ADVANCED STATISTICAL TECHNIQUES FOR TESTING BENFORD'S LAW

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Abstract: *The frequency of accounting data frauds has been increased in corporate environment. As a result of that, the research on detection of such irregularities in accounting and auditing is gaining researchers' focus. Bedford's law has been in the literature for the identification of data manipulation in accounting and auditing field. The application of this law in accounting fraud detection started in 1988 after the work of Carslaw (he observed a greater frequency of zeros and less frequency of nines in the second place in the reported earning numbers). The underlying idea about this technique is based on comparison of certain digit frequency to the expected digit pattern proposed by Bedford's law. Various goodness-of-fit test are used to analyze the data conformity to Benford's law based on a null hypothesis of conformity of empirical data to expected data pattern. This study addresses some of the most important goodness-of fit tests that can be used to analyze data pattern and digit behavior. Most importantly chi-square, Kolmogorov-Smirnov test (KS), Euclidean distance, Joenssen's JP-square, Freedman-Watson u-square, Chebyshev distance, Z-statistics and mean absolute deviation tests are discussed with expression to calculate test statistics. Tests like Chi-Square and KS are also sensitive to size of data set, so a combination of various goodness-of fit test is recommended in literature to make more accurate analysis of data conformity.*

Keywords: *Benford's Law; accounting and auditing; Goodness-of-fit; Conformity; Digit pattern.*

JEL Classification: *C58; M40; M41.*

Introduction

Simon Newcomb in 1881 wrote an article in American Journal of Mathematics stating that all digits don't appear equally in term of frequency. He noticed that first 2 or 3 pages of logarithmic books were more torn than later pages indicating the frequent use of those pages and confirmed an unequal frequency of the digit appearance (Newcomb 1881). However, he could not provide further evidence to this phenomenon and his findings remained unnoticed for almost sixty years. Apparently Frank Benford, unaware of Newcomb's theory, rediscovered this law in 1938 and it became Benford's Law (or the *first digit* law or *significant digit* law (Hill 1996) after the name of rediscoverer. In his famous paper 'Law of Anomalous Numbers', he used extensive dataset to analyze the digit frequency (Benford 1938). He begins by analyzing the same phenomenon as Newcomb; the first few pages of logarithmic books were more torn than latter. This implied that the first few pages were used more often than later, confirming that more numbers start with the lower digits like 1, 2 or 3 than higher digits (Benford 1938).
Benford used a comprehensive data analysis approach using 20 lists of large numbers (20,229 observations) and 10 lists of small numbers (2,968 observations). These lists include *independent* as well as *weakly dependent* data. Independent lists

contain data from sources such as street addresses, numbers from different issues of Reader's Digest, area of rivers, and population. Weakly dependent lists contain data from sources such as engineering handbooks and scientific constants. The results showed an approximate 30.6 percent of large numbers contain first digit 1 in contrast to only 4.7 percent of numbers having first digit 9. In the past 40 years, this law gained intensive research focus and is used to analyze data pattern in different fields of research i.e., image forensic (Qadir et al. 2010), computing (Hill 1996), social studies (Pericchi & Torres 2011) economics (Rauch et al. 2014) and accounting data (Nigrini 1996; Nigrini 1999). Essentially, this law is an effective tool to study financial irregularities and intentional data frauds. The notable contribution to this field is made by Carslaw 1988; Durtschi et al. 2004; Nigrini 1996; Nigrini & Mittermaier 1997 and Nigrini 2012.
This study aims to discuss various advanced statistical techniques that can be useful to check the conformity of data to Benford's law (Nigrini 2012). Particularly the main focus is to discuss various tests used in literature extensively to measure the goodness-of-fit (Joenssen 2014). Various classical and non-classical tests, such as Pearson Chi-square test, Kolmogorov–Smirnov goodness-of-fit test, mean absolute deviation test, z- statistic Euclidean distance test are discussed with their significance for empirical results and analysis.
The rest of this paper is organized as follows. Section II will outlines mathematical foundation of Benford's law, section III explains some empirical studies in detecting accounting anomalies, section IV explains various advanced statistical tools to check the conformity (non-conformity) of data. Section IV concludes.

Mathematical Foundation of Benford's Law

According to Benford's hypothesis, real-world number starting with 1 or 2 are more than numbers starting with 8 or 9. The analysis of diverse data sets confirmed that digit 1 appears 30.6% as first digit, whereas digit 2 appears 18.5% of times as first digit (Benford 1938). The general expression for this law can be derived using logarithmic pattern as

$$P(D_1 = d_1) = log\left(1 + \frac{1}{d1}\right) for\ d_1 \in \{1, \ldots, 9\} \quad (1)$$

$$P(D_2 = d_2) = \Sigma_{d_{1=1}}^{9} log\left(1 + \frac{1}{d1d2}\right)\ for\ d_2 \in \{1, \ldots, 9\} \quad (2)$$

$$P(D_1D_2 = d_1d_2) = log\left(1 + \frac{1}{d_1d_2}\right)\ for\ d_1\ d_2 \in \{1, \ldots, 9\}\ (3)$$

Where D_1 represents the first digit, D_2 the second digit and hence D_1D_2 represents first two digits. P represents the probability of occurrence of digit pattern. The expected probabilities of first four digits are given in *table 1*. For calculating probability of the joint-digit, for example 88 has a probability of 0.49% *[log (1+1/88)]*. The probability of first two digits as 10 can be calculated as:

$$P(D_1D_2 = 10) = log\left(1 + \frac{1}{10}\right) \cong 4.14\% \quad (4)$$

We can write probabilities in the form of Significant-digit law (Theodore P. Hill 1995b)

$$P(D_1D_2D_3 = 314) = log\left(1 + \frac{1}{314}\right) \cong .0014 \quad (5)$$

Table 1 shows that there is notable deviation in probabilities of smaller digits like 1 and 2 at first place. When we move towards the larger digits (8 or 9), the probabilities of digits become less evident (Nigrini & Mittermaier 1997; Alali & Romero 2013)

Table 1: Expected Digit Frequencies of Benford's Law

	Position in Number			
Digit	1st	2nd	3rd	4th
0		0.11968	0.10178	0.10018
1	0.30103	0.11389	0.10138	0.10014
2	0.17609	0.10882	0.10097	0.10010
3	0.12494	0.10433	0.10057	0.10006
4	0.09691	0.10031	0.10018	0.10002
5	0.07918	0.09668	0.09979	0.09998
6	0.06695	0.09337	0.09940	0.09994
7	0.05799	0.09035	0.09902	0.09990
8	0.05115	0.08757	0.09864	0.09986
9	0.04576	0.08500	0.09827	0.09982

Source:(Nigrini 1996)
Note: The number 199 has three digits, 1 being fist digit and 9 being 2nd and 3rd digit. The table shows that the probability of 2 as 1st digit is 17.6%, whereas the probability of 2 as 4th digit is 10.01%

Benford's law in Accounting and Auditing

The focus of researcher on use of Benford's law started in the beginning of 90's. Since the past two decades, hundreds of studies have been published to test the validity of this law in assessing accounting statement quality (Nigrini 2005). The pioneering research in finding accounting application of Benford's law was undertaken by Carslaw (1988). His proposition of rounding the second digit is based on psychological phenomenon; where income of firms proximate to a psychological limit, managers would tend to round up the income numbers. Performing second-digit analysis of New-Zealand based firms, he found that frequency of zeros is much higher than expected. Whereas expected frequency of nines is less than expected in second digits of reported income. Moreover, the income such as $*19.98* million is being rounded to $*20* million since there is a negligible difference. Thomas (1989) observed the same phenomenon for a sample of US firms, reporting less deviation in income numbers as compared to findings of Carslaw (1988). For the firms reported losses, he observed a reverse phenomenon with less zeros and more nines.

The other notable contribution is made by Mark J. Nigrini. Using Benford's law and distortion factor model, he considered the taxpayers' behavior and reported numbers to analyze the planned evasion and unintentional error (Nigrini 1996). Other studies such as the digital analysis to check conformity to Benford's law (Nigrini & Mittermaier 1997) and rounding of transaction level data (Nigrini & Miller 2009b) to show that Benford's law can be an efficient tool for analyzing manipulation. Hill (1995a) provided a formal mathematical proof to Benford's law and demonstrated how it actually works in stock market data and other accounting data. Durtschi et al. (2004) provided an empirical evidence to Benford's law using accounting data. They observed close conformity to Law for the first digit of chosen sample. Rauch et al.

(2011) used Benford's analysis to identify the deviations in macroeconomic data of European Union countries. Their finding suggested greater deviation to this law in macroeconomic data for Greece.
However, the accounting data conforms Benford's predicted pattern the most of the time (Van Caneghem 2016; Saville 2006; Sadaf 2016; Nigrini & Miller 2009a; Nigrini 2015; Joenssen & Vogel 2014; Alali & Romero 2013). But there are certain conditions when digital analysis is not suitable. A 'false negative analysis may not present the true picture. Durtschi et al. (2004) provided the details of conditions under which Benford's analysis is more useful. Those conditions are the following: set of numbers are resulting mathematical combination (*account receivable, number of units sold*price*), transaction level data *(expenses)*, large scale data *(a full-year record of transactions)* and data having mean value greater than median and is positively skewed. While Benford's analysis is not useful when data set comprised of invoices, checks, intentionally assigned numbers or where there is no record of transaction (Durtschi et al. 2004). Benford's law has certain important properties. If a series of number, showing conformity, is multiplied by a non-zero constant, the resulting series should also conform to Benford's pattern (Wójcik 2014). This characteristic is termed as scale-invariance (Pinkham 1961). The results will be the same if series in Euro is converted into Yen. The other property is the base-invariance, where the change of logarithmic base doesn't affect the expected probability of series (Hill 1995). This Law is also power-invariant; if Benford's series is increased with a certain power, the resultant series would also be a Benford's series (Nigrini 2015).
In order to make more precise analysis any data with expected Benford's series, various statistical techniques are used to measure goodness-of fit. The actual digit distribution is compared to Benford's distribution with the goodness-of-fit test (Archambault & Archambault 2011). Joenssen (2013) analyzed the conformity of accounting data to Benford's law by using four different goodness-of-fit tests.

Techniques for Assessing Conformity to Benford's Law

In this section, this paper discusses advanced statistical test widely used in literature for data conformity analysis to Benford's law.

Chi-Square Test

The most commonly used technique for comparing actual data with expected Benford's expected series is through Pearson Chi-Square goodness-of-fit test. Null hypothesis of this test is that *actual digit conforms to Benford's expected frequency* (Tam Cho & Gaines 2007). The test statistics can be calculated by following expression:

$$\chi 2 = \sum_{i=1}^{k} \frac{(P_i - \pi_i)^2}{P_i} \quad (6)$$

Where P_i is actual or observed frequency, π_i is expected frequency, K represents the number of observations and summation sign indicates that all calculations must be sum up together. This calculated $\chi 2$ value is compared with critical value. If calculated $\chi 2$ value exceeds the critical value, the null hypothesis of no significant difference between the two series is rejected (Joenssen 2014). Chi-square is also affected by size of data, so it should not be relied too much for making inference about data conformity. When data set becomes large (5000 observations), chi-square shows a higher than critical value leading us to conclusion of non- conformity (Nigrini 2012).

Kolmogorov-Smirnov Test (KS)

Another important test for analyzing the data conformity is *Kolmogorov-Smirnov test* (Kolmogorov 1956). This test is based on all digit analysis to check the pattern of data in comparison to Benford's pattern. KS statistics uses maximum deviation from Benford's distribution. KS test statistics *D* uses the largest of the absolute values, called supremum in statistics tem and can be calculated as,

$$D = \sup_{j = 1, \ldots, 9} \left| \sum_{i=1}^{j} P_i - \pi_i \right| . \sqrt{n} \tag{7}$$

We ca calculate critical value of K-S statistics using expression

$$Kolmogorv - Smirnnov\ critical\ valu = \frac{1.36}{\sqrt{N}} \tag{8}$$

Where 1.36 is constant for 5% level of significance and N is the number of observation (Nigrini 2012). KS statistics is also sensitive to sample size, hence its usefulness decreases when the sample size increases (Amiram et al. 2013). The applicability of this test to continuous data only and its sensitivity near center of distribution limits its usefulness (Lolbert 2006).

Euclidean Distance Test

Another important technique that is insensitive to sample size is Euclidean distance between Benford's distribution and test digit vector (Tam Cho & Gaines 2007).

$$d = \sqrt{\sum_{i=1}^{9} (P_i - \pi_i)^2} \tag{9}$$

Where P_i is actual proportion and π_i is expected Benford's distribution**.** This test was used for Benford hypothesis analysis by (Morrow 2010). Previous studies used it for purely descriptive purposes (Tam Cho & Gaines 2007).

Joenssen's JP-square Test

Considering the correlation between the first digit distribution and Benford's distribution, this test performs a goodness-of-fit test to confirm the data conformity to the expected Benford's distribution (Shapiro & Francia 1972; Joenssen 2013).

$$J_P^2 = sgn(cor\ (f^o, f^e)) \cdot cor\ (f^o, f^e)^2 \tag{10}$$

Where $f\text{-}^{0}$ represents the observed frequency and $f^{\,e}$ represents the expected frequency. For small sample size, testing only first digit is useful in this test. Otherwise, testing first two digits is useful for analysis of large sample (Joenssen 2013).

Freedman-Watson U-square Test

Freedman's (Freedman 1981) extension of Watson's (Watson 1961) U_n^2 statistics evaluates the difference between empirical distribution and Benford's distribution to assess the data conformity by goodness-of-fit. The test statistics can be calculated by following expression,

$$U^2 = \frac{n}{9*10^{k-1}} \cdot \left[\sum_{i=10^{k-1}}^{10^{k-2}} \left(\sum_{j=1}^{i}\left(f_j^o - f_j^e\right)\right)^2 - \frac{1}{9.10^{k-1}} \cdot \left(\sum_{i=10^{k-1}}^{10^{k-2}} \sum_{j=1}^{i}\left(f_i^o - f_i^e\right)\right)^2\right] \quad (11)$$

Where U^2 is the Freedman - Watson U^2 and f_i^o and f_i^e represents the same observed and expected frequency of digit *i* and n denotes the number of observations.

Chebyshev Distance Test

This test uses the *Chebyshev* distance between actual and observed digit proportion and performing a goodness-of-fit test by reducing the sample to specify significant. The purpose of test is to assess the data conformity to Benford's distribution (Morrow 2014). The test statistics can be calculated as,

$$m = \max_{i = 10^{k-1}, \ldots 10^k - 1} |f_i^o - f_i^e| . \sqrt{n} \quad (12)$$

Where f_i^o and f_i^e denote the actual and the expected digit proportion (Joenssen 2015).

Z-Statistics
Z- Stat is a popular measure that is actually based on degree of deviation from the Benford's proportion. Z-statistics considers the absolute difference between the actual frequency and the expected Benford's frequency.

$$Z = \frac{|f_i^o - f_i^e| - \left(\frac{1}{2N}\right)}{\sqrt{\frac{f_i^e(1-f_i^e)}{N}}} \quad (13)$$

Where f_i^o and f_i^e denote the actual and expected digit proportions, N denotes the number of observations. The term $\frac{1}{2N}$ is actually the correction term and we should use it when the difference between actual and expected proportions is larger than this term (Nigrini 2012). The value of Z-Stat increases as an increase in the difference between actual and expected frequency. As a rule, Z-Stat values of 1.96 or higher has *p*-value of 0.05. This is the cut-off level and hence our null hypothesis is rejected when the Z-value exceeds 1.96 (Sadaf 2016; Alali & Romero 2013).

Mean Absolute Deviation (MAD)
MAD considers the absolute value of difference between actual and expected proportions. We can calculate MAD as

$$MAD = \frac{\sum_{i=1}^{K} |AP - EP|}{K} \quad (14)$$

Where *AP* and *EP* are the actual and the expected proportions of digits. MAD test is preferred since it is not affected by number of observations.
MAD test overlooks the number of observations in calculating the test statistics.These critical test values are calculated by Nigrini (2012) on the basis of his own calculations. So, these values must be referred to with great attention (Lolbert 2006).

Table 2: Critical MAD Values and Results

Digits	Range	Results
First Digits	0.000 to 0.006	*Close conformity*
	0.006 to 0.012	*Acceptable conformity*
	0.012 to 0.015	*Marginally acceptable conformity*
	Above 0.015	*Nonconformity*
Second Digits	0.000 to 0.008	*Close conformity*
	0.008 to 0.010	*Acceptable conformity*
	0.010 to 0.012	*Marginally acceptable conformity*
	Above 0.012	*Nonconformity*
First-Two Digits	0.0000 to 0.0012	*Close conformity*
	0.0012 to 0.0018	*Acceptable conformity*
	0.0018 to 0.0022	*Marginally acceptable conformity*
	Above 0.0022	*Nonconformity*

Source: (Nigrini 2012)

Conclusion

The frequency of accounting data frauds has been increased in corporate environment. As a result of that, the research on detection of such irregularities in accounting and auditing is gaining researchers' focus. Benford's law is one of the effective tools used by researchers in order to understand the required data pattern. The use of Benford's law in empirical studies in examining financial data irregularities has gained enormous attention in past 30 years. The idea behind this approach is to compare the digit pattern of actual data with Benford's expected digit proportion. Various statistical tools are being used in literature for an effective analysis of data conformity and digit analysis as well. The test statistics, compared with critical values and p-values can show the degree of deviation (or conformity) to expected Benford's proportion. Most important advanced tests discussed in this study are *chi-square test, Kolmogorov-Smirnov test* (KS*), Euclidean distance test, Joenssen's jp-square test, Freedman-Watson u-square test, Chebyshev distance test, Z-statistics and mean absolute deviation* test. Additionally, for the future research directions, it is argue that the sectoral perspective should also be considered in order to gain a thorough analysis of financial frauds in various sector of economy (Máté 2014; Máté et al 2016) and to explain the role of actual and expected differences and how their interactions might influence accountancy frauds.

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