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Computational Section

The resolution of three exponential Diophantine equations in several variables [☆]

Csanád Bertók [a], Lajos Hajdu [b,c,*]

[a] Faculty of Informatics, University of Debrecen, H-4028 Debrecen, Kassai street 26, Hungary
[b] Institute of Mathematics, University of Debrecen, P. O. Box 400, H-4002 Debrecen, Hungary
[c] HUN-REN DE Equations, Functions, Curves and their Applications Research Group, Hungary

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ABSTRACT

We find all solutions of three exponential Diophantine equations, arising from certain quadratic, cubic and quartic identities. The first identity comes from a painting of the famous Russian painter Nikolay Bogdanov-Belsky, highlighted by Ja. I. Perelman. The equations have five, four and six terms, respectively, so they cannot be handled by classical tools based upon Baker's method. To solve the equations we use our method developed earlier, which is based upon Skolem's conjecture, local considerations and a computational approach.

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[*] Corresponding author.

E-mail addresses: bertok.csanad@inf.unideb.hu (Cs. Bertók), hajdul@science.unideb.hu (L. Hajdu).

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1. Introduction

In this note we provide all solutions of three exponential Diophantine equations, related to certain particular quadratic, cubic and quartic identities, respectively.

The first identity is the following:

$$10^2 + 11^2 + 12^2 = 13^2 + 14^2.$$

This is related to a painting of Nikolay Bogdanov-Belsky entitled ‘Mental Arithmetic, in the Rachinsky School’ [10] (or, ‘A Hard Problem’ according to Ja. I. Perelman, see Fig. 18. of [23] on page 167). In fact, the problem on the painting asks for calculating the value of

$$\frac{10^2 + 11^2 + 12^2 + 13^2 + 14^2}{365},$$

and in view of that both sides of the above identity equals 365, the answer is 2. The identity (and the solution) is highlighted by Ja. I. Perelman [23], see section ‘A Hard Problem’ on pages 166-168. Miklós Laczkovich (private communication) wondered whether it is possible to replace the exponents $(2, 2, 2, 2, 2)$ in the above identity by some different values. That is, we face the problem of resolving the equation

$$10^{a_1} + 11^{a_2} + 12^{a_3} = 13^{a_4} + 14^{a_5} \tag{1.1}$$

where the variables a_1, a_2, a_3, a_4, a_5 are non-negative integers.

The second identity is given by

$$1^3 + 12^3 = 9^3 + 10^3,$$

and this is the ‘simplest’ equality between sums of two cubes (see e.g. the online database of Wroblewski [29]). A natural problem is to find all solutions of the equation

$$1 + 12^{b_1} = 9^{b_2} + 10^{b_3} \tag{1.2}$$

in non-negative integers b_1, b_2, b_3.

Finally, the third identity we start from is

$$2^4 + 2^4 + 3^4 + 4^4 + 4^4 = 5^4,$$

being the ‘smallest’ instance where a sum of five fourth powers yields a fourth power (see e.g. [28] formula (101), or the database [29] again). As the implied exponential problem, here we consider the equation

$$2^{c_1} + 2^{c_2} + 3^{c_3} + 4^{c_4} + 4^{c_5} = 5^{c_6}, \tag{1.3}$$

in non-negative integers $c_1, c_2, c_3, c_4, c_5, c_6$.

As one can see, the equations to solve are exponential Diophantine equations, with five, four and six terms, respectively. To handle such equations (which may be considered to be special S-unit equations) one can use Baker's method - however, only up to number of terms at most three. (In some special cases, up to number of terms at most five; see [27] and [4].) When we have four or more terms, then by the help of Schmidt's deep subspace theorem, (in general) only the finiteness of the non-degenerate solutions can be established, together with a bound for the number of such solutions. Here we do not give more details: we suggest the interested reader to study the excellent book of Evertse and Győry [18] (and the references there) for history, many results and various important applications. For a detailed description of the situation, one can also check the Introduction of [7]. We only note that these deep, powerful methods are not capable even to give upper bounds for the sizes of the solutions, not to mention the complete solution of the equations.

To handle the equations, we shall use our method established in [7]. This method is based upon a variant of Skolem's conjecture, roughly saying that if an exponential Diophantine equation has no solutions, then it has no solutions modulo m, with some appropriately chosen modulus m. (Note that an algebraic version of our method is given in [9].) Our purpose to solve these equations were twofold: on the one hand, the equations are interesting in themselves, and on the other hand, in this way we can demonstrate that the method developed in [7] works indeed for exponential Diophantine equations 'out of the blue' - thus also supporting the validity of Skolem's conjecture and its variant in [7]. We mention that in the literature one can find several papers resolving exponential Diophantine equations with few terms and with small primes involved. For example, Brenner and Foster [12] solved several such equations up to five terms, while Alex and Foster [2,3] solved such equations with four terms, with primes involved being at most 5. Though their approach is similar to ours in principle, however, the moduli they used are just chosen 'accordingly', not by a general method. In particular, their method would certainly be sufficient to handle (1.2) (where the ultimate modulus is rather simple), but probably not (1.1) and (1.3) (where the ultimate moduli, as well as the ways to find them, are rather involved). In fact, we include (1.2) here only in order to illustrate the method in a simple way.

Finally, we mention that initially, beside (1.1) we wanted to solve the equation

$$3^{d_1} + 4^{d_2} + 5^{d_3} = 6^{d_4} \tag{1.4}$$

in non-negative integers d_1, d_2, d_3, d_4, arising from the identity

$$3^3 + 4^3 + 5^3 = 6^3$$

observed by Euler in 1770. (See [17] art. 249 and [13] for related results and history. Note that on page 139 of [23] the above identity is also mentioned.) However, putting together results from [2] (Theorems 2.A.3 and 3.D) and [3] (Theorem 3.12), equation (1.4) is solved. So, instead of (1.4) we take equations (1.2) and (1.3) which are similar to (1.4) - with (1.3) being significantly more difficult.

The structure of the paper is the following. In the next section we formulate our results about the solutions of the equations, and we briefly describe our approach. The final section is devoted to the proofs of our statements.

2. New results and sketch of the method of the proof

Concerning the first problem, we obtain the following theorem.

Theorem 2.1. *All solutions of the equation*

$$10^{a_1} + 11^{a_2} + 12^{a_3} = 13^{a_4} + 14^{a_5} \tag{2.1}$$

in non-negative integers a_1, a_2, a_3, a_4, a_5 *are given by*

$$(a_1, a_2, a_3, a_4, a_5) = (0, 0, 1, 1, 0),\ (2, 2, 2, 2, 2).$$

The next statement provides all solution to our second problem.

Theorem 2.2. *All solutions of the equation*

$$1 + 12^{b_1} = 9^{b_2} + 10^{b_3} \tag{2.2}$$

in non-negative integers b_1, b_2, b_3 *are given by*

$$(b_1, b_2, b_3) = (0, 0, 0),\ (3, 3, 3).$$

Our next theorem provides all solutions for a generalization of equation (1.3).

Theorem 2.3. *All solutions of the equation*

$$2^{e_1} + 2^{e_2} + 2^{e_3} + 2^{e_4} + 3^{e_5} = 5^{e_6} \tag{2.3}$$

in non-negative integers $e_1, e_2, e_3, e_4, e_5, e_6$ *with* $e_1 \geq \cdots \geq e_4$ *are given by*

$$\begin{gathered}(e_1, e_2, e_3, e_4, e_5, e_6) = (0, 0, 0, 0, 0, 1),\ (2, 2, 2, 2, 2, 2),\ (3, 2, 1, 1, 2, 2),\\ (3, 3, 2, 1, 1, 2),\ (3, 3, 2, 2, 0, 2),\ (4, 1, 1, 1, 1, 2),\ (4, 2, 0, 0, 1, 2),\\ (4, 2, 1, 1, 0, 2),\ (4, 4, 3, 2, 4, 3),\ (5, 2, 2, 2, 4, 3),\ (5, 3, 1, 1, 4, 3),\\ (5, 5, 5, 1, 3, 3),\ (6, 4, 4, 1, 3, 3),\ (6, 5, 0, 0, 3, 3),\ (6, 5, 4, 2, 2, 3),\\ (8, 7, 7, 5, 4, 4), (8, 8, 4, 4, 4, 4), (9, 4, 3, 3, 4, 4), (9, 6, 5, 3, 2, 4), (9, 6, 5, 4, 0, 4).\end{gathered}$$

From this we immediately obtain

Corollary 2.4. *All solutions of the equation*

$$2^{c_1} + 2^{c_2} + 3^{c_3} + 4^{c_4} + 4^{c_5} = 5^{c_6}$$

in non-negative integers $c_1, c_2, c_3, c_4, c_5, c_6$ *with* $c_1 \geq c_2$, $c_4 \geq c_5$ *are given by*

$$\begin{gathered}(c_1, c_2, c_3, c_4, c_5, c_6) = (0,0,0,0,0,1),\ (0,0,1,2,1,2),\ (1,1,0,2,1,2),\\ (2,0,1,2,0,2),\ (2,2,2,1,1,2),\ (3,2,4,2,2,3),\ (3,3,0,1,1,2),\\ (4,0,1,1,0,2),\ (4,1,3,3,2,3),\ (4,2,1,0,0,2),\ (4,3,4,2,1,3),\\ (4,4,4,4,4,4),\ (5,0,3,3,0,3),\ (5,2,2,3,2,3),\ (5,2,4,1,1,3),\\ (5,4,2,3,1,3),\ (6,1,3,2,2,3),\ (6,5,2,2,1,3),\ (6,5,3,0,0,3),\\ (8,4,4,4,2,4),\quad (8,8,4,2,2,4),\quad (9,5,0,3,2,4).\end{gathered}$$

Now we briefly sketch the method we use (developed in [7]) to prove Theorems 2.1 and 2.2. We do this in order to keep the presentation self-contained; however, we do not want to give details which are not important for our present purposes. In particular, we specify the presentation for the particular shape of the problems at hand. The interested reader may consult [7], where a general and much more detailed description of the method is given, with several examples.

Let $u_1, \dots, u_k$, $v_1, \dots, v_k$ be non-zero integers, c be an integer, and consider the exponential Diophantine equation

$$u_1 v_1^{\alpha_1} + \cdots + u_k v_k^{\alpha_k} = c \tag{2.4}$$

in non-negative integers $\alpha_1, \dots, \alpha_k$.

In [7] the following conjecture was proposed (in a more general form):

Conjecture. *Suppose that equation* (2.4) *has no solutions. Then there exists an integer* m *with* $m \geq 2$ *such that the congruence*

$$u_1 v_1^{\alpha_1} + \cdots + u_k v_k^{\alpha_k} \equiv c \pmod{m}$$

has no solutions in non-negative integers $\alpha_1, \dots, \alpha_k$.

The conjecture is a variant of a classical conjecture of Skolem [25] (see also [24], pp. 398–399). For some related history and results cf. [7]. We note that the conjecture has been supported by some theoretical and numerical results in [7], and has been proved in some cases with $k \leq 3$ in [24,22,5,19].

To find the solutions of an equation of the form (2.4), we perform the following steps. For simplicity, we assume that (2.4) has no solutions yielding vanishing subsums.

(i) Make a (supposedly complete) list of all solutions to equation (2.4). As we know by the general theory (see e.g. [18] and the references there), (2.4) has only finitely many solutions (without vanishing subsums). So this can be done by finding all solutions up to some 'large' bound.
(ii) Pick one of the unknown exponents, say α_1, and using the suspected list of all solutions define $\alpha_0 := \alpha_1 + 1$.
(iii) Instead of equation (2.4), consider the equation obtained by replacing the coefficient u_1 by $u_1 v_1^{\alpha_0}$. Observe that assuming that the list of solution was indeed complete, the new equation has no solutions.
(iv) Based upon the above Conjecture, find an m such that the new equation has no solution modulo m. Having such an m, conclude that $\alpha_1 < \alpha_0$ holds for all solutions of (2.4).

Then repeat the whole procedure for all α_1 with $0 \leq \alpha_1 < \alpha_0$ fixed for the implied equations which have one less variables.

Observe that though the strategy contains heuristic points, once we succeed to find appropriate moduli, we get all solutions - independently of any conjecture. For more and more precise details about the method, see [7]. For an algebraic version of the method, see [9].

We mention that in the literature one can find several sparse results of this type. See [7] for an account of the related literature. However, in these papers the way to find appropriate moduli m is rather ad-hoc, while in our method such moduli can be constructed systematically. Using our method, several similar equations have been solved e.g. in [7] and [6]. Note that it is a key problem how to find an appropriate modulus m. This has been analyzed in [7]; the starting point is finding small values of Carmichael's λ function based upon results generalizing theorems of Erdős, Pomerance and Schmutz [16]. (For related results cf. [1,14,20,21,7,9].) See also a recent paper of Dimitrov and Howe [15] for further analysis and considerations, where an efficient method has been worked out for the numerical solution up to some bound of a conjecture of Erdős related to the ternary representations of powers of 2, and also concerning binary representations of powers of 3.

Finally, we mention that we have implemented our algorithm for the solution of exponential Diophantine equations in Sage [26]. The program, together with a complete description can be downloaded from the web page of the second author [8].

3. Proofs

Since the proofs of our theorems are similar, we do the following. First we provide the proof of Theorem 2.2 in detail. (Since here we have a simple argument, it can be done easily.) Then, in the proofs of Theorems 2.1 and 2.3 we only indicate the critical steps, and give the data from which one can verify the calculations. (It seems to be a good choice also because the proofs of Theorems 2.1 and 2.3, that is, the constructions of appropriate moduli, are much more technical.) Further, since our ultimate goal is to solve (2.1), (2.2) and (2.3) completely, we shall make some simplifications in the general method outlined in the previous section, made available by the specific features of the equations considered. Note that beside Sage [26] computations, we also made some calculations in Magma [11].

We start with the proof of our second theorem, because this is the simplest technically, through which we can illustrate our method easily.

Proof of Theorem 2.2. First, according to step i) of the general algorithm, we make an exhaustive check of the exponent tuples (b_1, b_2, b_3) with $\max_{1\le i\le 3} b_i \le 100$, and we find only two such tuples which yield solutions, namely

$$(b_1, b_2, b_3) = (0, 0, 0),\ (3, 3, 3). \tag{3.1}$$

These are just those listed in the statement.

Now, following step ii) of the algorithm, instead of (2.2), we consider the equation

$$9^{b_2} + 10^{b_3} - 1 = 12^4 \cdot 12^{b_1^*} \tag{3.2}$$

in non-negative integers b_1^*, b_2, b_3. Here certainly $b_1 = b_1^* + 4$. If the list of solutions (3.1) is complete, then according to the Conjecture it is possible to find a modulus m such that (3.2) has no solutions already modulo m. To find such an m, we used our program [8] (though in this simple case it could be done easily by hand), and we obtained that

$$m = 2^4 \cdot 5 \tag{3.3}$$

is a good choice, assuming that $b_3 \ge 4$. Note that the program provides a modulus m such that the equation in question has no solutions modulo m with exponents satisfying the following property: if a prime p occurs on power $\nu_p(m)$ in m and p divides a basis v_i in (2.4), then $p^{\nu_p(m)}$ divides $v_i^{\alpha_i}$. (It is not a restriction: if $p^{\nu_p(m)} \nmid v_i^{\alpha_i}$, then α_i may assume only finitely many (small) values and we can loop over them - in an equation with one less variables.) In our case this means that (3.2) has no solutions with $b_3 \ge 4$, already modulo m with m given by (3.3). The remaining cases have to be checked separately. We summarize the cases and list appropriate moduli in Table 1. In general, one can use the strategy explained in [7] to find appropriate moduli.

Table 1
Moduli with no solutions to (3.2) modulo m.

exponent restrictions	moduli m
$b_3 \ge 4$	$2^4 \cdot 5$
$3 \ge b_3 \ge 1$	3^4
$b_3 = 0$	2

Now we give an argument which shows that the moduli appearing in Table 1 are appropriate, indeed. We do this for two reasons: it shows that our claim is valid independently of how m was found, and in this way we can show how to perform such a check (since a 'brute force' method checking all possibilities modulo m is not at all efficient in general).

We start with excluding the cases where $b_3 \leq 3$. If $b_3 = 0$ then modulo 2 we immediately see that equation (3.2) has no solutions. Suppose that $b_3 = 1, 2, 3$. Observe that then $\nu_3(10^{b_3} - 1) = 2, 2, 3$, respectively. In the last case we get a contradiction modulo 81. In the first two cases modulo 27 we see that $b_2 = 2$ must be valid, however, then we obtain contradictions again modulo 27.

Hence we may assume that $b_3 \geq 4$. Then modulo 5 we get that b_2 must be odd, while modulo 16 we obtain that b_2 should be even in (3.5). This is a contradiction, and the theorem is proved. □

Now we turn to the proof of our first theorem.

Proof of Theorem 2.1. The strategy of the proof is similar to that of Theorem 2.2. However, here the calculations are pretty much more involved. So we give only the critical points and provide the moduli obtained, but we suppress the details. To check that the exhibited moduli work, one can use arguments similar to those in the proof of Theorem 2.2 - however, now the use of a computer seems to be unavoidable.

First, following step i) of the general algorithm, performing an exhaustive search for $\max\limits_{1\leq i\leq 5} a_i \leq 100$ we find the solutions

$$(a_1, a_2, a_3, a_4, a_5) = (0, 0, 1, 1, 0),\ (2, 2, 2, 2, 2). \tag{3.4}$$

These are precisely the tuples appearing in the statement.

Now, according to step ii) of the method, in place of (2.2), we consider the equation

$$10^{a_1} + 11^{a_2} + 12^3 \cdot 12^{a_3^*} = 13^{a_4} + 14^{a_5} \tag{3.5}$$

in non-negative integers $a_1, a_2, a_3^*, a_4, a_5$, with $a_3 = a_3^* + 3$. Provided that the list of solutions (3.4) is complete, the Conjecture says that we can find a modulus m such that (3.5) has no solutions already modulo m. To find such a modulus, we used our program [8] again. We obtained that

$$m = 3^3 \cdot 19 \cdot 31 \cdot 37 \cdot 61 \cdot 73 \cdot 181 \cdot 193 \cdot 433 \cdot 1153 \tag{3.6}$$

is appropriate, that is, (3.5) has no solutions modulo m with m in (3.6).

This means that in all solutions of (2.2), we must have $a_3 \leq 2$. Since (2.2) with $a_3 = 0$ has no solutions modulo

$$19 \cdot 31 \cdot 37 \cdot 61 \cdot 73 \cdot 181,$$

we get that $a_3 = 1, 2$.

If $a_3 = 1$ then (2.2) is not possible modulo

$$5 \cdot 19 \cdot 31 \cdot 37 \cdot 61 \cdot 73 \cdot 181,$$

unless $a_1 = 0$. However, then modulo

$$3^3 \cdot 11 \cdot 31 \cdot 61$$

(2.2) yields a contradiction, unless $a_2 = 0$. In this case the left hand side of (2.2) equals 14.

If $a_3 = 2$ then modulo

$$5^3 \cdot 19 \cdot 61 \cdot 101 \cdot 151 \cdot 181$$

(2.2) implies that $a_1 = 0, 1, 2$. Then modulo

$$5^3 \cdot 101 \cdot 151$$

and

$$3^3 \cdot 19 \cdot 61 \cdot 181$$

we get that $a_1 = 0$ and $a_1 = 1$ are impossible, respectively. That is, we are left with $a_1 = 2$. However, then (2.2) modulo

$$3^3 \cdot 5^3 \cdot 11^3 \cdot 19 \cdot 23 \cdot 31 \cdot 727$$

has no solutions, except possibly with $a_2 \leq 2$. Altogether, we obtain that in this case the left hand side of (2.2) is at most 365.

Now a simple check reveals that the only solutions of the equation are given by (3.4). Hence our claim follows. We mention that to prove the theorem, altogether we used the modulus

$$3^3 \cdot 5^3 \cdot 11^3 \cdot 19 \cdot 23 \cdot 31 \cdot 37 \cdot 61 \cdot 73 \cdot 151 \cdot 181 \cdot 193 \cdot 433 \cdot 727 \cdot 1153. \quad \square$$

Finally, we give the proof of our third theorem.

Proof of Theorem 2.3. A search for the ‘small’ solutions with $\max_{1\leq i\leq 6} e_i \leq 30$ gives that (2.3) has no such solutions with $\max_{1\leq i\leq 6} e_i \geq 10$.

We shall focus on the exponents of the 2-s, and rewrite equation (1.4) as

$$2^{10} \cdot 2^{e_1^*} + 2^{10} \cdot 2^{e_2^*} + 2^{10} \cdot 2^{e_3^*} + 2^{10} \cdot 2^{e_4^*} + 3^{e_5} = 5^{e_6} \tag{3.7}$$

in non-negative integers $e_1^*, e_2^*, e_3^*, e_4^*, e_5, e_6$ with $e_1^* \geq e_2^* \geq e_3^* \geq e_4^*$. Using the modulus

$$m := 2^{10} \cdot 7 \cdot 13 \cdot 17 \cdot 97 \cdot 193 \cdot 257,$$

we obtain that (3.7) has no solutions modulo m. In other words, we get that $e_4 \leq 9$ must hold in (2.3). (Here we cannot give more details: there are just too many cases to consider. However, by our program, or using our algorithm from [7], one can easily check that the given modulus 'works' for (3.7)). Then the same modulus shows that none of the equations

$$2^{10} \cdot 2^{e_1^*} + 2^{10} \cdot 2^{e_2^*} + 2^{10} \cdot 2^{e_3^*} + 2^{e_4^*} + 3^{e_5} = 5^{e_6} \; (e_1^* \geq e_2^* \geq e_3^*, e_5, e_6 \in \mathbb{Z}_{\geq 0}, e_4^* \leq 9),$$
$$2^{10} \cdot 2^{e_1^*} + 2^{10} \cdot 2^{e_2^*} + 2^{e_3^*} + 2^{e_4^*} + 3^{e_5} = 5^{e_6} \; (e_1^* \geq e_2^*, e_5, e_6 \in \mathbb{Z}_{\geq 0}, e_4^* \leq e_3^* \leq 9),$$
$$2^{10} \cdot 2^{e_1^*} + 2^{e_2^*} + 2^{e_3^*} + 2^{e_4^*} + 3^{e_5} = 5^{e_6} \; (e_1^*, e_5, e_6 \in \mathbb{Z}_{\geq 0}, e_4^* \leq e_3^* \leq e_2^* \leq 9)$$

is solvable. Further, using the same modulus m once again, we obtain that the equation

$$2^{e_1^*} + 2^{e_2^*} + 2^{e_3^*} + 2^{e_4^*} + 3^{e_5} = 5^{e_6} \tag{3.8}$$

with $e_4^* \leq e_3^* \leq e_2^* \leq e_1^* \leq 9$ has no solutions in non-negative integers e_5, e_6, unless

$$(e_1^*, e_2^*, e_3^*, e_4^*) = (0,0,0,0), (2,2,2,2), (3,2,1,1), (3,3,2,1), (3,3,2,2), (4,1,1,1),$$
$$(4,2,0,0), (4,2,1,1), (4,4,3,2), (5,2,2,2), (5,3,1,1), (5,5,5,1), (6,4,4,1),$$
$$(6,5,0,0), (6,5,4,2), (8,7,7,5), (8,8,4,4), (9,4,3,3), (9,6,5,3), (9,6,5,4).$$

To handle (2.3) in the remaining cases, observe that we can rewrite (3.8) as

$$3^{e_5} - 5^{e_6} = e_0 \tag{3.9}$$

in non-negative integers e_5, e_6, with

$$e_0 \in \{4, 16, 22, 24, 44, 98, 116, 544, 616, 624\}.$$

Note that here e.g. the quadruples

$$(e_1^*, e_2^*, e_3^*, e_4^*) = (2,2,2,2), (3,2,1,1)$$

yield the same value $e_0 = 16$. Clearly, if any of the exponents e_5, e_6 is bounded, then we have a trivial situation. Further, in view of that the preliminary calculations up to $\max_{1 \leq i \leq 6} e_i \leq 30$ just yielded the solutions in the statement, we suspect that in (3.9) we should have $e_5 \leq 4$. Thus we consider the equation

$$3^5 \cdot 3^{e_5^*} - 5^{e_6} = e_0 \tag{3.10}$$

in non-negative integers e_5^*, e_6. A simple calculation yields that (3.10) has no solutions modulo

$$3^5 \cdot 7 \cdot 3889,$$

for any choice of e_0. Thus equation (3.9) has no solutions with $e_5 \geq 4$. Hence our claim easily follows. Note that to prove the statement, altogether we used the modulus

$$2^{10} \cdot 3^5 \cdot 7 \cdot 13 \cdot 17 \cdot 97 \cdot 193 \cdot 257 \cdot 3889. \quad \square$$

Data availability

The link to the program used is given in the manuscript.

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