

On the coefficients of power sums of arithmetic progressions

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Abstract

We investigate the coefficients of the polynomial

$$S_{m,r}^n(\ell) = r^n + (m+r)^n + (2m+r)^n + \cdots + ((\ell-1)m+r)^n.$$

We prove that these can be given in terms of Stirling numbers of the first kind and r -Whitney numbers of the second kind. Moreover, we prove a necessary and sufficient condition for the integrity of these coefficients.

Key words: arithmetic progressions, power sums, Stirling numbers, r -Whitney numbers, Bernoulli polynomials

1991 MSC: 11B25, 11B68, 11B73

1 Introduction

Let n be a positive integer, and let

$$S_n(\ell) = 1^n + 2^n + \cdots + (\ell-1)^n$$

be the power sum of the first $\ell-1$ positive integers. It is well known that $S_n(\ell)$ is strongly related to the Bernoulli polynomials $B_n(x)$ in the following

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way

$$S_n(\ell) = \frac{1}{n+1}(B_{n+1}(\ell) - B_{n+1}).$$

where the polynomials $B_n(x)$ are defined by the generating series

$$\frac{te^{tx}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}$$

and $B_n = B_n(0)$ is the n th Bernoulli number.

It is possible to find the explicit coefficients of ℓ in $S_n(\ell)$ [9]:

$$S_n(\ell) = \sum_{i=0}^{n+1} \ell^i \left(\sum_{k=0}^n S_2(n, k) S_1(k+1, i) \frac{1}{k+1} \right), \quad (1)$$

where $S_1(n, k)$ and $S_2(n, k)$ are the (signed) Stirling numbers of the first and second kind, respectively.

Recently, Bázsó et al. [1] considered the more general power sum

$$S_{m,r}^n(\ell) = r^n + (m+r)^n + (2m+r)^n + \cdots + ((\ell-1)m+r)^n,$$

where $m \neq 0, r$ are coprime integers. Obviously, $S_{1,0}^n(\ell) = S_n(\ell)$. They, among other things, proved that $S_{m,r}^n(\ell)$ is a polynomial of ℓ with the explicit expression

$$S_{m,r}^n(\ell) = \frac{m^n}{n+1} \left(B_{n+1} \left(\ell + \frac{r}{m} \right) - B_{n+1} \left(\frac{r}{m} \right) \right). \quad (2)$$

In [12], using a different approach, Howard also obtained the above relation via generating functions. Hirschhorn [11] and Chapman [8] deduced a longer expression which contains already just binomial coefficients and Bernoulli numbers.

For some related diophantine results on $S_{m,r}^n(\ell)$ see [3,10,15,16,2] and the references given there.

Our goal is to give the explicit form of the coefficients of the polynomial $S_{m,r}^n(\ell)$, thus generalizing (1). In this expression the Stirling numbers of the first kind also will appear, but, in place of the Stirling numbers of the second kind a more general class of numbers arises, the so-called r -Whitney numbers introduced by the second author [13].

The r -Whitney numbers $W_{m,r}(n, k)$ of the second kind are generalizations of the usual Stirling numbers of the second kind with the exponential generating function

$$\sum_{n=k}^{\infty} W_{m,r}(n, k) \frac{z^n}{n!} = \frac{e^{rz}}{k!} \left(\frac{e^{mz} - 1}{m} \right)^k.$$

For algebraic, combinatoric and analytic properties of these numbers see [5,14] and [6,7], respectively.

First, we prove the following.

Theorem 1 *For all parameters $\ell > 1, n, m > 0, r \geq 0$ we have*

$$S_{m,r}^n(\ell) = \sum_{i=0}^{n+1} \ell^i \left(\sum_{k=0}^n \frac{m^k W_{m,r}(n, k)}{k+1} S_1(k+1, i) \right).$$

Proof. The formula which connects the power sums and the r -Whitney numbers is the next one from [13]:

$$(mx + r)^n = \sum_{k=0}^n m^k W_{m,r}(n, k) x^{\underline{k}}.$$

Here $x^{\underline{k}} = x(x-1) \cdots (x-k+1)$ is the falling factorial. We can see that it is enough to sum from $x = 0, 1, \dots, \ell-1$ to get back $S_{m,r}^n(\ell)$. Hence

$$S_{m,r}^n(\ell) = \sum_{k=0}^n m^k W_{m,r}(n, k) \sum_{x=0}^{\ell-1} x^{\underline{k}}.$$

The inner sum can be determined easily (see [9]):

$$\sum_{x=0}^{\ell-1} x^{\underline{k}} = \frac{\ell^{\underline{k+1}}}{k+1} + \delta_{k,0}.$$

The Kronecker delta will never appear, because if $k = 0$ then the r -Whitney number is zero (unless the trivial case $n = 0$, which we excluded). Therefore, as an intermediate formula, we now have that

$$S_{m,r}^n(\ell) = \sum_{k=0}^n m^k W_{m,r}(n, k) \frac{\ell^{\underline{k+1}}}{k+1}.$$

The falling factorial $\ell^{\underline{k+1}}$ is a polynomial of ℓ with Stirling number coefficients:

$$\ell^{\underline{k+1}} = \sum_{i=0}^{k+1} S_1(k+1, i) \ell^i.$$

Substituting this to the formula above, we obtain:

$$S_{m,r}^n(\ell) = \sum_{k=0}^n \frac{m^k W_{m,r}(n, k)}{k+1} \sum_{i=0}^{k+1} S_1(k+1, i) \ell^i.$$

Since $S_1(k+1, i)$ is zero if $i > k+1$, we can run the inner summation up to $n+1$ (this is taken when $k = n$) to make the inner sum independent of k .

Altogether, we have that

$$S_{m,r}^n(\ell) = \sum_{i=0}^{n+1} \ell^i \sum_{k=0}^n \frac{m^k W_{m,r}(n, k)}{k+1} S_1(k+1, i).$$

This is exactly the formula that we wanted to prove. \square

Now we give some elementary consequences of the theorem. The proofs are trivial.

Remark. The next properties of the polynomial $S_{m,r}^n(\ell)$ hold true for all parameters $\ell > 1, n > 0, r, m \geq 0$:

- (i) The constant term of $S_{m,r}^n(\ell)$ is 0,
- (ii) The leading coefficient of $S_{m,r}^n(\ell)$ is $m^n/(n+1)$,
- (iii) $S_{m,r}^n(\ell)$ is a polynomial of ℓ of degree $n+1$ unless $m=0$; in this latter case the degree is n .

The above statements also follow from (2).

2 The integer property of the coefficients in $S_{m,r}^n(\ell)$

The coefficients of the polynomial $S_{m,r}^n(\ell)$ are not integer in the overwhelming majority of the cases:

$$S_{1,0}^1(\ell) = \frac{\ell(\ell-1)}{2},$$

$$S_{2,5}^2(\ell) = \frac{1}{3}\ell(47 + 24\ell + 4\ell^2),$$

etc.

However, we revealed that in special cases the polynomial $S_{m,r}^n(\ell)$ has integer coefficients. Several parameters are in the next table.

m	r	n
2	1	3
2	3	3
2	5	3
4	3	3
4	5	3

For example,

$$S_{2,1}^3(\ell) = \ell^2(2\ell^2 - 1),$$

or

$$S_{2,3}^3(\ell) = \ell(2 + \ell)(2\ell^2 + 4\ell + 3).$$

From the formula of Theorem 1 it can be seen that if

$$(k + 1) \mid m^k W_{m,r}(n, k) \quad (k = 1, 2, \dots, n),$$

then we get integer coefficients.

To find another condition which is necessary and sufficient for the integrity of the coefficients in $S_{m,r}^n(\ell)$, we recall the following well known properties of Bernoulli polynomials and Bernoulli numbers.

$$B_n(x + y) = \sum_{k=0}^n \binom{n}{k} B_k(x) y^{n-k} = \sum_{k=0}^n \binom{n}{k} B_k(y) x^{n-k}; \quad (3)$$

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k}; \quad (4)$$

$$B_3 = B_5 = B_7 = \dots = 0. \quad (5)$$

By the *denominator* of a rational number q we mean the smallest positive integer d such that dq is an integer. We recall also the von Staudt theorem

$$\Lambda_{2n} = \prod_{\substack{(p-1) \mid 2n \\ p \text{ prime}}} p, \quad (6)$$

where Λ_n is the denominator of B_n . In particular, Λ_n is a square-free integer, divisible by 6. For the proofs of (3)-(5) see e.g. the work of Brillhart [4].

Let $2 \leq j \leq n$ be an even number and put

$$f(n, j) := \text{lcm} \left(\frac{\Lambda_j}{\gcd(\Lambda_j, \binom{n+1}{j} \binom{j}{j})}, \frac{\Lambda_j}{\gcd(\Lambda_j, \binom{n+1}{j+1} \binom{j+1}{j})}, \dots, \frac{\Lambda_j}{\gcd(\Lambda_j, \binom{n+1}{n} \binom{n}{j})} \right). \quad (7)$$

Further, we define

$$F(n) := \begin{cases} \text{lcm}(\text{rad}(n+1), f(n, 2), f(n, 4), \dots, f(n, n)) & \text{if } n \text{ is even,} \\ \text{lcm}(\text{rad}(n+1), f(n, 2), f(n, 4), \dots, f(n, n-1)) & \text{if } n \text{ is odd,} \end{cases} \quad (8)$$

where

$$\text{rad}(n) = \prod_{\substack{p \mid n \\ p \text{ prime}}} p.$$

Theorem 2 *The polynomial $S_{m,r}^n(\ell)$ has integer coefficients if and only if $F(n) \mid m$.*

Proof. By relations (2), (3) and (4) we can rewrite $S_{m,r}^n(\ell)$ as follows:

$$S_{m,r}^n(\ell) = \frac{m^n}{n+1} \left(B_{n+1} \left(\ell + \frac{r}{m} \right) - B_{n+1} \left(\frac{r}{m} \right) \right) = \quad (9)$$

$$= \frac{m^n}{n+1} \left(\left(\sum_{k=0}^{n+1} \binom{n+1}{k} B_k \left(\frac{r}{m} \right) \ell^{n+1-k} \right) - B_{n+1} \left(\frac{r}{m} \right) \right) = \quad (10)$$

$$= \frac{m^n}{n+1} \sum_{k=0}^n \binom{n+1}{k} B_k \left(\frac{r}{m} \right) \ell^{n+1-k} = \quad (11)$$

$$= \frac{m^n}{n+1} \sum_{k=0}^n \binom{n+1}{k} \left(\sum_{j=0}^k \binom{k}{j} B_j \cdot \left(\frac{r}{m} \right)^{k-j} \right) \ell^{n+1-k} \quad (12)$$

We denote the common denominator of the coefficients of $S_{m,r}^n(\ell)$ by Q . One can see from (9) that the polynomial has integral coefficients if and only if m is divisible by Q . Thus we have to determine Q .

By (12) we observe that neither m nor r occurs in Q . Moreover, the only algebraic expressions which may affect Q in (12) are on one hand $n+1$ and on the other hand, the denominators of the Bernoulli numbers involved, which are $2, \Lambda_j (2 \leq j \leq n \text{ even})$ by (5) and the von Staudt theorem.

It can easily be seen that $n+1 \mid m^n$ if $\text{rad}(n+1) \mid m$. Indeed, supposing the contrary, i.e., that $\text{rad}(n+1) \mid m$ and $n+1 \nmid m^n$, it implies that there is a prime factor p of $n+1$ such that p^{n+1} divides $n+1$. Hence $2^{n+1} \leq p^{n+1} \leq n+1$, which is a contradiction.

Let $2 \leq j \leq n$ be an even index. It follows from (12) that the contribution of Λ_j to the common denominator Q is precisely $f(n, j)$ defined in (7). In other words, if $f(n, j) \mid m$, then every term of (12) containing the factor B_j has integer coefficients.

In conclusion, we obtained that Q is the least common multiple of $\text{rad}(n+1)$ and $f(n, j)$ for all even $j \in [2, n]$, which number we denoted in (8) by $F(n)$. The theorem is proved. \square

Remark. An easy consequence of our Theorem 2 is that $S_n(\ell) = S_{1,0}^n(\ell) \notin \mathbb{Z}[x]$ for any positive integer n .

Some small values of $F(n)$ are listed in the following table. These are results of an easy computation in MAPLE.

n	$F(n)$	n	$F(n)$	n	$F(n)$	n	$F(n)$
1	2	6	42	11	6	16	510
2	6	7	6	12	2730	17	30
3	2	8	30	13	210	18	3990
4	30	9	10	14	30	19	210
5	6	10	66	15	6	20	2310

Acknowledgements

This scientific work was financed by Proyecto Prometeo de la Secretaría Nacional de Ciencia, Tecnología e Innovación (Ecuador). The first author was supported by the Hungarian Academy of Sciences and by the OTKA grant NK104208.

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