{n+1}(x-y)}{2}\Big)\Big) = \Phi_q\Big(\frac{x-y}{2}\Big), \end{aligned}$$

which proves (1.39). □

In the next proposition we describe a decomposition property of the function Φ_q.

Proposition 1.20. (Makó–Páles [**48**])
Let $\phi \in \mathcal{H}$ and $q \in [1,2]$. Then, for all $u \in]0, \frac{1}{2}]$,

$$\Phi_q(u) = \Phi_q\Big(\frac{1}{2^{[\log_2 \frac{1}{u}]}} - u\Big) + \phi\Big(\frac{1}{2^{[\log_2 \frac{1}{u}]-1}}\Big)\Big((2^{[\log_2 \frac{1}{u}]}u)^q - (1 - 2^{[\log_2 \frac{1}{u}]}u)^q\Big). \tag{1.41}$$

Proof. For $u \in]0,1]$, denote $\ell(u) := [\log_2 \frac{1}{u}]$. Then by the definition of the integer-part function, we have $\ell(u) - 1 \le \log_2 \frac{1}{u} - 1 < \ell(u)$, hence

$$2^{\ell(u)-1}u \le \frac{1}{2} < 2^{\ell(u)}u.$$

Thus, we have

$$d_{\mathbb{Z}}(2^n u) = \begin{cases} 2^{n+1}u, & \text{if} \quad n \le \ell(u) - 1, \\ d_{\mathbb{Z}}\Big(2^n(\frac{1}{2^{\ell(u)}} - u)\Big), & \text{if} \quad n \ge \ell(u). \end{cases}$$

It can also be seen that

$$d_{\mathbb{Z}}\Big(2^n(\tfrac{1}{2^{\ell(u)}} - u)\Big) = 2^{n+1}(\tfrac{1}{2^{\ell(u)}} - u), \quad \text{if} \quad n \le \ell(u).$$

Using these formulas and applying (1.40), we get

$$\begin{aligned}
\Phi_q(u) = &\sum_{n=0}^{\ell(u)-2} \phi(\tfrac{1}{2^n})\Big((2^{n+1}u)^q - \frac{1}{2^q}(2^{n+2}u)^q\Big) + \phi(\tfrac{1}{2^{\ell(u)-1}})\Big((2^{\ell(u)}u)^q - \frac{1}{2^q}(2 - 2^{\ell(u)+1}u)^q\Big) \\
&+ \sum_{n=\ell(u)}^{\infty} \phi(\tfrac{1}{2^n})\Big(d_{\mathbb{Z}}^q\Big(2^n(\tfrac{1}{2^{\ell(u)}} - u)\Big) - \frac{1}{2^q}d_{\mathbb{Z}}^q\Big(2^{n+1}(\tfrac{1}{2^{\ell(u)}} - u)\Big)\Big) \\
=&\phi(\tfrac{1}{2^{\ell(u)-1}})\Big((2^{\ell(u)}u)^q - (1 - 2^{\ell(u)}u)^q\Big) \\
&+ \sum_{n=\ell(u)}^{\infty} \phi(\tfrac{1}{2^n})\Big(d_{\mathbb{Z}}^q\Big(2^n(\tfrac{1}{2^{\ell(u)}} - u)\Big) - \frac{1}{2^q}d_{\mathbb{Z}}^q\Big(2^{n+1}(\tfrac{1}{2^{\ell(u)}} - u)\Big)\Big) \\
=&\phi(\tfrac{1}{2^{\ell(u)-1}})\Big((2^{\ell(u)}u)^q - (1 - 2^{\ell(u)}u)^q\Big) \\
&+ \sum_{n=0}^{\infty} \phi(\tfrac{1}{2^n})\Big(d_{\mathbb{Z}}^q\Big(2^n(\tfrac{1}{2^{\ell(u)}} - u)\Big) - \frac{1}{2^q}d_{\mathbb{Z}}^q\Big(2^{n+1}(\tfrac{1}{2^{\ell(u)}} - u)\Big)\Big) \\
=&\phi(\tfrac{1}{2^{\ell(u)-1}})\Big((2^{\ell(u)}u)^q - (1 - 2^{\ell(u)}u)^q\Big) + \Phi_q\Big(\tfrac{1}{2^{\ell(u)}} - u\Big),
\end{aligned}$$

which means that (1.41) holds. □

In the next proposition an important class of functions ϕ from $\mathcal{H}$ will be described.

Proposition 1.21. (Makó–Páles [**48**])
Let $\phi : [0,1] \to \mathbb{R}_+$. *Assume that* $\phi(0) = 0$ *and the mapping* $x \mapsto \frac{\phi(x)}{x}$ *is concave on* $]0,1]$. *Then* $\phi|_P \in \mathcal{H}$, *the function* $x \mapsto \frac{\phi(x)}{x^2}$ *is decreasing on* $]0,1]$ *and* ϕ *is continuous on* $[0,1[$.

Proof. By the concavity of the function $x \mapsto \frac{\phi(x)}{x} =: \psi(x)$, there exists an affine function that majorizes ψ, hence ψ is bounded from above by a constant $C \geq 0$. Therefore,

$$\sum_{n=0}^{\infty} \phi|_P(\tfrac{1}{2^n}) = \sum_{n=0}^{\infty} \phi(\tfrac{1}{2^n}) = \sum_{n=0}^{\infty} \tfrac{1}{2^n}\psi(\tfrac{1}{2^n}) \leq C \sum_{n=0}^{\infty} \tfrac{1}{2^n} = 2C,$$

which proves that (1.37) holds, i.e., $\phi|_P \in \mathcal{H}$.

Let $0 < z < y < x \leq 1$. By the concavity of $x \mapsto \frac{\phi(x)}{x}$, we get

$$\frac{\phi(y)}{y} \geq \frac{x-y}{x-z} \cdot \frac{\phi(z)}{z} + \frac{y-z}{x-z} \cdot \frac{\phi(x)}{x}. \tag{1.42}$$

Since ϕ is nonnegative, by (1.42), we have that

$$\frac{\phi(y)}{y} \geq \frac{y-z}{x-z} \cdot \frac{\phi(x)}{x}.$$

Then, taking the limit $z \to 0$, we obtain $\frac{\phi(y)}{y^2} \geq \frac{\phi(x)}{x^2}$, which means that the mapping $x \mapsto \frac{\phi(x)}{x^2}$ is decreasing.

Rearranging the inequality in (1.42), we get

$$\phi(z) \leq z\Big(\frac{\phi(y)}{y} \cdot \frac{x-z}{x-y} - \frac{\phi(x)}{x} \cdot \frac{y-z}{x-y}\Big).$$

Upon taking the limit $z \to 0$ again, we get that ϕ is continuous at 0. By the concavity assumption, it follows that ϕ is also continuous on $]0,1[$. □

In the next result we show that, if ϕ is a cone combination of power functions, then $x \mapsto \frac{\phi(x)}{x}$ is concave.

Proposition 1.22. (Makó–Páles [**48**])
Let ν *be a nonnegative bounded Borel measure on* $[1,2]$. *Let the error function* $\phi_\nu : \mathbb{R}_+ \to \mathbb{R}_+$ *be given by*

$$\phi_\nu(x) := \int\limits_{[1,2]} x^q d\nu(q) \qquad (x \in \mathbb{R}_+). \tag{1.43}$$

Then $x \mapsto \frac{\phi_\nu(x)}{x}$ *is concave on* $]0,+\infty[$.

Proof. Define $\psi_\nu :]0,+\infty[\to \mathbb{R}_+$ by

$$\psi_\nu(x) := \frac{\phi_\nu(x)}{x} = \int_{[1,2]} x^{q-1} d\nu(q).$$

Applying standard calculus rules, we have

$$\psi_\nu''(x) = \int_{[1,2]} (q-1)(q-2)x^{q-3} d\nu(q),$$

for all $x \in]0,+\infty[$. By the nonnegativity of the measure and the nonpositivity of the integrand, we get that $\psi_\nu''(x) \le 0$, if $x > 0$. This implies that ψ_ν is concave on $]0,+\infty[$, which completes the proof. □

The next theorem has an important role in the proof of our subsequent main results.

Theorem 1.23. (Makó–Páles [**48**])
Let $\phi : [0,1] \to \mathbb{R}_+$. Assume that $\phi(0) = 0$ and the mapping $x \mapsto \frac{\phi(x)}{x}$ is concave on $]0,1]$, then, for all $u \in \mathbb{R}$,

$$\Phi_2(u) \le \phi \circ d_{\mathbb{Z}}(u). \tag{1.44}$$

Proof. By the 1-periodicity of Φ_2, in the proof of (1.44), we may assume that $|u| \le \frac{1}{2}$ and since Φ_2 is even, we may also assume that $u > 0$. We show, for every fixed $k \in \mathbb{N}$ that the theorem holds for all $u \in \mathbb{Z}/2^k$. Then the statement follows from the continuity of Φ_2 and ϕ and the denseness of dyadic rational numbers in $\mathbb{R}$.

Using induction on m, we prove that (1.44) holds for all $u = \frac{m}{2^k}$, where $m \in \{1,\dots,2^{k-1}\}$. First we consider the case when $m = 1$. Then, by (1.40),

$$\Phi_2(\tfrac{1}{2^k}) = \sum_{n=0}^{k-1} \phi(\tfrac{1}{2^n})\Big(d_{\mathbb{Z}}^2(\tfrac{2^n}{2^k}) - \frac{1}{4} d_{\mathbb{Z}}^2(\tfrac{2^{n+1}}{2^k})\Big) = \sum_{n=0}^{k-2} \phi(\tfrac{1}{2^n})\Big(\tfrac{4^{n+1}}{4^k} - \tfrac{4^{n+1}}{4^k}\Big) + \phi(\tfrac{1}{2^{k-1}}) = \phi(\tfrac{1}{2^{k-1}}),$$

which shows that in this case (1.44) holds with equality.

Now let $m \in \{2,\dots,2^{k-1}\}$ and assume that (1.44) has been verified for all $u = \frac{j}{2^k}$, where $j \in \{1,\dots,m-1\}$. With the notation $\ell(u) := [\log_2 \frac{1}{u}]$, it is easy to see that $\frac{1}{2^{\ell(u)}} - u \in \mathbb{Z}/2^k$ and $0 < \frac{1}{2^{\ell(u)}} - u < u$, thus by the inductive assumption, we have

$$\Phi_2(\tfrac{1}{2^{\ell(u)}} - u) \le \phi(\tfrac{1}{2^{\ell(u)-1}} - 2u). \tag{1.45}$$

In view of Proposition 1.20 with $q = 2$, the identity

$$\Phi_2(u) = \Phi_2(\tfrac{1}{2^{\ell(u)}} - u) + \phi(\tfrac{1}{2^{\ell(u)-1}})(2^{\ell(u)+1}u - 1) \tag{1.46}$$

holds. By (1.46) and (1.45) we have that

$$\Phi_2(u) \le \phi(\tfrac{1}{2^{\ell(u)-1}} - 2u) + \phi(\tfrac{1}{2^{\ell(u)-1}})(2^{\ell(u)+1}u - 1).$$

Thus to prove the assertion of the theorem it is enough to show that

$$\phi\big(\tfrac{1}{2^{\ell(u)-1}} - 2u\big) + \phi\big(\tfrac{1}{2^{\ell(u)-1}}\big)\big(2^{\ell(u)+1}u - 1\big) \le \phi(2u). \tag{1.47}$$

Let $\psi(x) = \frac{\phi(x)}{x}$, $(x \in]0,1])$. Then, by the concavity of ψ, we get

$$\begin{aligned}\big(\tfrac{1}{2^{\ell(u)}u} - 1\big)&\psi\big(\tfrac{1}{2^{\ell(u)-1}} - 2u\big) + \big(2 - \tfrac{1}{2^{\ell(u)}u}\big)\psi\big(\tfrac{1}{2^{\ell(u)-1}}\big)\\ &\le \psi\Big(\big(\tfrac{1}{2^{\ell(u)-1}} - 2u\big)\big(\tfrac{1}{2^{\ell(u)}u} - 1\big) + \tfrac{1}{2^{\ell(u)-1}}\big(2 - \tfrac{1}{2^{\ell(u)}u}\big)\Big) = \psi(2u),\end{aligned}$$

which is equivalent to (1.47) completing the proof of (1.44). □

The main result of this section is stated in the following theorem.

Theorem 1.24. (Makó–Páles [**48**])
Let $\phi : [0,1] \to \mathbb{R}_+$. *Assume that* $\phi(0) = 0$ *and the mapping* $x \mapsto \frac{\phi(x)}{x}$ *is concave on* $]0,1]$. *Then* S_ϕ *is approximately Jensen convex in the sense of* (1.36).

Proof. For an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ define $Jf : \mathbb{R}^2 \to \mathbb{R}$ as in the proof of Theorem 1.19. By Proposition 1.21, we have that $\phi|_P \in \mathcal{H}$, which, by Lemma 1.16, implies that S_ϕ is well-defined. By Proposition 1.21, we get that $x \mapsto \frac{\phi(x)}{x^2}$ is decreasing, which results, for all $u \in]0, \frac{1}{2}]$, that

$$\phi(2u) \le 4\phi(u).$$

This means that we can apply Theorem 1.19 with $q = 2$ for the function S_ϕ. Thus we obtain that for all $x, y \in \mathbb{R}$,

$$JS_\phi(x,y) \le \Phi_2\Big(\frac{x-y}{2}\Big). \tag{1.48}$$

In view of Theorem 1.23, we also have (1.44) for all $u \in \mathbb{R}$. Therefore, (1.48) and (1.44) imply that (1.36) holds for all $x, y \in \mathbb{R}$, which means that S_ϕ is approximately Jensen convex in the sense (1.36). □

The next result immediately follows from Proposition 1.22 and the above theorem. The particular case when ϕ is of the form $\phi(t) = \varepsilon t^q$ (where $\varepsilon \ge 0$ and $q \in [1,2]$ are constants) was discovered by Tabor and Tabor in [**66**] and was already stated in Theorem 1.19.

Corollary 1.25. *Let* ν *be a nonnegative bounded Borel measure on* $[1,2]$. *Let the error function* $\phi_\nu : \mathbb{R}_+ \to \mathbb{R}_+$ *be given by* (1.43). *Then, for all* $x, y \in \mathbb{R}$,

$$S_{\phi_\nu}\Big(\frac{x+y}{2}\Big) \le \frac{S_{\phi_\nu}(x) + S_{\phi_\nu}(y)}{2} + \phi_\nu \circ d_{\mathbb{Z}}\Big(\frac{x-y}{2}\Big).$$

We shall prove that the error terms $\mathcal{S}_\varphi(t, x-y)$ in (1.4) under certain assumptions on the error function φ is the smallest possible one. In other words, the next theorem will provide exact upper bound for the convexity-difference of φ-Jensen convex functions defined by (1.8).

Theorem 1.26. (Makó–Páles [**48**])
Let $\varphi : D^+ \to \mathbb{R}$ be nondecreasing and assume that $\varphi(0) = 0$ and the map $u \mapsto \frac{\varphi(u)}{u}$ is concave on $D^+ \setminus \{0\}$. Then, for all $x, y \in D$ and $t \in [0, 1]$,

$$C_\varphi(x, y, t) = \mathcal{S}_\varphi(t, \|x - y\|). \tag{1.49}$$

Proof. From the concavity of $u \mapsto \frac{\varphi(u)}{u}$ it follows (as in Proposition 1.21) that, for some $n_0 \in \mathbb{N}$, the series $\sum_{n=n_0}^{\infty} \varphi(2^{-n})$ is convergent. By Theorem 1.2, we get that

$$C_\varphi(x, y, t) \le \mathcal{S}_\varphi(t, \|x - y\|) \tag{1.50}$$

for all $x, y \in D$ and $t \in [0, 1]$.

Now let $x, y \in D$ be fixed. In order to show the reversed inequality (which then yields (1.49)), it suffices to prove that the function $t \mapsto \mathcal{S}_\varphi(t, \|x - y\|)$ is φ-Jensen convex in the sense of (1.10).

Observe that $\mathcal{S}_\varphi(0, u) = 0$ for all $u \in D^+$. Let $x, y \in D$ be fixed and the function $\phi : [0, 1] \to \mathbb{R}_+$ by

$$\phi(r) := \varphi(r\|x - y\|).$$

Then, we get that $r \mapsto \frac{\phi(r)}{r}$ is concave on $]0, 1]$. Therefore, by Theorem 1.24, the Takagi type function S_ϕ satisfies

$$S_\phi\Big(\frac{s + t}{2}\Big) \le \frac{S_\phi(s) + S_\phi(t)}{2} + \phi \circ d_{\mathbb{Z}}\Big(\frac{s - t}{2}\Big), \tag{1.51}$$

for all $t, s \in \mathbb{R}$. Observe that, for all $t \in \mathbb{R}$,

$$S_\phi(t) = \sum_{n=0}^{\infty} \phi(\tfrac{1}{2^n}) d_{\mathbb{Z}}(2^n t) = \sum_{n=0}^{\infty} \varphi(\tfrac{\|x-y\|}{2^n}) d_{\mathbb{Z}}(2^n t) = \mathcal{S}_\varphi(t, \|x - y\|).$$

On the other hand, using the increasingness of ϕ, for $s, t \in [0, 1]$, we have

$$\phi\Big(d_{\mathbb{Z}}\Big(\frac{s - t}{2}\Big)\Big) \le \phi(|s - t|) = \varphi(|s - t| \cdot \|x - y\|).$$

Thus, for $s, t \in [0, 1]$, (1.51) can be rewritten as

$$\mathcal{S}_\varphi\Big(\frac{s + t}{2}, \|x - y\|\Big) \le \frac{\mathcal{S}_\varphi(s, \|x - y\|) + \mathcal{S}_\varphi(t, \|x - y\|)}{2} + \varphi(|s - t| \cdot \|x - y\|).$$

which completes the proof of the approximate convexity of $\mathcal{S}_\varphi$ in the sense of (1.10). Hence we get that $\mathcal{S}_\varphi$ is optimal in Theorem 1.2 by Theorem 1.5. □

Taking an error function φ which is a combination of power functions of exponents from $[1, 2]$, we obtain the following result.

Theorem 1.27. (Makó–Páles [**48**])
Let ν *be a nonnegative bounded Borel measure on* $[0,1]$ *with* $\nu(\{0\}) = 0$*. Define the error function* $\varphi_\nu : D^+ \to \mathbb{R}_+$ *by*

$$\varphi_\nu(u) := \int_{[0,1]} u^q d\nu(q) \qquad (u \in D^+).$$

Then, for all $x, y \in D$ *and* $t \in [0,1]$*,*

$$C_{\varphi_\nu}(x,y,t) = \int_{[1,2]} S_q(t)\|x-y\|^q d\nu(q),$$

where $S_q : \mathbb{R} \to \mathbb{R}$ *is given by* (1.5)*.*

Proof. It is easy to see that φ_ν is nondecreasing and we also have that $\varphi_\nu(0) = 0$. By Proposition 1.22 we get that $t \mapsto \frac{\varphi_\nu(th)}{t}$ is concave on $]0,+\infty[$. Thus, Theorem 1.26 can be applied, and hence, for all $x, y \in D$ and $t \in [0,1]$,

$$C_{\varphi_\nu}(x,y,t) = \mathcal{S}_{\varphi_\nu}(t, \|x-y\|).$$

On the other hand, for all $x, y \in D$ and $t \in [0,1]$,

$$\begin{aligned}\mathcal{S}_{\varphi_\nu}(t,\|x-y\|) &= \sum_{n=0}^{\infty} \varphi_\nu\big(\tfrac{\|x-y\|}{2^n}\big) d_{\mathbb{Z}}(2^n t) = \sum_{n=0}^{\infty} \int_{[1,2]} \big(\tfrac{\|x-y\|}{2^n}\big)^q d\nu(q) d_{\mathbb{Z}}(2^n t)\\ &= \int_{[1,2]} \sum_{n=0}^{\infty} \tfrac{1}{2^{nq}} d_{\mathbb{Z}}(2^n t)\|x-y\|^q d\nu(q) = \int_{[1,2]} S_q(t)\|x-y\|^q d\nu(q),\end{aligned}$$

which completes the proof. □

Taking ν as the point measure $\varepsilon\delta_q$ where $q \in [1,2]$ in the above theorem, it reduces to a result by Jacek and Józef Tabor [**66**] which states that (1.7) is sharp, i.e., the term $\varepsilon S_q(t)$ is the smallest possible.

Corollary 1.28. (Makó–Páles [**48**])
Let $q \in [1,2]$ *and* $\varepsilon \geq 0$*. Define the error function* $\varphi : D^+ \to \mathbb{R}_+$ *by* $\varphi(u) := \varepsilon u^q$*. Then, for all* $x, y \in D$ *and* $t \in [0,1]$*,*

$$C_\varphi(x,y,t) = \varepsilon S_q(t)\|x-y\|^q.$$

2

Implications between upper Hermite–Hadamard and Jensen type inequalities

Let D be a nonempty, convex subset of the linear space X and let $\alpha : D^* \to \mathbb{R}$ be an even function. In this chapter we will investigate the connections between α-Jensen convex functions and functions which satisfy the following upper Hermite–Hadamard type inequality:

$$\int_0^1 f(tx + (1-t)y)\rho(t)dt \le \lambda f(x) + (1-\lambda)f(y) + \beta(x-y) \qquad (x, y \in D), \tag{2.1}$$

where $\alpha, \beta : D^* \to \mathbb{R}$ are given even functions, $\lambda \in [0, 1]$, and $\rho : [0, 1] \to \mathbb{R}_+$ is an integrable nonnegative function with $\int_0^1 \rho(t)dt = 1$.

2.1. From Jensen inequality to upper Hermite–Hadamard inequality

The following statement will be essential to obtain our first main result, Theorem 2.2.

Proposition 2.1. (Makó–Páles [**45**])
Let $\rho : [0, 1] \to \mathbb{R}_+$ be a Lebesgue integrable function. Then, the function $\psi : [0, 1] \to \mathbb{R}$ defined by

$$\psi(t) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{4^n}\Big(\sum_{k=0}^{2^n-1} \rho\big(\tfrac{t+k}{2^n}\big)\Big) \qquad (t \in [0, 1]), \tag{2.2}$$

is a nonnegative integrable solution of the functional equation

$$\rho(t) = 2\psi(t) - \frac{\psi(\frac{t}{2}) + \psi(\frac{t+1}{2})}{2} \qquad (t \in [0, 1]). \tag{2.3}$$

Furthermore,

$$\int_0^1 \psi = \int_0^1 \rho, \quad \int_{\frac{1}{2}}^1 \psi = \int_0^1 t\rho(t)dt \quad \textit{and} \quad \int_0^{\frac{1}{2}} \psi = \int_0^1 (1-t)\rho(t)dt. \tag{2.4}$$

Proof. Define the sequence $\psi_n : [0,1] \to \mathbb{R}$, by

$$\psi_0 := \tfrac{1}{2}\rho, \quad \psi_n(t) := \frac{1}{2}\rho(t) + \frac{1}{4}\big(\psi_{n-1}(\tfrac{t}{2}) + \psi_{n-1}(\tfrac{t+1}{2})\big) \quad (t \in [0,1],\ n \in \mathbb{N}). \tag{2.5}$$

Then the sequence (ψ_n) is nondecreasing, i.e.,

$$0 \le \psi_{n-1} \le \psi_n, \qquad \text{and} \qquad \int_0^1 \psi_n = \frac{2^{n+1}-1}{2^{n+1}} \int_0^1 \rho \qquad (n \in \mathbb{N}). \tag{2.6}$$

We prove (2.6) by induction on $n \in \mathbb{N}$. For $n = 1$, by the definition of ψ_1 and the nonnegativity of ρ, for $t \in [0,1]$, we have that

$$\psi_1(t) = \frac{1}{2}\rho(t) + \frac{1}{4}\big(\psi_0(\tfrac{t}{2}) + \psi_0(\tfrac{t+1}{2})\big) = \frac{1}{2}\rho(t) + \frac{1}{8}\big(\rho(\tfrac{t}{2}) + \rho(\tfrac{t+1}{2})\big) \ge \frac{1}{2}\rho(t) = \psi_0(t),$$

and

$$\int_0^1 \psi_0 = \frac{1}{2}\int_0^1 \rho.$$

Assume that, for some $n \in \mathbb{N}$, (2.6) holds and consider the case $n+1$. By the definition of ψ_{n+1}, the inductive assumption and the nonnegativity of ψ_n, for $t \in [0,1]$, yields

$$\psi_{n+1}(t) = \frac{1}{2}\rho(t) + \frac{1}{4}\big(\psi_n(\tfrac{t}{2}) + \psi_n(\tfrac{t+1}{2})\big) \ge \frac{1}{2}\rho(t) + \frac{1}{4}\big(\psi_{n-1}(\tfrac{t}{2}) + \psi_{n-1}(\tfrac{t+1}{2})\big) = \psi_n(t).$$

Using the definition ψ_{n+1}, the substitution $s := \frac{t}{2}$ and $s := \frac{t+1}{2}$, finally the inductive assumption, we get

$$\begin{aligned}\int_0^1 \psi_{n+1} &= \frac{1}{2}\int_0^1 \rho + \frac{1}{4}\Big(\int_0^1 \psi_n(\tfrac{t}{2})dt + \int_0^1 \psi_n(\tfrac{t+1}{2})dt\Big) = \frac{1}{2}\int_0^1 \rho + \frac{1}{2}\Big(\int_0^{\frac{1}{2}} \psi_n + \int_{\frac{1}{2}}^1 \psi_n\Big)\\ &= \frac{1}{2}\int_0^1 \rho + \frac{1}{2}\int_0^1 \psi_n = \Big(\frac{1}{2} + \frac{2^{n+1}-1}{2^{n+2}}\Big)\int_0^1 \rho = \frac{2^{n+2}-1}{2^{n+2}}\int_0^1 \rho.\end{aligned}$$

Denote by $L^1[0,1]$ the space of Lebesgue integrable functions $\chi : [0,1] \to \mathbb{R}$. Then $L^1[0,1]$ is a Banach-space with the standard norm $\|\chi\|_1 := \int_0^1 |\chi|$. Now we prove that (ψ_n) is a Cauchy sequence in $L^1[0,1]$. Using (2.6), for $n \le m$, we get that

$$\|\psi_m - \psi_n\|_1 = \int_0^1 (\psi_m - \psi_n) = \frac{2^m - 2^n}{2^{n+m+1}} \int_0^1 \rho \le \frac{1}{2^n}\int_0^1 \rho,$$

which implies that (ψ_n) is indeed a Cauchy sequence. Hence it converges to a function $\psi \in L^1[0,1]$. To prove (2.2), we show, by induction on $n \in \mathbb{N}$, that

$$\psi_n(t) = \frac{1}{2}\sum_{i=0}^{n} \frac{1}{4^i}\Big(\sum_{k=0}^{2^i-1} \rho(\tfrac{t+k}{2^i})\Big) \qquad (t \in [0,1]) \tag{2.7}$$

holds. For $n = 0$, we have an obvious identity. Assume that (2.7) holds some $n \in \mathbb{N}$. Using the definition of ψ_{n+1} and the inductive assumption, we obtain

$$\begin{aligned}\psi_{n+1}(t) =&\frac{1}{2}\rho(t) + \frac{1}{4}\big(\psi_n(\tfrac{t}{2}) + \psi_n(\tfrac{t+1}{2})\big)\\ =&\frac{1}{2}\rho(t) + \frac{1}{8}\sum_{i=0}^{n}\frac{1}{4^i}\Big(\sum_{k=0}^{2^i-1}\rho(\tfrac{t+2k}{2^{i+1}})\Big) + \frac{1}{8}\sum_{i=0}^{n}\frac{1}{4^i}\Big(\sum_{k=0}^{2^i-1}\rho(\tfrac{t+2k+1}{2^{i+1}})\Big)\\ =&\frac{1}{2}\rho(t) + \frac{1}{2}\sum_{i=0}^{n}\frac{1}{4^{i+1}}\sum_{k=0}^{2^i-1}\Big(\rho(\tfrac{t+2k}{2^{i+1}}) + \rho(\tfrac{t+2k+1}{2^{i+1}})\Big)\\ =&\frac{1}{2}\rho(t) + \frac{1}{2}\sum_{i=0}^{n}\frac{1}{4^{i+1}}\sum_{k=0}^{2^{i+1}-1}\rho(\tfrac{t+k}{2^{i+1}}) = \frac{1}{2}\sum_{i=0}^{n+1}\frac{1}{4^i}\Big(\sum_{k=0}^{2^i-1}\rho(\tfrac{t+k}{2^i})\Big),\end{aligned}$$

which proves (2.7). Thus, taking the limit $n \to \infty$ in (2.7), we obtain (2.2).

To prove the first expression in (2.4), integrate (2.3) on $[0, 1]$, then we get

$$\int_0^1 \rho = 2\int_0^1 \psi - \frac{\int_0^1 \psi(\frac{t}{2})dt + \int_0^1 \psi(\frac{t+1}{2})dt}{2} = 2\int_0^1 \psi - \Big(\int_0^{\frac{1}{2}} \psi + \int_{\frac{1}{2}}^1 \psi\Big) = \int_0^1 \psi.$$

To prove the second expression in (2.4), multiply (2.3) by t and integrate it on $[0, 1]$. Thus we get

$$\begin{aligned}\int_0^1 t\rho(t)dt =& 2\int_0^1 t\psi(t)dt - \Big(\int_0^1 \tfrac{t}{2}\psi(\tfrac{t}{2})dt + \int_0^1 \tfrac{t+1}{2}\psi(\tfrac{t+1}{2})dt\Big) + \frac{1}{2}\int_0^1 \psi(\tfrac{t+1}{2})dt\\ =& 2\int_0^1 t\psi(t)dt - 2\Big(\int_0^{\frac{1}{2}} s\psi(s)ds + \int_{\frac{1}{2}}^1 s\psi(s)ds\Big) + \int_{\frac{1}{2}}^1 \psi = \int_{\frac{1}{2}}^1 \psi.\end{aligned}$$

The last equality in (2.4) is a consequence of the first and second equalities. □

Theorem 2.2. (Makó–Páles [**45**])
Let $\alpha : D^* \to \mathbb{R}$ be radially bounded, measurable and $\rho : [0, 1] \to \mathbb{R}_+$ be a Lebesgue integrable function with $\int_0^1 \rho = 1$. Assume that $f : D \to \mathbb{R}$ is hemiintegrable and α-Jensen convex on D. Then f also satisfies the approximate upper Hermite–Hadamard inequality (2.1) *with $\lambda := \int_0^1 t\rho(t)dt$ and $\beta : D^* \to \mathbb{R}$ defined by*

$$\beta(u) := \sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 \alpha(d_{\mathbb{Z}}(2^n t)u)\rho(t)dt \quad (u \in D^*). \tag{2.8}$$

Proof. By Proposition 2.1, the function $\psi : [0, 1] \to \mathbb{R}_+$ defined by (2.2) is a Lebesgue integrable function satisfying the functional equation (2.3) for which (2.4) holds.

Let $f : D \to \mathbb{R}$ be an approximately Jensen convex function. Let $x, y \in D$ be arbitrary fixed. Then, by α-Jensen convexity of f, we have that

$$f(tx+(1-t)y) \le \begin{cases} \dfrac{f(2tx+(1-2t)y)+f(y)}{2} + \alpha(2t(x-y)) & (t \in [0,\frac{1}{2}]), \\ \dfrac{f(x)+f((2t-1)x+(2-2t)y)}{2} + \alpha((2-2t)(x-y)) & (t \in]\frac{1}{2},1]). \end{cases}$$

Multiplying the above inequality by $2\psi(t)$, taking the integral over $[0,1]$, we get

$$\begin{aligned} &\int_0^1 f(tx+(1-t)y)2\psi(t)dt \\ &\le \int_0^{\frac{1}{2}} \Big(f(2tx+(1-2t)y)+f(y)+2\alpha\big(2t(x-y)\big)\Big)\psi(t)dt \\ &+ \int_{\frac{1}{2}}^1 \Big(f(x)+f((2t-1)x+(2-2t)y)+2\alpha\big((2-2t)(x-y)\big)\Big)\psi(t)dt. \end{aligned} \tag{2.9}$$

Substituting $t := \frac{s}{2}$ and $t := \frac{1+s}{2}$ in the first and second terms on the right hand side of (2.9), using (2.4), and observing that $d_{\mathbb{Z}}(t) = \min(2t, 2-2t)$, we have that

$$\begin{aligned} &\int_0^{\frac{1}{2}} \Big(f(2tx+(1-2t)y)+f(y)+2\alpha\big(2t(x-y)\big)\Big)\psi(t)dt \\ &= f(y)\int_0^1 (1-t)\rho(t)dt + \frac{1}{2}\int_0^1 f(sx+(1-s)y)\psi(\tfrac{s}{2})ds \\ &+ 2\int_0^{\frac{1}{2}} \alpha\big(d_{\mathbb{Z}}(t)(x-y)\big)\psi(t)dt \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &\int_{\frac{1}{2}}^1 \Big(f(x)+f((2t-1)x+(2-2t)y)+2\alpha\big((2-2t)(x-y)\big)\Big)\psi(t)dt \\ &= f(x)\int_0^1 t\rho(t)dt + \frac{1}{2}\int_0^1 f(sx+(1-s)y)\psi(\tfrac{1+s}{2})ds \\ &+ 2\int_{\frac{1}{2}}^1 \alpha\big(d_{\mathbb{Z}}(t)(x-y)\big)\psi(t)dt. \end{aligned} \tag{2.11}$$

Combining (2.9), (2.10) and (2.11), we get that

$$\begin{aligned} \int_0^1 f(tx+(1-t)y)\rho(t)dt &= \int_0^1 f(tx+(1-t)y)\Big(2\psi(t) - \tfrac{1}{2}\psi(\tfrac{t}{2}) - \tfrac{1}{2}\psi(\tfrac{t+1}{2})\Big)dt \\ &\le f(x)\int_0^1 t\rho(t)dt + f(y)\int_0^1 (1-t)\rho(t)dt + 2\int_0^1 \alpha\big(d_{\mathbb{Z}}(t)(x-y)\big)\psi(t)dt. \end{aligned}$$

To complete the proof, it remains to show that the last term containing ψ equals $\beta(x-y)$. Indeed, applying formula (2.2) and the 1-periodicity of the function $d_{\mathbb{Z}}$, we get

$$\begin{aligned}
2\int_0^1 \alpha\big(d_{\mathbb{Z}}(t)(x-y)\big)\psi(t)dt &= 2\int_0^1 \alpha\big(d_{\mathbb{Z}}(t)(x-y)\big)\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{4^n}\Big(\sum_{k=0}^{2^n-1}\rho\big(\tfrac{t+k}{2^n}\big)\Big)dt \\
&= \sum_{n=0}^{\infty}\frac{1}{4^n}\Big(\sum_{k=0}^{2^n-1}\int_0^1 \alpha\big(d_{\mathbb{Z}}(t)(x-y)\big)\rho\big(\tfrac{t+k}{2^n}\big)dt\Big) \\
&= \sum_{n=0}^{\infty}\frac{1}{4^n}\Big(\sum_{k=0}^{2^n-1}2^n\int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}}\alpha\big(d_{\mathbb{Z}}(2^n s-k)(x-y)\big)\rho(s)ds\Big) \\
&= \sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 \alpha\big(d_{\mathbb{Z}}(2^n s)(x-y)\big)\rho(s)ds = \beta(x-y),
\end{aligned}$$

which proves the statement. □

The other form of the error function β stated in Theorem 2.4 can be obtained by using a generalized form of Theorem 1.2 by Jacek Tabor and Józef Tabor [**67**].

Theorem 2.3. (Tabor–Tabor [**67**])
Let $\alpha : D^* \to \mathbb{R}_+$ be radially increasing such that

$$\sum_{n=0}^{\infty}\alpha\Big(\frac{u}{2^n}\Big) < \infty \qquad (u \in D^*). \tag{2.12}$$

Then, an upper hemicontinuous function $f : D \to \mathbb{R}$ is α-Jensen convex on D if and only if

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)+\sum_{n=0}^{\infty}\alpha\Big(\frac{x-y}{2^n}\Big)d_{\mathbb{Z}}(2^n t) \tag{2.13}$$

for all $x, y \in D$ and $t \in [0,1]$.

Theorem 2.4. (Makó–Páles [**45**])
Let $\alpha : D^* \to \mathbb{R}_+$ be radially increasing such that (2.12) *holds. If $f : D \to \mathbb{R}$ is upper hemicontinuous and α-Jensen convex on D, then f also satisfies the Hermite–Hadamard inequality* (2.1) *with $\lambda := \int_0^1 t\rho(t)dt$ and $\beta : D^* \to \mathbb{R}$ is defined by*

$$\beta(u) := \sum_{n=0}^{\infty}\alpha\Big(\frac{u}{2^n}\Big)\int_0^1 d_{\mathbb{Z}}(2^n t)\rho(t)dt \qquad (u \in D^*). \tag{2.14}$$

Proof. If $f : D \to \mathbb{R}$ is upper hemicontinuous and α-Jensen convex on D, then (2.13) holds. Multiplying this inequality by $\rho(t)$ and then integrating with respect to t over $[0,1]$, (2.14) follows immediately. □

Let X be a normed space. Next, we consider the case, when α is a linear combination of the powers of the norm with positive exponents, i.e., if α is of the form

$$\alpha(u) := \int\limits_{]0,\infty[} \|u\|^q d\nu(q) \qquad (u \in D^*), \tag{2.15}$$

where ν is a nonnegative Borel measure on the interval $]0,\infty[$. An important particular case is when ν is of the form $\sum_{i=1}^k c_i\delta_{q_i}$, where $c_i \in \mathbb{R}_+$, $q_i > 0$ and δ_{q_i} stands for the Dirac measure concentrated at q_i for $i \in \{1,\dots,k\}$.

Theorem 2.5. (Makó–Páles [**45**])
Let $\rho : [0,1] \to \mathbb{R}_+$ be a Lebesgue integrable function with $\int_0^1 \rho = 1$ and let ν be a signed Borel measure on $]0,\infty[$ such that

$$\int\limits_{]0,\infty[} \|u\|^q d|\nu|(q) < \infty \qquad (u \in D^*).$$

Assume that $f : D \to \mathbb{R}$ is hemiintegrable on D and is approximately Jensen convex in the following sense

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2} + \int\limits_{]0,\infty[} \|x-y\|^q d\nu(q) \qquad (x,y \in D). \tag{2.16}$$

Then f also satisfies the approximate Hermite–Hadamard inequality

$$\begin{aligned}\int_0^1 f(tx+(1-t)y)\rho(t)dt &\le \lambda f(x) + (1-\lambda)f(y)\\ &\quad + \int\limits_{]0,\infty[}\int_0^1 T_q(t)\rho(t)dt\|x-y\|^q d\nu(q) \quad (x,y\in D),\end{aligned} \tag{2.17}$$

with $\lambda := \int_0^1 t\rho(t)dt$.

Proof. It is easy to see that α defined by (2.15) is radially bounded and measurable. Thus, by Theorem 2.2, it is enough to compute the error function β defined by (2.8). Hence, using (2.8), (2.15), Fubini's theorem and Lebesgue's theorem, we obtain that, for all $u \in D^*$,

$$\begin{aligned}\beta(u) &= \sum_{n=0}^{\infty} \frac{1}{2^n} \int_0^1 \int\limits_{]0,\infty[} \|(d_{\mathbb{Z}}(2^n t)u)\|^q d\nu(q)\rho(t)dt\\ &= \int\limits_{]0,\infty[}\int_0^1 \sum_{n=0}^{\infty} \frac{(d_{\mathbb{Z}}(2^n t))^q}{2^n}\rho(t)dt\|u\|^q d\nu(q) = \int\limits_{]0,\infty[}\int_0^1 T_q(t)\rho(t)dt\|u\|^q d\nu(q),\end{aligned}$$

which completes the proof. □

Theorem 2.6. (Makó–Páles [**45**])
Let $\rho : [0,1] \to \mathbb{R}_+$ *be a Lebesgue integrable function with* $\int_0^1 \rho = 1$ *and let* ν *be a nonnegative Borel measure on* $]0,\infty[$*, such that*

$$\int_{]0,\infty[} \|u\|^q d\nu(q) < \infty \qquad (u \in D^*) \tag{2.18}$$

and

$$\int_{]0,\infty[} \frac{2^q}{2^q-1} d\nu(q) < \infty. \tag{2.19}$$

Assume that $f : D \to \mathbb{R}$ *is upper hemicontinuous and approximately Jensen convex in the sense of* (2.16)*. Then* f *also satisfies the following approximate Hermite–Hadamard inequality*

$$\begin{aligned} \int_0^1 f(tx+(1-t)y)\rho(t)dt &\le \lambda f(x) + (1-\lambda) f(y) \\ &+ \int_{]0,\infty[} \int_0^1 S_q(t)\rho(t)dt\|x-y\|^q d\nu(q) \quad (x,y\in D), \end{aligned} \tag{2.20}$$

with $\lambda := \int_0^1 t\rho(t)dt$.

Proof. Consider the function α defined by (2.15). Then, for all $u \in D^*$, the mapping $t \mapsto \alpha(tu) = \int_{]0,\infty[} t^q\|u\|^q d\nu(q)$ is increasing on $[0,1]$ and, for all $u \in D^*$,

$$\sum_{n=0}^{\infty} \alpha\Big(\frac{u}{2^n}\Big) = \sum_{n=0}^{\infty} \int_{]0,\infty[} \|2^{-n}u\|^q d\nu(q) = \int_{]0,\infty[} \frac{2^q}{2^q-1}\|u\|^q d\nu(q).$$

If $\|u\| \le 1$, then the latter series is convergent in virtue of (2.19). For $\|u\| > 1$, we have

$$\int_{]0,\infty[} \frac{2^q}{2^q-1}\|u\|^q d\nu(q) \le \|u\| \int_{]0,1]} \frac{2^q}{2^q-1} d\nu(q) + 2\int_{]1,\infty[} \|u\|^q d\nu(q) < \infty,$$

which proves the convergence condition (2.12). Thus, by Theorem 2.4, it is enough to compute the error function β defined by (2.14). Hence, using (2.14), (2.15), Fubini's theorem and Lebesgue's theorem, we obtain that, for all $u \in D^*$

$$\begin{aligned} \beta(u) &= \sum_{n=0}^{\infty} \int_{]0,\infty[} \Big(\frac{\|u\|}{2^n}\Big)^q d\nu(q) \int_0^1 d_{\mathbb{Z}}(2^n t)\rho(t)dt \\ &= \int_{]0,\infty[} \int_0^1 \sum_{n=0}^{\infty} \frac{d_{\mathbb{Z}}(2^n t)}{2^{nq}} \rho(t)dt\|u\|^q d\nu(q) = \int_{]0,\infty[} \int_0^1 S_q(t)\rho(t)dt\|u\|^q d\nu(q), \end{aligned}$$

which completes the proof. □

Now we consider the case in the previous theorems when $\rho \equiv 1$ and the measure ν is the Dirac measure $\varepsilon\delta_q$.

Corollary 2.7. (Makó–Páles [**45**])
Let $\varepsilon \in \mathbb{R}_+$ and $q > 0$. Assume that $f : D \to \mathbb{R}$ is hemiintegrable and (ε, q)-Jensen convex. Then f also satisfies the following approximate Hermite–Hadamard inequality

$$\int_0^1 f(tx + (1-t)y)dt \le \frac{f(x) + f(y)}{2} + \frac{2\varepsilon}{q+1}\|x - y\|^q \qquad (x, y \in D). \tag{2.21}$$

Proof. The conditions of Theorem 2.5 hold with $\rho \equiv 1$ and $\nu := \varepsilon\delta_q$, Then, by (ε, q)-convexity implies that (2.16), hence to prove the statement, it is enough to compute the error term in (2.17). Using the definition of the T_q, the substitution $s := 2^n t$ and the 1-periodicity of $d_{\mathbb{Z}}^q$, we get

$$\begin{aligned}
\int_0^1 T_q(t)dt &= \int_0^1 \Big(\sum_{n=0}^{\infty} \frac{d_{\mathbb{Z}}^q(2^n t)}{2^n}\Big)dt = \sum_{n=0}^{\infty} \frac{1}{2^n} \int_0^1 d_{\mathbb{Z}}^q(2^n t)dt \\
&= \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \int_0^{2^n} d_{\mathbb{Z}}^q(s)ds = \sum_{n=0}^{\infty} \frac{1}{2^n} \int_0^1 d_{\mathbb{Z}}^q(s)ds \\
&= \sum_{n=0}^{\infty} \frac{2^q}{2^n}\Big(\int_0^{\frac{1}{2}} s^q ds + \int_{\frac{1}{2}}^1 (1-s)^q ds\Big) \\
&= \sum_{n=0}^{\infty} \frac{2^q}{2^n}\Big(\frac{(\frac{1}{2})^{q+1}}{q+1} + \frac{(\frac{1}{2})^{q+1}}{q+1}\Big) = \frac{1}{q+1}\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{2}{q+1}.
\end{aligned}$$

Thus, (2.17) reduces to (2.21), which proves the statement. □

Corollary 2.8. (Makó–Páles [**45**])
Let $\varepsilon \in \mathbb{R}_+$ and $q > 0$. Assume that $f : D \to \mathbb{R}$ is upper hemicontinuous and (ε, q)-Jensen convex. Then f also satisfies the following approximate Hermite–Hadamard inequality

$$\int_0^1 f(tx + (1-t)y)dt \le \frac{f(x) + f(y)}{2} + \frac{2^q\varepsilon}{2^{q+1} - 2}\|x - y\|^q \qquad (x, y \in D). \tag{2.22}$$

Proof. The conditions of Theorem 2.6 are satisfied with $\rho \equiv 1$ and $\nu := \varepsilon\delta_q$, Then, by (ε, q)-Jensen convexity of f, implies that (2.16) holds, hence to prove the statement, it is enough to compute the error term in (2.20). Using the definition of the

S_q, the substitution $s := 2^n t$ and the 1-periodicity of $d_{\mathbb{Z}}$, we get

$$\begin{aligned}\int_0^1 S_q(t)dt &= \int_0^1 \Big(\sum_{n=0}^{\infty} \frac{d_{\mathbb{Z}}(2^n t)}{2^{nq}}\Big)dt = \sum_{n=0}^{\infty} \frac{1}{2^{nq}} \int_0^1 d_{\mathbb{Z}}(2^n t)dt \\ &= \sum_{n=0}^{\infty} \frac{1}{2^{nq+n}} \int_0^{2^n} d_{\mathbb{Z}}(s)ds = \sum_{n=0}^{\infty} \frac{1}{2^{nq}} \int_0^1 d_{\mathbb{Z}}(s)ds \\ &= \sum_{n=0}^{\infty} \frac{1}{2^{nq}}\Big(\int_0^{\frac{1}{2}} 2sds + \int_{\frac{1}{2}}^1 2(1-s)ds\Big) \\ &= \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{2^{nq}} = \frac{2^q}{2^{q+1}-2}.\end{aligned}$$

Thus, (2.20) reduces to (2.22), which completes the proof. □

Remark 2.9. The constants in the two error terms obtained in (2.21) and (2.22) are comparable in the following way: for $q \in]0,1[\cup]2,\infty[$,

$$\frac{2}{q+1} < \frac{2^q}{2^{q+1}-2}$$

and the inequality reverses for $q \in]1,2[$.

2.2. From upper Hermite–Hadamard inequality to approximate Jensen convexity

For $p > 0$, define the class of functions Φ_p by

$$\Phi_p := \Big\{\phi :]0,1[\to \mathbb{R} \mid \phi \text{ is Lebesgue measurable and } |\!|\!|\phi|\!|\!|_p = \sup_{t\in]0,1[} |\ln t|^{1-p}|\phi(t)| < \infty\Big\}.$$

In the sequel, Γ denotes Euler's Gamma function.

Proposition 2.10. (Makó–Páles [**45**])
For all $p > 0$, the elements of Φ_p are Lebesgue integrable functions and

$$\|\phi\|_1 = \int_0^1 |\phi(t)|dt \le \Gamma(p)|\!|\!|\phi|\!|\!|_p \qquad (\phi \in \Phi_p). \tag{2.23}$$

Proof. Let $p > 0$ and $\phi \in \Phi_p$. From the definition of Φ_p, we get that

$$|\phi(t)| \le |\!|\!|\phi|\!|\!|_p(-\ln t)^{p-1} \qquad (t \in]0,1[).$$

Thus, with the substitution $s = -\ln t$, we get

$$\int_0^1 |\phi(t)|dt \le |\!|\!|\phi|\!|\!|_p \int_0^1 (-\ln t)^{p-1}dt = |\!|\!|\phi|\!|\!|_p \int_0^{\infty} s^{p-1}e^{-s}ds = \Gamma(p)|\!|\!|\phi|\!|\!|_p < \infty,$$

which proves the integrability of ϕ and (2.23). □

Proposition 2.11. (Makó–Páles [**45**])
For $p, q > 0$ and $\phi \in \Phi_p$, $\psi \in \Phi_q$, the function $\phi * \psi$ defined by

$$(\phi * \psi)(t) := \int_t^1 \tfrac{1}{\tau}\phi(\tfrac{t}{\tau})\psi(\tau)d\tau \qquad (t \in]0,1[).$$

is continuous on the open interval $]0,1[$, belongs to Φ_{p+q} and

$$\|\phi * \psi\|_{p+q} \leq \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\|\phi\|_p\|\psi\|_q. \tag{2.24}$$

Furthermore,

$$\int_0^1 (\phi * \psi) = \int_0^1 \phi \int_0^1 \psi. \tag{2.25}$$

Proof. Given $p, q > 0$, it is well known that the function

$$\tau \mapsto (1-\tau)^{p-1}\tau^{q-1}, \qquad (\tau \in]0,1[) \tag{2.26}$$

is integrable over $[0,1]$ and

$$B(p,q) := \int_0^1 (1-\tau)^{p-1}\tau^{q-1}d\tau = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

By the inclusions $\phi \in \Phi_p$, $\psi \in \Phi_q$, we have that

$$|\phi(t)| \leq \|\phi\|_p(-\ln t)^{p-1} \quad \text{and} \quad |\psi(t)| \leq \|\psi\|_q(-\ln t)^{q-1} \quad (t \in]0,1[). \tag{2.27}$$

To prove that $\phi * \psi$ is continuous at $t \in]0,1[$, let $\varepsilon > 0$. By the integrability of (2.26), there exists $\rho \in]0,1[$ such that, for every measurable subset $T \subseteq [0,1]$ with $\text{meas}(T) < \rho$,

$$\int\limits_T (1-\tau)^{p-1}\tau^{q-1}d\tau < \frac{\varepsilon}{6\|\phi\|_p\|\psi\|_q(-\ln t)^{p+q-1}+1}. \tag{2.28}$$

Define the function $\widetilde{\phi} :]0,\infty[\to \mathbb{R}$ by

$$\widetilde{\phi}(x) := \phi(e^{-x}).$$

Then, by the first inequality in (2.27), we have that

$$x^{1-p}|\widetilde{\phi}(x)| \leq \|\phi\|_p \qquad (x \in]0,\infty[).$$

Applying Luzin's theorem for the bounded measurable function $x \mapsto x^{1-p}\widetilde{\phi}(x)$, we can construct a measurable set $\widetilde{H} \subseteq]0,\infty[$ and a continuous function $\widetilde{f} :]0,\infty[\to \mathbb{R}$ such that

$$\text{meas}(]0,\infty[\setminus \widetilde{H}) < \frac{(-\ln t)\rho}{2}, \quad \widetilde{f}|_{\widetilde{H}} = \widetilde{\phi}|_{\widetilde{H}} \quad \text{and} \quad |\widetilde{f}(x)| \leq \|\phi\|_p x^{p-1} \tag{2.29}$$

for all $x \in]0,\infty[$. Now define $f :]0,1[\to \mathbb{R}$ and $H \subseteq]0,1[$ by

$$f(t) := \widetilde{f}(-\ln t) \qquad (t \in]0,1[) \qquad \text{and} \qquad H := \exp(-\widetilde{H}).$$

In a view of (2.29), we get that

$$f|_H = \phi|_H \qquad \text{and} \qquad |f(t)| \le \|\phi\|_p (-\ln t)^{p-1} \qquad (t \in]0,1[). \tag{2.30}$$

By the continuity of the logarithmic function, there exists $\delta \in]0, \min(t, 1-t)[$, such that, for all $s \in]t-\delta, t+\delta[$,

$$\|\phi\|_p \|\psi\|_q (-\ln(s))^{p+q-1} < \|\phi\|_p \|\psi\|_q (-\ln(t))^{p+q-1} + \frac{1}{6} \quad \text{and} \quad \left|1 - \frac{\ln t}{\ln s}\right| < \rho. \tag{2.31}$$

For $s \in]0,1[$, we have

$$\begin{aligned} |(\phi * \psi)(t) - (\phi * \psi)(s)| &= \Big| \int_t^1 \tfrac{1}{\tau}\phi(\tfrac{t}{\tau})\psi(\tau)d\tau - \int_s^1 \tfrac{1}{\tau}\phi(\tfrac{s}{\tau})\psi(\tau)d\tau \Big| \\ &\le \begin{cases} \int_s^t \tfrac{1}{\tau}\big|\phi(\tfrac{s}{\tau})\big||\psi(\tau)|d\tau + \int_t^1 \tfrac{1}{\tau}\big|\phi(\tfrac{t}{\tau}) - \phi(\tfrac{s}{\tau})\big||\psi(\tau)|d\tau & \text{if } s < t, \\ \int_t^s \tfrac{1}{\tau}\big|\phi(\tfrac{t}{\tau})\big||\psi(\tau)|d\tau + \int_s^1 \tfrac{1}{\tau}\big|\phi(\tfrac{t}{\tau}) - \phi(\tfrac{s}{\tau})\big||\psi(\tau)|d\tau & \text{if } t < s. \end{cases} \end{aligned} \tag{2.32}$$

Consider first the case $s \in]t-\delta, t[$. The second inequality in (2.31) implies that the measure of the interval $T :=]\frac{\ln t}{\ln s}, 1[$ is smaller than ρ. Thus, inequality (2.28) holds with this set T. Therefore, for the first term on the right hand side of (2.32), using the estimates (2.27), substituting $\tau = s^\sigma$, and using the first inequality in (2.31), we get

$$\begin{aligned} \int_s^t \tfrac{1}{\tau}\big|\phi(\tfrac{s}{\tau})\big||\psi(\tau)|d\tau &\le \|\phi\|_p \|\psi\|_q \int_s^t \tfrac{1}{\tau}(\ln\tau - \ln s)^{p-1}(-\ln\tau)^{q-1}d\tau \\ &\le \|\phi\|_p \|\psi\|_q (-\ln s)^{p+q-1} \int_{\frac{\ln t}{\ln s}}^1 (1-\sigma)^{p-1}\sigma^{q-1}d\sigma \\ &< \frac{\|\phi\|_p \|\psi\|_q (-\ln s)^{p+q-1}\varepsilon}{6\|\phi\|_p \|\psi\|_q (-\ln t)^{p+q-1} + 1} < \frac{\varepsilon}{6}. \end{aligned} \tag{2.33}$$

To obtain an estimate for the second term on the right hand side of (2.32) (when $s < t$), we use $\phi(x) = f(x)$ for $x \in H$ and obtain

$$\begin{aligned}
&\int_t^1 \tfrac{1}{\tau}\left|\phi(\tfrac{t}{\tau}) - \phi(\tfrac{s}{\tau})\right||\psi(\tau)|d\tau \\
&\le \int_t^1 \tfrac{1}{\tau}\Big(\left|\phi(\tfrac{t}{\tau}) - f(\tfrac{t}{\tau})\right| + \left|f(\tfrac{t}{\tau}) - f(\tfrac{s}{\tau})\right| + \left|f(\tfrac{s}{\tau}) - \phi(\tfrac{s}{\tau})\right|\Big)|\psi(\tau)|d\tau \\
&\le \int\limits_{]t,1[\setminus tH^{-1}} \tfrac{1}{\tau}\left|\phi(\tfrac{t}{\tau}) - f(\tfrac{t}{\tau})\right||\psi(\tau)|d\tau + \int\limits_{]t,1[} \tfrac{1}{\tau}\left|f(\tfrac{t}{\tau}) - f(\tfrac{s}{\tau})\right||\psi(\tau)|d\tau \\
&\qquad + \int\limits_{]t,1[\setminus sH^{-1}} \tfrac{1}{\tau}\left|f(\tfrac{s}{\tau}) - \phi(\tfrac{s}{\tau})\right||\psi(\tau)|d\tau.
\end{aligned} \tag{2.34}$$

The first inequality in (2.29) and the second estimate in (2.31) imply that, for $s \in]t-\delta, t]$,

$$\begin{aligned}
\text{meas}\,(]0,1[\setminus(1+(\ln s)^{-1}\widetilde{H})) &= (-\ln s)^{-1}\,\text{meas}(]0,-\ln s[\setminus\widetilde{H}) \\
&< \frac{\rho \ln t}{2\ln s} < \frac{\rho(1+\rho)}{2} < \rho.
\end{aligned} \tag{2.35}$$

Thus (2.28) holds with $T :=]0,1[\setminus(1+(\ln s)^{-1}\widetilde{H})$. Using (2.27) and (2.30), then substituting $\tau = s^{\sigma}$ and finally applying inequality (2.28), we get

$$\begin{aligned}
&\int\limits_{]t,1[\setminus sH^{-1}} \tfrac{1}{\tau}\left|f(\tfrac{s}{\tau}) - \phi(\tfrac{s}{\tau})\right||\psi(\tau)|d\tau \\
&\qquad\le \int\limits_{]s,1[\setminus sH^{-1}} \tfrac{1}{\tau}\Big(\left|f(\tfrac{s}{\tau})\right| + \left|\phi(\tfrac{s}{\tau})\right|\Big)|\psi(\tau)|d\tau \\
&\qquad\le 2\|\phi\|_p\|\psi\|_q \int\limits_{]s,1[\setminus sH^{-1}} \tfrac{1}{\tau}(\ln\tau - \ln s)^{p-1}(-\ln\tau)^{q-1}d\tau \\
&\qquad\le 2\|\phi\|_p\|\psi\|_q(-\ln s)^{p+q-1} \int\limits_{]0,1[\setminus(1+(\ln s)^{-1}\widetilde{H})} (1-\sigma)^{p-1}\sigma^{q-1}d\sigma \\
&\qquad\le \frac{2\|\phi\|_p\|\psi\|_q(-\ln s)^{p+q-1}\varepsilon}{6\|\phi\|_p\|\psi\|_q(-\ln t)^{p+q-1}+1} < \frac{\varepsilon}{3}.
\end{aligned} \tag{2.36}$$

Applying this inequality for $s = t$, we also get

$$\int\limits_{]t,1[\setminus tH^{-1}} \tfrac{1}{\tau}\left|f(\tfrac{s}{\tau}) - \phi(\tfrac{s}{\tau})\right||\psi(\tau)|d\tau < \frac{\varepsilon}{3}. \tag{2.37}$$

Consider the second expression on the right hand side of (2.34). We prove that

$$\lim_{s\to t-0}\int_{]t,1[}\tfrac{1}{\tau}\Big|f(\tfrac{t}{\tau})-f(\tfrac{s}{\tau})\Big||\psi(\tau)|d\tau=0. \tag{2.38}$$

By the continuity of f, the integrand pointwise converges to zero hence, in view of Lebesgue's dominated convergence theorem, it suffices to show that the integrand admits an integrable majorant which is independent of $s\in]t-\delta,t[$.

Using the inequality (2.27) and (2.30), we get that, for all $\tau\in[t,1[$ and $s\in]t-\delta,t[$,

$$\begin{aligned}\tfrac{1}{\tau}\Big|f(\tfrac{t}{\tau})-f(\tfrac{s}{\tau})\Big||\psi(\tau)| &\le \tfrac{1}{\tau}\Big(\Big|f(\tfrac{t}{\tau})\Big|+\Big|f(\tfrac{s}{\tau})\Big|\Big)|\psi(\tau)|\\ &\le \frac{\|\phi\|_p\|\psi\|_q}{\tau}\Big((\ln\tau-\ln t)^{p-1}+(\ln\tau-\ln s)^{p-1}\Big)(-\ln\tau)^{q-1}.\end{aligned}$$

Now there are two cases. If $p\le 1$ we have that the function $s\mapsto(\ln\tau-\ln s)^{p-1}$ is nondecreasing, hence

$$(\ln\tau-\ln s)^{p-1}\le(\ln\tau-\ln t)^{p-1}\quad(\tau\in[t,1[,\ s\in]0,t]).$$

This means that, in this case,

$$\tfrac{1}{\tau}\Big|f(\tfrac{t}{\tau})-f(\tfrac{s}{\tau})\Big||\psi(\tau)|\le\frac{2\|\phi\|_p\|\psi\|_q}{\tau}(\ln\tau-\ln t)^{p-1}(-\ln\tau)^{q-1}\quad(\tau\in[t,1[,\ s\in]0,t]).$$

Moreover, the right hand side is integrable with respect to τ because, with the substitution $\tau=t^{\sigma}$, it follows that

$$\int_t^1\tfrac{1}{\tau}(\ln\tau-\ln t)^{p-1}(-\ln\tau)^{q-1}d\tau=(-\ln t)^{p+q-1}B(p,q)<\infty. \tag{2.39}$$

When $p>1$, then the function $s\mapsto(\ln\tau-\ln s)^{p-1}$ is decreasing, hence, for all $\tau\in[t,1[,\ s\in]t-\delta,t]$,

$$(\ln\tau-\ln s)^{p-1}\le(\ln\tau-\ln(t-\delta))^{p-1}\le(-\ln(t-\delta))^{p-1}.$$

Thus, in this case, for all $\tau\in[t,1[,\ s\in]t-\delta,t]$,

$$\tfrac{1}{\tau}\Big|f(\tfrac{t}{\tau})-f(\tfrac{s}{\tau})\Big||\psi(\tau)|\le\frac{\|\phi\|_p\|\psi\|_q}{\tau}\Big((\ln\tau-\ln t)^{p-1}+(-\ln(t-\delta))^{p-1}\Big)(-\ln\tau)^{q-1}.$$

Again, the majorant is integrable because (2.39) holds, and (substituting $\tau=t^{\sigma}$)

$$\begin{aligned}\int_t^1\tfrac{1}{\tau}(-\ln(t-\delta))^{p-1}(-\ln\tau)^{q-1}d\tau&=(-\ln(t-\delta))^{p-1}\int_t^1\tfrac{1}{\tau}(-\ln\tau)^{q-1}d\tau\\&=(-\ln(t-\delta))^{p-1}(-\ln t)^{q}B(1,q)<\infty.\end{aligned}$$

Therefore, Lebesgue's Theorem can be applied and hence (2.38) holds. Thus there exists $\delta^* \in]0, \delta]$, such that, for all $s \in]t - \delta^*, t[$,

$$\int_{]t,1[} \tfrac{1}{\tau}\Big|f(\tfrac{t}{\tau}) - f(\tfrac{s}{\tau})\Big||\psi(\tau)|d\tau < \frac{\varepsilon}{6}. \tag{2.40}$$

Combining the inequalities (2.32), (2.33), (2.34), (2.36), (2.37), and (2.40), we get

$$|(\phi * \psi)(t) - (\phi * \psi)(s)| < \varepsilon \qquad (s \in]t - \delta^*, t[),$$

which proves the left-continuity of $\phi * \psi$ at t.

To prove the right-continuity of $\phi * \psi$ at t, we apply (2.32) for $s \in]t, t + \delta[$. The second inequality in (2.31) implies that

$$\text{meas}\,(]\tfrac{\ln s}{\ln t}, 1[) \le \frac{\rho}{1+\rho} < \rho.$$

Thus, inequality (2.28) holds with the interval $T :=]\tfrac{\ln s}{\ln t}, 1[$. Therefore, for the first term on the right hand side of (2.32) (using the estimates (2.27), and substituting $\tau = t^\sigma$ in the evaluation of the integral), we get

$$\int_t^s \tfrac{1}{\tau}\Big|\phi(\tfrac{t}{\tau})\Big||\psi(\tau)|d\tau < \frac{\varepsilon}{6}. \tag{2.41}$$

To obtain an estimate for the second term on the right hand side of (2.32) (when $t < s$), we use $\phi(x) = f(x)$ for $x \in H$ and obtain

$$\begin{aligned}
&\int_s^1 \tfrac{1}{\tau}\Big|\phi(\tfrac{t}{\tau}) - \phi(\tfrac{s}{\tau})\Big||\psi(\tau)|d\tau \\
&\quad\le \int_s^1 \tfrac{1}{\tau}\Big(\Big|\phi(\tfrac{t}{\tau}) - f(\tfrac{t}{\tau})\Big| + \Big|f(\tfrac{t}{\tau}) - f(\tfrac{s}{\tau})\Big| + \Big|f(\tfrac{s}{\tau}) - \phi(\tfrac{s}{\tau})\Big|\Big)|\psi(\tau)|d\tau \\
&\quad\le \int_{]s,1[\setminus tH^{-1}} \tfrac{1}{\tau}\Big|\phi(\tfrac{t}{\tau}) - f(\tfrac{t}{\tau})\Big||\psi(\tau)|d\tau + \int_{]s,1[} \tfrac{1}{\tau}\Big|f(\tfrac{t}{\tau}) - f(\tfrac{s}{\tau})\Big||\psi(\tau)|d\tau \\
&\qquad\qquad + \int_{]s,1[\setminus sH^{-1}} \tfrac{1}{\tau}\Big|f(\tfrac{s}{\tau}) - \phi(\tfrac{s}{\tau})\Big||\psi(\tau)|d\tau.
\end{aligned} \tag{2.42}$$

Applying an analogous argument as before, for $s \in]t, t+\delta[$, we can obtain the estimates

$$\int_{]s,1[\setminus sH^{-1}} \tfrac{1}{\tau}\Big|f(\tfrac{s}{\tau}) - \phi(\tfrac{s}{\tau})\Big||\psi(\tau)|d\tau < \frac{\varepsilon}{3} \quad \text{and} \quad \int_{]s,1[\setminus tH^{-1}} \tfrac{1}{\tau}\Big|f(\tfrac{s}{\tau}) - \phi(\tfrac{s}{\tau})\Big||\psi(\tau)|d\tau < \frac{\varepsilon}{3}. \tag{2.43}$$

Consider the second expression on the right hand side of (2.42). We will prove that

$$\lim_{s\to t+0}\int\limits_{]s,1[}\tfrac{1}{\tau}\Big|f(\tfrac{t}{\tau})-f(\tfrac{s}{\tau})\Big||\psi(\tau)|d\tau=0. \tag{2.44}$$

First, with the substitution $\tau=\sigma^{\frac{\ln s}{\ln t}}$, for $s\in]t,t+\delta[$, we can obtain

$$\begin{aligned}\int\limits_{]s,1[}\tfrac{1}{\tau}\Big|f(\tfrac{t}{\tau})-f(\tfrac{s}{\tau})\Big||\psi(\tau)|d\tau&=\frac{\ln s}{\ln t}\int\limits_{]t,1[}\tfrac{1}{\sigma}\Big|f(t\sigma^{-\frac{\ln s}{\ln t}})-f(s\sigma^{-\frac{\ln s}{\ln t}})\Big|\Big|\psi(\sigma^{\frac{\ln s}{\ln t}})\Big|d\sigma\\&\le\int\limits_{]t,1[}\tfrac{1}{\sigma}\Big|f(t\sigma^{-\frac{\ln s}{\ln t}})-f(s\sigma^{-\frac{\ln s}{\ln t}})\Big|\Big|\psi(\sigma^{\frac{\ln s}{\ln t}})\Big|d\sigma.\end{aligned} \tag{2.45}$$

By the continuity of f and the local boundedness of ψ (which is a consequence of the inequality (2.27)), the integrand on the right hand side of (2.45) pointwise converges to zero as $s\to t+0$, hence, in view of Lebesgue's dominated convergence theorem, it suffices to show that the integrand admits an integrable majorant which is independent of $s\in]t,t+\delta[$. Using the inequality (2.27) and (2.30), we get that, for all $\tau\in[t,1[$ and $s\in]t,t+\delta[$,

$$\begin{aligned}&\tfrac{1}{\sigma}\Big|f(t\sigma^{-\frac{\ln s}{\ln t}})-f(s\sigma^{-\frac{\ln s}{\ln t}})\Big|\Big|\psi(\sigma^{\frac{\ln s}{\ln t}})\Big|\\&\quad\le\|\phi\|_p\|\psi\|_q\,\tfrac{1}{\sigma}\Big(\big(-\ln(t\sigma^{-\frac{\ln s}{\ln t}})\big)^{p-1}+\big(-\ln(s\sigma^{-\frac{\ln s}{\ln t}})\big)^{p-1}\Big)\big(-\ln(\sigma^{\frac{\ln s}{\ln t}})\big)^{q-1}.\end{aligned}$$

Since $s\mapsto(-\ln s)^{q-1}$ is monotone on $]0,1[$, therefore, for $s\in]t,t+\delta[$, we have

$$\big(-\ln(\sigma^{\frac{\ln s}{\ln t}})\big)^{q-1}=\big(\tfrac{-\ln\sigma\ln s}{\ln t}\big)^{q-1}\le\max\{1,\big(\tfrac{\ln(t+\delta)}{\ln t}\big)^{q-1}\}(-\ln\sigma)^{q-1}.$$

Similarly, for $s\in]t,t+\delta[$ and $\sigma\in]t,1[$, we get

$$\begin{aligned}\big(-\ln(s\sigma^{-\frac{\ln s}{\ln t}})\big)^{p-1}&=\big(\tfrac{\ln s}{\ln t}\big)^{p-1}(-\ln t+\ln\sigma)^{p-1}\\&\le\begin{cases}\big(\frac{\ln(t+\delta)}{\ln t}\big)^{p-1}(-\ln t+\ln\sigma)^{p-1}&\text{if } p\le1,\\(-\ln t+\ln\sigma)^{p-1}&\text{if } p>1,\end{cases}\end{aligned}$$

and

$$\big(-\ln(t\sigma^{-\frac{\ln s}{\ln t}})\big)^{p-1}=\big(-\ln t+\tfrac{\ln s\ln\sigma}{\ln t}\big)^{p-1}\le\begin{cases}(-\ln t+\ln\sigma)^{p-1}&\text{if } p\le1,\\(-\ln t)^{p-1}&\text{if } p>1.\end{cases}$$

Combining these inequalities, for $s \in]t, t+\delta[$ and $\sigma \in]t, 1[$, we obtain

$$\tfrac{1}{\sigma}\Big|f(t\sigma^{-\frac{\ln s}{\ln t}}) - f(s\sigma^{-\frac{\ln s}{\ln t}})\Big|\Big|\psi(\sigma^{\frac{\ln s}{\ln t}})\Big|$$
$$\le \begin{cases} \|\phi\|_p\|\psi\|_q\big(1+(\frac{\ln(t+\delta)}{\ln t})^{p-1}\big)\max\{1,(\frac{\ln(t+\delta)}{\ln t})^{q-1}\}\,\frac{1}{\sigma}(-\ln t+\ln\sigma)^{p-1}(-\ln\sigma)^{q-1} \\ \|\phi\|_p\|\psi\|_q\max\{1,(\frac{\ln(t+\delta)}{\ln t})^{q-1}\}\,\frac{1}{\sigma}\big((-\ln t+\ln\sigma)^{p-1}+(-\ln t)^{p-1}\big)(-\ln\sigma)^{q-1} \end{cases}$$

if $p \le 1$ or $p > 1$, respectively. It is easy to check (by substituting $\sigma = t^\tau$) that the function on the right hand side of this inequality is integrable with respect to σ over $]t, 1[$. Therefore, Lebesgue's Theorem can be applied and hence (2.44) holds. Thus there exists $\delta^{**} \in]0, \delta^*]$, such that, for all $s \in]t, t+\delta^{**}[$,

$$\int\limits_{]s,1[} \tfrac{1}{\tau}\Big|f(\tfrac{t}{\tau}) - f(\tfrac{s}{\tau})\Big||\psi(\tau)|d\tau < \frac{\varepsilon}{6}. \tag{2.46}$$

By summing up the respective sides of the inequalities (2.41), (2.42), (2.43), and (2.46), for all $s \in]t, t+\delta^{**}[$, we get

$$|(\phi * \psi)(t) - (\phi * \psi)(s)| < \varepsilon,$$

which completes the proof of the right-continuity of $\phi * \psi$ at t.

To prove (2.24), let $t \in]0, 1]$ be fixed. Using (2.27) and substituting $\tau = t^s$, we get

$$\begin{aligned} |(\phi*\psi)(t)| &\le \int_t^1 \tfrac{1}{\tau}\Big|\phi(\tfrac{t}{\tau})\Big||\psi(\tau)|d\tau \le \|\phi\|_p\|\psi\|_q| \int_t^1 \tfrac{1}{\tau}\big(-\ln(\tfrac{t}{\tau})\big)^{p-1}(-\ln\tau)^{q-1}d\tau \\ &= \|\phi\|_p\|\psi\|_q(-\ln t)^{p+q-1}\int_0^1 (1-s)^{p-1}s^{q-1}ds \\ &= B(p,q)\|\phi\|_p\|\psi\|_q(-\ln t)^{p+q-1} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\|\phi\|_p\|\psi\|_q(-\ln t)^{p+q-1}, \end{aligned}$$

which proves the inclusion $\phi * \psi \in \Phi_{p+q}$ and the inequality (2.24).

In the proof of (2.25) first we use Fubini's theorem and then the variable t/τ is replaced by s:

$$\begin{aligned} \int_0^1 (\phi*\psi)(t)dt &= \int_0^1\int_t^1 \tfrac{1}{\tau}\phi(\tfrac{t}{\tau})\psi(\tau)d\tau dt = \int_0^1\int_0^\tau \tfrac{1}{\tau}\phi(\tfrac{t}{\tau})\psi(\tau)dtd\tau \\ &= \int_0^1 \psi(\tau)\Big(\tfrac{1}{\tau}\int_0^\tau \phi(\tfrac{t}{\tau})dt\Big)d\tau = \int_0^1 \psi(\tau)d\tau\int_0^1\phi(s)ds. \end{aligned}$$

□

Lemma 2.12. (Makó–Páles [**45**])
Let $p > 0$ be arbitrarily fixed, then, for all $x \in \mathbb{R}$,

$$\lim_{n\to\infty}\frac{x^n}{\Gamma(np)} = 0 \tag{2.47}$$

and the convergence is uniform on every compact interval of $\mathbb{R}$.

Proof. To prove the lemma, we will show that the series $\sum \frac{x^n}{\Gamma(np)}$ is convergent on $\mathbb{R}$. Using Cauchy's root test on this series and the Stirling formula for the Γ function, we get that

$$\lim_{n\to\infty} \sqrt[n]{\frac{|x|^n}{\Gamma(np)}} = \lim_{n\to\infty} \frac{|x|}{\sqrt[n]{\Gamma(np)}} = |x| \lim_{n\to\infty} \frac{1}{\sqrt[n]{\sqrt{\frac{2\pi}{np}}(\frac{np}{e})^{np}}}$$
$$= |x| \lim_{n\to\infty} \frac{e^p \sqrt[2n]{np}}{\sqrt[2n]{2\pi}(np)^p} = 0.$$

This means that the series is absolute convergent on $\mathbb{R}$ and hence the convergence is uniform on compact subsets of $\mathbb{R}$, which yields the statement. □

Proposition 2.13. *Let* $p > 0$ *and* $\phi \in \Phi_p$. *Define the sequence* $\phi_n :]0,1[\to \mathbb{R}$ *by the recursion*

$$\phi_1 := \phi, \qquad \phi_{n+1} := \phi * \phi_n \qquad (n \in \mathbb{N}). \tag{2.48}$$

Then, for all $n \in \mathbb{N}$,

$$\phi_n \in \Phi_{np}, \qquad |||\phi_n|||_{np} \le \frac{(\Gamma(p))^n}{\Gamma(np)} |||\phi|||_p^n, \qquad \int_0^1 \phi_n = \Big(\int_0^1 \phi\Big)^n \tag{2.49}$$

and, for all $s \in]0,1[$,

$$\lim_{n\to\infty} \phi_n(s) = 0 \tag{2.50}$$

furthermore, for all $\delta \in]0,1[$, *the convergence is uniform on* $[\delta, 1[$.

Proof. By the definition of ϕ_1, (2.49) holds trivially for $n = 1$. Assume that (2.49) is valid for some n. By Proposition 2.11 and the inductive assumption, $\phi_{n+1} = \phi * \phi_n \in \Phi_{p+np}$,

$$|||\phi_{n+1}|||_{(n+1)p} \le \frac{\Gamma(p)\Gamma(np)}{\Gamma((n+1)p)} |||\phi|||_p |||\phi_n|||_{np}$$
$$\le \frac{\Gamma(p)\Gamma(np)}{\Gamma((n+1)p)} |||\phi|||_p \frac{(\Gamma(p))^n}{\Gamma(np)} |||\phi|||_p^n = \frac{(\Gamma(p))^{n+1}}{\Gamma((n+1)p)} |||\phi|||_p^{n+1},$$

and

$$\int_0^1 \phi_{n+1} = \int_0^1 \phi * \phi_n = \int_0^1 \phi \int_0^1 \phi_n = \int_0^1 \phi\Big(\int_0^1 \phi\Big)^n = \Big(\int_0^1 \phi\Big)^{n+1},$$

which proves (2.49) for $n+1$. To prove (2.50), choose n_0 such that $n_0 p \geq 2$ and let $\delta \in]0,1[$. Since the Gamma function is increasing on the interval $[2,\infty[$, by (2.49), for all $n \geq n_0$ and $s \in [\delta, 1[$, we have

$$\begin{aligned} |\phi_{n+n_0}(s)| &\leq \|\phi_{n+n_0}\|_{(n+n_0)p}(-\ln s)^{(n+n_0)p-1} \\ &\leq \frac{(\Gamma(p))^{n+n_0}}{\Gamma((n+n_0)p)}\|\phi\|_p^{n+n_0}(-\ln s)^{(n+n_0)p-1} \\ &= (-\ln s)^{n_0 p-1}\Gamma(p)^{n_0}\|\phi\|_p^{n_0}\frac{\left(\Gamma(p)(-\ln s)^p\|\phi\|_p\right)^n}{\Gamma((n+n_0)p)} \\ &\leq (-\ln s)^{n_0 p-1}\Gamma(p)^{n_0}\|\phi\|_p^{n_0}\frac{\left(\Gamma(p)(-\ln s)^p\|\phi\|_p\right)^n}{\Gamma(np)} \\ &\leq (-\ln \delta)^{n_0 p-1}\Gamma(p)^{n_0}\|\phi\|_p^{n_0}\frac{\left(\Gamma(p)(-\ln \delta)^p\|\phi\|_p\right)^n}{\Gamma(np)}. \end{aligned} \tag{2.51}$$

Letting $n \to \infty$ in (2.51) and using Lemma 2.12, we get (2.50). The estimate (2.51) also ensures the uniformity of the convergence on the compact subsets of $]0,1]$. □

Proposition 2.14. (Makó–Páles [**45**])
Let $g : [0,1] \to \mathbb{R}$ be an upper bounded measurable function which is upper semicontinuous at 0. *Let $p > 0$, $\phi \in \Phi_p$ be a nonnegative function with $\int_0^1 \phi = 1$ and define the the sequence $\phi_n :]0,1[\to \mathbb{R}$ by* (2.48). *Then*

$$\limsup_{n\to\infty} \int_0^1 g(s)\phi_n(s)ds \leq g(0). \tag{2.52}$$

Proof. Let $\varepsilon > 0$. If g is upper semicontinuous at 0, then there exists $\delta \in]0,1[$, such that

$$g(s) < g(0) + \frac{\varepsilon}{2} \qquad \text{for all} \qquad s \in]0,\delta[, \tag{2.53}$$

and, by the upper boundedness, there exists $K > \max(0, -g(0))$, such that

$$g(s) \leq K \qquad (s \in [0,1]). \tag{2.54}$$

By Proposition 2.13, $\lim_{n\to\infty} \phi_n = 0$ and the convergence is uniform also on $[\delta, 1[$. Hence, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\phi_n(s) \leq \frac{\varepsilon}{4K} \qquad (s \in [\delta, 1[). \tag{2.55}$$

Using the third expression (2.49) and $\int_0^1 \phi = 1$, we have that

$$\int_0^1 \phi_n = \Big(\int_0^1 \phi\Big)^n = 1. \tag{2.56}$$

Applying (2.56) and the nonnegativity of ϕ_n, we obtain

$$\begin{aligned}\int_0^1 g(s)\phi_n(s)ds - g(0) \\ = \int_0^\delta (g(s)-g(0))\phi_n(s)ds + \int_\delta^1 (g(s)-g(0))\phi_n(s)ds\end{aligned} \tag{2.57}$$

To obtain an estimate for the first term of the right hand side, we use (2.53) and (2.56). Then, for all $n \in \mathbb{N}$,

$$\int_0^\delta (g(s)-g(0))\phi_n(s)ds \le \int_0^\delta \frac{\varepsilon}{2}\phi_n(s)ds \le \frac{\varepsilon}{2}\int_0^1 \phi_n(s)ds = \frac{\varepsilon}{2}. \tag{2.58}$$

Consider the second expression on the right hand side of (2.57). Then using (2.54) and (2.55), we obtain, for all $n \ge n_0$,

$$\begin{aligned}\int_\delta^1 (g(s)-g(0))\phi_n(s)ds &\le \int_\delta^1 2K\phi_n(s)ds \\ &\le \int_\delta^1 2K\frac{\varepsilon}{4K}ds = \frac{\varepsilon}{2}(1-\delta) < \frac{\varepsilon}{2}.\end{aligned} \tag{2.59}$$

Combining the inequalities (2.57), (2.58) and (2.59), we get that, for all $n \ge n_0$,

$$\int_0^1 g(s)\phi_n(s)ds - g(0) < \varepsilon,$$

which proves the statement. □

The main result of this section is stated in the following theorem.

Theorem 2.15. (Makó–Páles [**45**])
Let $\beta : D^* \to \mathbb{R}$ be even and radially upper semicontinuous, $\rho : [0,1] \to \mathbb{R}_+$ be integrable with $\int_0^1 \rho = 1$ and there exist $c \ge 0$ and $p > 0$ such that

$$\rho(t) \le c(-\ln|1-2t|)^{p-1} \qquad (t \in]0, \tfrac{1}{2}[\cup]\tfrac{1}{2}, 1[), \tag{2.60}$$

and $\lambda \in [0,1]$. Then every $f : D \to \mathbb{R}$ lower hemicontinuous function satisfying the approximate upper Hermite–Hadamard inequality (2.1)*, α-Jensen convex provided that $\alpha : D^* \to \mathbb{R}$ is a radially lower semicontinuous solution of the functional inequality*

$$\alpha(u) \ge \int_0^1 \alpha(|1-2t|u)\rho(t)dt + \beta(u) \qquad (u \in D^*) \tag{2.61}$$

and $\alpha(0) \ge \beta(0)$.

The proof of Theorem 2.15 is based on a sequence of lemmas.

Lemma 2.16. (Makó–Páles [**45**])
Let $\beta : D^* \to \mathbb{R}$ be even, $\rho : [0,1] \to \mathbb{R}$ be integrable and $\lambda \in]0,1[$. Then every $f : D \to \mathbb{R}$ lower hemicontinuous function satisfying the approximate Hermite–Hadamard inequality (2.1), *fulfills*

$$\int_0^1 \Big(f(\tfrac{1+s}{2}x + \tfrac{1-s}{2}y) + f(\tfrac{1-s}{2}x + \tfrac{1+s}{2}y)\Big)\frac{\rho(\frac{1+s}{2}) + \rho(\frac{1-s}{2})}{2}ds \\ \leq f(x) + f(y) + 2\beta(x-y) \qquad (x,y \in D). \tag{2.62}$$

Proof. Changing the role of x and y in (2.1), then adding the respective sides of the inequality so obtained and the original inequality (2.1), by the evenness of β, we get that, for all $x,y \in D$,

$$\int_0^1 (f(tx+(1-t)y) + f((1-t)x+ty))\rho(t)dt \leq f(x) + f(y) + 2\beta(x-y). \tag{2.63}$$

Replacing t by $1-t$ in the integral on the left hand side of (2.63), it follows that, for all $x,y \in D$,

$$\int_0^1 (f((1-t)x+ty) + f(tx+(1-t)y))\rho(1-t)dt \leq f(x) + f(y) + 2\beta(x-y), \tag{2.64}$$

hence, adding the respective sides of the inequalities (2.63) and (2.64), we obtain, for all $x,y \in D$,

$$\int_0^1 (f(tx+(1-t)y) + f((1-t)x+ty))\tfrac{\rho(t)+\rho(1-t)}{2}dt \leq f(x) + f(y) + 2\beta(x-y). \tag{2.65}$$

Finally, substituting $t := \frac{1+s}{2}$ in the integral on the left hand side of (2.65), we arrive at

$$\frac{1}{2}\int_{-1}^1 \Big(f(\tfrac{1+s}{2}x + \tfrac{1-s}{2}y) + f(\tfrac{1-s}{2}x + \tfrac{1+s}{2}y)\Big)\frac{\rho(\frac{1+s}{2}) + \rho(\frac{1-s}{2})}{2}ds \\ \leq f(x) + f(y) + 2\beta(x-y) \qquad (x,y \in D). \tag{2.66}$$

Since the integrand on the left hand side of (2.66) is even, this inequality reduces to (2.62), which completes the proof of the lemma. □

In what follows, we examine the Hermite–Hadamard inequality (2.62).

Lemma 2.17. (Makó–Páles [**45**])
Let $\rho : [0,1] \to \mathbb{R}_+$ be integrable with $\int_0^1 \rho = 1$ and there exist $c \geq 0$ and $p > 0$ such that (2.60) *holds. Define $\phi :]0,1[\to \mathbb{R}$ by*

$$\phi(s) := \frac{\rho(\frac{1+s}{2}) + \rho(\frac{1-s}{2})}{2} \qquad (s \in]0,1[). \tag{2.67}$$

Then $\phi \in \Phi_p$ and $\int_0^1 \phi = 1$.

PROOF. Inequality (2.60) results that

$$\phi(s) = \frac{\rho(\frac{1+s}{2}) + \rho(\frac{1-s}{2})}{2} \le c(-\ln s)^{p-1} \qquad (s \in]0,1[),$$

which proves $\phi \in \Phi_p$. The equality $\int_0^1 \phi = 1$ is an automatic consequence of the assumption $\int_0^1 \rho = 1$. □

LEMMA 2.18. (Makó–Páles [**45**])
Let $p, q > 0$ and $\phi \in \Phi_p$, $\psi \in \Phi_q$ be nonnegative functions. Let $\alpha : D^* \to \mathbb{R}$ and let $\beta : D^* \to \mathbb{R}$ be a radially upper semicontinuous function. Assume that a lower hemicontinuous function $f : D \to \mathbb{R}$ satisfies the approximate Hermite–Hadamard inequalities

$$\begin{aligned}\int_0^1 \Big(f(\tfrac{1+s}{2}x + \tfrac{1-s}{2}y) + f(\tfrac{1-s}{2}x + \tfrac{1+s}{2}y)\Big)\phi(s)ds \\ \le f(x) + f(y) + 2\alpha(x-y) \quad (x, y \in D),\end{aligned} \tag{2.68}$$

and

$$\begin{aligned}\int_0^1 \Big(f(\tfrac{1+s}{2}x + \tfrac{1-s}{2}y) + f(\tfrac{1-s}{2}x + \tfrac{1+s}{2}y)\Big)\psi(s)ds \\ \le f(x) + f(y) + 2\beta(x-y) \quad (x, y \in D).\end{aligned} \tag{2.69}$$

Then, for all $x, y \in D$, f also satisfies the inequality

$$\begin{aligned}\int_0^1 \Big(f(\tfrac{1+s}{2}x + \tfrac{1-s}{2}y) + f(\tfrac{1-s}{2}x + \tfrac{1+s}{2}y)\Big)(\phi * \psi)(s)ds \\ \le f(x) + f(y) + 2\alpha(x-y) + 2\int_0^1 \beta(t(x-y))\phi(t)dt.\end{aligned} \tag{2.70}$$

PROOF. Assume that $f : D \to \mathbb{R}$ satisfies the inequalities (2.68) and (2.69). To prove (2.70), let $x, y \in D$ be fixed. Applying (2.69) for the elements $\frac{1+t}{2}x + \frac{1-t}{2}y$, $\frac{1-t}{2}x + \frac{1+t}{2}y \in D$, we obtain

$$\begin{aligned}\int_0^1 \Big(f(\tfrac{1+ts}{2}x + \tfrac{1-ts}{2}y) + f(\tfrac{1-ts}{2}x + \tfrac{1+ts}{2}y)\Big)\psi(s)ds \\ \le f(\tfrac{1+t}{2}x + \tfrac{1-t}{2}y) + f(\tfrac{1-t}{2}x + \tfrac{1+t}{2}y) + 2\beta(t(x-y)).\end{aligned}$$

Multiplying this inequality by $\phi(t)$ and integrating the functions on both sides with respect to t on $]0,1[$, then using that f also satisfies (2.68), we get that

$$\begin{aligned}
&\int_0^1\int_0^1 \Big(f(\tfrac{1+ts}{2}x+\tfrac{1-ts}{2}y)+f(\tfrac{1-ts}{2}x+\tfrac{1+ts}{2}y)\Big)\psi(s)ds\phi(t)dt\\
&\le \int_0^1 \Big(f(\tfrac{1+t}{2}x+\tfrac{1-t}{2}y)+f(\tfrac{1-t}{2}x+\tfrac{1+t}{2}y)\Big)\phi(t)dt+2\int_0^1 \beta(t(x-y))\phi(t)dt\\
&\le f(x)+f(y)+2\alpha(x-y)+2\int_0^1\beta(t(x-y))\phi(t)dt.
\end{aligned}\tag{2.71}$$

Now we compute the left hand side of the previous inequality. Substituting $s=\frac{\tau}{t}$ and using also Fubini's theorem, we obtain

$$\begin{aligned}
&\int_0^1\int_0^1 \Big(f(\tfrac{1+ts}{2}x+\tfrac{1-ts}{2}y)+f(\tfrac{1-ts}{2}x+\tfrac{1+ts}{2}y)\Big)\psi(s)ds\phi(t)dt\\
&=\int_0^1\int_0^t \Big(f(\tfrac{1+\tau}{2}x+\tfrac{1-\tau}{2}y)+f(\tfrac{1-\tau}{2}x+\tfrac{1+\tau}{2}y)\Big)\psi(\tfrac{\tau}{t})\tfrac{1}{t}d\tau\phi(t)dt\\
&=\int_0^1 \bigg(\Big(f(\tfrac{1+\tau}{2}x+\tfrac{1-\tau}{2}y)+f(\tfrac{1-\tau}{2}x+\tfrac{1+\tau}{2}y)\Big)\int_\tau^1 \tfrac{1}{t}\psi(\tfrac{\tau}{t})\phi(t)dt\bigg)d\tau\\
&=\int_0^1 \Big(f(\tfrac{1+\tau}{2}x+\tfrac{1-\tau}{2}y)+f(\tfrac{1-\tau}{2}x+\tfrac{1+\tau}{2}y)\Big)(\psi*\phi)(\tau)d\tau.
\end{aligned}\tag{2.72}$$

Combining (2.71) and (2.72), the inequality (2.70) follows, which completes the proof. □

Lemma 2.19. (Makó–Páles [**45**])
Let $\phi:]0,1[\to\mathbb{R}_+$ be an integrable function and let $\gamma: D^*\to\mathbb{R}$ be a radially upper semicontinuous. Then, the function $\delta: D^*\to\mathbb{R}$ defined by

$$\delta(u):=\int_0^1\gamma(tu)\phi(t)dt\qquad(u\in D^*)\tag{2.73}$$

is also radially upper semicontinuous on D^*.

Proof. To prove that δ defined by (2.73) is radially upper semicontinuous at $u_0\in D^*$, let $s_n\to s_0$ be an arbitrary sequence in $[0,1]$. We have that

$$\gamma(ts_nu_0)\le\sup_{\tau\in[0,1]}\gamma(\tau u_0)=:K\qquad(t\in[0,1],\ n\in\mathbb{N}),$$

thus, $K\phi$ is an integrable majorant for the sequence of functions $t \mapsto \gamma(ts_n u_0)\phi(t)$. Using Fatou's lemma and the radial upper semicontinuity of γ, we get that

$$\begin{aligned}\limsup_{n\to\infty} \delta(s_n u_0) &= \limsup_{n\to\infty} \int_0^1 \gamma(ts_n u_0)\phi(t)dt \\ &\le \int_0^1 \limsup_{n\to\infty} \gamma(ts_n u_0)\phi(t)dt \le \int_0^1 \gamma(ts_0 u_0)\phi(t)dt = \delta(s_0 u_0),\end{aligned}$$

which proves the statement. □

Lemma 2.20. (Makó–Páles [**45**])
Let $p > 0$, $\phi \in \Phi_p$ be a nonnegative and $\beta : D^* \to \mathbb{R}$ be a radially upper semicontinuous function. If $f : D \to \mathbb{R}$ is lower hemicontinuous and fulfills the approximate Hermite–Hadamard inequality

$$\begin{aligned}\int_0^1 \Big(f\big(\tfrac{1+s}{2}x + \tfrac{1-s}{2}y\big) + f\big(\tfrac{1-s}{2}x + \tfrac{1+s}{2}y\big)\Big)\phi(s)ds \\ \le f(x) + f(y) + 2\beta(x-y) \quad (x, y \in D),\end{aligned} \tag{2.74}$$

then, for all $n \in \mathbb{N}$, the function f also satisfies the Hermite–Hadamard inequality

$$\begin{aligned}\int_0^1 \Big(f\big(\tfrac{1+s}{2}x + \tfrac{1-s}{2}y\big) + f\big(\tfrac{1-s}{2}x + \tfrac{1+s}{2}y\big)\Big)\phi_n(s)ds \\ \le f(x) + f(y) + 2\alpha_n(x-y) \quad (x, y \in D),\end{aligned} \tag{2.75}$$

where the sequences $\phi_n :]0,1[\to \mathbb{R}_+$ and $\alpha_n : D^* \to \mathbb{R}$ are defined by (2.48) *and*

$$\alpha_1 = \beta, \qquad \alpha_{n+1}(u) = \int_0^1 \alpha_n(tu)\phi(t)dt + \beta(u) \qquad (u \in D^*), \tag{2.76}$$

respectively.

Proof. We note that, by Lemma 2.19, the sequence of functions (α_n) is well-defined and α_n is radially lower semicontinuous for all $n \in N$.

Let $x, y \in D$ and $p > 0$. To prove (2.75), we use induction on $n \in \mathbb{N}$. For $n = 1$, we have (2.74). Assume that (2.75) holds for $n \in \mathbb{N}$. Since $\phi \in \Phi_p$, by Proposition 2.13, we have that $\phi_n \in \Phi_{np}$. The function f satisfies (2.74) and also (2.75), for $n \in \mathbb{N}$. Thus, in Lemma 2.18, (2.68) holds with the functions ϕ and $\alpha := \alpha_1$. Furthermore, by the inductive assumption, also in Lemma 2.18, (2.69) holds with the functions $\psi := \phi_n$ and $\beta := \alpha_n$, for $n \in \mathbb{N}$. Hence the function f also fulfills the Hermite–Hadamard inequality (2.70), which results,

$$\begin{aligned}\int_0^1 \Big(f\big(\tfrac{1+s}{2}x + \tfrac{1-s}{2}y\big) + f\big(\tfrac{1-s}{2}x + \tfrac{1+s}{2}y\big)\Big)(\phi * \phi_n)(s)ds \\ \le f(x) + f(y) + 2\beta(x-y) + 2\int_0^1 \alpha_n(t(x-y))\phi(t)dt,\end{aligned}$$

which is the case $n+1$. □

Lemma 2.21. (Makó–Páles [**45**])
Let $p>0$ and $\phi\in\Phi_p$ be a nonnegative function with $\int_0^1\phi(t)dt=1$ and $f:D\to\mathbb{R}$ be lower hemicontinuous. Then, for all $x,y\in D$,

$$\liminf_{n\to\infty}\int_0^1\Big(f\big(\tfrac{1+s}{2}x+\tfrac{1-s}{2}y\big)+f\big(\tfrac{1-s}{2}x+\tfrac{1+s}{2}y\big)\Big)\phi_n(s)ds\geq 2f\Big(\frac{x+y}{2}\Big). \tag{2.77}$$

Proof. To prove the statement, let $x,y\in D$ and $p>0$. Define $g_{x,y}:[0,1]\to\mathbb{R}$ by

$$g_{x,y}(s):=f\big(\tfrac{1+s}{2}x+\tfrac{1-s}{2}y\big)+f\big(\tfrac{1-s}{2}x+\tfrac{1+s}{2}y\big)\qquad(s\in[0,1]).$$

The lower hemicontinuity of f implies that $g_{x,y}$ is lower semicontinuous and hence lower bounded on $[0,1]$. Thus, we can apply Proposition 2.14, for $g:=-g_{x,y}$ and $\phi\in\Phi_p$, which yields that

$$\liminf_{n\to\infty}\int_0^1 g_{x,y}(s)\phi_n(s)ds\geq g_{x,y}(0).$$

This inequality is equivalent to (2.77). □

Lemma 2.22. (Makó–Páles [**45**])
Let $p>0$ and $\phi\in\Phi_p$ be a nonnegative function, and $\beta:D^*\to\mathbb{R}$ be a radially upper semicontinuous function. Then, for all $n\in\mathbb{N}$, the function $\alpha_n:D^*\to\mathbb{R}$ defined by (2.76) *is radially upper semicontinuous and the sequence (α_n) is nondecreasing [nonincreasing], whenever β is nonnegative [nonpositive]. Furthermore, if $\alpha:D^*\to\mathbb{R}$ is a radially lower semicontinuous solution of the functional inequality*

$$\alpha(u)\geq\int_0^1\alpha(su)\phi(s)ds+\beta(u)\qquad(u\in D^*), \tag{2.78}$$

then

$$\limsup_{n\to\infty}\alpha_n(u)\leq\alpha(u)-\alpha(0)+\beta(0)\qquad(u\in D^*). \tag{2.79}$$

Proof. The statement about the radial upper semicontinuity directly follows from Lemma 2.19.

Assume first that β is nonnegative. We will prove by induction on $n\in\mathbb{N}$, that the sequence (α_n) is nondecreasing, i.e.,

$$\alpha_{n+1}\geq\alpha_n\qquad(n\in\mathbb{N}). \tag{2.80}$$

For $n=1$, by the nonnegativity of $\alpha_1=\beta$, we have that

$$\alpha_2(u)=\int_0^1\alpha_1(su)\phi(s)ds+\beta(u)\geq\alpha_1(u)\qquad(u\in D^*).$$

Assume that (2.80) holds for some $n \in \mathbb{N}$ and consider the case $n+1$. Using the definition of α_{n+1}, the inductive assumption and the nonnegativity of α_n, we get that, for all $u \in D^*$,

$$\alpha_{n+2}(u) = \int_0^1 \alpha_{n+1}(su)\phi(s)ds + \beta(u) \geq \int_0^1 \alpha_n(su)\phi(s)ds + \beta(u) = \alpha_{n+1}(u).$$

Analogously, if β is nonpositive, we can obtain that the sequence (α_n) is nonincreasing.

To prove (2.79), let $\alpha : D^* \to \mathbb{R}$ be a radially lower semicontinuous solution of (2.78). Subtracting the respective sides of the inequalities (2.78) from (2.76), for the sequence of functions $g_n := \alpha_n - \alpha$, we obtain

$$g_{n+1}(u) \leq \int_0^1 g_n(su)\phi(s)ds \qquad (u \in D^*,\ n \in \mathbb{N}),$$

We obviously have that g_n is also radially upper semicontinuous.

Iterating this inequality, similarly as in Lemma 2.18 and Lemma 2.20, it can be proved that

$$g_{n+1}(u) \leq \int_0^1 g_1(su)\phi_n(s)ds \qquad (u \in D^*,\ n \in \mathbb{N}), \tag{2.81}$$

where ϕ_n is defined by (2.48). Taking the limsup as $n \to \infty$ in (2.81), by Proposition 2.14, we get that, for all $u \in D^*$

$$\limsup_{n\to\infty} g_{n+1}(u) \leq \limsup_{n\to\infty} \int_0^1 g_1(su)\phi_n(s)ds \leq g_1(0) = \beta(0) - \alpha(0),$$

which immediately yields (2.79). □

Proof of Theorem 2.15. Assume that the conditions of Theorem 2.15 hold and $f : D \to \mathbb{R}$ is an upper semicontinuous solution of (2.1). Then by Lemma 2.16, f also fulfills (2.74), where $\phi :]0,1[\to \mathbb{R}_+$ is defined by (2.67). Then, by Lemma 2.17, $\phi \in \Phi_p$ and $\int_0^1 \phi = 1$. Using Lemma 2.20, we get that (2.75) also holds, where, for all $n \in \mathbb{N}$, $\alpha_n : D^* \to \mathbb{R}$ is defined by (2.76). Since α satisfies the functional inequality (2.61), thus applying Fubini's theorem, then substituting $s := 1 - 2t$ and $s := 2t - 1$, we get that

$$\begin{aligned}\alpha(u) &\geq \int_0^{1/2} \alpha((1-2t)u)\rho(t)dt + \int_{1/2}^1 \alpha((2t-1)u)\rho(t)dt + \beta(u)\\ &= \int_0^1 \alpha(su)\rho(\tfrac{1-s}{2})\tfrac{1}{2}ds + \int_0^1 \alpha(su)\rho(\tfrac{1+s}{2})\tfrac{1}{2}ds + \beta(u)\\ &= \int_0^1 \alpha(su)\frac{\rho(\frac{1+s}{2}) + \rho(\frac{1-s}{2})}{2}ds + \beta(u) = \int_0^1 \alpha(su)\phi(s)ds + \beta(u),\end{aligned}$$

which means that (2.78) also holds. Taking the liminf as $n \to \infty$ in (2.75), using also Lemma 2.21, Lemma 2.22 and $\beta(0) \leq \alpha(0)$, we get that

$$2f\Big(\frac{x+y}{2}\Big) \leq f(x) + f(y) + 2\liminf_{n\to\infty} \alpha_n(x-y) \leq f(x) + f(y) + 2\limsup_{n\to\infty} \alpha_n(x-y)$$
$$\leq f(x) + f(y) + 2\big(\alpha(x-y) + \beta(0) - \alpha(0)\big) \leq f(x) + f(y) + 2\alpha(x-y).$$

Hence f is α-Jensen convex, which completes the proof of Theorem 2.15. □

In what follows, we examine the case, when X is a normed space and β is a linear combination of the powers of the norm with positive exponents, i.e., if β is of the form

$$\beta(u) := \int_{]0,\infty[} \|u\|^q d\nu(q) \qquad (u \in D^*), \tag{2.82}$$

where ν is a signed Borel measure on the interval $]0,\infty[$. An important particular case is when ν is of the form $\sum_{i=1}^k c_i\delta_{q_i}$, where $c_1,\dots,c_k \in \mathbb{R}$, $q_1,\dots,q_k > 0$ and δ_q denotes the Dirac measure concentrated at q.

Theorem 2.23. (Makó–Páles [**45**])
Let $\rho : [0,1] \to \mathbb{R}$ be integrable with $\int_0^1 \rho = 1$ and assume that there exist $c \geq 0$ and $p > 0$ such that (2.60) *holds. Let $\lambda \in [0,1]$ and ν be a signed Borel measure on* $]0,\infty[$ *such that*

$$\int_{]0,\infty[} \|u\|^q d|\nu|(q) < \infty \qquad (u \in D^*) \tag{2.83}$$

and

$$\int_{]0,\infty[} \Big(\int_0^1 (1 - |1-2t|^q)\rho(t)dt\Big)^{-1} d|\nu|(q) < \infty. \tag{2.84}$$

Assume that $f : D \to \mathbb{R}$ is lower hemicontinuous and satisfies the Hermite–Hadamard type inequality

$$\int_0^1 f(tx + (1-t)y)\rho(t)dt \leq \lambda f(x) + (1-\lambda)f(y) + \int_{]0,\infty[} \|x-y\|^q d\nu(q) \tag{2.85}$$

for all $x, y \in D$. Then, for all $x, y \in D$, f also fulfils the Jensen type inequality

$$f\Big(\frac{x+y}{2}\Big) \leq \frac{f(x)+f(y)}{2} + \int_{]0,\infty[} \Big(\int_0^1 (1 - |1-2t|^q)\rho(t)dt\Big)^{-1} \|x-y\|^q d\nu(q). \tag{2.86}$$

Proof. By Theorem 2.15, it suffices to show that the function

$$\alpha(u) := \int_{]0,\infty[} \Big(\int_0^1 (1 - |1 - 2t|^q)\rho(t)dt \Big)^{-1} \|u\|^q d\nu(q) \quad (u \in D^*)$$

is well-defined and satisfies (2.61) with equality where $\beta : D^* \to \mathbb{R}$ is defined by

$$\beta(u) := \int_{]0,\infty[} \|u\|^q d\nu(q) \qquad (u \in D^*).$$

To see that, for all $u \in D^*$, $\alpha(u)$ is finite, we distinguish two cases. If $\|u\| \le 1$, then $\|u\|^q \le 1$ for all $q > 0$, and hence, by assumption (2.84),

$$|\alpha(u)| \le \int_{]0,\infty[} \Big(\int_0^1 (1 - |1 - 2t|^q)\rho(t)dt \Big)^{-1} d|\nu|(q) < \infty.$$

Now let $\|u\| > 1$. Then, the functions $q \mapsto \|u\|^q$ and $q \mapsto \int_0^1 (1 - |1 - 2t|^q)\rho(t)dt$ are increasing functions, hence

$$|\alpha(u)| \le \|u\| \int_{]0,1]} \Big(\int_0^1 (1 - |1 - 2t|^q)\rho(t)dt \Big)^{-1} d|\nu|(q)$$
$$+ \Big(\int_0^1 (1 - |1 - 2t|)\rho(t)dt \Big)^{-1} \int_{]1,\infty[} \|u\|^q d|\nu|(q),$$

which is again finite by conditions (2.83) and (2.84).

To prove that α satisfies (2.61), using $\int_0^1 \rho = 1$, we compute

$$\int_0^1 \alpha(|1 - 2s|u)\rho(s)ds + \beta(u)$$
$$= \int_0^1 \int_{]0,\infty[} \Big(\int_0^1 (1 - |1 - 2t|^q)\rho(t)dt \Big)^{-1} \||1 - 2s|u\|^q d\nu(q)\rho(s)ds + \int_{]0,\infty[} \|u\|^q d\nu(q)$$
$$= \int_{]0,\infty[} \Big(\frac{\int_0^1 |1 - 2s|^q \rho(s)ds}{\int_0^1 (1 - |1 - 2t|^q)\rho(t)dt} + 1 \Big) \|u\|^q d\nu(q) = \int_{]0,\infty[} \frac{\|u\|^q}{\int_0^1 (1 - |1 - 2t|^q)\rho(t)dt} d\nu(q) = \alpha(u),$$

which proves that (2.61) holds with equality. □

Corollary 2.24. (Makó–Páles [**45**])
Let $\lambda \in [0, 1]$, $\varepsilon \in \mathbb{R}$ and $q > 0$. Assume that $f : D \to \mathbb{R}$ is lower hemicontinuous and

satisfies the Hermite–Hadamard type inequality

$$\int_0^1 f(tx+(1-t)y)dt \le \lambda f(x)+(1-\lambda)f(y)+\varepsilon\|x-y\|^q \qquad (x,y\in D).$$

Then f is $(\varepsilon\frac{q+1}{q},q)$-Jensen convex.

Proof. Observe that the constant weight function $\rho\equiv 1$ satisfies the assumptions of Theorem 2.23 with $c=p=1$. Also, with $\nu:=\varepsilon\delta_q$, conditions (2.83) and (2.84) hold trivially. Thus, for all $u\in D^*$ the conclusion of Theorem 2.23 is valid with

$$\alpha(u)=\frac{a\|u\|^q}{\int_0^1(1-|1-2t|^q)dt}=\frac{a\|u\|^q}{\int_0^{\frac{1}{2}}(1-(1-2t)^q)dt+\int_{\frac{1}{2}}^1(1-(2t-1)^q)dt}=a\frac{q+1}{q}\|u\|^q,$$

which proves the statement. □

3

Implications between lower Hermite–Hadamard and convexity inequalities

Given a nonempty open real interval I, denote by $\Delta(I)$ and $\Delta^\circ(I)$ the sets

$$\{(x,y) \in I \times I \mid x \le y\} \qquad \text{and} \qquad \{(x,y) \in I \times I \mid x < y\},$$

respectively.

To describe the regularity assumptions for our main results, we introduce the following terminology for real valued functions defined on $\Delta(I)$. Let $\Phi : \Delta(I) \to \mathbb{R}$. The function Φ is called *separately continuous (separately strictly increasing)* if, for all $z \in I$, the mappings $u \mapsto \Phi(u,z)$ and $u \mapsto \Phi(z,u)$ are continuous (strictly increasing) on the intervals $]-\infty, z] \cap I$ and on $[z, \infty[\cap I$, respectively. We say that Φ are *separately partially differentiable at the diagonal of* $I \times I$ if, for all $z \in I$, the mappings $u \mapsto \Phi(u,z)$ and $u \mapsto \Phi(z,u)$ are differentiable at z with derivatives denoted by $\partial_1\Phi(z,z)$ and $\partial_2\Phi(z,z)$, respectively. The function Φ is called a *two-variable mean on* I if, for all $(x,y) \in \Delta(I)$, the inequalities $x \le \Phi(x,y) \le y$ hold. A mean Φ is called *strict* if both of these inequalities are strict whenever $x < y$.

We say that a pair (ω_0, ω_1) is a *Chebyshev system* over I, if $\omega_0, \omega_1 : I \to \mathbb{R}$ are continuous functions and

$$\Omega(x,y) := \begin{vmatrix} \omega_0(x) & \omega_0(y) \\ \omega_1(x) & \omega_1(y) \end{vmatrix} > 0 \qquad ((x,y) \in \Delta^\circ(I)). \tag{3.1}$$

One can easily see, that if ω_0 is a positive function, then (3.1) holds if and only if ω_1/ω_0 is strictly increasing on I. In this latter case, (ω_0, ω_1) will be called a *positive Chebyshev system* over I. On the other hand, we can always assume that ω_0 is a positive function, because for every Chebyshev system (ω_0, ω_1), there exists $\alpha, \beta \in \mathbb{R}$ such that $\alpha\omega_0 + \beta\omega_1 > 0$ (cf. [**9**], [**10**]). In the sequel, for fixed $x, y \in I$, the partial functions $u \mapsto \Omega(u,y)$ and $u \mapsto \Omega(x,u)$ will be denoted by $\Omega(\cdot, y)$ and $\Omega(x, \cdot)$, respectively. An important property of Chebyshev systems is that for every two pairs

$(x,\xi),(y,\eta)\in I\times\mathbb{R}$ with $x\neq y$ the function ω defined as

$$\omega := \xi\frac{\Omega(\cdot,y)}{\Omega(x,y)} + \eta\frac{\Omega(x,\cdot)}{\Omega(x,y)}$$

is the unique linear combination of ω_0 and ω_1 such that $\omega(x)=\xi$ and $\omega(y)=\eta$ hold.

Given a positive Chebyshev system (ω_0,ω_1) over I and a proper subinterval J of I, a function $f: J\to\mathbb{R}$ is called (ω_0,ω_1)-*convex on* J if, for all $x<u<y$ from J,

$$\begin{vmatrix} f(x) & f(u) & f(y)\\ \omega_0(x) & \omega_0(u) & \omega_0(y)\\ \omega_1(x) & \omega_1(u) & \omega_1(y)\end{vmatrix} \geq 0, \tag{3.2}$$

or equivalently,

$$f(u) \leq \frac{\Omega(u,y)}{\Omega(x,y)}f(x) + \frac{\Omega(x,u)}{\Omega(x,y)}f(y). \tag{3.3}$$

If (3.2) holds with strict inequality sign ">", then f is said to be *strictly* (ω_0,ω_1)-*convex on* J. We also need the following lemma, which gives various characterizations of (ω_0,ω_1)-convex functions.

Lemma 3.1. (Bessenyei–Páles [**11**])
Let (ω_0,ω_1) be a positive Chebyshev system over I, $J\subset I$ be a proper subinterval and $f: J\to\mathbb{R}$. Then the following statements are equivalent:

(i) f is [strictly] (ω_0,ω_1)-convex on J;
(ii) $(\frac{f}{\omega_0})\circ(\frac{\omega_1}{\omega_0})^{-1}$ is [strictly] convex in the standard sense on $(\frac{\omega_1}{\omega_0})(J)$;
(iii) for every $u\in J^\circ$, there exist constants $\alpha,\beta\in\mathbb{R}$ such that $f(u)=\alpha\omega_0(u)+\beta\omega_1(u)$ and, for all $x\in J\setminus\{u\}$, the inequality $f(x)\geq\alpha\omega_0(x)+\beta\omega_1(x)$ [the strict inequality $f(x)>\alpha\omega_0(x)+\beta\omega_1(x)$] holds.

Let (ω_0,ω_1) be a positive Chebyshev-system. Assume that $f: I\to\mathbb{R}$ satisfies the following approximate (ω_0,ω_1)-convexity inequality

$$f(u) \leq \frac{\Omega(u,y)}{\Omega(x,y)}f(x) + \frac{\Omega(x,u)}{\Omega(x,y)}f(y) + \varepsilon_{x,y}(u) \qquad (u\in[x,y]), \tag{3.4}$$

where, for all $(x,y)\in\Delta^\circ(I)$, $\varepsilon_{x,y}:[x,y]\to\mathbb{R}$ is an error function. One can easily see that if f is (ω_0,ω_1)-convex, it satisfies (3.4) with the error function $\varepsilon_{x,y}:=0$.

In [**11**] and [**9**], the authors established the following connections between (ω_0,ω_1)-convexity and Hermite–Hadamard type inequality.

Theorem 3.2. (Bessenyei–Páles [**11**])
Let (ω_0,ω_1) be a positive Chebyshev system on an open interval I and let $\rho: I\to\mathbb{R}$ be a positive integrable function. Define, for all elements $x<y$ of I, the functions

$\xi(x,y)$, $c(x,y)$, $c_1(x,y)$ and $c_2(x,y)$ by the formulas

$$\xi(x,y) = \Big(\frac{\omega_1}{\omega_0}\Big)^{-1}\Big(\frac{\int_x^y \omega_1\rho}{\int_x^y \omega_0\rho}\Big) \quad \text{and} \quad c(x,y) = \frac{\int_x^y \omega_0\rho}{\omega_0(\xi(x,y))}. \tag{3.5}$$

A function $f : I \to \mathbb{R}$ is (ω_0, ω_1)-convex if and only if, for all $x < y$ of I, it satisfies the inequality

$$c(x,y)f(\xi(x,y)) \le \int_x^y f\rho. \tag{3.6}$$

In Theorem 3.7 and Theorem 3.30 below, this theorem will be generalized. Thus, we will investigate the connections between (3.4) and the following approximate lower Hermite–Hadamard type inequality

$$f(M_0(x,y)) \le \int_T \Lambda(t,x,y) f(M(t,x,y))d\mu(t) + \mathcal{E}(x,y) \quad ((x,y) \in \Delta(I)), \tag{3.7}$$

where $f : I \to \mathbb{R}$ is an unknown function, $\mathcal{E} : \Delta^\circ(I) \to \mathbb{R}$ and we make the following assumptions:

(i) $(T, \mathcal{A}, \mu)$ is a measure space,
(ii) $\Lambda : T \times \Delta^\circ(I) \to \mathbb{R}_+$,
(iii) $M : T \times \Delta^\circ(I) \to \mathbb{R}$ and for all $t \in T$, the map $(x,y) \mapsto M(t,x,y)$ is a two-variable mean on I. $M_0 : \Delta^\circ(I) \to I$ is a strict mean such that, for all $(x,y) \in \Delta^\circ(I)$,

$$\mu\{t \in T \mid \Lambda(t,x,y) > 0,\ M(t,x,y) \ne M_0(x,y)\} > 0. \tag{3.8}$$

(iv) There exist an (ω_0, ω_1)-Chebyshev system on I such that ω_0 is positive. Furthermore, for $i \in \{0,1\}$,

$$\omega_i(M_0(x,y)) = \int_T \Lambda(t,x,y)\omega_i(M(t,x,y))d\mu(t) \qquad ((x,y) \in \Delta^\circ(I)). \tag{3.9}$$

In [**34**], Házy and Páles also established a connection between an approximate lower Hermite–Hadamard type inequality and an approximate Jensen type inequality by proving the following results.

Theorem 3.3. (Házy–Páles [**34**])
Let $\alpha : D^* \to \mathbb{R}_+$ be a nonnegative radially Lebesgue integrable even function. Assume that $f : D \to \mathbb{R}$ is hemi-Lebesgue integrable and approximately α-Jensen convex in a sense of (5). *Then, for all $x, y \in D$, f also satisfies the approximate lower Hermite–Hadamard inequality*

$$f\Big(\frac{x+y}{2}\Big) \le \int_0^1 f(tx + (1-t)y)dt + \int_0^1 \alpha(t(x-y))dt. \tag{3.10}$$

Theorem 3.4. (Házy–Páles [**34**])
Let $\beta : D^* \to \mathbb{R}_+$ be a nonnegative even function. Assume that $f : D \to \mathbb{R}$ is an upper hemicontinuous function satisfying the approximate lower Hermite–Hadamard inequality

$$f\left(\frac{x+y}{2}\right) \le \int_0^1 f(tx+(1-t)y)dt + \beta(x-y) \qquad (x, y \in D). \tag{3.11}$$

Then f is α-Jensen convex where $\alpha : 2D^* \to \mathbb{R}_+$ is a nonnegative, radially increasing solution of the functional inequality

$$\alpha(u) \ge \int_0^1 \alpha(2tu)dt + \beta(u) \qquad (u \in D^*). \tag{3.12}$$

In Theorem 3.10 and Theorem 3.34 we will generalize these results. In Remark 3.13 and Remark 3.36, we will deduce these results (Theorem 3.3 and Theorem 3.4) as a consequence of Corollary 3.12 and Corollary 3.35, respectively. Consider the following approximate convexity type inequality

$$f((1-t)x+ty) \le (1-t)f(x) + tf(y) + e_{x,y}(t) \quad ((x, y) \in D^2,\ t \in [0, 1]), \tag{3.13}$$

where $f : D \to \mathbb{R}$ is an unknown function and for all $(x, y) \in D^2$, $e_{x,y} : [0, 1] \to \mathbb{R}$ is a given function. Thus, we will investigate connections between (3.13) and the following approximate lower Hermite–Hadamard type inequality

$$f((1-\mu_1)x+\mu_1 y) \le \int_{[0,1]} f((1-t)x+ty)d\mu(t) + E(x, y) \quad ((x, y) \in D^2), \tag{3.14}$$

where $f : D \to \mathbb{R}$ is an unknown function, μ is a measure on $[0, 1]$, $\mu_1 := \int_{[0,1]} td\mu(t)$ and $E : D^2 \to \mathbb{R}$.

The following theorem, which has been stated in [**43**], [**44**], [**67**] and [**66**], is also a consequence of our main result (Theorem 3.34). We will also derive it in Remark 3.38 by applying Corollary 3.37.

Theorem 3.5. (Tabor–Tabor [**67**], Makó–Páles [**44**])
Let $\alpha : D^* \to \mathbb{R}_+$ be a nonnegative even function. Assume that $f : D \to \mathbb{R}$ is an upper hemibounded α-Jensen function, then, for all $x, y \in D$ and $s \in [0, 1]$, f also satisfies the following convexity type inequality

$$f((1-s)x+sy) \le (1-s)f(x) + sf(y) + \sum_{n=0}^{\infty} \frac{1}{2^n}\alpha(d_{\mathbb{Z}}(2^n s)(x-y)). \tag{3.15}$$

3.1. From convexity type inequalities to lower Hermite–Hadamard inequalities

In this section we will investigate the implication from the (ω_0, ω_1)-convexity type inequality (3.4) to the lower Hermite–Hadamard inequality (3.7). Consider the following basic assumptions,

(A1) $(T, \mathcal{A}, \mu)$ is a measure space.
(A2) $\Lambda : T \times \Delta^\circ(I) \to \mathbb{R}_+$ is μ-integrable in its first variable.
(A3) $M : T \times \Delta^\circ(I) \to \mathbb{R}$ is $\mathcal{A}$-measurable in its first variable and for all $t \in T$, the map $(x, y) \mapsto M(t, x, y)$ is a two-variable mean on I. $M_0 : \Delta^\circ(I) \to I$ is a strict mean such that (3.8) holds.
(A4) There exist an (ω_0, ω_1)-Chebyshev system on I such that ω_0 is positive. Furthermore, for $i \in \{0, 1\}$ (3.9) holds.

For all $(x, y) \in \Delta^\circ(I)$, denote

$$T'_{x,y} := \{t \in T \mid \Lambda(t, x, y) > 0,\ M(t, x, y) < M_0(x, y)\},$$
$$T''_{x,y} := \{t \in T \mid \Lambda(t, x, y) > 0,\ M(t, x, y) \geq M_0(x, y)\}.$$

Observe that, for all $(x, y) \in \Delta^\circ(I)$, $T'_{x,y}$ and $T''_{x,y}$ are in $\mathcal{A}$, moreover, by (3.8), the μ-measure of $T'_{x,y} \cup T''_{x,y}$ is positive. Define, for all $(x, y) \in \Delta^\circ(I)$, $i \in \{0, 1\}$,

$$\begin{aligned} S'_i(x, y) &= \int_{T'_{x,y}} \Lambda(t, x, y)\omega_i(M(t, x, y))d\mu(t) \\ S''_i(x, y) &= \int_{T''_{x,y}} \Lambda(t, x, y)\omega_i(M(t, x, y))d\mu(t). \end{aligned} \tag{3.16}$$

The following proposition describes the properties of these sets and numbers.

Proposition 3.6. (Makó–Páles [**47**])
If (A1)–(A4) hold, then for all $(x, y) \in \Delta^\circ(I)$,

$$S'_i(x, y) + S''_i(x, y) = \omega_i(M_0(x, y)) \qquad (i \in \{0, 1\}). \tag{3.17}$$

Furthermore, the μ-measure of the sets $T'_{x,y}$ and $T''_{x,y}$ is positive.

Proof. Let $(x, y) \in \Delta^\circ(I)$. (3.9) implies that

$$\begin{aligned} \omega_i(M_0(x, y)) &= \int_T \Lambda(t, x, y)\omega_i(M(t, x, y))d\mu(t) \\ &= \int_{\{t\in T \mid \Lambda(t,x,y)>0\}} \Lambda(t, x, y)\omega_i(M(t, x, y))d\mu(t) = S'_i(x, y) + S''_i(x, y), \end{aligned}$$

for $i \in \{0, 1\}$. Hence (3.17) holds. To prove the positivity of $\mu(T'_{x,y})$ and $\mu(T''_{x,y})$, assume that $\mu(T'_{x,y}) = 0$. Then $S'_i(x, y) = 0$ and, in view of (3.8), it follows that

$\mu(T''_{x,y}) > 0$. Thus, by (3.17), we have that

$$\omega_i(M_0(x,y)) = S'_i(x,y) + S''_i(x,y) = S''_i(x,y) = \int_{T''_{x,y}} \Lambda(t,x,y)\omega_i(M(t,x,y))d\mu(t)$$

for $i \in \{0,1\}$. Dividing the above identities by each other and using also the positivity of ω_0, we get that

$$\frac{\int_{T''_{x,y}} \Lambda(t,x,y)\omega_1(M(t,x,y))d\mu(t)}{\int_{T''_{x,y}} \Lambda(t,x,y)\omega_0(M(t,x,y))d\mu(t)} = \frac{\omega_1(M_0(x,y))}{\omega_0(M_0(x,y))}.$$

Rearranging this equality, we obtain that

$$\int_{T''_{x,y}} \Lambda(t,x,y)\Omega(M_0(x,y), M(t,x,y))d\mu(t) = 0.$$

Hence,

$$\int_{\{t\in T \mid \Lambda(t,x,y)>0,\ M(t,x,y)>M_0(x,y)\}} \Lambda(t,x,y)\Omega(M_0(x,y), M(t,x,y))d\mu(t) = 0.$$

On the other hand, for all $t \in T$ with $M(t,x,y) > M_0(x,y)$, we have that

$$\Omega(M_0(x,y), M(t,x,y)) > 0$$

and, by (3.8), it follow that

$$\mu(\{t \in T \mid \Lambda(t,x,y) > 0,\ M(t,x,y) > M_0(x,y)\}) > 0.$$

This yields that

$$\int_{\{t\in T \mid \Lambda(t,x,y)>0,\ M(t,x,y)>M_0(x,y)\}} \Lambda(t,x,y)\Omega(M_0(x,y), M(t,x,y))d\mu(t) > 0,$$

which is a contradiction.

The proof for the case when $\mu(T''_{x,y}) = 0$ is analogous. □

One of the main result of this section is established in the following theorem.

Theorem 3.7. (Makó–Páles [**47**])
Assume that (A1)–(A4) hold. Let $f : I \to \mathbb{R}$ be a locally upper bounded Borel measurable solution of the approximate (ω_0, ω_1)-convexity type functional inequality (3.4)*, where for all $(x,y) \in \Delta^\circ(I)$ and $u \in]x,y[$, the function $(v,w) \mapsto \varepsilon_{v,w}(u)$ is bounded and Borel measurable for $(v,w) \in [x,u] \times [u,y]$. Then f also satisfies the approximate*

lower Hermite–Hadamard type inequality (3.7)*, where* $\mathcal{E} : \Delta^\circ(I) \to \mathbb{R}$ *is defined by the following way*

$$\mathcal{E}(x,y) := \frac{\int\limits_{T'_{x,y}} \int\limits_{T''_{x,y}} \Upsilon(t',t'',x,y)\varepsilon_{M(t',x,y),M(t'',x,y)}(M_0(x,y))d\mu(t'')d\mu(t')}{\int\limits_{T'_{x,y}} \int\limits_{T''_{x,y}} \Upsilon(t',t'',x,y)d\mu(t'')d\mu(t')}, \tag{3.18}$$

where, for all $(t',t'',x,y) \in T^2 \times \Delta^\circ(I)$*,*

$$\Upsilon(t',t'',x,y) := \Lambda(t',x,y)\Lambda(t'',x,y)\Omega(M(t',x,y),M(t'',x,y)).$$

Remark 3.8. (Makó–Páles [**47**])
In the above theorem, the regularity condition for f can be relaxed if the error function $\varepsilon_{x,y}$ enjoys boundedness and continuity properties. For instance, if $\varepsilon_{x,y}$ is bounded on $[x,y]$ for some $(x,y) \in \Delta^\circ(I)$, then (3.4) implies that f is bounded on $[x,y]$. Similarly, if $\limsup_{u\to x+0} \varepsilon_{x,y}(u) = 0$ for some $(x,y) \in \Delta^\circ(I)$, then (3.4) implies that f is upper semicontinuous at x from the right.

Proof. Let $(x,y) \in \Delta^\circ(I)$. Substituting in (3.4) x by $M(t',x,y)$ and y by $M(t'',x,y)$, and u by $M_0(x,y)$, where $t' \in T'_{x,y}$ and $t'' \in T''_{x,y}$, we get that

$$\begin{aligned}
&\Omega(M(t',x,y),M(t'',x,y))f(M_0(x,y))\\
&\quad\le \Omega(M_0(x,y),M(t'',x,y))f(M(t',x,y)) + \Omega(M(t',x,y),M_0(x,y))f(M(t'',x,y))\\
&\qquad + \Omega(M(t',x,y),M(t'',x,y))\varepsilon_{M(t',x,y),M(t'',x,y)}(M_0(x,y)).
\end{aligned}$$

Multiplying this inequality by $\Lambda(t',x,y)\Lambda(t'',x,y)$ and integrating with respect to $\mu\times\mu$ on $T'_{x,y} \times T''_{x,y}$, we get that

$$\begin{aligned}
&\int\limits_{T'_{x,y}}\int\limits_{T''_{x,y}} \Upsilon(t',t'',x,y)d\mu(t'')d\mu(t')f(M_0(x,y))\\
&\le \int\limits_{T'_{x,y}}\int\limits_{T''_{x,y}} \Lambda(t',x,y)\Lambda(t'',x,y)\Omega(M_0(x,y),M(t'',x,y))f(M(t',x,y))d\mu(t'')d\mu(t')\\
&+\int\limits_{T'_{x,y}}\int\limits_{T''_{x,y}} \Lambda(t',x,y)\Lambda(t'',x,y)\Omega(M(t',x,y),M_0(x,y))f(M(t'',x,y))d\mu(t'')d\mu(t')\\
&+\int\limits_{T'_{x,y}}\int\limits_{T''_{x,y}} \Upsilon(t',t'',x,y)\varepsilon_{M(t',x,y),M(t'',x,y)}(M_0(x,y))d\mu(t'')d\mu(t').
\end{aligned} \tag{3.19}$$

Applying Fubini's theorem and the notation of (3.16), we get that

$$\int_{T'_{x,y}}\int_{T''_{x,y}}\Upsilon(t',t'',x,y)d\mu(t'')d\mu(t') = S'_0(x,y)S''_1(x,y) - S'_1(x,y)S''_0(x,y). \tag{3.20}$$

Observe that $(S'_0(x,y)S''_1(x,y)-S'_1(x,y)S''_0(x,y))$ is positive. Indeed, by the definition of the Chebyshev-system, we have, for all $(t',t'') \in T'_{x,y} \times T''_{x,y}$,

$$\Omega(M(t',x,y), M(t'',x,y)) > 0.$$

By Proposition 3.6, the measure of $T'_{x,y} \times T''_{x,y}$ is positive. Hence the left hand side of (3.20) is positive. Using the identity (3.17), it follows that

$$\begin{aligned}(3.21)\quad &\int_{T'_{x,y}}\int_{T''_{x,y}} \Lambda(t',x,y)\Lambda(t'',x,y)\Omega(M_0(x,y),M(t'',x,y))f(M(t',x,y))d\mu(t'')d\mu(t')\\
&= (\omega_0(M_0(x,y))S''_1(x,y)-\omega_1(M_0(x,y))S''_0(x,y))\int_{T'_{x,y}}\Lambda(t',x,y)f(M(t',x,y))d\mu(t')\\
&= (S'_0(x,y)S''_1(x,y) - S'_1(x,y)S''_0(x,y))\int_{T'_{x,y}}\Lambda(t',x,y)f(M(t',x,y))d\mu(t'),\end{aligned}$$

and similarly,

$$\begin{aligned}&\int_{T'_{x,y}}\int_{T''_{x,y}} \Lambda(t',x,y)\Lambda(t'',x,y)\Omega(M(t',x,y),M_0(x,y))f(M(t'',x,y))d\mu(t'')d\mu(t')\\
&\quad = (S''_1(x,y)S'_0(x,y) - S''_0(x,y)S'_1(x,y))\int_{T''_{x,y}}\Lambda(t'',x,y)f(M(t'',x,y))d\mu(t'').\end{aligned}\tag{3.22}$$

Substituting the above formulas (3.20), (3.21), and (3.22) into (3.19) and dividing the inequality so obtained by $(S'_0(x,y)S''_1(x,y) - S'_1(x,y)S''_0(x,y))$, we get (3.7) with the error function $\mathcal{E} : \Delta^\circ(I) \to \mathbb{R}$ defined by (3.18), which completes the proof. □

Remark 3.9. (Makó–Páles [**47**])
A direct corollary of this theorem is the lower Hermite–Hadamard type inequality established by Theorem 3.2. Indeed, suppose that, with the notations introduced in (3.5), the assumptions of Theorem 3.2 hold. Then, the (ω_0,ω_2)-convexity of f implies that it is locally bounded and Borel measurable. We show first that the conditions of Theorem 3.7 are also valid. Let μ denote the Lebesgue measure on $[0,1]$ and define, for all $(x,y) \in \Delta^\circ(I)$, $t \in [0,1]$,

$$M_0(x,y) := \xi(x,y), \quad M(t,x,y) := (1-t)x+ty, \quad \Lambda(t,x,y) := \frac{(y-x)\rho((1-t)x+ty)}{c(x,y)}.$$

Since $M(t,x,y) = M_0(x,y)$ can hold only for one value of t, hence (3.8) holds trivially. We also have

$$\int_0^1 \Lambda(t,x,y)\omega_1(M(t,x,y))dt = \frac{y-x}{c(x,y)}\int_0^1 \rho((1-t)x+ty)\omega_1((1-t)x+ty)dt$$
$$= \frac{1}{c(x,y)}\int_x^y \rho\omega_1 = \omega_0(\xi(x,y))\frac{\int_x^y \omega_1\rho}{\int_x^y \omega_0\rho}$$
$$= \omega_0(\xi(x,y))\frac{\omega_1}{\omega_0}(\xi(x,y)) = \omega_1(\xi(x,y)) = \omega_1(M_0(x,y))$$

and, similarly,

$$\int_0^1 \Lambda(t,x,y)\omega_0(M(t,x,y))dt = \frac{1}{c(x,y)}\int_x^y \rho\omega_0 = \omega_0(\xi(x,y)),$$

which proves (3.9). Thus all the assumptions (A1)–A(4) are verified. Therefore, if a function $f: I \to \mathbb{R}$ is (ω_0,ω_1)-convex, i.e., satisfies (3.4) with $\varepsilon_{x,y} := 0$, then it fulfills (3.7) with $\mathcal{E} := 0$, which, by the obvious identity

$$\frac{1}{c(x,y)}\int_x^y f\rho = \int_0^1 \Lambda(t,x,y)f(M(t,x,y))dt$$

is equivalent to the left hand side inequality in (3.6).

The following result could be deduced form Theorem 3.7, however a direct proof is more convenient here. Given a set D, denote $\{(x,y) \mid x,y \in D,\ x \neq y\}$ by D^{2*}.

Theorem 3.10. (Makó–Páles [**47**])
Let D be a convex set of a linear space X. Let $\mathcal{A}$ be a sigma algebra containing the Borel subsets of $[0,1]$ *and μ be a probability measure on the measure space* $([0,1],\mathcal{A})$ *such that the support of μ is not a singleton. Denote*

$$S(\mu) := \mu([0,\mu_1])\int\limits_{]\mu_1,1]} td\mu(t) - \mu(]\mu_1,1])\int\limits_{[0,\mu_1]} td\mu(t).$$

Assume that $f: D \to \mathbb{R}$ is an hemi-μ-integrable solution of the functional inequality (3.13) *where, for all $(x,y) \in D^{2*}$, $e_{x,y}: [0,1] \to \mathbb{R}$ is a function such that*

$$I(x,y) := \int\limits_{]\mu_1,1]}\int\limits_{[0,\mu_1]} (t''-t')e_{(1-t')x+t'y,(1-t'')x+t''y}\Big(\frac{\mu_1-t'}{t''-t'}\Big)d\mu(t')d\mu(t'')$$

exists in $[-\infty,\infty]$ *for all $(x,y) \in D^{2*}$. Then, for all $(x,y) \in D^{2*}$, the function f also satisfies the lower Hermite–Hadamard type inequality* (3.14)*, where*

$$E(x,y) := \frac{I(x,y)}{S(\mu)} \qquad ((x,y) \in D^{2*}). \tag{3.23}$$

Remark 3.11. (Makó–Páles [**47**])
In the above theorem, the hemi-μ-integrability condition for f can be relaxed if the error function $e_{x,y}$ enjoys boundedness and continuity properties. For instance, if $e_{x,y}$ is upper bounded on $[x,y]$ for some $(x,y) \in D^{2*}$, then (3.4) implies that $f((1-t)x+ty)$ is upper bounded for $t \in [0,1]$. Similarly, if $\limsup_{t\to 0+0} e_{x,y}(t) = 0$ for some $(x,y) \in D^{2*}$, then (3.4) implies that $f((1-t)x+ty)$ is an upper semicontinuous function of t at zero from the right.

Proof. Let $(x,y) \in D^{2*}$. Substituting in (3.13) x by $(1-t')x+t'y$, y by $(1-t'')x+t''y$ and t by $\frac{\mu_1-t'}{t''-t'}$, where $0 \le t' \le \mu_1$ and $\mu_1 < t'' \le 1$, we get that

$$\begin{aligned} f((1-\mu_1)x+\mu_1 y) \le &\frac{t''-\mu_1}{t''-t'} f((1-t')x+t'y) + \frac{\mu_1-t'}{t''-t'} f((1-t'')x+t''y) \\ &+ e_{(1-t')x+t'y,(1-t'')x+t''y}\Big(\frac{\mu_1-t'}{t''-t'}\Big). \end{aligned} \tag{3.24}$$

Multiplying (3.24) by $t''-t'$ and integrating on $[0,\mu_1]\times]\mu_1,1]$ with respect to the product measure $\mu\times\mu$, we obtain

$$\begin{aligned} &\int\limits_{]\mu_1,1]}\int\limits_{[0,\mu_1]} (t''-t')d\mu(t')d\mu(t'') f((1-\mu_1)x+\mu_1 y) \\ &\le \int\limits_{]\mu_1,1]} (t''-\mu_1)d\mu(t'') \int\limits_{[0,\mu_1]} f((1-t')x+t'y)d\mu(t') \\ &\quad + \int\limits_{[0,\mu_1]} (\mu_1-t')d\mu(t') \int\limits_{]\mu_1,1]} f((1-t'')x+t''y)d\mu(t'') \\ &\quad + \int\limits_{]\mu_1,1]}\int\limits_{[0,\mu_1]} (t''-t') e_{(1-t')x+t'y,(1-t'')x+t''y}\Big(\frac{\mu_1-t'}{t''-t'}\Big)d\mu(t')d\mu(t''). \end{aligned} \tag{3.25}$$

Applying Fubini's theorem, we get that

$$\begin{aligned} &\int\limits_{]\mu_1,1]}\int\limits_{[0,\mu_1]} (t''-t')d\mu(t')d\mu(t'') \\ &\quad = \mu([0,\mu_1]) \int\limits_{]\mu_1,1]} t''d\mu(t'') - \mu(]\mu_1,1]) \int\limits_{[0,\mu_1]} t'd\mu(t') = S(\mu). \end{aligned} \tag{3.26}$$

Using that the support of μ is not a singleton, we can see that the left hand side of (3.26) is positive and hence so is $S(\mu)$.

Applying also Fubini's theorem, it follows that

$$(3.27)\qquad \int\limits_{]\mu_1,1]} (t''-\mu_1)d\mu(t'') = \mu([0,1])\int\limits_{]\mu_1,1]} t''d\mu(t'') - \mu(]\mu_1,1])\int\limits_{[0,1]} td\mu(t) = S(\mu)$$

and, similarly,

$$(3.28)\qquad \int\limits_{[0,\mu_1]} (\mu_1 - t')d\mu(t') = S(\mu).$$

Substituting the above formulas (3.26), (3.27), and (3.28) into (3.25) and dividing the inequality so obtained by $S(\mu)$, we arrive at (3.14), where E is defined by (3.23). This completes the proof. □

The following corollary is analogous to the result of [**34**].

Corollary 3.12. (Makó–Páles [**47**])
Assume that $f : D \to \mathbb{R}$ a hemi-Lebesgue integrable solution of the functional inequality (3.13)*, where, for all $(x,y) \in D^{2*}$, $e_{x,y} : [0,1] \to \mathbb{R}$ is a function, such that*

$$(3.29)\qquad I(x,y) := \int_{\frac12}^{1}\int_{0}^{\frac12}(t''-t')e_{(1-t')x+t'y,(1-t'')x+t''y}\Big(\frac{\frac12 - t'}{t''-t'}\Big)dt'dt''$$

exists in $[-\infty,\infty]$. *Then, for all $x,y \in D^{2*}$, the function f also satisfies*

$$(3.30)\qquad f\Big(\frac{x+y}{2}\Big) \le \int_0^1 f((1-t)x+ty)dt + 8I(x,y).$$

Proof. We apply Theorem 3.10, when $\mathcal{A}$ is the family of Lebesgue measurable subsets of $[0,1]$, μ is the Lebesgue measure. Then $\mu_1 = \frac12$ and $S(\mu) = \frac18$ and the result directly follows from Theorem 3.10. □

Remark 3.13. (Makó–Páles [**47**])
In what follows, we deduce the conclusion of Theorem 3.3 from the above corollary under stronger regularity assumption on f. Let $\alpha : D^* \to \mathbb{R}_+$ be a nonnegative radially Lebesgue integrable function and assume that $f : D \to \mathbb{R}$ is hemi-upper bounded and approximately α-Jensen convex. Then, by Theorem 3.5, f fulfils the approximate convexity inequality (3.15), i.e., (3.13) holds with $e_{x,y}$ defined as

$$e_{x,y}(t) := \sum_{n=0}^{\infty}\frac{\alpha(d_{\mathbb{Z}}(2^n t)(x-y))}{2^n} \qquad ((x,y)\in D^2,\ t\in[0,1]).$$

Thus, by Corollary 3.12, the inequality (3.30) holds with

$$I(x,y) = \int_{\frac{1}{2}}^{1} \int_{0}^{\frac{1}{2}} (t'' - t') e_{(1-t')x+t'y,(1-t'')x+t''y}\Big(\frac{\frac{1}{2} - t'}{t'' - t'}\Big) dt' dt''$$
$$= \sum_{n=0}^{\infty} \frac{1}{2^n} \int_{\frac{1}{2}}^{1} \int_{0}^{\frac{1}{2}} (t'' - t') \alpha\Big(d_{\mathbb{Z}}\Big(2^n \frac{\frac{1}{2} - t'}{t'' - t'}\Big)(t'' - t')(x - y)\Big) dt' dt''$$
$$= \sum_{n=0}^{\infty} \frac{1}{2^n} \int_{0}^{\frac{1}{2}} \int_{0}^{\frac{1}{2}} (t + s) \alpha\Big(d_{\mathbb{Z}}\Big(\frac{2^n t}{t + s}\Big)(t + s)(x - y)\Big) dt ds$$
$$= \sum_{n=0}^{\infty} \frac{2}{2^n} \int_{0}^{\frac{1}{2}} \int_{0}^{t} (t + s) \alpha\Big(d_{\mathbb{Z}}\Big(\frac{2^n t}{t + s}\Big)(t + s)(x - y)\Big) ds dt.$$

The last equality above is the consequence of the symmetry of the integrand with respect to the variables s, t. For $n = 0$,

$$\frac{2}{2^n} \int_{0}^{\frac{1}{2}} \int_{0}^{t} (t + s) \alpha\Big(d_{\mathbb{Z}}\Big(\frac{2^n t}{t + s}\Big)(t + s)(x - y)\Big) ds dt = 2 \int_{0}^{\frac{1}{2}} \int_{0}^{t} (t + s) \alpha(2s(x - y)) ds dt$$
$$= 2 \int_{0}^{\frac{1}{2}} \int_{s}^{\frac{1}{2}} (t + s) \alpha(2s(x - y)) dt ds = \int_{0}^{\frac{1}{2}} (1 - 2s)\Big(\frac{3}{2}s + \frac{1}{4}\Big) \alpha(2s(x - y)) ds$$
$$= \frac{1}{8} \int_{0}^{1} (1 - \sigma)(3\sigma + 1) \alpha(\sigma(x - y)) d\sigma.$$

To compute the double integral for $n \geq 1$, we will split its domain according to the position of $\frac{2^n t}{t+s}$ related to integer numbers. For all $n \in \mathbb{N}$ and $0 < s < t \leq \frac{1}{2}$, there exists a unique $m \in \{2^{n-1}, \dots, 2^n - 1\}$ (namely $m := \big[\frac{2^n t}{t+s}\big]$) such that

$$\text{either} \qquad m \leq \frac{2^n t}{t + s} < m + \frac{1}{2} \qquad \text{or} \qquad m + \frac{1}{2} \leq \frac{2^n t}{t + s} < m + 1.$$

This, for all $m \in \{2^{n-1}, \dots, 2^n - 1\}$, in terms of t yields the following inequalities for s:

$$\frac{2^n - m - \frac{1}{2}}{m + \frac{1}{2}} t < s \leq \frac{2^n - m}{m} t \qquad \text{and} \qquad \frac{2^n - m - 1}{m + 1} t < s \leq \frac{2^n - m - \frac{1}{2}}{m + \frac{1}{2}} t,$$

respectively. On these intervals, we have that

$$d_{\mathbb{Z}}\Big(\frac{2^n t}{t + s}\Big)(t + s) =$$
$$\begin{cases} 2(\frac{2^n t}{t+s} - m)(t + s) = 2(2^n - m)t - 2ms, & \frac{2^n - m - \frac{1}{2}}{m + \frac{1}{2}} t < s \leq \frac{2^n - m}{m} t, \\ 2(m + 1 - \frac{2^n t}{t+s})(t + s) = 2(m + 1 - 2^n)t + 2(m + 1)s, & \frac{2^n - m - 1}{m + 1} t < s \leq \frac{2^n - m - \frac{1}{2}}{m + \frac{1}{2}} t. \end{cases}$$

Thus, we get that

$$\begin{aligned}
&\int_0^{\frac12}\int_0^t (t+s)\alpha\Big(d_{\mathbb{Z}}\Big(2^n\frac{t}{t+s}\Big)(t+s)(x-y)\Big)dsdt\\
&=\int_0^{\frac12}\sum_{m=2^{n-1}}^{2^n-1}\Big(\int_{\frac{2^n-m-\frac12}{m+\frac12}t}^{\frac{2^n-m}{m}t}(t+s)\alpha(2((2^n-m)t-ms)(x-y))ds\\
&\qquad+\int_{\frac{2^n-m-1}{m-1}t}^{\frac{2^n-m-\frac12}{m+\frac12}t}(t+s)\alpha(2((m+1-2^n)t+(m+1)s)(x-y))ds\Big)dt\\
&=\sum_{m=2^{n-1}}^{2^n-1}\int_0^{\frac12}\Big(\int_0^{\frac{2^{n+1}t}{2m+1}}\alpha(\sigma(x-y))\Big(\frac{\sigma+2^{n+1}t}{(2m+2)^2}+\frac{2^{n+1}t-\sigma}{(2m)^2}\Big)d\sigma\Big)dt\\
&=\sum_{m=2^{n-1}}^{2^n-1}\int_0^{\frac{2^n}{2m+1}}\Big(\int_{\frac{(2m+1)\sigma}{2^{n+1}}}^{\frac12}\alpha(\sigma(x-y))\Big(\frac{\sigma+2^{n+1}t}{(2m+2)^2}+\frac{2^{n+1}t-\sigma}{(2m)^2}\Big)dt\Big)d\sigma\\
&=\frac{1}{16}\sum_{m=2^{n-1}}^{2^n-1}\int_0^{\frac{2^n}{2m+1}}\alpha(\sigma(x-y))\Big(1-\frac{2m+1}{2^n}\sigma\Big)\Big(\frac{\sigma(2m+3)+2^n}{(m+1)^2}+\frac{\sigma(2m-1)+2^n}{m^2}\Big)d\sigma\\
&=\frac{1}{16}\sum_{m=2^{n-1}}^{2^n-1}\int_0^{1}\alpha(\sigma(x-y))\Big(1-\frac{2m+1}{2^n}\sigma\Big)^+\Big(\frac{\sigma(2m+3)+2^n}{(m+1)^2}+\frac{\sigma(2m-1)+2^n}{m^2}\Big)d\sigma.
\end{aligned}$$

(Here x^+ stands for the positive part of x.) Summarizing our computations, for $8I(x,y)$, we get

$$8I(x,y)=\int_0^1\alpha(\sigma(x-y))\Phi(\sigma)d\sigma,$$

where

$$\Phi(\sigma):=(1-\sigma)(3\sigma+1)+\sum_{n=1}^{\infty}\sum_{m=2^{n-1}}^{2^n-1}\Big(1-\frac{2m+1}{2^n}\sigma\Big)^+\Big(\frac{\sigma(2m+3)+2^n}{2^n(m+1)^2}+\frac{\sigma(2m-1)+2^n}{2^nm^2}\Big).$$

One can easily see that Φ is a continuous function over $[0,1]$ with $\Phi(t)\ \geq\ 1$ for $0\ \leq\ t\ \leq\ \frac{2}{3}$ and $\Phi(1)=0$. Hence the error term $8I(x,y)$ obtained in (3.30) is not comparable with that in (3.11).

In what follows, we examine the case, when X is a normed space and $e_{x,y}(t)$ is a linear combination of the products of the powers of t, $1-t$, and of $\|x-y\|$, i.e., for all $(x,y)\in D^{2*}$, $e_{x,y}$ is of the form

$$e_{x,y}(t):=\int_{[0,\infty[^2}t^p(1-t)^q\|x-y\|^{p+q-1}d\nu(p,q)\qquad((x,y)\in D^{2*}), \tag{3.31}$$

where ν is a σ-finite Borel measure on $[0,\infty[^2$. An important particular case is when ν is of the form $\sum_{i=1}^k c_i\delta_{(p_i,q_i)}$, where $c_1,\dots,c_k>0$ and $(p_1,q_1),\dots,(p_k,q_k)\in[0,\infty[^2$.

Theorem 3.14. (Makó–Páles [**47**])
Let $\mathcal{A}$ be a sigma algebra containing the Borel subsets of $[0,1]$ and μ be a probability measure on the measure space $([0,1],\mathcal{A})$ such that the support of μ is not a singleton. Let ν be a σ-finite Borel measure on $[0,\infty[^2$ such that, for all $s\in\{\|x-y\|\mid(x,y)\in D^{2*}\}$,

$$J(s):=\int\limits_{[0,\infty[^2}\Big(\int\limits_{[0,\mu_1]}(\mu_1-t')^p d\mu(t')\int\limits_{]\mu_1,1]}(t''-\mu_1)^q d\mu(t'')\Big)s^{p+q-1}d\nu(p,q)$$

exists in $[-\infty,\infty]$. Assume that $f:D\to\mathbb{R}$ is a hemi-μ-integrable solution of the functional inequality

$$f((1-t)x+ty)\le(1-t)f(x)+tf(y)+\int\limits_{[0,\infty[^2}t^p(1-t)^q\|x-y\|^{p+q-1}d\nu(p,q) \tag{3.32}$$

for all $(x,y)\in D^{2*}$ and $t\in[0,1]$. Then, for all $(x,y)\in D^{2*}$, the function f also fulfils the Hermite–Hadamard type inequality

$$f((1-\mu_1)x+\mu_1 y)\le\int\limits_{[0,1]}f((1-t)x+ty)d\mu(t)+\frac{1}{S(\mu)}J(\|x-y\|). \tag{3.33}$$

Remark 3.15. (Makó–Páles [**47**])
In the above theorem, the hemi-μ-integrability condition for f can be relaxed if the measure ν is finite with compact support contained in $]0,\infty[^2$. Then the function $e_{x,y}$ defined by (3.31) is continuous on $[x,y]$ and $e_{x,y}(0)=e_{x,y}(1)=0$, hence (3.32) implies that $t\mapsto f((1-t)x+ty)$ is upper bounded on $[0,1]$ and upper semicontinuous at the endpoint of $[0,1]$. Thus f is hemi-upper bounded and upper hemicontinuous on D, which yields its hemi-μ-integrability.

Proof. We want to apply Theorem 3.10. Let $(x,y)\in D^{2*}$ be arbitrary and $e_{x,y}:[0,1]\to\mathbb{R}$ defined by (3.31). Then, (3.32) is equivalent to (3.13). To deduce (3.33), by Theorem 3.10, we obtain that

$$\begin{aligned}I(x,y)&=\int\limits_{]\mu_1,1]}\int\limits_{[0,\mu_1]}(t''-t')\int\limits_{[0,\infty[^2}\big(\tfrac{\mu_1-t'}{t''-t'}\big)^p\big(\tfrac{t''-\mu_1}{t''-t'}\big)^q\|(t''-t')(x-y)\|^{p+q-1}d\nu(p,q)d\mu(t')d\mu(t'')\\&=\int\limits_{[0,\infty[^2}\Big(\int\limits_{[0,\mu_1]}(\mu_1-t')^p d\mu(t')\int\limits_{]\mu_1,1]}(t''-\mu_1)^q d\mu(t'')\Big)\|x-y\|^{p+q-1}d\nu(p,q)=J(\|x-y\|),\end{aligned}$$

which proves the statement. □

Corollary 3.16. (Makó–Páles [**47**])
Let ν be a σ-finite Borel measure on $[0,\infty[^2$, such that for all $s \in \{\|x-y\| : (x,y) \in D^{2*}\}$,

$$\int\limits_{[0,\infty[^2} \frac{s^{p+q-1}}{2^{p+q-1}(p+1)(q+1)} d\nu(p,q)$$

exists in $[-\infty,\infty]$. Assume that $f : D \to \mathbb{R}$ is a hemi-Lebesgue integrable solution of the functional inequality

$$f((1-t)x+ty) \le (1-t)f(x) + tf(y) + \int\limits_{[0,\infty[^2} t^p(1-t)^q \|x-y\|^{p+q-1} d\nu(p,q), \tag{3.34}$$

where $(x,y) \in D^{2*}$ and $t \in [0,1]$. Then, for all $(x,y) \in D^{2*}$, f also satisfies the Hermite–Hadamard type inequality

$$f\Big(\frac{x+y}{2}\Big) \le \int\limits_{[0,1]} f((1-t)x+ty)dt + \int\limits_{[0,\infty[^2} \frac{\|x-y\|^{p+q-1}}{2^{p+q-1}(p+1)(q+1)} d\nu(p,q). \tag{3.35}$$

Proof. Observe that (3.34) is equivalent to (3.13), where for all $(x,y) \in D^{2*}$, $e_{x,y} : [0,1] \to \mathbb{R}$ is defined by (3.31). We have $S(\mu) = \frac{1}{8}$ and using Theorem 3.14, we obtain that

$$J(s) = \int\limits_{[0,\infty[^2} \int_0^{\frac{1}{2}} (\tfrac{1}{2}-t)^p dt \int_{\frac{1}{2}}^1 (t-\tfrac{1}{2})^q dt s^{p+q-1} d\nu(p,q) = \int\limits_{[0,\infty[^2} \frac{s^{p+q-1}}{2^{p+q+2}(p+1)(q+1)} d\nu(p,q),$$

which yields (3.35). □

3.2. Korovkin type theorems

The subsequent results are Korovkin type theorems. In the sequel, denote by $C([a,b])$ and $B([a,b])$ the space of continuous and bounded Borel measurable real valued functions defined on the interval $[a,b]$ equipped with the usual supremum norm.

Theorem 3.17. (Makó–Páles [**46**])
Let $a < b$, $\mathcal{T}_n : B([a,b]) \to B([a,b])$ $(n \in \mathbb{N})$ be a sequence of positive linear operators and let (ω_0, ω_1) be a positive Chebyshev system over $[a,b]$ such that, for all $u \in [a,b]$,

$$\lim_{n\to\infty} (\mathcal{T}_n\omega_0)(u) = \omega_0(u) \qquad \text{and} \qquad \lim_{n\to\infty} (\mathcal{T}_n\omega_1)(u) = \omega_1(u). \tag{3.36}$$

Suppose that there exists a function $g \in C([a,b])$ with $g(a) = g(b) = 0$ and $g > 0$ on $]a,b[$ such that $\lim_{n\to\infty}(\mathcal{T}_n g)(u) = 0$ for all $u \in [a,b]$. Then, for all bounded upper *semicontinuous function* $f : [a,b] \to \mathbb{R}$,

$$\limsup_{n\to\infty} \mathcal{T}_n f(u) \le f(a)\frac{\Omega(u,b)}{\Omega(a,b)} + f(b)\frac{\Omega(a,u)}{\Omega(a,b)} \qquad (u \in [a,b]). \tag{3.37}$$

Remark 3.18. (Makó–Páles [**46**])
It easily follows from the above theorem that, if f is continuous, then (3.37) holds with equality and the "limsup" can be replaced by "lim". By a generalization of the classical Korovkin Theorem if, for some continuous (ω_0, ω_1)-convex function $g : [a,b] \to \mathbb{R}$, $\mathcal{T}_n g$ converges to g, then $\mathcal{T}_n f$ converges to f for all $f \in C([a,b])$.

Proof. As a consequence of (3.36), it follows that

$$\lim_{n\to\infty} (\mathcal{T}_n \Omega(\cdot,b))(u) = \Omega(u,b) \quad \text{and} \quad \lim_{n\to\infty} (\mathcal{T}_n \Omega(a,u))(u) = \Omega(a,u) \tag{3.38}$$

for all $u \in [a,b]$. To prove (3.37), let $\varepsilon > 0$ be arbitrary and define $\phi :]a,b[\to \mathbb{R}$ by

$$\phi := \frac{\Omega(a,b)f - f(a)\Omega(\cdot,b) - f(b)\Omega(a,\cdot) - \varepsilon\Omega(a,b)\omega_0}{\Omega(a,b)g}.$$

Since f is upper semicontinuous at a and also at b, there exists $0 < \delta < \frac{b-a}{2}$ such that

$$\Omega(a,b)f(u) - f(a)\Omega(u,b) - f(b)\Omega(a,u) < \varepsilon\Omega(a,b)\omega_0(u) \quad \text{if} \quad a \le u < a+\delta,$$
$$\Omega(a,b)f(u) - f(a)\Omega(u,b) - f(b)\Omega(a,u) < \varepsilon\Omega(a,b)\omega_0(u) \quad \text{if} \quad b-\delta < u \le b.$$

Hence, the function ϕ is nonpositive on $[a, a+\delta[\cup]b-\delta, b]$. On the other hand, ϕ is upper semicontinuous on the compact interval $[a+\delta, b-\delta]$, therefore ϕ attains its maximum on $[a+\delta, b-\delta]$, which we denote by K. This implies that

$$f \le f(a)\frac{\Omega(\cdot,b)}{\Omega(a,b)} + f(b)\frac{\Omega(a,\cdot)}{\Omega(a,b)} + Kg + \varepsilon\omega_0 \qquad \text{on} \quad [a,b].$$

Applying the linearity and the monotonicity of $\mathcal{T}_n$, for all $n \in \mathbb{N}$ and $u \in [a,b]$, we get

$$(\mathcal{T}_n f)(u) \le \frac{f(a)}{\Omega(a,b)}(\mathcal{T}_n\Omega(\cdot,b))(u) + \frac{f(b)}{\Omega(a,b)}(\mathcal{T}_n\Omega(a,\cdot))(u) + K(\mathcal{T}_n g)(u) + \varepsilon(\mathcal{T}_n\omega_0)(u).$$

Upon taking the limit $n \to \infty$ and using (3.38), we obtain

$$\limsup_{n\to\infty}(\mathcal{T}_n f)(u) \le \frac{f(a)}{\Omega(a,b)}\Omega(u,b) + \frac{f(b)}{\Omega(a,b)}\Omega(a,u) + \varepsilon\omega_0(u) \quad (u \in [a,b]),$$

which results the statement. □

The following result offers a sufficient condition for (3.37) to hold when the sequence $(\mathcal{T}_n)$ is obtained as the sequence of iterates of a positive linear operator.

Corollary 3.19. (Makó–Páles [**46**])
Let $a < b$, $\mathcal{T} : B([a,b]) \to B([a,b])$ be a positive linear operator and let (ω_0, ω_1) be a positive Chebyshev system over $[a,b]$ such that ω_0 and ω_1 are fixed points of $\mathcal{T}$, i.e.,

$$\mathcal{T}\omega_0 = \omega_0 \qquad \text{and} \qquad \mathcal{T}\omega_1 = \omega_1. \tag{3.39}$$

Suppose that there exist a function $g \in C([a,b])$ with $g(a) = g(b) = 0$ and $g > 0$ on $]a,b[$ and a constant $q \in]0,1[$ such that

$$(\mathcal{T}g)(u) \le qg(u) \qquad (u \in [a,b]). \tag{3.40}$$

Then, for all upper semicontinuous $f \in B([a,b])$,

$$\limsup_{n\to\infty} \mathcal{T}^n f(u) \le f(a)\frac{\Omega(u,b)}{\Omega(a,b)} + f(b)\frac{\Omega(a,u)}{\Omega(a,b)} \qquad (u \in [a,b]). \tag{3.41}$$

Remark 3.20. (Makó–Páles [**46**])
It easily follows from the above corollary that, if f is continuous, then (3.41) holds with equality and the “limsup” can be replaced by “lim”.

Proof. We apply Theorem 3.17 to the sequence of operators $\mathcal{T}_n := \mathcal{T}^n$. Now (3.36) automatically holds by (3.39). On the other hand, by induction on $n \in \mathbb{N}$, we get $0 \le \mathcal{T}^n g \le q^n g$. Taking the limit $n \to \infty$, this inequality yields that $\lim_{n\to\infty} \mathcal{T}^n g = 0$. Hence, Theorem 3.17 applies and we obtain (3.37), which is now equivalent to (3.41). □

The following two result establish a connection between (ω_0, ω_1)-convexity and the inequality $f \le \mathcal{T}f$.

Proposition 3.21. (Makó–Páles [**46**])
Let $a < b$, $\mathcal{T} : B([a,b]) \to B([a,b])$ be a positive linear operator and let (ω_0, ω_1) be a positive Chebyshev system over $[a,b]$ such that ω_0 and ω_1 are fixed points of $\mathcal{T}$. Then, for all (ω_0, ω_1)-convex functions $f : [a,b] \to \mathbb{R}$,

$$f(u) \le (\mathcal{T}f)(u) \qquad (u \in]a,b[) \tag{3.42}$$

holds.

Proof. Assume that $f : [a,b] \to \mathbb{R}$ is a (ω_0, ω_1)-convex function. Then it is continuous on the interior of $[a,b]$ and upper semicontinuous at the endpoints of $[a,b]$, furthermore, it is also bounded. Hence $f \in B([a,b])$. In view of Lemma 3.1, for all $u \in]a,b[$, there exist constants α, β such that

$$\alpha\omega_0(u) + \beta\omega_1(u) = f(u) \qquad \text{and} \qquad \alpha\omega_0 + \beta\omega_1 \le f.$$

Applying $\mathcal{T}$ to the last inequality and using that ω_0 and ω_1 are fixed points of $\mathcal{T}$, we get

$$\alpha\omega_0 + \beta\omega_1 \le \mathcal{T}f.$$

Evaluating both sides at u, it follows that

$$f(u) = \alpha\omega_0(u) + \beta\omega_1(u) \le (\mathcal{T}f)(u).$$

This proves (3.42). □

Proposition 3.22. (Makó–Páles [**46**])
Let $a < b$, $\mathcal{T} : B([a,b]) \to B([a,b])$ be a positive linear operator and let (ω_0, ω_1) be a positive Chebyshev system over $[a,b]$ such that ω_0 and ω_1 are fixed points of $\mathcal{T}$. Assume that there exists a function $g \in C([a,b])$ with $g(a) = g(b) = 0$ and $g > 0$ on $]a,b[$ such that (3.40) *holds for some $q \in [0,1[$. Then, for all bounded upper semicontinuous function $f \in B([a,b])$ which satisfies $f \leq \mathcal{T}f$, the inequality*

$$f(u) \leq f(a)\frac{\Omega(u,b)}{\Omega(a,b)} + f(b)\frac{\Omega(a,u)}{\Omega(a,b)} \qquad (u \in [a,b]) \tag{3.43}$$

holds.

Proof. Assume that $f : [a,b] \to \mathbb{R}$ is a bounded upper semicontinuous function which satisfies $f \leq \mathcal{T}f$. Then, by induction, $\mathcal{T}^n f \leq \mathcal{T}^{n+1} f$ and hence $f \leq \mathcal{T}^n f$ follows for all $n \in \mathbb{N}$. On the other hand, by Corollary 3.19, (3.41) also holds. Therefore, upon taking the limit $n \to \infty$ in the inequality $f \leq \mathcal{T}^n f$, and using (3.41), the desired inequality (3.43) follows. □

In what follows, we construct a large family of positive linear operators on $B([a,b])$ which satisfies the assumptions of the previous results and will be instrumental in the investigation of approximate convexity.

For a triplet (μ, λ, m) define a linear operator $\mathcal{T}_{\mu,\lambda,m} : B([a,b]) \to B([a,b])$ by the following formula:

$$(\mathcal{T}_{\mu,\lambda,m} f)(u) := \int_T \lambda(t,u) f(m(t,u)) d\mu(t), \tag{3.44}$$

where, on the data (μ, λ, m), we make the following assumptions:

(H1) $(T, \mathcal{A}, \mu)$ is a measure space.

(H2) $\lambda : T \times [a,b] \to \mathbb{R}_+$ is measurable in its first variable and continuous in its second variable, furthermore, the function $\ell : T \to \mathbb{R}_+$ defined by

$$\ell(t) := \sup_{u \in [a,b]} \lambda(t,u)$$

is μ-integrable, i.e., $\int_T \ell(t) d\mu(t) < +\infty$.

(H3) $m : T \times [a,b] \to [a,b]$ is measurable in its first variable, continuous in its second variable on $[a,b]$ differentiable in its second variable at the endpoints of $[a,b]$ and it is uniformly calm at the endpoints of $[a,b]$, i.e., there exist constants $\delta > 0$ and $K \geq 0$ such that

$$|m(t,v) - m(t,u)| \leq K|v-u| \qquad \text{if} \quad u \in \{a,b\},\ v \in [a,b],\ |v-u| \leq \delta.$$

Furthermore, for all $t \in T$, $m(t,a) = a$, $m(t,b) = b$, and, for all $u \in [a,b]$,

$$\begin{aligned} &\mu\{t \in T \mid \lambda(t,u) \neq 0,\ m(t,u) \neq u\} > 0 \qquad \text{if} \quad u \notin \{a,b\}, \\ &\mu\{t \in T \mid \lambda(t,u) \neq 0,\ m'(t,u) \neq 1\} > 0 \qquad \text{if} \quad u \in \{a,b\}. \end{aligned} \tag{3.45}$$

(H4) There exists a positive Chebyshev system (ω_0, ω_1) over $[a, b]$ such that ω_0 is positive, $\frac{\omega_1}{\omega_0}$ is differentiable at a and at b with $(\frac{\omega_1}{\omega_0})'(a) > 0$ and $(\frac{\omega_1}{\omega_0})'(b) > 0$. Furthermore, for $i \in \{0, 1\}$,

$$\omega_i(u) = \int_T \lambda(t, u)\omega_i(m(t, u))d\mu(t) \qquad (u \in [a, b]). \tag{3.46}$$

Proposition 3.23. (Makó–Páles [**46**])
Assume that (μ, λ, m) fulfills hypotheses (H1)-(H4). Then

$$\int_T \lambda(t, a)d\mu(t) = 1, \qquad \int_T \lambda(t, b)d\mu(t) = 1, \tag{3.47}$$

$$\int_T \lambda(t, a)m'(t, a)d\mu(t) \le 1, \qquad \int_T \lambda(t, b)m'(t, b)d\mu(t) \le 1, \tag{3.48}$$

and, if $p \in]0, 1[$,

$$\int_T \lambda(t, a)(m'(t, a))^p d\mu(t) < 1, \quad \int_T \lambda(t, b)(m'(t, b))^p d\mu(t) < 1. \tag{3.49}$$

Proof. We note that $a \le m(t, u) \le b$ for all $(t, u) \in T \times [a, b]$, hence the assumptions $m(\cdot, a) = a$ and $m(\cdot, b) = b$ imply that $m'(t, a) \ge 0$ and $m'(t, b) \ge 0$ for all $t \in T$.

The substitution $u = a$ in (3.46) and $m(\cdot, a) = a$ yield that

$$\omega_0(a) = \int_T \lambda(t, a)\omega_0(m(t, a))d\mu(t) = \int_T \lambda(t, a)\omega_0(a)d\mu(t) = \omega_0(a)\int_T \lambda(t, a)d\mu(t),$$

which, by $\omega_0(a) > 0$, results the first formula in (3.47). The proof of the second formula is analogous.

To prove (3.48), let $u \in]a, b[$ be arbitrary. Using (3.46), we have that

$$\omega_0(u) \cdot (\tfrac{\omega_1}{\omega_0})(a) = \int_T \lambda(t, u) \cdot \omega_0(m(t, u)) \cdot (\tfrac{\omega_1}{\omega_0})(m(t, a))d\mu(t). \tag{3.50}$$

and

$$\omega_0(u) \cdot (\tfrac{\omega_1}{\omega_0})(u) = \int_T \lambda(t, u) \cdot \omega_0(m(t, u)) \cdot (\tfrac{\omega_1}{\omega_0})(m(t, u))d\mu(t). \tag{3.51}$$

Subtracting (3.50) from (3.51) and then dividing by $u - a$, we get that

$$\omega_0(u) \cdot \frac{(\frac{\omega_1}{\omega_0})(u) - (\frac{\omega_1}{\omega_0})(a)}{u - a} = \int_T \lambda(t, u) \cdot \omega_0(m(t, u)) \cdot \frac{(\frac{\omega_1}{\omega_0})(m(t, u)) - (\frac{\omega_1}{\omega_0})(m(t, a))}{u - a} d\mu(t). \tag{3.52}$$

Observe that the integrand is a nonnegative measurable function of the variable t for all $u \in [a,b]$. Now, let $u_n > a$ be an arbitrary sequence tending to a and substitute u by u_n in (3.52). Then, taking the limit $n \to \infty$ and using the Fatou Lemma, we get

$$\begin{aligned}
\omega_0(a)\cdot(\tfrac{\omega_1}{\omega_0})'(a) &= \liminf_{n\to\infty} \omega_0(u)\cdot \frac{(\frac{\omega_1}{\omega_0})(u_n) - (\frac{\omega_1}{\omega_0})(a)}{u_n - a} \\
&= \liminf_{n\to\infty} \int_T \lambda(t,u_n)\cdot\omega_0(m(t,u_n))\cdot \frac{(\frac{\omega_1}{\omega_0})(m(t,u_n)) - (\frac{\omega_1}{\omega_0})(m(t,a))}{u_n - a} d\mu(t) \\
&\geq \int_T \liminf_{n\to\infty} \lambda(t,u_n)\cdot\omega_0(m(t,u_n))\cdot \frac{(\frac{\omega_1}{\omega_0})(m(t,u_n)) - (\frac{\omega_1}{\omega_0})(m(t,a))}{u_n - a} d\mu(t) \\
&= \int_T \lambda(t,a)\cdot\omega_0(a)\cdot(\tfrac{\omega_1}{\omega_0})'(a)\cdot m'(t,a) d\mu(t)
\end{aligned}$$

Using $\omega_0(a)\cdot(\frac{\omega_1}{\omega_0})'(a) > 0$, the first inequality in (3.48) follows. The proof of the second inequality is analogous.

Let $p \in]0,1[$ be fixed. The function $s \mapsto s^p$ is strictly concave over $[0,\infty[$, therefore

$$s^p \leq p(s-1)+1 \qquad (s \in [0,\infty[)$$

and equality is valid if and only if $s = 1$. Hence,

$$(m'(t,a))^p \leq pm'(t,a) + 1 - p \qquad (t \in T), \tag{3.53}$$

and equality holds if and only if $m'(t,a) = 1$. Thus, by the second inequality in assumption (3.45), on a set of positive μ measure we have strict inequality in (3.53). Multiplying this inequality by $\lambda(t,a)$ and then integrating by t with respect to μ, and using (3.47) and (3.48), it follows that

$$\int_T \lambda(t,a)\cdot(m'(t,a))^p d\mu(t) < p\int_T \lambda(t,a)\cdot m'(t,a)d\mu(t) + (1-p)\int_T \lambda(t,a)d\mu(t) \leq 1.$$

The proof of the second inequality in (3.49) is completely similar. □

Proposition 3.24. (Makó–Páles [**46**])
Assume that (μ,λ,m) fulfills hypotheses (H1)-(H4) and define $\mathcal{T}_{\mu,\lambda,m}$ by (3.44). *Then $\mathcal{T}_{\mu,\lambda,m} : B([a,b]) \to B([a,b])$ is a bounded positive linear operator with*

$$\|\mathcal{T}_{\mu,\lambda,m}\| \leq \int_T \ell(t)d\mu(t). \tag{3.54}$$

In addition, $\mathcal{T}_{\mu,\lambda,m}$ has the following properties:

(i) For all $f \in B([a,b])$,

$$(\mathcal{T}_{\mu,\lambda,m}f)(a) = f(a) \quad \text{and} \quad (\mathcal{T}_{\mu,\lambda,m}f)(b) = f(b). \tag{3.55}$$

(ii) If $f : [a,b] \to \mathbb{R}$ is bounded and lower (upper) semicontinuous then $\mathcal{T}_{\mu,\lambda,m}f$ is also lower (upper) semicontinuous, respectively.

(iii) If $f_n \in B([a,b])$ is a norm-bounded sequence converging pointwise to $f_0 \in B([a,b])$ then

$$\lim_{n\to\infty}(\mathcal{T}_{\mu,\lambda,m}f_n)(u) = (\mathcal{T}_{\mu,\lambda,m}f_0)(u) \qquad (u \in [a,b]).$$

Proof. If $f \in B([a,b])$ then, for all fixed $u \in [a,b]$, the function $t \mapsto \lambda(t,u)f(m(t,u))$ is $\mathcal{A}$-measurable on T. On the other hand, $|\lambda(t,u)f(m(t,u))| \le \ell(t)\|f\|$, hence the integral on the right hand side of (3.44) is well-defined and

$$|(\mathcal{T}_{\mu,\lambda,m}f)(u)| \le \int_T |\lambda(t,u)f(m(t,u))|d\mu(t) \le \int_T \ell(t)d\mu(t)\|f\|,$$

which proves the boundedness of $\mathcal{T}_{\mu,\lambda,m}$ and (3.54). The linearity and positivity of $\mathcal{T}_{\mu,\lambda,m}$ is obvious. The endpoint properties (3.55) are consequences of hypothesis (H3) and Proposition 3.21.

Now we show that $\mathcal{T}_{\mu,\lambda,m}f$ is lower semicontinuous whenever $f \in B([a,b])$ is also lower semicontinuous. Let (u_n) be an arbitrary sequence in $[a,b]$ converging to $u_0 \in [a,b]$. By the continuity-semicontinuity properties of λ, m and f, and the uniform integrable bound estimate $|\lambda(t,u_n)f(m(t,u_n))| \le \ell(t)\|f\|$, Fatou's Lemma can be applied. Thus

$$\liminf_{n\to\infty}(\mathcal{T}_{\mu,\lambda,m}f)(u_n) = \liminf_{n\to\infty}\int_T \lambda(t,u_n)f(m(t,u_n))d\mu(t)$$
$$\ge \int_T \liminf_{n\to\infty}\lambda(t,u_n)f(m(t,u_n))d\mu(t) \ge \int_T \lambda(t,u_0)f(m(t,u_0))d\mu(t) = (\mathcal{T}_{\mu,\lambda,m}f)(u_0),$$

which proves the lower semicontinuity of $\mathcal{T}_{\mu,\lambda,m}f$ at u_0. Hence $\mathcal{T}_{\mu,\lambda,m}f$ is lower semicontinuous at every point of $[a,b]$ provided that f is also lower semicontinuous on $[a,b]$. The proof for the upper semicontinuity is analogous.

To verify (iii), assume that $f_n \in B([a,b])$ is a norm-bounded sequence converging pointwise to a function $f_0 : [a,b] \to \mathbb{R}$. Then $f_0 \in B([a,b])$ and, by Lebesgue's Dominated Convergence Theorem, for all $u \in [a,b]$,

$$\lim_{n\to\infty}(\mathcal{T}_{\mu,\lambda,m}f_n)(u) = \lim_{n\to\infty}\int_T \lambda(t,u)f_n(m(t,u))d\mu(t)$$
$$= \int_T \lim_{n\to\infty}(\lambda(t,u)f_n(m(t,u)))d\mu(t) = \int_T \lambda(t,u)f_0(m(t,u))d\mu(t) = (\mathcal{T}_{\mu,\lambda,m}f_0)(u),$$

which proves (iii).

Finally, we prove that $\mathcal{T}_{\mu,\lambda,m}f$ is a bounded Borel measurable for all $f \in B([a,b])$. For, define the class of functions $B_0([a,b])$ by

$$B_0([a,b]) := \{f \in B([a,b]) \mid \mathcal{T}_{\mu,\lambda,m}f \in B([a,b])\}.$$

Obviously, this class is a subspace of $B([a,b])$ which contains $C([a,b])$ because, by (ii), the images of continuous functions are continuous. On the other hand, if f_n is a norm-bounded sequence of functions from $B_0([a,b])$ converging to a function $f_0 \in B([a,b])$, then, by (iii), $\mathcal{T}_{\mu,\lambda,m}f_0$ is the pointwise limit of the sequence of bounded Borel functions $\mathcal{T}_{\mu,\lambda,m}f_n$, and hence, it $\mathcal{T}_{\mu,\lambda,m}f_0$ has to be also a Borel function, i.e., $f_0 \in B_0([a,b])$. In other words, the class $B_0([a,b])$ is closed with respect to the pointwise convergence and contains $C([a,b])$. Hence the equality $B_0([a,b]) = B([a,b])$ must hold (cf. [**15**]), which was to be proved. □

The following proposition is a counterpart of Proposition 3.21

Proposition 3.25. (Makó–Páles [**46**])
Assume that (μ,λ,m) *fulfills hypotheses (H1)-(H4) and define* $\mathcal{T}_{\mu,\lambda,m}$ *by* (3.44). *Then, for all strictly* (ω_0,ω_1)*-convex functions* $f:[a,b]\to\mathbb{R}$*,*

$$f(u) < (\mathcal{T}_{\mu,\lambda,m}f)(u) \qquad (u \in]a,b[). \tag{3.56}$$

Proof. Let $u \in]a,b[$ be a fixed element. Then, by Lemma 3.1, there exist constants $\alpha,\beta\in\mathbb{R}$ such that

$$\begin{aligned} &\alpha\omega_0(u)+\beta\omega_1(u) = f(u) \qquad \text{and} \\ &\alpha\omega_0(v)+\beta\omega_1(v) < f(v) \quad (v \in [a,b]\setminus\{u\}). \end{aligned} \tag{3.57}$$

We show that image of the function $f-(\alpha\omega_0+\beta\omega_1)$ by $\mathcal{T}_{\mu,\lambda,m}$ is everywhere positive on $[a,b]$. If this were not the case, then there would exist an element $v\in[a,b]$ such that

$$(\mathcal{T}_{\mu,\lambda,m}(f-\alpha\omega_0-\beta\omega_1))(v) = 0.$$

By the definition (3.44) of $\mathcal{T}_{\mu,\lambda,m}$ yields that

$$\int_T \lambda(t,v)\cdot(f-\alpha\omega_0-\beta\omega_1)(m(t,v))d\mu(t) = 0.$$

The function $t\mapsto(f-\alpha\omega_0-\beta\omega_1)(m(t,v))$ vanishes if and only if $m(t,v)=u$. Since the integrand is a nonnegative measurable function of the variable t, it can only vanish if and only if the set

$$S := \{t\in T \mid \lambda(t,v)>0,\ m(t,v)\neq u\}$$

has zero μ measure.

On the other hand, by assumption (H4), for $i \in \{0,1\}$, we also have that

$$\omega_i(v) = \int_T \lambda(t,v)\omega_i(m(t,v))d\mu(t) = \int_{T\setminus S} \lambda(t,v)\omega_i(m(t,v))d\mu(t)$$
$$= \int_{T\setminus S} \lambda(t,v)\omega_i(u)d\mu(t) = \omega_i(u)\int_{T\setminus S} \lambda(t,v)d\mu(t) = \omega_i(u)\int_T \lambda(t,v)d\mu(t).$$

Dividing the above identities by each other, we get that

$$\frac{\omega_1}{\omega_0}(v) = \frac{\omega_1}{\omega_0}(u).$$

This, by the strict monotonicity of $\frac{\omega_1}{\omega_0}$ contradicts $v \neq u$. Using also (3.46), this means that

$$\alpha\omega_0 + \beta\omega_1 = \mathcal{T}_{\mu,\lambda,m}(\alpha\omega_0 + \beta\omega_1) < \mathcal{T}_{\mu,\lambda,m}f \qquad \text{on} \qquad [a,b].$$

Substituting the fixed element $u \in]a,b[$ and using also (3.57), we get that

$$f(u) = \alpha\omega_0(u) + \beta\omega_1(u) < (\mathcal{T}_{\mu,\lambda,m}f)(u),$$

which proves (3.56). □

Theorem 3.26. (Makó–Páles [**46**])
Assume that (μ,λ,m) fulfills hypotheses (H1)–(H4) and define $\mathcal{T}_{\mu,\lambda,m}$ by (3.44)*. Then, there exists a (ω_0,ω_1)-concave function $g : [a,b] \to \mathbb{R}$, with $g(a) = g(b) = 0$, and a constant $q \in [0,1[$ such that*

$$(\mathcal{T}_{\mu,\lambda,m}g)(u) \le qg(u) \qquad (u \in [a,b]). \tag{3.58}$$

Proof. For $p \in]0,1[$, define $g_p : [a,b] \to \mathbb{R}$ by the following way:

$$g_p(u) := \omega_0(u)\min\Big(\big(\tfrac{\omega_1}{\omega_0}(u) - \tfrac{\omega_1}{\omega_0}(a)\big)^p, \big(\tfrac{\omega_1}{\omega_0}(b) - \tfrac{\omega_1}{\omega_0}(u)\big)^p\Big).$$

Observe that $g_p(a) = g_p(b) = 0$, $g_p(u) > 0$ for all $u \in]a,b[$, and

$$g_p(u) = \begin{cases} \omega_0(u)\big(\frac{\omega_1}{\omega_0}(u) - \frac{\omega_1}{\omega_0}(a)\big)^p & \text{if } u \in [a,c], \\ \omega_0(u)\big(\frac{\omega_1}{\omega_0}(b) - \frac{\omega_1}{\omega_0}(u)\big)^p & \text{if } u \in [c,b], \end{cases} \tag{3.59}$$

where $c := (\frac{\omega_1}{\omega_0})^{-1}(\frac{1}{2}\frac{\omega_1}{\omega_0}(a) + \frac{1}{2}\frac{\omega_1}{\omega_0}(b))$, furthermore

$$\big(\tfrac{g_p}{\omega_0}\big) \circ \big(\tfrac{\omega_1}{\omega_0}\big)^{-1}(v) = \min\Big(\big(v - \tfrac{\omega_1}{\omega_0}(a)\big)^p, \big(\tfrac{\omega_1}{\omega_0}(b) - v\big)^p\Big) \qquad \big(v \in [\tfrac{\omega_1}{\omega_0}(a), \tfrac{\omega_1}{\omega_0}(b)]\big),$$

which is the minimum of two strictly concave functions on $[\frac{\omega_1}{\omega_0}(a), \frac{\omega_1}{\omega_0}(b)]$ in the standard sense. Therefore, by Lemma 3.1, g_p is strictly (ω_0,ω_1)-concave (i.e., $(-g_p)$ is strictly (ω_0,ω_1)-convex). Using Proposition 3.25, we obviously have that

$$\mathcal{T}_{\mu,\lambda,m}g_p < g_p \quad \text{on} \quad]a,b[. \tag{3.60}$$

To prove (3.58), we will first show that

$$\limsup_{u\to a}\frac{(\mathcal{T}_{\mu,\lambda,m}g_p)(u)}{g_p(u)}<1 \qquad\text{and}\qquad \limsup_{u\to b}\frac{(\mathcal{T}_{\mu,\lambda,m}g_p)(u)}{g_p(u)}<1. \tag{3.61}$$

For the calculation of the first limit in (3.61), observe by (3.59) that, for $u\in]a,c]$,

$$\begin{aligned}\frac{(\mathcal{T}_{\mu,\lambda,m}g_p)(u)}{g_p(u)}&=\int_T\lambda(t,u)\frac{g_p(m(t,u))}{g_p(u)}d\mu(t)\\ &\le\int_T\lambda(t,u)\frac{\omega_0(m(t,u))}{\omega_0(u)}\bigg(\frac{\frac{\omega_1}{\omega_0}(m(t,u))-\frac{\omega_1}{\omega_0}(a)}{\frac{\omega_1}{\omega_0}(u)-\frac{\omega_1}{\omega_0}(a)}\bigg)^p d\mu(t).\end{aligned}\tag{3.62}$$

In what follows, to ensure the applicability of the Lebesgue Dominated Convergence Theorem, we show that the integrand in rightmost integral has a u-independent integrable upper bound.

Let $\varepsilon:=\frac{1}{2}(\frac{\omega_1}{\omega_0})'(a)$. By the differentiability of $\frac{\omega_1}{\omega_0}$, there exists $\eta>0$ such that

$$\left|\frac{\frac{\omega_1}{\omega_0}(u)-\frac{\omega_1}{\omega_0}(a)}{u-a}-(\tfrac{\omega_1}{\omega_0})'(a)\right|\le\tfrac{1}{2}(\tfrac{\omega_1}{\omega_0})'(a)\qquad(u\in]a,a+\eta]).$$

Therefore,

$$\begin{aligned}\tfrac{1}{2}(\tfrac{\omega_1}{\omega_0})'(a)\cdot(u-a)&\le\tfrac{\omega_1}{\omega_0}(u)-\tfrac{\omega_1}{\omega_0}(a)\\ &\le\tfrac{3}{2}(\tfrac{\omega_1}{\omega_0})'(a)\cdot(u-a)\qquad(u\in[a,a+\eta]).\end{aligned}\tag{3.63}$$

Let $\delta>0$ and $K\ge1$ be the constants required in hypotheses (H3) such that $K\delta\le\eta$. Define the function $\ell:T\to\mathbb{R}_+$ as in hypotheses (H2). Then ℓ is a μ-integrable function and

$$\lambda(t,u)\le\ell(t)\qquad((t,u)\in T\times[a,b]).$$

On the other hand, by the uniform calmness assumption on m, for $(t,u)\in T\times]a,a+\delta]$, we have that $|u-a|\le\eta$ and $|m(t,u)-m(t,a)|\le\eta$, hence, applying (3.63), we obtain

$$\begin{aligned}\frac{\frac{\omega_1}{\omega_0}(m(t,u))-\frac{\omega_1}{\omega_0}(a)}{\frac{\omega_1}{\omega_0}(u)-\frac{\omega_1}{\omega_0}(a)}&\le\frac{\frac{3}{2}(\frac{\omega_1}{\omega_0})'(a)\cdot(m(t,u)-a)}{\frac{1}{2}(\frac{\omega_1}{\omega_0})'(a)\cdot(u-a)}\\ &=3\frac{m(t,u)-m(t,a)}{u-a}\le3K\qquad((t,u)\in T\times]a,a+\delta]).\end{aligned}$$

Using the above estimates, it follows that

$$\lambda(t,u)\frac{\omega_0(m(t,u))}{\omega_0(u)}\bigg(\frac{\frac{\omega_1}{\omega_0}(m(t,u))-\frac{\omega_1}{\omega_0}(a)}{\frac{\omega_1}{\omega_0}(u)-\frac{\omega_1}{\omega_0}(a)}\bigg)^p\le\ell(t)\cdot\|\omega_0\|\cdot\|\omega_0^{-1}\|\cdot(3K)^p$$

for all $(t,u)\in T\times]a,a+\delta]$. Therefore, upon taking the limit $u\to a$ in (3.62), Lebesgue's Theorem can be applied. Let u_n be an arbitrary sequence in $]a,a+\delta]$

converging to a. By taking a subsequence if necessary, we may assume that the sequence $\frac{(\mathcal{T}_{\mu,\lambda,m}g_p)(u_n)}{g_p(u_n)}$ is convergent. Then we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{(\mathcal{T}_{\mu,\lambda,m}g_p)(u_n)}{g_p(u_n)} &\le \lim_{n\to\infty}\int_T \lambda(t,u_n)\frac{\omega_0(m(t,u_n))}{\omega_0(u_n)}\Big(\frac{\frac{\omega_1}{\omega_0}(m(t,u_n))-\frac{\omega_1}{\omega_0}(a)}{\frac{\omega_1}{\omega_0}(u_n)-\frac{\omega_1}{\omega_0}(a)}\Big)^p d\mu(t)\\ &\le \int_T \lim_{n\to\infty}\Big(\lambda(t,u_n)\frac{\omega_0(m(t,u_n))}{\omega_0(u_n)}\Big(\frac{\frac{\omega_1}{\omega_0}(m(t,u_n))-\frac{\omega_1}{\omega_0}(a)}{\frac{\omega_1}{\omega_0}(u_n)-\frac{\omega_1}{\omega_0}(a)}\Big)^p\Big)d\mu(t)\\ &= \int_T \lambda(t,a)(m'(t,a))^p d\mu(t).\end{aligned}$$

The choice of the sequence (u_n) being arbitrary, we get, by the first relation in (3.49), that

$$\limsup_{u\to a}\frac{(\mathcal{T}_{\mu,\lambda,m}g_p)(u)}{g_p(u)} \le \int_T \lambda(t,a)(m'(t,a))^p d\mu(t) < 1,$$

which results the first inequality of (3.61). The proof of second inequality is similar.

Let $p \in]0,1[$. Consider, the function $G_p :]a,b[\to \mathbb{R}$ defined by

$$G_p(u) := \Big(\frac{\mathcal{T}_{\mu,\lambda,m}g_p}{g_p}\Big)(u) \qquad (u \in]a,b[).$$

Then, by Proposition 3.24, G_p is continuous on $]a,b[$ and, by (3.60), we have that $G_p(u) < 1$ for all $u \in]a,b[$. Using (3.61), we can find constants $0 < \delta < \frac{b-a}{2}$ and $q_0 \in [0,1[$ such that

$$G_p(u) \le q_0 \qquad \text{if} \quad u \in]a, a+\delta] \cup [b-\delta, b[.$$

Since G_p is continuous on $]a,b[$,

$$q_1 := \sup_{u\in[a+\delta,b-\delta]} G_p(u) < 1.$$

Thus,

$$(\mathcal{T}_{\mu,\lambda,m}g_p)(u) \le \max(q_0,q_1)\cdot g_p(u) \qquad (u \in]a,b[).$$

On the other hand, by (3.55) in Proposition 3.24, we also have that

$$(\mathcal{T}_{\mu,\lambda,m}g_p)(a) = g_p(a) = 0 \qquad \text{and} \qquad (\mathcal{T}_{\mu,\lambda,m}g_p)(b) = g_p(b) = 0.$$

Then using the previous observations, we get (3.58) with $q := \max(q_0,q_1)$. □

The following theorem is the main result of this section.

Theorem 3.27. (Makó–Páles [**46**])
Assume that (μ, λ, m) fulfills hypotheses (H1)–(H4) and define $\mathcal{T}_{\mu,\lambda,m}$ by (3.44)*. Then $\mathcal{T}_{\mu,\lambda,m} : B([a,b]) \to B([a,b])$ is a bounded positive linear operator with the properties (i)–(iii) listed in Proposition 3.24. Furthermore, for all bounded upper semicontinuous function $f : [a,b] \to \mathbb{R}$,*

$$\limsup_{n\to\infty}(\mathcal{T}^n_{\mu,\lambda,m}f)(u) \le f(a)\frac{\Omega(u,b)}{\Omega(a,b)} + f(b)\frac{\Omega(a,u)}{\Omega(a,b)} \qquad (u \in [a,b]). \tag{3.64}$$

If, in addition, f satisfies the inequality $f \le \mathcal{T}_{\mu,\lambda,m}f$, then

$$f(u) \le f(a)\frac{\Omega(u,b)}{\Omega(a,b)} + f(b)\frac{\Omega(a,u)}{\Omega(a,b)} \qquad (u \in [a,b]). \tag{3.65}$$

Proof. The first part of this theorem is a consequence of Proposition 3.24. By (3.46) in hypothesis (H4), we obviously have that ω_0 and ω_1 are the fixed points of $\mathcal{T}_{\mu,\lambda,m}$. Using also Theorem 3.26, we get that there exists an (ω_0, ω_1)-concave function $g \in C([a,b])$ such that $g(a) = g(b) = 0$ and (3.58) holds for some $q \in [0, 1[$. Hence Corollary 3.19 can be applied with $\mathcal{T} := \mathcal{T}_{\mu,\lambda,m}$ and we obtain (3.41) which is equivalent to (3.64). If, in addition, f satisfies the inequality $f \le \mathcal{T}_{\mu,\lambda,m}f$, then Proposition 3.22 can be applied and (3.65) follows. □

Corollary 3.28. (Makó–Páles [**46**])
Let μ be a Borel probability measure on $[0,1]$, denote $\mu_1 = \int_{[0,1]} t d\mu(t)$ and assume that the support of μ is not a singleton, i.e., $\mu \ne \delta_{\mu_1}$. Define the operator $\mathcal{T}_\mu : B([a,b]) \to B([a,b])$ by the following formula:

$$(\mathcal{T}_\mu f)(u) := \begin{cases} \int\limits_{[0,1]} f(\frac{t}{\mu_1}(u-a)+a)d\mu(t) & \text{if} \quad a \le u \le (1-\mu_1)a + \mu_1 b, \\ \int\limits_{[0,1]} f(\frac{1-t}{1-\mu_1}(u-b)+b)d\mu(t) & \text{if} \quad (1-\mu_1)a + \mu_1 b \le u \le b. \end{cases} \tag{3.66}$$

Then $\mathcal{T}_\mu : B([a,b]) \to B([a,b])$ is a bounded positive linear operator with $\|\mathcal{T}_\mu\| = 1$, $\mathcal{T}_\mu(C([a,b])) \subseteq C([a,b])$ and

$$\begin{gathered} (\mathcal{T}_\mu f)(a) = f(a), \quad (\mathcal{T}_\mu f)(b) = f(b), \\ \text{and} \quad \inf_{[a,b]} f \le (\mathcal{T}_\mu f)(u) \le \sup_{[a,b]} f \quad (u \in [a,b]) \end{gathered} \tag{3.67}$$

for all $f \in B([a,b])$. Furthermore, for all bounded upper semicontinuous function $f : [a,b] \to \mathbb{R}$,

$$\limsup_{n\to\infty}(\mathcal{T}^n_\mu f)(ta + (1-t)b) \le tf(a) + (1-t)f(b) \qquad (t \in [0,1]). \tag{3.68}$$

If, in addition, f satisfies the inequality $f \le \mathcal{T}f$, then

$$f(ta + (1-t)b) \le tf(a) + (1-t)f(b) \qquad (t \in [0,1]). \tag{3.69}$$

Proof. The measure μ is not a Dirac measure, therefore $0 < \mu_1 < 1$ and hence $\mathcal{T}_\mu$ is well-defined. Define $\lambda : [0,1] \times [a,b] \to \mathbb{R}_+$ and $m : [0,1] \times [a,b] \to [a,b]$ by the following way:

$$\lambda(t,u) = 1, \quad m(t,u) := \begin{cases} \frac{t}{\mu_1}(u-a)+a, & \text{if} \quad a \le u \le (1-\mu_1)a+\mu_1 b, \\ \frac{1-t}{1-\mu_1}(u-b)+b, & \text{if} \quad (1-\mu_1)a+\mu_1 b \le u \le b. \end{cases} \tag{3.70}$$

One can see that $\mathcal{T}_\mu$ defined in (3.66) equals $\mathcal{T}_{\lambda,\mu,m}$ defined in (3.44) with λ and μ from (3.70). In order to make Theorem 3.27 appliable, it is enough to show that the triplet (λ,μ,m) satisfies the conditions (H1)–(H4). It is obvious that (H1) and (H2) hold. Furthermore, the equalities $m(\cdot,a) = a$ and $m(\cdot,b) = b$ are also trivial. Observe that m defined in (3.70) is continuous in its first variable and differentiable in its second variable at every element $u \in [a,b] \setminus \{(1-\mu_1)a+\mu_1 b\}$ and

$$m'(t,u) = \begin{cases} \frac{t}{\mu_1} & \text{if} \quad a \le u < (1-\mu_1)a+\mu_1 b, \\ \frac{1-t}{1-\mu_1} & \text{if} \quad (1-\mu_1)a+\mu_1 b < u \le b. \end{cases}$$

On the other hand, for all $t \in [a,b]$

$$|m(t,u)-a| = \frac{t}{\mu_1}(u-a) \le \frac{1}{\mu_1}(u-a) \quad \text{if} \quad 0 \le u-a < \mu_1(b-a),$$

and

$$|m(t,u)-b| = \frac{1-t}{1-\mu_1}(b-a) \le \frac{1}{1-\mu_1}(b-a) \quad \text{if} \quad 0 \le b-u < (1-\mu_1)(b-a).$$

This shows that m is uniformly calm at the endpoints of $[a,b]$. (3.45) also holds, because if $u \notin \{a,b\}$, then $m(t,u) = u$ can hold if and only if $t = \mu_1$ and the support of μ is not the singleton μ_1. Similarly, if $u \in \{a,b\}$, then $m'(t,u) = 1$ can hold if and only if $t = \mu_1$, which yields the second condition in (3.45). To show (H4), let (ω_0,ω_1) be the standard Chebyshev system, i.e., $\omega_0(u) = 1$ and $\omega_0(u) = u$, if $u \in [a,b]$. Using that μ is a Borel probability measure and the definition of μ_1, we get that

$$1 = \int_{[0,1]} 1 d\mu(t) \qquad \text{and} \qquad u = \int_{[0,1]} m(t,u) d\mu(t) \qquad (u \in [a,b]),$$

which proves that ω_0 and ω_1 are fixed points of $\mathcal{T}_\mu$ and hence (H4) follows. Thus, all the hypotheses (H1)–(H4) hold and hence Theorem 3.27 can be applied.

Therefore, $\mathcal{T}_\mu$ is a bounded positive linear operator with the properties listed in the theorem, (3.67) can be checked directly. Using the formula $\Omega(x,y) = y-x$, and putting $u = ta+(1-t)b$ into (3.64) and (3.65), the inequalities (3.68) and (3.69) follow, respectively. □

Remark 3.29. (Makó–Páles [**46**])
The assumption in Corollary 3.28 that μ is not a Dirac measure is essential, because if

$\mu = \delta_\tau$ with $0 < \tau < 1$, then $\mu_1 = \tau$ and $\mathcal{T}_\mu$ is the identity operator on $B([a,b])$. Then, (3.68) and (3.69) cannot hold for all upper semicontinuous functions.

3.3. From lower Hermite–Hadamard inequalities to convexity type inequalities

Consider the approximate lower Hermite–Hadamard type functional inequality (3.7), for the unknown function $f : I \to \mathbb{R}$, where we make the following assumption:

(B1) $(T, \mathcal{A}, \mu)$ is a measure space.

(B2) $\Lambda : T\times\Delta(I) \to \mathbb{R}_+$ is measurable in its first variable and separately continuous in its second variable; furthermore, for all $(x,y) \in \Delta(I)$, the function $L_{x,y} : T \to \mathbb{R}_+$ defined by

$$L_{x,y}(t) := \sup_{u\in[x,y]} \max\left(\Lambda(t,x,u), \Lambda(t,u,y)\right)$$

is μ-integrable, i.e., $\int_T L_{x,y}(t)d\mu(t) < +\infty$.

(B3) $M : T \times \Delta(I) \to \mathbb{R}$ is measurable in its first variable and for all $t \in T$, the map $(x,y) \mapsto M(t,x,y)$ is separately continuous and partially differentiable at the diagonal of $I \times I$. $M_0 : \Delta(I) \to I$ is a separately continuous, strictly increasing and partially differentiable at the diagonal of $I \times I$ with $\partial_1 M_0(z,z) > 0$ and $\partial_2 M_0(z,z) > 0$ for all $z \in I$. Furthermore, $M(t,\cdot)$ is separately uniformly calm with respect to M_0 at the diagonal of $I\times I$, i.e., for all $z \in I$, there exist constants $\delta > 0$ and $K \geq 0$ such that, for all $t \in T$,

$$z - M(t,u,z) \leq K(z - M_0(u,z)) \qquad (u \in [z-\delta, z]),$$
$$M(t,z,u) - z \leq K(M_0(z,u) - z) \qquad (u \in [z, z+\delta]),$$

and (3.8) holds and, for all $z \in I$ and $i \in \{0,1\}$,

$$\mu\{t \in T \mid \Lambda(t,z,z) \neq 0,\ \partial_i M(t,z,z) \neq \partial_i M_0(z,z)\} > 0. \tag{3.71}$$

(B4) There exist functions $\omega_0, \omega_1 : I \to \mathbb{R}$ such that ω_0 is positive, $\frac{\omega_1}{\omega_0}$ is differentiable on I, with $(\frac{\omega_1}{\omega_0})' > 0$. Furthermore, for $i \in \{0,1\}$, (3.9) hold.

Theorem 3.30. (Makó–Páles [**46**])
Suppose that conditions (B1)–(B4) hold and assume that $f : I \to \mathbb{R}$ is an upper semicontinuous solution of the functional inequality (3.7)*, where $\mathcal{E} : \Delta(I) \to \mathbb{R}$ is a function. Assume that, for all $(x,y) \in \Delta^\circ(I)$, $\varepsilon_{x,y} : [x,y] \to \mathbb{R}$ is a lower semicontinuous function with $\varepsilon_{x,y}(x) = \varepsilon_{x,y}(y) = 0$ satisfying the following system of inequalities:*

$$\begin{aligned} \varepsilon_{x,y}(M_0(x,u)) &\geq \int_T \Lambda(t,x,u)\varepsilon_{x,y}(M(t,x,u))d\mu(t) + \mathcal{E}(x,u) \quad (u \in [x,y]),\\ \varepsilon_{x,y}(M_0(u,y)) &\geq \int_T \Lambda(t,u,y)\varepsilon_{x,y}(M(t,u,y))d\mu(t) + \mathcal{E}(u,y) \quad (u \in [x,y]). \end{aligned} \tag{3.72}$$

Then, for all fixed $(x, y) \in \Delta^\circ(I)$*, the function* f *also satisfies the approximate* (ω_0, ω_1)*-convexity inequality* (3.4).

Proof. To prove (3.4), let $(x, y) \in \Delta^\circ(I)$ be fixed. In what follows, we will apply Theorem 3.27 for functions defined on the interval $[a, b] := [x, y]$. We are going to construct the functions λ and m so that all hypotheses of Theorem 3.27 be satisfied.

By the separate continuity and strict increasingness of the mean M_0, the functions $u \mapsto M_0(u, y)$ and $u \mapsto M_0(x, u)$ are continuous, strictly increasing and map $[x, y]$ onto $[M_0(x, y), y]$ and $[x, M_0(x, y)]$, respectively. Therefore they are continuously invertible and hence there exist continuous functions $\alpha : [M_0(x, y), y] \to [x, y]$ and $\beta : [x, M_0(x, y)] \to [x, y]$ such that

$$M_0(\alpha(u), y) = u \quad (u \in [M_0(x, y), y]) \quad \text{and} \quad M_0(x, \beta(u)) = u \quad (u \in [x, M_0(x, y)]).$$

Define the functions $\lambda, m : T \times [x, y] \to \mathbb{R}$ and $\eta : [x, y] \to \mathbb{R}$ by the following formulas:

$$\lambda(t, u) := \lambda_{x,y}(t, u) := \begin{cases} \Lambda(t, x, \beta(u)) & \text{if } u \in [x, M_0(x, y)], \\ \Lambda(t, \alpha(u), y) & \text{if } u \in [M_0(x, y), y], \end{cases}$$

$$m(t, u) := m_{x,y}(t, u) := \begin{cases} M(t, x, \beta(u)) & \text{if } u \in [x, M_0(x, y)], \\ M(t, \alpha(u), y) & \text{if } u \in [M_0(x, y), y], \end{cases} \tag{3.73}$$

$$\eta(u) := \eta_{x,y}(u) := \begin{cases} \mathcal{E}(x, \beta(u)) & \text{if } u \in [x, M_0(x, y)], \\ \mathcal{E}(\alpha(u), y) & \text{if } u \in [M_0(x, y), y], \end{cases}$$

respectively. To see that λ, m and η are correctly defined, observe that $(x, \beta(u)) = (\alpha(u), y)$ for $u = M_0(x, y)$. By $\alpha(y) = y$ and $\beta(x) = x$, we can see that $m(t, x) = x$, $m(t, y) = y$, for all $t \in T$.

It immediately follows from the definition of λ and m and from the measurability and separate continuity assumptions in (B2) and (B3) that λ and m are measurable in their first variable and continuous in their second variable. Using (B2), we can also see that λ fulfills the integrability condition in hypothesis (H2) with $\ell(t) := L_{x,y}(t)$.

Applying the partial differentiability assumptions of (B3) at $z = x$ and $z = y$ and the chain rule and elementary calculus rule for the differentiation of real inverse functions, it follows that, for all $t \in T$, m is differentiable at the endpoints of $[x, y]$ with

$$m'(t, x) = \frac{\partial_2 M(t, x, x)}{\partial_2 M_0(x, x)} \qquad \text{and} \qquad m'(t, y) = \frac{\partial_1 M(t, y, y)}{\partial_2 M_0(y, y)}. \tag{3.74}$$

To prove the uniform calmness property of m, we apply the uniform calmness of M with respect to M_0 at $z = x$ and $z = y$ from (B3). Then there exist constants

$0 < \delta \le y - x$ and $K \ge 0$ such that

$$y - M(t, v, y) \le K(y - M_0(v, y)) \qquad (v \in [y - \delta, y]),$$
$$M(t, x, v) - x \le K(M_0(x, v) - x) \qquad (v \in [x, x + \delta]).$$

Substituting $v = \alpha(u)$ and $v = \beta(u)$ into these inequalities, we get

$$y - m(t, u) \le K(y - u) \qquad (u \in [M_0(y - \delta, y), y]),$$
$$m(t, u) - x \le K(u - x) \qquad (u \in [x, M_0(x, x + \delta)]).$$

Choose $\zeta := \min(M_0(x, x + \delta) - x, y - M_0(y - \delta, y))$. Then, $\zeta > 0$ and

$$[x, x + \zeta] \subseteq [x, M_0(x, x + \delta)] \quad \text{and} \quad [y - \zeta, y] \subseteq [M_0(y - \delta, y), y].$$

Then, we obtain that

$$y - m(t, u) \le K(y - u) \qquad (u \in [y - \zeta, y]),$$
$$m(t, u) - x \le K(u - x) \qquad (u \in [x, x + \zeta]),$$

which yields the uniform calmness of m at the endpoints of $[a, b]$.

To check condition (3.8), let $u \in]x, y[$. In the case $u \le M_0(x, y)$ the equalities $\lambda(t, u) = 0$ and $m(t, u) = u$ can hold if and only if $\Lambda(t, v) = 0$ and $M(t, x, v) = M_0(x, v)$ are valid with $v = \beta(u)$, respectively. Therefore, applying the first inequality for $(v, y) \in \Delta^\circ(I)$, we get (3.8) is satisfied. The case $M_0(x, y) \le u$ is analogous.

Using formulas (3.74), it is completely similar to show that the condition (3.45) in (H3) is a consequence of inequality in (3.71).

By condition (B4), for $i \in \{0, 1\}$, we have that

$$\omega_i(M_0(x, v)) = \int_T \Lambda(t, x, v)\omega_i(M(t, x, v))d\mu(t) \qquad (v \in [x, y]),$$
$$\omega_i(M_0(v, y)) = \int_T \Lambda(t, v, y)\omega_i(M(t, v, y))d\mu(t) \qquad (v \in [x, y]).$$

Substituting $v = \beta(u)$ and $v = \alpha(u)$ into the above identities, respectively, we get that

$$\omega_i(u) = \int_T \lambda(t, u)\omega_i(m(t, u))d\mu(t) \qquad (u \in [x, y]).$$

This proves that hypothesis (H4) is also satisfied.

Thus, we have verified that all the hypotheses (H1)–(H4) of Theorem 3.27 are fulfilled.

Now define the operator $\mathcal{T}_{\mu,\lambda,m} : B([x,y]) \to B([x,y])$ by (3.44). Using (3.7), we have that

$$
\begin{aligned}
f(M_0(x,v)) &\le \int_T \Lambda(t,x,v) f(M(t,x,v)) d\mu(t) + \mathcal{E}(x,v) \quad (v \in [x,y]),\\
f(M_0(v,y)) &\le \int_T \Lambda(t,v,y) f(M(t,v,y)) d\mu(t) + \mathcal{E}(v,y) \quad (v \in [x,y]).
\end{aligned} \tag{3.75}
$$

The function f is upper semicontinuous, therefore, the restriction $f|_{[x,y]}$ is bounded from above (but maybe unbounded from below). Thus the integrals on the right hand side of (3.75) do exist and, by the inequality (3.75), are also finite for all $v \in [x,y]$. With the substitutions $v = \beta(u)$ and $v = \alpha(u)$, respectively, we get that

$$f(u) \le \int_T \lambda(t,u) f(m(t,u)) d\mu(t) + \eta(u) \qquad (u \in [x,y]).$$

Therefore, we obtain that $f|_{[x,y]}$ satisfies the following inequality

$$f|_{[x,y]} \le \mathcal{T}_{\mu,\lambda,m}(f|_{[x,y]}) + \eta. \tag{3.76}$$

Similarly, due to the separate lower semicontinuity of $\varepsilon_{x,y}$, the integrals on the right hand side of (3.72) exist and are also finite. Thus, replacing u by $\beta(u)$ and $\alpha(u)$ in inequality (3.72), respectively, it follows that

$$\varepsilon_{x,y} \ge \mathcal{T}_{\mu,\lambda,m}(\varepsilon_{x,y}) + \eta. \tag{3.77}$$

Combining (3.76) and (3.77), we get that

$$f|_{[x,y]} - \varepsilon_{x,y} \le \mathcal{T}_{\mu,\lambda,m}(f|_{[x,y]} - \varepsilon_{x,y}). \tag{3.78}$$

By our semicontinuity assumptions on f and $\varepsilon_{x,y}$, the function $h := f|_{[x,y]} - \varepsilon_{x,y}$ is upper semicontinuous, and (3.78) is equivalent to the inequality $h \le \mathcal{T}_{\mu,\lambda,m}h$. In order to make Theorem 3.27 appliable, we approximate h by a sequence of upper semicontinuous and bounded functions. For $n \in \mathbb{N}$, define

$$h_n := \max(h, (-n)\omega_0|_{[x,y]}).$$

Then, one can see that (h_n) is a sequence of upper semicontinuous functions which pointwise converges to h. On the other hand,

$$(-n)\omega_0|_{[x,y]} = \mathcal{T}_{\mu,\lambda,m}(-n\omega_0|_{[x,y]}) \le \mathcal{T}_{\mu,\lambda,m}h_n \quad \text{and} \quad h \le \mathcal{T}_{\mu,\lambda,m}h \le \mathcal{T}_{\mu,\lambda,m}h_n.$$

Therefore, we get that

$$h_n \le \mathcal{T}_{\mu,\lambda,m}h_n \qquad (n \in \mathbb{N}).$$

Now, applying the last statement of Theorem 3.27, for $u \in [x,y]$, we obtain that

$$h_n(u) \le h_n(x)\frac{\Omega(u,y)}{\Omega(x,y)} + h_n(y)\frac{\Omega(x,u)}{\Omega(x,y)} \qquad (n \in \mathbb{N}).$$

Upon taking the limit $n \to \infty$, it follows that

$$f(u) - \varepsilon_{x,y}(u) = h(u) \le h(x)\frac{\Omega(u,y)}{\Omega(x,y)} + h(y)\frac{\Omega(x,u)}{\Omega(x,y)} = f(x)\frac{\Omega(u,y)}{\Omega(x,y)} + f(y)\frac{\Omega(x,u)}{\Omega(x,y)},$$

which results (3.4). □

Proposition 3.31. (Makó–Páles [**46**])
Suppose that conditions (B1)–(B4) hold and, in addition, $\mathcal{E} : \Delta(I) \to \mathbb{R}_+$ is a nonnegative separately lower semicontinuous function with $\mathcal{E}(z,z) = 0$, for all $z \in I$. Then the following three statements are equivalent.

(i) For all $(x,y) \in \Delta^\circ(I)$, there exists a nonnegative bounded lower semicontinuous function $\varepsilon^*_{x,y} : [x,y] \to \mathbb{R}$ satisfying $\varepsilon^*_{x,y}(x) = \varepsilon^*_{x,y}(y) = 0$ and condition (3.72) *with equality.*

(ii) For all $(x,y) \in \Delta^\circ(I)$, there exists a bounded lower semicontinuous function $\varepsilon_{x,y} : [x,y] \to \mathbb{R}$ satisfying $\varepsilon_{x,y}(x) = \varepsilon_{x,y}(y) = 0$ and inequality (3.72).

(iii) For all $(x,y) \in \Delta^\circ(I)$, the series

$$\varepsilon^*_{x,y} := \sum_{n=0}^{\infty} \mathcal{T}^n_{\mu,\lambda_{x,y},m_{x,y}} \eta_{x,y}$$

is pointwise convergent and bounded on $[x,y]$, where the functions $\lambda_{x,y}, m_{x,y}$, and $\eta_{x,y}$ are defined by (3.73).

Furthermore, if the above equivalent conditions are satisfied, then, for all $(x,y) \in \Delta^\circ(I)$, the function $\varepsilon^*_{x,y}$ is bounded lower semicontinuous and satisfies $\varepsilon^*_{x,y}(x) = \varepsilon^*_{x,y}(y) = 0$ and condition (3.72) *with equality.*

Proof. The implication from (i) to (ii) holds trivially.

Consider the implication from (ii) to (iii). Let $(x,y) \in \Delta^\circ(I)$ and assume that there exists a bounded lower semicontinuous solution $\varepsilon_{x,y} : [x,y] \to \mathbb{R}$ of (3.72) satisfying $\varepsilon_{x,y}(x) = \varepsilon_{x,y}(y) = 0$. Define the function $\eta_{x,y} : [x,y] \to \mathbb{R}$ by (3.73). Since E is nonnegative, separately bounded and separately lower semicontinuous, we have that $\eta_{x,y}$ is nonnegative, bounded and lower semicontinuous. As we have seen in the proof of Theorem 3.30, the functions $\eta_{x,y}$ and $\varepsilon_{x,y}$ satisfy the functional inequality (3.77) and hence

$$\eta_{x,y} \le \varepsilon_{x,y} - \mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}} \varepsilon_{x,y}.$$

Applying the positivity of $\mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}}$, we get by induction on $i \in \mathbb{N}$, that

$$\mathcal{T}^{i-1}_{\mu,\lambda_{x,y},m_{x,y}} \eta_{x,y} \le \mathcal{T}^{i-1}_{\mu,\lambda,m} \varepsilon_{x,y} - \mathcal{T}^{i}_{\mu,\lambda,m} \varepsilon_{x,y}.$$

Summing these equalities from 1 to n, we get

$$\sum_{i=1}^{n} \mathcal{T}^{i-1}_{\mu,\lambda_{x,y},m_{x,y}} \eta_{x,y} \le \varepsilon_{x,y} - \mathcal{T}^{n}_{\mu,\lambda,m} \varepsilon_{x,y} \le \varepsilon_{x,y} \quad (n \in \mathbb{N}). \tag{3.79}$$

Hence, by the nonnegativity of $\eta_{x,y}$, the series $\sum_{i=1}^{\infty} \mathcal{T}^i_{\mu,\lambda_{x,y},m_{x,y}} \eta_{x,y}$ is pointwise convergent and, by (3.79), is also bounded on $[x,y]$. This proves (iii).

Finally, assume that (iii) holds. By the lower semicontinuity and nonnegativity assumptions, the series $\sum_{i=1}^{\infty} \mathcal{T}^i_{\mu,\lambda_{x,y},m_{x,y}} \eta_{x,y}$ has nonnegative and lower semicontinuous terms. Therefore, its sum, denoted by $\varepsilon^*_{x,y}$ is also a lower semicontinuous function on $[x,y]$. To complete the proof, we need to show that $\varepsilon^*_{x,y}$ satisfies (3.72) with equality. Indeed,

$$\sum_{i=0}^{n} \mathcal{T}^i_{\mu,\lambda_{x,y},m_{x,y}} \eta_{x,y} = \eta_{x,y} + \mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}} \Big(\sum_{i=0}^{n-1} \mathcal{T}^i_{\mu,\lambda_{x,y},m_{x,y}} \eta_{x,y} \Big).$$

Taking the limit $n \to \infty$ and applying the pointwise convergence property of the operator $\mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}}$, we get (i). □

Now we consider the most important particular cases of Theorem 3.30.

Corollary 3.32. (Makó–Páles [**46**])
Suppose that conditions (B1)–(B4) hold and let $f : I \to \mathbb{R}$ be an upper semicontinuous function. Then f is a solution of the functional inequality

$$f(M_0(x,y)) \le \int_T \Lambda(t,x,y) f(M(t,x,y)) d\mu(t) \qquad ((x,y) \in \Delta(I)) \tag{3.80}$$

if and only if f is (ω_0,ω_1)-convex.

Proof. Assume that $f : I \to \mathbb{R}$ is an upper semicontinuous solution of the functional inequality (3.80). Then f also fulfills (3.7) with $E \equiv 0$. Then, for all $(x,y) \in \Delta(I)$, $\varepsilon_{x,y} : [x,y] \to \mathbb{R}$ defined by $\varepsilon_{x,y} \equiv 0$ is a lower semicontinuous solution of (3.72). Hence Theorem 3.30 can be applied, which results (3.4) holds $\varepsilon_{x,y} \equiv 0$. Thus f is (ω_0,ω_1)-convex.

Conversely, assume that $f : I \to \mathbb{R}$ is (ω_0,ω_1)-convex. Then $f|_{[x,y]}$ is upper semicontinuous and bounded on $[x,y]$. The inequality (3.80) trivially holds if $x = y \in I$. Let $(x,y) \in \Delta^\circ(I)$. Define the triplet $(\mu, \lambda_{x,y}, m_{x,y})$ by (3.73) as in the proof of Theorem 3.30. Then, by Proposition 3.24 we have that $(\mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}} f)(x) = f(x)$ and $(\mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}} f)(y) = f(y)$. On the other hand, $\mathcal{T} := \mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}}$ is a positive linear operator with fixed points $\omega_0|_{[x,y]}$ and $\omega_1|_{[x,y]}$. Hence, by the (ω_0,ω_1)-convexity of f and Proposition 3.21, we get

$$f|_{[x,y]} \le (\mathcal{T}_{\mu,\lambda_{x,y},m_{x,y}} f|_{[x,y]}),$$

which means that

$$f(u) \le \int_T \lambda_{x,y}(t,u) f(m_{x,y}(t,u)) d\mu(t) \qquad (u \in [x,y]).$$

Substituting $u = M_0(x, y)$ into the above inequality and using the definition of $\lambda_{x,y}$ and $m_{x,y}$, we get (3.80), which completes the proof. □

Remark 3.33. (Makó–Páles [**46**])
A direct consequence of this corollary is the reversed implication of Theorem 3.2, with a stronger assumptions. Indeed, suppose that, with the notations introduced in (3.5), the assumptions of Theorem 3.2 hold and ω_0 is positive and (ω_1/ω_0) is differentiable. We show first that the conditions of Corollary 3.32 are also valid. Let μ denote the Lebesgue measure on $[0, 1]$ and define, for all $(x, y) \in \Delta^\circ(I)$ and $t \in [0, 1]$,

$$M_0(x, y) := \xi(x, y), \quad M(t, x, y) := (1 - t)x + ty,$$
$$\text{and} \quad \Lambda(t, x, y) := \frac{(y - x)\rho((1 - t)x + ty)}{c(x, y)}.$$

It can be also seen that (B3) holds. Similar as in Remark 3.9 we also have that (3.9) hold. Thus all the assumptions (B1)–(B4) are verified. Therefore, if a function $f : I \to \mathbb{R}$ satisfies (3.6), then it is (ω_0, ω_1)-convex.

Theorem 3.34. (Makó–Páles [**46**])
Let μ be a Borel probability measure on $[0, 1]$, denote $\mu_1 := \int_{[0,1]} t d\mu(t)$ and assume that the support of μ is not a singleton, i.e., $\mu \neq \delta_{\mu_1}$. Assume that, for all $(x, y) \in D^2$, $f : D \to \mathbb{R}$ is an upper hemicontinuous solution of the functional inequality (3.14)*, where $E : D^2 \to \mathbb{R}$. Assume that, for all $(x, y) \in D^2$, $e_{x,y} : [0, 1] \to \mathbb{R}$ is a lower semicontinuous function with $e_{x,y}(0) = e_{x,y}(1) = 0$ satisfying the following system of inequalities:*

(3.81)
$$e_{x,y}(s) \geq \begin{cases} \int\limits_{[0,1]} e_{x,y}(\frac{st}{\mu_1})d\mu(t) + E(x, (1 - \frac{s}{\mu_1})x + \frac{s}{\mu_1}y) & (s \in [0, \mu_1]), \\ \int\limits_{[0,1]} e_{x,y}(1 - \frac{(1-s)(1-t)}{1-\mu_1})d\mu(t) + E(\frac{1-s}{1-\mu_1}x + (1 - \frac{1-s}{1-\mu_1})y, y) & (s \in [\mu_1, 1]). \end{cases}$$

Then, for all $(x, y) \in D^2$ and $t \in [0, 1]$, the function f also satisfies the approximate convexity inequality (3.13)*.*

Proof. Assume that (3.14) holds. Let (x, y) be arbitrarily fixed elements of D^2. Substituting x by $ux + (1 - u)y$ and y by $vx + (1 - v)y$, where $u \leq v$ from $[0, 1]$, we get that

$$\begin{aligned} &f(((1 - \mu_1)u + \mu_1 v)x + (1 - (1 - \mu_1)u - \mu_1 v)y) \\ &\qquad \leq \int\limits_{[0,1]} f(((1 - t)u + tv)x + (1 - (1 - t)u - tv)y)d\mu(t) \\ &\qquad\qquad + E(ux + (1 - u)y, vx + (1 - v)y). \end{aligned} \tag{3.82}$$

Define $g := g_{x,y} : [0,1] \to \mathbb{R}$ by the following way

$$g(u) := g_{x,y}(u) = f(ux + (1-u)y) \quad (u \in [0,1])$$

and

$$F(u,v) := E(ux + (1-u)y, vx + (1-v)y) \quad ((u,v) \in \Delta([0,1])).$$

Then, (3.82) reduces to

$$g((1-\mu_1)u + \mu_1 v) \le \int\limits_{[0,1]} g((1-t)u + tv)d\mu(t) + F(u,v) \quad ((u,v) \in \Delta([0,1])).$$

Define the functions $\Lambda : [0,1] \times \Delta([0,1]) \to \mathbb{R}_+$, $M : [0,1] \times \Delta([0,1]) \to [0,1]$ and $M_0 : \Delta([0,1]) \to [0,1]$ as follows:

$$\Lambda(t,u,v) := 1, \quad M(t,u,v) := (1-t)u + tv \quad \text{and} \quad M_0(u,v) := (1-\mu_1)u + \mu_1 v. \tag{3.83}$$

In order to make Theorem 3.30 appliable for the function g, it is enough to show that Λ, μ, M, and M_0 satisfy the conditions (B1)–(B4). It is obvious that (B1) and (B2) hold. Observe that M defined in (3.83) is measurable in its first variable and, for all $t \in [0,1]$, the map $(u,v) \mapsto M(t,u,v)$ is a two-variable mean on $[0,1]$, which is separately continuous and partially differentiable at the diagonal of $[0,1] \times [0,1]$, i.e., for all $t \in [0,1]$

$$\partial_1 M(t,z,z) = 1 - t \qquad \text{and} \qquad \partial_2 M(t,z,z) = t \qquad (z \in [0,1]).$$

The function $M_0 : \Delta([0,1]) \to [0,1]$ is also separately continuous, strictly increasing and partially differentiable at the diagonal of $[0,1] \times [0,1]$, i.e.,

$$\partial_1 M_0(z,z) = 1 - \mu_1 \qquad \text{and} \qquad \partial_2 M_0(z,z) = \mu_1 \qquad (z \in [0,1]).$$

Observe that these derivatives are positive for all $z \in [0,1]$. The uniform calmness of M with respect to M_0 also holds with $K := \max\left(\frac{1}{\mu_1}, \frac{1}{1-\mu_1}\right)$. The equalities

$$M(t,u,v) = M_0(u,v) \quad ((u,v) \in \Delta^\circ([0,1])) \quad \text{and} \quad \partial_i M(t,z,z) = \partial_i M_0(z,z) \quad (z \in [0,1])$$

can hold if and only if $t = \mu_1$, hence $\mu \neq \delta_{\mu_1}$ yields that (3.45) holds. To show (B4), let (ω_0, ω_1) be the standard Chebyshev system, i.e., $\omega_0(u) = 1$ and $\omega_0(u) = u$, if $u \in [0,1]$. Using that μ is a Borel probability measure and the definition of μ_1, we get that

$$1 = \int\limits_{[0,1]} 1 d\mu(t), \quad (1-\mu_1)u + \mu_1 v = \int\limits_{[0,1]} (1-t)u + tv d\mu(t) \quad ((u,v) \in \Delta([0,1])),$$

which proves that ω_0 and ω_1 fulfill (3.9) and hence (B4) follows.

Let $(u,v) \in \Delta^\circ([0,1])$ be a fixed element and define $\varepsilon_{u,v} : [u,v] \to \mathbb{R}$ in the following way:

$$\varepsilon_{u,v}(w) = e_{ux+(1-u)y, vx+(1-v)y}\left(\tfrac{w-u}{v-u}\right) \qquad (w \in [u,v]).$$

Then the definition of $\varepsilon_{u,v}$ is correct and by the lower semicontinuity of the function $e_{ux+(1-u)y,vx+(1-v)y}$, $\varepsilon_{u,v}$ is also lower semicontinuous. We show that $\varepsilon_{u,v}$ is a solution of (3.72). Using (3.81) and the previous notations, for $w \in [u, v]$, we have that

$$\begin{aligned}\varepsilon_{u,v}(M_0(u,w)) &= \varepsilon_{u,v}((1-\mu_1)u + \mu_1 w) = e_{ux+(1-u)y,vx+(1-v)y}\big(\mu_1 \tfrac{w-u}{v-u}\big)\\ &\geq \int\limits_{[0,1]} e_{ux+(1-u)y,vx+(1-v)y}\big(t\tfrac{w-u}{v-u}\big)d\mu(t) + E(ux+(1-u)y, wx+(1-w)y)\\ &= \int\limits_{[0,1]} \varepsilon_{u,v}((1-t)u+tw)d\mu(t) + F(u,w) = \int\limits_{[0,1]} \varepsilon_{u,v}(M(t,u,w))d\mu(t) + F(u,w)\end{aligned}$$

and

$$\begin{aligned}&\varepsilon_{u,v}(M_0(w,v)) = \varepsilon_{u,v}((1-\mu_1)w + \mu_1 v) = e_{ux+(1-u)y,vx+(1-v)y}\big(1-(1-\mu_1)(1-\tfrac{w-u}{v-u})\big)\\ &\geq \int\limits_{[0,1]} e_{ux+(1-u)y,vx+(1-v)y}\big(1-(1-t)(1-\tfrac{w-u}{v-u})\big)d\mu(t) + E(wx+(1-w)y, vx+(1-v)y)\\ &= \int\limits_{[0,1]} \varepsilon_{u,v}((1-t)w+tv)d\mu(t) + F(w,v) = \int\limits_{[0,1]} \varepsilon_{u,v}(M(t,w,v))d\mu(t) + F(w,v).\end{aligned}$$

Then, $\varepsilon_{u,v}$ satisfies the condition (3.72) in Theorem 3.30. On the other hand $\varepsilon_{u,v}(u) = e_{ux+(1-u)y,vx+(1-v)y}(0) = 0$ and $\varepsilon_{u,v}(v) = e_{ux+(1-u)y,vx+(1-v)y}(1) = 0$. Thus, we can apply Theorem 3.30 for the function g, which yields that for all $s \in [u, v]$,

$$g(s) \leq \frac{v-s}{v-u}g(u) + \frac{s-u}{v-u}g(v) + \varepsilon_{u,v}(s).$$

Substituting $u = 0$ and $v = 1$, we get that, for all $s \in [0, 1]$,

$$g(s) = f(sx+(1-s)y) \leq (1-s)g(0) + sg(1) + \varepsilon_{0,1}(s) = (1-s)f(y) + sf(x) + e_{x,y}(s),$$

which means that (3.13) holds. □

The following corollary is a generalization of the result of [**34**] recalled in Theorem 3.4.

Corollary 3.35. (Makó–Páles [**46**])
Let $\beta : D^* \to \mathbb{R}$ and assume that $f : D \to \mathbb{R}$ is an upper hemicontinuous function satisfying the approximate lower Hermite–Hadamard inequality (3.14)*. Then, for all $x, y \in D$ and $s \in [0, 1]$, f satisfies the approximate convexity inequality*

$$f((1-s)x+sy) \leq (1-s)f(x) + sf(y) + \alpha(\min(2s, 2(1-s))(x-y)), \tag{3.84}$$

where $\alpha : D^* \to \mathbb{R}$ is a radially lower semicontinuous solution of the functional inequality

$$\alpha(su) \geq \frac{1}{s}\int_0^s \alpha(\min(2t, 2(1-t))u)dt + \beta(su) \quad (u \in D^*,\ s \in]0,1]) \tag{3.85}$$

with $\alpha(0) = 0$.

Proof. We apply Theorem 3.34, when μ is the Lebesgue measure on $[0,1]$. Then we obviously have that $\mu_1 = \int_0^1 t dt = \frac{1}{2}$. Then, it can be seen that (3.81) is equivalent to the following system of inequalities:

$$(3.86) \qquad e_{x,y}(s) \geq \begin{cases} \frac{1}{2s}\int_0^{2s} e_{x,y}(t)dt + E(x,(1-2s)x+2sy) & (s \in]0,\frac{1}{2}]), \\ \frac{1}{2-2s}\int_{2s-1}^{1} e_{x,y}(t)dt + E((2-2s)x+(2s-1)y,y) & (]\frac{1}{2},1[). \end{cases}$$

Let $\beta : D^* \to \mathbb{R}$ be an even function and $\alpha : D^* \to \mathbb{R}_+$ be a lower semicontinuous solution of (3.85). Define $E : D^2 \to \mathbb{R}$ and, for all $x, y \in D$, $e_{x,y} : [0,1] \to \mathbb{R}$ by the following formulas:

$$E(x,y) := \beta(x-y) \qquad (x,y \in D),$$
$$e_{x,y}(s) := \alpha(\min(2s, 2(1-s))(x-y)) \qquad (s \in [0,1]).$$

We show that E and $e_{x,y}$ satisfy (3.86). First, let $s \in]0,\frac{1}{2}]$, then

$$\begin{aligned} e_{x,y}(s) = \alpha(2s(x-y)) &\geq \frac{1}{2s}\int_0^{2s} \alpha(\min(2t, 2(1-t))(x-y))dt + \beta(2s(x-y)) \\ &= \frac{1}{2s}\int_0^{2s} e_{x,y}(t)dt + E(x,(1-2s)x+2sy). \end{aligned}$$

On the other hand, if $s \in]\frac{1}{2},1[$, we obtain that

$$\begin{aligned} e_{x,y}(s) &= \alpha(2(1-s)(x-y)) \\ &\geq \frac{1}{2(1-s)}\int_0^{2(1-s)} \alpha(\min(2t, 2(1-t))(x-y))dt + \beta(2(1-s)(x-y)) \\ &= \frac{1}{2(1-s)}\int_{2s-1}^{1} \alpha(\min(2t, 2(1-t))(x-y))dt + \beta(2(1-s)(x-y)) \\ &= \frac{1}{2-2s}\int_{2s-1}^{1} e_{x,y}(t)dt + E((2-2s)x+(2s-1)y,y). \end{aligned}$$

Hence (3.86) holds, which means that Theorem 3.34 can be applied. Thus, f satisfies (3.13), which is equivalent to (3.84). □

Remark 3.36. (Makó–Páles [**46**])
In what follows, we deduce the conclusion of Theorem 3.4 from the above corollary. Let f be an upper hemicontinuous solution of (3.7) and assume that $\beta : D^* \to \mathbb{R}$ is nonnegative and even. Let $\alpha : 2D^* \to \mathbb{R}$ be a nonnegative radially increasing, radially lower semicontinuous solution of (3.12). First, we show that the assumption (3.85) of

Corollary 3.35 hold for α and β. Indeed, for $u \in D^*$ and $s \in]0,1]$, using (3.12), we get

$$\alpha(su) \geq \int_0^1 \alpha(2tsu)dt + \beta(su) = \frac{1}{s}\int_0^s \alpha(2\tau u)d\tau + \beta(su)$$
$$\geq \frac{1}{s}\int_0^s \alpha(\min(2\tau, 2(1-\tau))u)d\tau + \beta(su),$$

which means that (3.85) holds. Thus, by Corollary 3.35, we get that (3.84) holds. This, with the substitution $s = \frac{1}{2}$, implies the α-Jensen convexity of f.

The following result is related to the main result of the paper [**33**, Theorem 6].

Corollary 3.37. (Makó–Páles [**46**])
Let $\tau \in]0,1[$ and $E : D^2 \to \mathbb{R}$. Assume that $f : D \to \mathbb{R}$ is an upper hemicontinuous solution of the functional inequality

$$f((1-\tau)x + \tau y) \leq (1-\tau)f(x) + \tau f(y) + E(x,y) \qquad ((x,y) \in D^2). \tag{3.87}$$

Assume that, for all $x, y \in D$, $e_{x,y} : [0,1] \to \mathbb{R}$ is a lower semicontinuous function with $e_{x,y}(0) = e_{x,y}(1) = 0$, satisfying the following system of inequalities:

$$e_{x,y}(s) \geq \begin{cases} \tau e_{x,y}(\frac{s}{\tau}) + E(x, (1-\frac{s}{\tau})x + \frac{s}{\tau}y) & (s \in [0,\tau]), \\ (1-\tau)e_{x,y}(\frac{s-\tau}{1-\tau}) + E(\frac{1-s}{1-\tau}x + (1-\frac{1-s}{1-\tau})y, y) & (s \in [\tau,1]). \end{cases} \tag{3.88}$$

Then, for all $x, y \in D$ and $t \in [0,1]$, the function f also satisfies the approximate convexity inequality (3.13).

Proof. We apply Theorem 3.34, when μ is a measure on $[0,1]$ defined by $\mu := (1-\tau)\delta_0 + \tau\delta_1$. Then we obviously have that $\mu_1 = \int_{[0,1]} t d\mu(t) = \tau$. Now we can see that (3.87) and (3.88) are equivalent to (3.14) and (3.81), respectively. □

Remark 3.38. (Makó–Páles [**46**])
In what follows, we deduce the conclusion of Theorem 3.5 from the above corollary under stronger regularity assumption on α and f. Let $\alpha : D^* \to \mathbb{R}_+$ be a nonnegative, even and radially lower semicontinuous function with $\alpha(0) = 0$ and assume that $f : D \to \mathbb{R}$ is upper hemicontinuous and α-Jensen convex in a sense of (5). For all $x, y \in D$, define $E : D^2 \to \mathbb{R}$ and $e_{x,y} : [0,1] \to \mathbb{R}$, by

$$E(x,y) := \alpha(x-y) \quad \text{and} \quad e_{x,y}(s) := \sum_{n=0}^{\infty} \frac{1}{2^n}\alpha(d_{\mathbb{Z}}(2^n s)(x-y)) \quad (s \in [0,1]).$$

Then, f is also a solution of (3.87) with $\tau = \frac{1}{2}$. By the nonnegativity and radial lower semicontinuity of α, for all $x, y \in D$, we get that $e_{x,y}$ is lower semicontinuous on $[0,1]$.

Obviously, $e_{x,y}(0) = e_{x,y}(1) = 0$. We show that $e_{x,y}$ satisfies the functional equation (3.88) with equality, which can equivalently be written as

$$
(3.89)\qquad e_{x,y}(s) = \begin{cases} \frac{1}{2}e_{x,y}(2s) + \alpha(2s(x-y)) & \text{if} \quad s \in [0, \frac{1}{2}], \\ \frac{1}{2}e_{x,y}(2s-1) + \alpha((2-2s)(x-y)) & \text{if} \quad s \in [\frac{1}{2}, 1]. \end{cases}
$$

Let $s \in [0, \frac{1}{2}]$, then

$$
\begin{aligned}
e_{x,y}(s) &= \alpha(2s(x-y)) + \sum_{n=1}^{\infty} \frac{1}{2^n}\alpha(d_{\mathbb{Z}}(2^n s)(x-y)) \\
&= \alpha(2s(x-y)) + \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{2^n}\alpha(d_{\mathbb{Z}}(2^n 2s)(x-y)) = \alpha(2s(x-y)) + \frac{1}{2}e_{x,y}(2s).
\end{aligned}
$$

The case $s \in [\frac{1}{2}, 1]$ is similar. Thus, Corollary 3.37 yields that f is approximately convex in the sense of (3.13) which is equivalent to (3.15).

In what follows, we examine the case, when X is a normed space and E is a linear combination of the powers of the norm with positive exponents, i.e., if E is of the form

$$
(3.90)\qquad E(x,y) := \int_{]0,1[} \|x-y\|^q d\nu(q) \qquad (x, y \in D),
$$

where ν is a Borel measure on the open interval $]0, 1[$. An important particular case is when ν is of the form $\sum_{i=1}^{k} c_i\delta_{q_i}$, where $c_1, \dots, c_k > 0$, $q_1, \dots, q_k \in]0, 1[$.

Theorem 3.39. (Makó–Páles [**46**])
Let μ be a Borel probability measure on $[0, 1]$, denote $\mu_1 := \int_{[0,1]} t d\mu(t)$ and assume that the support of μ is not a singleton, i.e., $\mu \neq \delta_{\mu_1}$. Let ν be a Borel measure on $]0, 1[$ such that

$$
(3.91)\qquad \int_{]0,1[} \gamma(q) d\nu(q) < \infty,
$$

where

$$
\gamma(q) := \max\left(\frac{\mu_1^q}{\mu_1^q - \int\limits_{[0,1]} \tau^q d\mu(\tau)}, \frac{(1-\mu_1)^q}{(1-\mu_1)^q - \int\limits_{[0,1]} (1-\tau)^q d\mu(\tau)}\right) \qquad (q \in]0, 1[).
$$

Assume that, for all $(x, y) \in D^2$, $f : D \to \mathbb{R}$ is an upper hemicontinuous solution of the functional inequality

$$
(3.92)\qquad f((1-\mu_1)x + \mu_1 y) \leq \int_{[0,1]} f((1-t)x + ty) d\mu(t) + \int_{]0,1[} \|x-y\|^q d\nu(q).
$$

Then, for all $x, y \in D$*,* f *also fulfills the convexity type inequality* (3.13) *where, for all* $x, y \in D$*,* $e_{x,y} : [0,1] \to \mathbb{R}$ *is defined by*

$$e_{x,y}(s) := \int_{]0,1[} \gamma(q) \min\Big(\big(\tfrac{s}{\mu_1}\big)^q, \big(\tfrac{1-s}{1-\mu_1}\big)^q\Big) \|x-y\|^q d\nu(q) \quad (s \in [0,1]). \tag{3.93}$$

Proof. Let E be defined by (3.90). Then (3.92) is equivalent to (3.14). To deduce (3.13) using Theorem 3.34, it suffices to show that, for all $x, y \in D$, the function $e_{x,y}$ defined by (3.93) satisfies (3.81). Observe that, for all $x, y \in D$, $e_{x,y}$ is finite-valued and continuous since (3.91) holds. It is also obvious that $e_{x,y}(0) = e_{x,y}(1) = 0$. For all $q \in]0,1[$, let $\phi_q : [0,1] \to \mathbb{R}$ be defined by

$$\phi_q(s) := \gamma(q) \min\Big(\big(\tfrac{s}{\mu_1}\big)^q, \Big(\tfrac{1-s}{1-\mu_1}\Big)^q\Big).$$

Observe that $e_{x,y}(s) = \int_{]0,1[} \phi_q(s)\|x-y\|^q d\nu(q)$. First we show that ϕ_q satisfies the following functional inequality,

$$\phi_q(s) \geq \begin{cases} \int_{[0,1]} \phi_q\big(\tfrac{s}{\mu_1}t\big) d\mu(t) + \big(\tfrac{s}{\mu_1}\big)^q & (0 \leq s \leq \mu_1), \\ \int_{[0,1]} \phi_q\big(1 - \tfrac{1-s}{1-\mu_1}(1-t)\big) d\mu(t) + \big(\tfrac{1-s}{1-\mu_1}\big)^q & (\mu_1 < s \leq 1). \end{cases} \tag{3.94}$$

To see that (3.94) holds, let $0 \leq s \leq \mu_1$ be arbitrarily fixed. Using the definition of γ, we have that, for all $q \in]0,1[$,

$$\gamma(q) \geq \frac{\mu_1^q}{\mu_1^q - \int_{[0,1]} t^q d\mu(t)}.$$

Rearranging the previous inequality, we get that

$$\gamma(q) \geq \gamma(q) \frac{\int_{[0,1]} t^q d\mu(t)}{\mu_1^q} + 1 \qquad (q \in]0,1[).$$

Multiplying this inequality by $\big(\frac{s}{\mu_1}\big)^q$, we get that

$$\gamma(q)\big(\tfrac{s}{\mu_1}\big)^q \geq \int_{[0,1]} \gamma(q)\big(\tfrac{s}{\mu_1^2}t\big)^q d\mu(t) + \big(\tfrac{s}{\mu_1}\big)^q \qquad (q \in]0,1[).$$

which implies that (3.94) holds for $s \in [0, \mu_1]$. The proof in the case $\mu_1 \leq s \leq 1$ is similar.

Multiplying (3.94) by $\|x-y\|^q$, then integrating on $]0,1[$ for q with respect to ν, and finally using Fubini's theorem, we get that

$$\begin{aligned}
e_{x,y}(s) &= \int\limits_{]0,1[} \phi_q(s)\|x-y\|^q d\nu(q) \\
&\geq \int\limits_{]0,1[}\int\limits_{[0,1]} \phi_q(\tfrac{s}{\mu_1}t)\|x-y\|^q d\mu(t)d\nu(q) + \int\limits_{]0,1[} (\tfrac{s}{\mu_1})^q\|x-y\|^q d\nu(q) \\
&= \int\limits_{[0,1]}\int\limits_{]0,1[} \phi_q(\tfrac{s}{\mu_1}t)\|x-y\|^q d\nu(q)d\mu(t) + \int\limits_{]0,1[} (\tfrac{s}{\mu_1})^q\|x-y\|^q d\nu(q) \\
&= \int\limits_{[0,1]} e_{x,y}(\tfrac{s}{\mu_1}t)d\mu(t) + \int\limits_{]0,1[} (\tfrac{s}{\mu_1})^q\|x-y\|^q d\nu(q), \quad \text{if} \quad 0 \leq s \leq \mu_1.
\end{aligned}$$

In the case $\mu_1 < s \leq 1$, we similarly obtain that

$$e_{x,y}(s) \geq \int\limits_{[0,1]} e_{x,y}(1-\tfrac{1-s}{1-\mu_1}(1-t))d\mu(t) + \int\limits_{]0,1[} (\tfrac{1-s}{1-\mu_1})^q\|x-y\|^q d\nu(q).$$

Thus, $e_{x,y}$ satisfies (3.81). Therefore, by Theorem 3.34, the statement follows. □

Corollary 3.40. (Makó–Páles [**46**])
Let ν be a Borel measure on $]0,1[$ such that

$$\int\limits_{]0,1[} \tfrac{q+1}{q+1-2^q} d\nu(q) < \infty. \tag{3.95}$$

Assume that, for all $(x,y) \in D^2$, $f : D \to \mathbb{R}$ is an upper hemicontinuous solution of the functional inequality

$$f\Big(\frac{x+y}{2}\Big) \leq \int_0^1 f((1-t)x+ty)dt + \int\limits_{]0,1[} \|x-y\|^q d\nu(q).$$

Then, for all $x,y \in D$ and $s \in [0,1]$, f also fulfills the convexity type inequality

$$f((1-s)x+sy) \leq (1-s)f(x)+sf(y) + \int\limits_{]0,1[} \tfrac{(q+1)2^q}{q+1-2^q} \min\big(s^q,(1-s)^q\big)\|x-y\|^q d\nu(q).$$

Proof. We want to apply Theorem 3.39. Let μ be the Lebesgue measure, then $\mu_1 = \frac{1}{2}$ and

$$\gamma(q) = \max\bigg(\frac{(\frac{1}{2})^q}{(\frac{1}{2})^q - \int_0^1 \tau^q d\tau}, \frac{(\frac{1}{2})^q}{(\frac{1}{2})^q - \int_0^1 (1-\tau)^q d\tau}\bigg) = \frac{1}{1-2^q\int_0^1 \tau^q d\tau} = \frac{q+1}{q+1-2^q}.$$

By (3.95), we get that (3.91) holds in Theorem 3.39. For all $x, y \in D$ and $s \in [0,1]$, we obtain that

$$\begin{aligned} e_{x,y}(s) &= \int\limits_{]0,1[} \tfrac{q+1}{q+1-2^q} \min\Big(\Big(\tfrac{s}{1/2}\Big)^q, \Big(\tfrac{1-s}{1/2}\Big)^q\Big)\|x-y\|^q d\nu(q) \\ &= \int\limits_{]0,1[} \tfrac{(q+1)2^q}{q+1-2^q} \min\big(s^q, (1-s)^q\big)\|x-y\|^q d\nu(q), \end{aligned}$$

which proves the statement. □

Corollary 3.41. (Makó–Páles [**46**])
Let $\tau \in]0,1[$ and let ν be a Borel measure on $]0,1[$ such that

$$\int\limits_{]0,1[} \frac{1}{1-\big(\max(\tau,1-\tau)\big)^{1-q}} d\nu(q) < \infty.$$

Assume that, for all $(x,y) \in D^2$, $f : D \to \mathbb{R}$ is an upper hemicontinuous solution of the functional inequality

$$f((1-\tau)x+\tau y) \le \tau f(x) + (1-\tau)f(y) + \int\limits_{]0,1[} \|x-y\|^q d\nu(q).$$

Then, for all $x, y \in D$ and $s \in [0,1]$, f also fulfills the convexity type inequality

$$\begin{aligned} f((1-s)x+sy) &\le (1-s)f(x) + sf(y) \\ &\quad + \int\limits_{]0,1[} \frac{1}{1-\big(\max(\tau,1-\tau)\big)^{1-q}} \min\Big(\big(\tfrac{s}{\tau}\big)^q, \big(\tfrac{1-s}{1-\tau}\big)^q\Big)\|x-y\|^q d\nu(q). \end{aligned}$$

Proof. We apply Theorem 3.39, when μ is a measure on $[0,1]$ defined by $\mu := (1-\tau)\delta_0 + \tau\delta_1$. Then, we obviously have that $\mu_1 = \tau$ and

$$\begin{aligned} \gamma(q) &= \max\Big(\frac{\tau^q}{\tau^q - \int\limits_{[0,1]} \sigma^q d\mu(\sigma)}, \frac{(1-\tau)^q}{(1-\tau)^q - \int\limits_{[0,1]} (1-\sigma)^q d\mu(\sigma)}\Big) \\ &= \max\Big(\frac{\tau^q}{\tau^q-\tau}, \frac{(1-\tau)^q}{(1-\tau)^q-(1-\tau)}\Big) = \frac{1}{1-\big(\max(\tau,1-\tau)\big)^{1-q}}. \end{aligned}$$

Thus, for all $x, y \in D$ and $s \in [0,1]$,

$$e_{x,y}(s) = \int\limits_{]0,1[} \frac{1}{1-\max(\tau,1-\tau)^{1-q}} \min\Big(\big(\tfrac{s}{\tau}\big)^q, \big(\tfrac{1-s}{1-\tau}\big)^q\Big)\|x-y\|^q d\nu(q),$$

which proves the statement. □

Summary

This PhD dissertation contains new results related to the theory of approximate convexity. It consists of three main parts.

In the introduction the preliminary results are reviewed and the main objectives of this work are formulated. In Chapter 1, we introduce Takagi-like functions, which appear naturally in the investigation of approximate convexity. To recall their definition, let $\phi : [0,1] \to \mathbb{R}$ and introduce the following Takagi type functions.

$$T_\phi(x) := \sum_{n=0}^{\infty} \frac{\phi(d_{\mathbb{Z}}(2^n x))}{2^n} \quad \text{and} \quad S_\phi(x) := \sum_{n=0}^{\infty} \phi\Big(\frac{1}{2^n}\Big) d_{\mathbb{Z}}(2^n x) \qquad (x \in \mathbb{R}),$$

where

$$d_{\mathbb{Z}}(x) := 2\,\mathrm{dist}(x, \mathbb{Z}) := 2\min\{|x - z| : z \in \mathbb{Z}\} \qquad (x \in \mathbb{R}).$$

Our main results state that, under some natural regularity assumptions for ϕ, the functions T_ϕ and S_ϕ are approximately Jensen convex. To describe the details, we shall need to recall the notion of higher-order monotonicity and convexity. Let $I \subseteq \mathbb{R}$ be a proper interval and $\phi : I \to \mathbb{R}$. Given $h \in \mathbb{R}$, we use the notation $\Delta_h \phi(x) := \phi(x+h) - \phi(x)$ whenever $x \in I \cap (I - h)$. We say that a function ϕ is *n-monotone* (*$(n-1)$-Wright-convex) on I* if, for all $h_1, \dots, h_n \geq 0$ and for all $x \in I \cap (I - h_1 - \cdots - h_n)$, the inequality

$$\Delta_{h_1} \cdots \Delta_{h_n} \phi(x) \geq 0$$

holds.

Theorem. (Makó–Páles [**43**])
Let $\phi : [0,1] \to \mathbb{R}_+$ be a continuous function. Assume that $\phi(0) = 0$ and ϕ is 1- and 3-monotone, and $(-\phi)$ is 2-monotone. Then T_ϕ is approximately Jensen convex in the following sense

$$T_\phi\Big(\frac{x+y}{2}\Big) \leq \frac{T_\phi(x) + T_\phi(y)}{2} + \phi \circ d_{\mathbb{Z}}\Big(\frac{x-y}{2}\Big) \qquad (x, y \in \mathbb{R}).$$

THEOREM. (Makó–Páles [**48**])
Let $\phi : [0,1] \to \mathbb{R}_+$. Assume that $\phi(0) = 0$ and the mapping $x \mapsto \frac{\phi(x)}{x}$ is concave on $]0,1]$. Then S_ϕ is approximately Jensen convex in the following sense

$$S_\phi\Big(\frac{x+y}{2}\Big) \le \frac{S_\phi(x) + S_\phi(y)}{2} + \phi \circ d_{\mathbb{Z}}\Big(\frac{x-y}{2}\Big) \qquad (x, y \in \mathbb{R}).$$

Applying these theorems, we can get the approximate Jensen convexity of the following Takagi type functions:

$$T_q(x) := \sum_{n=0}^{\infty} \frac{(d_{\mathbb{Z}}(2^n x))^q}{2^n} \quad \text{and} \quad S_q(x) := \sum_{n=0}^{\infty} \frac{1}{2^{nq}} d_{\mathbb{Z}}(2^n x) \qquad (x \in \mathbb{R}),$$

where $q > 0$.

COROLLARY. (Makó–Páles [**43**])
For $0 < q \le 1$, the Takagi type function T_q satisfies, for all $x, y \in \mathbb{R}$,

$$T_q\Big(\frac{x+y}{2}\Big) \le \frac{T_q(x) + T_q(y)}{2} + d_{\mathbb{Z}}^q\Big(\frac{x-y}{2}\Big).$$

COROLLARY. (Tabor–Tabor [**66**])
For $1 \le q \le 2$ the Takagi type function S_q satisfies, for all $x, y \in \mathbb{R}$,

$$S_q\Big(\frac{x+y}{2}\Big) \le \frac{S_p(x) + S_q(y)}{2} + d_{\mathbb{Z}}^q\Big(\frac{x-y}{2}\Big).$$

These results can be applied in theory of approximate convexity. Let D be a nonempty convex set of the normed space X and denote $D^+ := \{\|x-y\| : x, y \in D\}$. Let $\varphi : D^+ \to \mathbb{R}_+$ be a given error function. We say that $f : D \to \mathbb{R}$ is *φ-Jensen convex on D*, if for all $x, y \in D$,

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x) + f(y)}{2} + \varphi(\|x-y\|).$$

An important particular case occurs when $\varphi : D^+ \to \mathbb{R}_+$ is of the form $\varphi(u) := \varepsilon u^q$, where $q, \varepsilon \ge 0$ are arbitrary constants and $u \in D^+$. In this case, a φ-Jensen convex function f is called *(ε, q)-Jensen convex* on D.

Define, for all $(t, u) \in \mathbb{R} \times D^+$,

$$\mathcal{T}_\varphi(t, u) := \sum_{n=0}^{\infty} \frac{\varphi(d_{\mathbb{Z}}(2^n t)u)}{2^n} \quad \text{and} \quad \mathcal{S}_\varphi(t, u) := \sum_{n=0}^{\infty} \varphi\Big(\frac{u}{2^n}\Big) d_{\mathbb{Z}}(2^n t).$$

Hereafter we shall formulate some Bernstein–Doetsch type theorems.

THEOREM. (Makó–Páles [**44**], Tabor–Tabor [**67**])
Let $f : D \to \mathbb{R}$ be locally upper bounded on D. Assume that $\varphi : D^+ \to \mathbb{R}_+$ is a

continuous function, such that $\varphi(0) = 0$. Then f is φ-Jensen convex on D, if and only if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \mathcal{T}_\varphi(t, \|x-y\|)$$

for all $x, y \in D$ and $t \in [0,1]$.

Theorem. (Tabor–Tabor [**67**])
Let $f : D \to \mathbb{R}$ be upper semicontinuous on D and let $\varphi : D^+ \to \mathbb{R}_+$ be nondecreasing such that $\sum_{n=n_0}^{\infty} \varphi(2^{-n}) < \infty$ for some $n_0 \in \mathbb{N}$. Then f is φ-Jensen convex on D, if and only if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \mathcal{S}_\varphi(t, \|x-y\|)$$

for all $x, y \in D$ and $t \in [0,1]$.

Some immediate consequences of these theorems can be formulated in the following corollaries.

Corollary. (Házy [**29**])
Let $f : D \to \mathbb{R}$ be locally upper bounded on D and $\varepsilon \ge 0, q > 0$ Then f is (ε, q)-Jensen convex on D, if and only if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \varepsilon T_q(t)\|x-y\|^q$$

for all $x, y \in D$ and $t \in [0,1]$.

Corollary. (Tabor–Tabor [**67**])
Let $f : D \to \mathbb{R}$ be upper semicontinuous on D and $\varepsilon \ge 0, q > 0$ Then f is (ε, q)-Jensen convex on D if and only if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \varepsilon S_q(t)\|x-y\|^q$$

for all $x, y \in D$ and $t \in [0,1]$.

In [**14**] Boros proved that if $q = 1$ and $t \in [0,1]$ is fixed, then $S_1(t) = T_1(t)$ is the smallest possible. In [**66**] Tabor and Tabor showed that if $1 \le q \le 2$ and $t \in [0,1]$ is fixed, then $S_q(t)$ is the smallest possible value.

It is also an important question whether the error terms $\mathcal{T}_\varphi(t, \|x-y\|)$, $\mathcal{S}_\varphi(t, \|x-y\|)$ and $T_q(t)$ are the smallest possible ones. In other words, for all fixed $x, y \in D$, we want to obtain the exact upper bound of the convexity-difference of φ-Jensen convex functions defined by

$$C_\varphi(x, y, t) := \sup_{f \in \mathcal{JC}_\varphi(D)} \{f(tx + (1-t)y) - tf(x) - (1-t)f(y)\},$$

where

$$\mathcal{JC}_\varphi(D) := \{f : D \to \mathbb{R} \mid f \text{ is } \varphi\text{-Jensen convex on } D\}.$$

Theorem. (Makó–Páles [**43**], [**48**])
Let $\varphi : D^+ \to \mathbb{R}_+$ be a continuous nondecreasing function with $\varphi(0) = 0$. Then, for all fixed $x, y \in D$,

$$C_\varphi(x, y, t) = \mathcal{T}_\varphi(t, \|x - y\|) \qquad (t \in [0, 1])$$

provided that φ is 3-monotone, and $(-\varphi)$ is 2-monotone on D^+. Furthermore, for all fixed $x, y \in D$,

$$C_\varphi(x, y, t) = \mathcal{S}_\varphi(t, \|x - y\|) \qquad (t \in [0, 1]),$$

provided that $u \mapsto \frac{\varphi(u)}{u}$ is concave on $D^+ \setminus \{0\}$.

Applying the previous result to the function $\varphi(u) := \varepsilon u^q$. where $\varepsilon, q \in \mathbb{R}_+$, we obtain the following results.

Corollary. (Makó–Páles [**43**],[**48**] Tabor–Tabor [**66**])
Let $\varepsilon \in \mathbb{R}_+$, $q > 0$. Then, with $\varphi(u) := \varepsilon u^q$, for all $x, y \in D$ and $t \in [0, 1]$,

$$C_\varphi(x, y, t) = \begin{cases} \varepsilon T_q(t)\|x - y\|^q & \text{if} \quad 0 < q \le 1, \\ \varepsilon S_q(t)\|x - y\|^q & \text{if} \quad 1 \le q \le 2. \end{cases}$$

In what follows, let D be a nonempty, convex subset of the linear space X. Denote $D^* := (D - D)$ and $D^{2*} := \{(x, y) \in D^2 \mid x \ne y\}$. Let $\alpha : D^* \to \mathbb{R}$ be a given even function (we note that α need not be nonnegative). We say that $f : D \to \mathbb{R}$ is *α-Jensen convex on D*, if, for all $x, y \in D$,

$$f\Big(\frac{x + y}{2}\Big) \le \frac{f(x) + f(y)}{2} + \alpha(x - y).$$

An important particular case occurs when X is a normed space and $\alpha : D^* \to \mathbb{R}$ is of the form $\alpha(u) := \varepsilon\|u\|^q$, where $\varepsilon \in \mathbb{R}$ and $q \ge 0$ are arbitrary constants and $u \in D^*$. In this case, a α-Jensen convexity is equivalent to (ε, q)-Jensen convexity on D.

In Chapter 2, we investigate connections between α-Jensen convex functions and functions which satisfy the following upper Hermite–Hadamard type inequality:

$$(1) \qquad \int_0^1 f(tx + (1 - t)y)\rho(t)dt \le \lambda f(x) + (1 - \lambda)f(y) + \beta(x - y) \qquad (x, y \in D),$$

where $\beta : D^* \to \mathbb{R}$ is a given even functions, $\lambda \in [0, 1]$, and $\rho : [0, 1] \to \mathbb{R}_+$ is an integrable nonnegative function with $\int_0^1 \rho = 1$.

Two implications from α-Jensen convexity to an upper Hermite–Hadamard type inequality are stated in the following two theorems.

Theorem. (Makó–Páles [**45**])
Let $\alpha : D^* \to \mathbb{R}$ be radially bounded and measurable and $\rho : [0, 1] \to \mathbb{R}_+$ be

a Lebesgue integrable function with $\int_0^1 \rho = 1$. *Assume that* $f : D \to \mathbb{R}$ *is hemi-integrable and* α*-Jensen convex on* D. *Then* f *also satisfies the approximate upper Hermite–Hadamard inequality* (1) *with* $\lambda := \int_0^1 t\rho(t)dt$ *and* $\beta : D^* \to \mathbb{R}$ *defined by*

$$\beta(u) := \sum_{n=0}^{\infty} \frac{1}{2^n} \int_0^1 \alpha\big(d_{\mathbb{Z}}(2^n t)u\big)\rho(t)dt \quad (u \in D^*).$$

THEOREM. (Makó–Páles [**45**])
Let $\alpha : D^* \to \mathbb{R}_+$ *be nonnegative, radially increasing such that* $\sum_{n=0}^{\infty} \alpha(\frac{u}{2^n}) < \infty$ *if* $u \in D^*$. *If* $f : D \to \mathbb{R}$ *is upper hemicontinuous and* α*-Jensen convex on* D, *then* f *also satisfies the Hermite–Hadamard inequality* (1) *with* $\lambda := \int_0^1 t\rho(t)dt$ *and* $\beta : D^* \to \mathbb{R}$ *is defined by*

$$\beta(u) := \sum_{n=0}^{\infty} \alpha\Big(\frac{u}{2^n}\Big) \int_0^1 d_{\mathbb{Z}}(2^n t)\rho(t)dt \qquad (u \in D^*).$$

Now, if we consider the case when $\rho \equiv 1$ and f is (ε, q)-Jensen convex, then these theorems reduce to the following corollaries.

COROLLARY. (Makó–Páles [**45**])
Let $\varepsilon \in \mathbb{R}$ *and* $q > 0$. *Assume that* $f : D \to \mathbb{R}$ *is hemiintegrable and* (ε, q)*-Jensen convex. Then* f *also satisfies the following approximate Hermite–Hadamard inequality*

$$\int_0^1 f(tx + (1-t)y)dt \le \frac{f(x) + f(y)}{2} + \frac{2\varepsilon}{q+1}\|x - y\|^q \qquad (x, y \in D).$$

COROLLARY. (Makó–Páles [**45**])
Let $\varepsilon \in \mathbb{R}_+$ *and* $q > 0$. *Assume that* $f : D \to \mathbb{R}$ *is upper hemicontinuous and* (ε, q)*-Jensen convex. Then* f *also satisfies the following approximate Hermite–Hadamard inequality*

$$\int_0^1 f(tx + (1-t)y)dt \le \frac{f(x) + f(y)}{2} + \frac{2^q\varepsilon}{2^{q+1} - 2}\|x - y\|^q \qquad (x, y \in D).$$

The following theorem states that approximate upper Hermite–Hadamard inequality can also imply α-Jensen convexity with a properly chosen function α. The core of our approach is a multiplicative type convolution and its asymptotic properties.

THEOREM. (Makó–Páles [**45**])
Let $\beta : D^* \to \mathbb{R}$ *be even and radially upper semicontinuous,* $\rho : [0, 1] \to \mathbb{R}_+$ *be integrable with* $\int_0^1 \rho = 1$ *and there exist* $c \ge 0$ *and* $p > 0$ *such that*

$$\rho(t) \le c(-\ln|1 - 2t|)^{p-1} \qquad (t \in]0, \tfrac{1}{2}[\cup]\tfrac{1}{2}, 1[),$$

and $\lambda \in [0,1]$. *Then every* $f : D \to \mathbb{R}$ *lower hemicontinuous function satisfying the approximate upper Hermite–Hadamard inequality* (1)*, is* α*-Jensen convex provided that* $\alpha : D^* \to \mathbb{R}$ *is a radially lower semicontinuous solution of the functional inequality*

$$\alpha(u) \geq \int_0^1 \alpha(|1-2t|u)\rho(t)dt + \beta(u) \qquad (u \in D^*)$$

and $\alpha(0) \geq \beta(0)$*.*

Now we consider the case in the previous theorem when $\rho \equiv 1$ and β is of the form $\varepsilon\|\cdot\|^q$.

Corollary. (Makó–Páles [**45**])
Let $\lambda \in [0,1]$*,* $\varepsilon \in \mathbb{R}$ *and* $q > 0$*. Assume that* $f : D \to \mathbb{R}$ *is lower hemicontinuous and satisfies the Hermite–Hadamard type inequality*

$$\int_0^1 f(tx+(1-t)y)dt \leq \lambda f(x) + (1-\lambda)f(y) + \varepsilon\|x-y\|^q \qquad (x,y \in D).$$

Then f *is* $(\varepsilon\frac{q+1}{q}, q)$*-Jensen convex.*

Let (ω_0, ω_1) be a positive Chebyshev system on the real interval I. In Chapter 3, we establish connections between an approximate (ω_0, ω_1)-convexity inequality and an approximate lower Hermite–Hadamard type inequality. First, we investigate implication from approximate (ω_0, ω_1)-convexity to approximate lower Hermite–Hadamard type inequality. Consider the following basic assumptions.

(A1) $(T, \mathcal{A}, \mu)$ is a measure space.
(A2) $\Lambda : T \times \Delta^\circ(I) \to \mathbb{R}_+$ is μ-integrable in its first variable.
(A3) $M : T \times \Delta^\circ(I) \to \mathbb{R}$ is $\mathcal{A}$-measurable in its first variable and for all $t \in T$, the map $(x,y) \mapsto M(t,x,y)$ is a two-variable mean on I. $M_0 : \Delta^\circ(I) \to I$ is a strict mean such that

$$\mu\{t \in T \mid \Lambda(t,x,y) > 0,\ M(t,x,y) \neq M_0(x,y)\} > 0 \tag{2}$$

holds.
(A4) There exists an (ω_0, ω_1)-Chebyshev system on I such that ω_0 is positive. Furthermore, for $i \in \{0,1\}$

$$\omega_i(M_0(x,y)) = \int_T \Lambda(t,x,y)\omega_i(M(t,x,y))d\mu(t) \qquad ((x,y) \in \Delta^\circ(I)). \tag{3}$$

holds.

Theorem. (Makó–Páles [**47**])
Assume that (A1)–(A4) hold. Let $f : I \to \mathbb{R}$ be a locally upper bounded Borel measurable solution of the approximate (ω_0, ω_1)-convexity type functional inequality

$$f(u) \le \frac{\Omega(u,y)}{\Omega(x,y)} f(x) + \frac{\Omega(x,u)}{\Omega(x,y)} f(y) + \varepsilon_{x,y}(u) \qquad (u \in [x,y]), \tag{4}$$

where for all $(x,y) \in \Delta^\circ(I)$ and $u \in]x,y[$, the function $(v,w) \mapsto \varepsilon_{v,w}(u)$ is bounded and Borel measurable for $(v,w) \in [x,u] \times [u,y]$. Then f also satisfies the approximate lower Hermite–Hadamard type inequality

$$f(M_0(x,y)) \le \int_T \Lambda(t,x,y) f(M(t,x,y)) d\mu(t) + \mathcal{E}(x,y) \quad ((x,y) \in \Delta(I)), \tag{5}$$

where $\mathcal{E} : \Delta^\circ(I) \to \mathbb{R}$ is defined by

$$\mathcal{E}(x,y) := \frac{\int\limits_{T'_{x,y}} \int\limits_{T''_{x,y}} \Upsilon(t',t'',x,y)\varepsilon_{M(t',x,y),M(t'',x,y)}(M_0(x,y)) d\mu(t'') d\mu(t')}{\int\limits_{T'_{x,y}} \int\limits_{T''_{x,y}} \Upsilon(t',t'',x,y) d\mu(t'') d\mu(t')},$$

where, for all $(t',t'',x,y) \in T^2 \times \Delta^\circ(I)$,

$$\begin{aligned} \Upsilon(t',t'',x,y) &:= \Lambda(t',x,y)\Lambda(t'',x,y)\Omega(M(t',x,y), M(t'',x,y)), \\ T'_{x,y} &:= \{t \in T \mid \Lambda(t,x,y) > 0,\ M(t,x,y) < M_0(x,y)\}, \\ T''_{x,y} &:= \{t \in T \mid \Lambda(t,x,y) > 0,\ M(t,x,y) > M_0(x,y)\}. \end{aligned}$$

The reversed implication will be stated in the theorem below, which was the main result of this chapter. The key for the proof of this result is a Korovkin type theorem which enables us to deduce the approximate convexity property from the approximate lower Hermite–Hadamard type inequality via an iteration process. Consider the following basic assumptions:

(B1) $(T, \mathcal{A}, \mu)$ is a measure space.

(B2) $\Lambda : T \times \Delta(I) \to \mathbb{R}_+$ is measurable in its first variable and separately continuous in its second variable; furthermore, for all $(x,y) \in \Delta(I)$, the function $L_{x,y} : T \to \mathbb{R}_+$ defined by

$$L_{x,y}(t) := \sup_{u \in [x,y]} \max\left(\Lambda(t,x,u), \Lambda(t,u,y)\right)$$

is μ-integrable, i.e., $\int_T L_{x,y}(t) d\mu(t) < +\infty$.

(B3) $M : T \times \Delta(I) \to \mathbb{R}$ is measurable in its first variable and for all $t \in T$, the map $(x,y) \mapsto M(t,x,y)$ is separately continuous and partially differentiable at the diagonal of $I \times I$. $M_0 : \Delta(I) \to I$ is a separately continuous, strictly increasing and partially differentiable at the diagonal of $I \times I$ with $\partial_1 M_0(z,z) > 0$ and $\partial_2 M_0(z,z) > 0$ for all $z \in I$. Furthermore, $M(t,\cdot)$ is separately uniformly calm with respect to M_0 at the diagonal of $I\times I$, i.e., for all $z\in I$, there exist constants $\delta>0$ and $K\geq 0$ such that, for all $t\in T$,

$$\begin{aligned} z-M(t,u,z) &\leq K(z-M_0(u,z)) \qquad (u\in[z-\delta,z]),\\ M(t,z,u)-z &\leq K(M_0(z,u)-z) \qquad (u\in[z,z+\delta]), \end{aligned}$$

and (2) holds and, for all $z\in I$ and $i\in\{0,1\}$,

$$\mu\{t\in T \mid \Lambda(t,z,z)\neq 0,\ \partial_i M(t,z,z)\neq \partial_i M_0(z,z)\}>0.$$

(B4) There exist functions $\omega_0,\omega_1 : I\to\mathbb{R}$ such that ω_0 is positive, $\frac{\omega_1}{\omega_0}$ is differentiable on I, with $(\frac{\omega_1}{\omega_0})'>0$. Furthermore, for $i\in\{0,1\}$, (3) hold.

Theorem. (Makó–Páles [**46**])
Suppose that conditions (B1)–(B4) hold and assume that $f : I\to\mathbb{R}$ is an upper semicontinuous solution of the functional inequality (5)*, where $\mathcal{E} : \Delta(I)\to\mathbb{R}$ is an arbitrary function. Assume that, for all $(x,y)\in\Delta^\circ(I)$, $\varepsilon_{x,y} : [x,y]\to\mathbb{R}$ is a lower semicontinuous function with $\varepsilon_{x,y}(x)=\varepsilon_{x,y}(y)=0$ satisfying the following system of inequalities:*

$$\varepsilon_{x,y}(M_0(x,u))\geq\int_T \Lambda(t,x,u)\varepsilon_{x,y}(M(t,x,u))d\mu(t)+\mathcal{E}(x,u) \quad (u\in[x,y]),$$

$$\varepsilon_{x,y}(M_0(u,y))\geq\int_T \Lambda(t,u,y)\varepsilon_{x,y}(M(t,u,y))d\mu(t)+\mathcal{E}(u,y) \quad (u\in[x,y]).$$

Then, for all fixed $(x,y)\in\Delta^\circ(I)$, the function f also satisfies the approximate (ω_0,ω_1)-convexity inequality (4)*.*

Important applications of these theorems are related to the investigations of the connections of the following inequalities.

Consider the following approximate convexity type inequality:

$$f((1-t)x+ty)\leq(1-t)f(x)+tf(y)+e_{x,y}(t) \quad ((x,y)\in D^2,\ t\in[0,1]), \tag{6}$$

where $f : D\to\mathbb{R}$ is an unknown function and for all $(x,y)\in D^2$, $e_{x,y} : [0,1]\to\mathbb{R}$ is a given function. In Chapter 3, we also investigated the connections between (6) and the following approximate lower Hermite–Hadamard type inequality

$$f((1-\mu_1)x+\mu_1 y)\leq\int_{[0,1]} f((1-t)x+ty)d\mu(t)+E(x,y) \quad ((x,y)\in D^2), \tag{7}$$

where $f : D\to\mathbb{R}$ is an unknown function, μ is a probability measure on $[0,1]$, $\mu_1 := \int_{[0,1]} t d\mu(t)$ and $E : D^2\to\mathbb{R}$.

Theorem. (Makó–Páles [**47**])
Let $\mathcal{A}$ be a σ-algebra containing the Borel subsets of $[0,1]$ and μ be a probability measure on the measurable space $([0,1],\mathcal{A})$ such that the support of μ is not a singleton. Denote

$$S(\mu) := \mu([0,\mu_1]) \int\limits_{]\mu_1,1]} t d\mu(t) - \mu(]\mu_1,1]) \int\limits_{[0,\mu_1]} t d\mu(t).$$

Assume that $f : D \to \mathbb{R}$ is an hemi-μ-integrable solution of the functional inequality (6) *where, for all $(x,y) \in D^{2*}$, $e_{x,y} : [0,1] \to \mathbb{R}$ is a function such that*

$$I(x,y) := \int\limits_{]\mu_1,1]} \int\limits_{[0,\mu_1]} (t''-t') e_{(1-t')x+t'y,(1-t'')x+t''y}\Big(\frac{\mu_1 - t'}{t''-t'}\Big) d\mu(t') d\mu(t'')$$

exists in $[-\infty,\infty]$. Then, for all $(x,y) \in D^{2*}$, the function f also satisfies the lower Hermite–Hadamard type inequality (7)*, where $E(x,y) := \frac{I(x,y)}{S(\mu)}$.*

Theorem. (Makó–Páles [**46**])
Let μ be a Borel probability measure on $[0,1]$, denote $\mu_1 := \int_{[0,1]} t d\mu(t)$ and assume that the support of μ is not a singleton, i.e., $\mu \neq \delta_{\mu_1}$. Assume that, for all $(x,y) \in D^2$, $f : D \to \mathbb{R}$ is an upper hemicontinuous solution of the functional inequality (7)*, where $E : D^2 \to \mathbb{R}$. Assume that, for all $(x,y) \in D^2$, $e_{x,y} : [0,1] \to \mathbb{R}$ is a lower semicontinuous function with $e_{x,y}(0) = e_{x,y}(1) = 0$ satisfying the following system of inequalities:*

$$e_{x,y}(s) \geq \begin{cases} \int\limits_{[0,1]} e_{x,y}(\frac{st}{\mu_1}) d\mu(t) + E(x, (1-\frac{s}{\mu_1})x + \frac{s}{\mu_1}y) & (s \in [0,\mu_1]), \\ \int\limits_{[0,1]} e_{x,y}(1 - \frac{(1-s)(1-t)}{1-\mu_1}) d\mu(t) + E(\frac{1-s}{1-\mu_1}x + (1-\frac{1-s}{1-\mu_1})y, y) & (s \in [\mu_1,1]). \end{cases}$$

Then, for all $(x,y) \in D^2$ and $t \in [0,1]$, the function f also satisfies the approximate convexity inequality (6)*.*

An important application of this result is a Bernstein–Doetsch type theorem (see: [**8**]).

Corollary. (Makó–Páles [**46**])
Let $\tau \in]0,1[$ and $E : D^2 \to \mathbb{R}$. Assume that $f : D \to \mathbb{R}$ is an upper hemicontinuous solution of the functional inequality

$$f((1-\tau)x + \tau y) \leq (1-\tau)f(x) + \tau f(y) + E(x,y) \qquad ((x,y) \in D^2).$$

Assume that, for all $x, y \in D$, $e_{x,y} : [0,1] \to \mathbb{R}$ is a lower semicontinuous function with $e_{x,y}(0) = e_{x,y}(1) = 0$, satisfying the following system of inequalities:

$$e_{x,y}(s) \geq \begin{cases} \tau e_{x,y}(\frac{s}{\tau}) + E(x, (1-\frac{s}{\tau})x + \frac{s}{\tau}y) \quad (s \in [0,\tau]), \\ (1-\tau)e_{x,y}(\frac{s-\tau}{1-\tau}) + E(\frac{1-s}{1-\tau}x + (1-\frac{1-s}{1-\tau})y, y) \quad (s \in [\tau, 1]). \end{cases}$$

Then, for all $x, y \in D$ and $t \in [0,1]$, the function f also satisfies the approximate convexity inequality (6).

Összefoglalás

Ez a PhD értekezés a közelítőleg konvex függvények vizsgálatával foglalkozik. Három fő részből áll.

A bevezetésben áttekintjük a korábbi eredményeket és megfogalmazzuk az értekezés célkitűzéseit. Az első fejezetben Takagi-típusú függvényekkel foglalkozunk. Ezek a függvények ugyanis fontos szerepet játszanak a közelítőleg konvex függvények elméletében. Adott $\phi : [0,1] \to \mathbb{R}$ folytonos függvény esetén tekintsük a következő Takagi-típusú függvényeket

$$T_\phi(x) := \sum_{n=0}^{\infty} \frac{\phi(d_{\mathbb{Z}}(2^n x))}{2^n} \quad \text{és} \quad S_\phi(x) := \sum_{n=0}^{\infty} \phi\Big(\frac{1}{2^n}\Big) d_{\mathbb{Z}}(2^n x) \qquad (x \in \mathbb{R}),$$

ahol

$$d_{\mathbb{Z}}(x) := 2\,\mathrm{dist}(x,\mathbb{Z}) := 2\min\{|x-z| : z \in \mathbb{Z}\} \qquad (x \in \mathbb{R}).$$

A fejezet fő eredménye az, hogy T_ϕ és S_ϕ közelítőleg Jensen-konvexek, ha ϕ rendelkezik bizonyos regularitási tulajdonságokkal. Az eredményeink ismertetéséhez szükségünk van a magasabb rendben monoton függvények definíciójára. Legyen $I \subseteq \mathbb{R}$ egy valódi intervallum és $\phi : I \to \mathbb{R}$. Adott $h \in \mathbb{R}$ esetén jelölje $\Delta_h\phi(x) := \phi(x+h) - \phi(x)$ ha $x \in I \cap (I-h)$. Azt mondjuk, hogy a ϕ függvény *n-monoton ($(n-1)$-Wright-konvex) I-n* ha, bármely $h_1, \dots, h_n \geq 0$ és bármely $x \in I \cap (I - h_1 - \cdots - h_n)$ esetén a következő

$$\Delta_{h_1} \cdots \Delta_{h_n}\phi(x) \geq 0$$

egyenlőtlenség teljesül.

Tétel. (Makó–Páles [**43**])
Legyen $\phi : [0,1] \to \mathbb{R}_+$ olyan folytonos függvény, hogy $\phi(0) = 0$, ϕ 1- és 3-monoton, illetve $(-\phi)$ 2-monoton. Ekkor T_ϕ a következő értelemben Jensen-konvex:

$$T_\phi\Big(\frac{x+y}{2}\Big) \leq \frac{T_\phi(x) + T_\phi(y)}{2} + \phi \circ d_{\mathbb{Z}}\Big(\frac{x-y}{2}\Big) \qquad (x, y \in \mathbb{R}).$$

Tétel. (Makó–Páles [**48**])
Legyen $\phi : [0,1] \to \mathbb{R}_+$ *olyan függvény, hogy* $\phi(0) = 0$ *és a* $x \mapsto \frac{\phi(x)}{x}$ *leképezés konkáv* $]0,1]$*-n. Ekkor* S_ϕ *a következő értelemben Jensen-konvex*

$$S_\phi\Big(\frac{x+y}{2}\Big) \le \frac{S_\phi(x)+S_\phi(y)}{2} + \phi \circ d_{\mathbb{Z}}\Big(\frac{x-y}{2}\Big) \qquad (x,y \in \mathbb{R}).$$

Alkalmazva ezeket a tételeket, a következő Takagi-típusú függvények közelítő Jensen-konvexitását is kapjuk:

$$T_q(x) := \sum_{n=0}^{\infty} \frac{(d_{\mathbb{Z}}(2^n x))^q}{2^n} \quad \text{és} \quad S_q(x) := \sum_{n=0}^{\infty} \frac{1}{2^{nq}} d_{\mathbb{Z}}(2^n x) \qquad (x \in \mathbb{R}),$$

ahol $q \in \mathbb{R}_+$.

Következmény. (Makó–Páles [**43**])
Ha $0 < q \le 1$ *és* $x, y \in \mathbb{R}$, *akkor*

$$T_q\Big(\frac{x+y}{2}\Big) \le \frac{T_q(x)+T_q(y)}{2} + d_{\mathbb{Z}}^q\Big(\frac{x-y}{2}\Big).$$

Következmény. (Tabor–Tabor [**66**])
Ha $1 \le q \le 2$ *és* $x, y \in \mathbb{R}$, *akkor*

$$S_q\Big(\frac{x+y}{2}\Big) \le \frac{S_p(x)+S_q(y)}{2} + d_{\mathbb{Z}}^q\Big(\frac{x-y}{2}\Big).$$

Ezek az eredmények alkalmazhatóak a közelítőleg konvex függvények elméletében. Legyen D egy nemüres, konvex részhalmaza az X normált térnek és vezessük be a következő jelölést: $D^+ := \{\|x-y\| : x,y \in D\}$. Legyen $\varphi : D^+ \to \mathbb{R}_+$ egy adott hibafüggvény. Azt mondjuk, hogy az $f : D \to \mathbb{R}$ függvény *φ-Jensen-konvex D-n*, ha bármely $x, y \in D$ esetén

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2} + \varphi(\|x-y\|).$$

A φ-Jensen-konvex függvények fontos osztályát alkotják az *(ε, q)-Jensen-konvex függvények*. Ebben az esetben $\varphi : D^+ \to \mathbb{R}_+$ függvény a $\varphi(u) := \varepsilon u^q$ módon van értelmezve, ahol $q, \varepsilon \ge 0$ tetszőleges konstansok és $u \in D^+$. Definiáljuk minden $(t,u) \in \mathbb{R} \times D^+$ esetén a

$$\mathcal{T}_\varphi(t,u) := \sum_{n=0}^{\infty} \frac{\varphi(d_{\mathbb{Z}}(2^n t)u)}{2^n} \quad \text{és} \quad \mathcal{S}_\varphi(t,u) := \sum_{n=0}^{\infty} \varphi\Big(\frac{u}{2^n}\Big) d_{\mathbb{Z}}(2^n t)$$

függvényeket. A következő tételek Bernstein–Doetsch típusúak.

Tétel. (Makó–Páles [**44**], Tabor–Tabor [**67**])
Legyen $f : D \to \mathbb{R}$ lokálisan felülről korlátos D-n és $\varphi : D^+ \to \mathbb{R}_+$ egy olyan folytonos függvény, amelyre $\varphi(0) = 0$. Ekkor f pontosan akkor φ-Jensen-konvex D-n, ha

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \mathcal{T}_\varphi(t, \|x-y\|)$$

teljesül minden $x, y \in D$ és $t \in [0,1]$ esetén.

Tétel. (Tabor–Tabor [**67**])
Legyen $f : D \to \mathbb{R}$ felülről félig folytonos D-n és tegyük fel, hogy $\varphi : D^+ \to \mathbb{R}_+$ monoton növekvő, úgy hogy $\sum_{n=n_0}^{\infty} \varphi(2^{-n}) < \infty$ valamely $n_0 \in \mathbb{N}$ esetén. Ekkor f pontosan akkor φ-Jensen-konvex D-n, ha

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \mathcal{S}_\varphi(t, \|x-y\|)$$

teljesül minden $x, y \in D$ és $t \in [0,1]$ esetén.

Az előző tételekből azonnal adódnak a következők.

Következmény. (Házy [**29**])
Legyen $f : D \to \mathbb{R}$ lokálisan felülről korlátos D-n és és $\varepsilon \geq 0, q > 0$ Ekkor f pontosan akkor (ε, q)-Jensen-konvex D-n, ha

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon T_q(t)\|x-y\|^q$$

minden $x, y \in D$ és $t \in [0,1]$ esetén.

Következmény. (Tabor–Tabor [**67**])
Legyen $f : D \to \mathbb{R}$ felülről félig folytonos D-n és $\varepsilon \geq 0, q > 0$. Ekkor f pontosan akkor (ε, q)-Jensen-konvex D-n, ha

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon S_q(t)\|x-y\|^q$$

minden $x, y \in D$ és $t \in [0,1]$ esetén.

A [**14**] cikkben Boros megmutatta, hogy $q = 1$ esetén $S_1(t) = T_1(t)$ a lehető legkisebb, amit hibafüggvényként az előző egyenlőtlenségbe írhatunk. [**66**]-ben Tabor és Tabor bebizonyították, hogy $1 \leq q \leq 2$ esetén $S_q(t)$ a lehető legkisebb.

Az első fejezetben arra keressük a választ, hogy az $\mathcal{T}_\varphi(t, \|x-y\|)$, $\mathcal{S}_\varphi(t, \|x-y\|)$ és $T_q(t)$ milyen φ hibafüggvény és q konstans esetén lesznek a legoptimálisabb választás. Más szavakkal, minden rögzített $x, y \in D$ esetén a φ-Jensen-konvex függvények konvexitási differenciájának keressük a pontos felső korlátját. Azaz $C_\varphi(x, y, t)$-t, ahol

$$C_\varphi(x, y, t) := \sup_{f \in \mathcal{JC}_\varphi(D)} \{f(tx + (1-t)y) - tf(x) - (1-t)f(y)\},$$

és

$$\mathcal{JC}_\varphi(D) := \{f : D \to \mathbb{R} \mid f \ \varphi\text{-Jensen-konvex } D\text{-n}\}.$$

Tétel. (Makó–Páles [**43**], [**48**])
Legyen $\varphi : D^+ \to \mathbb{R}_+$ *olyan folytonos, monoton növekvő függvény, hogy* $\varphi(0) = 0$. *Ekkor, minden rögzített* $x, y \in D$ *esetén*

$$C_\varphi(x, y, t) = \mathcal{T}_\varphi(t, \|x - y\|) \qquad (t \in [0, 1])$$

ha φ *3-monoton, és* $(-\varphi)$ *2-monoton* D^+*-n. Továbbá, minden* $x, y \in D$ *esetén*

$$C_\varphi(x, y, t) = \mathcal{S}_\varphi(t, \|x - y\|) \qquad (t \in [0, 1]),$$

ha, $u \mapsto \frac{\varphi(u)}{u}$ *konkáv* $D^+ \setminus \{0\}$*-n.*

Alkalmazva az előző eredményt (ε, q)-Jensen-konvex függvényekre, a következőket kapjuk.

Következmény. (Makó–Páles [**43**],[**48**] Tabor–Tabor [**66**])
Legyen $\varepsilon \in \mathbb{R}_+$, $q > 0$. *Ha* $\varphi(u) := \varepsilon u^q$, *akkor bármely rögzített* $x, y \in D$ *és* $t \in [0, 1]$ *esetén*

$$C_\varphi(x, y, t) = \begin{cases} \varepsilon T_q(t)\|x - y\|^q & ha \quad 0 < q \le 1, \\ \varepsilon S_q(t)\|x - y\|^q & ha \quad 1 \le q \le 2. \end{cases}$$

Az alábbiakban legyen D egy nemüres konvex részhalmaza az X lineáris térnek. Legyen $D^* := (D - D)$ és $D^{2*} := \{(x, y) \in D^2 \mid x \neq y\}$. Tekintsük az $\alpha : D^* \to \mathbb{R}$ páros hibafüggvényt. (Vegyük észre, hogy α-ról nincs megkövetelve a nemnegativitás.) Azt mondjuk, hogy $f : D \to \mathbb{R}$ *α-Jensen-konvex*, ha, minden $x, y \in D$ esetén

$$f\Big(\frac{x + y}{2}\Big) \le \frac{f(x) + f(y)}{2} + \alpha(x - y).$$

A második fejezetben α-Jensen-konvex függvények kapcsolatát vizsgáljuk, olyan függvényekkel, amelyek teljesítik a következő felső Hermite–Hadamard-féle egyenlőtlenséget:

$$(1) \qquad \int_0^1 f(tx + (1 - t)y)\rho(t)dt \le \lambda f(x) + (1 - \lambda)f(y) + \beta(x - y) \qquad (x, y \in D),$$

ahol $\beta : D^* \to \mathbb{R}$ a hibafüggvény, $\lambda \in [0, 1]$, és $\rho : [0, 1] \to \mathbb{R}_+$ egy nemnegatív integrálható függvény, amelyre $\int_0^1 \rho = 1$.

A következő két tétel alsó Hermite–Hadamard-féle egyenlőtlenségeket állít α-Jensen-konvex függvényekre.

Tétel. (Makó–Páles [**45**])
Legyen $\alpha : D^* \to \mathbb{R}_+$ *irányonként korlátos, mérhető és* $\rho : [0, 1] \to \mathbb{R}_+$ *egy Lebesgue-integrálható függvény, amely* $\int_0^1 \rho = 1$. *Tegyük fel, hogy* $f : D \to \mathbb{R}$ *hemi-integrálható*

és α-Jensen-konvex D-n. Ekkor az f függvény $\lambda := \int_0^1 t\rho(t)dt$-val és a $\beta : D^* \to \mathbb{R}_+$ hibafüggvénnyel, ahol

$$\beta(u) := \sum_{n=0}^{\infty} \frac{1}{2^n} \int_0^1 \alpha\big(d_{\mathbb{Z}}(2^n t)u\big)\rho(t)dt \quad (u \in D^*).$$

teljesíti a felső Hermite–Hadamard típusú (1) *egyenlőtlenséget.*

Tétel. (Makó–Páles [**45**])
Legyen $\alpha : D^* \to \mathbb{R}_+$ olyan nemnegatív, irányonként monoton növekvő függvény, amelyre $\sum_{n=0}^{\infty} \alpha(\frac{u}{2^n}) < \infty$ ha $u \in D^*$. Tegyük fel, hogy $f : D \to \mathbb{R}$ felülről hemi-folytonos és α-Jensen-konvex D-n. Ekkor az f függvény $\lambda := \int_0^1 t\rho(t)dt$-val és a $\beta : D^* \to \mathbb{R}_+$ hibafüggvénnyel, ahol

$$\beta(u) := \sum_{n=0}^{\infty} \alpha\Big(\frac{u}{2^n}\Big) \int_0^1 d_{\mathbb{Z}}(2^n t)\rho(t)dt \qquad (u \in D^*)$$

teljesíti a felső Hermite–Hadamard-féle (1) *egyenlőtlenséget.*

A következőkben azt az esetet tekintjük, amikor az α hibafüggvény $\varepsilon\|\cdot\|^q$ alakú és $\rho \equiv 1$.

Következmény. (Makó–Páles [**45**])
Legyen $\varepsilon \in \mathbb{R}$ és $q \in \mathbb{R}_+$. Tegyük fel, hogy $f : D \to \mathbb{R}$ hemi-integrálható és (ε, q)-Jensen-konvex D-n. Ekkor f teljesíti a következő felső Hermite–Hadamard-féle egyenlőtlenséget:

$$\int_0^1 f(tx + (1-t)y)dt \le \frac{f(x) + f(y)}{2} + \frac{2\varepsilon}{q+1}\|x-y\|^q \qquad (x, y \in D).$$

Következmény. (Makó–Páles [**45**])
Legyen $\varepsilon, q \in \mathbb{R}_+$. Tegyük fel, hogy $f : D \to \mathbb{R}$ felülről hemi-folytonos és (ε, q)-Jensen-konvex D-n. Ekkor, f teljesíti a következő felső Hermite–Hadamard-féle egyenlőtlenséget:

$$\int_0^1 f(tx + (1-t)y)dt \le \frac{f(x) + f(y)}{2} + \frac{2^q\varepsilon}{2^{q+1}-2}\|x-y\|^q \qquad (x, y \in D).$$

A következő tétel azt állítja, hogy a felső Hermite–Hadamard-féle egyenlőtlenségből is következik α-Jensen-konvexitás típusú egyenlőtlenség, ha az α-t megfelelő módon választjuk. A tétel bizonyítása egy multiplikatív típusú konvolúció bevezetésén és tulajdonságain alapszik.

Tétel. (Makó–Páles [**45**])
Legyen $\lambda \in [0, 1]$, $\beta : D^* \to \mathbb{R}$ páros és irányonként felülről félig folytonos, ρ :

$[0,1] \to \mathbb{R}_+$ *integrálható, amelyre* $\int_0^1 \rho = 1$. *Tegyük fel továbbá, hogy létezik* $c \geq 0$ *és* $p > 0$ *úgy, hogy*

$$\rho(t) \leq c(-\ln|1-2t|)^{p-1} \qquad (t \in]0, \tfrac{1}{2}[\cup]\tfrac{1}{2}, 1[).$$

Ekkor, ha az alulról hemi-folytonos $f : D \to \mathbb{R}$ *teljesíti a Hermite–Hadamard-féle* (1) *egyenlőtlenséget, akkor* α*-Jensen-konvex, ahol* $\alpha : D^* \to \mathbb{R}$ *irányonként alulról félig folytonos megoldása a*

$$\alpha(u) \geq \int_0^1 \alpha(|1-2t|u)\rho(t)dt + \beta(u) \qquad (u \in D^*)$$

függvényegyenlőtlenségnek és $\alpha(0) \geq \beta(0)$.

Most tekintsük azt a speciális esetet, amikor $\rho \equiv 1$ és a β függvény $\varepsilon\|\cdot\|^q$ alakban írható.

Következmény. (Makó–Páles [**45**])
Legyen $\lambda \in [0,1]$, $\varepsilon \in \mathbb{R}$ *és* $q > 0$. *Tegyük fel, hogy az alulról hemi-folytonos* $f : D \to \mathbb{R}$ *teljesíti a*

$$\int_0^1 f(tx+(1-t)y)dt \leq \lambda f(x) + (1-\lambda)f(y) + \varepsilon\|x-y\|^q \qquad (x, y \in D)$$

Hermite–Hadamard-féle egyenlőtlenséget. Ekkor az f *függvény* $(\varepsilon\frac{q+1}{q}, q)$*-Jensen-konvex* D*-n.*

Legyen (ω_0, ω_1) egy pozitív Csebisev rendszer az I intervallumon. A harmadik fejezetben (ω_0, ω_1)-konvexitás típusú egyenlőtlenségek és alsó Hermite–Hadamard-féle egyenlőtlenségek kapcsolatával foglalkozunk. Először azt vizsgáljuk, hogy egy közelítőleg (ω_0, ω_1)-konvex függvényre milyen alsó Hermite–Hadamard-féle egyenlőtlenség igaz. Tekintsük a következő feltételeket.

(A1) Legyen $(T, \mathcal{A}, \mu)$ egy mértéktér.
(A2) Legyen $\Lambda : T \times \Delta^\circ(I) \to \mathbb{R}_+$ az első változójában μ-integrálható függvény.
(A3) Legyen $M : T \times \Delta^\circ(I) \to \mathbb{R}$ az első változójában $\mathcal{A}$-mérhető és minden $t \in T$-re, $(x, y) \mapsto M(t, x, y)$ egy kétváltozós közép I-n. Továbbá legyen $M_0 : \Delta^\circ(I) \to I$ egy szigorú közép I-n és tegyük fel, hogy

$$\mu\{t \in T \mid \Lambda(t, x, y) > 0,\ M(t, x, y) \neq M_0(x, y)\} > 0. \tag{2}$$

(A4) Tegyük fel, hogy létezik egy (ω_0, ω_1)-Csebisev rendszer I-n, úgy hogy $\omega_0 > 0$ és $i \in \{0, 1\}$ esetén

$$\omega_i(M_0(x, y)) = \int_T \Lambda(t, x, y)\omega_i(M(t, x, y))d\mu(t) \qquad ((x, y) \in \Delta^\circ(I)). \tag{3}$$

Tétel. (Makó–Páles [**47**])
Tegyük fel, hogy (A1)–(A4) teljesül és minden $(x,y)\in\Delta^\circ(I)$ esetén $\varepsilon_{x,y}:[x,y]\to\mathbb{R}$ egy olyan függvény, hogy tetszőleges $u\in]x,y[$ esetén a $(v,w)\mapsto\varepsilon_{v,w}(u)$ függvény korlátos és Borel-mérhető $[x,u]\times[u,y]$-en. Ekkor, ha a lokálisan felülről korlátos Borel mérhető $f:I\to\mathbb{R}$ függvény megoldása az

$$f(u)\leq\frac{\Omega(u,y)}{\Omega(x,y)}f(x)+\frac{\Omega(x,u)}{\Omega(x,y)}f(y)+\varepsilon_{x,y}(u)\qquad(u\in[x,y])\tag{4}$$

(ω_0,ω_1)-konvexitás típusú egyenlőtlenségnek, akkor f teljesíti a

$$f(M_0(x,y))\leq\int_T\Lambda(t,x,y)f(M(t,x,y))d\mu(t)+\mathcal{E}(x,y)\quad((x,y)\in\Delta(I))\tag{5}$$

alsó Hermite–Hadamrd-féle egyenlőtlenséget, ahol $\mathcal{E}:\Delta^\circ(I)\to\mathbb{R}$

$$\mathcal{E}(x,y):=\frac{\int\limits_{T'_{x,y}}\int\limits_{T''_{x,y}}\Upsilon(t',t'',x,y)\varepsilon_{M(t',x,y),M(t'',x,y)}(M_0(x,y))d\mu(t'')d\mu(t')}{\int\limits_{T'_{x,y}}\int\limits_{T''_{x,y}}\Upsilon(t',t'',x,y)d\mu(t'')d\mu(t')}$$

módon van értelmezve és minden $(t',t'',x,y)\in T^2\times\Delta^\circ(I)$ esetén

$$\begin{aligned}\Upsilon(t',t'',x,y)&:=\Lambda(t',x,y)\Lambda(t'',x,y)\Omega(M(t',x,y),M(t'',x,y)),\\ T'_{x,y}&:=\{t\in T\mid\Lambda(t,x,y)>0,\ M(t,x,y)<M_0(x,y)\},\\ T''_{x,y}&:=\{t\in T\mid\Lambda(t,x,y)>0,\ M(t,x,y)>M_0(x,y)\}.\end{aligned}$$

A másik irányú implikáció bizonyítása egy Korovkin-féle tételen alapszik, amely segítségével közelítő konvexitás típusú egyenlőtlenséget kaphatunk alsó Hermite–Hadamard-féle egyenlőtlenségből. Tekintsük a következő feltételeket:

(B1) $(T,\mathcal{A},\mu)$ egy mértéktér.
(B2) $\Lambda:T\times\Delta(I)\to\mathbb{R}_+$ az első változójában mérhető és minden $t\in T$ esetén a $(x,y)\mapsto\Lambda(t,x,y)$ leképezés változónként folytonos; továbbá, minden $(x,y)\in\Delta(I)$ esetén a

$$L_{x,y}(t):=\sup_{u\in[x,y]}\max\left(\Lambda(t,x,u),\Lambda(t,u,y)\right)$$

módon definiált $L_{x,y}:T\to\mathbb{R}$ függvény μ-integrálható, azaz $\int_T L_{x,y}(t)d\mu(t)<+\infty$.
(B3) $M:T\times\Delta(I)\to\mathbb{R}$ az első változójában mérhető és minden $t\in T$ esetén az $(x,y)\mapsto M(t,x,y)$ leképezés változónként folytonos és változónként parciálisan differenciálható $I\times I$ átlóján. $M_0:\Delta(I)\to I$ változónként folytonos, monoton növekvő és parciálisan differenciálható $I\times I$ átlóján és $\partial_1M_0(z,z)>0$, $\partial_2M_0(z,z)>0$ minden $z\in I$ esetén. Továbbá, minden $t\in T$ esetén $M(t,\cdot,\cdot)$

változónként egyenletesen nyugodt M_0-ra nézve $I\times I$ átlóján, azaz minden $z\in I$ esetén léteznek $\delta>0$ és $K\geq 0$ konstansok úgy, hogy bármely $t\in T$ esetén

$$z-M(t,u,z)\leq K(z-M_0(u,z)) \qquad (u\in[z-\delta,z]),$$
$$M(t,z,u)-z\leq K(M_0(z,u)-z) \qquad (u\in[z,z+\delta]).$$

Továbbá, (2) teljesül és, minden $z\in I$ és $i\in\{0,1\}$ esetén

$$\mu\{t\in T\mid \Lambda(t,z,z)\neq 0,\ \partial_i M(t,z,z)\neq\partial_i M_0(z,z)\}>0.$$

(B4) Léteznek $\omega_0,\omega_1: I\to\mathbb{R}$ függvények, hogy ω_0 pozitív, $\frac{\omega_1}{\omega_0}$ differenciálható I-n, és $(\frac{\omega_1}{\omega_0})'>0$. Továbbá, ha $i\in\{0,1\}$ akkor (3) is teljesül.

Tétel. (Makó–Páles [**46**])
Tegyük fel, hogy (B1)–(B4) teljesül és $\mathcal{E}:\Delta(I)\to\mathbb{R}$ *esetén az* $f: I\to\mathbb{R}$ *függvény felülről félig folytonos megoldása az* (5) *egyenlőtlenségnek. Minden* $(x,y)\in\Delta^\circ(I)$ *esetén legyen* $\varepsilon_{x,y}:[x,y]\to\mathbb{R}$ *egy olyan alulról félig folytonos függvény, amely megoldása a*

$$\varepsilon_{x,y}(M_0(x,u))\geq\int_T \Lambda(t,x,u)\varepsilon_{x,y}(M(t,x,u))d\mu(t)+\mathcal{E}(x,u) \qquad (u\in[x,y]),$$
$$\varepsilon_{x,y}(M_0(u,y))\geq\int_T \Lambda(t,u,y)\varepsilon_{x,y}(M(t,u,y))d\mu(t)+\mathcal{E}(u,y) \qquad (u\in[x,y])$$

függvényegyenlőtlenség-rendszernek és $\varepsilon_{x,y}(x)=\varepsilon_{x,y}(y)=0$*. Ekkor, minden rögzített* $(x,y)\in\Delta^\circ(I)$ *esetén az* f *függvény teljesíti a* (ω_0,ω_1)*-konvexitás típusú* (4) *egyenlőtlenséget.*

A következő függvényegyenlőtlenségek kapcsolatának vizsgálata az előző tételek alkalmazása segítségével történhet. Tekintsük a:

(6) $$f((1-t)x+ty)\leq(1-t)f(x)+tf(y)+e_{x,y}(t) \quad ((x,y)\in D^2,\ t\in[0,1])$$

konvexitás típusú egyenlőtlenséget, ahol $f: D\to\mathbb{R}$ az ismeretlen függvény és minden $(x,y)\in D^2$ esetén $e_{x,y}:[0,1]\to\mathbb{R}$ a hibafüggvény. Illetve tekintsük a

(7) $$f((1-\mu_1)x+\mu_1 y)\leq\int_{[0,1]} f((1-t)x+ty)d\mu(t)+E(x,y) \quad ((x,y)\in D^2)$$

alsó Hermite–Hadamard-féle egyenlőtlenséget, ahol $f: D\to\mathbb{R}$ az ismeretlen függvény, μ valószínűségi mérték $[0,1]$-n, $\mu_1:=\int_{[0,1]}td\mu(t)$ és $E: D^2\to\mathbb{R}$. A harmadik fejezet másik eredménye tehát (6) és (7) összehasonlítása.

Tétel. (Makó–Páles [**47**])
Legyen $\mathcal{A}$ *egy olyan* σ*-algebra, amely tartalmazza* $[0,1]$ *Borel-mérhető részhalmazait*

és μ egy olyan valószínűségi mérték a $([0,1],\mathcal{A})$ mérhető téren, hogy μ tartója nem egyelemű. Jelölje $S(\mu)$ a

$$S(\mu) := \mu([0,\mu_1]) \int\limits_{]\mu_1,1]} t d\mu(t) - \mu(]\mu_1,1]) \int\limits_{[0,\mu_1]} t d\mu(t)$$

kifejezést. Tegyük fel, hogy $f : D \to \mathbb{R}$ hemi-μ-integrálható megoldása (6)*-nak ahol, minden $(x,y) \in D^{2*}$, $e_{x,y} : [0,1] \to \mathbb{R}$ egy olyan függvény, hogy*

$$I(x,y) := \int\limits_{]\mu_1,1]} \int\limits_{[0,\mu_1]} (t'' - t') e_{(1-t')x+t'y,(1-t'')x+t''y}\Big(\frac{\mu_1 - t'}{t'' - t'}\Big) d\mu(t') d\mu(t'')$$

a kifejezés létezik $[-\infty,\infty]$-ben. Ekkor, minden $(x,y) \in D^{2*}$ esetén az f függvény megoldása az alsó Hermite–Hadamard-féle (7) *egyenlőtlenségnek, ahol*

$$E(x,y) := \frac{I(x,y)}{S(\mu)}.$$

A fordított irányú implikációt a következő tétel tartalmazza.

Tétel. (Makó–Páles [**46**])
Legyen μ egy olyan Borel-valószínűségi mérték $[0,1]$-n, hogy μ tartója nem egyelemű. Jelölje $\mu_1 := \int_{[0,1]} t d\mu(t)$. Tegyük fel, hogy minden $(x,y) \in D^2$ esetén a felülről hemi-folytonos $f : D \to \mathbb{R}$ megoldása a Hermite–Hadamard-féle (7) *egyenlőtlenségnek, ahol $E : D^2 \to \mathbb{R}$. Tegyük fel továbbá, hogy minden $(x,y) \in D^2$ esetén az alulról félig folytonos $e_{x,y} : [0,1] \to \mathbb{R}$ megoldása a*

$$e_{x,y}(s) \geq \begin{cases} \int\limits_{[0,1]} e_{x,y}\big(\frac{st}{\mu_1}\big) d\mu(t) + E\big(x, (1 - \frac{s}{\mu_1})x + \frac{s}{\mu_1} y\big) & (s \in [0,\mu_1]), \\ \int\limits_{[0,1]} e_{x,y}\big(1 - \frac{(1-s)(1-t)}{1-\mu_1}\big) d\mu(t) + E\big(\frac{1-s}{1-\mu_1} x + (1 - \frac{1-s}{1-\mu_1}) y, y\big) & (s \in [\mu_1, 1]) \end{cases}$$

függvényegyenlőtlenség-rendszernek és $e_{x,y}(0) = e_{x,y}(1) = 0$. Ekkor, minden $(x,y) \in D^2$ és $t \in [0,1]$ esetén az f függvény teljesíti a közelítő konvexitás típusú (6) *egyenlőtlenséget.*

Ennek az eredménynek egy fontos alkalmazása a következő Bernstein–Doetsch-féle tétel.

Következmény. (Makó–Páles [**46**])
Legyen $\tau \in]0,1[$ és $E : D^2 \to \mathbb{R}$. Tegyük fel, hogy $f : D \to \mathbb{R}$ felülről hemi-folytonos megoldása a

$$f((1-\tau)x + \tau y) \leq (1-\tau) f(x) + \tau f(y) + E(x,y) \qquad ((x,y) \in D^2)$$

Jensen-féle egyenlőtlenségnek. Tegyük fel továbbá, hogy minden $(x, y) \in D^2$ esetén az alulról félig folytonos $e_{x,y} : [0, 1] \to \mathbb{R}$ függvény teljesíti a

$$e_{x,y}(s) \geq \begin{cases} \tau e_{x,y}(\frac{s}{\tau}) + E(x, (1 - \frac{s}{\tau})x + \frac{s}{\tau}y) \quad (s \in [0, \tau]), \\ (1 - \tau)e_{x,y}(\frac{s-\tau}{1-\tau}) + E(\frac{1-s}{1-\tau}x + (1 - \frac{1-s}{1-\tau})y, y) \quad (s \in [\tau, 1]) \end{cases}$$

függvényegyenlőtlenség-rendszert és $e_{x,y}(0) = e_{x,y}(1) = 0$. Ekkor, minden $(x, y) \in D^2$ és $t \in [0, 1]$ esetén az f függvény teljesíti a közelítő konvexitás típusú (6) egyenlőtlenséget.

Publications of the author

1. J. Makó, Zs. Páles, *Approximate convexity of Takagi type functions*, J. Math. Anal. Appl., 369:545–554, 2010.
2. J. Makó, Zs. Páles, *Strengthening of strong and approximate convexity*, Acta Math. Hungar., 132:78–91, 2011.
3. J. Makó, Zs. Páles, *On φ-convexity*, Publ. Math. Debrecen, 80:107–126, 2012.
4. J. Makó, Zs. Páles, *Implications between approximate convexity properties and approximate Hermite–Hadamard inequalities*, Cent. Eur. J. Math., 10:1017–1041, 2012.
5. J. Makó, K. Nikodem, Zs. Páles, *On strong $(\alpha,\mathbb{F})$-convexity*, Math. Inequal. Appl., 15:289–299, 2012.
6. J. Makó, Zs. Páles, *Korovkin type theorems and approximate Hermite–Hadamard inequalities*, J. Approx. Theory, 164:1111-1142, 2012.
7. J. Makó, Zs. Páles, *Approximate Hermite–Hadamard type inequalities for approximately convex functions*, Math. Inequal. Appl., 16:507–526, 2013.
8. J. Makó, Zs. Páles, *On approximately convex Takagi type functions*, Amer. Math. Soc., 141:2069-2080 , 2013.

The list of the talks

(1) On φ-convexity. *13th International Conference on Functional Equations and Inequalities,* Male Ciche, Poland, 2009, September 13–19;

(2) φ-konvex függvények, *Miskolci Függvényegyenletek Nap Dr. Vincze Endre emlékére,* Miskolc, Hungary, 2009, November 27;

(3) Approximate convexity of Takagi type functions, *International Students' Conference on Analysis,* Síkfőkút, Hungary, 2010, January 30–February 3;

(4) Approximate convexity of Takagi type functions, *Debrecen-Katowice Winter Seminar,* Zamárdi, 2010, February 3–6;

(5) Takagi típusú függvények konvexitási tulajdonságai, *Síkfőkúti Analízis Szeminárium,* Síkfőkút, Hungary, 2010, May 21–24;

(6) On φ-convexity, *Bolyai János Emlékkonferencia,* Budapest-Marosvásárhely, Hungary-Romania, 2010, August 30–September 4;

(7) On convexity properties of Takagi type functions, *Convexity and applications,* Iwonicz Zdroj, Poland, 2010, September 5–10;

(8) Approximate convexity of Takagi type functions, *Conference of Inequalities and Applications,* Hajdúszoboszló, Hungary, 2010, September 19–25;

(9) Approximate Hermite–Hadamard inequality, *Katowice–Debrecen Winter Seminar,* Wisła-Malinka, Poland, 2011, February 2–5;

(10) On strong $(\alpha, \mathbb{F})$-convexity, *International Students' Conference on Analysis,* Wisła, Poland, 2011, February 5–8;

(11) Egy Korovkin típusú tétel alkalmazása a konvex analízisben, *Síkfőkúti Analízis Szeminárium,* Síkfőkút, Hungary, 2010, May 27–29;

(12) On strong $(\alpha, \mathbb{F})$-convexity, *10th International Symposium on Generalized Convexity and Monotonicity,* Kolozsvár, Romania, 2011, August 22-27;

(13) Implications between approximate convexity properties and approximate Hermite–Hadamard inequalities, *The 14th International Conference on Functional Equations and Inequalities,* Będlewo, Poland, 2011, September 11–17;

(14) A Takagi függvény konvexitási tulajdonságai *Miskolci Egyetem, Matematika Intézet, Intézeti Szeminárium,* Miskolc, Hungary, 2011, november 9;

(15) Approximate Hermite–Hadamard type inequalities for approximately convex functions *12th Debrecen-Katowice Winter Seminar on Functional Equations and Inequalities,* Hajdúszoboszló, Hungary, 2012, January 25–28;

(16) Approximate Hermite–Hadamard type inequalities, *International Students' Conference on Analysis,* Síkfőkút, Hungary, 2012, January 28–31;

(17) On Hermite–Hadamard type inequalities *50th International Symposium on Functional Equations,* Hajdúszoboszló, Hungary, 2012, June 17-24;

(18) On Hermite–Hadamard type inequalities *The International Conference on Mathematical Inequalities and Nonlinear Functional Analysis with Applications* Gyeongsang National University, Jinju, Korea, 2012, July 25–29;

(19) Strengthening of approximate convexity *13th Debrecen-Katowice Winter Seminar on Functional Equations and Inequalities,* Zakopane, Poland, 2013, January 30–February 2;

(20) Strengthening of approximate convexity *International Students' Conference on Analysis,* Ustron, Poland, 2013, February 2–5;

Bibliography

[1] J. Aczél. A generalization of the notion of convex functions. *Norske Vid. Selsk. Forh., Trondhjem*, 19(24):87–90, 1947.

[2] P. C. Allaart. An inequality for sums of binary digits, with application to Takagi functions. *J. Math. Anal. Appl.*, 381(2):689–694, 2011.

[3] P. C. Allaart. The finite cardinalities of level sets of the Takagi function. *J. Math. Anal. Appl.*, 388(2):1117–1129, 2012.

[4] P. C. Allaart and K. Kawamura. The improper infinite derivatives of Takagi's nowhere-differentiable function. *J. Math. Anal. Appl.*, 372(2):656–665, 2010.

[5] P. C. Allaart and K. Kawamura. On the distribution of the cardinalities of level sets of the Takagi function. *arXiv.org*, (1107.0712), 2011.

[6] P. C. Allaart and K. Kawamura. The Takagi function: a survey. *arXiv.org*, (1110.1691), 2011.

[7] F. Altomare and M. Campiti. Korovkin-type approximation theory and its applications *de Gruyter Studies in Mathematics, Walter de Gruyter & Co., Berlin*, 17, , 1994.

[8] F. Bernstein and G. Doetsch. Zur Theorie der konvexen Funktionen. *Math. Ann.*, 76(4):514–526, 1915.

[9] M. Bessenyei. Hermite–Hadamard-type inequalities for generalized convex functions. *J. Inequal. Pure Appl. Math.*, 9(3):Article 63, pp. 51 (electronic), 2008.

[10] M. Bessenyei and Zs. Páles. Hadamard-type inequalities for generalized convex functions. *Math. Inequal. Appl.*, 6(3):379–392, 2003.

[11] M. Bessenyei and Zs. Páles. Characterizations of convexity via Hadamard's inequality. *Math. Inequal. Appl.*, 9(1):53–62, 2006.

[12] M. Bessenyei and Zs. Páles. Characterization of higher-order monotonicity via integral inequalities. Proc. R. Soc. Edinburgh Sect. A,140A:723–736, 2010.

[13] P. Billingsley, Notes: Van Der Waerden's Continuous Nowhere Differentiable Function. *Amer. Math. Monthly*, 89(9):691, 1982.

[14] Z. Boros. An inequality for the Takagi function. *Math. Inequal. Appl.*, 11(4):757–765, 2008.

[15] A.M. Bruckner, J.B. Bruckner and B.S. Thomson. Real Analysis. *International Edition. Upper Saddle River, NJ: Prentice-Hall International*, 1997.

[16] P. Cannarsa and C. Sinestrari. Semiconcave functions, Hamilton-Jacobi equations, and optimal control. *Progress in Nonlinear Differential Equations and their Applications, 58. Birkhäuser Boston Inc., Boston, MA*, 2004.

[17] F. S. Cater, On van der Waerden's nowhere differentiable function. *Amer. Math. Monthly*, 91:307–308, 1984.

[18] E. de Amo, I. Bhouri, M. Díaz Carrillo, and J. Fernández-Sánchez. The Hausdorff dimension of the level sets of Takagi's function. *Nonlinear Anal., Theory Methods Appl., Ser. A*, 74(15):5081–5087, 2011.

[19] E. de Amo and J. Fernández-Sánchez. Takagi's function revisited from an arithmetical point of view. *Int. J. Pure Appl. Math.*, 54(3):407–427, 2009.

[20] S. J. Dilworth, R. Howard, and J. W. Roberts. Extremal approximately convex functions and estimating the size of convex hulls. *Adv. Math.*, 148(1):1–43, 1999.

[21] S. J. Dilworth, R. Howard, and J. W. Roberts. Extremal approximately convex functions and the best constants in a theorem of Hyers and Ulam. *Adv. Math.*, 172(1):1–14, 2002.

[22] S. J. Dilworth, R. Howard, and J. W. Roberts. A general theory of almost convex functions. *Trans. Amer. Math. Soc.*, 358(8):3413–3445 (electronic), 2006.

[23] S. S. Dragomir and C. E. M. Pearce. Selected Topics on Hermite-Hadamard Inequalities. RGMIA Monographs (`http://rgmia.vu.edu.au/monographs/hermite_hadamard.html`), Victoria University, 2000.

[24] R. Ger. Almost approximately convex functions. *Math. Slovaca*, 38(1):61–78, 1988.

[25] A. Gilányi and Zs. Páles. On Dinghas-type derivatives and convex functions of higher order. *Real Anal. Exchange*, 27(2):485–493, 2001.

[26] A. Gilányi and Zs. Páles. On convex functions of higher order. *Math. Inequal. Appl.*, 11(2):271–282, 2008.

[27] J. W. Green. Approximately convex functions. *Duke Math. J.*, 19:499–504, 1952.

[28] J. Hadamard. Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. *J. Math. Pures Appl.*, 58:171–215, 1893.

[29] A. Házy. On approximate t-convexity. *Math. Inequal. Appl.*, 8(3):389–402, 2005.

[30] A. Házy. On stability of t-convexity. In *Proc. MicroCAD 2007 Int. Sci. Conf.*, volume G, pages 23–28, 2007.

[31] A. Házy. On the stability of t-convex functions. *Aequationes Math.*, 74(3):210–218, 2007.

[32] A. Házy and Zs. Páles. On approximately midconvex functions. *Bull. London Math. Soc.*, 36(3):339–350, 2004.

[33] A. Házy and Zs. Páles. On approximately t-convex functions. *Publ. Math. Debrecen*, 66:489–501, 2005.

[34] A. Házy and Zs. Páles. On a certain stability of the Hermite–Hadamard inequality. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 465(2102):571–583, 2009.

[35] D. H. Hyers and S. M. Ulam. Approximately convex functions. *Proc. Amer. Math. Soc.*, 3:821–828, 1952.

[36] H.-H. Kairies. Takagi's function and its functional equations. *Rocznik Nauk.-Dydakt. Prace Mat.*, (15): 73–83, 1998.

[37] K. Knopp. Ein einfaches Verfahren zur Bildüng stetiger nirgends differenzierbarer Funktionen. *Math. Z.*, 2(1-2):1–26, 1918.

[38] M. Krüppel. On the extrema and the improper derivatives of Takagi's continuous nowhere differentiable function. *Rostocker Math. Kolloq.*, 62:41–59, 2007.

[39] M. Krüppel. Takagi's continuous nowhere differentiable function and binary digital sums. *Rostocker Math. Kolloq.*, 63:37–54, 2008.

[40] P. P. Korovkin On convergence of linear positive operators in the space of continuous functions *Doklady Akad. Nauk SSSR (N.S.)*, 90:961–964, 1953.

[41] M. Kuczma. An Introduction to the Theory of Functional Equations and Inequalities, volume 489 of *Prace Naukowe Uniwersytetu Śląskiego w Katowicach*. Państwowe Wydawnictwo Naukowe — Uniwersytet Śląski. Warszawa-Kraków-Katowice, 1985.

[42] M. Laczkovich. The local stability of convexity, affinity and of the Jensen equation. *Aequationes Math.*, 58:135–142, 1999.

[43] J. Makó and Zs. Páles. Approximate convexity of Takagi type functions. *J. Math. Anal. Appl.*, 369:545–554, 2010.

[44] J. Makó and Zs. Páles. On φ-convexity. *Publ. Math. Debrecen*, 80:107–126, 2012.

[45] J. Makó and Zs. Páles. Implications between approximate convexity properties and approximate Hermite–Hadamard inequalities. *Cent. Eur. J. Math.*, 10:1017–1041, 2012.

[46] J. Makó and Zs. Páles. Korovkin type theorems and approximate Hermite–Hadamard inequalities. *J. Approx. Theory*, 164:1111-1142, 2012.

[47] J. Makó and Zs. Páles. Approximate Hermite–Hadamard type inequalities for approximately convex functions. *Math. Inequal. Appl.*, 16:507–526, 2013.

[48] J. Makó and Zs. Páles. On approximately convex Takagi type functions. *Proc. Amer. Math. Soc.*, 141: 2069-2080 , 2013.

[49] D. S. Mitrinović and I. B. Lacković. Hermite and convexity. *Aequationes Math.*, 28:229–232, 1985.

[50] J. Mrowiec, Ja. Tabor, and Jó. Tabor. Approximately midconvex functions. In C. Bandle, A. Gilányi, L. Losonczi, M. Plum, and Zs. Páles, editors, *Inequalities and Applications (Noszvaj, 2007)*, volume 157 of *International Series of Numerical Mathematics*, pages 261–267. Birkhäuser Verlag, 2008.

[51] A. Mureńko, Ja. Tabor, and Jó. Tabor. Applications of de Rham Theorem in approximate midconvexity. *J. Diff. Equat. Appl.*, 18:335-344, 2012.

[52] C. T. Ng and K. Nikodem. On approximately convex functions. *Proc. Amer. Math. Soc.*, 118(1):103–108, 1993.

[53] H. V. Ngai, D. T. Luc, and M. Théra. Approximate convex functions. *J. Nonlinear Convex Anal.*, 1(2):155–176, 2000.

[54] H. V. Ngai and J.-P. Penot. Approximately convex functions and approximately monotonic operators. *Nonlin. Anal.*, 66:547–567, 2007.

[55] C. P. Niculescu and L.-E. Persson. Old and new on the Hermite-Hadamard inequality. *Real Anal. Exchange*, 29(2):663–685, 2003/04.

[56] C. P. Niculescu and L.-E. Persson. *Convex Functions and Their Applications*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 23. Springer-Verlag, New York, 2006. A contemporary approach.

[57] K. Nikodem, T. Riedel, and P. K. Sahoo. The stability problem of the Hermite-Hadamard inequality. *Math. Inequal. Appl.*, 10(2):359–363, 2007.

[58] Zs. Páles. On approximately convex functions. *Proc. Amer. Math. Soc.*, 131(1):243–252, 2003.

[59] Zs. Páles. The Forty-first International Symposium on Functional Equations, June 8–15, 2003, Noszvaj, Hungary. *Aequationes Math.*, 67:285–320, 2004.

[60] T. Popoviciu. *Les fonctions convexes*. Hermann et Cie, Paris, 1944.

[61] A. W. Roberts and D. E. Varberg. *Convex Functions*, volume 57 of *Pure and Applied Mathematics*. Academic Press, New York–London, 1973.

[62] S. Rolewicz. On paraconvex multifunctions. In *Third Symposium on Operations Research (Univ. Mannheim, Mannheim, 1978), Section I*, volume 31 of *Operations Res. Verfahren*, pages 539–546. Hain, Königstein/Ts., 1979.

[63] S. Rolewicz. On γ-paraconvex multifunctions. *Math. Japon.*, 24(3):293–300, 1979/80.

[64] S. Rolewicz. On $\alpha(\cdot)$-paraconvex and strongly $\alpha(\cdot)$-paraconvex functions. *Control Cybernet.*, 29(1):367–377, 2000.

[65] S. Rolewicz. Paraconvex analysis. *Control Cybernet.*, 34(3):951–965, 2005.

[66] Ja. Tabor and Jó. Tabor. Generalized approximate midconvexity. *Control Cybernet.*, 38(3):655–669, 2009.

[67] Ja. Tabor and Jó. Tabor. Takagi functions and approximate midconvexity. *J. Math. Anal. Appl.*, 356(2):729–737, 2009.

[68] Ja. Tabor, Jó. Tabor, and M. Żołdak. Approximately convex functions on topological vector spaces. *Publ. Math. Debrecen*, 77:115–123, 2010.

[69] Ja. Tabor, Jó. Tabor, and M. Żołdak. Optimality estimations for approximately midconvex functions. *Aequationes Math.*, 80:227–237, 2010.

[70] T. Takagi. A simple example of the continuous function without derivative. *J. Phys. Math. Soc. Japan*, 1:176–177, 1903.

[71] B. L. van der Waerden. Ein einfaches Beispiel einer nichtdifferenzierbaren stetigen Funktion. *Math. Z.*, 32:474–475, 1930.

[72] E. M. Wright. An inequality for convex functions. *Amer. Math. Monthly*, 61:620–622, 1954.