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Conditional equations for monomial functions

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Hereby I declare that I prepared this thesis within the Doctoral Council of Natural Sciences and Information Technology, Doctoral School of Mathematical and Computational Sciences, University of Debrecen in order to obtain a PhD Degree in Natural Sciences at Debrecen University.
The results published in the thesis are not reported in any other PhD theses.

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I support the acceptance of the thesis.

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Conditional equations for monomial functions

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Contents

1. Introduction

1.1 Investigated problems

The existence of discontinuous additive functions (or higher order monomial functions) was an open problem for many years. Mathematicians could neither prove that all additive functions are continuous, nor give an example to a discontinuous additive function. G. Hamel [24] succeded in proving that there exist discontinuous additive functions.

In this dissertation we study monomial functions f of degree $2 \leqslant n \in \mathbb{N}$, defined as diagonalizations of n-additive functions (functions that are additive with respect to each of their n variables) in Section 2.1. It is well known that such a function f is continuous if, and only if, it can be given as

$$f(x) = cx^n \qquad (x \in \mathbb{R}) \tag{1.1}$$

with some real number c. The existence of discontinuous monomial functions follows, for example, from the above cited theorem of Hamel. Clearly, any function of the form (1.1) satisfies the identity

$$y^n f(x) = x^n f(y) \tag{1.2}$$

for all $x, y \in \mathbb{R}$. Conversely, identity (1.2) implies (1.1) with $c = f(1)$ even if we assume only that (1.2) is fulfilled for $y = 1$ (and every $x \in \mathbb{R}$). On the other hand, if we assume (1.2) only for $y = x$, we do not obtain any information about the function f at all. Our main purpose is to answer the following question for various particular algebraic (or some specific transcendental) curves $S \subset \mathbb{R}^2$ and reasonably small n:

Let us suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a monomial function of degree n that satisfies the additional equation (1.2) for every $(x, y) \in S$. Does it imply that f is continuous? We provide affirmative answers in several particular cases. However, for a natural choice of S, we obtain a counterexample.

Since the calculation becomes more difficult as n increases, we obtain the continuity of f in the following particular cases:

- $2 \leqslant n \in \mathbb{N}$ and S is given by $y = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ with $m \in \mathbb{N}$, $a_i \in \mathbb{R}$, $i = 0, \ldots, m$, $a_0 \neq 0$ (the curve does not pass through the origin);
- $n \in \{2, 3\}$ and S is given by $y = x^m$ with $m \in \mathbb{Z}$, $|m| \geq 2$;
- $n = 2$ and S is given by $x^2 \pm y^2 = 1$ (two cases);
- $n \in \{2, 3\}$ and S is given by $y = e^x$ (or $x = e^y$).

When S denotes the hyperbola given by the equation $xy = 1$, we obtain counterexamples for any $2 \leqslant n \in \mathbb{N}$. For this curve, in case $n = 2$ (i.e., when f is quadratic), a considerably non-trivial necessary condition is obtained.

Generalizing the problem to a pair of monomial functions f, g of degree $n \in \mathbb{N}$, $n \geqslant 2$ related by the functional equation

$$y^n f(x) = x^n g(y) \tag{1.3}$$

under the condition $P(x, y) = 0$ for some fixed polynomial P of two variables, we find that in most (but not all) examined cases f and g are equal and continuous.

1.2 Motivation

In this section we introduce the necessary notations and we present some preliminary results.

Let $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ denote the set of all real numbers, rationals, integers and positive integers, respectively. Let $\mathbb{R}^+$ denote the set of positive real numbers. We call a function $f : \mathbb{R} \to \mathbb{R}$ *additive* if

$f(x+y) = f(x) + f(y)$ holds for all $x, y \in \mathbb{R}$. The function f is called $\mathbb{Q}$-*homogeneous* if the equation $f(qx) = qf(x)$ is fulfilled by every $q \in \mathbb{Q}$ and $x \in \mathbb{R}$. As it is also well-known [29, Theorem 5.2.1], if $f : \mathbb{R} \to \mathbb{R}$ is additive, then f is $\mathbb{Q}$-homogeneous as well.
We define the following sets:

$$\begin{aligned}
S_0 =& \{(x, y) \in \mathbb{R}^2 \,|\, a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0 = y\,\} \\
& \quad \text{with} \;\; m \in \mathbb{N},\; a_i \in \mathbb{R},\; i = 0, \ldots, m,\; a_m \neq 0,\, a_0 \neq 0\,, \\
S_1 =& \{(x, y) \in \mathbb{R}^2 \,|\, x^m = y\} \quad \text{with } m \in \mathbb{Z}\,, \quad |m| \geqslant 2\,, \\
S_2 =& \{(x, y) \in \mathbb{R}^2 \,|\, xy = 1\}, \\
S_3 =& \{(x, y) \in \mathbb{R}^2 \,|\, x^2 - y^2 = 1\}, \\
S_4 =& \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}, \\
S_5 =& \{(x, y) \in \mathbb{R}^2 \,|\, x > 0 \;\; \text{and} \;\; \log x = y\,\}, \\
S_6 =& \{(x, y) \in \mathbb{R}^2 \,|\, e^x = y\}.
\end{aligned}$$

We note that the sets S_0 and S_1 depend on some parameters (the positive integer m, the real numbers $a_i, i = 0, \ldots, m$, and the integer m, respectively), so our statements with respect to these sets are valid for all admitted values of these parameters, unless otherwise stated.

The motivation for our investigations are some problems solved for additive functions.
The problem is the following:
Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an additive function. If f satisfies the additional equation

$$xf(y) = yf(x) \tag{1.4}$$

for the pairs $(x, y) \in S_i$, $i = 0, 1, 2, 4$, does it imply that f is continuous (i.e., linear)? In these cases the continuity of the additive function was proved.
Case $(x, y) \in S_0$: the affirmative answer can be found in [12].
Case $(x, y) \in S_1$: in 1968 A. Nishiyama and S. Horinouchi [35] proved that every additive mapping $f : \mathbb{R} \to \mathbb{R}$, which satisfies (1.4) for all

$(x, y) \in S_1$ is of the form $f(x) = f(1)x$ for all $x \in \mathbb{R}$.
Case $(x, y) \in S_2$: the problem was posed by I. Halperin in 1963 (communicated in J. Aczél [1]). To Halperin's question S. Kurepa [30] has given the answer in the affirmative by proving, among others, a theorem which contains a more general result and leads to $f(x) = f(1)x$. W. B. Jurkat [25] has obtained independently the same result. Several authors extended this result in various directions. Among numerous further publications they provided generalizations in [29, Theorem 14.3.3], [27], and [31].
Case $(x, y) \in S_4$: the problem was formulated by W. Benz [5] in 1989. This question, together with a similar one for derivations, was answered in the affirmative by Z. Boros and P. Erdei [6].

B. Ebanks [12] generalizes the problem to a pair of additive functions f, g related by the functional equation

$$yf(x) = xg(y) \tag{1.5}$$

for all points (x, y) on a specified curve. He finds that for many (but not all) types of curves this forces f and g to be equal and linear (including S_i, $i = 0, 4, 5, 6$).

The motivation of such investigations is the possible representation of the solutions of various functional equations in terms of additive functions. When the investigated functional equation is obtained from the axiomatic description of certain mathematical models in applied mathematics, some additional algebraic condition might be obtained from the same theory as well. In such a case, it is an essential question whether the additional condition implies the linearity of the additive function appearing in the aforementioned representation.

As it was shown by M. A. McKiernan [34] for real functions and by L. Székelyhidi [37] in quite general context, solutions of a wide class of linear functional equations are generalized polynomials, which can be represented as the sum of generalized monomials. It is therefore reasonable to extend the above cited investigations to generalized monomials, that can be considered as generalizations of additive functions.

Z. Kominek, L. Reich and J. Schwaiger [28] investigated additive functions that satisfy the additional equation

$$f(x)f(y) = 0 \tag{1.6}$$

for every $(x, y) \in S$, considering various subsets S of $\mathbb{R}^2$. In several cases, involving $S = S_4$, they obtained $f(x) = 0$ for every $x \in \mathbb{R}$. This particular result was extended by Z. Boros and W. Fechner [7] to the situation when f is a generalized polynomial. On the other hand, P. Kutas [32] has recently established the existence of a nonzero additive function $f : \mathbb{R} \to \mathbb{R}$ fulfilling (1.6) for all $(x, y) \in S_2$. The case of bounded $f(x)f(y)$ on S_4 was investigated by these authors [8].

1.3 Structure of the dissertation

The dissertation is divided into six structural units.

The first chapter includes the general problem statement and gives a brief overview of relating results: we investigate monomial functions $f, g \colon \mathbb{R} \to \mathbb{R}$ of degree $n \in \mathbb{N}$, $n \geqslant 2$ which satisfy the additional equation (1.2) or (1.3) for all points (x, y) on a specified curve.
The second chapter contains the display of tools and we present some related concepts and results.
The next four chapter contain the main results, classified by curves:

- In Chapter 3 we investigate monomial functions that satisfy additional equations involving polynomial functions whose graphs do not pass through the origin: we find that if f, g are monomial functions of degree $n \in \mathbb{N}$, $n \geqslant 2$ which satisfy the additional equation (1.3) for all points $(x, y) \in S_0$, then f and g are identical and continuous.

- Chapter 4 contains results for quadratic and cubic functions satisfying conditional equations involving the power function: we obtain that if f is a monomial function of degree $n \in \{2, 3\}$ and f satisfies the additional equation (1.2) for the pairs $(x, y) \in S_1$ then f is continuous. In the particular case $n = 2$, $m = 2$, a modified version of the condition (1.2) admits a discontinuous quadratic solution f. Introducing a second quadratic function in case $n = 2$ and $m = 2$ we find that there exist discontinuous solutions.

- In Chapter 5 we study additive, quadratic and higher order monomial functions that satisfy subsidiary equations along hyperbolas or the unit circle. We give counterexamples to demonstrate that there exist discontinuous solutions f, when f is a monomial function of degree $n \in \mathbb{N}$, $n \geqslant 2$ which satisfies the additional equation (1.2) under the condition $xy = 1$. Nonetheless, we prove that if f, g satisfy the additional equation (1.3) for all points $(x, y) \in S_3$ (f, g are additive or quadratic) or $(x, y) \in S_4$ (f, g are quadratic), then f and g are identical and continuous. Furthermore, we prove an interesting necessary condition for a quadratic function f which satisfies the additional equation (1.2) under the condition $xy = 1$.

- Chapter 6 contains results for quadratic and cubic functions that satisfy conditional equations involving logarithmic (exponential) function: we find that if f, g satisfy the additional equation (1.3) for all points $(x, y) \in S_5$ or $(x, y) \in S_6$, then f and g are identical and continuous.

2. Preliminaries

2.1 Multiadditive functions and generalized monomials

Let $n \in \mathbb{N}$. A function $F : \mathbb{R}^n \to \mathbb{R}$ is called *n–additive* if, for every $i \in \{ 1 , 2 , \dots , n \}$ and for every $x_1, \dots, x_n, y_i \in \mathbb{R}$,

$$F(x_1, \dots, x_{i-1}, x_i + y_i, x_{i+1}, \dots, x_n)$$
$$= F(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) + F(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n),$$

i.e., F is additive in each of its variables $x_i \in \mathbb{R}$, $i = 1, \dots, n$. We call 1-additive functions simply additive, 2-additive functions biadditive. Further, constant functions will be called 0-additive.

Clearly, an n–additive function is also $\mathbb{Q}$-homogeneous in each variable.

Given a function $F : \mathbb{R}^n \to \mathbb{R}$, by the *diagonalization (or trace)* of F we understand the function $f : \mathbb{R} \to \mathbb{R}$ arising from F by putting all the variables (from $\mathbb{R}$) equal:

$$f(x) = F(x, \dots, x) \qquad (x \in \mathbb{R}). \tag{2.1}$$

If, in particular, f is the diagonalization of an n–additive function $F : \mathbb{R}^n \to \mathbb{R}$, we say that f is a *monomial function (or generalized monomial) of degree n*. In such a case we obtain $f(rx) = r^n f(x)$ whenever $x \in \mathbb{R}$ and $r \in \mathbb{Q}$. Monomial functions of degree 3 are called cubic functions, quadratic functions are generalized monomials of degree 2. Further, additive functions are generalized monomials of degree 1 and real constants are generalized monomials of degree 0. For $y \in \mathbb{R}$ we define ([29], Chapter 15) the linear difference operator $\triangle_y$

on $\mathbb{R}$ by

$$\triangle_y f(x) = f(x+y) - f(x), \quad \text{for all} \quad f : \mathbb{R} \to \mathbb{R} \text{ and } x \in \mathbb{R}$$

and, for an arbitrary natural number $n \geqslant 2$

$$\triangle_{y_1,\ldots,y_{n-1},y_n} f(x) = \triangle_{y_1,\ldots,y_{n-1}} \left(\triangle_{y_n} f(x)\right).$$

If $y_1 = y_2 = ... = y_n = y$, instead of $\triangle_{y,y,\ldots,y} f(x)$ we write $\triangle_y^n f(x)$. As one can easily verify by induction,

$$\triangle_y^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+ky), \quad \text{for all } f : \mathbb{R} \to \mathbb{R} \text{ and } x, y \in \mathbb{R}.$$

Let F_s denote the symmetric part of $F : \mathbb{R}^n \to \mathbb{R}$, i.e.,

$$F_s(x_1, x_2, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma \in P_n} F\left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}\right)$$

for every $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, where P_n denotes the set of all permutations of the index set $\{1, 2, \ldots, n\}$. Clearly, if f is defined by (2.1), then we have $f(x) = F_s(x, x, \ldots, x)$ (for all $x \in \mathbb{R}$) as well. Moreover, if F is n–additive, then F_s is also n–additive. Therefore, if $f \colon \mathbb{R} \to \mathbb{R}$ is a generalized monomial of degree n, then there exists a symmetric n–additive function $F : \mathbb{R}^n \to \mathbb{R}$ such that (2.1) holds. Furthermore, as it is also well known [29, Lemma 15.9.2], in that case we have

$$\Delta_{y_1,\ldots,y_{n-1},y_n} f(x) = n! F(y_1, \ldots, y_{n-1}, y_n) \tag{2.2}$$

for every $x, y_1, \ldots, y_n \in \mathbb{R}$. This shows the uniqueness of F.

It is a consequence of the identity (2.2) that any generalized monomial $f \colon \mathbb{R} \to \mathbb{R}$ of degree n satisfies the n–monomial functional equation

$$\triangle_y^n f(x) = n! f(y) \quad (x, y \in \mathbb{R}). \tag{2.3}$$

In fact ([29, Chapter 15], [38, Chapter 1]), generalized monomials of degree n are characterized as the solutions of the n–monomial functional equation (2.3).

In case $n = 2$, writing $x - y$ in place of x, equation (2.3) can be formulated as

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \qquad (x, y \in \mathbb{R}), \tag{2.4}$$

which is the so called *norm square equation* or *parallelogram law.* Therefore, quadratic functions are characterized by the functional equation (2.4).

As it is well known ([2], [3, Section 11.1]), we can (and, in the rest of this dissertation, we shall) associate with a quadratic function $f\colon \mathbb{R} \to \mathbb{R}$ the biadditive and symmetric functional $F\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, given by the formula

$$F(x, y) = \frac{1}{2}[f(x + y) - f(x) - f(y)] \tag{2.5}$$

for all $x, y \in \mathbb{R}$. It is not very difficult to verify that F is, in fact, biadditive (i.e., the mappings

$$t \mapsto F(t, x) \quad \text{and} \quad t \mapsto F(x, t) \quad (t \in \mathbb{R})$$

are additive for each $x \in \mathbb{R}$), and f is obtained as the diagonalization of F (i.e., $f(x) = F(x, x)$ for all $x \in \mathbb{R}$). Applying the $\mathbb{Q}$-homogeneity of additive functions, we have

$$F(rx, sy) = rsF(x, y) \quad \text{and} \quad f(rx) = F(rx, rx) = r^2 F(x, x) = r^2 f(x)$$

for every $r, s \in \mathbb{Q}$ and $x, y \in \mathbb{R}$. In particular, $f(0) = 0$. On the other hand, applying equation (2.5) and induction on n, one can easily prove the identity

$$f\left(\sum_{k=0}^{n} u_k\right) = \sum_{k=0}^{n} f(u_k) + 2 \sum_{0\le i<j\le n} F(u_i\,,\, u_j) \tag{2.6}$$

for every $n \in \mathbb{N}$ and $u_0\,,\, u_1\,,\, \ldots\,,\, u_n \in \mathbb{R}$.

In order to present our tools and arguments, it is convenient to introduce the following notations. If $F : \mathbb{R}^n \to \mathbb{R}$ is symmetric, $x\,,\, y \in \mathbb{R}$ and $k \in \mathbb{Z}$ such that $0 \leqslant k \leqslant n$, let

$$F\left([x]_k, [y]_{n-k}\right) = F(\underbrace{x, \ldots, x}_{k}, \underbrace{y, \ldots, y}_{n-k}).$$

In particular, let

$$F\left([x]_0, [y]_n\right) = F(\underbrace{y, \ldots, y}_{n}) = f(y)$$

and

$$F\left([x]_n, [y]_0\right) = F(\underbrace{x, \ldots, x}_{n}) = f(x)\,.$$

Now we can formulate the well known analogue of the celebrated binomial theorem and its extension to several variables for generalized monomials (established, for instance, in the above cited monograph by L. Székelyhidi [38, Chapter 1]).

Lemma 2.1. (Binomial Theorem)*: If $n \in \mathbb{N}$, $F : \mathbb{R}^n \to \mathbb{R}$ is a symmetric n–additive function and $f\colon \mathbb{R} \to \mathbb{R}$ is defined by* (2.1)*, then*

$$f(x + y) = \sum_{k=0}^{n} \binom{n}{k} F\left([x]_k, [y]_{n-k}\right) \tag{2.7}$$

for all $x, y \in \mathbb{R}$.

Lemma 2.2. (Polynomial Theorem)*: If $n \in \mathbb{N}$, $F : \mathbb{R}^n \to \mathbb{R}$ is a symmetric n–additive function and $f\colon \mathbb{R} \to \mathbb{R}$ is defined by* (2.1)*,* *then*

$$f(x_1 + x_2 + \cdots + x_m) =$$
$$= \sum_{k_1+k_2+\cdots+k_m=n} \frac{n!}{k_1!k_2!\cdots k_m!} F\left([x_1]_{k_1}, [x_2]_{k_2}, \cdots, [x_m]_{k_m}\right) \quad (2.8)$$

for any positive integer m and for all $x_1, x_2, \cdots, x_m \in \mathbb{R}$.

Clearly, we extended our former notations such that, in the last term of equation (2.8), the argument of the function F in n variables consists of k_j copies of the real variable x_j $(j = 1, 2, \ldots, m)$.

We shall also make use of the following observation, motivated by Ebanks [12, Lemma 7.3].

Lemma 2.3. *(Z. Boros and E. Garda-Mátyás [9]). If $\mathbb{F}$ is a field, $n \in \mathbb{N}$, X is an arbitrary set, $V \subset \mathbb{F}$ contains at least $n+1$ elements, and the functions $H_k \colon X \to \mathbb{F}$ $(k = 0, 1, \ldots, n)$ satisfy the equation*

$$\sum_{k=0}^{n} H_k(x) r^k = 0 \quad (2.9)$$

for every $x \in X$ and $r \in V$, then $H_k(x) = 0$ for every $x \in X$ and $k \in \{0, 1, \ldots, n\}$.

Proof. For each fixed $x \in X$, $\sum_{k=0}^{n} H_k(x) r^k$ is a polynomial of degree at most n, with the coefficients $H_k(x) \in \mathbb{F}$ $(k = 0, 1, \ldots, n)$, with respect to the variable r. According to equation (2.9), this polynomial takes the value zero at each $r \in V$. Since V contains at least $n+1$ elements, this polynomial has to be identically zero. □

In this dissertation, we shall apply Lemma 2.3 for $X = \mathbb{F} = \mathbb{R}$ and $V = \mathbb{Q}$.

2.2 Derivations

We say that $f\colon \mathbb{R} \to \mathbb{R}$ is a *derivation* if f satisfies the system of functional equations

$$
\begin{aligned}
f(x+y) &= f(x)+f(y) \qquad (2.10)\\
f(xy) &= f(x)y + xf(y) \qquad (2.11)
\end{aligned}
$$

for every $x, y \in \mathbb{R}$. The family of derivations $f\colon \mathbb{R} \to \mathbb{R}$ is denoted by $\mathcal{D}(\mathbb{R})$ in the sequel.

Clearly, equation (2.10) expresses that f is additive, while equation (2.11) is motivated by the differentiation rule for the product of two differentiable functions (however, here the arguments are real numbers instead of functions). It is an immediate consequence of the definition that any derivation f fulfills $f(x^2) = 2xf(x)$ for all $x \in \mathbb{R}$. Obviously, equation (2.11) implies $f(1) = 0$. Hence, any linear derivation is identically zero. On the other hand, it is also well known (and easy to prove) that the graph of any non-linear additive function $f\colon \mathbb{R} \to \mathbb{R}$ is dense in $\mathbb{R}^2$. In particular, the graph of any non-trivial (i.e., not identically zero) derivation $f\colon \mathbb{R} \to \mathbb{R}$ has to be dense in $\mathbb{R}^2$. The existence of such functions is established, in a more general setting, for instance, in [40] (and in [29, Section 14.2]). Moreover, one can easily prove the following statement.

Proposition 2.1. *If $K \in \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ is an additive mapping that satisfies the equation*

$$
f(x^2) = Kxf(x) \qquad (x \in \mathbb{R}) \qquad (2.12)
$$

as well, then either $f = 0$ or $K \in \{1, 2\}$. Moreover, if $K = 1$, then f is linear, while, if $K = 2$, then f is a derivation.

The characterization of linear functions or derivations among additive mappings via functional equations in a single variable (which are generalizations of (2.12)) is the main topic of articles by Nishiyama and Horinouchi [35], Kannappan and Kurepa [26], Grząślewicz [17], Halter-Koch [21, 22], and some recent papers by Ebanks [13], [14] and Gselmann [19]. A characterization of derivations via a single functional equation has been provided by Gselmann [18]. A functional $B\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is called a *bi-derivation* if the mappings $t\mapsto B(t,x)$ and $t\mapsto B(x,t)$ $(t\in\mathbb{R})$ are derivations for each $x\in\mathbb{R}$. An additive mapping $f\colon\mathbb{R}\to\mathbb{R}$ is called a *derivation of order* 2, if there exists a (symmetric) bi-derivation $B\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that

$$f(xy)-xf(y)-f(x)y=B(x,y)\qquad (x,y\in\mathbb{R})\,.$$

The set of derivations of order 2 will be denoted by $\mathcal{D}_2(\mathbb{R})$. Since the identically zero mapping from $\mathbb{R}\times\mathbb{R}$ into $\mathbb{R}$ is a bi-derivation, we have $\mathcal{D}(\mathbb{R})\subset\mathcal{D}_2(\mathbb{R})$. The terminology is motivated by the observation that the composition of two derivations on $\mathbb{R}$ belongs to the class $\mathcal{D}_2(\mathbb{R})$. In particular, if $d\colon\mathbb{R}\to\mathbb{R}$ is a not identically zero derivation and $f=d\circ d$, then

$$\begin{aligned} B(x,y)\colon =&f(xy)-xf(y)-f(x)y=\\ &=d(xd(y)+d(x)y)-xd(d(y))-yd(d(x))=2d(x)d(y)\end{aligned}$$

$((x,y)\in\mathbb{R}\times\mathbb{R})$ is a not identically zero bi-derivation, hence $f\in\mathcal{D}_2(\mathbb{R})\setminus\mathcal{D}(\mathbb{R})$.

The concept of derivations of higher order was introduced and characterized via functional equations by Unger and Reich [39]. The theory has been developed by Reich [36], Halter-Koch and Reich [23], Ebanks [11], and quite recently by Gselmann, Vincze and Kiss [20].

As consequences of [15, Proposition 2.2] and [11, Proposition 4.6] (the same equations are listed as equivalent conditions for additive mappings to belong to the class of second order derivations), and as a particular case of [20, Proposition 3], we have the following characterization of $\mathcal{D}_2(\mathbb{R})$.

Proposition 2.2. *Let* $\varphi\colon \mathbb{R} \to \mathbb{R}$ *be an additive mapping. Then* $\varphi \in \mathcal{D}_2(\mathbb{R})$ *if, and only if,*

$$\varphi\left(x^3\right) - 3x\varphi\left(x^2\right) + 3x^2\varphi\left(x\right) = 0 \qquad (x \in \mathbb{R})\,. \tag{2.13}$$

As an application of Proposition 2.2, we can prove the following lemma, which appears in [11, Proposition 4.6], as well.

Lemma 2.4. *Let* $\varphi\colon \mathbb{R} \to \mathbb{R}$ *be an additive function. Then* $\varphi \in \mathcal{D}_2(\mathbb{R})$ *if, and only if,*

$$\varphi(x^4) = 6x^2\varphi(x^2) - 8x^3\varphi(x) \qquad (x \in \mathbb{R})\,. \tag{2.14}$$

Proof. If $\varphi \in \mathcal{D}_2(\mathbb{R})$, then there exists a symmetric bi-derivation $B\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$\varphi\left(x^2\right) - 2x\varphi(x) = B(x,x) \qquad (x \in \mathbb{R})\,.$$

Substituting x^2 in place of x in the latter equation and applying the identity $f\left(x^2\right) = 2xf(x)$ fulfilled by any real derivation f (cf. (2.11)), we obtain

$$\begin{aligned}\varphi\left(x^4\right) - 2x^2\varphi\left(x^2\right) &= B\left(x^2,x^2\right) = (2x)^2B(x,x) = 4x^2B(x,x)\\ &= 4x^2\left(\varphi\left(x^2\right) - 2x\varphi(x)\right) = 4x^2\varphi\left(x^2\right) - 8x^3\varphi(x),\end{aligned}$$

which yields (2.14).

Now we assume that $\varphi\colon \mathbb{R} \to \mathbb{R}$ is additive and it satisfies equation (2.14) as well. Substituting $x = 1$ into (2.14) we obtain $\varphi(1) = 0$. Let $x \in \mathbb{R}$ and $r \in \mathbb{Q}$. Replacing x with $x + r$ in equation (2.14) we have

$$\begin{aligned} 0 &= \varphi\left((x+r)^4\right) - 6(x+r)^2\varphi\left((x+r)^2\right) + 8(x+r)^3\varphi(x+r)\\ &= \varphi(x^4) + 4r\varphi(x^3) + 6r^2\varphi(x^2) + 4r^3\varphi(x) + r^4\varphi(1)\\ &- 6\left(x^2 + 2rx + r^2\right)\left(\varphi(x^2) + 2r\varphi(x) + r^2\varphi(1)\right)\\ &+ 8\left(x^3 + 3rx^2 + 3r^2x + r^3\right)\left(\varphi(x) + r\varphi(1)\right)\,.\end{aligned}$$

Here, for each fixed $x \in \mathbb{R}$, we obtain a polynomial with respect to r on the right side of the equation. Hence we may apply Lemma 2.3. The equality of the coefficient of r with zero for every $x \in \mathbb{R}$, in view of $\varphi(1) = 0$, finally leads to equation (2.13). Therefore, the statement follows from Proposition 2.2. $\square$

3. Conditional equations involving polynomial functions

In this chapter we investigate monomial functions $f, g\colon \mathbb{R} \to \mathbb{R}$ of degree $n \in \mathbb{N}$, $n \geqslant 2$ that satisfy conditional equations involving polynomial functions:

$$\begin{aligned} S_0 = &\{(x,y) \in \mathbb{R}^2 \mid a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 = y\,\} \\ &\quad \text{with} \;\; m \in \mathbb{N},\; a_i \in \mathbb{R},\; i = 0, \ldots, m,\; a_m \neq 0,\, a_0 \neq 0\,. \end{aligned}$$

We know that if $f, g : \mathbb{R} \to \mathbb{R}$ are additive functions and satisfy the additional equation (1.5) for the pairs $(x, y) \in S_0$, then f and g are the same linear function ([Theorem 4.3] in [12]).

Theorem 3.1. *Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are generalized monomials of degree $n \in \mathbb{N}$ that satisfy the additional equation $y^n f(x) = x^n g(y)$ for the pairs $(x, y) \in S_0$. Then $f(x) = g(x) = x^n f(1)$ for all $x \in \mathbb{R}$.*

Proof. We can associate with the generalized monomial g an n–additive and symmetric functional $G : \mathbb{R}^n \to \mathbb{R}$, such that

$$g(x) = G(x, x, \ldots, x) \quad (x \in \mathbb{R})$$

holds. The additional equation is:

$$\begin{aligned} (a_m x^m + &a_{m-1}x^{m-1} + \cdots + a_1 x + a_0)^n f(x) = \\ &= x^n g\left(a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0\right). \end{aligned} \tag{3.1}$$

We know that

$$\begin{aligned}&\big(a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0\big)^n = \\ &= \sum_{i_0+i_1+\cdots+i_m=n} \frac{n!}{i_0! i_1! \cdots i_m!} a_m^{i_0} x^{m i_0} \cdots a_1^{i_{m-1}} x^{i_{m-1}} a_0^{i_m}.\end{aligned} \tag{3.2}$$

Due to the Polynomial Theorem, we have

$$\begin{aligned}g(a_m x^m &+ a_{m-1}x^{m-1} + \ldots + a_1 x + a_0) = \\ &= \sum_{k_0+k_1+\ldots+k_m=n} \frac{n!}{k_0! k_1! \cdots k_m!} \cdot \\ &\quad \cdot G\left([a_m x^m]_{k_0} \,,\, \left[a_{m-1}x^{m-1}\right]_{k_1} \,, \ldots ,\, [a_0]_{k_m} \right).\end{aligned} \tag{3.3}$$

With (3.2) and (3.3), the additional equation (3.1) can be written in the following form:

$$\begin{aligned}&\sum_{i_0+i_1+\ldots+i_m=n} \frac{n!}{i_0! i_1! \cdots i_m!} a_m^{i_0} x^{m i_0} \cdots a_1^{i_{m-1}} x^{i_{m-1}} a_0^{i_m} f(x) = \\ &\qquad = x^n \sum_{k_0+k_1+\ldots+k_m=n} \frac{n!}{k_0! k_1! \cdots k_m!} \cdot \\ &\qquad \cdot G\left([a_m x^m]_{k_0} \,,\, \left[a_{m-1}x^{m-1}\right]_{k_1} \,, \ldots ,\, [a_0]_{k_m} \right).\end{aligned} \tag{3.4}$$

If $x \in \mathbb{R}$, $r \in \mathbb{Q}$, and we replace x with rx in equation (3.4), we get

$$\begin{aligned}&\sum_{i_0+i_1+\ldots+i_m=n} \frac{n!}{i_0! i_1! \cdots i_m!} a_m^{i_0} (rx)^{m i_0} \cdots a_1^{i_{m-1}} (rx)^{i_{m-1}} a_0^{i_m} f(rx) = \\ &\qquad = (rx)^n \sum_{k_0+k_1+\ldots+k_m=n} \frac{n!}{k_0! k_1! \cdots k_m!} \cdot \\ &\qquad \cdot G\left([a_m (rx)^m]_{k_0} \,,\, \left[a_{m-1}(rx)^{m-1}\right]_{k_1} \,, \ldots ,\, [a_0]_{k_m} \right).\end{aligned}$$

Using $f(rx) = r^n f(x)$ and

$$G\left(\left[a_m(rx)^m\right]_{k_0}, \left[a_{m-1}(rx)^{m-1}\right]_{k_1}, \ldots, \left[a_0\right]_{k_m}\right) = \\ = r^{mk_0} r^{(m-1)k_1} \cdots r^{k_{m-1}} G\left(\left[a_m x^m\right]_{k_0}, \left[a_{m-1}x^{m-1}\right]_{k_1}, \ldots, \left[a_0\right]_{k_m}\right),$$

we obtain

$$\sum_{i_0+i_1+\ldots+i_m=n} \frac{n!}{i_0! i_1! \cdots i_m!} r^{n+mi_0+(m-1)i_1+\ldots+i_{m-1}} \cdot \\ \cdot a_m^{i_0} x^{mi_0} \cdots a_1^{i_{m-1}} x^{i_{m-1}} a_0^{i_m} f(x) = \\ = \sum_{k_0+k_1+\ldots+k_m=n} \frac{n!}{k_0! k_1! \cdots k_m!} r^{n+mk_0+(m-1)k_1+\ldots+k_{m-1}} x^n \cdot \\ \cdot G\left(\left[a_m x^m\right]_{k_0}, \left[a_{m-1}x^{m-1}\right]_{k_1}, \ldots, \left[a_0\right]_{k_m}\right),$$

and thus

$$0 = \sum_{i_0+i_1+\ldots+i_m=n} \frac{n!}{i_0! i_1! \cdots i_m!} r^{n+mi_0+(m-1)i_1+\ldots+i_{m-1}} \cdot \\ \cdot a_m^{i_0} x^{mi_0} \cdots a_1^{i_{m-1}} x^{i_{m-1}} a_0^{i_m} f(x) - \\ - \sum_{k_0+k_1+\ldots+k_m=n} \frac{n!}{k_0! k_1! \cdots k_m!} r^{n+mk_0+(m-1)k_1+\ldots+k_{m-1}} x^n \cdot \\ \cdot G\left(\left[a_m x^m\right]_{k_0}, \left[a_{m-1}x^{m-1}\right]_{k_1}, \ldots, \left[a_0\right]_{k_m}\right). \tag{3.5}$$

On the right side of equation (3.5), for each fixed $x \in \mathbb{R}$, with respect to the variable r, we obtain a polynomial (of degree at most $(m+1)n$) that equals zero for every $r \in \mathbb{Q}$. According to Lemma 2.3, each coefficient of this polynomial must be zero (for each $x \in \mathbb{R}$).

For the coefficient of r^n this observation yields

$$0 = a_0^n f(x) - x^n g(a_0).$$

Hence
$$f(x) = x^n \frac{g(a_0)}{a_0^n}.$$
For $x = 1$ we have $f(1) = \frac{g(a_0)}{a_0^n}$, therefore $f(x) = x^n f(1)$ for all $x \in \mathbb{R}$.

Considering the coefficient of $r^{(m+1)n}$ we obtain

$$0 = a_m^n x^{mn} f(x) - x^n g(a_m x^m).$$

Substituting $f(x) = x^n f(1)$ in the latter equation then dividing the equation by x^n if $x \neq 0$ we get

$$g(a_m x^m) = (a_m x^m)^n f(1).$$

If m is odd, then the range of $a_m x^m$ (with $a_m \neq 0$) is the set of real numbers. If m is even, then the range of $a_m x^m$ (with $a_m \neq 0$) is the set of non-negative or non-positive real numbers, depending on the sign of a_m. But since g is rationally homogeneous of degree n, then $g(-u) = (-1)^n g(u)$, consequently

$$g(x) = x^n f(1) = f(x)$$

for all $x \in \mathbb{R}$. □

Remark 3.1. If $a_m = a_{m-1} = ... = a_1 = 0$ then $y = a_0$ is constant. Therefore we have
$$a_0^n f(x) = x^n g(a_0)$$
and thus
$$f(x) = x^n \frac{g(a_0)}{a_0^n} = x^n f(1),$$
but we have no further information about g other than $g(a_0) = a_0^n f(1)$.

Remark 3.2. In the particular case $y = a_1 x$ ($a_0 = 0$) the conditional equation has the form

$$a_1^n x^n f\left(x\right) = x^n g\left(a_1 x\right),$$

i.e., $g\left(a_1 x\right) = a_1^n f\left(x\right)$. We note that, if $a_1 = 0$ then this equation yields no information, so f and g can be any monomial functions. In the case $a_1 \neq 0$, let f be any discontinuous real monomial function. Then it follows from the conditional equation that g is also discontinuous.

If we tighten the study to a single function $f\colon \mathbb{R} \to \mathbb{R}$, we have an immediate consequence of the above theorem, without the restriction $a_m \neq 0$:

Corollary 3.1. *If a monomial function $f\colon \mathbb{R} \to \mathbb{R}$ of degree $n \in \mathbb{N}$ satisfies the additional equation $y^n f(x) = x^n f(y)$ for the pairs $(x, y) \in S_0$, then $f(x) = x^n f(1)$ for all $x \in \mathbb{R}$.*

Proof. If $a_m = a_{m-1} = \ldots = a_1 = 0$ the implication is trivial.
If $a_m \neq 0$, we can apply the above theorem. □

Remark 3.3. In case $m = 1$, the implication in Corollary 3.1 does not hold if $a_0 = 0$. In this case, if, for instance, $a_1 \neq 0$, the conditional equation (1.2) takes the form

$$a_1^n x^n f\left(x\right) = x^n f\left(a_1 x\right),$$

i.e., $f\left(a_1 x\right) = a_1^n f\left(x\right)$. Indeed, there exists a discontinuous example of the form $f(x) = \left(h(x)\right)^n$ $(x \in \mathbb{R})$, where $h : \mathbb{R} \to \mathbb{R}$ is a discontinuous additive function, such that the homogeneity field of h contains a_1.

Though in this project we mainly concentrate on functions defined on the real line, we would like to note that monomial functions of degree n can be defined in more abstract settings in the same way, i.e., as diagonalizations of n-additive mappings. The proof of Theorem 3.1 (and Corollary 3.1) can be applied to the somewhat more general case when the domain of f and g is a subring E of real numbers such that

E contains all rational numbers (for instance, an arbitrary subfield of $\mathbb{R}$), assuming that all coefficients $a_j \in E$ $(j = 0, 1, ..., m)$.

In the particular case $m = 1$ we can give an independent, algorithmic proof for the Corollary 3.1. In this case we have $y = a_1x + a_0$, $a_1, a_0 \in \mathbb{R} \setminus \{0\}$, and (1.2) takes the form

$$(a_1x + a_0)^n f(x) = x^n f(a_1x + a_0). \tag{3.6}$$

This proof admits an even more general domain for f.

Theorem 3.2. *Let D be a subring of $\mathbb{R}$ such that $1/2 \in D$ (i.e., D is divisible by 2) and let $a_0, a_1 \in D \setminus \{0\}$. If $f : D \to \mathbb{R}$ is a monomial of order n that satisfies the additional equation (3.6) for every $x \in D$, then $f(x) = x^n f(1)$ for all $x \in D$.*

Proof. We know that

$$(a_1x + a_0)^n = \sum_{k=0}^{n} \binom{n}{k} a_1^{n-k} x^{n-k} a_0^k.$$

Based on this and from (2.7), the equation (3.6) can be written in the following form:

$$\sum_{k=0}^{n} \binom{n}{k} a_1^{n-k} x^{n-k} a_0^k f(x) = x^n \sum_{k=0}^{n} \binom{n}{k} F\left([a_1x]_k, [a_0]_{n-k}\right). \tag{3.7}$$

The following algorithm is used for the proof:
Step 1: Let be $p = 0$.
Step 2: Replace x by $\frac{x}{2}$ in last equation.
Step 3: Multiply the resulting equation with 2^{2n-p}.
Step 4: From the resulting equation subtract the last numbered equation.
Step 5: Omit the zero value members of the sum
Step 6: Number the last equation

Step 7: If $p < n-1$ then $p = p+1$ and go to Step 2, otherwise end of the algorithm.
It can be observed, that steps 2 through 7 are performed n times.
Let us see how the algorithm works:

1). $p = 0$
If we replace x by $\frac{x}{2}$ in equation (3.7), we obtain

$$\sum_{k=0}^{n} \binom{n}{k} a_1^{n-k} \frac{x^{n-k}}{2^{n-k}} a_0^k f\left(\frac{x}{2}\right) = \frac{x^n}{2^n} \sum_{k=0}^{n} \binom{n}{k} F\left(\left[a_1 \frac{x}{2}\right]_k, [a_0]_{n-k}\right),$$

i.e.

$$\sum_{k=0}^{n} \binom{n}{k} a_1^{n-k} \frac{x^{n-k}}{2^{n-k}} a_0^k \frac{1}{2^n} f(x) = \frac{x^n}{2^n} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^k} F\left([a_1 x]_k, [a_0]_{n-k}\right).$$

Multiplying the last equation with 2^{2n}, we obtain

$$\sum_{k=0}^{n} \binom{n}{k} a_1^{n-k} 2^k x^{n-k} a_0^k f(x) = x^n \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} F\left([a_1 x]_k, [a_0]_{n-k}\right).$$

Subtracting the equation (3.7) from the last equation, we get

$$\sum_{k=0}^{n} \binom{n}{k} a_1^{n-k} x^{n-k} a_0^k f(x) \left(2^k - 1\right) =$$
$$= x^n \sum_{k=0}^{n} \binom{n}{k} F\left([a_1 x]_k, [a_0]_{n-k}\right) \left(2^{n-k} - 1\right).$$

Observe that on the left hand side for $k = 0$, we have $\left(2^k - 1\right) = 0$, on the right hand side for $k = n$, we have $\left(2^{n-k} - 1\right) = 0$.

Therefore

$$\sum_{k=1}^{n} \binom{n}{k} a_1^{n-k} x^{n-k} a_0^k f(x) \left(2^k - 1\right) = \\ = x^n \sum_{k=0}^{n-1} \binom{n}{k} F\left([a_1 x]_k, [a_0]_{n-k}\right) \left(2^{n-k} - 1\right). \tag{3.8}$$

2). $p = 1$

If we replace x by $\frac{x}{2}$ in equation (3.8), we obtain

$$\sum_{k=1}^{n} \binom{n}{k} a_1^{n-k} \frac{x^{n-k}}{2^{n-k}} a_0^k f\left(\frac{x}{2}\right) \left(2^k - 1\right) = \\ = \frac{x^n}{2^n} \sum_{k=0}^{n-1} \binom{n}{k} F\left(\left[a_1 \frac{x}{2}\right]_k, [a_0]_{n-k}\right) \left(2^{n-k} - 1\right),$$

i.e.

$$\sum_{k=1}^{n} \binom{n}{k} a_1^{n-k} \frac{x^{n-k}}{2^{n-k}} a_0^k \frac{1}{2^n} f(x) \left(2^k - 1\right) = \\ = \frac{x^n}{2^n} \sum_{k=0}^{n-1} \binom{n}{k} \frac{1}{2^k} F\left([a_1 x]_k, [a_0]_{n-k}\right) \left(2^{n-k} - 1\right).$$

Multiplying the last equation with 2^{2n-1}, we obtain

$$\sum_{k=1}^{n} \binom{n}{k} a_1^{n-k} 2^{k-1} x^{n-k} a_0^k f(x) \left(2^k - 1\right) = \\ = x^n \sum_{k=0}^{n-1} \binom{n}{k} 2^{n-1-k} F\left([a_1 x]_k, [a_0]_{n-k}\right) \left(2^{n-k} - 1\right).$$

Subtracting the equation (3.8) from the last equation, we get

$$\sum_{k=1}^{n}\binom{n}{k}a_1^{n-k}x^{n-k}a_0^k f(x)\left(2^k-1\right)\left(2^{k-1}-1\right)=$$
$$=x^n\sum_{k=0}^{n-1}\binom{n}{k}F\left(\left[a_1x\right]_k,\left[a_0\right]_{n-k}\right)\left(2^{n-k}-1\right)\left(2^{n-k-1}-1\right).$$

Observe that on the left hand side for $k=1$, we have $\left(2^{k-1}-1\right)=0$, on the right hand side for $k=n-1$ we have $\left(2^{n-k-1}-1\right)=0$. Therefore

$$\sum_{k=2}^{n}\binom{n}{k}a_1^{n-k}x^{n-k}a_0^k f(x)\left(2^k-1\right)\left(2^{k-1}-1\right)=$$
$$=x^n\sum_{k=0}^{n-2}\binom{n}{k}F\left(\left[a_1x\right]_k,\left[a_0\right]_{n-k}\right)\left(2^{n-k}-1\right)\left(2^{n-k-1}-1\right).$$

Increasing the value of p by 1 each time, the algorithm runs similarly to the previous cases.
Having $p=n-1$, we obtain the final equation:

$$a_0^n f(x)\left(2^n-1\right)\left(2^{n-1}-1\right)\left(2^{n-2}-1\right)\ldots(2-1)=$$
$$=x^nF\left(\left[a_1x\right]_0,\left[a_0\right]_{n-0}\right)\left(2^n-1\right)\left(2^{n-1}-1\right)\left(2^{n-2}-1\right)...(2-1).$$

It follows that

$$a_0^n f(x)=x^nF\left(\left[a_1x\right]_0,\left[a_0\right]_{n-0}\right).$$

But $F\left(\left[a_1x\right]_0,\left[a_0\right]_{n-0}\right)=f\left(a_0\right)$. Thus

$$f(x)=x^n\frac{f\left(a_0\right)}{a_0^n}.$$

For $x = 1$,

$$f(1) = \frac{f(a_0)}{a_0^n} \text{ hence } f(x) = x^n f(1).$$

□

Note that the constraint $a_0 \neq 0$, i.e. that the curve does not pass through the origin, plays an important role. Otherwise, it brings a lot of complications even in a simple case, as we can see in the next chapter.

4. Conditional equations involving the power function

In this chapter we investigate quadratic functions $f, g\colon \mathbb{R} \to \mathbb{R}$ and cubic functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy conditional equations involving the power function:

$$S_1 = \{(x, y) \in \mathbb{R}^2 \mid x^m = y\} \quad \text{with } m \in \mathbb{Z}\,, \quad |m| \geqslant 2.$$

We know that all additive functions $f\colon \mathbb{R} \to \mathbb{R}$ fulfilling the condition (1.4) for all points $(x, y) \in S_1$ are linear ([35]). If the additive functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the equation (1.5) for all points $(x, y) \in S_1$, then there exist $c \in \mathbb{R}$ and a derivation $d : \mathbb{R} \to \mathbb{R}$ such that $f(x) = d(x) + cx$, $g(x) = \frac{1}{m}d(x) + cx$ $(x \in \mathbb{R})$ ([26]).

4.1 Equations for quadratic functions

In this section we investigate quadratic real functions f that satisfy the additional equation (1.2) for the pairs $(x, y) \in S_1$.

In this case the additional equation has the form $x^{2m}f(x) = x^2 f(x^m)$, with $|m| \geqslant 2,\ m \in \mathbb{Z}$. Dividing this equation by x^2 if $x \neq 0$, and taking $f(0) = 0$ (fulfilled by any quadratic function f) into consideration as well, we obtain

$$f(x^m) = x^{2m-2} f(x) \tag{4.1}$$

for every $x \in \mathbb{R}$.

Theorem 4.1. *If $2 \leqslant |m|,\ m \in \mathbb{Z}$ and the quadratic function $f : \mathbb{R} \to \mathbb{R}$ satisfies (4.1) for every $x \in \mathbb{R}$, then there exists $C \in \mathbb{R}$ such that*

$$f(x) = C \cdot x^2 \quad (x \in \mathbb{R}).$$

Proof. First we prove it for $2 \leqslant m \in \mathbb{N}$:
Let $x \in \mathbb{R}$ and $r \in \mathbb{Q}$. Replacing x with $x+r$ in equation (4.1) we obtain

$$f\left((x+r)^m\right) = (x+r)^{2m-2} f(x+r)\,. \tag{4.2}$$

Let $F\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be given by (2.5). Expanding the powers of sums on both sides, equation (4.2) can be written as

$$f\left(\sum_{k=0}^{m}\binom{m}{k}x^k r^{m-k}\right) = \\ = \left(\sum_{l=0}^{2m-2}\binom{2m-2}{l}x^l r^{2m-2-l}\right)\cdot\left(f(x)+f(r)+2F(x,r)\right).$$

Applying the identity (2.6) to the left side of this equation and the rational homogeneity properties of F and f to the right side, we obtain

$$\sum_{k=0}^{m} f\left(\binom{m}{k}x^k r^{m-k}\right) + 2\sum_{0\le i<j\le m} F\left(\binom{m}{i}x^i r^{m-i},\, \binom{m}{j}x^j r^{m-j}\right) = \\ = \sum_{l=0}^{2m-2}\binom{2m-2}{l}x^l r^{2m-2-l}\left(f(x)+r^2 f(1)+2rF(x,1)\right),$$

and thus

$$\begin{aligned} 0 &= \sum_{k=0}^{m}\binom{m}{k}^2 r^{2(m-k)} f(x^k) \\ &+ 2\sum_{0\le i<j\le m}\binom{m}{i}\binom{m}{j} r^{2m-i-j} F(x^i,\, x^j) \\ &- \sum_{l=0}^{2m-2}\binom{2m-2}{l} r^{2m-2-l} x^l \left(r^2 f(1)+2rF(x,1)+f(x)\right). \end{aligned} \tag{4.3}$$

It is clear that on the right side of equation (4.3), for each fixed $x \in \mathbb{R}$, with respect to the variable r, we obtain a polynomial (of degree at most $2m$) that equals zero for every $r \in \mathbb{Q}$. According to Lemma 2.3, each coefficient of this polynomial has to equal zero (for each $x \in \mathbb{R}$). The coefficient of r^{2m} equals $f(1) - f(1) = 0$, so the degree is, in fact, smaller than $2m$. However, for the coefficient of r^{2m-1} this observation yields

$$\begin{aligned} 0 &= 2\binom{m}{0}\binom{m}{1}F(1,x) - \binom{2m-2}{0}2F(x,1) - \binom{2m-2}{1}xf(1) \\ &= (2m-2)(F(1,x) - xf(1)). \end{aligned}$$

Since $m \geqslant 2$ implies $2m - 2 > 0$, we obtain

$$F(1,x) = f(1) \cdot x \tag{4.4}$$

for every $x \in \mathbb{R}$. The equality of the coefficient of r^{2m-2} to zero can be written as

$$\begin{aligned} 0 &= \binom{n}{1}^2 f(x) + 2\binom{m}{0}\binom{m}{2}F(1,x^2) - \\ &- \binom{2m-2}{0}f(x) - \binom{2m-2}{1}x \cdot 2F(x,1) - \binom{2m-2}{2}x^2 f(1) = \\ &= (m^2-1)f(x) + m(m-1)F(1,x^2) - 4(m-1)xF(1,x) - \\ &- (m-1)(2m-3)f(1)x^2 . \end{aligned}$$

Applying property (4.4), this equation can be reformulated as

$$0 = (m-1)(m+1)\left(f(x) - f(1)x^2\right),$$

which implies $f(x) = f(1) \cdot x^2$ for every $x \in \mathbb{R}$.
Case $-2 \geqslant m \in \mathbb{Z}$:
Let $s = -m$, whence $s \in \mathbb{N}$, $s \geqslant 2$. Then $y = x^{-s} = \dfrac{1}{x^s}$.

Hence equation (1.2) with $n = 2$ has the form

$$\frac{1}{x^{2s}} f(x) = x^2 f\left(\frac{1}{x^s}\right)$$

for $x \neq 0$. Multiplying the last equation with x^{-2}, we obtain

$$\frac{1}{x^{2s+2}} f(x) = f\left(\frac{1}{x^s}\right). \tag{4.5}$$

Replacing x with x^{-s} in equation (4.5) we have

$$x^{s(2s+2)} f\left(\frac{1}{x^s}\right) = f\left(x^{s^2}\right).$$

Now from (4.5), we get

$$x^{2s^2+2s} \frac{1}{x^{2s+2}} f(x) = f\left(x^{s^2}\right),$$

i.e.

$$x^{2s^2-2} f(x) = f\left(x^{s^2}\right). \tag{4.6}$$

For every $2 \leqslant s \in \mathbb{N}$ there exists $p \in \mathbb{N}, p \geqslant 2$ such that $p = s^2$. Replacing s^2 with p in equation (4.6) we get the condition (4.1) for $2 \leqslant p \in \mathbb{N}$ in place of m, and we have already proved this case.

Hence $f(x) = x^2 f(1)$ for all $x \in \mathbb{R}$. □

We note that in case $m = 0$ the same implication is trivial, while in case $m = 1$ condition (4.1) becomes a trivial identity that does not imply any restriction for f (hence f can be discontinuous as well).

4.2 Related problems, admitting quadratic functions generated by derivations

In this section we discuss, in the particular case $m = 2$, a modified version of the condition (4.1), admitting a discontinuous quadratic solution f. Then we examine quadratic real functions f, g that satisfy the conditional equation (1.3) for the pairs $(x, y) \in S_1$ with $m = 2$.

We can formulate an analogy of Proposition 2.1 for quadratic mappings.

Theorem 4.2. *Let $K \in \mathbb{R}$. If a quadratic function $f : \mathbb{R} \to \mathbb{R}$ satisfies the additional equation*

$$f(x^2) = Kx^2 f(x) \tag{4.7}$$

for every $x \in \mathbb{R}$, then either $f = 0$ or $K \in \{1, 2, 4\}$. In the latter cases, we have the following representations for f.

- *A quadratic mapping $f\colon \mathbb{R} \to \mathbb{R}$ fulfills (4.7) with $K = 1$ if, and only if,*
$$f(x) = f(1) \cdot x^2 \qquad (x \in \mathbb{R}).$$
- *A quadratic mapping $f\colon \mathbb{R} \to \mathbb{R}$ fulfills (4.7) with $K = 2$ if, and only if, there exists $\varphi \in \mathcal{D}_2(\mathbb{R})$ such that*
$$f(x) = 4x\varphi(x) - \varphi(x^2) \qquad (x \in \mathbb{R}). \tag{4.8}$$
- *If $B\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a symmetric bi-derivation, then*
$$f(x) = B(x, x) \qquad (x \in \mathbb{R})$$
is a quadratic solution of the equation (4.7) with $K = 4$.

Proof. Let $x \in \mathbb{R}$ and $r \in \mathbb{Q}$. Replacing x with $x + r$ in equation (4.7) we obtain

$$f\left((x+r)^2\right) = K(x+r)^2 f(x+r)\,. \tag{4.9}$$

Let $F\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be given by (2.5). Expanding the powers of sums on both sides, equation (4.9) can be written as

$$f\left(x^2+2rx+r^2\right) = K\left(x^2+2rx+r^2\right)\cdot\left(f(x)+f(r)+2F(x,r)\right).$$

Applying the identity (2.6) to the left side of this equation and the rational homogeneity properties of F and f, we obtain

$$\begin{aligned} f(x^2)+4r^2f(x)+r^4f(1)+4rF(x^2,x)+2r^2F(x^2,1)+4r^3F(x,1) = \\ = Kx^2f(x)+2Krxf(x)+Kr^2f(x)+Kr^2x^2f(1)+2Kr^3xf(1)+ \\ + Kr^4f(1)+2Krx^2F(x,1)+4Kr^2xF(x,1)+2Kr^3F(x,1)\,, \end{aligned}$$

and thus

$$\begin{aligned} 0 &= (K-1)f(1)r^4 + 2\left[(K-2)F(x,1)+Kf(1)x\right]r^3 \\ &+ \left[(K-4)f(x)+Kf(1)x^2+4KxF(x,1)-2F(x^2,1)\right]r^2 \\ &+ 2\left[Kxf(x)+Kx^2F(x,1)-2F(x^2,x)\right]r + \left[Kx^2f(x)-f(x^2)\right]. \end{aligned}$$

Applying Lemma 2.3, we obtain, besides (4.7), the equations

$$(K-1)f(1) = 0\,, \tag{4.10}$$
$$(K-2)F(x,1)+Kf(1)x = 0\,, \tag{4.11}$$
$$(K-4)f(x)+Kf(1)x^2+4KxF(x,1)-2F(x^2,1) = 0\,, \tag{4.12}$$
$$Kxf(x)+Kx^2F(x,1)-2F(x^2,x) = 0\,. \tag{4.13}$$

for every $x\in\mathbb{R}$.

Equation (4.10) implies $K=1$ or $f(1)=0$. If $K=1$, we may apply Theorem 4.1 with $n=2$ (or directly (4.11) to get $F(x,1)=f(1)x$ and then (4.12)) to obtain $f(x)=f(1)\cdot x^2$ for all $x\in\mathbb{R}$. Clearly, $f(x)=Cx^2$ $(x\in\mathbb{R})$ is, in fact, a quadratic solution of (4.7) with

$K = 1$ for any real coefficient C.

In the rest of this proof, we consider the case $K \neq 1$, hence we have $f(1) = 0$. Then equation (4.11) has the form

$$(K-2)F(x,1) = 0 \qquad (x \in \mathbb{R}). \tag{4.14}$$

This implies $K = 2$ or $F(x,1) = 0$ $(x \in \mathbb{R})$. If $K = 2$, equation (4.12) implies (4.8) with the additive mapping

$$\varphi(x) = F(x,1) \qquad (x \in \mathbb{R}).$$

Substituting (4.8) into $f(x^2) = 2x^2 f(x)$ $(x \in \mathbb{R})$, we obtain (2.14). Hence, due to Lemma 2.4, we have $\varphi \in \mathcal{D}_2(\mathbb{R})$.

It is easy to verify that, for each $\varphi \in \mathcal{D}_2(\mathbb{R})$, the function f defined by (4.8) is quadratic. In order to show that f fulfills equation (4.7) with $K = 2$, let $B\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ denote the symmetric bi-derivation satisfying

$$\varphi(xy) - x\varphi(y) - \varphi(x)y = B(x,y)$$

for every x, $y \in \mathbb{R}$. In particular, $B(x,x) = \varphi(x^2) - 2x\varphi(x)$ $(x \in \mathbb{R})$, hence we have

$$f(x) = 4x\varphi(x) - \varphi(x^2) = 2x\varphi(x) - B(x,x) = \varphi(x^2) - 2B(x,x) \quad (x \in \mathbb{R}).$$

Since B is a bi-derivation, it fulfills

$$B(x^2,y^2) = 4xyB(x,y) \qquad (x,y \in \mathbb{R}), \tag{4.15}$$

and thus

$$f(x^2) = 2x^2\varphi(x^2) - B(x^2,x^2) = 2x^2\varphi(x^2) - 4x^2B(x,x) = 2x^2 f(x)$$

for every $x \in \mathbb{R}$.

If $K \in \mathbb{R} \setminus \{1, 2\}$, equation (4.14) yields $F(x,1) = 0$ $(x \in \mathbb{R})$.

Hence equation (4.12) has the short form

$$(K-4)f(x)=0 \qquad (x\in\mathbb{R}). \tag{4.16}$$

If f is not identically equal to zero, this implies $K=4$. Now, it follows from property (4.15) of bi-derivations that the trace $f(x)=B(x,x)$ $(x\in\mathbb{R})$ of a (symmetric) bi-derivation B fulfills $f\left(x^2\right)=4x^2f(x)$ for every $x\in\mathbb{R}$ and, of course, such a function f is also quadratic. □

Remark 4.1. If $\varphi\in\mathcal{D}(\mathbb{R})$, then equation (4.8) yields $f(x)=\varphi(x^2)$ $(x\in\mathbb{R})$. This observation ensures the existence of a non-zero quadratic solution f of (4.7) for $K=2$. The existence of such solutions in the cases $K=1$ and $K=4$ is an obvious consequence of Theorem 4.2.

Remark 4.2. We can observe that, in case $K=4$, Theorem 4.2 provides only a sufficient condition for f to satisfy equations (2.4) and (4.7). It is an open question whether this condition is necessary.

However, we can prove a somewhat weaker necessary condition in that case.

Theorem 4.3. *If a quadratic function $f\colon\mathbb{R}\to\mathbb{R}$ satisfies the additional equation*

$$f\left(x^2\right)=4x^2f(x) \tag{4.17}$$

for every $x\in\mathbb{R}$, then f is the trace of a symmetric bi-derivation of order 2.

Proof. Equation (4.14) yields

$$F(x,1)=0, \tag{4.18}$$

for all $x\in\mathbb{R}$.
Putting $x+1$ in place of x in equation (4.17), we have

$$f\left(x^2+2x+1\right)=4\left(x^2+2x+1\right)f(x+1).$$

Applying the identity (2.6) to the left side of this equation and the rational homogeneity properties of F and f, we obtain

$$\begin{aligned} f\left(x^2\right) + 4f(x) + f(1) + 4F\left(x^2, x\right) + 2F\left(x^2, 1\right) + 4F(x, 1) = \\ = 4\left(x^2 + 2x + 1\right)\left[f(x) + f(1) + 2F(x, 1)\right]. \end{aligned}$$

Substituting (4.17) and (4.18) into the latter equation, we obtain

$$F\left(x^2, x\right) = 2xf(x). \tag{4.19}$$

Let $x, y \in \mathbb{R}$ and $r \in \mathbb{Q}$. Substituting $x + ry$ in place of x in equation (4.19), we get

$$F\left(x^2 + 2rxy + r^2y^2, x + ry\right) = 2(x + ry)f(x + ry).$$

Rearranging the latter equation and using (4.19) we obtain

$$\begin{aligned} 0 =& 2rF(xy, x) + r^2F\left(y^2, x\right) + rF\left(x^2, y\right) + 2r^2F(xy, y) - \\ &- 2r^2xf(y) - 4rxF(x, y) - 2ryf(x) - 4r^2yF(x, y). \end{aligned}$$

Thus we get a polynomial in r. The coefficient of r^1 equals

$$2F(xy, x) = 4xF(x, y) + 2yf(x) - F\left(x^2, y\right). \tag{4.20}$$

Replacing x by x^2 in equation (4.20), we have

$$2F\left(x^2y, x^2\right) = 4x^2F\left(x^2, y\right) + 2yf\left(x^2\right) - F\left(x^4, y\right).$$

Applying (4.17), this equation can be reformulated as

$$2F\left(x^2y, x^2\right) = 4x^2F\left(x^2, y\right) + 8x^2yf(x) - F\left(x^4, y\right). \tag{4.21}$$

Let $x, y \in \mathbb{R}$ and $r \in \mathbb{Q}$. Replacing x with $x + ry$ in equation (4.17) we obtain

$$f\left((x+ry)^2\right) = 4(x+ry)^2 f(x+ry)\,. \tag{4.22}$$

Expanding the powers of sums on both sides, equation (4.22) can be written as

$$\begin{aligned} &f\left(x^2+2rxy+r^2y^2\right) = \\ &\qquad = 4\left(x^2+2rxy+r^2y^2\right)\cdot\left[f(x)+f(ry)+2F(x,ry)\right]. \end{aligned}$$

Applying the identity (2.6) to the left side of this equation and the rational homogeneity properties of F and f, we obtain

$$\begin{aligned} &f\left(x^2\right)+4r^2f(xy)+r^4f\left(y^2\right)+ \\ &\quad +4rF\left(x^2,xy\right)+2r^2F\left(x^2,y^2\right)+4r^3F\left(xy,y^2\right)= \\ &\quad =4x^2f(x)+8rxyf(x)+4r^2y^2f(x)+4r^2x^2f(y)+8r^3xyf(y)+ \\ &\quad +4r^4y^2f(y)+8rx^2F(x,y)+16r^2xyF(x,y)+8r^3y^2F(x,y)\,, \end{aligned}$$

and thus

$$\begin{aligned} 0 = &\left[4y^2f(y)-f\left(y^2\right)\right]r^4+\left[8xyf(y)+8y^2F(x,y)-4F\left(xy,y^2\right)\right]r^3 \\ &+\left[4y^2f(x)+4x^2f(y)+16xyF(x,y)-4f(xy)-2F\left(x^2,y^2\right)\right]r^2 \\ &+\left[8xyf(x)+8x^2F(x,y)-4F\left(x^2,xy\right)\right]r+\left[4x^2f(x)-f\left(x^2\right)\right]. \end{aligned}$$

Applying Lemma 2.3 for the coefficient of r^1 we obtain

$$4F\left(x^2,xy\right)=8xyf(x)+8x^2F(x,y).$$

Replacing y by xy in the latter equation, then dividing the resulting equation by 2, we get

$$2F\left(x^2,x^2y\right)=4x^2yf(x)+4x^2F(x,xy).$$

Substituting (4.20) into the latter equation, we obtain

$$2F\left(x^2, x^2y\right) = 4x^2yf(x) + 2x^2\left[4xF(x,y) + 2yf(x) - F\left(x^2,y\right)\right] = \\ = 4x^2yf(x) + 4x^2yf(x) + 8x^3F(x,y) - 2x^2F\left(x^2,y\right),$$

i.e.,

$$2F\left(x^2, x^2y\right) = 8x^2yf(x) + 8x^3F(x,y) - 2x^2F\left(x^2,y\right). \tag{4.23}$$

From the equality of the left sides of (4.21) and (4.23), we have

$$4x^2F\left(x^2,y\right) + 8x^2yf\left(x\right) - F\left(x^4,y\right) = \\ = 8x^2yf(x) + 8x^3F(x,y) - 2x^2F\left(x^2,y\right),$$

therefore

$$F\left(x^4,y\right) = 6x^2F\left(x^2,y\right) - 8x^3F(x,y). \tag{4.24}$$

Equation (4.24) holds for a fixed $y \in \mathbb{R}$, for each $x \in \mathbb{R}$. By Lemma 2.4 F is a derivation of order 2 in x. Since F is a symmetric, bi-additive function, it follows that F is a derivation of order 2 in each variable, so F is a symmetric bi-derivation of order 2. □

The significance of the previous results is highlighted by the following theorem, where two quadratic functions are involved.

Theorem 4.4. *The quadratic functions* $f, g: \mathbb{R} \to \mathbb{R}$ *satisfy the additional equation* $y^2f(x) = x^2g(y)$ *for the pairs* $(x,y) \in S_1$ *with* $m = 2$ *if, and only if, there exist an additive function* $\varphi\colon \mathbb{R} \to \mathbb{R}$ *and a quadratic function* $h\colon \mathbb{R} \to \mathbb{R}$ *satisfying the condition*

$$h(x^2) = 4x^2h(x) \qquad (x \in \mathbb{R})$$

such that

$$f(x) = h(x) + \varphi(x^2) \quad \textit{and} \quad g(x) = \frac{1}{4}h(x) + x\varphi(x)$$

for all $x \in \mathbb{R}$.

Proof. In this case the conditional equation takes the form

$$x^2 f(x) = g\left(x^2\right). \tag{4.25}$$

Let $x \in \mathbb{R}$ and $r \in \mathbb{Q}$. Replacing x with $x+r$ in equation (4.25) we obtain

$$(x+r)^2 f(x+r) = g\left((x+r)^2\right). \tag{4.26}$$

Expanding the powers of sums on both sides, equation (4.26) can be written as

$$\begin{aligned}&\left(x^2+2rx+r^2\right)(f(x)+f(r)+2F(x,r)) = \\ &\quad = g(x^2)+4r^2 g(x)+r^4 g(1)+4rG(x^2,x)+2r^2 G(x^2,1)+4r^3 G(x,1).\end{aligned}$$

In view of Lemma 2.3, the coefficients of each power of r have to be equal on the two sides. For the coefficients of r^4, we get $f(1) = g(1)$. Considering the coefficient of r^3 we obtain $2F(x,1)+2xf(1) = 4G(x,1)$, that is

$$2G(x,1) = F(x,1) + xf(1).$$

Hence

$$2G\left(x^2,1\right) = F\left(x^2,1\right) + x^2 f(1). \tag{4.27}$$

The equality of the coefficients of r^2 can be written as

$$f(x) + 4xF(x,1) + x^2 f(1) = 4g(x) + 2G\left(x^2,1\right).$$

Substituting (4.27) in the latter equation we obtain

$$4g(x) = f(x) - F\left(x^2,1\right) + 4xF(x,1). \tag{4.28}$$

Substituting x^2 in place of x in equation (4.28) and applying the identity (4.25), we obtain

$$f\left(x^2\right) = 4x^2 f(x) - 4x^2 F\left(x^2, 1\right) + F\left(x^4, 1\right),$$

i.e.

$$f\left(x^2\right) - F\left(x^4, 1\right) = 4x^2\left[f(x) - F\left(x^2, 1\right)\right].$$

Now, we define a map $h\colon \mathbb{R} \to \mathbb{R}$ by $h(x) := f(x) - F\left(x^2, 1\right)$.

$$h\left(x^2\right) = f\left(x^2\right) - F\left(x^4, 1\right) = 4x^2\left[f(x) - F\left(x^2, 1\right)\right] = 4x^2 h(x).$$

Substituting h in (4.28), we have

$$g(x) = \frac{1}{4}h(x) + xF(x, 1).$$

The converse is easily verified. □

4.3 Equation for cubic functions

In this section we investigate cubic functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy the additional equation (1.2) for the pairs $(x, y) \in S_1$.

Theorem 4.5. *If $f : \mathbb{R} \to \mathbb{R}$ is a generalized monomial of degree 3 that satisfies the additional equation $y^3 f(x) = x^3 f(y)$ under the condition $(x, y) \in S_1$, then $f(x) = x^3 f(1)$ for all $x \in \mathbb{R}$.*

Proof. First, we prove the theorem in case of positive m.
Let $m \geqslant 2,\ m \in \mathbb{N}$. The additional equation is: $x^{3m} f(x) = x^3 f\left(x^m\right)$. Dividing this equation by x^3 if $x \neq 0$, and taking $f(0) = 0$ into consideration as well, we obtain

$$f\left(x^m\right) = x^{3m-3} f(x) \tag{4.29}$$

for all $x \in \mathbb{R}$.

Replacing x with $x+r$ in equation (4.29) we obtain

$$f\left((x+r)^m\right) = (x+r)^{3m-3} f(x+r). \tag{4.30}$$

Expanding the powers of sums on both sides, equation (4.30) can be written as

$$f\left(\sum_{k=0}^{m}\binom{m}{k}x^k r^{m-k}\right) = \sum_{l=0}^{3m-3}\binom{3m-3}{l}x^l r^{3m-3-l} f(x+r). \tag{4.31}$$

Applying the identity (2.8) to the left side of this equation and the rational homogeneity properties of F, we have

$$\begin{aligned}
&f\left(\sum_{k=0}^{m}\binom{m}{k}x^k r^{m-k}\right) = \\
&= \sum_{k_0+k_1+\cdots+k_m=3}\frac{3!}{k_0!k_1!\cdots k_m!}\cdot \\
&\quad \cdot F\left([r^m]_{k_0}, \left[\binom{m}{1}xr^{m-1}\right]_{k_1}, \ldots, \left[\binom{m}{m}x^m r^0\right]_{k_m}\right) = \\
&= \sum_{k_0+\cdots+k_m=3}\frac{3!}{k_0!k_1!\cdots k_m!} r^{mk_0} r^{(m-1)k_1}\cdots r^{(m-m)k_m}\cdot \\
&\quad \cdot F\left([1]_{k_0}, \left[\binom{m}{1}x\right]_{k_1}, \ldots, [x^m]_{k_m}\right) = \\
&= \sum_{k_0+k_1+\cdots+k_m=3}\frac{3!}{k_0!k_1!\cdots k_m!} r^{3m-(k_1+2k_2+\cdots+mk_m)}\cdot \\
&\quad \cdot F\left([1]_{k_0}, \left[\binom{m}{1}x\right]_{k_1}, \ldots, [x^m]_{k_m}\right).
\end{aligned}$$

Then applying (2.7) to the right side of equation (4.31) and the rational homogeneity properties of F, we have

$$\sum_{l=0}^{3m-3} \binom{3m-3}{l} x^l r^{3m-3-l} f(x+r) =$$
$$= \sum_{l=0}^{3m-3} \binom{3m-3}{l} x^l r^{3m-3-l} \sum_{p=0}^{3} \binom{3}{p} F\left([x]_p, [r]_{3-p}\right) =$$
$$= \sum_{l=0}^{3m-3} \binom{3m-3}{l} x^l r^{3m-3-l} \sum_{p=0}^{3} \binom{3}{p} r^{3-p} F\left([x]_p, [1]_{3-p}\right).$$

Thus, the equation (4.31) can be written in the form

$$\begin{aligned} 0 = &\sum_{k_0+k_1+\cdots+k_m=3} \frac{3!}{k_0!k_1!\cdots k_m!} r^{3m-(k_1+2k_2+\cdots+mk_m)}\cdot \\ &\cdot F\left([1]_{k_0}, \left[\binom{m}{1}x\right]_{k_1}, \ldots, [x^m]_{k_m}\right) - \\ &- \sum_{l=0}^{3m-3} \binom{3m-3}{l} x^l r^{3m-3-l} \sum_{p=0}^{3} \binom{3}{p} r^{3-p} F\left([x]_p, [1]_{3-p}\right). \end{aligned} \tag{4.32}$$

On the right side of equation (4.32), for each fixed $x \in \mathbb{R}$, with respect to the variable r, we obtain a polynomial (of degree at most $3m$) that equals zero for every $r \in \mathbb{Q}$. According to Lemma 2.3, each coefficient of this polynomial has to equal zero (for each $x \in \mathbb{R}$). The coefficient of r^{3m} equals $f(1) - f(1) = 0$, so the degree is, in fact, smaller than $3m$. However, for the coefficient of r^{3m-1} this observation yields

$$0 = \frac{3!}{2!} F(1,1,mx) - \left[3F(x,1,1) + (3m-3)xf(1)\right].$$

Thus

$$3(m-1)F(x,1,1) = 3(m-1)xf(1).$$

Since $m \geqslant 2$ implies $m-1>0$, we obtain

$$F(x,1,1) = xf(1) \tag{4.33}$$

for all $x \in \mathbb{R}$. The equality of the coefficient of r^{3m-2} to zero can be written as

$$0 = \frac{3!}{2!}m^2 F(1,x,x) + \frac{3!}{2!}\frac{m(m-1)}{2}F\left(1,1,x^2\right) - \\ - \left[3F(x,x,1) + 3(m-1)3xF(x,1,1) + \frac{(3m-3)(3m-4)}{2}x^2 f(1)\right].$$

Applying property (4.33), after some computation we have

$$3\left(m^2-1\right)F(x,x,1) = 3\left(m^2-1\right)x^2 f(1),$$

i.e.

$$F(x,x,1) = x^2 f(1) \tag{4.34}$$

for all $x \in \mathbb{R}$. The equality of the coefficient of r^{3m-3} to zero can be written as

$$0 = \frac{3!}{3!}m^3 F(x,x,x) + 3!\frac{m^2(m-1)}{2}F\left(1,x,x^2\right) + \\ + \frac{3!}{2!}\frac{m(m-1)(m-2)}{3!}F\left(1,1,x^3\right) - \\ - f(x) - (3m-3)3xF(x,x,1) - \frac{(3m-3)(3m-4)}{2}3x^2F(x,1,1) - \\ - \frac{(3m-3)(3m-4)(3m-5)}{3!}x^3 f(1).$$

In case $m<3$, the third member of the right side of equality is eliminated.

Applying properties (4.33) and (4.34), this equation can be written in the form

$$m^3 f(x) + 3m^2(m-1)F(1\left(1, x, x^2\right) + \frac{m(m-1)(m-2)}{2}x^3 f(1) =$$
$$= f(x) + (9m-9)x^3 f(1) + \frac{3\left(9m^2 - 21m + 12\right)}{2}x^3 f(1) +$$
$$+ \frac{\left(9m^2 - 21m + 12\right)(3m-5)}{6}x^3 f(1).$$

After some computation we have

$$\begin{aligned} &\left(3m^3 - 3m^2\right)F\left(1, x, x^2\right) = \\ &\quad = -\left(m^3 - 1\right)f(x) + \left(4m^3 - 3m^2 - 1\right)x^3 f(1). \end{aligned} \tag{4.35}$$

The equality of the coefficient of r^{3m-4} to zero can be written as

$$0 = 3!\frac{m^2(m-1)(m-2)}{3!}F\left(1, x, x^3\right) + \frac{3!}{2!}\frac{m^3(m-1)}{2}F\left(x, x, x^2\right) +$$
$$+ \frac{3!}{2!}\frac{m^2(m-1)^2}{4}F\left(1, x^2, x^2\right) + \frac{3!}{2!}\frac{m(m-1)(m-2)(m-3)}{4!}\cdot$$
$$\cdot F\left(x^4, 1, 1\right) - (3m-3)xf(x) - \frac{(3m-3)(3m-4)}{2}3x^2 F(x, x, 1) -$$
$$- \frac{(3m-3)(3m-4)(3m-5)}{3!}3x^3 F(x, 1, 1) -$$
$$- \frac{(3m-3)(3m-4)(3m-5)(3m-6)}{4!}x^4 f(1).$$

In case $m < 4$, the first member and the fourth member of the right side of equality are eliminated.

Applying properties (4.33) and (4.34), this equation can be written in the form

$$\begin{aligned}
&m^2(m-1)(m-2)F\left(1,x,x^3\right)+\frac{3}{2}m^3(m-1)F\left(x,x,x^2\right)+\\
&\quad+\frac{3}{4}m^2(m-1)^2x^4f(1)+\frac{3m(m-1)(m-2)(m-3)}{24}x^4f(1)=\\
&\quad=3(m-1)xf(x)+\frac{9(m-1)(3m-4)}{2}x^4f(1)+\\
&\quad+\frac{3(m-1)(3m-4)(3m-5)}{2}x^4f(1)+\\
&\quad+\frac{(m-1)(3m-4)(3m-5)(3m-6)}{8}x^4f(1).
\end{aligned}$$

After some compuation we get

$$\begin{aligned}
6xf(x)- \quad 3m^3F\left(x,x,x^2\right)-\left(2m^3-4m^2\right)F\left(x,x^3,1\right)=\\
=\left(-5m^3+4m^2+6\right)x^4f(1).
\end{aligned} \tag{4.36}$$

Replacing x with $x+1$ in equation (4.36) we have

$$\begin{aligned}
&6(x+1)f(x+1)-3m^3F\left(x+1,x+1,x^2+2x+1\right)-\\
&\quad-\left(2m^3-4m^2\right)F\left(x+1,x^3+3x^2+3x+1,1\right)=\\
&\quad=\left(-5m^3+4m^2+6\right)(x+1)^4f(1).
\end{aligned} \tag{4.37}$$

$$f(x+1)=f(x)+\left(3x^2+3x+1\right)f(1),$$

$$\begin{aligned}
&F\left(x+1,x+1,x^2+2x+1\right)=\\
&\quad=F\left(x,x,x^2\right)+2F\left(1,x,x^2\right)+2f(x)+\left(6x^2+4x+1\right)f(1),
\end{aligned}$$

$$\begin{aligned}
&F\left(x+1,x^3+3x^2+3x+1,1\right)=\\
&\quad=F\left(x,x^3,1\right)+3F\left(x,x^2,1\right)+\left(x^3+6x^2+4x+1\right)f(1).
\end{aligned}$$

Substituting these in equation (4.37), we obtain

$$\begin{aligned}
&6(x+1)\left[f(x)+\left(3x^2+3x+1\right)f(1)\right]-\\
&-3m^3\left[F\left(x,x,x^2\right)+2F\left(1,x,x^2\right)+2f(x)+\left(6x^2+4x+1\right)f(1)\right]-\\
&-\left(2m^3-4m^2\right)\left[F\left(x,x^3,1\right)+3F\left(x,x^2,1\right)+\right.\\
&\left.+\left(x^3+6x^2+4x+1\right)f(1)\right]=\left(-5m^3+4m^2+6\right)(x+1)^4f(1).
\end{aligned}$$

Using equation (4.36), we get

$$\begin{aligned}
&6f(x)+6(x+1)\left(3x^2+3x+1\right)f(1)-\\
&-3m^3\left[2F\left(1,x,x^2\right)+2f(x)+\left(6x^2+4x+1\right)f(1)\right]-\\
&-\left(2m^3-4m^2\right)\left[3F\left(x,x^2,1\right)+\left(x^3+6x^2+4x+1\right)f(1)\right]=\\
&=\left(-5m^3+4m^2+6\right)x^4f(1)+\left(-5m^3+4m^2+6\right)(x+1)^4f(1).
\end{aligned}$$

After some computation we have

$$\begin{aligned}
-6\left(m^3-1\right)f(x)-4\left(3m^3-3m^2\right)F\left(1,x,x^2\right)=&\\
=\left(6x^3-18m^3x^3+12m^2x^3\right)f(1).&
\end{aligned}$$

Now, we use equation (4.35),

$$\begin{aligned}
-6\left(m^3-1\right)f(x)-4\left[-\left(m^3-1\right)f(x)+\left(4m^3-3m^2-1\right)x^3f(1)\right]=&\\
=\left(6x^3-18m^3x^3+12m^2x^3\right)f(1),&
\end{aligned}$$

i.e.

$$-2\left(m^3-1\right)f(x)=\left(-2m^3x^3+2x^3\right)f(1).$$

Hence, in case $m\geqslant 2$

$$f(x)=x^3f(1)$$

for all $x\in\mathbb{R}$.

If $m\leqslant -2$, then let $l=-m$, whence $l\in\mathbb{N}$, $l\geqslant 2$. Then $y=x^{-l}=\dfrac{1}{x^l}$. Hence equation (4.29) has the form

$$f\left(x^{-l}\right)=x^{-3l-3}f(x). \tag{4.38}$$

Replacing x with x^{-l} in equation (4.38) we have

$$f\left(x^{l^2}\right)=x^{-l(-3l-3)}f\left(x^{-l}\right).$$

Now from (4.38), we get

$$f\left(x^{l^2}\right)=x^{-l(-3l-3)}x^{-3l-3}f(x),$$

i.e.

$$f\left(x^{l^2}\right)=x^{3l^2-3}f(x). \tag{4.39}$$

For every $2\leqslant l\in\mathbb{N}$ there exists $p\in\mathbb{N}, p\geqslant 2$ such that $p=l^2$. Replacing l^2 with p in equation (4.39) we get equation (4.29). Hence $f(x)=x^3f(1)$ for all $x\in\mathbb{R}$. □

5. Equations along conic sections

In this chapter we investigate additive, quadratic and higher order monomial functions $f, g\colon \mathbb{R} \to \mathbb{R}$ that satisfy conditional equations along hyperbolas or the unit circle:

$$\begin{aligned} S_2 &= \{(x,y) \in \mathbb{R}^2 \,|\, xy = 1\}, \\ S_3 &= \{(x,y) \in \mathbb{R}^2 \,|\, x^2 - y^2 = 1\}, \\ S_4 &= \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}. \end{aligned}$$

We begin with a summary of what is already known about additive functions satisfying conditional equations along hyperbolas or the unit circle.

Every additive mapping $f : \mathbb{R} \to \mathbb{R}$, which satisfies

$$f\left(\frac{1}{x}\right) = \frac{1}{x^2} f(x)$$

for all $x \neq 0$ is of the form $f(x) = f(1)x$ for all $x \in \mathbb{R}$ (Theorem II. in [25] and Corollary 2 in [30]). If $f, g\colon \mathbb{R} \to \mathbb{R}$ are additive functions and satisfy (1.5) for all $x \neq 0$ with $(x,y) \in S_2$, then there exists a derivation $d\colon \mathbb{R} \to \mathbb{R}$ for which $f(x) = d(x) + xf(1)$ and $g(x) = -d(x) + xf(1)$ (Theorem 4 in [30]). If the real additive map f satisfies (1.4) for all points $(x,y) \in S_4$ then f is linear (Corollary 2.2 in [6]). Theorem 3.1 in [12] generalizes the latter case by introducing a second additive mapping, that is every additive functions $f, g : \mathbb{R} \to \mathbb{R}$, which satisfy (1.5) for all points $(x,y) \in S_4$ are equal and linear.

5.1 Counterexamples for the hyperbola $xy = 1$

If $f\colon \mathbb{R} \to \mathbb{R}$ is a generalized monomial of degree $n \in \mathbb{N}$, $n \geqslant 2$, f satisfies (1.2) for all $(x, y) \in S_2$, it is easy to see, that there exist discontinuous solutions.

The conditional equation is

$$f(x) = x^{2n} f\left(\frac{1}{x}\right), \qquad (\forall\; x \neq 0).$$

1. Example. If $d\colon \mathbb{R} \to \mathbb{R}$ is a not identically zero derivation, then a discontinuous solution f can be given in the following way:

$$f(x) = \begin{cases} x\,(d(x))^{n-1} & \text{if } n \text{ is an odd number,} \\ (d(x))^{n} & \text{if } n \text{ is an even number.} \end{cases}$$

If n is an odd number, then

$$f(x) = x\,(d(x))^{n-1}.$$

Putting $\dfrac{1}{x}$ in place of x, we get

$$f\left(\frac{1}{x}\right) = \frac{1}{x}\left(d\left(\frac{1}{x}\right)\right)^{n-1} = \frac{1}{x}\left(-\frac{1}{x^2}d\,(x)\right)^{n-1} =$$
$$= \frac{1}{x}\frac{1}{x^{2n-2}}\,(d\,(x))^{n-1} = \frac{x}{x^{2n}}\,(d\,(x))^{n-1} = \frac{1}{x^{2n}}f(x).$$

If n is an even number, then

$$f(x) = (d(x))^{n},$$

$$f\left(\frac{1}{x}\right) = \left(d\left(\frac{1}{x}\right)\right)^n = \left(-\frac{1}{x^2}d\left(x\right)\right)^n = \\ = \frac{1}{x^{2n}}\left(d\left(x\right)\right)^n = \frac{1}{x^{2n}}f(x).$$

2. A larger family of examples of discontinuous generalized monomials of degree $n \in \mathbb{N}$, $n \geqslant 2$ is given by:

$$f(x) = x^{n-2k}\left(d(x)\right)^{2k} \quad (x \in \mathbb{R}),$$

where $d\colon \mathbb{R} \to \mathbb{R}$ is a not identically zero derivation, $k \in \{1, \ldots, [n/2]\}$.
Putting $\frac{1}{x}$ in place of x, we get

$$f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^{n-2k}\left(d\left(\frac{1}{x}\right)\right)^{2k} = \frac{1}{x^{n-2k}}\left(-\frac{1}{x^2}d\left(x\right)\right)^{2k} = \\ = \frac{1}{x^{n-2k}}\frac{1}{x^{4k}}\left(d\left(x\right)\right)^{2k} = \frac{1}{x^{n+2k}}\left(d\left(x\right)\right)^{2k} = \\ = \frac{1}{x^{2n}}x^{n-2k}\left(d\left(x\right)\right)^{2k} = \frac{1}{x^{2n}}f(x).$$

It is an open problem, what is the general monomial solution $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy the additional assumption $y^n f(x) = x^n f(y)$, for all $(x, y) \in S_2$.

5.2 Partial results for the hyperbola $xy = 1$

In this section we study quadratic real functions that satisfy the additional assumption (1.2) for all $(x, y) \in S_2$. Though the continuity of f does not follow from this assumption, we can obtain some interesting results for the mappings $x \mapsto F(x, 1)$ and $x \mapsto F(x, 1/x)$.

Lemma 5.1. *If a quadratic function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the additional* *equation $y^2 f(x) = x^2 f(y)$ for the pairs $(x, y) \in S_2$ then*

$$F(x,1) = xf(1) \tag{5.1}$$

for all $x \in \mathbb{R}$.

Proof. The additional equation is

$$f(x) = x^4 f\left(\frac{1}{x}\right), \quad (\forall\, x \neq 0). \tag{5.2}$$

Using this equality for $x \neq -1, 0, 1$ we write $f\left(\dfrac{x}{x-1}\right)$ in two ways:

$$\begin{aligned} f\left(\frac{x}{x-1}\right) &= \left(\frac{x}{x-1}\right)^4 f\left(\frac{x-1}{x}\right) = \left(\frac{x}{x-1}\right)^4 f\left(1-\frac{1}{x}\right) = \\ &= \left(\frac{x}{x-1}\right)^4 \left[f(1) + \frac{1}{x^4} f(x) - 2F\left(\frac{1}{x}, 1\right)\right] = \\ &= \frac{1}{(x-1)^4}\left[x^4 f(1) + f(x) - 2x^4 F\left(\frac{1}{x}, 1\right)\right], \end{aligned}$$

but

$$\begin{aligned} f\left(\frac{x}{x-1}\right) &= f\left(1 + \frac{1}{x-1}\right) = \\ &= f(1) + f\left(\frac{1}{x-1}\right) + 2F\left(\frac{1}{x-1}, 1\right) = \\ &= f(1) + \frac{1}{(x-1)^4} f(x-1) + 2F\left(\frac{1}{x-1}, 1\right) = \\ &= \frac{1}{(x-1)^4}\left[(x-1)^4 f(1) + f(x) + f(1) - 2F(x,1) + \right. \\ &\quad \left. +2(x-1)^4 F\left(\frac{1}{x-1}, 1\right)\right]. \end{aligned}$$

From the equality of the two last equations, it follows that

$$x^4 f(1) + f(x) - 2x^4 F\left(\frac{1}{x}, 1\right) = \\ = (x-1)^4 f(1) + f(1) + f(x) - 2F(x,1) + 2(x-1)^4 F\left(\frac{1}{x-1}, 1\right),$$

thus

$$(x-1)^4 F\left(\frac{1}{x-1}, 1\right) - F(x,1) + x^4 F\left(\frac{1}{x}, 1\right) = \\ = \left(2x^3 - 3x^2 + 2x - 1\right) f(1). \tag{5.3}$$

Putting $x+1$ in place of x in last equation, and rearranging the equation, we get

$$(x+1)^4 F\left(\frac{1}{x+1}, 1\right) - F(x,1) + x^4 F\left(\frac{1}{x}, 1\right) = \\ = \left(2x^3 + 3x^2 + 2x + 1\right) f(1).$$

Adding the last two equations, we obtain

$$(x-1)^4 F\left(\frac{1}{x-1}, 1\right) + (x+1)^4 F\left(\frac{1}{x+1}, 1\right) = \\ = 2F(x,1) - 2x^4 F\left(\frac{1}{x}, 1\right) + \left(4x^3 + 4x\right) f(1). \tag{5.4}$$

On the other hand,

$$f\left(\frac{x+1}{x-1}\right) = \left(\frac{x+1}{x-1}\right)^4 f\left(\frac{x-1}{x+1}\right) = \left(\frac{x+1}{x-1}\right)^4 f\left(1 - \frac{2}{x+1}\right) = \\ = \left(\frac{x+1}{x-1}\right)^4 \left[f(1) + 4f\left(\frac{1}{x+1}\right) - 4F\left(\frac{1}{x+1}, 1\right)\right] =$$

$$= \left(\frac{x+1}{x-1}\right)^4 \left[f(1) + \frac{4}{(x+1)^4} f\left(x+1\right) - 4F\left(\frac{1}{x+1}, 1\right)\right] =$$
$$= \left(\frac{1}{x-1}\right)^4 \left\{(x+1)^4 f(1) + 4\left[f(x) + f(1) + 2F(x,1)\right] - \right.$$
$$\left. - 4(x+1)^4 F\left(\frac{1}{x+1}, 1\right)\right\}.$$

However, we may also write

$$f\left(\frac{x+1}{x-1}\right) = f\left(1 + \frac{2}{x-1}\right) = f(1) + f\left(\frac{2}{x-1}\right) + 2F\left(\frac{2}{x-1}, 1\right) =$$
$$= f(1) + \frac{4}{(x-1)^4} f(x-1) + 4F\left(\frac{1}{x-1}, 1\right) =$$
$$= \left(\frac{1}{x-1}\right)^4 \left\{(x-1)^4 f(1) + 4\left[f(x) + f(1) - 2F(x,1)\right] + \right.$$
$$\left. + 4(x-1)^4 F\left(\frac{1}{x-1}, 1\right)\right\}.$$

Hence, it follows that

$$(x+1)^4 f(1) + 4\left[f(x) + f(1) + 2F(x,1)\right] - 4(x+1)^4 F\left(\frac{1}{x+1}, 1\right) =$$
$$= (x-1)^4 f(1) + 4\left[f(x) + f(1) - 2F(x,1)\right] + 4(x-1)^4 F\left(\frac{1}{x-1}, 1\right),$$

thus

$$(x-1)^4 F\left(\frac{1}{x-1}, 1\right) + (x+1)^4 F\left(\frac{1}{x+1}, 1\right) = \\ = 4F(x,1) + \left(2x^3 + 2x\right) f(1). \tag{5.5}$$

From the equality of the left sides of (5.4) and (5.5) we obtain

$$2F(x,1) - 2x^4 F\left(\frac{1}{x}, 1\right) + \left(4x^3 + 4x\right) f(1) = 4F(x,1) + \left(2x^3 + 2x\right) f(1),$$

therefore

$$x^4 F\left(\frac{1}{x}, 1\right) = -F(x,1) + \left(x^3 + x\right) f(1). \tag{5.6}$$

Putting $x - 1$ in place of x in this equality, we get

$$(x-1)^4 F\left(\frac{1}{x-1}, 1\right) = \\ = -F(x,1) + f(1) + \left(x^3 - 3x^2 + 3x - 1 + x - 1\right) f(1),$$

i.e.

$$(x-1)^4 F\left(\frac{1}{x-1}, 1\right) = -F(x,1) + \left(x^3 - 3x^2 + 4x - 1\right) f(1). \tag{5.7}$$

Substituting the equations (5.6) and (5.7) in (5.3), we get

$$-F(x,1) + \left(x^3 - 3x^2 + 4x - 1\right) f(1) - F(x,1) - F(x,1) + \\ + \left(x^3 + x\right) f(1) = \left(2x^3 - 3x^2 + 2x - 1\right) f(1),$$

so

$$F(x,1) = x f(1).$$

The last equation is also true for $x = -1, 0$, or $x = 1$. □

Lemma 5.2. *If a quadratic function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the additional equation $y^2 f(x) = x^2 f(y)$ for the pairs $(x,y) \in S_2$, then*

$$f\left(x^2\right) = 2x^4 F\left(x, \frac{1}{x}\right) + 6x^2 f(x) - 7x^4 f(1) \tag{5.8}$$

for all $x \in \mathbb{R} \setminus \{0\}$.

Proof. Obviously, (5.8) holds for $x = -1$ and $x = 1$.
For $x \neq -1, 1$ we have

$$\frac{1}{x^2 - 1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$

and

$$\frac{x}{x^2-1} = \frac{1}{2}\left(\frac{1}{x-1} + \frac{1}{x+1}\right).$$

According to (2.4)

$$\begin{aligned} f\left(\frac{1}{x^2-1}\right) + f\left(\frac{x}{x^2-1}\right) &= \\ &= \frac{1}{4} f\left(\frac{1}{x-1} - \frac{1}{x+1}\right) + \frac{1}{4} f\left(\frac{1}{x-1} + \frac{1}{x+1}\right) = \\ &= \frac{1}{2}\left[f\left(\frac{1}{x-1}\right) + f\left(\frac{1}{x+1}\right)\right]. \end{aligned} \tag{5.9}$$

According to Lemma 5.1 and using (5.2)

$$\begin{aligned} f\left(\frac{1}{x-1}\right) &= \frac{1}{(x-1)^4} f(x-1) = \frac{1}{(x-1)^4}\left[f(x) + (1-2x)f(1)\right], \\ f\left(\frac{1}{x+1}\right) &= \frac{1}{(x+1)^4} f(x+1) = \frac{1}{(x+1)^4}\left[f(x) + (1+2x)f(1)\right], \\ f\left(\frac{1}{x^2-1}\right) &= \frac{1}{(x^2-1)^4} f\left(x^2-1\right) = \\ &= \frac{1}{(x^2-1)^4}\left[f\left(x^2\right) + \left(1-2x^2\right) f(1)\right], \\ f\left(\frac{x}{x^2-1}\right) &= \frac{x^4}{(x^2-1)^4} f\left(x - \frac{1}{x}\right) = \\ &= \frac{1}{(x^2-1)^4}\left[\left(x^4+1\right) f(x) - 2x^4 F\left(x, \frac{1}{x}\right)\right]. \end{aligned}$$

Substituting these in equation (5.9), we obtain

$$\begin{aligned} &\frac{1}{(x^2-1)^4}\left[f\left(x^2\right) + \left(1-2x^2\right) f(1) + \left(x^4+1\right) f(x) - 2x^4 F\left(x, \frac{1}{x}\right)\right] = \\ &= \frac{1}{2}\frac{1}{(x-1)^4}\left[f(x) + (1-2x)f(1)\right] + \frac{1}{2}\frac{1}{(x+1)^4}\left[f(x) + (1+2x)f(1)\right]. \end{aligned}$$

Multiplying the last equation with $(x^2-1)^4$:

$$f\left(x^2\right) + \left(1-2x^2\right) f(1) + \left(x^4+1\right) f(x) - 2x^4 F\left(x, \frac{1}{x}\right) =$$
$$=\frac{1}{2}(x+1)^4\left[f(x) + (1-2x)f(1)\right] + \frac{1}{2}(x-1)^4\left[f(x) + (1+2x)f(1)\right].$$

After some computation, we get

$$f\left(x^2\right) + \left(1-2x^2\right) f(1) + \left(x^4+1\right) f(x) - 2x^4 F\left(x, \frac{1}{x}\right) =$$
$$= \left(x^4+6x^2+1\right) f(x) + \left(-7x^4-2x^2+1\right) f(1),$$

and thus

$$f\left(x^2\right) = 2x^4 F\left(x, \frac{1}{x}\right) + 6x^2 f(x) - 7x^4 f(1).$$

□

Lemma 5.3. *If a quadratic function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies*

$$F\left(x, \frac{1}{x}\right) = \frac{f(x)}{x^2} \tag{5.10}$$

for every $x \neq 0$, then

$$f(x) = x^2 f(1)$$

for all $x \in \mathbb{R}$.

Proof. Replacing x by $\frac{1}{x}$ in (5.10), we get

$$F\left(\frac{1}{x}, x\right) = x^2 f\left(\frac{1}{x}\right).$$

From the equality of the left sides of the last equation and (5.10) we obtain that f satisfies the additional equation (1.2) ($n = 2$) for the pairs $(x, y) \in S_2$.

According to Lemma 5.2 and using (5.10) for every $x \neq 0$,

$$f\left(x^2\right) = 2x^4 F\left(x, \frac{1}{x}\right) + 6x^2 f(x) - 7x^4 f(1) =$$
$$= 2x^2 f(x) + 6x^2 f(x) - 7x^4 f(1),$$

thus

$$f\left(x^2\right) = 8x^2 f(x) - 7x^4 f(1).$$

This equation is also true for $x = 0$.
Putting $x - 1$ in place of x in this equality, then $x + 1$ in place of x, and using Lemma 5.1 we get

$$f\left((x-1)^2\right) = 8(x-1)^2 f(x-1) - 7(x-1)^4 f(1) =$$
$$= 8(x-1)^2 \left[f(x) + (1-2x)f(1)\right] - 7(x-1)^4 f(1) =$$
$$= 8(x-1)^2 f(x) + 8(x-1)^2(1-2x)f(1) - 7(x-1)^4 f(1) =$$
$$= (8x^2 - 16x + 8)f(x) + \left(-7x^4 + 12x^3 - 2x^2 - 4x + 1\right) f(1),$$

$$f\left((x+1)^2\right) = 8(x+1)^2 f(x+1) - 7(x+1)^4 f(1) =$$
$$= 8(x+1)^2 \left[f(x) + (1+2x)f(1)\right] - 7(x+1)^4 f(1) =$$
$$= 8(x+1)^2 f(x) + 8(x+1)^2(1+2x)f(1) - 7(x+1)^4 f(1) =$$
$$= (8x^2 + 16x + 8)f(x) + \left(-7x^4 - 12x^3 - 2x^2 + 4x + 1\right) f(1),$$

$$F\left((x+1)^2, (x-1)^2\right) = F\left(x^2 + 1 + 2x, x^2 + 1 - 2x\right) =$$
$$= f\left(x^2 + 1\right) - f(2x) =$$
$$= f\left(x^2\right) + \left(1 + 2x^2\right) f(1) - 4f(x) =$$
$$= 8x^2 f(x) + \left(-7x^4 + 1 + 2x^2\right) f(1) - 4f(x) =$$
$$= (8x^2 - 4)f(x) + \left(-7x^4 + 2x^2 + 1\right) f(1).$$

Now use $4x = (x+1)^2 - (x-1)^2$ to obtain

$$\begin{aligned} 16f(x) &= f(4x) = f\left((x+1)^2 - (x-1)^2\right) = \\ &= f\left((x+1)^2\right) + f\left((x-1)^2\right) - 2F\left((x+1)^2, (x-1)^2\right) = \\ &= \left(8x^2 - 16x + 8\right) f(x) + \left(-7x^4 + 12x^3 - 2x^2 - 4x + 1\right) f(1) + \\ &+ \left(8x^2 + 16x + 8\right) f(x) + \left(-7x^4 - 12x^3 - 2x^2 + 4x + 1\right) f(1) - \\ &- \left(16x^2 - 8\right) f(x) - \left(-14x^4 + 4x^2 + 2\right) f(1) = \\ &= 24f(x) - 8x^2 f(1). \end{aligned}$$

Hence

$$f(x) = x^2 f(1)$$

for all $x \in \mathbb{R}$. □

5.3 Equations along the hyperbola $x^2 - y^2 = 1$

In this section we investigate additive and quadratic functions f, g that satisfy additional equations along the hyperbola $x^2 - y^2 = 1$. We start with the additive case:

Theorem 5.1. *Let* $f, g\colon \mathbb{R} \to \mathbb{R}$ *be additive functions. If* f, g *satisfy the additional equation* (1.5) *for the pairs* $(x, y) \in S_3$, *then* $f(x) = g(x) = xf(1)$ *for all* $x \in \mathbb{R}$.

Proof. Setting

$$u = \frac{5x + 3y}{4}, \qquad v = \frac{3x + 5y}{4},$$

we obtain $(u, v) \in S_3$. Thus we have $vf(u) = ug(v)$. Expanding the latter equation we get

$$(3x + 5y)\left[5f(x) + 3f(y)\right] = (5x + 3y)\left[3g(x) + 5g(y\right]. \tag{5.11}$$

Then setting

$$z = \frac{5x - 3y}{4}, \qquad w = \frac{3x - 5y}{4},$$

we obtain $(z, w) \in S_3$. Thus we have $wf(z) = zg(w)$. Expanding the latter equation we get

$$(3x - 5y)\left[5f(x) - 3f(y)\right] = (5x - 3y)\left[3g(x) - 5g(y)\right]. \tag{5.12}$$

Subtracting (5.12) from (5.11), then dividing by 2, we get

$$25yf(x) + 9xf(y) = 9yg(x) + 25xg(y).$$

Now using (1.5) to replace $xg(y)$, we get

$$xf(y) = yg(x). \tag{5.13}$$

Adding (5.12) and (5.11), then dividing by 30 we have

$$xf(x) + yf(y) = xg(x) + yg(y).$$

Using (1.5) and (5.13) to replace $g(x)$ and $g(y)$ in the latest equation, we obtain

$$xf(y) = yf(x). \tag{5.14}$$

Since $x^2 - y^2 = 1$, it follows

$$x + y = \frac{1}{x - y}.$$

Therefore, by (5.14) we have

$$f(x + y) = f(x) + f(y) = f(x) + \frac{y}{x}f(x) = \frac{x + y}{x}f(x).$$

Since

$$x - y = \frac{1}{x+y},$$

thus

$$f\left(\frac{1}{x+y}\right) = f(x-y) = f(x) - f(y) = \\ = (x-y)\frac{f(x)}{x} = \frac{x-y}{x+y}f(x+y),$$

i.e.

$$f\left(\frac{1}{x+y}\right) = \frac{1}{(x+y)^2}f(x+y).$$

Putting t instead of $x+y$ in the latest equation, we get

$$f\left(\frac{1}{t}\right) = \frac{1}{t^2}f(t)$$

for all $t \neq 0$. By Theorem II. in [25] and Corollary 2 in [30] we know that $f(x) = xf(1)$. Substituting this in equation (1.5), we obtain $g(y) = f(1)y = f(y)$. □

Now let us examine the quadratic case with a single quadratic function:

Theorem 5.2. *If a quadratic function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies* (1.2) *for all $(x, y) \in S_3$, then $f(x) = x^2 f(1)$ for all $x \in \mathbb{R}$.*

Proof. The additional equation is

$$f\left(\sqrt{x^2-1}\right) = \frac{x^2-1}{x^2}f(x), \quad \forall x \in \mathbb{R} \setminus (-1,1). \tag{5.15}$$

Take an arbitrary $x \in \mathbb{R} \setminus (-1, 1)$ and choose a y such that $x^2 - y^2 = 1$. Setting

$$u = \frac{5x+3y}{4}, \qquad v = \frac{3x+5y}{4}$$

we observe that

$$u^2 - v^2 = \frac{(5x+3y)^2 - (3x+5y)^2}{4^2} = \frac{16x^2 - 16y^2}{16} = x^2 - y^2 = 1.$$

Thus we have $y^2 f(x) = x^2 f(y)$ and $u^2 f(v) = v^2 f(u)$. Clearly, the latter equation implies

$$\frac{(5x+3y)^2}{16} f\left(\frac{3x+5y}{4}\right) = \frac{(3x+5y)^2}{16} f\left(\frac{5x+3y}{4}\right),$$

i.e.

$$(5x+3y)^2 f(3x+5y) = (3x+5y)^2 f(5x+3y),$$

which yields

$$\begin{aligned}\left(25x^2+9y^2+30xy\right)[9f(x)+25f(y)+30F(x,y)] = \\ = \left(9x^2+25y^2+30xy\right)[25f(x)+9f(y)+30F(x,y)]\end{aligned}$$

and thus

$$F(x,y) = xy\,[f(x) - f(y)] = xy\left[f(x) - \frac{y^2}{x^2} f(x)\right]$$

and hence $F(x,y) = \dfrac{y}{x} f(x)$, i.e.

$$F\left(x, \sqrt{x^2-1}\right) = \frac{\sqrt{x^2-1}}{x} f(x). \tag{5.16}$$

Applying (5.15) and (5.16), we get

$$\begin{aligned} f\left(x+\sqrt{x^2-1}\right) &= f(x) + f\left(\sqrt{x^2-1}\right) + 2F\left(x,\sqrt{x^2-1}\right) = \\ &= f(x) + \frac{x^2-1}{x^2} f(x) + 2\frac{\sqrt{x^2-1}}{x} f(x) = \\ &= \frac{\left(x+\sqrt{x^2-1}\right)^2}{x^2} f(x), \end{aligned}$$

and

$$\begin{aligned} f\left(x-\sqrt{x^2-1}\right) &= f(x)+f\left(\sqrt{x^2-1}\right)-2F\left(x,\sqrt{x^2-1}\right)= \\ &= f(x)+\frac{x^2-1}{x^2}f(x)-2\frac{\sqrt{x^2-1}}{x}f(x)= \\ &= \frac{\left(x-\sqrt{x^2-1}\right)^2}{x^2}f(x). \end{aligned}$$

Hence

$$\begin{aligned} &\left(x-\sqrt{x^2-1}\right)^2 f\left(x+\sqrt{x^2-1}\right)= \\ &\qquad = \left(x+\sqrt{x^2-1}\right)^2 f\left(x-\sqrt{x^2-1}\right). \end{aligned} \tag{5.17}$$

Observing the identity

$$x+\sqrt{x^2-1}=\frac{1}{x-\sqrt{x^2-1}}$$

and taking $u = x - \sqrt{x^2-1}$, (5.17) yields

$$f\left(\frac{1}{u}\right)=\frac{1}{u^4}f(u), \quad u\in(-\infty,-1]\cup(0,1]. \tag{5.18}$$

If $u = x - \sqrt{x^2-1}$, then

$$\frac{1}{u}=\frac{1}{x-\sqrt{x^2-1}}=x+\sqrt{x^2-1} \qquad \in[-1,0)\cup[1,\infty).$$

Replacing u by $\dfrac{1}{u}$, if necessary we can verify (5.18) for every $u \neq 0$. From

$$\begin{aligned} F\left(x-\sqrt{x^2-1},x+\sqrt{x^2-1}\right) &= f(x)-f\left(\sqrt{x^2-1}\right)= \\ &= f(x)-\frac{x^2-1}{x^2}f(x)=\frac{f(x)}{x^2} \end{aligned}$$

and
$$x = \frac{u^2+1}{2u} = \frac{u}{2} + \frac{1}{2u},$$
we obtain
$$F\left(u, \frac{1}{u}\right) = \frac{4u^2}{(u^2+1)^2} f\left(\frac{u^2+1}{2u}\right).$$
Due to equation (5.18) we get
$$\begin{aligned}
\left(u^2+1\right)^2 F\left(u, \frac{1}{u}\right) &= 4u^2 f\left(\frac{u}{2} + \frac{1}{2u}\right) = \\
&= 4u^2 \left[\frac{1}{4} f(u) + \frac{1}{4u^4} f(u) + \frac{1}{2} F\left(u, \frac{1}{u}\right)\right] = \\
&= u^2 f(u) + \frac{1}{u^2} f(u) + 2u^2 F\left(u, \frac{1}{u}\right).
\end{aligned}$$

So
$$\left(u^4+1\right) F\left(u, \frac{1}{u}\right) = \frac{1}{u^2}\left(u^4+1\right) f(u),$$
i.e.,
$$F\left(u, \frac{1}{u}\right) = \frac{f(u)}{u^2}, \quad \forall u \neq 0.$$
Hence, according to Lemma 5.3
$$f(x) = x^2 f(1)$$
for all $x \in \mathbb{R}$. □

We generalize this result by using a second quadratic function.

Theorem 5.3. *Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be quadratic functions. If f, g satisfy the additional equation* (1.3) *for the pairs $(x, y) \in S_3$, then $f(x) = g(x) = x^2 f(1)$ for all $x \in \mathbb{R}$.*

Proof. The additional equation is
$$y^2 f(x) = x^2 g(y), \tag{5.19}$$

for $x, y \in \mathbb{R}$ fulfilling $x^2 - y^2 = 1$. Setting

$$u = \frac{5x + 3y}{4}, \qquad v = \frac{3x + 5y}{4}$$

we observe that $u^2 - v^2 = 1$.
Thus we have $v^2 f(u) = u^2 g(v)$. Expanding the latter equation we get

$$\begin{aligned}(9x^2{+}25y^2 + 30xy)\,[25f(x) + 9f(y) + 30F(x,y)] = \\ = \left(25x^2 + 9y^2 + 30xy\right)[9g(x) + 25g(y) + 30G(x,y)]\,.\end{aligned} \tag{5.20}$$

Let

$$z = \frac{5x - 3y}{4}, \qquad w = \frac{3x - 5y}{4}$$

we observe that $z^2 - w^2 = 1$.
Thus

$$\begin{aligned}(9x^2{+}25y^2 - 30xy)\,[25f(x) + 9f(y) - 30F(x,y)] = \\ = \left(25x^2 + 9y^2 - 30xy\right)[9g(x) + 25g(y) - 30G(x,y)]\,.\end{aligned} \tag{5.21}$$

Subtracting (5.21) from (5.20), then dividing by 60, we get

$$\begin{aligned}25xy\,(f(x) - g(y)) + 9xy\,(f(y) - g(x)) = \\ = 25\left(x^2G(x,y) - y^2F(x,y)\right) + 9\left(y^2G(x,y) - x^2F(x,y)\right).\end{aligned} \tag{5.22}$$

Similarly let

$$u_1 = \frac{5x + 4y}{3}, \qquad v_1 = \frac{4x + 5y}{3}$$

we observe that $u_1^2 - v_1^2 = 1$.
Therefore

$$\begin{aligned}(16x^2 + 25y^2 + 40xy)\,[25f(x) + 16f(y) + 40F(x,y)] = \\ = \left(25x^2 + 16y^2 + 40xy\right)[16g(x) + 25g(y) + 40G(x,y)]\,.\end{aligned} \tag{5.23}$$

Let

$$z_1 = \frac{5x - 4y}{3}, \qquad w_1 = \frac{4x - 5y}{3}$$

we observe that $z_1^2 - w_1^2 = 1$.
Thus

$$\begin{aligned}(16x^2 + 25y^2 - 40xy)\,[25f(x) + 16f(y) - 40F(x,y)] =\\ = \left(25x^2 + 16y^2 - 40xy\right)[16g(x) + 25g(y) - 40G(x,y)]\,.\end{aligned} \tag{5.24}$$

Subtracting (5.24) from (5.23), then dividing by 80, we get

$$\begin{aligned}&25xy\,(f(x) - g(y)) + 16xy\,(f(y) - g(x)) =\\ &= 25\left(x^2G(x,y) - y^2F(x,y)\right) + 16\left(y^2G(x,y) - x^2F(x,y)\right).\end{aligned} \tag{5.25}$$

Subtracting (5.25) from (5.22) we have

$$xy\,(f(y) - g(x)) = y^2G(x,y) - x^2F(x,y). \tag{5.26}$$

Substituting equation (5.26) in (5.22) we get

$$xy\,(f(x) - g(y)) = x^2G(x,y) - y^2F(x,y). \tag{5.27}$$

Adding the last two equations, we obtain

$$xy\,(f(x) + f(y) - g(x) - g(y)) = \left(x^2 + y^2\right)[G(x,y) - F(x,y)]\,. \tag{5.28}$$

Let $h = f - g$. Then h is quadratic and from (5.28) we have

$$xy\,(h(x) + h(y)) = -\left(x^2 + y^2\right)H(x,y). \tag{5.29}$$

Substituting the previously used u and v into equation (5.29), we get

$$\left(15x^2 + 15y^2 + 34xy\right)[34h(x) + 34f(y) + 60H(x,y)] = \\ = -\left(34x^2 + 34y^2 + 60xy\right)[15h(x) + 15h(y) + 34H(x,y)].$$

After some computation and using (5.29), we obtain

$$\left(x^2 + y^2\right)(h(x) + h(y)) = -4xyH(x,y).$$

Substituting $h(x) + h(y)$ from (5.29) in the latest equation, we have $H(x,y) = 0$, i. e. $F(x,y) - G(x,y) = 0$. It follows

$$G(x,y) = F(x,y). \tag{5.30}$$

Substituting (5.29) in (5.27)

$$xy\,(f(x) - g(y)) = \left(x^2 - y^2\right)F(x,y) = F(x,y). \tag{5.31}$$

Using (5.19), we get from (5.31)

$$F(x,y) = \frac{y}{x}f(x). \tag{5.32}$$

Substituting (5.30) and (5.32) in equation (5.26), we obtain

$$x^2 g(x) = f(x) + x^2 f(y). \tag{5.33}$$

Substituting the previously used u and v into equation (5.33), then multiplying with 16^2, we get

$$\left(25x^2 + 9y^2 + 30xy\right)[25g(x) + 9g(y) + 30G(x,y)] = \\ = 400f(x) + 144f(y) + 480F(x,y) + \\ + \left(25x^2 + 9y^2 + 30xy\right)[9f(x) + 25f(y) + 30F(x,y)].$$

Using (5.19), (5.30), (5.32) and (5.33), after some computation we have

$$y^2 f(x) = x^2 f(y).$$

Therefore by Theorem 5.2 $f(x) = x^2 f(1)$. Substituing f in equation (5.19), we get

$$g(y) = y^2 f(1) = f(y).$$

□

5.4 Equations along the unit circle

In this section we investigate quadratic real functions that satisfy conditional equations for the pairs $(x, y) \in S_4$. First we examine the case, when the conditional equation is with a single quadratic function, then we generalize the problem by introducing a second quadratic function.

We shall also make use of the following observation.

Lemma 5.4. *(Z. Boros and P. Erdei [6]) Let p be an integer fulfilling $p > 1$. Then, for every $x \in \mathbb{R} \setminus \{0\}$, there exist $r \in \mathbb{Q} \setminus \{0\}$ and $t \in (0, 1)$ such that*

$$rx = t + \sqrt{p^2 - 1 + t^2}. \tag{5.34}$$

Theorem 5.4. *If a quadratic function $f : \mathbb{R} \to \mathbb{R}$ satisfies $y^2 f(x) = x^2 f(y)$ for the pairs $(x, y) \in S_4$, then $f(x) = x^2 f(1)$ for all $x \in \mathbb{R}$.*

Proof. The additional equation is

$$x^2 f\left(\sqrt{1 - x^2}\right) = (1 - x^2) f(x), \quad \forall x \in (0, 1). \tag{5.35}$$

Let $x \in \mathbb{R} \setminus \{0\}$. According to Lemma 5.4, there exist $r \in \mathbb{Q} \setminus \{0\}$ and $t \in (0, 1)$ such that $rx = t + \sqrt{p^2 - 1 + t^2}$ holds. We will use Lemma 5.4 to $p = 2$.

Applying (5.35) for the pairs

$$\left(t,\sqrt{1-t^2}\right),\left(\frac{1}{2}\sqrt{3+t^2},\frac{1}{2}\sqrt{1-t^2}\right)\in S_4,$$

$$\frac{1-t^2}{t^2}f(t)=f\left(\sqrt{1-t^2}\right)=4f\left(\frac{1}{2}\sqrt{1-t^2}\right)=\frac{1-t^2}{3+t^2}f\left(\sqrt{3+t^2}\right),$$

therefore

$$\frac{f(t)}{t^2}=\frac{f\left(\sqrt{3+t^2}\right)}{3+t^2}.$$

Hence

$$\begin{aligned}F\left(t-\sqrt{3+t^2},t+\sqrt{3+t^2}\right)&=f(t)-f\left(\sqrt{3+t^2}\right)=\\&=f(t)-\frac{3+t^2}{t^2}f(t)=\frac{-3}{t^2}f(t).\end{aligned}\tag{5.36}$$

We observe the identity

$$t-\sqrt{3+t^2}=\frac{-3}{t+\sqrt{3+t^2}}.$$

Take $rx=t+\sqrt{3+t^2}$, then $t=\dfrac{r^2x^2-3}{2rx}=\dfrac{rx}{2}-\dfrac{3}{2rx}$. Substituting these in (5.36), we get

$$-3F\left(x,\frac{1}{x}\right)=F\left(rx,\frac{-3}{rx}\right)=-3\left(\frac{2rx}{r^2x^2-3}\right)^2f\left(\frac{rx}{2}-\frac{3}{2rx}\right).$$

Therefore

$$F\left(x,\frac{1}{x}\right)=\left(\frac{2rx}{r^2x^2-3}\right)^2\left[\frac{r^2}{4}f(x)+\frac{9}{4r^2}f\left(\frac{1}{x}\right)-2\frac{r}{2}\frac{3}{2r}F\left(x,\frac{1}{x}\right)\right],$$

i.e.

$$\left(r^2x^2-3\right)^2 F\left(x,\frac{1}{x}\right)=r^4x^2f(x)+9x^2\left(\frac{1}{x}\right)-6r^2x^2F\left(x,\frac{1}{x}\right),$$

thus

$$F\left(x,\frac{1}{x}\right)=\frac{r^4x^2}{r^4x^4+9}f(x)+\frac{9x^2}{r^4x^4+9}f\left(\frac{1}{x}\right). \tag{5.37}$$

Putting $\frac{1}{x}$ in place of x in last equation, we get

$$F\left(x,\frac{1}{x}\right)=\frac{r^4x^2}{r^4+9x^4}f\left(\frac{1}{x}\right)+\frac{9x^2}{r^4+9x^4}f(x).$$

From the equality of the left sides of the last two equations we obtain

$$\begin{aligned}\left(r^4+9x^4\right)&\left[r^4x^2f(x)+9x^2f\left(\frac{1}{x}\right)\right]=\\&=\left(r^4x^4+9\right)\left[r^4x^2f\left(\frac{1}{x}\right)+9x^2f(x)\right],\end{aligned}$$

i.e.

$$\begin{aligned}r^8x^2f(x)+9r^4x^6f(x)+9r^4x^2f\left(\frac{1}{x}\right)+81x^6f\left(\frac{1}{x}\right)=\\=r^8x^6f\left(\frac{1}{x}\right)+9r^4x^2f\left(\frac{1}{x}\right)+9r^4x^6f(x)+81x^2f(x),\end{aligned}$$

thus

$$\left(r^8x^2-81x^2\right)f(x)=x^4\left(r^8x^2-81x^2\right)f\left(\frac{1}{x}\right),$$

which implies the identity (5.2).

Substituting (5.2) in (5.37) we obtain

$$F\left(x,\frac{1}{x}\right)=\frac{r^4x^2}{r^4x^4+9}f(x)+\frac{9x^2}{r^4x^4+9}\,\frac{1}{x^4}f(x)=\frac{f(x)}{x^2}.$$

Hence, according to Lemma 5.3,

$$f(x) = x^2 f(1)$$

for all $x \in \mathbb{R}$. □

We generalize this result by using a second quadratic function.

Theorem 5.5. *If* $f, g\colon \mathbb{R} \to \mathbb{R}$ *are quadratic functions that satisfy the additional equation* (1.3) *for the pairs* $(x, y) \in S_4$, *then* $f(x) = g(x) = x^2 f(1)$ *for all* $x \in \mathbb{R}$.

Proof. For $n = 2$ the additional equation is

$$y^2 f(x) = x^2 g(y), \tag{5.38}$$

for $x, y \in \mathbb{R}$ fulfilling $x^2 + y^2 = 1$.
Interchanging x and y in equation (5.38) we have

$$x^2 f(y) = y^2 g(x). \tag{5.39}$$

Adding (5.38) and (5.39) we have

$$x^2 \left(f(y) + g(y)\right) = y^2 \left(f(x) + g(x)\right).$$

Let $h = f + g$. Then h is quadratic and we have

$$x^2 h(y) = y^2 h(x)$$

for the pairs $(x, y) \in S_4$. By Theorem 5.4 $h(x) = x^2 h(1)$ with $h(1) = f(1) + g(1)$.
Subtracting (5.39) from (5.38)

$$y^2 \left(f(x) - g(x)\right) = -x^2 \left(f(y) - g(y)\right).$$

Let $d = f - g$. Then d is quadratic and we have

$$y^2 d(x) = -x^2 d(y)\,, \quad (x,y) \in S_4\,,$$

that is, for every $x \in (0,1)$,

$$(x^2-1)d(x) = x^2 d\left(\sqrt{1-x^2}\right) \tag{5.40}$$

holds.
Applying (5.40) for the pairs

$$\left(t, \sqrt{1-t^2}\right), \left(\frac{1}{2}\sqrt{3+t^2}, \frac{1}{2}\sqrt{1-t^2}\right) \in S_4,$$

$$\frac{t^2-1}{t^2} d(t) = d\left(\sqrt{1-t^2}\right) = 4d\left(\frac{1}{2}\sqrt{1-t^2}\right) = \frac{t^2-1}{3+t^2} d\left(\sqrt{3+t^2}\right),$$

therefore

$$\frac{d(t)}{t^2} = \frac{d\left(\sqrt{3+t^2}\right)}{3+t^2}.$$

Introducing the biadditive mapping

$$D(u,w) = \frac{1}{2}\left(d(u+w) - d(u) - d(w)\right) \quad \left((u,w) \in \mathbb{R}^2\right),$$

we obtain $D\left(t - \sqrt{3+t^2}, t + \sqrt{3+t^2}\right) = \frac{-3}{t^2} d(t)$, which is equation (5.36). Hence, by the proof of the previous theorem, it follows that $d(x) = x^2 d(1)$, with $d(1) = f(1) - g(1)$.
From $h(x) + d(x)$ we have $f(x) = x^2 f(1)$ and from $h(x) - d(x)$ we have $g(x) = x^2 g(1)$. Substituting these in equation (5.38), we get $f(1) = g(1)$. Hence $f(x) = g(x) = x^2 f(1)$. □

5.5 Further results for the hyperbola $xy = 1$

In Section 5.2 we proved some initial results for quadratic functions $f\colon \mathbb{R} \to \mathbb{R}$ that fulfill the additional equation (5.2). Then we applied those statements in Sections 5.3 and 5.4, where we elaborated the investigations of quadratic functions satisfying an additional condition for $x^2 \pm y^2 = 1$. In this section we consider further tools from the literature of functional equations, involving very recent results as well, that finally make us possible to prove an interesting necessary condition for a quadratic function f to satisfy the additional condition (5.2).

Definition 5.1. The identically zero map is the only derivation of order zero. For each $n \in \mathbb{N}$, an additive mapping $\varphi\colon \mathbb{R} \to \mathbb{R}$ is called a derivation of order n, if there exists $B\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that B is a (symmetric) bi-derivation of order $n-1$ (that is, B is a derivation of order $n-1$ in each variable) and

$$\varphi(xy) - x\varphi(y) - \varphi(x)y = B(x,y) \qquad (x, y \in \mathbb{R}).$$

The set of derivations of order n will be denoted by $\mathcal{D}_n(\mathbb{R})$.

For every $n \in \mathbb{N}$, it is seen, that $\mathcal{D}_{n-1}(\mathbb{R}) \subset \mathcal{D}_n(\mathbb{R})$.

Proposition 5.1. *Derivations of order 3 (Unger and Reich [39] and Ebanks [11]).*
Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be an additive function. Then $\varphi \in \mathcal{D}_3(\mathbb{R})$ if and only if

$$\varphi\left(x^4\right) - 4x\varphi\left(x^3\right) + 6x^2\varphi\left(x^2\right) - 4x^3\varphi\left(x\right) = 0$$

for all $x \in \mathbb{R}$.

We also need the following Lemma:

Lemma 5.5. *(Amou [4]). Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be an additive function such that*

$$\varphi\left(x^8\right) - 14x^4\varphi\left(x^4\right) + 56x^6\varphi\left(x^2\right) - 64x^7\varphi\left(x\right) = 0,\ x \in \mathbb{R}. \tag{5.41}$$

Then $\varphi \in \mathcal{D}_3(\mathbb{R})$.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a quadratic function, which satisfies the additional equation $y^2 f(x) = x^2 f(y)$ under the condition $xy = 1$. We define a map $H\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$H(x, y) := F(x, y) - xyf(1). \tag{5.42}$$

H is certainly symmetric and biadditive, with the trace $h(x) := H(x, x)$. Then

$$h(x) = f(x) - x^2 f(1). \tag{5.43}$$

Replacing y with $\frac{1}{x}$ in equation (5.42) we have

$$H\left(x, \frac{1}{x}\right) = F\left(x, \frac{1}{x}\right) - f(1). \tag{5.44}$$

From (5.43) we have $h(1) = 0$, then from Lemma 5.1 and (5.42) we obtain

$$H(x, 1) = F(x, 1) - xf(1) = 0. \tag{5.45}$$

The conditional equation (5.2) has the form

$$h(x) + x^2 f(1) = x^4 \left[h\left(\frac{1}{x}\right) + \frac{1}{x^2} f(1)\right], \quad (\forall\ x \neq 0),$$

i.e.

$$h(x) = x^4 h\left(\frac{1}{x}\right), \quad (\forall\ x \neq 0). \tag{5.46}$$

Lemma 5.6. *If a quadratic function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the additional equation $y^2 f(x) = x^2 f(y)$ under the condition $xy = 1$, then*

$$F\left(x^2, x\right) = 2xf(x) - x^3 f(1). \tag{5.47}$$

for all $x \in \mathbb{R}$.

Proof. By Lemma 5.2 we have (5.8) for all $x \in \mathbb{R} \setminus \{0\}$. We rearrange equation (5.8) in the following form

$$f\left(x^2\right) - x^4 f(1) = 2x^4\left[F\left(x, \frac{1}{x}\right) - f(1)\right] + 6x^2\left[f(x) - x^2 f(1)\right].$$

Let $h(x) = f(x) - x^2 f(1)$. Then $h\left(x^2\right) = f\left(x^2\right) - x^4 f(1)$. Then with (5.44), equation (5.8) has the form

$$h\left(x^2\right) = 2x^4 H\left(x, \frac{1}{x}\right) + 6x^2 h(x). \tag{5.48}$$

Using (5.48) for $x \neq 0, 1$, we write $h\left((x-1)^2\right)$ in two ways:

$$h\left((x-1)^2\right) = 2(x-1)^4 H\left(x-1, \frac{1}{x-1}\right) + 6(x-1)^2 h(x-1).$$

From (5.45) we have $h(x-1) = h(x)$ and $H\left(x-1, \frac{1}{x-1}\right) = H\left(x, \frac{1}{x-1}\right)$, so

$$h\left((x-1)^2\right) = 2(x-1)^4 H\left(x, \frac{1}{x-1}\right) + 6(x-1)^2 h(x),$$

but

$$\begin{aligned} h\left((x-1)^2\right) &= h\left(x^2 - 2x + 1\right) = h\left(x^2 - 2x\right) = \\ &= h\left(x^2\right) + 4h(x) - 4H\left(x^2, x\right). \end{aligned}$$

From the equality of the two last equations, it follows that

$$\begin{aligned} &(x-1)^4 H\left(x, \frac{1}{x-1}\right) = \\ &\qquad = \frac{1}{2} h\left(x^2\right) + \left(-3x^2 + 6x - 1\right) h(x) - 2H\left(x^2, x\right). \end{aligned} \tag{5.49}$$

Using (5.46) for $x \neq 0, 1$, now we write $h\left(x^2 - x\right)$ in two ways:

$$h\left(x^2-x\right) = \left(x^2-x\right)^4 h\left(\frac{1}{x^2-x}\right) = x^4(x-1)^4 h\left(\frac{1}{x-1}-\frac{1}{x}\right) =$$
$$= x^4(x-1)^4\left[h\left(\frac{1}{x-1}\right)+h\left(\frac{1}{x}\right)-2H\left(\frac{1}{x},\frac{1}{x-1}\right)\right] =$$
$$= x^4(x-1)^4\left[\frac{1}{(x-1)^4}h\left(x-1\right)+\frac{1}{x^4}h\left(x\right)-2H\left(\frac{1}{x},\frac{1}{x-1}\right)\right] =$$
$$= x^4h(x)+(x-1)^4h(x)-2x^4(x-1)^4H\left(\frac{1}{x},\frac{1}{x-1}\right).$$

But

$$h\left(x^2-x\right) = h\left(x^2\right)+h(x)-2H\left(x^2,x\right).$$

From the equality of the two last equations, it follows that

$$\begin{aligned} x^4(x-1)^4 H\left(\frac{1}{x},\frac{1}{x-1}\right) &= \\ = -\frac{1}{2}h\left(x^2\right)+\left(x^4-2x^3+3x^2-2x\right)h(x)+H\left(x^2,x\right). \end{aligned} \tag{5.50}$$

Replacing x with $\frac{1}{x}$ in equation (5.50),

$$-\frac{1}{x^4}\frac{(x-1)^4}{x^4}H\left(x,\frac{1}{x-1}\right) =$$
$$= -\frac{1}{2x^8}h\left(x^2\right)+\frac{\left(1-2x+3x^2-2x^3\right)}{x^8}h(x)+H\left(\frac{1}{x^2},\frac{1}{x}\right).$$

Multiplying the latter equation by $-x^8$, we get

$$\begin{aligned} (x-1)^4 H\left(x,\frac{1}{x-1}\right) &= \\ = \frac{1}{2}h\left(x^2\right)+\left(2x^3-3x^2+2x-1\right)h(x)-x^8H\left(\frac{1}{x^2},\frac{1}{x}\right). \end{aligned} \tag{5.51}$$

From the equality of the left sides of (5.49) and (5.51) we obtain

$$\left(-3x^2+6x-1\right)h(x)-2H\left(x^2,x\right)=$$
$$=\left(2x^3-3x^2+2x-1\right)h(x)-x^8H\left(\frac{1}{x^2},\frac{1}{x}\right),$$

therefore

$$2H\left(x^2,x\right)=\left(-2x^3+4x\right)h(x)+x^8H\left(\frac{1}{x^2},\frac{1}{x}\right). \tag{5.52}$$

Putting $\frac{1}{x}$ in place of x in this equality, we get

$$2H\left(\frac{1}{x^2},\frac{1}{x}\right)=\left(\frac{-2}{x^3}+\frac{4}{x}\right)\frac{1}{x^4}h(x)+\frac{1}{x^8}H\left(x^2,x\right).$$

Substituting $H\left(\frac{1}{x^2},\frac{1}{x}\right)$ in equation (5.52), we obtain

$$2H\left(x^2,x\right)=\left(-2x^3+4x\right)h(x)+(-x+2x^3)h(x)+\frac{1}{2}H\left(x^2,x\right),$$

therefore

$$H\left(x^2,x\right)=2xh(x), \tag{5.53}$$

i.e.

$$F\left(x^2,x\right)=2xf(x)-x^3f(1).$$

□

Lemma 5.7. *If a quadratic function $f\colon \mathbb{R}\to\mathbb{R}$ satisfies the additional equation $y^2f(x)=x^2f(y)$ under the condition $xy=1$, then*

$$f\left(x^4\right)=20x^4f\left(x^2\right)-64x^6f(x)+45x^8f(1). \tag{5.54}$$

for all $x\in\mathbb{R}$.

Proof. Let $h(x)=f(x)-x^2f(1)$.

Replacing x with $x - \frac{1}{x}$ in equation (5.48), we obtain

$$h\left(\left(x-\frac{1}{x}\right)^2\right) = 2\left(x-\frac{1}{x}\right)^4 H\left(x-\frac{1}{x}, \frac{1}{x-\frac{1}{x}}\right) + \\ + 6\left(x-\frac{1}{x}\right)^2 h\left(x-\frac{1}{x}\right).$$

On the other hand,

$$h\left(\left(x-\frac{1}{x}\right)^2\right) = h\left(x^2+\frac{1}{x^2}-2\right) = h\left(x^2+\frac{1}{x^2}\right) = \\ = h\left(x^2\right) + h\left(\frac{1}{x^2}\right) + 2H\left(x^2, \frac{1}{x^2}\right) = \\ = \left(1+\frac{1}{x^8}\right) h\left(x^2\right) + 2H\left(x^2, \frac{1}{x^2}\right).$$

From the equality of the two last equations, it follows that

$$2H\left(x^2, \frac{1}{x^2}\right) = 2\left(x-\frac{1}{x}\right)^4 H\left(x-\frac{1}{x}, \frac{1}{x-\frac{1}{x}}\right) + \\ + 6\left(x-\frac{1}{x}\right)^2 h\left(x-\frac{1}{x}\right) - \left(1+\frac{1}{x^8}\right) h\left(x^2\right). \tag{5.55}$$

Using (5.48) for $x \neq -1, 0$, we write $h\left((x+1)^2\right)$ in two ways:

$$h\left((x+1)^2\right) = 2(x+1)^4 H\left(x+1, \frac{1}{x+1}\right) + 6(x+1)^2 h(x+1).$$

From (5.45) we have $h(x+1) = h(x)$ and $H\left(x+1, \frac{1}{x+1}\right) = H\left(x, \frac{1}{x+1}\right)$, so

$$h\left((x+1)^2\right) = 2(x+1)^4 H\left(x, \frac{1}{x+1}\right) + 6(x+1)^2 h(x),$$

but using (5.53), we have

$$h\left((x+1)^2\right) = h\left(x^2+2x+1\right) = h\left(x^2+2x\right) = \\ = h\left(x^2\right) + 4h(x) + 4H\left(x^2, x\right) = h\left(x^2\right) + (8x+4)h(x).$$

From the equality of the left sides of the last two equations we obtain

$$(x+1)^4 H\left(x, \frac{1}{x+1}\right) = \frac{1}{2}h\left(x^2\right) - \left(3x^2+2x+1\right)h(x). \qquad (5.56)$$

Using (5.46) for $x \neq -1, 0$, now we write $h\left(x^2+x\right)$ in two ways:

$$h\left(x^2+x\right) = \left(x^2+x\right)^4 h\left(\frac{1}{x^2+x}\right) = x^4(x+1)^4 h\left(\frac{1}{x} - \frac{1}{x+1}\right) = \\ = x^4(x+1)^4\left[h\left(\frac{1}{x}\right) + h\left(\frac{1}{x+1}\right) - 2H\left(\frac{1}{x}, \frac{1}{x+1}\right)\right] = \\ = x^4(x+1)^4\left[\frac{1}{x^4}h(x) + \frac{1}{(x+1)^4}h(x+1) - 2H\left(\frac{1}{x}, \frac{1}{x+1}\right)\right] = \\ = (x+1)^4 h(x) + x^4 h(x) - 2x^4(x+1)^4 H\left(\frac{1}{x}, \frac{1}{x+1}\right).$$

But using (5.53), we have

$$h\left(x^2+x\right) = h\left(x^2\right) + h(x) + 2H\left(x^2, x\right) = h\left(x^2\right) + h(x) + 4xh(x).$$

From the equality of the two last equations, it follows that

$$x^4(x+1)^4 H\left(\frac{1}{x}, \frac{1}{x+1}\right) = -\frac{1}{2}h\left(x^2\right) + \left(x^4+2x^3+3x^2\right)h(x). \qquad (5.57)$$

Using (5.53) in equation (5.49), we obtain

$$(x-1)^4 H\left(x, \frac{1}{x-1}\right) = \frac{1}{2}h\left(x^2\right) + \left(-3x^2+2x-1\right)h(x). \qquad (5.58)$$

Using (5.53) in equation (5.50), we have

$$x^4(x-1)^4 H\left(\frac{1}{x},\frac{1}{x-1}\right) = -\frac{1}{2}h\left(x^2\right)+\left(x^4-2x^3+3x^2\right)h(x). \quad (5.59)$$

Now we write

$$\begin{aligned}
2\left(x^2-1\right)^4 H\left(x-\frac{1}{x},\frac{1}{x-\frac{1}{x}}\right) &= 2\left(x^2-1\right)^4 H\left(x-\frac{1}{x},\frac{x}{x^2-1}\right) = \\
&=2\left(x^2-1\right)^4 H\left(x-\frac{1}{x},\frac{1}{2}\left(\frac{1}{x-1}+\frac{1}{x+1}\right)\right) = \\
&=\left(x^2-1\right)^4 H\left(x,\frac{1}{x-1}\right)+\left(x^2-1\right)^4 H\left(x,\frac{1}{x+1}\right)- \\
&-\left(x^2-1\right)^4 H\left(\frac{1}{x},\frac{1}{x-1}\right)-\left(x^2-1\right)^4 H\left(\frac{1}{x},\frac{1}{x+1}\right).
\end{aligned}$$

Substituting (5.56),(5.57),(5.58) and (5.59) into the latter equation, after some computation, we get

$$\begin{aligned}
2\left(x^2-1\right)^4 H\left(x-\frac{1}{x},\frac{1}{x-\frac{1}{x}}\right) &= \\
&=\frac{\left(x^4+6x^2+1\right)\left(x^4+1\right)}{x^4}h\left(x^2\right)- \\
&-\frac{2\left(3x^8+12x^6+2x^4+12x^2+3\right)}{x^2}h(x).
\end{aligned} \quad (5.60)$$

Substituting (5.60) in equation (5.55), we have

$$\begin{aligned}
2H\left(x^2,\frac{1}{x^2}\right) = &\frac{\left(x^4+6x^2+1\right)\left(x^4+1\right)}{x^8}h\left(x^2\right)- \\
&-\frac{2\left(3x^8+12x^6+2x^4+12x^2+3\right)}{x^6}h(x)+ \\
&+6\left(x-\frac{1}{x}\right)^2 h\left(x-\frac{1}{x}\right)-\left(1+\frac{1}{x^8}\right)h\left(x^2\right).
\end{aligned} \quad (5.61)$$

Expressing $H\left(x, \frac{1}{x}\right)$ from equation (5.48), we get

$$\begin{aligned} h\left(x - \frac{1}{x}\right) &= h(x) + h\left(\frac{1}{x}\right) - 2H\left(x, \frac{1}{x}\right) = \\ &= h(x) + \frac{1}{x^4}h(x) - \frac{1}{x^4}h\left(x^2\right) + \frac{6}{x^2}h(x) = \\ &= \frac{x^4 + 6x^2 + 1}{x^4}h(x) - \frac{1}{x^4}h\left(x^2\right). \end{aligned}$$

Substituting this in equation (5.61), after some computation we obtain

$$H\left(x^2, \frac{1}{x^2}\right) = \frac{7}{x^4}h\left(x^2\right) - \frac{32}{x^2}h(x). \tag{5.62}$$

Replacing x with x^2 in equation (5.48), we have

$$h\left(x^4\right) = 2x^8 H\left(x^2, \frac{1}{x^2}\right) + 6x^4 h\left(x^2\right).$$

Finally we substitute (5.62) in the latter equation to obtain

$$h\left(x^4\right) = 20x^4 h\left(x^2\right) - 64x^6 h(x).$$

The statement of the Lemma follows from this equation and the definition of h, i.e.

$$f\left(x^4\right) - x^8 f(1) = 20x^4\left[f\left(x^2\right) - x^4 f(1)\right] - 64x^6\left[f(x) - x^2 f(1)\right],$$

hence

$$f\left(x^4\right) = 20x^4 f\left(x^2\right) - 64x^6 f(x) + 45x^8 f(1).$$

□

Theorem 5.6. *If a quadratic function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the additional equation $y^2 f(x) = x^2 f(y)$ under the condition $xy = 1$, then there exists a symmetric bi-derivation H of order 3 for which $f(x) = H(x, x) + x^2 f(1)$.*

Proof. Let $h(x) = f(x) - x^2 f(1)$. From Lemma 5.6 we have

$$H\left(x^2, x\right) = 2xh(x), \tag{5.63}$$

Let $x, y \in \mathbb{R}$ and $r \in \mathbb{Q}$. Substituting $x + ry$ in place of x in equation (5.63), we get

$$H\left(x^2 + 2rxy + r^2y^2, x + ry\right) = 2(x + ry)h(x + ry).$$

Rearranging the latter equation and using (5.63) we obtain

$$\begin{aligned}0 =& 2rH(xy, x) + r^2H\left(y^2, x\right) + rH\left(x^2, y\right) + 2r^2H(xy, y) - \\ & - 2r^2xh(y) - 4rxH(x, y) - 2ryh(x) - 4r^2yH(x, y).\end{aligned}$$

Thus we get a polynomial in r. The coefficient of r^1 equals zero, hence we obtain

$$2H(xy, x) + H\left(x^2, y\right) = 4xH(x, y) + 2yh(x). \tag{5.64}$$

From Lemma 5.7 we have

$$h\left(x^4\right) = 20x^4h\left(x^2\right) - 64x^6h(x). \tag{5.65}$$

Let $x, y \in \mathbb{R}$ and $r \in \mathbb{Q}$. Replacing x with $x + ry$ in equation (5.65) we obtain

$$h\left((x + ry)^4\right) = 20(x+ry)^4h\left((x + ry)^2\right) - 64(x+ry)^6h(x+ry). \tag{5.66}$$

Expanding the powers of sums on both sides, equation (5.66) can be written as

$$h\left(\sum_{k=0}^{4}\binom{4}{k}x^k r^{n-k} y^{n-k}\right) = 20\left(\sum_{l=0}^{4}\binom{4}{l}x^l r^{4-l} y^{4-l}\right)\cdot$$
$$\cdot\, h\left(x^2+2rxy+r^2y^2\right) - 64\sum_{q=0}^{6}\binom{6}{q}x^q r^{n-q} y^{n-q} h(x+ry).$$

Applying the identity (2.6), the rational homogeneity properties of H and h, equation (5.45), and using $h(1) = 0$, we obtain

$$\begin{aligned}0 = &\sum_{k=0}^{4}\binom{4}{k}^2 r^{8-2k} h\left(x^k y^{4-k}\right) + \\ &+ 2\sum_{0\le i<j\le 4}\binom{4}{i}\binom{4}{j} r^{8-(i+j)} H\left(x^i y^{4-i},\, x^j y^{4-j}\right) - \\ &- 20\sum_{l=0}^{4}\binom{4}{l}x^l r^{4-l} y^{4-l}\cdot\left[h\left(x^2\right) + 4r^2 h(xy) + r^4 h\left(y^2\right) + \right. \\ &\left. + 4rH\left(x^2, xy\right) + 4r^3 H\left(y^2, xy\right) + 2r^2 H\left(x^2, y^2\right)\right] + \\ &+ 64\sum_{q=0}^{6}\binom{6}{q}x^q r^{6-q} y^{6-q}\left[h(x) + 2rH(x,y) + r^2 h(y)\right].\end{aligned} \tag{5.67}$$

The coefficient of r^1 equals zero, hence we obtain

$$\begin{aligned}0 =& 2\binom{4}{3}\binom{4}{4}H\left(x^3y, x^4\right) - 20\left[\binom{4}{3}x^3 y h\left(x^2\right) + \binom{4}{4}x^4 4H\left(x^2, xy\right)\right] + \\ &+ 64\binom{6}{5}x^5 y h(x) + 64\binom{6}{6}x^6 2H(x,y) = \\ &= 8H\left(x^3y, x^4\right) - 80x^3 y h\left(x^2\right) - 80x^4 H\left(x^2, xy\right) + \\ &+ 384x^5 y h(x) + 128x^6 H(x,y).\end{aligned}$$

Thus

$$H\left(x^3y,x^4\right)=10x^3yh\left(x^2\right)+ \\ +10x^4H\left(x^2,xy\right)-48x^5yh(x)-16x^6H(x,y) \tag{5.68}$$

Replacing y with xy in equation (5.68), we get

$$H\left(x^4y,x^4\right)=10x^4yh\left(x^2\right)+ \\ +10x^4H\left(x^2,x^2y\right)-48x^6yh(x)-16x^6H(x,xy) \tag{5.69}$$

Putting x^2 in place of x in equation (5.64) we have

$$2H\left(x^2y,x^2\right)=-H\left(x^4,y\right)+4x^2H\left(x^2,y\right)+2yh\left(x^2\right). \tag{5.70}$$

Substituting $H\left(x^2y,x^2\right)$ into the equation (5.69), we obtain

$$H\left(x^4y,x^4\right)=20x^4yh\left(x^2\right)+20x^6H\left(x^2,y\right)- \\ -5x^4H\left(x^4,y\right)-96x^6yh(x)-32x^6H(x,xy). \tag{5.71}$$

Expressing $H(x,xy)$ from equation (5.64), then substituting into equation (5.71), we get

$$H\left(x^4y,x^4\right)=20x^4yh\left(x^2\right)+28x^6H\left(x^2,y\right)- \\ -5x^4H\left(x^4,y\right)-64x^6yh(x)-32x^7H(x,y). \tag{5.72}$$

Now we replace x with x^4 in equation (5.64)

$$2H\left(x^4y,x^4\right)=-H\left(x^8,y\right)+4x^4H\left(x^4,y\right)+2yh\left(x^4\right). \tag{5.73}$$

And finally, from the equality of the left sides of (5.72) and (5.73), with (5.65), we obtain

$$H\left(x^8,y\right)-14x^4H\left(x^4,y\right)+56x^6H\left(x^2,y\right)-64x^7H(x,y)=0.$$

The latter equation holds for a fixed $y \in \mathbb{R}$, for each $x \in \mathbb{R}$. By Lemma 5.5 H is a derivation of order 3 in x. Since H is a symmetric, biadditive function, it follows that H is a derivation of order 3 in each variable, so H is a symmetric bi-derivation of order 3. $\square$

6. Conditional equations involving transcendental functions

In this chapter we investigate quadratic and cubic functions $f, g\colon \mathbb{R} \to \mathbb{R}$ that satisfy conditional equations involving logarithmic and exponential functions:

$$S_5 = \{(x, y) \in \mathbb{R}^2 \mid x > 0 \ \text{ and } \ \log x = y\},$$
$$S_6 = \{(x, y) \in \mathbb{R}^2 \mid e^x = y\}.$$

All additive functions $f, g\colon \mathbb{R} \to \mathbb{R}$ fulfilling the condition (1.5) on $\mathbb{R}^+$ for all points $(x, y) \in S_5$ respectively $(x, y) \in S_6$ are identical and linear (Theorem 6.1 and 6.2 in [12]).

Theorem 6.1. *Suppose $f, g\colon \mathbb{R} \to \mathbb{R}$ are monomial functions of degree $n \in \{2, 3\}$ and f, g satisfy the additional equation* (1.3) *on $\mathbb{R}^+$ for the pairs $(x, y) \in S_5$, then $f(x) = g(x) = x^n f(1)$.*

Proof. The additional equation is

$$(\log x)^n f(x) = x^n g(\log x), \quad x \in \mathbb{R}^+. \tag{6.1}$$

Replacing x by x^2 in (6.1) and using properties of logarithmic and monomial functions, we have

$$2^n (\log x)^n f\left(x^2\right) = 2^n x^{2n} g(\log x), \quad x \in \mathbb{R}^+.$$

Dividing by 2^n and using (6.1), we obtain

$$(\log x)^n f\left(x^2\right) = x^n (\log x)^n f(x),$$

therefore

$$f\left(x^2\right) = x^n f(x), \qquad x \in \mathbb{R}^+.$$

This equation holds also for $x = 0$. For $x = -t < 0$:

$$\begin{aligned} f(x^2) &= f(t^2) = t^n f(t) = (-1)^n(-t)^n(-1)^n f(-t) \\ &= (-1)^{2n} x^n f(x) = x^n f(x)\,. \end{aligned}$$

Therefore $f\left(x^2\right) = x^n f(x)$ for all $x \in \mathbb{R}$. By Theorem 4.1 ($n = 2$, $m = 2$) and Theorem 4.5 ($n = 3$, $m = 2$) we know that $f(x) = x^n f(1)$. From equation (6.1) we get

$$g(\log x) = (\log x)^n f(1), \qquad x \in \mathbb{R}^+,$$

therefore $g(t) = t^n f(1) = f(t)$ for all $t \in \mathbb{R}$. □

Remark 6.1. The results of the above theorem can be transferred to the case of exponential functions, that is $(x, y) \in S_6$, since the exponential and logarithmic functions of the same basis are inverses of each other.

We note that both the base of the logarithm, and the base of the exponential can be any positive real number except 1.

7. Summary

In this PhD dissertation we study monomial functions $f, g\colon \mathbb{R} \to \mathbb{R}$ of degree $n \in \mathbb{N}$, $n \geqslant 2$ which satisfy the conditional equation $y^n f(x) = x^n f(y)$ or $y^n f(x) = x^n g(y)$ for all points (x, y) on a specified curve. The above question was motivated by similar problems solved for additive functions, see the papers [1, 5, 6, 12, 30, 25, 35].
Our investigations were carried out along the following curves:

$$\begin{aligned}
&S_0 = \{(x,y) \in \mathbb{R}^2 \mid a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 = y\} \\
&\quad \text{with } \; m \in \mathbb{N},\; a_i \in \mathbb{R},\; i = 0, \ldots, m,\; a_m \neq 0,\, a_0 \neq 0\,, \\
&S_1 = \{(x,y) \in \mathbb{R}^2 \mid x^m = y\} \quad \text{with } m \in \mathbb{Z}\,, \quad |m| \geqslant 2\,, \\
&S_2 = \{(x,y) \in \mathbb{R}^2 \mid xy = 1\}, \\
&S_3 = \{(x,y) \in \mathbb{R}^2 \mid x^2 - y^2 = 1\}, \\
&S_4 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}, \\
&S_5 = \{(x,y) \in \mathbb{R}^2 \mid x > 0 \;\text{ and }\; \log x = y\}, \\
&S_6 = \{(x,y) \in \mathbb{R}^2 \mid e^x = y\}.
\end{aligned}$$

Before summarizing our theorems, let us look at the necessary terminology.
We call a function $f : \mathbb{R} \to \mathbb{R}$ *additive* if $f(x + y) = f(x) + f(y)$ holds for all $x, y \in \mathbb{R}$. A function $F : \mathbb{R}^n \to \mathbb{R}$ ($n \in \mathbb{N}$) is called *n–additive* if F is additive in each of its variables. Given a function $F : \mathbb{R}^n \to \mathbb{R}$, by the *diagonalization* of F we understand the function $f : \mathbb{R} \to \mathbb{R}$ arising from F by putting all the variables (from $\mathbb{R}$) equal. If, in particular, f is the diagonalization of an n–additive function $F : \mathbb{R}^n \to \mathbb{R}$, we say that f is a *monomial function (or generalized monomial) of degree n*. Generalized monomials of degree 2 are

called quadratic functions, cubic functions are generalized monomials of degree 3. Quadratic functions are characterized by the functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \qquad (x,y\in\mathbb{R}), \tag{7.1}$$

which is the so called *norm square equation* or *parallelogram law.*
The biadditive symmetric functional F that generates the quadratic function f is given by the formula

$$F(x,y)=\frac{1}{2}[f(x+y)-f(x)-f(y)]$$

for all $x,y\in\mathbb{R}$.
We say that $f\colon\mathbb{R}\to\mathbb{R}$ is a *derivation* if f is additive and satisfies the functional equation $f(xy)=f(x)y+xf(y)$ for every $x,y\in\mathbb{R}$. The family of derivations $f\colon\mathbb{R}\to\mathbb{R}$ is denoted by $\mathcal{D}(\mathbb{R})$. A functional $B\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is called a *bi-derivation* if the mappings $t\mapsto B(t,x)$ and $t\mapsto B(x,t)$ $(t\in\mathbb{R})$ are derivations for each $x\in\mathbb{R}$. For each $n\in\mathbb{N}$, an additive mapping $f\colon\mathbb{R}\to\mathbb{R}$ is called a derivation of order n, if there exists $B\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that B is a (symmetric) bi-derivation of order $n-1$ (that is, B is a derivation of order $n-1$ in each variable) and $f(xy)-xf(y)-f(x)y=B(x,y)$ $(x,y\in\mathbb{R})$. The identically zero map is the only derivation of order zero. The set of derivations of order n will be denoted by $\mathcal{D}_n(\mathbb{R})$.

Our main results are presented in four chapters, classified by curves. In Chapter 3 we investigate the continuity of monomial functions satisfying additional equations involving polynomial functions whose graphs do not pass through the origin. We get the following result.

Theorem 7.1. *Suppose that $f\colon\mathbb{R}\to\mathbb{R}$ and $g\colon\mathbb{R}\to\mathbb{R}$ are generalized monomials of degree $n\in\mathbb{N}$ that satisfy the additional equation $y^nf(x)=x^ng(y)$ for the pairs $(x,y)\in S_0$. Then $f(x)=g(x)=x^nf(1)$*

for all $x \in \mathbb{R}$ *.*

Remark 7.1. If $a_m = a_{m-1} = \ldots = a_1 = 0$ then $y = a_0$ is constant. Therefore we have

$$a_0^n f(x) = x^n g(a_0)$$

and thus

$$f(x) = x^n \frac{g(a_0)}{a_0^n} = x^n f(1),$$

but we have no further information about g other than $g(a_0) = a_0^n f(1)$.

Remark 7.2. In the particular case $y = a_1 x$ ($a_0 = 0$) the conditional equation has the form

$$a_1^n x^n f(x) = x^n g(a_1 x),$$

i.e., $g(a_1 x) = a_1^n f(x)$. We note that, if $a_1 = 0$ then this equation yields no information, so f and g can be any monomial functions. In the case $a_1 \neq 0$, let f be any discontinuous real monomial function. Then it follows from the conditional equation that g is also discontinuous.

If we tighten the study to a single monomial function $f \colon \mathbb{R} \to \mathbb{R}$, we have an immediate consequence of the above theorem, without the restriction $a_m \neq 0$:

Corollary 7.1. *(Z. Boros and E. Garda-Mátyás [10]). If a monomial function* $f \colon \mathbb{R} \to \mathbb{R}$ *of degree* $n \in \mathbb{N}$ *satisfies the additional equation* $y^n f(x) = x^n f(y)$ *for the pairs* $(x, y) \in S_0$*, then* $f(x) = x^n f(1)$ *for all* $x \in \mathbb{R}$ *.*

Remark 7.3. In case $m = 1$, the implication in Corollary 7.1 does not hold if $a_0 = 0$. In this case, if, for instance, $a_1 \neq 0$, the conditional equation takes the form

$$a_1^n x^n f(x) = x^n f(a_1 x),$$

i.e., $f(a_1 x) = a_1^n f(x)$. Indeed, there exists a discontinuous example of the form $f(x) = (h(x))^n$ $(x \in \mathbb{R})$, where $h : \mathbb{R} \to \mathbb{R}$ is a discontinuous additive function, such that the homogeneity field of h contains a_1.

Chapter 4 contains results for quadratic and cubic functions satisfying conditional equations involving the power function. First we study the case when there is a single quadratic function in the conditional equation.

Theorem 7.2. *(Z. Boros and E. Garda-Mátyás [9], E. Garda-Mátyás [16]). If $2 \leqslant |m|$, $m \in \mathbb{Z}$ and the quadratic function $f : \mathbb{R} \to \mathbb{R}$ satisfies*

$$f(x^m) = x^{2m-2} f(x)$$

for every $x \in \mathbb{R}$, then there exists $C \in \mathbb{R}$ such that

$$f(x) = C \cdot x^2 \quad (x \in \mathbb{R}).$$

We note that in case $m = 0$ the same implication is trivial, while in case $m = 1$ the additional equation becomes a trivial identity that does not imply any restriction for f (hence f can be discontinuous as well).

In the particular case $m = 2$, but with a modified version of the additional equation, we find discontinuous solutions.

Theorem 7.3. *(Z. Boros and E. Garda-Mátyás [9]). Let $K \in \mathbb{R}$. If a quadratic function $f : \mathbb{R} \to \mathbb{R}$ satisfies the additional equation*

$$f(x^2) = K x^2 f(x) \tag{7.2}$$

for every $x \in \mathbb{R}$, then either $f = 0$ or $K \in \{1, 2, 4\}$. In the latter cases, we have the following representations for f.

- *A quadratic mapping $f : \mathbb{R} \to \mathbb{R}$ fulfills (7.2) with $K = 1$ if, and only if,*

$$f(x) = f(1) \cdot x^2 \qquad (x \in \mathbb{R}).$$

- *A quadratic mapping* $f\colon \mathbb{R} \to \mathbb{R}$ *fulfills* (7.2) *with* $K = 2$ *if, and only if, there exists* $\varphi \in \mathcal{D}_2(\mathbb{R})$ *such that*

$$f(x) = 4x\varphi(x) - \varphi(x^2) \qquad (x \in \mathbb{R}). \tag{7.3}$$

- *If* $B\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a symmetric bi-derivation, then*

$$f(x) = B(x, x) \qquad (x \in \mathbb{R})$$

is a quadratic solution of the equation (7.2) *with* $K = 4$ *.*

Remark 7.4. If $\varphi \in \mathcal{D}(\mathbb{R})$, then equation (7.3) yields $f(x) = \varphi(x^2)$ $(x \in \mathbb{R})$. This observation ensures the existence of a non-zero quadratic solution f of (7.2) for $K = 2$. The existence of such solutions in the cases $K = 1$ and $K = 4$ is an obvious consequence of the last theorem.

Remark 7.5. We can observe that, in case $K = 4$, this theorem provides only a sufficient condition for f to satisfy equations (7.1) and (7.2). It is an open question whether this condition is necessary.

However, we can prove a somewhat weaker necessary condition in that case.

Theorem 7.4. *If a quadratic function* $f\colon \mathbb{R} \to \mathbb{R}$ *satisfies the additional equation*

$$f\left(x^2\right) = 4x^2 f(x)$$

for every $x \in \mathbb{R}$*, then* f *is the trace of a symmetric bi-derivation of order 2.*

The significance of the previous results is highlighted by the following theorem, where two quadratic functions are involved.

Theorem 7.5. *The quadratic functions* $f, g : \mathbb{R} \to \mathbb{R}$ *satisfy the additional equation* $y^2 f(x) = x^2 g(y)$ *for the pairs* $(x, y) \in S_1$ *with* $m = 2$ *if, and only if, there exist an additive function* $\varphi\colon \mathbb{R} \to \mathbb{R}$ *and* *a quadratic function* $h\colon \mathbb{R} \to \mathbb{R}$ *satisfying the condition*

$$h(x^2) = 4x^2 h(x) \qquad (x \in \mathbb{R})$$

such that

$$f(x) = h(x) + \varphi(x^2) \quad \textit{and} \quad g(x) = \frac{1}{4}h(x) + x\varphi(x)$$

for all $x \in \mathbb{R}$ *.*

And finally, extending the study to cubic functions, we get the following result.

Theorem 7.6. *(Z. Boros and E. Garda-Mátyás [10]). If* $f\colon \mathbb{R} \to \mathbb{R}$ *is a generalized monomial of degree 3 that satisfies the additional equation* $y^3 f(x) = x^3 f(y)$ *under the condition* $(x, y) \in S_1$*, then* $f(x) = x^3 f(1)$ *for all* $x \in \mathbb{R}$ *.*

In Chapter 5 we investigate the continuity of additive, quadratic and higher order monomial functions that satisfy subsidiary equations along hyperbolas or the unit circle.
We start with a negative result along the hyperbola given by the equation $xy = 1$. If $f\colon \mathbb{R} \to \mathbb{R}$ is a generalized monomial of degree $n \in \mathbb{N}, n \geqslant 2$, f satisfies the additional equation

$$f(x) = x^{2n} f\left(\frac{1}{x}\right), \quad (\forall\ x \neq 0),$$

it is easy to see, that there exist discontinuous solutions.
For example, if $d\colon \mathbb{R} \to \mathbb{R}$ is a not identically zero derivation, then a discontinuous solution f is

$$f(x) = x^{n-2k} \left(d(x)\right)^{2k} \qquad (x \in \mathbb{R}),$$

where $k \in \{ 1\, , \, 2\, , \, \ldots\, , \, [n/2] \}$.
Although we know that for all $n \geqslant 2$ there are discontinuous solutions

of monomial functions satisfying the additional equation $y^n f(x) = x^n f(y)$ for all $(x, y) \in S_2$, we continue our investigations for quadratic functions. In this case, the conditional equation has the form

$$f(x) = x^4 f\left(\frac{1}{x}\right), \quad (\forall\ x \neq 0). \tag{7.4}$$

Despite the fact that the continuity of f does not follow from this assumption, we can obtain some interesting and important results for the mappings $x \mapsto F(x, 1)$ and $x \mapsto F(x, 1/x)$. Using these results hereinafter, we prove the continuity of quadratic functions in several related cases.

Lemma 7.1. *(E. Garda-Mátyás [16]). If a quadratic function* $f : \mathbb{R} \to \mathbb{R}$ *satisfies the additional equation* (7.4) *then*

$$F(x, 1) = x f(1)$$

for all $x \in \mathbb{R}$ *.*

Lemma 7.2. *(E. Garda-Mátyás [16]). If a quadratic function* $f : \mathbb{R} \to \mathbb{R}$ *satisfies the additional equation* $y^2 f(x) = x^2 f(y)$ *for the pairs* $(x, y) \in S_2$*, then*

$$f\left(x^2\right) = 2x^4 F\left(x, \frac{1}{x}\right) + 6x^2 f(x) - 7x^4 f(1)$$

for all $x \in \mathbb{R} \setminus \{0\}$ *.*

Lemma 7.3. *(E. Garda-Mátyás [16]). If a quadratic function* $f : \mathbb{R} \to \mathbb{R}$ *satisfies*

$$F\left(x, \frac{1}{x}\right) = \frac{f(x)}{x^2}$$

for every $x \neq 0$ *, then* $f(x) = x^2 f(1)$ *for all* $x \in \mathbb{R}$ *.*

Thereafter we investigate the continuity of additive and quadratic functions satisfying additional equations along the hyperbola given by the equation $x^2 - y^2 = 1$. Our first result relates to the additive case.

Theorem 7.7. *Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be additive functions. If f, g satisfy the additional equation $yf(x) = xg(y)$ for the pairs $(x, y) \in S_3$, then $f(x) = g(x) = xf(1)$ for all $x \in \mathbb{R}$.*

Our next result applies to the quadratic case with a single quadratic function.

Theorem 7.8. *(E. Garda-Mátyás [16]). If $f\colon \mathbb{R} \to \mathbb{R}$ is a quadratic function that satisfies the conditional equation $y^2 f(x) = x^2 f(y)$ for all $(x, y) \in S_3$, then $f(x) = x^2 f(1)$ for all $x \in \mathbb{R}$.*

We generalize this result by using a second quadratic function.

Theorem 7.9. *Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be quadratic functions. If f, g satisfy the additional equation $y^2 f(x) = x^2 g(y)$ for the pairs $(x, y) \in S_3$, then $f(x) = g(x) = x^2 f(1)$ for all $x \in \mathbb{R}$.*

We continue our investigations with quadratic real functions that satisfy conditional equations along the unit circle. When the conditional equation is with a single quadratic function, we have the following result.

Theorem 7.10. *(E. Garda-Mátyás [16]). If a quadratic function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies $y^2 f(x) = x^2 f(y)$ for the pairs $(x, y) \in S_4$, then $f(x) = x^2 f(1)$ for all $x \in \mathbb{R}$.*

Generalizing this result by using a second quadratic function, we obtain the following theorem.

Theorem 7.11. *If $f, g\colon \mathbb{R} \to \mathbb{R}$ are quadratic functions that satisfy the additional equation $y^2 f(x) = x^2 g(y)$ for the pairs $(x, y) \in S_4$, then $f(x) = g(x) = x^2 f(1)$ for all $x \in \mathbb{R}$.*

Finally, considering further tools from the literature of functional equations, which include very recent results as well, we get an interesting necessary condition for quadratic functions that satisfy the additional equation (7.4).

Theorem 7.12. *If a quadratic function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the additional equation $y^2 f(x) = x^2 f(y)$ under the condition $xy = 1$, then there exists a symmetric bi-derivation H of order 3 for which $f(x) = H(x,x) + x^2 f(1)$.*

In Chapter 6 we investigate quadratic and cubic functions $f,\ g\colon \mathbb{R} \to \mathbb{R}$ that satisfy conditional equations involving logarithmic and exponential functions. In these cases, we prove the equality and continuity of the quadratic and cubic functions f, g.

Theorem 7.13. *Suppose $f, g\colon \mathbb{R} \to \mathbb{R}$ are monomial functions of degree $n \in \{2, 3\}$ and f, g satisfy the additional equation $y^n f(x) = x^n g(y)$ on $\mathbb{R}^+$ for the pairs $(x, y) \in S_5$, then $f(x) = g(x) = x^n f(1)$.*

Remark 7.6. The results of the above theorem can be transferred to the case of exponential functions, that is $(x, y) \in S_6$, since the exponential and logarithmic functions of the same basis are inverses of each other.

We note that both the base of the logarithm, and the base of the exponential can be any positive real number except 1.

8. Összefoglaló

Ebben a PhD értekezésben olyan $f, g\colon \mathbb{R} \to \mathbb{R}$ n-edfokú monom függvényeket tanulmányozunk ($n \geqslant 2$), amelyek teljesítik az $y^n f(x) = x^n f(y)$ vagy $y^n f(x) = x^n g(y)$ feltételes egyenletet egy adott görbe összes (x, y) pontjára.
A fenti kérdést az additív függvényekkel kapcsolatban megoldott hasonló problémák indokolták, lásd [1, 5, 6, 12, 30, 25, 35].
Vizsgálatainkat a következő görbék mentén végeztük:

$$
\begin{aligned}
&S_0 = \{(x, y) \in \mathbb{R}^2 \,|\, a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0 = y\,\},\\
&\qquad m \in \mathbb{N},\ a_i \in \mathbb{R},\ i = 0, \ldots, m,\ a_m \neq 0,\ a_0 \neq 0\,,\\
&S_1 = \{(x, y) \in \mathbb{R}^2 \,|\, x^m = y\},\ m \in \mathbb{Z}\,,\quad |m| \geqslant 2\,,\\
&S_2 = \{(x, y) \in \mathbb{R}^2 \,|\, xy = 1\},\\
&S_3 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 - y^2 = 1\},\\
&S_4 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\},\\
&S_5 = \{(x, y) \in \mathbb{R}^2 \,|\, x > 0 \ \text{ és } \ \log x = y\,\},\\
&S_6 = \{(x, y) \in \mathbb{R}^2 \,|\, e^x = y\}.
\end{aligned}
$$

Tételeink összefoglalása előtt felidézzük az eredmények megfogalmazásához szükséges terminológia fontosabb elemeit.
Az $f : \mathbb{R} \to \mathbb{R}$ függvényt *additív* függvénynek nevezzük, ha bármely valós x, y esetén $f(x + y) = f(x) + f(y)$ teljesül. Az $F : \mathbb{R}^n \to \mathbb{R}$ ($n \in \mathbb{N}$) függvényt n*–additív* függvénynek nevezzük, ha F minden változójában additív. Adott $F : \mathbb{R}^n \to \mathbb{R}$ függvény esetén az F *diagonalizáltjának* nevezzük azt az $f : \mathbb{R} \to \mathbb{R}$ függvényt, amelyet az F-ből kapunk az összes ($\mathbb{R}$-beli) változó egyenlővé tételével. Sajátos esetben, ha f az $F : \mathbb{R}^n \to \mathbb{R}$ n–additív függvény diago-

nalizáltja, akkor azt mondjuk, hogy f *általánosított* n *-edfokú monom.* A másodfokú általánosított monomokat kvadratikus függvényeknek nevezzük, a köbfüggvények a harmadfokú általánosított monomok. A kvadratikus függvényeket az

$$f(x+y)+f(x-y)=2f(x)+2f(y) \qquad (x,y\in\mathbb{R}), \tag{8.1}$$

függvényegyenlet jellemzi, amely az úgynevezett *norma-négyzet egyenlet.*
Az f kvadratikus függvényt generáló biadditív szimmetrikus F függvényt a következő képlet adja:

$$F(x,y)=\frac{1}{2}[f(x+y)-f(x)-f(y)]$$

bármely $x,y\in\mathbb{R}$ esetén.
Az $f\colon\mathbb{R}\to\mathbb{R}$ függvény *deriváció*, ha f additív és teljesíti az $f(xy)=f(x)y+xf(y)$ függvényegyenletet bármely $x,y\in\mathbb{R}$ esetén. Az $f\colon\mathbb{R}\to\mathbb{R}$ derivációk halmazát $\mathcal{D}(\mathbb{R})$-rel jelöljük. A $B\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ függvényt *bi-derivációnak* nevezzük, ha a $t\mapsto B(t,x)$ és $t\mapsto B(x,t)$ $(t\in\mathbb{R})$ leképezések derivációk minden $x\in\mathbb{R}$ esetén. Minden $n\in\mathbb{N}$ esetén, egy $f\colon\mathbb{R}\to\mathbb{R}$ additív leképezést n-edrendű derivációnak nevezünk, ha létezik $B\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ úgy, hogy B egy $(n-1)$-edrendű (szimmetrikus) bi-deriváció (vagyis B $(n-1)$-edrendű deriváció mindkét változójában) és $f(xy)-xf(y)-f(x)y=B(x,y)\,(x,y\in\mathbb{R})$. Az azonosan nulla leképezés az egyetlen nulladrendű deriváció. Az n-edrendű derivációk halmazát $\mathcal{D}_n(\mathbb{R})$-nel jelöljük.

Legfontosabb eredményeinket négy fejezetben mutatjuk be, görbék szerint csoportosítva.

A 3. fejezetben olyan monom függvények folytonosságát vizsgáljuk, amelyek nem nulla konstans taggal rendelkező polinomfüggvényeket tartalmazó feltételes egyenleteket teljesítenek.

A következő eredményeket kapjuk.

8.1 Tétel. *Tegyük fel, hogy* $f\colon \mathbb{R} \to \mathbb{R}$ *és* $g\colon \mathbb{R} \to \mathbb{R}$ *általánosított* n*-edfokú monomok* ($n \in \mathbb{N}$)*, amelyek teljesítik az* $y^n f(x) = x^n g(y)$ *kiegészítő egyenletet az* $(x, y) \in S_0$ *párokra. Akkor* $f(x) = g(x) = x^n f(1)$ *minden* $x \in \mathbb{R}$ *esetén.*

8.1 Megjegyzés. Ha $a_m = a_{m-1} = \ldots = a_1 = 0$, akkor $y = a_0$ konstans. Ekkor a feltételes egyenlet

$$a_0^n f(x) = x^n g(a_0)$$

alakú, és így

$$f(x) = x^n \frac{g(a_0)}{a_0^n} = x^n f(1),$$

de a g függvényről nincs további információnk a $g(a_0) = a_0^n f(1)$-en kívül.

8.2 Megjegyzés. Az $m = 1,\ a_0 = 0$ sajátos esetben, vagyis ha $y = a_1 x$, a feltételes egyenlet

$$a_1^n x^n f(x) = x^n g(a_1 x)$$

alakú, azaz $g(a_1 x) = a_1^n f(x)$.
Ha $a_1 = 0$, akkor ez az egyenlet nem nyújt információt, így f és g bármilyen monom függvény lehet.
Abban az esetben, ha $a_1 \neq 0$, legyen f bármilyen nem folytonos valós monom függvény. Akkor a feltételes egyenletből adódik, hogy g sem folytonos.

Ha a vizsgálatot egyetlen $f\colon \mathbb{R} \to \mathbb{R}$ monom függvényre szűkítjük, akkor a fenti tétel azonnali következményeként kapjuk:

8.1 Következmény. *(Z. Boros és E. Garda-Mátyás [10]). Ha az* $f\colon \mathbb{R} \to \mathbb{R}$ n*-edfokú* ($n \in \mathbb{N}$) *monom függvény teljesíti az* $y^n f(x) = x^n f(y)$ *kiegészítő egyenletet az* $(x, y) \in S_0$ *párokra, akkor* $f(x) = x^n f(1)$ *bármely* $x \in \mathbb{R}$ *esetén.*

A fenti következményben nincs szükség az $a_m \neq 0$ korlátozásra. $a_m = a_{m-1} = \ldots = a_1 = 0$ esetén az állítás triviálisan igaz.

8.3 Megjegyzés. A 8.1. Következményben $m = 1$, $a_0 = 0$ esetén az implikáció nem áll fenn. Ebben az esetben, ha például $a_1 \neq 0$, a feltételes egyelet

$$a_1^n x^n f(x) = x^n f(a_1 x)$$

alakú, vagyis $f(a_1 x) = a_1^n f(x)$. Valóban, létezik $f(x) = (h(x))^n$ $(x \in \mathbb{R})$ alakú nem folytonos példa, ahol $h : \mathbb{R} \to \mathbb{R}$ egy nem folytonos additív függvény, úgy, hogy a h homogenitási teste tartalmazza a_1-et.

Megjegyezzük, hogy az $a_0 \neq 0$ korlátozás, vagyis hogy a görbe nem halad át az origón, fontos szerepet játszik. Egyébként még egyszerű esetben is sok komplikáció lép fel, ezt láthatjuk a következő fejezetben.

A 4. fejezet a hatványfüggvényt tartalmazó feltételes egyenleteket teljesítő kvadratikus és köbfüggvények eredményeit tartalmazza. Először azt az esetet tanulmányozzuk, amikor a kiegészítő egyenletben csak egy kvadratikus függvény található.

8.2 Tétel. *(Z. Boros és E. Garda-Mátyás [9], E. Garda-Mátyás [16]). Ha $2 \leqslant |m|$, $m \in \mathbb{Z}$ és az $f : \mathbb{R} \to \mathbb{R}$ kvadratikus függvény teljesíti az*

$$f(x^m) = x^{2m-2} f(x)$$

egyenletet bármely $x \in \mathbb{R}$ esetén, akkor létezik $C \in \mathbb{R}$ úgy, hogy

$$f(x) = C \cdot x^2 \quad (x \in \mathbb{R}).$$

Megjegyezzük, hogy az $m = 0$ esetben ez a következtetés triviális, vagyis maga a kiegészítő egyenlet az f folytonosságát adja, míg $m = 1$ esetén maga a feltétel triviális azonossága válik, azaz nem jelent semmilyen korlátozást az f számára (ezért f lehet nem folytonos is).

Az $m = 2$ sajátos esetben, de a kiegészítő egyenlet módosított változatával nem folytonos megoldásokat találunk.

8.3 Tétel. *(Z. Boros és E. Garda-Mátyás [9]). Legyen $K \in \mathbb{R}$. Ha egy $f : \mathbb{R} \to \mathbb{R}$ kvadratikus függvény teljesíti az*

$$f(x^2) = Kx^2 f(x) \tag{8.2}$$

kiegészítő egyenletet bármely $x \in \mathbb{R}$ esetén, akkor vagy $f = 0$, vagy $K \in \{1, 2, 4\}$. Ez utóbbi esetekben f-nek a következő reprezentációi vannak.

- *Az $f\colon \mathbb{R} \to \mathbb{R}$ kvadratikus leképezés a $K = 1$ esetben akkor és csak akkor tesz eleget a* (8.2) *feltételnek, ha*

$$f(x) = f(1) \cdot x^2 \qquad (x \in \mathbb{R}).$$

- *Az $f\colon \mathbb{R} \to \mathbb{R}$ kvadratikus leképezés a $K = 2$ esetben akkor és csak akkor tesz eleget a* (8.2) *feltételnek, ha létezik $\varphi \in \mathcal{D}_2(\mathbb{R})$ úgy, hogy*

$$f(x) = 4x\varphi(x) - \varphi(x^2) \qquad (x \in \mathbb{R}). \tag{8.3}$$

- *Ha $B\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ egy szimmetrikus bi-deriváció, akkor*

$$f(x) = B(x, x) \qquad (x \in \mathbb{R})$$

  a (8.2) *egyenlet egy kvadratikus megoldása $K = 4$ esetén.*

8.4 Megjegyzés. Ha $\varphi \in \mathcal{D}(\mathbb{R})$, akkor a (8.3) egyenletből $f(x) = \varphi(x^2)$ $(x \in \mathbb{R})$ következik. Ez a megfigyelés biztosítja a (8.2) egyenlet egy nem nulla kvadratikus f megoldásának létezését $K = 2$ esetén. Az ilyen megoldások létezése $K = 1$ és $K = 4$ esetén az utolsó tétel nyilvánvaló következménye.

8.5 Megjegyzés. Megfigyelhetjük, hogy $K = 4$ esetén ez a tétel csak elégséges feltételt biztosít az f számára ahhoz, hogy teljesítse a (8.1) és a (8.2) egyenleteket. Nyitott kérdés, hogy ez szükséges feltétel-e.

Ebben az esetben azonban valamivel gyengébb szükséges feltételt tudunk bizonyítani.

8.4 Tétel. *Ha egy* $f\colon \mathbb{R} \to \mathbb{R}$ *kvadratikus függgvény teljesíti az*

$$f\left(x^2\right) = 4x^2 f(x)$$

kiegészítő egyenletet bármely $x \in \mathbb{R}$ *esetén, akkor* f *egy másodfokú szimmetrikus bi-deriváció diagonalizáltja.*

Az előző eredmények jelentőségét a következő tétel emeli ki, ahol két kvadratikus függvény szerepel.

8.5 Tétel. *Az* $f\,,\, g : \mathbb{R} \to \mathbb{R}$ *kvadratikus függvények akkor és csakis akkor teljesítik az* $y^2 f(x) = x^2 g(y)$ *kiegészítő egyenletet az* $(x, y) \in S_1$ *párokra* $m = 2$ *esetén, ha létezik egy* $\varphi\colon \mathbb{R} \to \mathbb{R}$ *additív függvény és egy* $h\colon \mathbb{R} \to \mathbb{R}$ *kvadratikus függvény, mely teljesíti a*

$$h(x^2) = 4x^2 h(x) \qquad (x \in \mathbb{R})$$

egyenletet úgy, hogy

$$f(x) = h(x) + \varphi(x^2) \quad \text{és} \quad g(x) = \frac{1}{4}h(x) + x\varphi(x)$$

minden $x \in \mathbb{R}$ *esetén.*

És végül, köbfüggvényekre kiterjesztve a vizsgálatot a következő eredményt kapjuk.

8.6 Tétel. *(Z. Boros és E. Garda-Mátyás [10]). Ha* $f\colon \mathbb{R} \to \mathbb{R}$ *egy általánosított harmadfokú monom, amely teljesíti az* $y^3 f(x) = x^3 f(y)$ *kiegészítő egyenletet* $(x, y) \in S_1$ *feltétel mellett, akkor* $f(x) = x^3 f(1)$ *minden* $x \in \mathbb{R}$ *esetén.*

Az 5. fejezetben olyan additív, kvadratikus és magasabb rendű monom függvények folytonosságát vizsgáljuk, amelyek a hiperbolák

vagy az egységkör mentén teljesítenek kiegészítő egyenleteket.
Negatív eredménnyel kezdünk az $xy = 1$ egyenlet által adott hiperbola mentén. Ha $f\colon \mathbb{R} \to \mathbb{R}$ egy n-edfokú általánosított monom függvény ($2 \leqslant n \in \mathbb{N}$), f eleget tesz az

$$f(x) = x^{2n} f\left(\frac{1}{x}\right) \qquad (\forall\ x \neq 0)$$

kiegészítő egyenletnek, könnyen belátható, hogy léteznek nem folytonos megoldások.
Például, ha $d\colon \mathbb{R} \to \mathbb{R}$ egy nem azonosan nulla deriváció, akkor egy nem folytonos f megoldás az

$$f(x) = x^{n-2k} \left(d(x)\right)^{2k} \qquad (x \in \mathbb{R}),$$

ahol $k \in \{ 1\,, 2\,, \dots\,, [n/2] \}$.
Habár tudjuk, hogy minden $n \geqslant 2$ esetén vannak nem folytonos megoldásai az $y^n f(x) = x^n f(y)$ kiegészítő egyenletet teljesítő monom függvényeknek az $(x, y) \in S_2$ feltétel mellett, folytatjuk a vizsgálatainkat kvadratikus függvényekkel. Ebben az esetben a feltételes egyenlet

$$f(x) = x^4 f\left(\frac{1}{x}\right) \qquad (\forall\ x \neq 0) \tag{8.4}$$

alakú. Annak ellenére, hogy az f folytonossága nem következik ebből a feltevésből, érdekes és fontos eredményeket kaphatunk az $x \mapsto F(x, 1)$ és $x \mapsto F(x, 1/x)$ leképezésekre. Ezeket az eredményeket felhasználva a továbbiakban igazoljuk a kvadratikus függvények folytonosságát több kapcsolódó esetben.

8.1 Lemma. *(E. Garda-Mátyás [16]). Ha egy $f\colon \mathbb{R} \to \mathbb{R}$ kvadratikus függvény teljesíti a* (8.4) *kiegészítő egyenletet, akkor*

$$F(x, 1) = x f(1)$$

minden $x \in \mathbb{R}$ *esetén.*

8.2 Lemma. *(E. Garda-Mátyás [16]). Ha egy* $f\colon \mathbb{R} \to \mathbb{R}$ *kvadratikus függvény teljesíti az* $y^2 f(x) = x^2 f(y)$ *kiegészítő egyenletet az* $(x, y) \in S_2$ *párok esetén, akkor*

$$f\left(x^2\right) = 2x^4 F\left(x, \frac{1}{x}\right) + 6x^2 f(x) - 7x^4 f(1)$$

minden $x \in \mathbb{R} \setminus \{0\}$ *-ra.*

8.3 Lemma. *(E. Garda-Mátyás [16]). Ha egy* $f\colon \mathbb{R} \to \mathbb{R}$ *kvadratikus függvény eleget tesz az*

$$F\left(x, \frac{1}{x}\right) = \frac{f(x)}{x^2}$$

feltételnek bármely $x \neq 0$ *esetén, akkor* $f(x) = x^2 f(1)$ *minden* $x \in \mathbb{R}$ *-re.*

Ezután a kiegészítő egyenleteket az $x^2 - y^2 = 1$ egyenletű hiperbola mentén teljesítő additív és kvadratikus függvények folytonosságát vizsgáljuk. Első eredményünk az additív esetre vonatkozik.

8.7 Tétel. *Legyenek* $f, g\colon \mathbb{R} \to \mathbb{R}$ *additív függvények. Ha* f, g *teljesítik az* $yf(x) = xg(y)$ *kiegészítő egyenletet az* $(x, y) \in S_3$ *párok esetén, akkor* $f(x) = g(x) = xf(1)$ *minden* $x \in \mathbb{R}$ *-re.*

A következő eredményünk a kvadratikus esetre vonatkozik, egyetlen kvadratikus függvénnyel.

8.8 Tétel. *(E. Garda-Mátyás [16]). Ha egy* $f\colon \mathbb{R} \to \mathbb{R}$ *kvadratikus függvény eleget tesz az* $y^2 f(x) = x^2 f(y)$ *kiegészítő egyenletnek az* $(x, y) \in S_3$ *feltétel mellett, akkor* $f(x) = x^2 f(1)$ *minden* $x \in \mathbb{R}$ *esetén.*

Ezt az eredményt általánosítjuk egy második kvadratikus függvény használatával.

8.9 Tétel. *Legyenek* $f, g\colon \mathbb{R} \to \mathbb{R}$ *kvadratikus függvények. Ha* f, g *teljesítik az* $y^2 f(x) = x^2 g(y)$ *kiegészítő egyenletet az* $(x, y) \in S_3$ *párok esetén, akkor* $f(x) = g(x) = x^2 f(1)$ *minden* $x \in \mathbb{R}$ *-re.*

A kiegészítő egyenleteket az egységkör mentén teljesítő kvadratikus valós függvényekkel folytatjuk vizsgálatainkat. Amikor a feltételes egyenletben egyetlen kvadratikus függvény van, a következő eredményt kapjuk.

8.10 Tétel. *(E. Garda-Mátyás [16]). Ha egy* $f : \mathbb{R} \to \mathbb{R}$ *kvadratikus függvény eleget tesz az* $y^2 f(x) = x^2 f(y)$ *kiegészítő egyenletnek az* $(x, y) \in S_4$ *párok esetén, akkor* $f(x) = x^2 f(1)$ *minden* $x \in \mathbb{R}$*-re.*

Általánosítva ezt az eredményt egy második kvadratikus függvény használatával, a következő tételt kapjuk.

8.11 Tétel. *Ha* $f, g\colon \mathbb{R} \to \mathbb{R}$ *kvadratikus függvények teljesítik az* $y^2 f(x) = x^2 g(y)$ *kiegészítő egyenletet az* $(x, y) \in S_4$ *párok esetén, akkor* $f(x) = g(x) = x^2 f(1)$ *minden* $x \in \mathbb{R}$ *-re.*

Végül, figyelembe véve a függvényegyenletek irodalmának további eszközeit, amelyek nagyon friss eredményeket is tartalmaznak, érdekes szükséges feltételt kapunk a (8.4) kiegészítő egyenletet teljesítő kvadratikus függvényekre.

8.12 Tétel. *Ha egy* $f\colon \mathbb{R} \to \mathbb{R}$ *kvadratikus függvény eleget tesz az* $y^2 f(x) = x^2 f(y)$ *kiegészítő egyenletnek az* $xy = 1$ *feltétel mellett, akkor létezik egy* H *harmadrendű szimmetrikus bi-deriváció, amelyre* $f(x) = H(x, x) + x^2 f(1)$.

A 6. fejezetben olyan $f, g\colon \mathbb{R} \to \mathbb{R}$ kvadratikus és köbfüggvényeket vizsgálunk, amelyek logaritmus illetve exponenciális függvényeket tartalmazó feltételes egyenleteket teljesítenek. Ezekben az esetben bebizonyítjuk az f, g kvadratikus és köbfüggvények egyenlőségét és folytonosságát.

8.13 Tétel. *Tegyük fel, hogy* $f, g\colon \mathbb{R} \to \mathbb{R}$ *n-edfokú monom függvények* ($n \in \{2, 3\}$) *és* f, g *eleget tesznek az* $y^n f(x) = x^n g(y)$ *kiegészítő egyenletnek* $\mathbb{R}^+$*-on minden* $(x, y) \in S_5$ *esetén. Ekkor* $f(x) = g(x) = x^n f(1)$.

8.6 Megjegyzés. A fenti tétel eredményei átvihetők az exponenciális függvény esetére, amikor $(x, y) \in S_6$, mivel az azonos alapú exponenciális és logaritmus függvények egymás inverzei.

Megjegyezzük, hogy mind a logaritmus, mind az exponenciális függvény alapja bármely pozitív valós szám lehet, az 1-et kivéve.

Talks held by the author

1. Conditional equations for monomial functions, 14th Annual International Conference on Economics and Business - Challenges in the Carpathian Basin, Innovation and technology in the knowledge based economy, Miercurea Ciuc, Romania, 2018, May 10-12;

2. Quadratic functions fulfilling an additional condition along hyperbolas or the unit circle, Analysis Researchers' Seminar 2017 fall, Debrecen, Hungary, 2017, November 29;

3. Quadratic functions fulfilling an additional condition along hyperbolas or the unit circle, 6th Int. Conf. on Mathematics and Informatics, Târgu Mureș, Romania, 2017, September 7-9;

4. Conditional equations for quadratic functions, 13th Annual International Conference on Economics and Business – Challenges in the Carpathian Basin, Miercurea Ciuc, Romania, 2016, October 20-22;

5. Algoritmustervezési stratégiák gráfelméleti háttere, "Az EME 150 éves" – Emlékkonferencia, Miercurea Ciuc, Romania, 2009, November 6–7;

6. The Chaotic Variation of Capture Effect in the N-Body Problem, 4th International Conference of PHD Students, University of Miskolc, Miskolc, Hungary, 2003, August 11-17;

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1. E. Garda-Mátyás, *Quadratic functions fulfilling an additional condition along hyperbolas or the unit circle*, Aequationes Math., **93 (2)** (2019), 451–465,

2. Z. Boros and E. Garda-Mátyás, *Conditional equations for quadratic functions*, Acta Math. Hungar., **154 (2)** (2018), 389–401,

3. Z. Boros and E. Garda-Mátyás, *Conditional equations for monomial functions*, Publ. Math. Debrecen (accepted).

Manuscripts

1. Linked pairs of monomial functions,

2. Quadratic functions fulfilling an additional condition along the hyperbola $xy = 1$.

Further publications of the author

1. E. Garda-Mátyás and G. Zsombori, *VII. Sapi tehetségnap*, Matlap - Ifjúsági matematikai lapok **5** (2017), 161–173. (ISSN: 1224-3140)

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