Egyetemi doktori (Ph.D.) értekezés

On the equality and invariance problem of two variable means and perturbation of monotonic functions

Makó Zita

Témavezető: Dr. Páles Zsolt

DEBRECENI EGYETEM

Matematikai és Számítástudományi Doktori Iskola

Debrecen, 2009

Ezen értekezést a Debreceni Egyetem Matematikai és Számítástudományi Doktori Iskolájának Analízis programja keretében készítettem 2003-2008 között és ezúton nyújtom be a Debreceni Egyetem doktori (Ph.D.) fokozatának elnyerése céljából.

Debrecen, 2009. március 3.

. . . .
Makó Zita
jelölt

Tanúsítom, hogy Makó Zita doktorjelölt 2003-2006 között a fent nevezett doktori program keretében irányításommal végezte munkáját. Az értekezésben foglaltak a jelölt önálló munkáján alapulnak, az eredményekhez önálló alkotó tevékenységével meghatározó módon járult hozzá. Az értekezés elfogadását javaslom.

Debrecen, 2009. március 3.

. . . .
Dr. Páles Zsolt
témavezető

On the equality and invariance problem
of two variable means
and perturbation of monotonic function

Értekezés a doktori (Ph.D.) fokozat megszerzése érdekében
a matematika tudományágban.

Írta: Makó Zita okleveles matematikus

Készült a Debreceni Egyetem Matematikai és Számítástudományi Doktori
Iskolájának Analízis programja keretében.

Témavezető: Dr. Páles Zsolt

A doktori szigorlati bizottság:

elnök: Dr. ....

tagok: Dr. ....

Dr. ....

A doktori szigorlat időpontja: 2009. ....

Az értekezés bírálói:

Dr. ....

Dr. ....

Dr. ....

A bírálóbizottság:

elnök: Dr. ....

tagok: Dr. ....

Dr. ....

Dr. ....

Dr. ....

Az értekezés védésének időpontja: 2009. ....

Acknowledgements

I would like to thank all those who conduce the completion of my dissertation.

First I wish to express my deep gratitude to my supervisor, Dr. Zsolt Páles for the conscientious guidance during my PhD years. His useful advice and excellent ideas were great help for me during writing the present dissertation.

I am greatful to my teachers of the Department of Analysis from those I could learn a lot during my academic and Ph.D. studies.

I thank my parents and my family for their patinence.

Contents

Introduction

In this dissertation we deal with the equality and invariance problem for two variable means and Lipschitz perturbation of monotonic functions.

1. A. On the equality problem for two variable means

In this section we define various classes of means. These classes are defined with the help of generating functions, weight functions and parameters.

Let I be an open real interval. A two-variable function $M : I^2 \to I$ is called a *mean* on the interval I if

$$\min(x, y) \le M(x, y) \le \max(x, y) \qquad (x, y \in I)$$

holds. The mean $M : I^2 \to I$ is called a *symmetric mean* if

$$M(x, y) = M(y, x) \qquad (x, y \in I)$$

holds. The most widely known mean, on a nonvoid open interval $I \subset \mathbb{R}$, is the arithmetic mean

$$A(x, y) := \frac{x + y}{2} \qquad (x, y \in I).$$

A generalization of this mean is the well-known concept of the quasi-arithmetic means. In what follows, $\mathcal{CM}(I)$ will denote the class of real valued continuous strictly monotone functions defined on I.

DEFINITION A. Given $\varphi \in \mathcal{CM}(I)$, the *two variable quasi-arithmetic mean* generated by φ is the function $M_\varphi : I^2 \to I$ defined by

$$\mathcal{M}_\varphi(x, y) := \varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right) \qquad (x, y \in I), \tag{1}$$

where φ^{-1} denotes the inverse of the function φ.

This inverse function exists since the function φ is a continuous and strictly monotone on I. Thus $\varphi(I) \subseteq \mathbb{R}$ is nonempty open interval.

The systematic treatment of quasi-arithmetic means was first given by Hardy, Littlewood and Pólya [**40**]. The most basic problem, the characterization of the equality of these means, is solved by the following theorem.

THEOREM A. *(Hardy–Littlewood–Pólya) Let $\varphi, \psi \in \mathcal{CM}(I)$. Then the means $\mathcal{M}_\varphi$ and $\mathcal{M}_\psi$ are equal to each other if and only if there exist two real constants $a \ne 0$ and b such that $\psi = a\varphi + b$.*

Moreover, in this monograph also the comparison problem and the homogeneity problem of quasi-arithmetic means is considered and discussed.

The characterization of quasi-arithmetic means was solved independently by Kolmogorov [**46**], Nagumo [**65**], de Finetti [**33**] for the case when the number of variables is non-fixed. For the two-variable case, Aczél [**1**], [**2**], [**3**], [**4**], proved the following characterization theorem involving the notion of bisymmetry.

THEOREM B. *Let $M : I^2 \to I$ be a continuous function having the following properties:*

- $M(x,x) = x$, *if* $x \in I$
- $M(x,y) = M(y,x)$, *if* $x, y \in I$
- $x \mapsto M(x,y)$ *is strictly monotone increasing on* I *for every fixed* $y \in I$,
- *for all* $x, y, u, v \in I$, *the bisymmetry equation holds*

$$M(M(x,y), M(u,v)) = M(M(x,u), M(y,v)).$$

Then there exists $\varphi \in \mathcal{CM}(I)$ *such that* M *is of the form* (1), *i.e.,* M *is a quasi-arithmetic mean. Conversely, if* M *is a quasi-arithmetic mean on* I, *then the properties (i), (ii), (iii) and (iv) hold and* M *is continuous.*

This result was extended to the n-variable case by Maksa–Münnich–Mokken [**64**]. Another characterization is due to Matkowski [**60**].

A somewhat more general class of means is the class of weighted quasi-arithmetic means.

DEFINITION B. A two-variable function $M : I^2 \to I$ is called a *weighted quasi-arithmetic mean* on I if there exists a continuous, strictly monotone function $\varphi : I \to \mathbb{R}$ and a constant $\lambda \in]0,1[$ such that

$$M(x,y) = \mathcal{M}_\varphi(x,y;\lambda) := \varphi^{-1}\big(\lambda\varphi(x) + (1-\lambda)\varphi(y)\big) \qquad (x, y \in I).$$

Then λ is called a weight and $\varphi \in \mathcal{CM}(I)$ is said to be the *generating function*.

Another class of means whose definition is related to the Lagrange mean value theorem was introduced by Berrone and Moro [**6**], [**5**].

DEFINITION C. A two-variable function $M : I^2 \to I$ is called a *Lagrangian mean* on I if there exists a continuous strictly monotone function $\varphi : I \to \mathbb{R}$ such that

$$M(x,y) = \mathcal{L}_\varphi(x,y) := \begin{cases} \varphi^{-1}\left(\dfrac{1}{y-x}\displaystyle\int_x^y \varphi(t)dt\right), & \text{if } x \neq y \\ x, & \text{if } x = y \end{cases} \qquad (x, y \in I).$$

Both classes of means have a rich literature, see, e.g., the monographs of Borwein–Borwein [**8**], Mitrinović–Pečarić–Fink [**62**], [**63**], Niculescu–Persson [**66**].

The equality of Lagrangian means is characterized by the following result.

THEOREM C. *(Berrone–Moro) Let $\varphi, \psi \in \mathcal{CM}(I)$. Then the means $\mathcal{L}_\varphi$ and $\mathcal{L}_\psi$ are equal to each other if and only if there exist two real constants $a \neq 0$ and b such that $\psi = a\varphi + b$.*

In the paper [**6**] the comparison problem and the homogeneity problem for Lagrangian mean is also solved.

The equality problem of means in various classes of two-variable means has been solved. We refer here to Losonczi's works [**48**], [**49**], [**50**], [**51**], [**52**] where the equality of two-variable, so-called Cauchy means and Bajraktarević means is characterized. A key idea in these papers, under high order differentiability assumptions, is to calculate and then compare the partial derivatives of the means at points of the form (x, x).

A paper where also the regularity properties are proved (not just assumed) is Daróczy–Maksa–Páles [**23**], where means that are simultaneously quasi-arithmetic and arithmetic means weighted by a weight function are determined without assuming any regularity properties of the data. A similar problem, the mixed equality problem of quasi-arithmetic and Lagrangian means has been recently solved by Páles [**67**] also without any differentiability assumptions.

THEOREM D. *(Páles) Let $\varphi, \psi \in \mathcal{CM}(I)$. Then the means M_φ and L_ψ are equal to each other if and only if one of the following cases holds for all $x \in I$:*

(i) either there exist real constants a, b, c, d with $ac \neq 0$ such that

$$\varphi(x) = ax + b, \qquad \textit{and} \qquad \psi(x) = cx + d;$$

(ii) or there exist real constants a, b, c, d with $ac \neq 0$, and $q \notin I$ such that

$$\varphi(x) = a \ln|x - q| + b, \qquad \textit{and} \qquad \psi(x) = \frac{c}{(x-q)^2} + d;$$

(iii) or there exist real constants a, b, c, d with $ac \neq 0$, and $q \notin I$ such that

$$\varphi(x) = a\sqrt{|x - q|} + b, \qquad \textit{and} \qquad \psi(x) = \frac{c}{\sqrt{|x-q|}} + d;$$

(iv) or there exist real constants a, b, c, d, p, q with $ac \neq 0$ and $p > 0$ such that

$$\varphi(x) = a \operatorname{arsinh}(p(x - q)) + b \qquad \textit{and} \qquad \psi(x) = \frac{c(x-q)}{\sqrt{1 + p^2(x-q)^2}} + d;$$

(v) or there exist real constants a, b, c, d, p, q with $ac \neq 0$, $p > 0$, and $I \cap [q - 1/p, q + 1/p] = \emptyset$ such that

$$\varphi(x) = a \operatorname{arcosh}(p(x - q)) + b \qquad \textit{and} \qquad \psi(x) = \frac{c(x-q)}{\sqrt{p^2(x-q)^2 - 1}} + d;$$

(vi) or there exist real constants a, b, c, d, p, q with $ac \neq 0$, $p > 0$, and $I \subseteq [q - 1/p, q + 1/p]$ such that

$$\varphi(x) = a \arcsin(p(x - q)) + b \qquad \textit{and} \qquad \psi(x) = \frac{c(x-q)}{\sqrt{1 - p^2(x-q)^2}} + d.$$

1. B. Invariance equations for two variable means

A recently rediscovered and blossoming subject is the investigation of the so-called invariance equation and the Gauss-iteration related to quasi-arithmetic means: Gauss [**35**], Błasińska-Lesk–Głazowska–Matkowski [**7**], Burai [**9**], [**10**], Daróczy [**13**], [**14**], [**15**], [**16**], [**17**], Daróczy–Hajdu [**18**], Daróczy–Hajdu–Ng [**19**], Daróczy–Lajkó–Lovas–Maksa–Páles [**20**], Daróczy–Maksa [**21**], Daróczy–Maksa–Páles [**22**], [**24**], Daróczy–Ng [**25**], Daróczy–Páles [**27**], [**29**], [**28**], [**30**], [**32**], [**26**], [**31**], Domsta–Matkowski [**34**], Głazowska–Jarczyk–Matkowski [**37**], Hajdu [**39**], Haruki–Rassias [**41**], Jarczyk–Matkowski [**44**], Jarczyk [**43**], Matkowski [**56**], [**59**], [**61**],

Let $x, y \in \mathbb{R}_+$ be arbitrary and

$$x_1 := x, \quad y_1 := y, \quad x_{n+1} := \frac{x_n + y_n}{2}, \quad y_{n+1} := \sqrt{x_n y_n} \quad (n \in \mathbb{N}).$$

Then the common limit exists and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n =: \mathcal{A} \otimes \mathcal{G}(x, y),$$

which defines the *arithmetic-geometric mean* on the set of the positive real numbers $\mathbb{R}_+$.

Gauss [**35**] found the following astonishing formula for $\mathcal{A} \otimes \mathcal{G}$

$$\mathcal{A} \otimes \mathcal{G}(x, y) = \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{x^2 \cos^2 t + y^2 \sin^2 t}} \right)^{-1} \qquad (x, y > 0).$$

Let $M_i : I^2 \to I (i = 1, 2)$ be given means on I. Moreover let $(x, y) \in I^2$ be arbitrary. Then the iteration sequence

$$x_1 := x, \quad y_1 := y, \quad x_{n+1} := M_1(x_n, y_n), \quad y_{n+1} := M_2(x_n, y_n) \quad (n \in \mathbb{N})$$

is said to be the *Gauss-iteration* determined by the pair (M_1, M_2) with the initial values $(x, y) \in I^2$.

Let I_n be the closed interval determined by x_n and y_n. Then, because of property of means, we have

$$I_{n+1} \subseteq I_n \quad (n \in \mathbb{N}).$$

The Gauss-iteration is said to be *convergent* if the set $\cap_{n=1}^{\infty} I_n$ is a singleton for any initial value $(x, y) \in I^2$. By Cantor's theorem, this is true if and only if

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n =: M_1 \otimes M_2(x, y),$$

where $M_1 \otimes M_2 : I^2 \to I$ is a function.

THEOREM E. *(Daróczy–Páles) If M_1 and M_2 are means on I and the Gauss-iteration determined by the pair (M_1, M_2) is convergent, then the function $M_1 \otimes M_2 : I^2 \to I$ is a mean on I.*

DEFINITION D. If M_1 and M_2 are means on I and the Gauss-iteration determined by the pair (M_1, M_2) is convergent, then the uniquely determined mean $M_1 \otimes M_2 : I^2 \to I$ is said to be the *Gauss-composition* of M_1 and M_2.

THEOREM F. *(Daróczy–Páles) If M_1 and M_2 are means on I and one of them is a strict mean on I, then the Gauss-iteration determined by the pair (M_1, M_2) is convergent.*

THEOREM G. *(Daróczy–Páles) Let M_1 and M_2 be means on I, and suppose that the Gauss-iteration determined by the pair (M_1, M_2) is convergent. Then the Gauss-composition $M_1 \otimes M_2$ satisfies the invariance equation*

$$M_1 \otimes M_2(M_1(x, y), M_2(x, y)) = M_1 \otimes M_2(x, y)$$

for all $x, y \in I$. Furthermore, if $F : I^2 \to \mathbb{R}$ is such a continuous function for which $F(x, x) = x, (x \in I)$ and it satisfies the functional equation

$$F(M_1(x, y), M_2(x, y)) = F(x, y)$$

for every $x, y \in I$, then

$$F(x, y) = M_1 \otimes M_2(x, y)$$

for all $x, y \in I$.

The simplest example when the invariance equation holds is the well-known identity

$$\mathcal{G}(x, y) = \mathcal{G}\big(\mathcal{A}(x, y), \mathcal{H}(x, y)\big) \qquad (x, y > 0),$$

where $\mathcal{A}$, $\mathcal{G}$, and $\mathcal{H}$ stand for the two-variable arithmetic, geometric, and harmonic means, respectively. Another less trivial invariance equation is the identity

$$\mathcal{A} \otimes \mathcal{G}(x, y) = \mathcal{A} \otimes \mathcal{G}\big(\mathcal{A}(x, y), \mathcal{G}(x, y)\big) \qquad (x, y > 0).$$

We note that the quasi-arithmetic means are *strict*, so, for any two such means, does exist the Gauss-composition. At the same time the arithmetic-geometric mean illustrates that the class of the quasi-arithmetic means is *not closed* for the Gauss-composition, since $\mathcal{A} \otimes \mathcal{G}$ is not a quasi-arithmetic mean.(It can be shown that $\mathcal{A} \otimes \mathcal{G}$ is not bisymmetric, and hence, by Theorem B, it cannot be quasi-arithmetic.)

The *invariance equation* involving three two-variable means $M, N, K : I^2 \to I$ is the following identity

$$K\big(M(x, y), N(x, y)\big) = K(x, y) \qquad (x, y \in I). \tag{2}$$

If (2) holds then we say that K *is invariant with respect to the means* M, N. The particular case, when K is the arithmetic mean and M, N are quasi-arithmetic mean, i.e., when

$$M(x, y) + N(x, y) = x + y \qquad (x, y \in I) \tag{3}$$

holds, was investigated by Sutô **[71, 72]** and Matkowski in several papers **[56, 57]**, therefore (3) will be called the Matkowski–Sutô equation in the sequel.

In more details, this latter means finding functions $\varphi, \psi \in \mathcal{CM}(I)$ which satisfy the following functional equation

$$\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)+\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)=x+y \qquad (x, y \in I). \tag{4}$$

The solution of (4) was first found by Matkowski [**56**] under twice continuous differentiability assumptions concerning the generating functions of the quasi-arithmetic means. These regularity assumptions were weakened step-by-step by Daróczy, Maksa and Páles in the papers [**22**], [**27**] and finally in 2002 the following result was proved [**28**]:

THEOREM H. *(Daróczy–Páles) The strictly monotone, continuous functions* $\varphi, \psi : I \to \mathbb{R}$ *satisfy the functional equation*

$$\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)+\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)=x+y \qquad (x, y \in I)$$

if and only if

(i) either there exist non-zero real constants a, c *and constants* b, d *such that*

$$\varphi(x)=ax+b, \qquad \psi(x)=cx+d \qquad (x \in I);$$

(ii) or there exist real constants p, a, b, c, d *with* $acp \neq 0$ *such that*

$$\varphi(x)=ae^{px}+b, \qquad \psi(x)=ce^{-px}+d \qquad (x \in I).$$

The invariance of the arithmetic mean with respect to Lagrangian means was the subject of investigation of the paper [**61**] by Matkowski. He proved, without any regularity assumptions on the generators φ and ψ of the Lagrangian means, that they are also of the forms presented in Theorem H. The invariance of the arithmetic, geometric, and harmonic means with respect to the so-called Beckenbach–Gini means was studied by Matkowski in [**58**]. Pairs of Stolarsky means for which the geometric mean is invariant were determined by Błasińska-Lesk–Głazowska–Matkowski [**7**]. The invariance of the arithmetic mean with respect to further means was studied by Głazowska–Jarczyk–Matkowski [**37**] and Domsta–Matkowski [**34**]. The invariance equation involving three weighted quasi-arithmetic means was studied by Burai [**9**], [**10**] and Jarczyk–Matkowski [**44**], Jarczyk [**43**]. The final answer (where no additional regularity assumptions are required) has been obtained in [**43**].

1. C. Generalized quasi-arithmetic means

We consider the following common generalization of quasi-arithmetic and Lagrangian means.

DEFINITION E. Given a continuous strictly monotone function $\varphi : I \to \mathbb{R}$ and a probability measure μ on the Borel subsets of $[0,1]$, the two variable mean $\mathcal{M}_{\varphi,\mu} : I^2 \to I$ is defined by

$$\mathcal{M}_{\varphi,\mu}(x,y) := \varphi^{-1}\Big(\int_0^1 \varphi\big(tx+(1-t)y\big)d\mu(t)\Big) \qquad (x,y \in I). \tag{5}$$

If $\mu = \dfrac{\delta_0+\delta_1}{2}$, then $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_\varphi$, where δ_t is the Dirac measure concentrated at the point $t \in [0,1]$.

If μ is the Lebesgue measure on $[0,1]$, then $\mathcal{M}_{\varphi,\mu} = \mathcal{L}_\varphi$. Therefore, quasi-arithmetic and Lagrangian means are of the form (5) under the proper choice of the measure μ.

The one of the aims of this dissertation is to study the *equality and the Matkowski-Sutô problem of generalized quasi-arithmetic means*, i.e., to characterize those pairs (φ,μ) and (ψ,ν) such that

$$\mathcal{M}_{\varphi,\mu}(x,y) = \mathcal{M}_{\psi,\nu}(x,y) \qquad (x,y \in I)$$

and

$$\mathcal{M}_{\varphi,\mu}(x,y) + \mathcal{M}_{\psi,\nu}(x,y) = x+y \qquad (x,y \in I)$$

holds, respectively. Under at most fourth-order differentiability assumptions for the unknown functions φ and ψ, a complete description of the solution set of the above functional equations is obtained. The results of CHAPTER 1 can be found in the papers [**54**], [**55**].

2. Lipschitz perturbation

The stability theory of functional inequalities started with the paper of Hyers and Ulam [**42**] (cf. also [**38**]). They introduced the notion of δ-convex function: If D is a convex subset of a real linear space X and δ is a nonnegative number, then a function $f : D \to \mathbb{R}$ is called δ-convex if

$$f(tx + ((1-t)y) \le tf(x) + (1-t)f(y) + \delta$$

for all $x, y \in D, t \in [0,1]$. The basic result obtained by Hyers and Ulam states that if the underlying space X is of finite dimension, then f can be written as $f = g + h$, where g is a convex function and h is a bounded function whose supremum norm is not larger than $k_n\delta$, where the positive constant k_n depends only on the dimension n of the underlying space X. Hyers and Ulam proved that $k_n \le (n(n+3)) = (4(n+1))$. Green [**38**], Cholewa [**12**] obtained much better estimations of k_n showing that asymptotically k_n is not bigger than $(\log_2(n))/2$. Laczkovich [**47**] compared this constant to several other dimension-depending stability constants and proved that it is not less than $(\log_2(n/2))/4$. This result shows that there is no stability results for infinite dimensional spaces X. A counter-example in this direction was earlier constructed by Casini and Papini [**11**]. The stability aspects of δ-convexity are discussed by Ger [**36**]. A more general form of this stability theorem has recently been obtained in [**69**], where the stability of convex functions was investigated under Lipschitz perturbations. A useful auxiliary concept introduced in [**69**] was the notion of ϵ-monotonicity which leaded to the stability properties of monotonic functions. A function $p : I \to \mathbb{R}$ is called ϵ-increasing if

$$p(x) \le p(y) + \epsilon$$

holds for all $x \le y$. It turned out in [**69**] that ϵ-increasing functions are closely related to increasing functions, more precisely, p is ϵ-increasing if and only if $p = q + h$, where q is an increasing function and h is a bounded function with $||h|| \le \epsilon/2$.

Motivated by the above theorem, the another aim of this dissertation is to investigate when a function p can be written in the form $p = q + \ell$, where q is increasing and ℓ is d-Lipschitz (i.e., it satisfies

$$|\ell(x) - \ell(y)| \le d(x, y)$$

for $x, y \in I$.) Here $d : I^2 \to \mathbb{R}$ is assumed to be a semimetric on I. Our main results in CHAPTER 2 offer necessary and sufficient conditions for the above decomposability in the cases of general semimetrics and concave semimetrics. The results of CHAPTER 2 can be found in the paper [**53**].

CHAPTER 1

On the equality and invariance problem of generalized quasi-arithmetic means

1.1. Notations and basic results

This section contains the basic notations and lemmas, which we need to present our results.

Given a Borel probability measure μ on the interval $[0,1]$, we define the *kth moment* and the *kth centralized moment* of μ by

$$\widehat{\mu}_k := \int_0^1 t^k d\mu(t) \qquad \text{and} \qquad \mu_k := \int_0^1 (t-\widehat{\mu}_1)^k d\mu(t) \qquad (k \in \mathbb{N} \cup \{0\}).$$

Clearly, $\widehat{\mu}_0 = \mu_0 = 1$ and $\mu_1 = 0$. In view of the binomial theorem, we easily obtain

$$\begin{aligned} \mu_k &= \int_0^1 (t-\widehat{\mu}_1)^k d\mu(t) = \int_0^1 \sum_{i=0}^{k} (-1)^i \binom{k}{i} t^i \widehat{\mu}_1^{k-i} d\mu(t) \\ &= \sum_{i=0}^{k} (-1)^k \binom{k}{i} \widehat{\mu}_i \widehat{\mu}_1^{k-i} \qquad (k \in \mathbb{N}) \end{aligned} \tag{1.1.1}$$

and

$$\begin{aligned} \widehat{\mu}_k &= \int_0^1 \big((t-\widehat{\mu}_1)+\widehat{\mu}_1\big)^k d\mu(t) = \int_0^1 \sum_{i=0}^{k} \binom{k}{i} (t-\widehat{\mu}_1)^i \widehat{\mu}_1^{k-i} d\mu(t) \\ &= \sum_{i=0}^{k} \binom{k}{i} \mu_i \widehat{\mu}_1^{k-i} \qquad (k \in \mathbb{N}). \end{aligned} \tag{1.1.2}$$

In particular, we have that

$$\widehat{\mu}_2 = \binom{2}{0}\mu_0\widehat{\mu}_1^2 + \binom{2}{1}\mu_1\widehat{\mu}_1 + \binom{2}{2}\mu_2 = \widehat{\mu}_1^2 + \mu_2, \tag{1.1.3}$$

(1.1.4)
$$\widehat{\mu}_3 = \binom{3}{0}\mu_0\widehat{\mu}_1^3 + \binom{3}{1}\mu_1\widehat{\mu}_1^2 + \binom{3}{2}\mu_2\widehat{\mu}_1 + \binom{3}{3}\mu_3 = \widehat{\mu}_1^3 + 3\mu_2\widehat{\mu}_1 + \mu_3,$$

$$
\begin{aligned}
\widehat{\mu}_4 &= \binom{4}{0}\mu_0\widehat{\mu}_1^4 + \binom{4}{1}\mu_1\widehat{\mu}_1^3 + \binom{4}{2}\mu_2\widehat{\mu}_1^2 + \binom{4}{3}\mu_3\widehat{\mu}_1 + \binom{4}{4}\mu_4 \\
&= \widehat{\mu}_1^4 + 6\mu_2\widehat{\mu}_1^2 + 4\mu_3\widehat{\mu}_1 + \mu_4.
\end{aligned} \tag{1.1.5}
$$

The statement of the following lemma is obvious.

LEMMA 1.1.1. *Let μ be a Borel probability measure on $[0,1]$ and $k \in \mathbb{N}$. Then $\mu_{2k} \geq 0$ and equality can hold if and only if μ is the Dirac measure $\delta_{\widehat{\mu}_1}$.*

(In the sequel, δ_τ will denote the Dirac measure concentrated at the point $\tau \in [0,1]$.)

On the other hand, the odd-order centralized moments can be zero. One can prove that $\mu_{2k-1} = 0$ holds for all $k \in \mathbb{N}$ if and only if μ is symmetric with respect to its first moment $\widehat{\mu}_1$, i.e., if $\mu(A) = \mu\big((2\mu_1 - A) \cap [0,1]\big)$ for all Borel sets $A \subseteq [0,1]$.

The reflection of the measure μ with respect to the point $1/2$ is defined by

$$\widetilde{\mu}(A) = \mu(\widetilde{A}),$$

where A is an arbitrary Borel subset of $[0,1]$ and $\widetilde{A} := 1 - A := \{1 - x \mid x \in A\}$. The following lemma characterizes the reflection of a measure in terms of the moments.

LEMMA 1.1.2. *Let μ, ν be Borel probability measures over $[0,1]$. Then $\nu = \widetilde{\mu}$ if and only if*

$$\widehat{\mu}_1 + \widehat{\nu}_1 = 1 \qquad \textit{and} \qquad \nu_k = (-1)^k \mu_k \qquad (k \in \mathbb{N}). \tag{1.1.6}$$

PROOF. Assume first that $\nu = \widetilde{\mu}$. Then

$$\widehat{\nu}_1 = \int_0^1 s d\nu(s) = \int_0^1 s d\widetilde{\mu}(s) = \int_0^1 (1-t) d\mu(t) = 1 - \widehat{\mu}_1.$$

Furthermore,

$$
\begin{aligned}
\nu_k &= \int_0^1 (s - \widehat{\nu}_1)^k d\nu(s) = \int_0^1 (s - \widehat{\nu}_1)^k d\widetilde{\mu}(s) = \int_0^1 (1 - t - \widehat{\nu}_1)^k d\mu(t) \\
&= \int_0^1 (\widehat{\mu}_1 - t)^k d\mu(t) = (-1)^k \int_0^1 (t - \widehat{\mu}_1)^k d\mu(t) = (-1)^k \mu_k.
\end{aligned}
$$

Conversely, assume that (1.1.6) holds. Let $\widetilde{\mu}$ be the reflection of μ with respect to the point $1/2$. Then, it follows from (1.1.6) that

$$\widehat{\widetilde{\mu}}_1 = \widehat{\nu}_1 \qquad \text{and} \qquad \widetilde{\mu}_k = \nu_k \qquad (k \in \mathbb{N}),$$

i.e., all the moments of $\widetilde{\mu}$ and ν coincide. Hence, these two measures are identical. □

To formulate the main results of this chapter, we consider the cases when the first n moments of the measures μ and ν involved in (1.2.1) are identical. For $n \in \mathbb{N} \cup \{0, \infty\}$, we say that the *nth-order moment condition* $\mathcal{M}_n$ *holds* if μ, ν are Borel probability measures on $[0, 1]$, furthermore,

$$\widehat{\mu}_k = \widehat{\nu}_k \quad \text{for all} \quad 1 \leq k \leq n. \tag{1.1.7}$$

Thus the $\mathcal{M}_\infty$ condition means that all the moments of μ and ν are equal, whence, by well-known results of measure and approximation theory, the equality of the two measure μ and ν follows. On the other hand, the condition $\mathcal{M}_0$ simply means that μ, ν are probability measures on the Borel subsets of $[0, 1]$. For $n \in \mathbb{N} \cup \{0\}$, we say that the *exact nth-order moment condition* $\mathcal{M}_n^*$ *holds* if $\mathcal{M}_n$ is valid but $\mathcal{M}_{n+1}$ fails, i.e.,

$$\widehat{\mu}_k = \widehat{\nu}_k \quad \text{for all} \quad 1 \leq k \leq n \qquad \text{and} \qquad \widehat{\mu}_{n+1} \neq \widehat{\nu}_{n+1}. \tag{1.1.8}$$

It is obvious that, for all pairs of measures μ, ν, exactly one of the conditions $\mathcal{M}_0^*, \mathcal{M}_1^*, \mathcal{M}_2^*, \dots, \mathcal{M}_\infty$ can hold, i.e., $\mathcal{M}_0$ is the union of the pairwise exclusive cases $\mathcal{M}_0^*, \mathcal{M}_1^*, \mathcal{M}_2^*, \dots, \mathcal{M}_\infty$.

In view of the formulae (1.1.1) and (1.1.2), it is immediate to see that, for $n \geq 2$, $\mathcal{M}_n$ holds if and only if $\widehat{\mu}_1 = \widehat{\nu}_1$ and $\mu_k = \nu_k$ for $2 \leq k \leq n$.

In order to describe the various regularity conditions on the two unknown functions φ and ψ, for $\in \mathbb{N} \cup \{\infty\}$, we say that the *nth-order regularity condition* $\mathcal{C}_n$ *holds* if $\varphi, \psi : I \to \mathbb{R}$ are n-times continuously differentiable functions with non-vanishing first-order derivatives. For convenience, we also say that $\mathcal{C}_0$ holds if $\varphi, \psi : I \to \mathbb{R}$ are just continuous strictly monotone functions.

In our first result, we compute the first partial derivatives of the mean $M_{\varphi,\mu}$ at a point of the diagonal of $I \times I$ under a weak regularity assumption. We note that, by Lebesgue theorem, φ is differentiable almost everywhere in I, however the derivative of φ can be equal to zero almost everywhere even if φ is strictly increasing.

LEMMA 1.1.3. *Let μ be a Borel probability measure, let $\varphi : I \to \mathbb{R}$ be a continuous strictly monotone function and assume that φ is differentiable at a point $p \in I$ and $\varphi'(p) \neq 0$. Then $\partial_1 \mathcal{M}_{\varphi,\mu}(p, p) = \widehat{\mu}_1$.*

PROOF. Using the differentiability of φ at p, one can easily see that the function $f : I \to \mathbb{R}$ defined by

$$f(x) := \int_0^1 \varphi\big(tx + (1-t)p\big) d\mu(t) \qquad (x \in I)$$

is differentiable at p and $f'(p) = \int_0^1 t\varphi'(p) d\mu(t) = \varphi'(p)\widehat{\mu}_1$. We have that $\mathcal{M}_{\varphi,\mu}(x, p) = \varphi^{-1}\big(f(x)\big)$ and $\varphi'(p) \neq 0$ implies that φ^{-1} is differentiable at $\varphi(p) = f(p)$. Therefore, by the standard chain rule,

$$\partial_1 \mathcal{M}_{\varphi,\mu}(p, p) = \big(\varphi^{-1}\big)'\big(f(p)\big) \cdot f'(p) = \frac{1}{\varphi'(p)} \cdot \varphi'(p)\widehat{\mu}_1 = \widehat{\mu}_1.$$

□

To obtain necessary conditions of higher-order, we need the following result.

LEMMA 1.1.4. *Let μ be a Borel probability measure. For $k \geq 1$, $\mathcal{M}_{\varphi,\mu}$ is k-times continuously differentiable if $\mathcal{C}_k$ holds. If $\mathcal{C}_2$ is valid then, with the notation $\Phi(x) := \partial_1^2 \mathcal{M}_{\varphi,\mu}(x,x)$, we have*

$$\Phi(x) = (\widehat{\mu}_2 - \widehat{\mu}_1^2)\frac{\varphi''}{\varphi'}(x) = \mu_2 \frac{\varphi''}{\varphi'}(x) \qquad (x \in I). \tag{1.1.9}$$

If $\mathcal{C}_3$ and $\mu_2 \neq 0$ hold, then

$$\partial_1^3 \mathcal{M}_{\varphi,\mu}(x,x) = \frac{3\widehat{\mu}_1\mu_2 + \mu_3}{\mu_2}\Phi'(x) + \frac{\mu_3}{\mu_2^2}\Phi^2(x) \qquad (x \in I). \tag{1.1.10}$$

Finally, if $\mathcal{C}_4$ and $\mu_2 \neq 0$ hold, then

$$\begin{aligned} \partial_1^4 \mathcal{M}_{\varphi,\mu}(x,x) =& \frac{6\widehat{\mu}_1^2\mu_2 + 4\widehat{\mu}_1\mu_3 + \mu_4}{\mu_2}\Phi''(x) + \frac{8\widehat{\mu}_1\mu_3 + 3\mu_4}{\mu_2^2}\Phi(x)\Phi'(x) \\ &+ \frac{\mu_4 - 3\mu_2^2}{\mu_2^3}\Phi^3(x) \quad (x \in I). \end{aligned} \tag{1.1.11}$$

PROOF. The k times continuous differentiability of $\mathcal{M}_{\varphi,\mu}$ follows from $\mathcal{C}_k$ by the standard calculus rules. By the definition of the mean $\mathcal{M}_{\varphi,\mu}$, we get that

$$\varphi\big(\mathcal{M}_{\varphi,\mu}(x,y)\big) = \int_0^1 \varphi\big(tx + (1-t)y\big)d\mu(t) \qquad (x, y \in I). \tag{1.1.12}$$

Now assume that $\mathcal{C}_2$ holds. Differentiating the equation (1.1.12) twice with respect to x, we have

$$\begin{aligned} \varphi''\big(\mathcal{M}_{\varphi,\mu}(x,y)\big)\big(\partial_1\mathcal{M}_{\varphi,\mu}(x,y)\big)^2 + \varphi'\big(\mathcal{M}_{\varphi,\mu}(x,y)\big)\partial_1^2\mathcal{M}_{\varphi,\mu}(x,y) \\ = \int_0^1 t^2\varphi''\big(tx + (1-t)y\big)d\mu(t). \end{aligned} \tag{1.1.13}$$

Substituting $y := x$ and applying $\mathcal{M}_{\varphi,\mu}(x,x) = x$, we obtain

$$\varphi''(x)\big(\partial_1\mathcal{M}_{\varphi,\mu}(x,x)\big)^2 + \varphi'(x)\partial_1^2\mathcal{M}_{\varphi,\mu}(x,x) = \int_0^1 t^2\varphi''(x)d\mu(t).$$

Using that $\partial_1\mathcal{M}_{\varphi,\mu}(x,x) = \widehat{\mu}_1$ and $\int_0^1 t^2 d\mu(t) = \widehat{\mu}_2$, we get

$$\widehat{\mu}_1^2\varphi''(x) + \varphi'(x)\partial_1^2\mathcal{M}_{\varphi,\mu}(x,x) = \widehat{\mu}_2\varphi''(x).$$

It follows from this equation that

$$\partial_1^2\mathcal{M}_{\varphi,\mu}(x,x) = (\widehat{\mu}_2 - \widehat{\mu}_1^2)\frac{\varphi''}{\varphi'}(x) = \mu_2\frac{\varphi''}{\varphi'}(x), \tag{1.1.14}$$

thus, with the given notation, we get (1.1.9).

To prove (1.1.10), suppose that $\mathcal{C}_3$ holds. Differentiating (1.1.13) with respect to x, we have

(1.1.15)
$$\varphi'''\big(\mathcal{M}_{\varphi,\mu}(x,y)\big)\big(\partial_1\mathcal{M}_{\varphi,\mu}(x,y)\big)^3+3\varphi''\big(\mathcal{M}_{\varphi,\mu}(x,y)\big)\partial_1\mathcal{M}_{\varphi,\mu}(x,y)\partial_1^2\mathcal{M}_{\varphi,\mu}(x,y)$$
$$+\varphi'\big(\mathcal{M}_{\varphi,\mu}(x,y)\big)\partial_1^3\mathcal{M}_{\varphi,\mu}(x,y)=\int_0^1 t^3\varphi'''\big(tx+(1-t)y\big)d\mu(t).$$

Substituting $y:=x$ and using $\mathcal{M}_{\varphi,\mu}(x,x)=x$, we obtain

$$\varphi'''(x)\big(\partial_1\mathcal{M}_{\varphi,\mu}(x,x)\big)^3+3\varphi''(x)\partial_1\mathcal{M}_{\varphi,\mu}(x,x)\partial_1^2\mathcal{M}_{\varphi,\mu}(x,x)$$
$$+\varphi'(x)\partial_1^3\mathcal{M}_{\varphi,\mu}(x,x)=\widehat{\mu}_3\varphi'''(x).$$

Applying $\partial_1\mathcal{M}_{\varphi,\mu}(x,x)=\widehat{\mu}_1$ and (1.1.9), this simplifies to

$$\widehat{\mu}_1^3\varphi'''(x)+3\widehat{\mu}_1\mu_2\frac{(\varphi'')^2}{\varphi'}(x)+\varphi'(x)\partial_1^3\mathcal{M}_{\varphi,\mu}(x,x)=\widehat{\mu}_3\varphi'''(x).$$

Using the identity (1.1.4), we get

$$\partial_1^3\mathcal{M}_{\varphi,\mu}(x,x)=(3\widehat{\mu}_1\mu_2+\mu_3)\frac{\varphi'''}{\varphi'}(x)-3\widehat{\mu}_1\mu_2\Big(\frac{\varphi''}{\varphi'}\Big)^2(x). \tag{1.1.16}$$

By the definition of the function Φ, we have that

$$\frac{\varphi''}{\varphi'}=\frac{\Phi}{\mu_2}. \tag{1.1.17}$$

Differentiating this equality, it follows that

$$\frac{\varphi'''}{\varphi'}=\frac{\Phi'}{\mu_2}+\frac{\Phi^2}{\mu_2^2}. \tag{1.1.18}$$

Using the identities (1.1.17) and (1.1.18), the equation (1.1.16) reduces to (1.1.10).

In order to obtain (1.1.11), assume that $\mathcal{C}_4$ holds. Differentiating (1.1.15) with respect to x, we get

$$\varphi''''\big(\mathcal{M}_{\varphi,\mu}(x,y)\big)\big(\partial_1\mathcal{M}_{\varphi,\mu}(x,y)\big)^4$$
$$+6\varphi'''\big(\mathcal{M}_{\varphi,\mu}(x,y)\big)\big(\partial_1\mathcal{M}_{\varphi,\mu}(x,y)\big)^2\partial_1^2\mathcal{M}_{\varphi,\mu}(x,y)$$
$$+3\varphi''\big(\mathcal{M}_{\varphi,\mu}(x,y)\big)\big(\partial_1^2\mathcal{M}_{\varphi,\mu}(x,y)\big)^2+4\varphi''\big(\mathcal{M}_{\varphi,\mu}(x,y)\big)\partial_1\mathcal{M}_{\varphi,\mu}(x,y)\partial_1^3\mathcal{M}_{\varphi,\mu}(x,y)$$
$$+\varphi'\big(\mathcal{M}_{\varphi,\mu}(x,y)\big)\partial_1^4\mathcal{M}_{\varphi,\mu}(x,y)=\int_0^1 t^4\varphi''''\big(tx+(1-t)y\big)d\mu(t).$$

Substituting $y:=x$ and using $\mathcal{M}_{\varphi,\mu}(x,x)=x$ and $\int t^4d\mu(t)=\widehat{\mu}_4$, we obtain

$$\varphi''''(x)\big(\partial_1\mathcal{M}_{\varphi,\mu}(x,x)\big)^4+6\varphi'''(x)\big(\partial_1\mathcal{M}_{\varphi,\mu}(x,x)\big)^2\partial_1^2\mathcal{M}_{\varphi,\mu}(x,x)$$
$$+3\varphi''(x)\big(\partial_1^2\mathcal{M}_{\varphi,\mu}(x,x)\big)^2+4\varphi''(x)\partial_1\mathcal{M}_{\varphi,\mu}(x,x)\partial_1^3\mathcal{M}_{\varphi,\mu}(x,x)$$
$$+\varphi'(x)\partial_1^4\mathcal{M}_{\varphi,\mu}(x,x)=\widehat{\mu}_4\varphi''''(x).$$

Applying $\partial_1 \mathcal{M}_{\varphi,\mu}(x,x) = \widehat{\mu}_1$, (1.1.14) and (1.1.16), this simplifies to

$$\begin{aligned}\widehat{\mu}_1^4 \varphi''''(x) &+ 6\widehat{\mu}_1^2\mu_2\varphi'''(x)\frac{\varphi''}{\varphi'}(x) + 3\mu_2^2\varphi''(x)\Big(\frac{\varphi''}{\varphi'}(x)\Big)^2 \\ &+ 4\varphi''(x)\widehat{\mu}_1\left((3\widehat{\mu}_1\mu_2+\mu_3)\frac{\varphi'''}{\varphi'}(x) - 3\widehat{\mu}_1\mu_2\Big(\frac{\varphi''}{\varphi'}\Big)^2(x)\right) \\ &+ \varphi'(x)\partial_1^4\mathcal{M}_{\varphi,\mu}(x,x) = \widehat{\mu}_4\varphi''''(x).\end{aligned}$$

Using the identity (1.1.5), we get

$$\begin{aligned}\partial_1^4\mathcal{M}_{\varphi,\mu}(x,x) &= (6\widehat{\mu}_1^2\mu_2 + 4\widehat{\mu}_1\mu_3 + \mu_4)\frac{\varphi''''}{\varphi'}(x) \\ &- (18\widehat{\mu}_1^2\mu_2 + 4\widehat{\mu}_1\mu_3)\frac{\varphi''}{\varphi'}(x)\frac{\varphi'''}{\varphi'}(x) + (12\widehat{\mu}_1^2\mu_2 - 3\mu_2^2)\Big(\frac{\varphi''}{\varphi'}\Big)^3(x).\end{aligned} \tag{1.1.19}$$

Differentiating (1.1.18), it follows that

$$\frac{\varphi''''}{\varphi'} = \frac{\Phi''}{\mu_2} + 3\frac{\Phi\Phi'}{\mu_2^2} + \frac{\Phi^3}{\mu_2^3}. \tag{1.1.20}$$

Using the identities (1.1.17), (1.1.18) and (1.1.20), after a simple computation, the equation, (1.1.19) reduces to (1.1.11). □

1.2. The equality problem

In this section first we characterize those pairs (φ,μ) and (ψ,ν) such that

$$\mathcal{M}_{\varphi,\mu}(x,y) = \mathcal{M}_{\psi,\nu}(x,y) \quad (x,y\in I) \tag{1.2.1}$$

holds.

As an immediate consequence of the Lemma 1.1.3, we obtain the first necessary condition for the equality of the generalized quasi-arithmetic means. This shows that, under weak regularity assumptions, there is no solution of the equality problem if the exact moment condition $\mathcal{M}_0^*$ holds.

COROLLARY 1.2.1. *Assume $\mathcal{C}_0$ and $\mathcal{M}_0$. Suppose that there exists a point $p\in I$ such that φ and ψ are differentiable at p and $\varphi'(p)\psi'(p)\neq 0$. Then, in order that $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ be valid, it is necessary that*

$$\widehat{\mu}_1 = \widehat{\nu}_1, \tag{1.2.2}$$

i.e., $\mathcal{M}_1$ be satisfied.

PROOF. Using Lemma 1.1.3 and the equality of the means $\mathcal{M}_{\varphi,\mu}$ and $\mathcal{M}_{\psi,\nu}$, we get

$$\widehat{\mu}_1 = \partial_1\mathcal{M}_{\varphi,\mu}(p,p) = \partial_1\mathcal{M}_{\psi,\nu}(p,p) = \widehat{\nu}_1.$$

□

The necessary condition (1.2.2) does not involve the derivatives of φ and ψ explicitly. It remains an open problem to derive the necessity of (1.2.2) assuming only the continuity and monotonicity of the functions φ and ψ.

In view of Corollary 1.2.1, in the rest of the paper, we may assume that the first-order moment condition $\mathcal{M}_1$ holds.

In our next result, assuming $\mathcal{C}_1$, we obtain a characterization of the equality (1.2.1) that does not involve the inverses of the unknown functions φ and ψ.

THEOREM 1.2.2. *Assume $\mathcal{C}_1$ and $\mathcal{M}_1$. Then $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ holds for all $x, y \in I$ if and only if*

$$\int_0^1 \int_0^1 (t-s)\varphi'\big(tx+(1-t)y\big)\psi'\big(sx+(1-s)y\big)d\mu(t)d\nu(s) = 0. \tag{1.2.3}$$

PROOF. *Necessity.*

In view of the continuous differentiability of $\varphi, \psi : I \to \mathbb{R}$ and that φ' and ψ' do not vanish anywhere, the means $\mathcal{M}_{\varphi,\mu}$ and $\mathcal{M}_{\psi,\nu}$ are continuously partially differentiable with respect to their variables. Thus, (1.2.1) yields for all $x, y \in I$

$$\partial_1\mathcal{M}_{\varphi,\mu}(x,y) = \partial_1\mathcal{M}_{\psi,\nu}(x,y) \quad \text{and} \quad \partial_2\mathcal{M}_{\varphi,\mu}(x,y) = \partial_2\mathcal{M}_{\psi,\nu}(x,y).$$

Hence,

$$\partial_1\mathcal{M}_{\varphi,\mu}(x,y)\partial_2\mathcal{M}_{\psi,\nu}(x,y) = \partial_1\mathcal{M}_{\psi,\nu}(x,y)\partial_2\mathcal{M}_{\varphi,\mu}(x,y) \qquad (x,y \in I). \tag{1.2.4}$$

By an elementary calculation, (1.2.4) can be rewritten as for all $x, y \in I$

$$\begin{aligned}
&\frac{\int_0^1 t\varphi'\big(tx+(1-t)y\big)d\mu(t)}{\varphi'\big(\mathcal{M}_{\varphi,\mu}(x,y)\big)} \cdot \frac{\int_0^1 (1-s)\psi'\big(sx+(1-s)y\big)d\nu(s)}{\psi'\big(\mathcal{M}_{\psi,\nu}(x,y)\big)} \\
&= \frac{\int_0^1 (1-t)\varphi'\big(tx+(1-t)y\big)d\mu(t)}{\varphi'\big(\mathcal{M}_{\varphi,\mu}(x,y)\big)} \cdot \frac{\int_0^1 s\psi'\big(sx+(1-s)y\big)d\nu(s)}{\psi'\big(\mathcal{M}_{\psi,\nu}(x,y)\big)},
\end{aligned}$$

which simplifies to for all $x, y \in I$

$$\begin{aligned}
&\int_0^1 t\varphi'\big(tx+(1-t)y\big)d\mu(t) \int_0^1 (1-s)\psi'\big(sx+(1-s)y\big)d\nu(s) \\
&= \int_0^1 (1-t)\varphi'\big(tx+(1-t)y\big)d\mu(t) \int_0^1 s\psi'\big(sx+(1-s)y\big)d\nu(s).
\end{aligned} \tag{1.2.5}$$

One can easily see that (1.2.5) is equivalent to (1.2.3).

Sufficiency. We have that (1.2.3) is equivalent to (1.2.5), which easily yields (1.2.4). Therefore, it suffices to prove that (1.2.4) implies (1.2.1). For the sake of simplicity, denote

$$F(x,y) := \mathcal{M}_{\varphi,\mu}(x,y), \qquad G(x,y) := \mathcal{M}_{\psi,\nu}(x,y) \qquad (x,y \in I).$$

Due to the mean value property, we have

$$F(x,x) = x = G(x,x) \qquad (x \in I).$$

Thus it remains to prove $F(x,y)=G(x,y)$ for $x\neq y$. Without loss of generality, we can assume that $x<y$. Set $z:=F(x,y)$. Then $x<z<y$. By the continuity and strict monotonicity of φ, we have that the mapping $s\mapsto F(t,s)$ is continuous and strictly increasing on I for all fixed $t\in I$. Thus, for $t\in[x,z]$,

$$F(t,z)\leq F(z,z)=z=F(x,y)\leq F(t,y).$$

Therefore, for all $t\in[x,z]$, there exists a unique element $s\in[z,y]$ such that $F(t,s)=z$. Denote this element s by $f(t)$. Then f is a function mapping $[x,z]$ into $[z,y]$ and satisfying the identity

$$F(t,f(t))=z \qquad (t\in[x,z]) \tag{1.2.6}$$

and the boundary value conditions

$$f(x)=y \qquad \text{and} \qquad f(z)=z. \tag{1.2.7}$$

Due to the implicit function theorem, f is continuously differentiable on $[x,z]$. Differentiating (1.2.6) with respect to the variable t, it follows that

$$f'(t)=-\frac{\partial_1F(t,f(t))}{\partial_2F(t,f(t))} \qquad (t\in[x,z]).$$

On the other hand, by (1.2.4), we have

$$\frac{\partial_1F(t,f(t))}{\partial_2F(t,f(t))}=\frac{\partial_1G(t,f(t))}{\partial_2G(t,f(t))} \qquad (t\in[x,z]),$$

whence it follows that

$$\partial_1G(t,f(t))+f'(t)\partial_2G(t,f(t))=0 \qquad (t\in[x,z]).$$

Therefore, the mapping $t\mapsto G(t,f(t))$ is constant on $[x,z]$. Thus, by (1.2.7) and the definition of z,

$$G(x,y)=G(x,f(x))=G(z,f(z))=G(z,z)=z=F(x,y).$$

This proves the equality of $F(x,y)$ and $G(x,y)$, i.e., the equality of $\mathcal{M}_{\varphi,\mu}(x,y)$ and $\mathcal{M}_{\psi,\nu}(x,y)$, too. □

Substituting $x=y$ into (1.2.3) we get the condition

$$\big(\widehat{\mu}_1\widehat{\nu}_0-\widehat{\mu}_0\widehat{\nu}_1\big)\varphi'\psi'=0,$$

which simplifies to (1.2.2) because φ' and ψ' do not vanish anywhere. The result of Corollary 1.2.1 states the same condition under a weaker regularity assumption.

Assuming $\mathcal{C}_{n+1}$, we now deduce further conditions that are necessary for the equality (1.2.1).

THEOREM 1.2.3. *Assume $\mathcal{C}_{n+1}$ for some $n\in\mathbb{N}$ and $\mathcal{M}_1$. Then, in order that $\mathcal{M}_{\varphi,\mu}=\mathcal{M}_{\psi,\nu}$ be valid, it is necessary that*

$$\sum_{i=0}^{n}\binom{n}{i}\big(\mu_{i+1}\nu_{n-i}-\mu_i\nu_{n+1-i}\big)\frac{\varphi^{(i+1)}}{\varphi'}\cdot\frac{\psi^{(n+1-i)}}{\psi'}=0. \tag{1.2.8}$$

Conversely, if φ,ψ are analytic functions and (1.2.8) *holds for all $n\in\mathbb{N}$, then $\mathcal{M}_{\varphi,\mu}=\mathcal{M}_{\psi,\nu}$ is satisfied.*

PROOF. Denote by m the joint value of $\widehat{\mu}_1$ and $\widehat{\nu}_1$. Substituting $x:=u+(1-m)v$ and $y:=u-mv$ into (1.2.3), in view of Theorem 1.2.2, we can see that (1.2.3) holds for all $x,y\in I$ if and only if

(1.2.9)
$$\begin{aligned}&F_u(v)\\&:=\int_0^1\int_0^1(t-s)\varphi'\big(u+(t-m)v\big)\psi'\big(u+(s-m)v\big)d\mu(t)d\nu(s)=0,\end{aligned}$$

for all $u\in I$, $v\in I_u$, where $I_u:=\{v\in\mathbb{R}\mid(1-m)v,-mv\in I-u\}$ (which is a neighborhood of the origin). If $\mathcal{C}_{n+1}$ holds then, for all fixed $u\in I$, the function F_u is n-times continuously differentiable on I_u. Differentiating F_u n-times by applying the Leibniz rule, we obtain

$$\begin{aligned}&F_u^{(n)}(v)\\&=\int_0^1\int_0^1\sum_{i=0}^n\binom{n}{i}\varphi^{(i+1)}\big(u+(t-m)v\big)\psi^{(n+1-i)}\big(u+(s-m)v\big)\\&\cdot(t-s)(t-m)^i(s-m)^{n-i}d\mu(t)d\nu(s).\end{aligned}$$

Now substituting $v:=0$, we get

$$\begin{aligned}&F_u^{(n)}(0)\\&=\int_0^1\!\!\int_0^1\sum_{i=0}^n\binom{n}{i}\varphi^{(i+1)}(u)\psi^{(n+1-i)}(u)(t-s)(t-m)^i(s-m)^{n-i}d\mu(t)d\nu(s)\\&=\sum_{i=0}^n\binom{n}{i}\int_0^1\!\!\int_0^1(t-s)(t-m)^i(s-m)^{n-i}d\mu(t)d\nu(s)\varphi^{(i+1)}(u)\psi^{(n+1-i)}(u)\\&=\sum_{i=0}^n\binom{n}{i}\int_0^1\!\!\int_0^1\big((t-m)^{i+1}(s-m)^{n-i}-(t-m)^i(s-m)^{n-i+1}\big)d\mu(t)d\nu(s)\\&\qquad\qquad\cdot\varphi^{(i+1)}(u)\psi^{(n+1-i)}(u)\\&=\sum_{i=0}^n\binom{n}{i}\big(\mu_{i+1}\nu_{n-i}-\mu_i\nu_{n-i+1}\big)\varphi^{(i+1)}(u)\psi^{(n+1-i)}(u).\end{aligned}$$

If (1.2.9) holds, then $F_u^{(n)}(0)=0$, whence the above formula for $F_u^{(n)}(0)$ divided by $\varphi'(u)\psi'(u)$ yields (1.2.8).

Conversely, assume that φ and ψ are analytic and (1.2.8) holds for all $n\in\mathbb{N}$. Then, for all fixed $u\in I$, the function F_u is analytic on the open interval I_u. On the other hand, (1.2.8) shows that $F_u^{(n)}(0)=0$ for all $n\in\mathbb{N}$. The equality $F_u(0)=0$ is a consequence of $\widehat{\mu}_1=\widehat{\nu}_1$. Therefore, due to its analyticity, the function F_u

is identically zero over I_u. Thus (1.2.9) holds, whence the equality of the means $\mathcal{M}_{\varphi,\mu}$ and $\mathcal{M}_{\psi,\nu}$ follows. $\square$

In the particular case $n = 1$, the above theorem yields the following result.

COROLLARY 1.2.4. *Assume $\mathcal{C}_2$ and $\mathcal{M}_1$. Then, in order that $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ be valid, it is necessary that*

$$|\psi'|^{\nu_2} = \alpha|\varphi'|^{\mu_2} \tag{1.2.10}$$

for some constant $\alpha > 0$.

PROOF. In the case $n = 1$, condition (1.2.8) of Theorem 1.2.3 results

$$\big(\mu_1\nu_1 - \mu_0\nu_2\big)\frac{\psi''}{\psi'} + \big(\mu_2\nu_0 - \mu_1\nu_1\big)\frac{\varphi''}{\varphi'} = 0.$$

Using $\mu_0 = \nu_0 = 1$ and $\mu_1 = \nu_1 = 0$, the above equation can be rewritten as

$$-\nu_2\frac{\psi''}{\psi'} + \mu_2\frac{\varphi''}{\varphi'} = 0.$$

After integration, it follows that

$$-\nu_2 \ln|\psi'| + \mu_2 \ln|\varphi'| = \ln\Big(|\psi'|^{-\nu_2} \cdot |\varphi'|^{\mu_2}\Big)$$

is a constant function, which yields (1.2.10). $\square$

Though we assumed $\mathcal{C}_2$ in Corollary 1.2.4, the necessary condition (1.2.10) involves only the first-order derivatives of φ and ψ. It remains an open problem to derive the necessity of (1.2.10) under first-order continuous differentiability.

1.2.1. The case when $\mathcal{M}_\infty$ holds. In this section we solve the equality problem (1.2.1) if the two measures μ and ν coincide.

THEOREM 1.2.5. *Assume $\mathcal{C}_0$ and $\mathcal{M}_\infty$. Then $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ holds if and only if*

(i) either $\mu = \nu = \delta_\tau$ for some $\tau \in [0,1]$ and φ, ψ are arbitrary,
(ii) or $\mu = \nu$ is not a Dirac measure and there exist constants $a \neq 0$ and b such that

$$\psi = a\varphi + b. \tag{1.2.11}$$

PROOF. If $\mu = \nu = \delta_\tau$, then one can easily check that both sides of (1.2.1) are equal to $\tau x + (1-\tau)y$, hence (1.2.1) is satisfied for any functions φ and ψ.

It is also elementary to see that condition (ii) is sufficient for the equality of the means $\mathcal{M}_{\varphi,\mu}$ and $\mathcal{M}_{\psi,\mu}$.

To show the necessity of (ii), assume that $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ and $\mu = \nu$ is not a Dirac measure. Define now the function $f : \varphi(I) \to \mathbb{R}$ by $f := \psi \circ \varphi^{-1}$. To prove that (1.2.11) holds for some constants $a \neq 0$ and b, it suffices to show that f is affine (i.e., convex and concave at the same time). Indeed, if f is affine then

$f(t) = at + b$ for some constants a and b. Substituting $t = \varphi(x)$, (1.2.11) follows. (Note that, by the strict monotonicity of f, a cannot be zero.)

If f is not affine then either it is non-convex or non-concave over $J := \varphi(I)$. Without loss of generality, we can assume that f is non-convex and φ, ψ are strictly increasing functions. Applying the characterization of non-convexity obtained in [**68**], it follows that there exist a point $q \in J$ such that f is strictly concave at q, i.e., there exists a positive number δ and a constant a such that, for $t \in]q - \delta, q[$ and $s \in]q, q + \delta[$,

$$f(t) < f(q) + a(t - q) \qquad \text{and} \qquad f(s) \le f(q) + a(s - q).$$

Substituting $t := \varphi(u)$, $s := \varphi(v)$, and denoting $p := \varphi^{-1}(q)$, it follows that there exists $\eta > 0$ such that, for $u \in]p - \eta, p[$ and $v \in]p, p + \eta[$,

(1.2.12)
$$\psi(u) < \psi(p) + a(\varphi(u) - \varphi(p)) \qquad \text{and} \qquad \psi(v) \le \psi(p) + a(\varphi(v) - \varphi(p)).$$

Introduce the function $\widetilde{\varphi}$ by $\widetilde{\varphi}(u) := \psi(p) + a(\varphi(u) - \varphi(p))$. Then $\widetilde{\varphi}$ is an affine transform of φ, hence we have the identity $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\widetilde{\varphi},\mu}$. On the other hand, by (1.2.12), for $u \in]p - \eta, p[$ and $v \in]p, p + \eta[$,

$$\psi(u) < \widetilde{\varphi}(u), \qquad \psi(v) \le \widetilde{\varphi}(v) \qquad \text{and} \qquad \psi(p) = \widetilde{\varphi}(p). \tag{1.2.13}$$

By our assumption, μ is not a Dirac measure, hence $\mathcal{M}_{\psi,\mu}$ is strictly increasing in both variables. Using also its continuity, we can easily find $x \in]p - \eta, p[$ and $y \in]p, p + \eta[$ such that $\mathcal{M}_{\psi,\mu}(x, y) = p$. Define $\tau \in [0, 1]$ by the equality $\tau x + (1 - \tau)y = p$. Using that μ is not the Dirac measure δ_τ, we show that $\mu(]\tau, 1]) > 0$. Indeed, if $\mu(]\tau, 1]) = 0$, then $\mu([0, \tau[) > 0$. If $t \in [0, \tau[$ then $tx + (1 - t)y > \tau x + (1 - \tau)y = p$, hence, by the strict monotonicity of ψ,

$$\begin{aligned}
\psi(p) = \psi\big(\mathcal{M}_{\psi,\mu}(x, y)\big) &= \int_0^1 \psi\big(tx + (1 - t)y\big) d\mu(t) \\
&= \int_{[0,\tau]} \psi\big(tx + (1 - t)y\big) d\mu(t) > \int_{[0,\tau]} \psi\big(\tau x + (1 - \tau)y\big) d\mu(t) \\
&= \int_{[0,\tau]} \psi(p) d\mu(t) = \mu([0, \tau])\psi(p) = \psi(p),
\end{aligned}$$

which is a contradiction. Thus $\mu(]\tau, 1]) > 0$ must be valid. On the other hand, if $t \in]\tau, 1]$ then $p - \eta < x \le tx + (1 - t)y < \tau x + (1 - \tau)y = p$. Hence, by the first inequality in (1.2.13), we have

$$\psi\big(tx + (1 - t)y\big) < \widetilde{\varphi}\big(tx + (1 - t)y\big) \qquad (t \in]\tau, 1])$$

and, using the second inequality in (1.2.13), we also get

$$\psi\big(tx + (1 - t)y\big) \le \widetilde{\varphi}\big(tx + (1 - t)y\big) \qquad (t \in [0, \tau]).$$

Using these inequalities, $\mu(]\tau,1]) > 0$, and $\mathcal{M}_{\widetilde{\varphi},\mu} = M_{\varphi,\mu} = M_{\psi,\mu}$ we finally obtain

$$\begin{aligned}\psi(p) = \psi\big(\mathcal{M}_{\psi,\mu}(x,y)\big) &= \int_0^1 \psi\big(tx+(1-t)y\big)d\mu(t)\\ &< \int_0^1 \widetilde{\varphi}\big(tx+(1-t)y\big)d\mu(t) = \widetilde{\varphi}\big(\mathcal{M}_{\widetilde{\varphi},\mu}(x,y)\big)\\ &= \widetilde{\varphi}\big(\mathcal{M}_{\psi,\mu}(x,y)\big) = \widetilde{\varphi}(p),\end{aligned}$$

which contradicts the last equality in (1.2.13). This contradiction proves that f is affine. □

1.2.2. The case when $\mathcal{M}_n^*$ holds for some $2 \le n < \infty$. In this section we characterize the equality problem (1.2.1) assuming that at least the first two moments of the measures μ and ν are the same but the measures are not identical. The investigation of this case requires twice continuous differentiability of the unknown functions φ and ψ.

THEOREM 1.2.6. *Assume $\mathcal{C}_2$ and $\mathcal{M}_n^*$ for some $2 \le n < \infty$. Then $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ holds if and only if there exist constants $a \ne 0$ and b such that*

$$\psi = a\varphi + b \tag{1.2.14}$$

and φ is a polynomial with $\deg\varphi \le n$.

PROOF. Since $n \ge 2$, condition $\mathcal{M}_n^*$ implies that

$$\mu_2 = \widehat{\mu}_2 - \widehat{\mu}_1^2 = \widehat{\nu}_2 - \widehat{\nu}_1^2 = \nu_2 =: \beta.$$

If β were zero, then, by Lemma 1.1.1, μ and ν are equal to some Dirac measures δ_τ and δ_σ ($\tau, \sigma \in [0,1]$), respectively. By Corollary 1.2.1, we have $\widehat{\mu}_1 = \widehat{\nu}_1$ which yields that $\tau = \sigma$. Hence $\mu = \nu$ follows, which is impossible in the case when $\mathcal{M}_n^*$ holds for some $2 \le n < \infty$. Consequently, β cannot be zero.

By Corollary 1.2.4, we have (1.2.10), which can be rewritten as $|\psi'|^\beta = \alpha|\varphi'|^\beta$. Hence, $\psi' = a\varphi'$ for some nonzero constant a which proves (1.2.14).

Using (1.2.14), we have the identity $\mathcal{M}_{\psi,\nu} = \mathcal{M}_{\varphi,\nu}$, therefore (1.2.1) is equivalent to the following equation

$$\mathcal{M}_{\varphi,\mu}(x,y) = \mathcal{M}_{\varphi,\nu}(x,y) \qquad (x,y \in I). \tag{1.2.15}$$

Applying the function φ to both sides, we get

$$\int_0^1 \varphi\big(tx+(1-t)y\big)d(\mu-\nu)(t) = 0 \qquad (x,y \in I). \tag{1.2.16}$$

Using a recent result of Páles [**70**], it follows that a function φ satisfying the linear functional equation (1.2.16) must be a polynomial, therefore it is infinitely many times continuously differentiable on I. Differentiating (1.2.16) $(n+1)$-times with respect to x and then substituting $y := x$, we obtain

$$\int_0^1 t^{n+1}\varphi^{(n+1)}(x)d(\mu-\nu)(t) = 0 \qquad (x \in I),$$

which yields $(\widehat{\mu}_{n+1} - \widehat{\nu}_{n+1})\varphi^{(n+1)} = 0$. By assumption $\mathcal{M}_n^*$, $\widehat{\mu}_{n+1} - \widehat{\nu}_{n+1}$ cannot be zero, hence $\varphi^{(n+1)} = 0$. Therefore, φ must be a polynomial with $\deg \varphi \leq n$.

Now assume that φ is a polynomial with $\deg \varphi \leq n$. Then, for fixed $x, y \in I$, the function $f(t) := \varphi\big(tx + (1-t)y\big)$ is again a polynomial of degree not bigger than n. Thus, by $\mathcal{M}_n^*$, (1.2.16) and hence (1.2.15) follows. Now using (1.2.14), we can see that (1.2.1) holds. □

1.2.3. The case when $\mathcal{M}_1^*$ holds. In the investigation of this case we consider two subcases.
Subcase 1: $\mu_2\nu_2 = 0$.

THEOREM 1.2.7. *Assume $\mathcal{C}_2$ and $\mathcal{M}_1^*$ with $\mu_2\nu_2 = 0$. Then $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ holds if and only if*

(i) either μ and ψ are arbitrary, $\nu = \delta_{\widehat{\mu}_1}$, and there exist constants $a \neq 0$ and b such that

$$\varphi(x) = ax + b \qquad (x \in I); \tag{1.2.17}$$

(ii) or ν and φ are arbitrary, $\mu = \delta_{\widehat{\nu}_1}$, and there exist constants $c \neq 0$ and d such that

$$\psi(x) = cx + d \qquad (x \in I). \tag{1.2.18}$$

PROOF. If $\mu_2 = \nu_2 = 0$, then $\mu_2 = \nu_2$, which contradicts $\mathcal{M}_1^*$. Thus, only one of the values μ_2 and ν_2 can be equal to zero.

In the first case, μ is equal to a Dirac measure δ_τ for some $\tau \in [0,1]$. By $\widehat{\mu}_1 = \widehat{\nu}_1$, it follows that $\tau = \widehat{\nu}_1$. Now (1.2.10) can be rewritten as $|\psi'|^{\nu_2} = \alpha$, which results that ψ' is a constant function. Hence (1.2.18) follows for some constants $a \neq 0$ and b.

Conversely, one can easily check that if condition (ii) holds, then (1.2.1) is indeed satisfied.

The case $\nu_2 = 0$ is analogous. □

Subcase 2: $\mu_2\nu_2 \neq 0$.

In our first result, applying Theorem 1.2.3, we derive further necessary conditions for the equality (1.2.1).

THEOREM 1.2.8. *Assume $\mathcal{C}_2$ and $\mathcal{M}_1$ with $\mu_2\nu_2 \neq 0$ and assume that $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ holds. Then*

$$\nu_2\frac{\psi''}{\psi'} = \mu_2\frac{\varphi''}{\varphi'} =: \Phi. \tag{1.2.19}$$

If $\mathcal{C}_3$ is valid then the function $\Phi : I \to \mathbb{R}$ introduced in (1.2.19) *satisfies the differential equation*

$$\left(\frac{\mu_3}{\mu_2}-\frac{\nu_3}{\nu_2}\right)\Phi'+\left(\frac{\mu_3}{\mu_2^2}-\frac{\nu_3}{\nu_2^2}\right)\Phi^2=0. \tag{1.2.20}$$

If $\mathcal{C}_4$ is also valid, then φ and ψ are analytic functions and Φ satisfies the differential equations

(1.2.21)

$$\left(\frac{\mu_4}{\mu_2}-\frac{\nu_4}{\nu_2}\right)\Phi''+\left(\frac{3\mu_4}{\mu_2^2}-\frac{3\nu_4}{\nu_2^2}\right)\Phi\Phi'+\left(\frac{\mu_4-3\mu_2^2}{\mu_2^3}-\frac{\nu_4-3\nu_2^2}{\nu_2^3}\right)\Phi^3=0.$$

If $\mathcal{M}_1$ holds then the three coefficients in this equation do not vanish simultaneously.

PROOF. If $\mathcal{C}_2$ is valid then, from (1.2.10), we get that (1.2.19) holds. By this definition of the function Φ, we have that

$$\frac{\varphi''}{\varphi'}=\frac{\Phi}{\mu_2} \qquad \text{and} \qquad \frac{\psi''}{\psi'}=\frac{\Phi}{\nu_2}. \tag{1.2.22}$$

To show (1.2.20), assume $\mathcal{C}_3$. Differentiating the equalities in (1.2.22), it follows that

$$\frac{\varphi'''}{\varphi'}=\frac{\Phi'}{\mu_2}+\frac{\Phi^2}{\mu_2^2} \qquad \text{and} \qquad \frac{\psi'''}{\psi'}=\frac{\Phi'}{\nu_2}+\frac{\Phi^2}{\nu_2^2}. \tag{1.2.23}$$

In the particular case $n=2$, condition (1.2.8) of Theorem 1.2.3 yields

(1.2.24)

$$\big(\mu_1\nu_2-\mu_0\nu_3\big)\frac{\psi'''}{\psi'}+2\big(\mu_2\nu_1-\mu_1\nu_2\big)\frac{\varphi''}{\varphi'}\cdot\frac{\psi''}{\psi'}+\big(\mu_3\nu_0-\mu_2\nu_1\big)\frac{\varphi'''}{\varphi'}=0.$$

Using $\mu_1=\nu_1=0$ and the identities (1.2.22), (1.2.23), equation (1.2.24) can be rewritten as

$$-\nu_3\left(\frac{\Phi'}{\nu_2}+\frac{\Phi^2}{\nu_2^2}\right)+\mu_3\left(\frac{\Phi'}{\mu_2}+\frac{\Phi^2}{\mu_2^2}\right)=0.$$

which results the differential equation (1.2.20).

If the regularity assumption $\mathcal{C}_4$ holds, then differentiating (1.2.23) again, one obtains

$$\frac{\varphi''''}{\varphi'}=\frac{\Phi''}{\mu_2}+3\frac{\Phi\Phi'}{\mu_2^2}+\frac{\Phi^3}{\mu_2^3} \qquad \text{and} \qquad \frac{\psi''''}{\psi'}=\frac{\Phi''}{\nu_2}+3\frac{\Phi\Phi'}{\nu_2^2}+\frac{\Phi^3}{\nu_2^3}. \tag{1.2.25}$$

On the other hand, in the particular case $n=3$, condition (1.2.8) of Theorem 1.2.3 yields

$$\begin{aligned}&\big(\mu_1\nu_3-\mu_0\nu_4\big)\frac{\psi''''}{\psi'}+3\big(\mu_2\nu_2-\mu_1\nu_3\big)\frac{\varphi''}{\varphi'}\cdot\frac{\psi'''}{\psi'}\\&\quad+3\big(\mu_3\nu_1-\mu_2\nu_2\big)\frac{\varphi'''}{\varphi'}\cdot\frac{\psi''}{\psi'}+\big(\mu_4\nu_0-\mu_3\nu_1\big)\frac{\varphi''''}{\varphi'}=0.\end{aligned} \tag{1.2.26}$$

Using $\mu_1 = \nu_1 = 0$, applying (1.2.22), (1.2.23), and (1.2.25), equation (1.2.26) can be rewritten in the following form:

$$-\nu_4\left(\frac{\Phi''}{\nu_2}+3\frac{\Phi\Phi'}{\nu_2^2}+\frac{\Phi^3}{\nu_2^3}\right)+3\mu_2\nu_2\frac{\Phi}{\mu_2}\left(\frac{\Phi'}{\nu_2}+\frac{\Phi^2}{\nu_2^2}\right)$$
$$-3\mu_2\nu_2\left(\frac{\Phi'}{\mu_2}+\frac{\Phi^2}{\mu_2^2}\right)\frac{\Phi}{\nu_2}+\mu_4\left(\frac{\Phi''}{\mu_2}+3\frac{\Phi\Phi'}{\mu_2^2}+\frac{\Phi^3}{\mu_2^3}\right)=0,$$

which results (1.2.21), an at most second-order differential equation for Φ. Introduce the notations

(1.2.27)
$$\eta := \frac{\mu_4}{\mu_2}-\frac{\nu_4}{\nu_2}, \qquad \gamma := \frac{3\mu_4}{\mu_2^2}-\frac{3\nu_4}{\nu_2^2}, \qquad \delta := \frac{\mu_4-3\mu_2^2}{\mu_2^3}-\frac{\nu_4-3\nu_2^2}{\nu_2^3}.$$

First we show that the constants η, γ, and δ, cannot be simultaneously zero. The equations $\eta = 0$ and $\gamma = 0$ form a system of linear equations for the unknowns μ_4, ν_4. The determinant of this system is nonzero because $\mu_2 - \nu_2 \neq 0$ by $\mathcal{M}_1^*$. Thus $\mu_4 = \nu_4 = 0$. Then the equation $\delta = 0$ yields $\mu_2 = \nu_2$, which again contradicts $\mathcal{M}_1^*$. Therefore, the coefficients in (1.2.21) do not vanish simultaneously.

To show that φ and ψ are analytic, in view of (1.2.22), it suffices to show that Φ is analytic.

If $\eta \neq 0$, then (1.2.21) is an explicit second-order differential equation for Φ. Applying the results on the analyticity of the solutions of such equations, it follows that Φ is analytic.

If $\eta = 0$, then (1.2.21) could be rewritten as

(1.2.28)
$$\Phi\big(\gamma\Phi' + \delta\Phi^2\big) = 0.$$

We show that this equation is satisfied if and only if

(1.2.29)
$$\gamma\Phi' + \delta\Phi^2 = 0.$$

Denote

$$J := \big\{t \in I : \gamma\Phi'(t) + \delta\Phi^2(t) \neq 0\big\}.$$

Then J is an open subset of I. By (1.2.28), Φ has to be zero on J. By the openness of J, it follows that Φ' is also zero on J. Hence J must be empty which means that (1.2.29) holds. If $\gamma \neq 0$, then (1.2.29) is a first-order explicit differential equation for Φ. Thus, it follows that Φ is analytic. If $\gamma = 0$, then δ cannot be zero, therefore $\Phi = 0$, which again yields the analyticity of Φ. $\square$

In our second result, we obtain a necessary and sufficient condition for the equality problem (1.2.1) under the additional assumption that Φ satisfies a first-order polynomial differential equation.

THEOREM 1.2.9. *Assume $\mathcal{C}_3$ and $\mathcal{M}_1$ with $\mu_2\nu_2 \neq 0$. Suppose that* (1.2.19) *holds and that there exists integer numbers $0 \leq 2n \leq k$ and a constant vector*

$(c_0,\dots,c_n)\neq(0,\dots,0)$ *such that the function* $\Phi: I\to\mathbb{R}$ *introduced in* (1.2.19) *satisfies the following first-order polynomial differential equation*

$$\sum_{i=0}^{n} c_i\Phi^{k-2i}\big(\Phi'\big)^i=0. \tag{1.2.30}$$

Then $\mathcal{M}_{\varphi,\mu}=\mathcal{M}_{\psi,\nu}$ *holds if and only if*

(i) either there exist real constants a,b,c,d *with* $ac\neq 0$ *such that*

$$\varphi(x)=ax+b,\qquad \text{and}\qquad \psi(x)=cx+d\qquad (x\in I); \tag{1.2.31}$$

(ii) or there exist real constants a,b,c,d,p,q *with* $ac(p-q)\neq 0$*,* $pq>0$ *such that*

$$\varphi(x)=ae^{px}+b\qquad \text{and}\qquad \psi(x)=ce^{qx}+d\qquad (x\in I) \tag{1.2.32}$$

and, for $n\in\mathbb{N}$*,*

$$\sum_{i=0}^{n}\binom{n}{i}p^iq^{n-i}\big(\mu_{i+1}\nu_{n-i}-\mu_i\nu_{n+1-i}\big)=0; \tag{1.2.33}$$

(iii) or there exist real constants a,b,c,d,p,q *with* $ac(p-q)\neq 0$*,* $(p-1)(q-1)>0$*, and* $x_0\notin I$ *such that, for* $x\in I$*,*

$$\begin{aligned}\varphi(x)&=\begin{cases}a|x-x_0|^p+b, & \text{if } p\neq 0\\ a\ln|x-x_0|+b, & \text{if } p=0,\end{cases}\\ \psi(x)&=\begin{cases}c|x-x_0|^q+d, & \text{if } q\neq 0\\ c\ln|x-x_0|+d, & \text{if } q=0\end{cases}\end{aligned} \tag{1.2.34}$$

and for $n\in\mathbb{N}$*,*

$$\sum_{i=0}^{n}\binom{p-1}{i}\binom{q-1}{n-i}\big(\mu_{i+1}\nu_{n-i}-\mu_i\nu_{n+1-i}\big)=0. \tag{1.2.35}$$

PROOF. To solve (1.2.30), we distinguish three cases.

Case 1: $\Phi=0$ (which is trivially a solution of (1.2.30)). Then $\varphi''=0$, whence $\varphi'=a$, and by (1.2.10), also $\psi'=c$ for some nonzero constants a and c. Therefore, in this case, statement (i) of the theorem must be valid.

Conversely, if (i) holds, then, for all $x,y\in I$,

$$\mathcal{M}_{\varphi,\mu}(x,y)=\widehat{\mu}_1x+(1-\widehat{\mu}_1)y\qquad\text{and}\qquad \mathcal{M}_{\psi,\nu}(x,y)=\widehat{\nu}_1x+(1-\widehat{\nu}_1)y,$$

hence the equality of the means follows from $\widehat{\mu}_1=\widehat{\nu}_1$.

In the rest of the proof we may assume that Φ is not identically zero. Denote by J a maximal subinterval of I where Φ does not vanish. Clearly, J is open and nonempty and (1.2.30) can be rewritten as

$$\sum_{i=0}^{n} c_i\left(\frac{\Phi'(x)}{\Phi^2(x)}\right)^i=0\qquad (x\in J). \tag{1.2.36}$$

Therefore, the values of the function $\frac{\Phi'}{\Phi^2}$ on J are equal to the roots of the polynomial $P(x) := \sum_{i=0}^{n} c_i x^i$. Due to the continuity, we get that

$$\frac{\Phi'(x)}{\Phi^2(x)} = c \qquad (x \in J), \tag{1.2.37}$$

where the constant c is one of the roots of the polynomial P. Now we can consider the cases $c = 0$ and $c \neq 0$.

Case 2: $c = 0$. Then, (1.2.37) says that $\Phi' = 0$ on J. Thus, there exists a nonzero constant p such that $\Phi = \mu_2 p$ on J. If J were a proper subinterval of I, then one of the endpoints of J, say α, would be contained in I. By the continuity, we have $\Phi(\alpha) = \mu_2 p \neq 0$, which results that J is not maximal. The contradiction so obtained shows that $J = I$.

Using the definition of Φ, we get that $\varphi'' = p\varphi'$. Integrating this equality, we can find a constant b such that $\varphi' = p(\varphi - b)$. This is a first-order linear differential equation for φ, whose general solution is of the form $\varphi(x) = ae^{px} + b$ for some constant a. Of course, ap cannot be zero, otherwise φ is not strictly monotone. Using (1.2.10), it follows that ψ is also of the form stated in (1.2.32) of (ii), where $q = (\mu_2/\nu_2)p$. Clearly $pq = (\mu_2/\nu_2)p^2 > 0$. The condition $\mu_2 \neq \nu_2$ implies that $q \neq p$. The functions φ and ψ are obviously analytic, hence, Theorem 1.2.3 can be applied. Using

$$\frac{\varphi^{(j)}(x)}{\varphi'(x)} = p^{j-1}, \qquad \frac{\psi^{(j)}(x)}{\psi'(x)} = q^{j-1}, \qquad (x \in I,\ j \in \mathbb{N}),$$

one can see that (1.2.8) is equivalent to (1.2.33), therefore, by Theorem 1.2.3, the means $\mathcal{M}_{\varphi,\mu}$ and $\mathcal{M}_{\psi,\nu}$ are identical if and only if (1.2.33) holds for all $n \in \mathbb{N}$.

Case 3: $c \neq 0$. Then, with the notation $p := 1 + 1/(\mu_2 c) \neq 1$, (1.2.37) can be rewritten as

$$\frac{\Phi'(x)}{\Phi^2(x)} = \frac{1}{\mu_2(p-1)} \qquad (x \in J).$$

Integrating this equality, it follows, for some x_0, that

$$\frac{1}{\Phi(x)} = \frac{x - x_0}{\mu_2(p-1)} \qquad (x \in J). \tag{1.2.38}$$

Hence x_0 cannot be in J. If J were a proper subinterval of I, then one of the endpoints of J, say α, would be an element of I. By taking the limit $x \to \alpha$ in the above equation, it follows that Φ has a finite nonzero limit at α. By continuity, this yields that $\Phi(\alpha) = \frac{\mu_2(p-1)}{\alpha - x_0} \neq 0$. showing that J is not maximal. The contradiction so obtained proves that $J = I$. Applying (1.2.38) and the definition (1.2.19) of the function Φ, we get

$$\frac{\varphi''(x)}{\varphi'(x)} = \frac{\Phi(x)}{\mu_2} = \frac{p-1}{x - x_0} \qquad (x \in J).$$

Integrating this equation, it results that

$$\varphi'(x) = \begin{cases} ap|x-x_0|^{p-1}, & \text{if } p \neq 0 \\ a|x-x_0|^{-1}, & \text{if } p \neq 0 \end{cases}$$

for some constant a. After integration this yields that φ is of the form (1.2.34). Using (1.2.10), we get that ψ is also of the form (1.2.34) with $q := 1+(\mu_2/\nu_2)(p-1)$. Obviously, $(p-1)(q-1) = (\mu_2/\nu_2)(p-1)^2 > 0$. We also have $ac \neq 0$ otherwise φ or ψ is not strictly monotone. The condition $p \neq q$ follows from $\mu_2 \neq \nu_2$.

Now assume that $x_0 \leq \inf I$ (the case $x_0 \geq \sup I$ is analogous). In view of (1.2.34), the functions φ and ψ are analytic and we have

$$\begin{aligned} \frac{\varphi^{(j)}(x)}{\varphi'(x)} &= (j-1)!\binom{p-1}{j-1}(x-x_0)^{1-j}, \\ \frac{\psi^{(j)}(x)}{\psi'(x)} &= (j-1)!\binom{q-1}{j-1}(x-x_0)^{1-j}, \qquad (x \in I,\ j \in \mathbb{N}). \end{aligned}$$

Using these formulae, we can see that (1.2.8) is valid if and only if (1.2.35) holds. Therefore, by Theorem 1.2.3, the equality of the means $\mathcal{M}_{\varphi,\mu}$ and $\mathcal{M}_{\psi,\nu}$ is equivalent to the validity of condition (1.2.35) for all $n \in \mathbb{N}$. □

Subcase 2.A: $\mu_2\nu_2 \neq 0$ and $(\mu_3, \nu_3) \neq (0,0)$.

THEOREM 1.2.10. *Assume $\mathcal{C}_3$ and $\mathcal{M}_1^*$ with $\mu_2\nu_2 \neq 0$ and $(\mu_3, \nu_3) \neq (0,0)$. Then $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ holds if and only if one of the alternatives (i), (ii), or (iii) of Theorem 1.2.9 is satisfied.*

PROOF. By Theorem 1.2.8, we have that (1.2.20) holds. We show that (1.2.20) is not a trivial equation, i.e., one of the coefficients different from zero. Indeed, if both coefficients were zero, then we would get a homogeneous system of linear equations for the unknowns μ_3 and ν_3. Since the determinant of this linear system is $(\mu_2 - \nu_2)/(\mu_2\nu_2)^2 \neq 0$ hence $\mu_3 = \nu_3 = 0$, which contradicts the assumption $(\mu_3, \nu_3) \neq (0,0)$ of the theorem.

Thus (1.2.20) is a nontrivial first-order polynomial differential equation for Φ. The statement now follows from Theorem 1.2.8. □

If $\mu_3 = \nu_3 = 0$, then the necessary condition (1.2.20) of Theorem 1.2.8 does not result any information, Thus, we may apply differential equation (1.2.21). Unfortunately, this equation can be solved explicitly if $\mu_2\nu_4 = \nu_2\mu_4$. In the remaining cases, we shall use again the necessary condition (1.2.8) of Theorem 1.2.3 in the cases $n = 4$ and $n = 5$.

Subcase 2.B: $\mu_2\nu_2 \neq 0$, $(\mu_3, \nu_3) = (0,0)$, and $\mu_2\nu_4 = \nu_2\mu_4$.

THEOREM 1.2.11. *Assume $\mathcal{C}_4$ and $\mathcal{M}_1^*$ with $\mu_2\nu_2 \neq 0$, $(\mu_3, \nu_3) = (0,0)$, and $\mu_2\nu_4 = \nu_2\mu_4$. Then $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ holds if and only if one of the alternatives (i), (ii), or (iii) of Theorem 1.2.9 is satisfied.*

PROOF. As we have shown in Theorem 1.2.8, the functions φ and ψ are analytic on I and Φ defined by (1.2.19) satisfies

$$\eta\Phi'' + \gamma\Phi\Phi' + \delta\Phi^3 = 0, \tag{1.2.39}$$

where the constants η, γ, δ are defined by (1.2.27).

Now, by $\mu_2\nu_4 = \nu_2\mu_4$, we have that $\eta = 0$ then, (1.2.39) is an equation of the form (1.2.30). Thus, by Theorem 1.2.8, one of the alternatives (i), (ii), or (iii) must be valid. □

Subcase 2.C: $\mu_2\nu_2 \neq 0$, $(\mu_3, \nu_3) = (0,0)$, and $\mu_2\nu_4 \neq \nu_2\mu_4$.

THEOREM 1.2.12. *Assume $\mathcal{C}_4$ and $\mathcal{M}_1^*$ with $\mu_2\nu_2 \neq 0$, $(\mu_3, \nu_3) = (0,0)$, $\mu_2\nu_4 \neq \nu_2\mu_4$, $(\mu_5, \nu_5) \neq (0,0)$, and*

$$(\mu_5 - \nu_5)^2 + (\mu_4 - 3\mu_2\nu_2)^2 + (\nu_4 - 3\mu_2\nu_2)^2 \neq 0. \tag{1.2.40}$$

Then $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ holds if and only if one of the alternatives (i), (ii), or (iii) of Theorem 1.2.9 is satisfied.

PROOF. As we have shown in Theorem 1.2.8, the functions φ and ψ are analytic on I and Φ defined by (1.2.19) satisfies (1.2.39), where the constants η, γ, δ are defined by (1.2.27).

By condition $\mu_2\nu_4 \neq \nu_2\mu_4$, we have that $\eta \neq 0$, therefore (1.2.39) is a second-order differential equation that cannot be solved explicitly. However, using this equation, the second and third (an also higher-order) derivatives of Φ can be expressed as a polynomial of Φ and Φ'. With the notations $\alpha := -\gamma/\eta$ and $\beta := -\delta/\eta$, easily follows from (1.2.39) that

$$\Phi'' = \alpha\Phi\Phi' + \beta\Phi^3 \qquad \text{and} \qquad \Phi''' = \alpha(\Phi')^2 + (\alpha^2 + 3\beta)\Phi^2\Phi' + \alpha\beta\Phi^4. \tag{1.2.41}$$

In the particular case $n = 4$, condition (1.2.8) of Theorem 1.2.3 yields

$$\begin{aligned}(\mu_1\nu_4 - \mu_0\nu_5)\frac{\psi'''''}{\psi'} &+ 4(\mu_2\nu_3 - \mu_1\nu_4)\frac{\varphi''}{\varphi'}\cdot\frac{\psi''''}{\psi'} + 6(\mu_3\nu_2 - \mu_2\nu_3)\frac{\varphi'''}{\varphi'}\cdot\frac{\psi'''}{\psi'} \\ &+ 4(\mu_4\nu_1 - \mu_3\nu_2)\frac{\varphi''''}{\varphi'}\cdot\frac{\psi''}{\psi'} + (\mu_5\nu_0 - \mu_4\nu_1)\frac{\varphi'''''}{\varphi'} = 0.\end{aligned} \tag{1.2.42}$$

Differentiating (1.2.25), we get that

$$\begin{aligned}\frac{\varphi'''''}{\varphi'} &= \frac{\Phi'''}{\mu_2} + 4\frac{\Phi\Phi''}{\mu_2^2} + 3\frac{(\Phi')^2}{\mu_2^2} + 6\frac{\Phi^2\Phi'}{\mu_2^3} + \frac{\Phi^4}{\mu_2^4}, \\ \frac{\psi'''''}{\psi'} &= \frac{\Phi'''}{\nu_2} + 4\frac{\Phi\Phi''}{\nu_2^2} + 3\frac{(\Phi')^2}{\nu_2^2} + 6\frac{\Phi^2\Phi'}{\nu_2^3} + \frac{\Phi^4}{\nu_2^4}.\end{aligned} \tag{1.2.43}$$

Now using $\mu_1 = \nu_1 = \mu_3 = \nu_3 = 0$, and (1.2.43), equation (1.2.42) simplifies to the following (at most) third-order differential equation for Φ:

$$\begin{aligned}\left(\frac{\mu_5}{\mu_2} - \frac{\nu_5}{\nu_2}\right)\Phi''' + 4\left(\frac{\mu_5}{\mu_2^2} - \frac{\nu_5}{\nu_2^2}\right)\Phi\Phi'' + 3\left(\frac{\mu_5}{\mu_2^2} - \frac{\nu_5}{\nu_2^2}\right)(\Phi')^2 \\ + 6\left(\frac{\mu_5}{\mu_2^3} - \frac{\nu_5}{\nu_2^3}\right)\Phi^2\Phi' + \left(\frac{\mu_5}{\mu_2^4} - \frac{\nu_5}{\nu_2^4}\right)\Phi^4 = 0.\end{aligned}$$

Substituting the formulae from (1.2.41) into the above equation, we get

$$\begin{aligned}\left(\frac{\mu_5}{\mu_2} - \frac{\nu_5}{\nu_2}\right)\big(\alpha(\Phi')^2 + (\alpha^2+3\beta)\Phi^2\Phi' + \alpha\beta\Phi^4\big) \\ +4\left(\frac{\mu_5}{\mu_2^2} - \frac{\nu_5}{\nu_2^2}\right)\big(\alpha\Phi^2\Phi' + \beta\Phi^4\big) + 3\left(\frac{\mu_5}{\mu_2^2} - \frac{\nu_5}{\nu_2^2}\right)(\Phi')^2 \\ +6\left(\frac{\mu_5}{\mu_2^3} - \frac{\nu_5}{\nu_2^3}\right)\Phi^2\Phi' + \left(\frac{\mu_5}{\mu_2^4} - \frac{\nu_5}{\nu_2^4}\right)\Phi^4 = 0.\end{aligned}$$

Finally, we obtain the following (at most) first-order differential equation for Φ:
(1.2.44)

$$\begin{aligned}&\left(\frac{3+\alpha\mu_2}{\mu_2^2}\mu_5 - \frac{3+\alpha\nu_2}{\nu_2^2}\nu_5\right)(\Phi')^2 \\ &+ \left(\frac{6+4\alpha\mu_2+(\alpha^2+3\beta)\mu_2^2}{\mu_2^3}\mu_5 - \frac{6+4\alpha\nu_2+(\alpha^2+3\beta)\nu_2^2}{\nu_2^3}\nu_5\right)\Phi^2\Phi' \\ &+ \left(\frac{1+4\beta\mu_2^2+\alpha\beta\mu_2^3}{\mu_2^4}\mu_5 - \frac{1+4\beta\nu_2^2+\alpha\beta\nu_2^3}{\nu_2^4}\nu_5\right)\Phi^4 = 0.\end{aligned}$$

In the next step we show that the three constant coefficients in this equation cannot be simultaneously zero. Indeed, if all these coefficients are zero then, using that $(\mu_5, \nu_5) \neq (0,0)$, we can see that the following two vectors in $\mathbb{R}^3$ are linearly dependent:
(1.2.45)

$$\begin{aligned}u = (u_1,u_2,u_3) &:= \left(\frac{3+\alpha\mu_2}{\mu_2^2}, \frac{6+4\alpha\mu_2+(\alpha^2+3\beta)\mu_2^2}{\mu_2^3}, \frac{1+4\beta\mu_2^2+\alpha\beta\mu_2^3}{\mu_2^4}\right) \\ v = (v_1,v_2,v_3) &:= \left(\frac{3+\alpha\nu_2}{\nu_2^2}, \frac{6+4\alpha\nu_2+(\alpha^2+3\beta)\nu_2^2}{\nu_2^3}, \frac{1+4\beta\nu_2^2+\alpha\beta\nu_2^3}{\nu_2^4}\right).\end{aligned}$$

Therefore, their vectorial product is zero, i.e., $u_iv_j = u_jv_i$ for all $1 \le i < j \le 3$. The equations corresponding to the cases $(i,j) = (1,2)$ and $(i,j) = (1,3)$ are

$$\begin{aligned}&\mu_2\big(3+\alpha\mu_2\big)\big(6+4\alpha\nu_2+(\alpha^2+3\beta)\nu_2^2\big) \\ &= \nu_2\big(3+\alpha\nu_2\big)\big(6+4\alpha\mu_2+(\alpha^2+3\beta)\mu_2^2\big)\end{aligned}$$

and

$$\mu_2^2\big(3+\alpha\mu_2\big)\big(1+4\beta\nu_2^2+\alpha\beta\nu_2^3\big) = \nu_2^2\big(3+\alpha\nu_2\big)\big(1+4\beta\mu_2^2+\alpha\beta\mu_2^3\big).$$

After some calculations, simplifying also by the factor $\mu_2 - \nu_2 \neq 0$, we arrive at

$$6\alpha(\mu_2 + \nu_2) + (\alpha^2 - 9\beta)\mu_2\nu_2 + 18 = 0 \tag{1.2.46}$$

and

$$\alpha(\mu_2^2 + \nu_2^2 + \mu_2\nu_2) + \alpha\beta\mu_2^2\nu_2^2 + 3(\mu_2 + \nu_2) = 0. \tag{1.2.47}$$

Multiplying (1.2.46) by $(\mu_2 + \nu_2)$, (1.2.47) by 6, and subtracting the equations so obtained, finally dividing by $\mu_2\nu_2 \neq 0$, we get

$$(\alpha^2 - 9\beta)(\mu_2 + \nu_2) - 6\alpha\beta\mu_2\nu_2 + 6\alpha = 0. \tag{1.2.48}$$

Using the formulae

$$\begin{aligned} \alpha &= -\frac{\gamma}{\eta} = -\frac{3(\mu_4\nu_2^2 - \nu_4\mu_2^2)}{\mu_2\nu_2(\mu_4\nu_2 - \nu_4\mu_2)}, \\ \beta &= -\frac{\delta}{\eta} = -\frac{\mu_4\nu_2^3 - \nu_4\mu_2^3 + 3\mu_2^2\nu_2^2(\mu_2 - \nu_2)}{\mu_2^2\nu_2^2(\mu_4\nu_2 - \nu_4\mu_2)}, \end{aligned} \tag{1.2.49}$$

equations (1.2.46) and (1.2.48) can be rewritten in the form

$$\frac{9(\mu_2 - \nu_2)\big(3\mu_2\nu_2(\mu_4\nu_2 - \nu_4\mu_2) + \mu_4\nu_4(\mu_2 - \nu_2)\big)}{(\mu_4\nu_2 - \nu_4\mu_2)^2} = 0,$$

$$\frac{9(\mu_2 - \nu_2)^2\big(3\mu_2\nu_2(\mu_4\nu_2 + \nu_4\mu_2) - \mu_4\nu_4(\mu_2 + \nu_2)\big)}{\mu_2\nu_2(\mu_4\nu_2 - \nu_4\mu_2)^2} = 0,$$

respectively. Using $\mu_2 - \nu_2 \neq 0$ and $\mu_4\nu_2 - \nu_4\mu_2 \neq 0$, we get

$$\begin{aligned} 3\mu_2\nu_2(\mu_4\nu_2 - \nu_4\mu_2) + \mu_4\nu_4(\mu_2 - \nu_2) &= 0, \\ 3\mu_2\nu_2(\mu_4\nu_2 + \nu_4\mu_2) - \mu_4\nu_4(\mu_2 + \nu_2) &= 0. \end{aligned}$$

Adding up, and subtracting these two equations, we obtain

$$6\mu_4\mu_2\nu_2^2 - 2\mu_4\nu_4\nu_2 = 0, \qquad 6\nu_4\mu_2^2\nu_2 - 2\mu_4\nu_4\mu_2 = 0,$$

whence it follows that

$$\mu_4 = \nu_4 = 3\mu_2\nu_2. \tag{1.2.50}$$

In this case, (1.2.49) simplifies to

$$\alpha = -3\left(\frac{1}{\mu_2} + \frac{1}{\nu_2}\right), \qquad \beta = -\left(\frac{1}{\mu_2^2} + \frac{1}{\nu_2^2}\right).$$

Therefore, for the vectors u and v defined in (1.2.45), we get

$$u = v = \left(\frac{-3}{\mu_2\nu_2}, \frac{6(\mu_2 + \nu_2)}{\mu_2^2\nu_2^2}, \frac{3(\mu_2^2 + \nu_2^2) - \mu_2^2\nu_2^2}{\mu_2^3\nu_2^3}\right).$$

Thus, the differential equation (1.2.44) reduces to the following form

$$\begin{aligned}&\frac{-3}{\mu_2\nu_2}(\mu_5-\nu_5)(\Phi')^2+\frac{6(\mu_2+\nu_2)}{\mu_2^2\nu_2^2}(\mu_5-\nu_5)\Phi'\Phi^2\\&+\frac{3(\mu_2^2+\nu_2^2)-\mu_2^2\nu_2^2}{\mu_2^3\nu_2^3}(\mu_5-\nu_5)\Phi^4=0.\end{aligned}$$

The coefficients of this equation can simultaneously vanish if and only if $\mu_5=\nu_5$. However, this equality, together with (1.2.50) contradicts the condition (1.2.40) of the theorem. The contradiction so obtained shows that the coefficients of (1.2.44) cannot be identically zero under the assumptions of the theorem. Thus, (1.2.44) is a nontrivial first-order polynomial differential equation of the form (1.2.30). Therefore, it follows that φ and ψ satisfy one of the alternatives of Theorem 1.2.9. □

If either $\mu_5=\nu_5=0$ or $\mu_5=\nu_5$ and $\mu_4=\nu_4=3\mu_2\nu_2$, then (1.2.44) is useless, thus we need to apply the necessary condition (1.2.8) of Theorem 1.2.3 in the case $n=5$.

THEOREM 1.2.13. *Assume* $\mathcal{C}_4$ *and* $\mathcal{M}_1^*$ *with* $\mu_2\nu_2\neq 0$, $(\mu_3,\nu_3)=(0,0)$, $\mu_2\nu_4\neq\nu_2\mu_4$,

(1.2.51)
$$\big(\mu_6,\ \nu_6,\ 0\big)\neq\left(\frac{5\mu_2\mu_4^2}{6\mu_2^2-\mu_4},\ \frac{5\nu_2\nu_4^2}{6\nu_2^2-\nu_4},\ 3\mu_2\nu_2(\nu_2\mu_4-\mu_2\nu_4)-(\mu_2-\nu_2)\mu_4\nu_4\right)$$

and

$$\big(\mu_6,\ \nu_6,\ 0\big)\neq\left(\frac{F}{E},\ \frac{G}{E},\ D\right), \tag{1.2.52}$$

where

(1.2.53)
$$\begin{aligned}D:=&45\mu_2^7\nu_2^7(\mu_2-\nu_2)^3(\mu_2\nu_4-\mu_4\nu_2)^4\big((\mu_2-\nu_2)\mu_4\nu_4+3\mu_2\nu_2(\mu_2\nu_4-\mu_4\nu_2)\big)\\&\big(\nu_2(\mu_2-\nu_2)(2\mu_2-\nu_2)(\mu_2-2\nu_2)(7\mu_2-8\nu_2)\mu_4^2\nu_4\\&+\mu_2(\mu_2-\nu_2)(2\mu_2-\nu_2)(\mu_2-2\nu_2)(8\mu_2-7\nu_2)\mu_4\nu_4^2\\&-6\mu_2^2\nu_2^2(\mu_2+\nu_2)(6\mu_2^2-5\mu_2\nu_2+6\nu_2^2)\mu_4\nu_4\\&-6\mu_2\nu_2^3(\mu_2-2\nu_2)(7\mu_2^2+\mu_2\nu_2-\nu_2^2)\mu_4^2\\&-6\mu_2^3\nu_2(2\mu_2-\nu_2)(\mu_2^2-\mu_2\nu_2-7\nu_2^2)\nu_4^2\\&+45\mu_2^3\nu_2^4(\mu_2-\nu_2)(7\mu_2-2\nu_2)\mu_4+45\mu_2^4\nu_2^3(\mu_2-\nu_2)(2\mu_2-7\nu_2)\nu_4\big),\end{aligned}$$

$$
\begin{aligned}
E :=& 4\nu_2^2(\mu_2-\nu_2)(\mu_2-2\nu_2)^2\mu_4^3\nu_4 + 4\mu_2^2(\mu_2-\nu_2)(2\mu_2-\nu_2)^2\mu_4\nu_4^3\\
&+ \mu_2\nu_2(\mu_2-\nu_2)(7\nu_2^2-22\mu_2\nu_2+7\mu_2^2)\mu_4^2\nu_4^2\\
&- 6\mu_2\nu_2^4(2\mu_2+\nu_2)(\mu_2-2\nu_2)\mu_4^3 - 6\mu_2^4\nu_2(\mu_2+2\nu_2)(2\mu_2-\nu_2)\nu_4^3\\
&+ 3\mu_2^2\nu_2^3(7\mu_2^2-24\mu_2\nu_2-\nu_2^2)\mu_4^2\nu_4 + 3\mu_2^3\nu_2^2(\mu_2^2+24\mu_2\nu_2-7\nu_2^2)\mu_4\nu_4^2\\
&+ 90\mu_2^3\nu_2^3(\mu_2-\nu_2)(\mu_2\nu_4-\nu_2\mu_4)^2,\\
F :=& 3\mu_2\nu_2(\mu_2\nu_4-\mu_4\nu_2)\big(\mu_2^2\nu_2\nu_4-\mu_2\mu_4\nu_2^2+(3\nu_2-3\mu_2)\mu_4\nu_4\big)\\
&\quad\big(6\mu_2^3\nu_2\nu_4-6\mu_2^2\mu_4\nu_2^2+\mu_2(7\mu_2-2\nu_2)\mu_4\nu_4+5\nu_2(\mu_2-2\nu_2)\mu_4^2\big),\\
G :=& 3\mu_2\nu_2(\mu_2\nu_4-\mu_4\nu_2)\big(\mu_2^2\nu_2\nu_4-\mu_2\mu_4\nu_2^2+(3\nu_2-3\mu_2)\mu_4\nu_4\big)\\
&\quad(6\mu_2^2\nu_2^2\nu_4-6\mu_2\mu_4\nu_2^3+\nu_2(2\mu_2-7\nu_2)\mu_4\nu_4+5\mu_2(2\mu_2-\nu_2)\nu_4^2).
\end{aligned}
$$

Then $\mathcal{M}_{\varphi,\mu}=\mathcal{M}_{\psi,\nu}$ holds if and only if either one of the alternatives (i), (ii), (iii) of Theorem 1.2.9 is satisfied.

PROOF. Following the argument of the proof of the previous theorem, we get that the functions φ and ψ are analytic on I and Φ defined by (1.2.19) satisfies (1.2.39), where the constants η,γ,δ are defined by (1.2.27). We have $\eta\neq 0$, and, with the notations $\alpha := -\gamma/\eta$ and $\beta := -\delta/\eta$, (1.2.39) yields (1.2.41) and

$$\Phi'''' = (4\alpha^2+6\beta)\Phi(\Phi')^2 + (\alpha^3+9\alpha\beta)\Phi^3\Phi' + (\alpha^2\beta+3\beta^2)\Phi^5. \tag{1.2.54}$$

Differentiating (1.2.43), we get that

(1.2.55)
$$
\begin{aligned}
\frac{\varphi''''''}{\varphi'} &= \frac{\Phi''''}{\mu_2} + 5\frac{\Phi\Phi'''}{\mu_2^2} + 10\frac{\Phi'\Phi''}{\mu_2^2} + 10\frac{\Phi^2\Phi''}{\mu_2^3} + 15\frac{\Phi(\Phi')^2}{\mu_2^3} + 10\frac{\Phi^3\Phi'}{\mu_2^4} + \frac{\Phi^5}{\mu_2^5},\\
\frac{\psi''''''}{\psi'} &= \frac{\Phi''''}{\nu_2} + 5\frac{\Phi\Phi'''}{\nu_2^2} + 10\frac{\Phi'\Phi''}{\nu_2^2} + 10\frac{\Phi^2\Phi''}{\nu_2^3} + 15\frac{\Phi(\Phi')^2}{\nu_2^3} + 10\frac{\Phi^3\Phi'}{\nu_2^4} + \frac{\Phi^5}{\nu_2^5}.
\end{aligned}
$$

In the particular case $n=5$, condition (1.2.8) of Theorem 1.2.3 yields

$$
\begin{aligned}
&(\mu_1\nu_5-\mu_0\nu_6)\frac{\psi''''''}{\psi'} + 5(\mu_2\nu_4-\mu_1\nu_5)\frac{\varphi''}{\varphi'}\cdot\frac{\psi'''''}{\psi'}\\
&+ 10(\mu_3\nu_3-\mu_2\nu_4)\frac{\varphi'''}{\varphi'}\cdot\frac{\psi''''}{\psi'} + 10(\mu_4\nu_2-\mu_3\nu_3)\frac{\varphi''''}{\varphi'}\cdot\frac{\psi'''}{\psi'}\\
&+ 5(\mu_5\nu_1-\mu_4\nu_2)\frac{\varphi'''''}{\varphi'}\cdot\frac{\psi''}{\psi'} + (\mu_6\nu_0-\mu_5\nu_1)\frac{\varphi''''''}{\varphi'} = 0.
\end{aligned}
$$

Now using $\mu_1 = \nu_1 = \mu_3 = \nu_3 = 0$, and the identities (1.2.22), (1.2.23), (1.2.25), (1.2.41), (1.2.43), (1.2.54), and (1.2.55), we obtain

$$
\begin{aligned}
-\nu_6\Big(&\frac{\Phi''''}{\nu_2} + 5\frac{\Phi\Phi'''}{\nu_2^2} + 10\frac{\Phi'\Phi''}{\nu_2^2} + 10\frac{\Phi^2\Phi''}{\nu_2^3} + 15\frac{\Phi(\Phi')^2}{\nu_2^3} + 10\frac{\Phi^3\Phi'}{\nu_2^4} + \frac{\Phi^5}{\nu_2^5}\Big)\\
&+ 5\mu_2\nu_4\frac{\Phi}{\mu_2}\cdot\Big(\frac{\Phi'''}{\nu_2} + 4\frac{\Phi\Phi''}{\nu_2^2} + 3\frac{(\Phi')^2}{\nu_2^2} + 6\frac{\Phi^2\Phi'}{\nu_2^3} + \frac{\Phi^4}{\nu_2^4}\Big)\\
&- 10\mu_2\nu_4\Big(\frac{\Phi'}{\mu_2} + \frac{\Phi^2}{\mu_2^2}\Big)\cdot\Big(\frac{\Phi''}{\nu_2} + 3\frac{\Phi\Phi'}{\nu_2^2} + \frac{\Phi^3}{\nu_2^3}\Big)\\
&+ 10\mu_4\nu_2\Big(\frac{\Phi''}{\mu_2} + 3\frac{\Phi\Phi'}{\mu_2^2} + \frac{\Phi^3}{\mu_2^3}\Big)\cdot\Big(\frac{\Phi'}{\nu_2} + \frac{\Phi^2}{\nu_2^2}\Big)\\
&- 5\mu_4\nu_2\Big(\frac{\Phi'''}{\mu_2} + 4\frac{\Phi\Phi''}{\mu_2^2} + 3\frac{(\Phi')^2}{\mu_2^2} + 6\frac{\Phi^2\Phi'}{\mu_2^3} + \frac{\Phi^4}{\mu_2^4}\Big)\cdot\frac{\Phi}{\nu_2}\\
+\mu_6\Big(&\frac{\Phi''''}{\mu_2} + 5\frac{\Phi\Phi'''}{\mu_2^2} + 10\frac{\Phi'\Phi''}{\mu_2^2} + 10\frac{\Phi^2\Phi''}{\mu_2^3} + 15\frac{\Phi(\Phi')^2}{\mu_2^3} + 10\frac{\Phi^3\Phi'}{\mu_2^4} + \frac{\Phi^5}{\mu_2^5}\Big) = 0.
\end{aligned}
$$

Using now the formulae (1.2.41) and (1.2.54), the above equation reduces to the following first-order differential equation for Φ:

$$
\begin{aligned}
(A_1\mu_6 + A_2\nu_6 + A_3)\Phi(\Phi')^2 + (B_1\mu_6 + B_2\nu_6 + B_3)\Phi^3\Phi'&\\
+ (C_1\mu_6 + C_2\nu_6 + C_3)\Phi^5 = 0,&
\end{aligned}
\tag{1.2.56}
$$

where

$$
\begin{aligned}
A_1 &:= \mu_2^2\nu_2^5\big((4\alpha^2 + 6\beta)\mu_2^2 + 15\alpha\mu_2 + 15\big)\\
A_2 &:= -\mu_2^5\nu_2^2\big((4\alpha^2 + 6\beta)\nu_2^2 + 15\alpha\nu_2 + 15\big)\\
A_3 &:= -5\mu_2^3\nu_2^3\big((\alpha\nu_2 + 3)\mu_2^2\nu_4 - (\alpha\mu_2 + 3)\mu_4\nu_2^2\big)\\
B_1 &:= \mu_2\nu_2^5\big((\alpha^3 + 9\alpha\beta)\mu_2^3 + (5\alpha^2 + 25\beta)\mu_2^2 + 10\alpha\mu_2 + 10\big)\\
B_2 &:= -\mu_2^5\nu_2\big((\alpha^3 + 9\alpha\beta)\nu_2^3 + (5\alpha^2 + 25\beta)\nu_2^2 + 10\alpha\nu_2 + 10\big)\\
B_3 &:= 5\mu_2^2\nu_2^2\big((\alpha^2\nu_2^2 + 4\alpha\nu_2 + \beta\nu_2^2 + 4)\mu_2^3\nu_4 - (\alpha^2\mu_2^2 + 4\alpha\mu_2 + \beta\mu_2^2 + 4)\mu_4\nu_2^3\\
&\qquad + (2\alpha\mu_2^2 + 6\mu_2)\mu_4\nu_2^2 - (2\alpha\nu_2^2 + 6\nu_2)\mu_2^2\nu_4\big)\\
C_1 &:= \nu_2^5\big((\alpha^2\beta + 3\beta^2)\mu_2^4 + 5\alpha\beta\mu_2^3 + 10\beta\mu_2^2 + 1\big)\\
C_2 &:= -\mu_2^5\big((\alpha^2\beta + 3\beta^2)\nu_2^4 + 5\alpha\beta\nu_2^3 + 10\beta\nu_2^2 + 1\big)\\
C_3 &:= 5\mu_2\nu_2\big((\alpha\beta\nu_2^3 + 4\beta\nu_2^2 + 1)\mu_2^4\nu_4 - (\alpha\beta\mu_2^3 + 4\beta\mu_2^2 + 1)\mu_4\nu_2^4\\
&\qquad + (2\mu_2^3\beta + 2\mu_2)\mu_4\nu_2^3 - (2\nu_2^3\beta + 2\nu_2)\mu_2^3\nu_4\big).
\end{aligned}
$$

If the coefficients in equation (1.2.56) vanish simultaneously, then μ_6, ν_6 and $\xi = 1$ is a nontrivial solution of the following system of homogeneous linear equations

$$
\begin{aligned}
A_1\mu_6 + A_2\nu_6 + A_3\xi = 0, \qquad B_1\mu_6 + B_2\nu_6 + B_3\xi = 0,&\\
C_1\mu_6 + C_2\nu_6 + C_3\xi = 0.&
\end{aligned}
\tag{1.2.57}
$$

Therefore, the value D defined in (1.2.53), which is the determinant D of this system has to be zero. The constant D was factorized by using the Maple 9 symbolic package. Thus, in order that D be zero, we have two possibilities. The first (simpler) case is when

$$(\mu_2 - \nu_2)\mu_4\nu_4 + 3\mu_2\nu_2(\mu_2\nu_4 - \mu_4\nu_2) = 0.$$

Then, again using Maple 9, we get the following values for the solutions μ_6 and ν_6 of the linear system (1.2.57):

$$\mu_6 = \frac{5\mu_2\mu_4^2}{6\mu_2^2 - \mu_4}, \qquad \nu_6 = \frac{5\nu_2\nu_4^2}{6\nu_2^2 - \nu_4}.$$

This, however, contradicts the assumption (1.2.51). Thus, in this case the three coefficients of (1.2.56) cannot vanish simultaneously.

The second case is when the last factor of D is zero, i.e., when

$$\begin{aligned}
&\nu_2(\mu_2 - \nu_2)(2\mu_2 - \nu_2)(\mu_2 - 2\nu_2)(7\mu_2 - 8\nu_2)\mu_4^2\nu_4 \\
&+ \mu_2(\mu_2 - \nu_2)(2\mu_2 - \nu_2)(\mu_2 - 2\nu_2)(8\mu_2 - 7\nu_2)\mu_4\nu_4^2 \\
&- 6\mu_2^2\nu_2^2(\mu_2 + \nu_2)(6\mu_2^2 - 5\mu_2\nu_2 + 6\nu_2^2)\mu_4\nu_4 \\
&- 6\mu_2\nu_2^3(\mu_2 - 2\nu_2)(7\mu_2^2 + \mu_2\nu_2 - \nu_2^2)\mu_4^2 \\
&- 6\mu_2^3\nu_2(2\mu_2 - \nu_2)(\mu_2^2 - \mu_2\nu_2 - 7\nu_2^2)\nu_4^2 + 45\mu_2^3\nu_2^4(7\mu_2 - 2\nu_2)(\mu_2 - \nu_2)\mu_4 \\
&+ 45\mu_2^4\nu_2^3(\mu_2 - \nu_2)(2\mu_2 - 7\nu_2)\nu_4 = 0.
\end{aligned}$$

Calculating with the help of the Maple 9 package, we get the following values for the unknowns μ_6 and ν_6:

$$\mu_6 = \frac{F}{E}, \qquad \nu_6 = \frac{G}{E},$$

where E, F, and G are given by (1.2.53). In view of condition (1.2.52), we get again a contradiction. Thus, in this case, the three coefficients of (1.2.56) cannot be simultaneously zero

Therefore, in each case, Φ satisfies a nontrivial first order polynomial differential equation of the form (1.2.30). Hence, one of the alternatives of Theorem 1.2.9 must be valid. □

1.2.4. Applications. In this section we demonstrate some possible applications of our results.

EXAMPLE 1. Consider the functional equation

$$\varphi^{-1}\left(\frac{\varphi\left(\frac{2x+y}{3}\right) + \varphi\left(\frac{x+2y}{3}\right)}{2}\right) = \psi^{-1}\left(\frac{\psi(x) + 16\psi\left(\frac{x+y}{2}\right) + \psi(y)}{18}\right), \tag{1.2.58}$$

where $\varphi, \psi : I \to \mathbb{R}$ are continuous strictly monotone functions.

Equation (1.2.58) is an obvious particular case of the equality problem (1.2.1), where the measures μ and ν are given by

$$\mu = \frac{\delta_{1/3} + \delta_{2/3}}{2} \qquad \text{and} \qquad \nu = \frac{\delta_0 + 16\delta_{1/2} + \delta_1}{18}.$$

Then, $\widehat{\mu}_1 = \widehat{\nu}_1 = \frac{1}{2}$ and, for $k \in \mathbb{N}$,

$$\mu_k = \frac{(-1)^k + 1}{2 \cdot 6^k} \qquad \text{and} \qquad \nu_k = \frac{(-1)^k + 1}{18 \cdot 2^k}.$$

Hence

$$\begin{aligned} &\mu_1 = 0, && \mu_2 = \frac{1}{36}, && \mu_3 = 0, && \mu_4 = \frac{1}{1296}, && \dots, \\ &\nu_1 = 0, && \nu_2 = \frac{1}{36}, && \nu_3 = 0, && \nu_4 = \frac{1}{144}, && \dots. \end{aligned}$$

Thus the exact moment condition $\mathcal{M}_3^*$ holds. If $\mathcal{C}_4$ is assumed, then, by Theorem 1.2.6, $\varphi, \psi : I \to \mathbb{R}$ satisfy (1.2.58) if and only if there exist constants $a \neq 0$ and b such that

$$\psi = a\varphi + b$$

and φ is an arbitrary strictly monotone polynomial with $\deg \varphi \leq 3$.

It remains an open problem to find the solutions of (1.2.58) under the regularity assumption $\mathcal{C}_0$ only.

EXAMPLE 2. Consider the functional equation

$$\varphi^{-1}\left(\frac{2\varphi(x) + \varphi(y)}{3}\right) = \psi^{-1}\left(\int_0^1 2t\psi(tx + (1-t)y)dt\right), \tag{1.2.59}$$

where $\varphi, \psi : I \to \mathbb{R}$ are continuous strictly monotone functions.

Equation (1.2.59) is also a particular case of the equality problem (1.2.1), where the measures μ and ν are now given by

$$\mu = \frac{\delta_0 + 2\delta_1}{3} \qquad \text{and} \qquad d\nu(t) = 2tdt.$$

Then, $\widehat{\mu}_1 = \widehat{\nu}_1 = \frac{2}{3}$ and, for $k \in \mathbb{N}$, we have

$$\mu_k = \int_0^1 \left(t - \frac{2}{3}\right)^k d\mu(t) = \frac{(-2)^k + 2}{3^{k+1}} \qquad \text{and}$$

$$\nu_k = \int_0^1 2t\left(t - \frac{2}{3}\right)^k dt = \frac{6k + 10 - (-2)^{k+3}}{(k+1)(k+2)3^{k+2}}.$$

Hence

$$\begin{aligned} &\mu_1 = 0, && \mu_2 = \frac{2}{9}, && \mu_3 = -\frac{2}{27}, && \mu_4 = \frac{2}{27}, && \dots, \\ &\nu_1 = 0, && \nu_2 = \frac{1}{18}, && \nu_3 = -\frac{1}{135}, && \nu_4 = \frac{1}{135}, && \dots. \end{aligned}$$

Thus the exact moment condition $\mathcal{M}_1^*$ holds. Since $\mu_3 \neq 0 \neq \nu_3$, Theorem 1.2.10 can be applied. If $\mathcal{C}_3$ is assumed, then, one of the alternatives (i), (ii), and (iii) of Theorem 1.2.9 holds.

If the alternative (i) is valid then there exist real constants a, b, c, d with $ac \neq 0$ such that φ and ψ are given by (1.2.31), i.e., they are affine functions. In this case, the means $\mathcal{M}_{\varphi,\mu}(x, y)$ and $\mathcal{M}_{\psi,\nu}(x, y)$ are equal to the weighted arithmetic mean $\dfrac{2x + y}{3}$.

If (ii) were valid, then there exist real constants a, b, c, d, p, q with $acpq(p - q) \neq 0$ such that (1.2.32) and (1.2.33) hold for all $n \in \mathbb{N}$. In the case $n = 1$, (1.2.33) yields

$$q(\mu_1\nu_1 - \mu_0\nu_2) + p(\mu_2\nu_0 - \mu_1\nu_1) = 0, \tag{1.2.60}$$

whence $q = 4p$. If $n = 2$, then (1.2.33) implies

$$q^2(\mu_1\nu_2 - \mu_0\nu_3) + pq(\mu_2\nu_1 - \mu_1\nu_2) + p^2(\mu_3\nu_0 - \mu_2\nu_1) = 0, \tag{1.2.61}$$

resulting $q^2 = 10p^2$, which contradicts $q = 4p$.

If (iii) is valid then there exist real constants a, b, c, d, p, q with $ac(p - 1)(q - 1)(p - q) \neq 0$ and $x_0 \notin I$ such that (1.2.34) and (1.2.35) hold for all $n \in \mathbb{N}$. In the case $n = 1$, (1.2.35) yields

$$(q - 1)(\mu_1\nu_1 - \mu_0\nu_2) + (p - 1)(\mu_2\nu_0 - \mu_1\nu_1) = 0, \tag{1.2.62}$$

whence $q = 4p - 3$. If $n = 2$, then (1.2.35) implies

$$\begin{aligned} &\frac{(q-1)(q-2)}{2}(\mu_1\nu_2 - \mu_0\nu_3) + (p - 1)(q - 1)(\mu_2\nu_1 - \mu_1\nu_2) \\ &+ \frac{(p-1)(p-2)}{2}(\mu_3\nu_0 - \mu_2\nu_1) = 0, \end{aligned} \tag{1.2.63}$$

which results $p = 0$ and $q = 4p - 3 = -3$. Instead of showing now that (1.2.35) holds for all $n \geq 3$, we prove that the functions $\varphi, \psi : I \to \mathbb{R}$ given by (1.2.34) satisfy (1.2.59). For simplicity, we assume that $x_0 = 0 \leq \inf I$. Then $\varphi(x) = a \ln x + b$ and $\psi(x) = cx^{-3} + d$.

On one hand, we have

$$\varphi^{-1}\left(\frac{2\varphi(x) + \varphi(y)}{3}\right) = \sqrt[3]{x^2y}.$$

On the other hand,

$$\begin{aligned} &\psi^{-1}\left(\int_0^1 2t\psi(tx + (1 - t)y)dt\right) = \left(\int_0^1 \frac{2t}{(tx + (1 - t)y)^3}dt\right)^{-\frac{1}{3}} \\ &= \left(\frac{2t(y - x) - y}{(y - x)^2(tx + (1 - t)y)^2}\bigg|_{t=0}^{t=1}\right)^{-\frac{1}{3}} = \left(\frac{2(y - x) - y}{(y - x)^2x^2} - \frac{-y}{(y - x)^2y^2}\right)^{-\frac{1}{3}} \\ &= \left(\frac{1}{x^2y}\right)^{-\frac{1}{3}} = \sqrt[3]{x^2y}, \end{aligned}$$

which proves the equality in (1.2.59).

EXAMPLE 3. Consider the functional equation

$$\varphi^{-1}\left(\frac{2\varphi(x)+\varphi(y)}{3}\right)=\psi^{-1}\left(\frac{4\psi(x)+4\psi\left(\frac{x+y}{2}\right)+\psi(y)}{9}\right), \tag{1.2.64}$$

where $\varphi, \psi : I \to \mathbb{R}$ are continuous strictly monotone functions.

Equation (1.2.64) is an obvious particular case of the equality problem (1.2.1), where the measures μ and ν are given by

$$\mu=\frac{\delta_0+2\delta_1}{3} \qquad \text{and} \qquad \nu=\frac{\delta_0+4\delta_{1/2}+4\delta_1}{9}.$$

Then, $\widehat{\mu}_1=\widehat{\nu}_1=\frac{2}{3}$ and, for $k\in\mathbb{N}$, we have

$$\mu_k=\int_0^1\left(t-\frac{2}{3}\right)^k d\mu(t)=\frac{(-2)^k+2}{3^{k+1}} \qquad \text{and}$$
$$\nu_k=\int_0^1\left(t-\frac{2}{3}\right)^k d\nu(t)=\frac{(-4)^k+4(-1)^k+4\cdot 2^k}{9\cdot 6^k}.$$

Hence

$$\mu_1=0, \qquad \mu_2=\frac{2}{9}, \qquad \mu_3=-\frac{2}{27}, \qquad \mu_4=\frac{2}{27}, \quad \dots,$$
$$\nu_1=0, \qquad \nu_2=\frac{1}{9}, \qquad \nu_3=-\frac{1}{54}, \qquad \nu_4=\frac{1}{36}, \quad \dots.$$

Thus the exact moment condition $\mathcal{M}_1^*$ holds. Since $\mu_3\neq 0\neq \nu_3$, Theorem 1.2.10 can be applied. If $\mathcal{C}_3$ is assumed, then, one of the alternatives (i), (ii), and (iii) of Theorem 1.2.10 holds.

Clearly, if (i) holds, then φ and ψ are affine functions and the two means on the left and right hand sides of (1.2.64) are equal to the weighted arithmetic mean $\dfrac{2x+y}{3}$.

If (ii) holds, then there exist constants a, b, c, d, p, q with $acpq(p-q)\neq 0$ such that (1.2.32) and (1.2.33) are satisfied for all $n\in\mathbb{N}$. In the case $n=1$, (1.2.33) simplifies to (1.2.60), which results $q=2p$. Instead of showing that (1.2.33) holds for all $n\geq 2$, we prove that the functions φ and ψ given by (1.2.32) are solutions of (1.2.64). Indeed,

$$\begin{aligned}
&\varphi^{-1}\left(\frac{2\varphi(x)+\varphi(y)}{3}\right)=\frac{1}{p}\ln\left(\frac{2e^{px}+e^{py}}{3}\right)\\
&=\frac{1}{2p}\ln\left(\frac{2e^{px}+e^{py}}{3}\right)^2=\frac{1}{2p}\ln\left(\frac{4e^{2px}+4e^{2p\frac{x+y}{2}}+e^{2py}}{9}\right)\\
&=\frac{1}{q}\ln\left(\frac{4e^{qx}+4e^{q\frac{x+y}{2}}+e^{qy}}{9}\right)=\psi^{-1}\left(\frac{4\psi(x)+4\psi\left(\frac{x+y}{2}\right)+\psi(y)}{9}\right).
\end{aligned}$$

In this case, we can also see that the means on the two sides of (1.2.35) are weighted exponential means.

If (iii) were valid then there exist real constants a, b, c, d, p, q with $ac(p-1)(q-1)(p-q) \neq 0$ and $x_0 \notin I$ such that (1.2.34) and (1.2.35) hold for all $n \in \mathbb{N}$. In the case $n = 1$, (1.2.35) simplifies to (1.2.62) whence $q = 2p - 1$ follows. If $n = 2$, then (1.2.35) yields (1.2.63) which results $p = 1$ contradicting the conditions on the parameters. Therefore, there is no solution of (1.2.64) in the case (iii).

1.3. The invariance problem

Now we characterize the continuous strictly monotone functions φ, ψ and Borel probability measures μ, ν such that

$$\mathcal{M}_{\varphi,\mu}(x, y) + \mathcal{M}_{\psi,\nu}(x, y) = x + y \quad (x, y \in I) \tag{1.3.1}$$

holds.

COROLLARY 1.3.1. *Let μ and ν be a Borel probability measures. Assume $\mathcal{C}_0$. Suppose that there exists a point $p \in I$ such that φ and ψ are differentiable at p and $\varphi'(p)\psi'(p) \neq 0$. Then, in order that* (1.3.1) *be valid, it is necessary that*

$$\widehat{\mu}_1 + \widehat{\nu}_1 = 1. \tag{1.3.2}$$

PROOF. Using Lemma 1.1.3 twice and the equality of the means $\mathcal{M}_{\varphi,\mu}$ and $\mathcal{M}_{\psi,\nu}$, we get

$$\widehat{\mu}_1 + \widehat{\nu}_1 = \partial_1\mathcal{M}_{\varphi,\mu}(p, p) + \partial_1\mathcal{M}_{\psi,\nu}(p, p) = 1.$$

□

COROLLARY 1.3.2. *Let μ and ν be a Borel probability measures. Assume $\mathcal{C}_2$. Then, in order that invariance equation* (1.3.1) *be valid, it is necessary that*

$$|\varphi'|^{\mu_2}|\psi'|^{\nu_2} = \alpha \tag{1.3.3}$$

for some constant $\alpha > 0$.

PROOF. If $\mathcal{C}_2$ is valid, then differentiating (1.3.1) twice with respect to x, we get that

$$\partial_1^2\mathcal{M}_{\varphi,\mu}(x, x) + \partial_1^2\mathcal{M}_{\psi,\nu}(x, x) = 0.$$

Using the Lemma 1.1.4, it follows that

$$\mu_2\frac{\varphi''}{\varphi'} + \nu_2\frac{\psi''}{\psi'} = 0. \tag{1.3.4}$$

After integration, this yields (1.3.3). □

In the solution of the invariance equation (1.3.1), we consider two subcases.
Subcase 1: $\mu_2\nu_2 = 0$.

THEOREM 1.3.3. *Let μ and ν be a Borel probability measures with $\mu_2\nu_2 = 0$. Assume $\mathcal{C}_2$. Then the invariance equation* (1.3.1) *holds if and only if* *(i) either* $\mu = \delta_\tau$, $\nu = \delta_{1-\tau}$ *for some* $\tau \in [0,1]$ *and* φ, ψ *are arbitrary,*

(ii) or $\mu = \delta_\tau$ *for some* $\tau \in [0,1]$, $\nu_2 \neq 0$, $\widehat{\nu}_1 = 1-\tau$, φ *is arbitrary and there exist constants* $a \neq 0$ *and* b *such that*

$$\psi(x) = ax + b \qquad (x \in I), \tag{1.3.5}$$

(iii) or $\nu = \delta_{1-\tau}$ *for some* $\tau \in [0,1]$, $\mu_2 \neq 0$, $\widehat{\mu}_1 = \tau$, ψ *is arbitrary and there exist constants* $a \neq 0$ *and* b *such that*

$$\varphi(x) = ax + b \qquad (x \in I). \tag{1.3.6}$$

PROOF. (i): If $\mu_2 = 0$ and $\nu_2 = 0$, then $\mu = \delta_\tau$, $\nu = \delta_{1-\tau}$ and

$$\mathcal{M}_{\varphi,\mu}(x,y) = \widehat{\mu}_1 x + (1-\widehat{\mu}_1)y = \tau x + (1-\tau)y \tag{1.3.7}$$

and

$$\mathcal{M}_{\psi,\nu}(x,y) = \widehat{\nu}_1 x + (1-\widehat{\nu}_1)y = (1-\tau)x + \tau y.$$

Hence, the left side of the equation (1.3.1) is equal to $x+y$, which implies that (1.3.1) is satisfied for any functions φ and ψ.

(ii): If $\mu_2 = 0$ and $\nu_2 \neq 0$. Then $\mu = \delta_\tau$ for some $\tau \in [0,1]$ and, by (1.3.3), it follows that ψ' is constant. Therefore, ψ is of the form (1.3.5). Conversely, if (ii) holds, then we have (1.3.7) and

$$\mathcal{M}_{\psi,\nu}(x,y) = \int_0^1 \big(tx + (1-t)y\big) d\nu(t) = \widehat{\nu}_1 x + (1-\widehat{\nu}_1)y = (1-\tau)x + \tau y.$$

Hence, (1.3.1) is satisfied.

(iii): The case $\mu_2 \neq 0$ and $\nu_2 = 0$ is analogous to the case (ii). □

Subcase 2: $\mu_2\nu_2 \neq 0$.

Our first main result offers a necessary condition for the validity of the invariance equation (1.3.1) in terms of two differential equations for the second-order partial derivative $\partial_1^2 \mathcal{M}_{\varphi,\mu}$ of the mean $\mathcal{M}_{\varphi,\mu}$.

THEOREM 1.3.4. *Let* μ *and* ν *be a Borel probability measures with* $\mu_2\nu_2 \neq 0$ *and assume that the invariance equation* (1.3.1) *is satisfied. If* $\mathcal{C}_3$ *holds then the function* $\Phi : I \to \mathbb{R}$ *defined by*

$$\Phi(x) := \partial_1^2 \mathcal{M}_{\varphi,\mu}(x,x) \tag{1.3.8}$$

satisfies the differential equation

$$\left(\frac{3\widehat{\mu}_1\mu_2 + \mu_3}{\mu_2} - \frac{3\widehat{\nu}_1\nu_2 + \nu_3}{\nu_2} \right)\Phi' + \left(\frac{\mu_3}{\mu_2^2} + \frac{\nu_3}{\nu_2^2} \right)\Phi^2 = 0 \tag{1.3.9}$$

and if

$$(\mu_3, \nu_3) \neq \frac{3(\widehat{\mu}_1 - \widehat{\nu}_1)}{\mu_2 + \nu_2}(-\mu_2^2, \nu_2^2), \tag{1.3.10}$$

then the coefficients in equation (1.3.9) *do not vanish simultaneously.*
If, in addition, $\mathcal{C}_4$ *holds then* Φ *also satisfies the differential equation*
(1.3.11)
$$\Big(\frac{6\widehat{\mu}_1^2\mu_2+4\widehat{\mu}_1\mu_3+\mu_4}{\mu_2}-\frac{6\widehat{\nu}_1^2\nu_2+4\widehat{\nu}_1\nu_3+\nu_4}{\nu_2}\Big)\Phi''$$
$$+\Big(\frac{8\widehat{\mu}_1\mu_3+3\mu_4}{\mu_2^2}+\frac{8\widehat{\nu}_1\nu_3+3\nu_4}{\nu_2^2}\Big)\Phi\Phi'+\Big(\frac{\mu_4-3\mu_2^2}{\mu_2^3}-\frac{\nu_4-3\nu_2^2}{\nu_2^3}\Big)\Phi^3=0.$$

PROOF. By Lemma 1.1.4, for $x\in I$, we have that
$$\Phi(x)=\mu_2\frac{\varphi''}{\varphi'}(x)\qquad\text{and}\qquad\partial_1^2\mathcal{M}_{\psi,\nu}(x,x)=\nu_2\frac{\psi''}{\psi'}(x)=:\Psi(x).$$
Using (1.3.1), it follows that

(1.3.12)
$$\Phi(x)+\Psi(x)=\partial_1^2\mathcal{M}_{\varphi,\mu}(x,x)+\partial_1^2\mathcal{M}_{\psi,\nu}(x,x)=0.$$

In order to prove (1.3.9) suppose that $\mathcal{C}_3$ is valid. Differentiating (1.3.1) three times with respect x and putting $y:=x$, we have
$$\partial_1^3\mathcal{M}_{\varphi,\mu}(x,x)+\partial_1^3\mathcal{M}_{\psi,\nu}(x,x)=0.$$
Using the equation (1.1.10)
$$\partial_1^3\mathcal{M}_{\varphi,\mu}(x,x)=\frac{3\widehat{\mu}_1\mu_2+\mu_3}{\mu_2}\Phi'(x)+\frac{\mu_3}{\mu_2^2}\Phi^2(x)$$
and
$$\partial_1^3\mathcal{M}_{\psi,\nu}(x,x)=\frac{3\widehat{\nu}_1\nu_2+\nu_3}{\nu_2}\Psi'(x)+\frac{\nu_3}{\nu_2^2}\Psi^2(x),$$
we get the following differential equation

(1.3.13)
$$\frac{3\widehat{\mu}_1\mu_2+\mu_3}{\mu_2}\Phi'+\frac{\mu_3}{\mu_2^2}\Phi^2+\frac{3\widehat{\nu}_1\nu_2+\nu_3}{\nu_2}\Psi'+\frac{\nu_3}{\nu_2^2}\Psi^2=0.$$
From (1.3.12) it follows that
$$\Phi^2=\Psi^2\quad\text{and}\quad\Psi'=-\Phi'.$$
Applying these connections and reducing (1.3.13), we get the differential equation (1.3.9) for the function Φ.

If the coefficients in the equation (1.3.9) vanish simultaneously, then μ_3,ν_3 is a solution of the following system of linear equations
$$\frac{3\widehat{\mu}_1\mu_2+\mu_3}{\mu_2}-\frac{3\widehat{\nu}_1\nu_2+\nu_3}{\nu_2}=0,\qquad\frac{\mu_3}{\mu_2^2}+\frac{\nu_3}{\nu_2^2}=0.$$
Solving this system of equations we get the following solutions for μ_3 and ν_3
$$\mu_3=-\frac{3(\widehat{\mu}_1-\widehat{\nu}_1)\mu_2^2}{\mu_2+\nu_2},\qquad\nu_3=\frac{3(\widehat{\mu}_1-\widehat{\nu}_1)\nu_2^2}{\mu_2+\nu_2}.$$
Therefore, if (1.3.10) holds, then (1.3.9) cannot be a trivial equation.

If $\mathcal{C}_4$ is valid, then differentiating (1.3.1) four times with respect x and putting $y := x$, we have

$$\partial_1^4\mathcal{M}_{\varphi,\mu}(x,x) + \partial_1^4\mathcal{M}_{\psi,\nu}(x,x) = 0.$$

By Lemma 1.1.4, we have that

$$\begin{aligned}\partial_1^4\mathcal{M}_{\varphi,\mu}(x,x) = {} & \frac{6\widehat{\mu}_1^2\mu_2 + 4\widehat{\mu}_1\mu_3 + \mu_4}{\mu_2}\Phi''(x) + \frac{8\widehat{\mu}_1\mu_3 + 3\mu_4}{\mu_2^2}\Phi(x)\Phi'(x) \\ & + \frac{\mu_4 - 3\mu_2^2}{\mu_2^3}\Phi^3(x)\end{aligned}$$

and

$$\begin{aligned}\partial_1^4\mathcal{M}_{\psi,\nu}(x,x) = {} & \frac{6\widehat{\nu}_1^2\nu_2 + 4\widehat{\nu}_1\nu_3 + \nu_4}{\nu_2}\Psi''(x) + \frac{8\widehat{\nu}_1\nu_3 + 3\nu_4}{\nu_2^2}\Psi(x)\Psi'(x) \\ & + \frac{\nu_4 - 3\nu_2^2}{\nu_2^3}\Psi^3(x),\end{aligned}$$

thus we get the following differential equation

(1.3.14)

$$\begin{aligned}&\frac{6\widehat{\mu}_1^2\mu_2 + 4\widehat{\mu}_1\mu_3 + \mu_4}{\mu_2}\Phi''(x) + \frac{8\widehat{\mu}_1\mu_3 + 3\mu_4}{\mu_2^2}\Phi(x)\Phi'(x) + \frac{\mu_4 - 3\mu_2^2}{\mu_2^3}\Phi^3(x) \\ &+ \frac{6\widehat{\nu}_1^2\nu_2 + 4\widehat{\nu}_1\nu_3 + \nu_4}{\nu_2}\Psi''(x) + \frac{8\widehat{\nu}_1\nu_3 + 3\nu_4}{\nu_2^2}\Psi(x)\Psi'(x) + \frac{\nu_4 - 3\nu_2^2}{\nu_2^3}\Psi^3(x)=0.\end{aligned}$$

From (1.3.12) it follows that

$$\Psi^3 = -\Phi^3, \quad \Psi' = -\Phi' \quad \text{and} \quad \Psi'' = -\Phi''.$$

Applying these connections and reducing (1.3.14), we get the differential equation (1.3.11) for the function Φ. □

By our second main result, under three times continuous differentiability assumptions and certain non-degeneracy conditions on the second and third centralized moments of the two measures, the solutions of the invariance equation (1.3.1) fall into three different classes. The unknown generator functions φ and ψ are either linear, or exponential or power functions.

THEOREM 1.3.5. *Let μ and ν be a Borel probability measures with $\mu_2\nu_2 \neq 0$ and satisfying* (1.3.10). *Assume also $\mathcal{C}_3$. Then the invariance equation* (1.3.1) *holds if and only if $\widehat{\mu}_1 + \widehat{\nu}_1 = 1$ and*

(i) either there exist real constants a, b, c, d with $ac \neq 0$ such that

(1.3.15) $$\varphi(x) = ax + b \quad \textit{and} \quad \psi(x) = cx + d \quad (x \in I);$$

(ii) or there exist real constants a, b, c, d, p, q with $ac \neq 0$, $pq < 0$ such that

(1.3.16) $$\varphi(x) = ae^{px} + b \quad \textit{and} \quad \psi(x) = ce^{qx} + d \quad (x \in I)$$

and, for $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} \big(\mu_{i+1}\nu_{n-i} + \mu_i \nu_{n+1-i}\big) = 0; \tag{1.3.17}$$

(iii) or there exist real constants a, b, c, d, p, q *with* $ac \neq 0$, $(p-1)(q-1) < 0$, *and* $x_0 \notin I$ *such that, for* $x \in I$,

$$\begin{aligned} \varphi(x) &= \begin{cases} a|x-x_0|^p + b, & \text{if } p \neq 0 \\ a \ln|x-x_0| + b, & \text{if } p = 0, \end{cases} \\ \psi(x) &= \begin{cases} c|x-x_0|^q + d, & \text{if } q \neq 0 \\ c \ln|x-x_0| + d, & \text{if } q = 0 \end{cases} \end{aligned} \tag{1.3.18}$$

and, with the notation

$$F_{p,\mu}(z) := \begin{cases} \Big(\int_0^1 (1+tz)^p d\mu(t) \Big)^{\frac{1}{p}}, & \text{if } p \neq 0 \\ \exp\Big(\int_0^1 \ln(1+tz) d\mu(t) \Big), & \text{if } p = 0 \end{cases} \qquad (z > -1), \tag{1.3.19}$$

the identity

$$F_{p,\mu}(z) + F_{q,\nu}(z) = 2 + z \qquad (z > -1) \tag{1.3.20}$$

holds.

PROOF. First we show that the equation (1.3.1) implies that one of the conditions (i), (ii), (iii) must be valid. By the assumptions on the moments of μ and ν, and Theorem 1.3.4, the function Φ introduced in (1.3.8), satisfies (1.3.9) which is non trivial. Then the differential equation (1.3.9) is solvable. This equation is the form

$$\alpha\Phi' + \beta\Phi^2 = 0 \tag{1.3.21}$$

where $(\alpha, \beta) \neq (0,0)$. To examine this differential equation, we distinguish three cases. In all of the three cases we can apply the Theorem 11 by [**54**].

Case 1: $\Phi = 0$. Then $\varphi'' = 0$, whence $\varphi' = a$, and $\psi' = c$ for some nonzero constants a and c. Therefore, in this case, statement (i) of the theorem must be valid.

If (i) holds and $\widehat{\mu}_1 + \widehat{\nu}_1 = 1$, then for all $x, y \in I$

$$\mathcal{M}_{\varphi,\mu}(x,y) = \widehat{\mu}_1 x + (1-\widehat{\mu}_1)y \qquad \text{and} \qquad \mathcal{M}_{\psi,\nu}(x,y) = \widehat{\nu}_1 x + (1-\widehat{\nu}_1)y.$$

Then after a short calculation it follows, that

$$\widehat{\mu}_1 x + (1-\widehat{\mu}_1)y + \widehat{\nu}_1 x + (1-\widehat{\nu}_1)y = x + y$$

that is, the equation (1.3.1) is true.

Case 2: $\beta = 0$ and Φ is not identically zero. Then (1.3.21) says that $\Phi' = 0$. Thus, there exists a nonzero constant p such that $\Phi = \mu_2 p$. Using the definition of Φ, we get that $\varphi'' = p\varphi'$. In this case the general solution is of the form $\varphi(x) = ae^{px} + b$ for some constant $a \neq 0$ and b. By Corollary 1.3.2, we have that

$$\psi'(x) = \gamma|\varphi'(x)|^{-\frac{\mu_2}{\nu_2}} = \gamma|ap|^{-\frac{\mu_2}{\nu_2}} e^{-\frac{p\mu_2}{\nu_2}x} = ce^{qx}$$

where $q := -\frac{p\mu_2}{\nu_2}$. This implies that ψ is also of the stated form and $pq < 0$.

If (ii) holds then (1.3.1) can be written in the form

$$\frac{1}{p}\ln\Big(\int_0^1 e^{p(tx+(1-t)y)}d\mu(t)\Big) + \frac{1}{q}\ln\Big(\int_0^1 e^{q(tx+(1-t)y)}d\nu(t)\Big) = x + y.$$

We have that

$$\begin{aligned}\frac{1}{p}\ln\left(\int_0^1 e^{p(tx+(1-t)y)}d\mu(t)\right) =& x + y - \frac{1}{q}\ln\left(\int_0^1 e^{q(tx+(1-t)y)}d\nu(t)\right)\\ =&\frac{1}{q}\ln\left(e^{-qx-qy}\int_0^1 e^{q((1-t)x+ty)}d\nu(t)\right)\\ =&-\frac{1}{q}\ln\left(\int_0^1 e^{-q((1-t)x+ty)}d\nu(t)\right)\\ =&-\frac{1}{q}\ln\int_0^1 e^{-q(sx+(1-s)y)}d\widetilde{\nu}(s).\end{aligned}$$

Thus we get the following equality

$$\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\widetilde{\psi},\widetilde{\nu}}, \tag{1.3.22}$$

where $\varphi(x) = ae^{px} + b$ and $\widetilde{\psi}(x) = ce^{-qx} + d$ and $\widetilde{\nu}$ denotes the reflection of the measure ν with respect to the point $\frac{1}{2}$. By [**54**], (1.3.22) holds if and only if, for all $n \in \mathbb{N}$,

$$0 = \sum_{i=0}^{n}\binom{n}{i}\big(\mu_{i+1}\widetilde{\nu}_{n-i} - \mu_i\widetilde{\nu}_{n+1-i}\big)\frac{\varphi^{(i+1)}}{\varphi'}\cdot\frac{\widetilde{\psi}^{(n+1-i)}}{\widetilde{\psi}'}.$$

In other words,

$$0 = \sum_{i=0}^{n}\binom{n}{i}p^i(-q)^{n-i}\big(\mu_{i+1}\widetilde{\nu}_{n-i} - \mu_i\widetilde{\nu}_{n+1-i}\big). \tag{1.3.23}$$

Then, by Lemma 1.1.2, $\widehat{\widehat{\nu}}_1 = 1 - \widehat{\nu}_1$ and, for $k \in \mathbb{N}$, we have $\widetilde{\nu}_k = (-1)^k \nu_k$. Using these equalities, condition (1.3.23) can be rewritten as

$$\begin{aligned}
0 &= \sum_{i=0}^{n} \binom{n}{i} p^i (-q)^{n-i} \big(\mu_{i+1}\widetilde{\nu}_{n-i} - \mu_i \widetilde{\nu}_{n+1-i}\big) \\
&= \sum_{i=0}^{n} \binom{n}{i} p^i (-1)^{n-i} q^{n-i} \big(\mu_{i+1}(-1)^{n-i}\nu_{n-i} - \mu_i (-1)^{n+1-i}\nu_{n+1-i}\big) \\
&= \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} \big((-1)^{n-i}\big)^2 \big(\mu_{i+1}\nu_{n-i} + \mu_i \nu_{n+1-i}\big) \\
&= \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} \big(\mu_{i+1}\nu_{n-i} + \mu_i \nu_{n+1-i}\big).
\end{aligned}$$

This proves that (1.3.17) is necessary and sufficient for the validity of the invariance equation.

Case 3: $\beta \neq 0$ and Φ is not identically zero. Denote by L a maximal open subinterval of I where Φ does not vanish. Then, with the notation $p := 1 - \alpha/(\mu_2\beta)$, (1.3.21) can be rewritten as

$$\frac{\Phi'(x)}{\Phi^2(x)} = \frac{1}{\mu_2(p-1)} \qquad (x \in L).$$

Integrating this equality, it follows, for some x_0, that

$$\frac{1}{\Phi(x)} = \frac{x - x_0}{\mu_2(p-1)} \qquad (x \in L). \tag{1.3.24}$$

Hence x_0 cannot be in L. If L were a proper subinterval of I, then one of the endpoints of L, say α, would be an element of I. By taking the limit $x \to \alpha$ in the above equation, it follows that Φ has a finite nonzero limit at α. By continuity, this yields that $\Phi(\alpha) = \frac{\mu_2(p-1)}{\alpha - x_0} \neq 0$. Showing that L is not maximal. The contradiction so obtained proves that $L = I$.

Applying (1.3.24) and the definition (1.3.8) of the function Φ, we get

$$\frac{\varphi''(x)}{\varphi'(x)} = \frac{\Phi(x)}{\mu_2} = \frac{p-1}{x - x_0} \qquad (x \in I).$$

Integrating this equation, it results that

$$\varphi'(x) = \begin{cases} ap|x - x_0|^{p-1}, & \text{if } p \neq 0 \\ a|x - x_0|^{-1}, & \text{if } p = 0 \end{cases}$$

for some constant a. After integration this yields that φ is of the form (1.3.18). Using (1.3.3), we get that Ψ is also of the form (1.3.18) with $q := -(p-1)\frac{\mu_2}{\nu_2}$. Obviously, $(p-1)(q-1) = -(\mu_2/\nu_2)(p-1)^2 < 0$.

If (iii) holds and $pq \neq 0$ then (1.3.1) holds for all $x, y \in I$ if and only if

$$\begin{aligned}(1.3.25)\qquad &\left(\int_0^1 |tx+(1-t)y-x_0|^p d\mu(t)\right)^{\frac{1}{p}} + x_0 \\ &+ \left(\int_0^1 |tx+(1-t)y-x_0|^q d\nu(t)\right)^{\frac{1}{q}} + x_0 = x+y.\end{aligned}$$

Let $x - x_0 =: u$, $y - x_0 =: v$ and $x_0 \leq \inf I$ (the case $x_0 \geq \sup I$ is analogous). Then $u, v \in I - x_0 \subseteqq \mathbb{R}_+$ and (1.3.25) is equivalent to

$$\begin{aligned}\left(\int_0^1 (tu+(1-t)v)^p d\mu(t)\right)^{\frac{1}{p}} + \left(\int_0^1 (tu+(1-t)v)^q d\nu(t)\right)^{\frac{1}{q}} \\ = u+v = (u-v)+2v.\end{aligned}$$

With the notation $\frac{u-v}{v} = z$ we get that (1.3.25) holds for all $x, y \in I$ if and only if (1.3.26)

$$\left(\int_0^1 (1+tz)^p d\mu(t)\right)^{\frac{1}{p}} + \left(\int_0^1 (1+tz)^q d\nu(t)\right)^{\frac{1}{q}} = 2+z \qquad (z \in J),$$

where $J = \{\frac{u}{v} - 1 | u, v \in I - x_0\}$, which is an open neighborhood of zero contained in the interval $]-1, \infty[$. The left hand side of (1.3.26) is an analytic function of z, therefore (1.3.26) is valid for all $z \in J$ if and only if it holds for all $z \in]-1, \infty[$.

The proof of (1.3.26) in the case $pq = 0$ is similar. □

As we see from Theorem 1.3.5, the solutions of the invariance equation (1.3.1) may have three different forms. They are either linear, or exponential, or power functions. The following result formulates necessary conditions for the existence of exponential solutions of the invariance equation (1.3.1).

PROPOSITION 1.3.6. Let μ and ν be a Borel probability measures with $\mu_2\nu_2 \neq 0$. If $pq \neq 0$ and there exists a solution of the invariance equation (1.3.1) of the form (ii) in Theorem 1.3.5, then

$$(1.3.27)\qquad \frac{p}{q} = -\frac{\nu_2}{\mu_2}$$

and the following condition must be valid

$$(1.3.28)\qquad \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{\mu_{i+1}\nu_{n-i} + \mu_i\nu_{n+1-i}}{\mu_2^i \nu_2^{n-i}} = 0 \qquad (n \in \mathbb{N}).$$

In particular, for $n = 2, 3, 4$, condition (1.3.28) can be written in the form

$$(1.3.29)\qquad \frac{\mu_3}{\mu_2^2} + \frac{\nu_3}{\nu_2^2} = 0,$$

$$(1.3.30)\qquad \left(\frac{\mu_4}{\mu_2^3} - \frac{3}{\mu_2}\right) - \left(\frac{\nu_4}{\nu_2^3} - \frac{3}{\nu_2}\right) = 0,$$

(1.3.31)
$$\left(\frac{\mu_5}{\mu_2^4}-10\frac{\mu_3}{\mu_2^3}\right)+\left(\frac{\nu_5}{\nu_2^4}-10\frac{\nu_3}{\nu_2^3}\right)=0.$$

PROOF. If there exists a solution of the invariance equation (1.3.1) of the form (ii) in Theorem 1.3.5, then equation (1.3.17) holds for all $n\in\mathbb{N}$. In the case $n=1$ this equation is of the form

$$q(\mu_1\nu_1+\mu_0\nu_2)+p(\mu_2\nu_0+\mu_1\nu_1)=0.$$

This equation simplifies to $\nu_2 q+\mu_2 p=0$, whence we get (1.3.27). Using (1.3.27), the equation (1.3.17) divided by $(q\nu_2)^n$ reduces to (1.3.28). In the particular cases $n=2$ and $n=3$, from the equation (1.3.28) we obtain (1.3.29) and (1.3.30), respectively. If $n=4$, then (1.3.28) is of the form

(1.3.32)
$$\frac{\nu_5}{\nu_2^4}-4\frac{\nu_3}{\nu_2^3}+\frac{6}{\mu_2\nu_2}\left(\frac{\mu_3}{\mu_2}+\frac{\nu_3}{\nu_2}\right)-4\frac{\mu_3}{\mu_2^3}+\frac{\mu_5}{\mu_2^4}=0.$$

Applying (1.3.29), we have

$$\frac{6}{\mu_2\nu_2}\left(\frac{\mu_3}{\mu_2}+\frac{\nu_3}{\nu_2}\right)=\frac{6}{\mu_2\nu_2}\left(\frac{-\nu_3\mu_2}{\nu_2^2}+\frac{-\mu_3\nu_2}{\mu_2^2}\right)=-6\frac{\nu_3}{\nu_2^3}-6\frac{\mu_3}{\mu_2^3},$$

whence (1.3.32) simplifies to (1.3.31). □

The next result formulates necessary conditions for the existence of power function solutions of the invariance equation (1.3.1).

PROPOSITION 1.3.7. Let μ and ν be a Borel probability measures with $\mu_2\nu_2\neq 0$ and (1.3.10). If $(p-1)(q-1)<0$ and there exists a solution of the invariance equation (1.3.1) of the form (iii) in Theorem 1.3.5, then $\mu_3\nu_2^2+\nu_3\mu_2^2\neq 0$,

(1.3.33)
$$\begin{aligned} p&=\frac{2\mu_3\nu_2^2+\nu_3\mu_2(\mu_2-\nu_2)+3(\widehat{\mu}_1-\widehat{\nu}_1)\mu_2\nu_2^2}{\mu_3\nu_2^2+\nu_3\mu_2^2},\\ q&=\frac{2\nu_3\mu_2^2+\mu_3\nu_2(\nu_2-\mu_2)+3(\widehat{\nu}_1-\widehat{\mu}_1)\mu_2^2\nu_2}{\mu_3\nu_2^2+\nu_3\mu_2^2}\end{aligned}$$

and the following condition must hold

(1.3.34)
$$\begin{aligned} &27\mu_2^3\nu_2^3(\widehat{\mu}_1-\widehat{\nu}_1)^2(\mu_2-\nu_2)+6\mu_2^2\nu_2^2(\nu_2-\mu_2)\mu_3\nu_3\\ &+18\mu_2^2\nu_2^2(\mu_2-\nu_2)(\widehat{\mu}_1-\widehat{\nu}_1)(\mu_3\nu_2-\nu_3\mu_2)\\ &+\Big(-12\nu_2^2(\widehat{\mu}_1-\widehat{\nu}_1)^2+3\mu_2\nu_2(\mu_2-\nu_2)+8(\widehat{\mu}_1-\widehat{\nu}_1)(\mu_2+\nu_2)\nu_3\Big)\nu_2^2\mu_3^2\\ &-\Big(-12\mu_2^2(\widehat{\mu}_1-\widehat{\nu}_1)^2+3\mu_2\nu_2(\nu_2-\mu_2)+8(\widehat{\nu}_1-\widehat{\mu}_1)(\mu_2+\nu_2)\mu_3\Big)\mu_2^2\nu_3^2\\ &+\bigg(\Big(3\nu_2^2(\widehat{\nu}_1-\widehat{\mu}_1)+\nu_3(\mu_2+\nu_2)\Big)\Big(\nu_2^2(-3\widehat{\mu}_1\mu_2+3\mu_2\widehat{\nu}_1+\mu_3)+\nu_3(\mu_2\nu_2+2\mu_2^2)\Big)\bigg)\mu_4\\ &-\bigg(\Big(3\mu_2^2(\widehat{\mu}_1-\widehat{\nu}_1)+\mu_3(\mu_2+\nu_2)\Big)\Big(\mu_2^2(3\widehat{\mu}_1\nu_2-3\widehat{\nu}_1\nu_2+\nu_3)+\mu_3(\mu_2\nu_2+2\nu_2^2)\Big)\bigg)\nu_4=0.\end{aligned}$$

PROOF. If φ and ψ are of the form (1.3.18), then we have

$$\Phi(x) = \mu_2 \frac{\varphi''(x)}{\varphi'(x)} = \mu_2 \frac{p-1}{x-x_0}, \qquad \Psi(x) = \nu_2 \frac{\psi''(x)}{\psi'(x)} = \nu_2 \frac{q-1}{x-x_0}. \tag{1.3.35}$$

By Corollary 1.3.2, we have (1.3.3), which is equivalent to (1.3.4). This equation then yields

$$(p-1)\mu_2 + (q-1)\nu_2 = 0. \tag{1.3.36}$$

From (1.3.35) we obtain

$$\Phi'(x) = -\mu_2 \frac{p-1}{(x-x_0)^2}, \qquad \Phi''(x) = 2\mu_2 \frac{p-1}{(x-x_0)^3}. \tag{1.3.37}$$

Therefore, equation (1.3.9) multiplied by the expression $\frac{(x-x_0)^2}{\mu_2(p-1)}$ reduces to

$$-\left(\frac{3\widehat{\mu}_1\mu_2 + \mu_3}{\mu_2} - \frac{3\widehat{\nu}_1\nu_2 + \nu_3}{\nu_2}\right) + \left(\frac{\mu_3}{\mu_2^2} + \frac{\nu_3}{\nu_2^2}\right)\mu_2(p-1) = 0. \tag{1.3.38}$$

Here the coefficient of $(p-1)$ cannot be zero, otherwise both coefficients in (1.3.9) are zero which contradicts condition (1.3.10). Solving (1.3.38) for p, we get the first formula in (1.3.33). Using (1.3.36), the formula for q also follows.

Using (1.3.37) the equation (1.3.11) multiplied by the expression $\frac{(x-x_0)^3}{\mu_2(p-1)}$ simplifies to

$$\begin{aligned} &2\Big(\frac{6\widehat{\mu}_1^2\mu_2 + 4\widehat{\mu}_1\mu_3 + \mu_4}{\mu_2} - \frac{6\widehat{\nu}_1^2\nu_2 + 4\widehat{\nu}_1\nu_3 + \nu_4}{\nu_2}\Big) \\ &- \Big(\frac{8\widehat{\mu}_1\mu_3 + 3\mu_4}{\mu_2^2} + \frac{8\widehat{\nu}_1\widehat{\nu}_3 + 3\nu_4}{\nu_2^2}\Big)\mu_2(p-1) \\ &+ \Big(\frac{\mu_4 - 3\mu_2^2}{\mu_2^3} - \frac{\nu_4 - 3\nu_2^2}{\nu_2^3}\Big)\mu_2^2(p-1)^2 = 0. \end{aligned} \tag{1.3.39}$$

Substituting the values of p in the equation (1.3.39) we obtain the condition (1.3.34). $\square$

THEOREM 1.3.8. *Let μ, ν be a Borel probability measures with $\widehat{\mu}_1 + \widehat{\nu}_1 = 1$, $\mu_2 = \nu_2 \neq 0$, $\mu_3 = -\nu_3$, such that*

$$\mu_3 \neq 3\Big(\frac{1}{2} - \widehat{\mu}_1\Big)\mu_2. \tag{1.3.40}$$

Assume also $\mathcal{C}_3$. Then the invariance equation (1.3.1) *is satisfied if and only if*

(i) either there exist real constants a, b, c, d with $ac \neq 0$ such that

$$\varphi(x) = ax + b \qquad \text{and} \qquad \psi(x) = cx + d \qquad (x \in I); \tag{1.3.41}$$

(ii) or there exist real constants a, b, c, d, p with $acp \neq 0$, such that

$$\varphi(x) = ae^{px} + b \qquad \text{and} \qquad \psi(x) = ce^{-px} + d \qquad (x \in I) \tag{1.3.42}$$

and ν is the reflection of μ with respect to the point $1/2$.

PROOF. Assume first that (1.3.1) holds. Then, by Theorem 1.3.4 the function Φ defined by (1.3.8) satisfies differential equations (1.3.9). In view of the conditions of this theorem on the moments of the measures μ and ν, this differential equation simplifies to

$$\Big(3\Big(2\widehat{\mu}_1 - 1\Big) + 2\frac{\mu_3}{\mu_2}\Big)\Phi' = 0$$

which, in view of condition (1.3.40), yields that $\Phi' = 0$, i.e., Φ is a constant.

If Φ is identically zero then, following the argument of Case 1 of the proof of Theorem 1.3.5, we obtain that φ and ψ are of the form (1.3.41). Conversely, if (1.3.41) holds then one can easily see that (1.3.1) is satisfied.

If Φ is a nonzero constant, then we can follow the argument of Case 2 of the proof of Theorem 1.3.5 and obtain that φ and ψ are of the form (1.3.16) and condition (1.3.17) holds. Using Proposition 1.3.6, form formula (1.3.27), we get that

$$\frac{p}{q} = -\frac{\nu_2}{\mu_2} = -1.$$

Now, (1.3.28) reduces to the following condition:

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\big(\mu_{i+1}\nu_{n-i} + \mu_i\nu_{n+1-i}\big) = 0 \qquad (n \in \mathbb{N}). \tag{1.3.43}$$

We show that these equalities imply that ν is the reflection of μ with respect to the point $\frac{1}{2}$. In view of Lemma 1.1.2, it suffices to show that, for all $k \in \mathbb{N}$,

$$\nu_k = (-1)^k\mu_k. \tag{1.3.44}$$

These equalities hold true for $k = 1, 2, 3$ by the assumptions of this theorem. Assume that (1.3.44) holds for $k \in \{1, \dots, n\}$, where $n \geq 3$. Using (1.3.43) and (1.3.44) for $k \in \{1, \dots, n\}$, we obtain that

$$\begin{aligned}
0 &= \sum_{i=0}^{n}(-1)^i\binom{n}{i}\big(\mu_{i+1}\nu_{n-i} + \mu_i\nu_{n+1-i}\big)\\
&\quad + \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{n-i}\big(\mu_{n+1-i}\nu_i + \mu_{n-i}\nu_{i+1}\big)\\
&= \sum_{i=0}^{n}(-1)^i\binom{n}{i}\Big(\big(\mu_{i+1}\nu_{n-i} + \mu_i\nu_{n+1-i}\big) + (-1)^n\big(\mu_{n+1-i}\nu_i + \mu_{n-i}\nu_{i+1}\big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n-1} (-1)^i \binom{n}{i} \Big((-1)^{n-i} \big(\mu_{i+1}\mu_{n-i} - \mu_i \mu_{n+1-i}\big) \\
&\quad + (-1)^{n+i} \big(\mu_{n+1-i}\mu_i - \mu_{n-i}\mu_{i+1}\big)\Big) \\
&\quad + 2\Big(\big(\mu_1\nu_n + \mu_0\nu_{n+1}\big) + (-1)^n \big(\mu_{n+1}\nu_0 + \mu_n \nu_1\big)\Big) \\
&= 2\big(\nu_{n+1} + (-1)^n \mu_{n+1}\big).
\end{aligned}
$$

Therefore, (1.3.44) holds also for $k = n+1$.

Conversely, if (ii) holds then

$$
\begin{aligned}
&\mathcal{M}_{\varphi,\mu}(x,y) + \mathcal{M}_{\psi,\nu}(x,y) \\
&\quad = \frac{1}{p} \ln \Big(\int_0^1 e^{p(tx+(1-t)y)} d\mu(t) \Big) - \frac{1}{p} \ln \Big(\int_0^1 e^{-p(tx+(1-t)y)} d\nu(t) \Big) \\
&\quad = \frac{1}{p} \ln \Big(\int_0^1 e^{p(tx+(1-t)y)} d\mu(t) \Big) - \frac{1}{p} \ln \Big(\int_0^1 e^{-p((1-t)x+ty)} d\mu(t) \Big) \\
&\quad = \frac{1}{p} \ln \Big(\int_0^1 e^{p(tx+(1-t)y)} d\mu(t) \Big) - \frac{1}{p} \ln \Big(e^{-px-py} \int_0^1 e^{p(tx+(1-t)y)} d\mu(t) \Big) \\
&\quad = \frac{1}{p} \ln \left(\frac{\int_0^1 e^{p(tx+(1-t)y)} d\mu(t)}{e^{-px-py} \int_0^1 e^{p(tx+(1-t)y)} d\mu(t)} \right) = x + y,
\end{aligned}
$$

which proves that (1.3.1) is satisfied. □

THEOREM 1.3.9. *Let μ, ν be a Borel probability measures with $\widehat{\mu}_1 = \widehat{\nu}_1 = \frac{1}{2}$, $\mu_2 = \nu_2 \neq 0$, $\mu_3 = -\nu_3$, $\mu_4 = \nu_4$. Assume also $\mathcal{C}_4$. Then the invariance equation* (1.3.1) *is satisfied if and only if one of the alternatives of Theorem 1.3.8 holds.*

PROOF. Assume first that (1.3.1) holds. Then, by Theorem 1.3.4 the function Φ defined by (1.3.8) satisfies differential equations (1.3.9) and (1.3.11). In view of the conditions of this theorem on the moments of the measures μ and ν, these two differential equations simplify to

$$2\frac{\mu_3}{\mu_2}\Phi' = 0 \tag{1.3.45}$$

and

$$4\frac{\mu_3}{\mu_2}\Phi'' + 6\frac{\mu_4}{\mu_2^2}\Phi'\Phi = 0, \tag{1.3.46}$$

respectively. If $\mu_3 \neq 0$ then, by (1.3.45), we have that $\Phi' = 0$, hence Φ is a constant. If $\mu_3 = 0$ then, by (1.3.46), the equality $\Phi'\Phi = 0$ follows. Hence $\big(\Phi^2\big)' = 0$, which implies that Φ^2 is a constant. By the continuity of Φ, this yields that Φ is also a constant.

Now, following the argument of the proof of the previous theorem, the result follows. □

In the next result we consider the particular case of Theorem 1.3.9 when $\mu = \nu$ is a symmetric measure.

COROLLARY 1.3.10. *Let μ be a Borel probability measure with $\mu_2 \neq 0$ which is symmetric with respect to the point $1/2$. Assume also $\mathcal{C}_4$. Then the invariance equation*

$$\mathcal{M}_{\varphi,\mu}(x,y) + \mathcal{M}_{\psi,\mu}(x,y) = x + y \qquad (x, y \in I) \tag{1.3.47}$$

is satisfied if and only if

(i) either there exist real constants a, b, c, d with $ac \neq 0$ such that

$$\varphi(x) = ax + b \qquad \textit{and} \qquad \psi(x) = cx + d \qquad (x \in I);$$

(ii) or there exist real constants a, b, c, d, p with $acp \neq 0$, such that

$$\varphi(x) = ae^{px} + b \qquad \textit{and} \qquad \psi(x) = ce^{-px} + d \qquad (x \in I).$$

1.3.1. Examples and Applications. In the subsequent examples we demonstrate how some known results of the literature follow from ours.

EXAMPLE 4. Consider the functional equation

$$\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right) + \psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right) = x + y \qquad (x, y \in I), \tag{1.3.48}$$

where $\varphi, \psi : I \to \mathbb{R}$ are continuous strictly monotone functions.

If the measures μ and ν are choosen as,

$$\mu = \nu = \frac{\delta_0 + \delta_1}{2},$$

then (1.3.1) simplifies to (1.3.48). Observe that $\mu = \nu$ is a symmetric measure, furthermore, $\mu_2 = \nu_2 = \frac{1}{4} \neq 0$. Therefore, we can apply Corollary 1.3.10. If $\mathcal{C}_4$ is assumed, then we get that one of the alternatives (i), (ii) of Corollary 1.3.10 holds and we deduce — under four times continuous differentiability assumptions — the result formulated in the theorem by Daróczy–Páles. This statement was first proved by Sutô [**71**],[**72**] assuming analyticity and by Matkowski [**57**] who supposed twice continuous differentiability. After some preliminary regularity improving steps (cf. [**22**],[**27**]), the main goal of the paper [**28**] was to show that the same conclusion can be obtained without any superflouos differentiability assumptions.

EXAMPLE 5. Consider the functional equation

$$\varphi^{-1}\big(\lambda\varphi(x) + (1-\lambda)\varphi(y)\big) + \psi^{-1}\big((1-\lambda)\psi(x) + \lambda\psi(y)\big) = x + y, \tag{1.3.49}$$

for all $x, y \in I$, where $\varphi, \psi : I \to \mathbb{R}$ are continuous strictly monotone functions and $\lambda \in [0,1] \setminus \left\{0, \frac{1}{2}, 1\right\}$.

Defining the measures μ and ν by

$$\mu = (1-\lambda)\delta_0 + \lambda\delta_1 \qquad \text{and} \qquad \nu = \lambda\delta_0 + (1-\lambda)\delta_1,$$

we see that (1.3.49) is a particular case of (1.3.1). Then, $\nu = \widetilde{\mu}$, furthermore, $\widehat{\mu}_1 = \lambda$, $\widehat{\nu}_1 = 1-\lambda$ and

$$\mu_2 = \nu_2 = \lambda(1-\lambda) \neq 0, \qquad \mu_3 = -\nu_3 = \lambda(1-\lambda)(1-2\lambda).$$

Observe that now (1.3.40) is satisfied because $\lambda \notin \{0, \frac{1}{2}, 1\}$. Therefore, we can apply Theorem 1.3.8. If $\mathcal{C}_3$ is assumed, then, we obtain that one of the alternatives (i), (ii) of Theorem 1.3.8 holds. The result so obtained has been discovered by Jarczyk and Matkowski [**44**] and has recently been proved without any continuous differentiability assumptions by Jarczyk [**43**].

EXAMPLE 6. Consider the functional equation

$$\varphi^{-1}\left(\frac{1}{y-x}\int_x^y \varphi(t)dt\right) + \psi^{-1}\left(\frac{1}{y-x}\int_x^y \psi(t)dt\right) = x+y, \tag{1.3.50}$$

for all $x, y \in I$, $x \neq y$, where $\varphi, \psi : I \to \mathbb{R}$ are continuous strictly monotone functions.

With an obvious substitution, (1.3.50) can be rewritten as

$$\varphi^{-1}\left(\int_0^1 \varphi\big(tx+(1-t)y\big)dt\right) + \psi^{-1}\left(\int_0^1 \psi\big(tx+(1-t)y\big)dt\right) = x+y, \tag{1.3.51}$$

for all $x, y \in I$, $x \neq y$. If μ and ν are equal to the Lebesgue measure, then (1.3.51) becomes a particular case of the invariance equation (1.3.1). Obviously, $\mu = \nu$ is a symmetric measure, furthermore, $\mu_2 = \nu_2 = \frac{1}{12} \neq 0$. Therefore, we can apply the Corollary 1.3.10. If $\mathcal{C}_4$ is assumed, then we obtain that one of the alternatives (i), (ii) of Corollary 1.3.10 holds and we deduce the result of Matkowski [**61**] (with stronger regularity assumptions).

EXAMPLE 7. Consider the functional equation

$$\varphi^{-1}\left(\frac{2\varphi(x)+\varphi(y)}{3}\right) + \psi^{-1}\left(\frac{\psi(x)+4\psi\big(\frac{x+y}{2}\big)+4\psi(y)}{9}\right) = x+y, \tag{1.3.52}$$

where $\varphi, \psi : I \to \mathbb{R}$ are continuous strictly monotone functions.

The measures μ and ν are given by

$$\mu = \frac{\delta_0 + 2\delta_1}{3} \qquad \text{and} \qquad \nu = \frac{4\delta_0 + 4\delta_{1/2} + \delta_1}{9}.$$

Then, $\widehat{\mu}_1 = \frac{2}{3}$ and $\widehat{\nu}_1 = \frac{1}{3}$ and, for $k \in \mathbb{N}$, we have

$$\mu_k = \int_0^1 \left(t - \frac{2}{3}\right)^k d\mu(t) = \frac{(-2)^k + 2}{3^{k+1}}$$

and

$$\nu_k = \int_0^1 \left(t - \frac{2}{3}\right)^k d\nu(t) = \frac{(-4)^k + 4(-1)^k + 4 \cdot 2^k}{9 \cdot 6^k}.$$

Hence

$$\mu_1 = 0, \qquad \mu_2 = \frac{2}{9}, \qquad \mu_3 = -\frac{2}{27}, \qquad \mu_4 = \frac{2}{27}, \quad \dots ,$$
$$\nu_1 = 0, \qquad \nu_2 = \frac{1}{9}, \qquad \nu_3 = -\frac{1}{54}, \qquad \nu_4 = \frac{1}{36}, \quad \dots .$$

We can apply the Theorem 1.3.5. If $\mathcal{C}_3$ is assumed, then, one of the alternatives (i), (ii), and (iii) of Theorem 1.3.5 holds.

If (i) holds then φ and ψ are nonconstant linear functions and, indeed, they are solutions of (1.3.52).

If (ii) holds, then there exist constants a, b, c, d, p, q with $acpq(p-q) \neq 0$ such that (1.3.16) and (1.3.17) are satisfied for all $n \in \mathbb{N}$. In the case $n = 1$, (1.3.17) simplifies to

$$q(\mu_1\nu_1 + \mu_0\nu_2) + p(\mu_2\nu_0 + \mu_1\nu_1) = 0,$$

which results $q = -2p$. Instead of showing that (1.3.17) holds for all $n \geq 2$, we prove that the functions φ and ψ given by (1.3.16) are solutions of (1.2.64). Indeed,

$$\begin{aligned}
&\varphi^{-1}\left(\frac{2\varphi(x) + \varphi(y)}{3}\right) + \psi^{-1}\left(\frac{\psi(x) + 4\psi\left(\frac{x+y}{2}\right) + 4\psi(y)}{9}\right) \\
&= \frac{1}{p} \ln\left(\frac{2e^{px} + e^{py}}{3}\right) + \frac{1}{q} \ln\left(\frac{e^{qx} + 4e^{q\frac{x+y}{2}} + 4e^{qy}}{9}\right) \\
&= \frac{1}{p} \ln\left(\frac{2e^{px} + e^{py}}{3}\right) - \frac{1}{2p} \ln\left(\frac{e^{-2px} + 4e^{-p(x+y)} + 4e^{-2py}}{9}\right) \\
&= \ln\left(\frac{2e^{px} + e^{py}}{3}\right)^{\frac{1}{p}} - \ln\left(\left(\frac{e^{-px} + 2e^{-py}}{3}\right)^2\right)^{\frac{1}{2p}} \\
&= \ln\left(\frac{2e^{px} + e^{py}}{e^{-px} + 2e^{-py}}\right)^{\frac{1}{p}} = \ln\left(e^{px+py}\frac{2e^{px} + e^{py}}{e^{py} + 2e^{px}}\right)^{\frac{1}{p}} = x + y.
\end{aligned}$$

If (iii) holds, then there exist real constants a, b, c, d, p, q with $ac \neq 0$, $(p-1)(q-1) < 0$, and $x_0 \notin I$ such that (1.3.18) and (1.3.20) hold. By Proposition 1.3.7, in order that solutions of (1.3.52) of this form exist, it is necessary that (1.3.34) be valid. Substituting the values of the moments of μ and ν into (1.3.34), we obtain a contradiction, which shows that there are no solutions of (1.3.52) of the form (iii).

CHAPTER 2

On the Lipschitz perturbation of monotone functions

2.1. An Auxiliary Result

Denote by $\mathbb{R}^*$ the set $\mathbb{R} \cup \{+\infty\}$. Let Ω be a non-void set and $\emptyset \in \mathcal{P} \subset 2^\Omega$, where 2^Ω denotes the power set of Ω. A function $\mu : \mathcal{P} \to \mathbb{R}^*$ with $\mu(\emptyset) = 0$ will be called *set function*.

We define the relation $\preceq$ among two set functions μ, ν as follows:

$$\mu \preceq \nu \quad \text{if and only if} \quad \sum_{i=1}^{n} \mu(A_i) \leq \sum_{j=1}^{m} \nu(B_j)$$

for all systems of sets $A_1, \ldots, A_n, B_1, \ldots, B_m \in \mathcal{P}$ with $\sum_{i=1}^{n} 1_{A_i} = \sum_{j=1}^{m} 1_{B_j}$, where 1_S stands for the characteristic function of a subset $S \subseteq \Omega$.

Obviously (with $n = m = 1$, $A_1 = B_1$), it follows that $\mu \preceq \nu$ implies $\mu \leq \nu$, but the converse is not necessarily true. It is also easy to see that $\preceq$ is transitive and antisymmetric. However, this relation is not reflexive in general, therefore $\preceq$ is *not a partial order* on the family of set functions. The relation $\mu \preceq \mu$ has nontrivial consequences. E.g., (by interchanging the roles of A_i and B_j), one gets that

$$\sum_{i=1}^{n} \mu(A_i) = \sum_{j=1}^{m} \mu(B_j)$$

for all systems of sets $A_1, \ldots, A_n, B_1, \ldots, B_m \in \mathcal{P}$ with $\sum_{i=1}^{n} 1_{A_i} = \sum_{j=1}^{m} 1_{B_j}$. Taking pairwise disjoint sets $A_1, \ldots, A_n \in \mathcal{P}$ with $B = A_1 \cup \cdots \cup A_n \in \mathcal{P}$, we trivially have $\sum_{i=1}^{n} 1_{A_i} = 1_B$, therefore

$$\sum_{i=1}^{n} \mu(A_i) = \mu(B).$$

Hence μ is additive on $\mathcal{P}$.

In the sequel, the following theorem of Kindler [**45**] will play a crucial role. This theorem characterizes the situation when two set functions can be separated by a set function μ with the property $\mu \preceq \mu$.

THEOREM I. *[Kin88] Let $\alpha : \mathcal{P} \to \mathbb{R}^*$ and $\beta : \mathcal{P} \to \mathbb{R}^*$ be set functions. Then there is a set function $\mu : \mathcal{P} \to \mathbb{R}^*$ such that $\mu \preceq \mu$ and $\alpha \leq \mu \leq \beta$ if and only if $\alpha \preceq \beta$.*

In the proof of our main results we shall also need the next lemma.

LEMMA 2.1.1. *Let I be an interval of $\mathbb{R}$, $t_1 < s_1, \dots, t_n < s_n$ and $u_1 < v_1, \dots, u_m < v_m$ be real numbers in I such that*

$$\sum_{i=1}^{n} 1_{]t_i, s_i]} = \sum_{i=1}^{m} 1_{]u_i, v_i]} \tag{2.1.1}$$

be fulfilled. Then the following equality is true:

$$\sum_{i=1}^{n} \big(q(s_i) - q(t_i)\big) = \sum_{i=1}^{m} \big(q(v_i) - q(u_i)\big). \tag{2.1.2}$$

PROOF. Let $t_1 < s_1, \dots, t_n < s_n, u_1 < v_1, \dots, u_m < v_m$ be real numbers in I such that (2.1.1) is fulfilled. Let the set A be defined by

$$A := \{t_1, s_1, \dots, t_n, s_n, u_1, v_1, \dots, u_m, v_m\} =: \{w_1, \dots, w_k\},$$

where $w_1 < \dots < w_k$. Then, for each interval $]w_j, w_{j+1}]$ there exists a natural number c_j such that

$$\sum_{i=1}^{n} 1_{]t_i, s_i]} = \sum_{j=1}^{k-1} c_j 1_{]w_j, w_{j+1}]}.$$

This number c_j shows that how many times is the interval $]w_j, w_{j+1}]$ contained in one of the intervals $]t_i, s_i]$, that is, $c_j = \sharp\Big\{ i \in \{1, \dots, n\} \Big|]w_j, w_{j+1}] \subseteq]t_i, s_i] \Big\}$. (Here $\sharp S$ denotes the cardinality of the set S.) We intend to prove that

$$\sum_{i=1}^{n} \big(q(s_i) - q(t_i)\big) = \sum_{j=1}^{k-1} c_j \big(q(w_{j+1}) - q(w_j)\big).$$

Indeed,

$$\begin{aligned}
\sum_{i=1}^{n} \big(q(s_i) - q(t_i)\big) &= \sum_{i=1}^{n} \sum_{]w_j, w_{j+1}] \subseteq]t_i, s_i]} \big(q(w_{j+1}) - q(w_j)\big) \\
&= \sum_{j=1}^{k-1} \sharp\big\{ i \in \{1, \dots, n\} \big|]w_j, w_{j+1}] \subseteq]t_i, s_i] \big\} \cdot \big(q(w_{j+1}) - q(w_j)\big) \\
&= \sum_{j=1}^{k-1} c_j \big(q(w_{j+1}) - q(w_j)\big).
\end{aligned} \tag{2.1.3}$$

However due to (2.1.1), we have that

$$\sum_{i=1}^{m} 1_{]u_i, v_i]} = \sum_{j=1}^{m} c_j 1_{]w_j, w_{j+1}]}.$$

We can similarly prove that

$$\sum_{i=1}^{m}\big(q(v_i)-q(u_i)\big)=\sum_{j=1}^{k-1}c_j\big(q(w_{j+1})-q(w_j)\big).$$

This equation combined with (2.1.3) results that

$$\sum_{i=1}^{n}\big(q(s_i)-q(t_i)\big)=\sum_{i=1}^{m}\big(q(v_i)-q(u_i)\big).$$

□

We can interpret the essence of the previous lemma that the left hand side of (2.1.2) depends only on the sum of the corresponding characteristic functions (but it is independent of its concrete form). Motivated by this, denote by $\mathcal{F}(I)$ the class of those functions that are of the following form

$$f=\sum_{i=1}^{n}1_{]t_i,s_i]}-\sum_{j=1}^{m}1_{]u_j,v_j]},\tag{2.1.4}$$

where $t_1<s_1,\dots,t_n<s_n,u_1<v_1,\dots,u_m<v_m$ are in I. Then $\mathcal{F}(I)$ is closed under the usual pointwise addition.

Given an arbitrary function $q:I\to\mathbb{R}$, define now a functional $\mathcal{I}_q(f):\mathcal{F}(I)\to\mathbb{R}$ by

$$\mathcal{I}_q(f)=\sum_{i=1}^{n}\big(q(s_i)-q(t_i)\big)-\sum_{j=1}^{m}\big(q(v_j)-q(u_j)\big),$$

where f is given by (2.1.4). If f is also represented in the form

$$f=\sum_{i=1}^{n'}1_{]t'_i,s'_i]}-\sum_{j=1}^{m'}1_{]u'_j,v'_j]},$$

then

$$\sum_{i=1}^{n}1_{]t_i,s_i]}+\sum_{j=1}^{m'}1_{]u'_j,v'_j]}=\sum_{i=1}^{n'}1_{]t'_i,s'_i]}+\sum_{j=1}^{m}1_{]u_j,v_j]}.$$

By Lemma 2.1.1, we have

$$\sum_{i=1}^{n}\big(q(s_i)-q(t_i)\big)+\sum_{j=1}^{m'}\big(q(v'_j)-q(u'_j)\big)=$$
$$\sum_{i=1}^{n'}\big(q(s'_i)-q(t'_i)\big)+\sum_{j=1}^{m}\big(q(v_j)-q(u_j)\big),$$

that is,

$$\sum_{i=1}^{n}\big(q(s_i)-q(t_i)\big)-\sum_{j=1}^{m}\big(q(v_j)-q(u_j)\big)=$$
$$\sum_{i=1}^{n'}\big(q(s_i')-q(t_i')\big)-\sum_{j=1}^{m'}\big(q(v_j')-q(u_j')\big),$$

which shows that the value of $\mathcal{I}_q(f)$ does not depend on the representation of f. It also easily follows from the definition that $\mathcal{I}_q$ is an additive function on $\mathcal{F}(I)$.

2.2. The case of general semimetrics

Let $d : I^2 \to I$ be a semimetric throughout the rest of the paper (i.e., d is a nonnegative symmetric two variable function satisfying also the triangle inequality). A function $\ell : I \to \mathbb{R}$ is said to be *d-Lipschitz* if $|\ell(x)-\ell(y)| \le d(x,y)$ for $x, y \in I$. The notation x^+ will stand for the positive part of $x \in \mathbb{R}$, i.e., $x^+ := \max(0, x)$.

The next theorem contains our main result that gives the first characterization for Lipschitz perturbations of increasing functions. The proof of the sufficiency will directly utilize the theorem of Kindler quoted in the previous section.

THEOREM 2.2.1. *A function $p : I \to \mathbb{R}$ can be written in the form $p = q + \ell$, where q is increasing and ℓ is d-Lipschitz if and only if*

(2.2.1)
$$\sum_{i=1}^{n}\big(p(s_i)-p(t_i)-d(t_i,s_i)\big)^+ \le \sum_{j=1}^{m}\big(p(v_j)-p(u_j)+d(u_j,v_j)\big)$$

is fulfilled for all real numbers $t_1 < s_1, \dots, t_n < s_n$ and $u_1 < v_1, \dots, u_m < v_m$ in I satisfying (2.1.1).

PROOF. We prove first the necessity of the condition. When $p = q + \ell$ and $x \le y$ then $q(y) - q(x) \ge 0$ since q is monotone increasing furthermore ℓ is d-Lipschitz, hence

$$q(y)-q(x) = p(y)-l(y)-p(x)+l(x) \ge p(y)-p(x)-d(x,y),$$

therefore

$$q(y)-q(x) \ge \big(p(y)-p(x)-d(x,y)\big)^+.$$

On the other hand,

$$q(y)-q(x) = p(y)-l(y)-p(x)+l(x) \le p(y)-p(x)+d(x,y).$$

Thus, for all $x < y$ in I,

$$\big(p(y)-p(x)-d(x,y)\big)^+ \le q(y)-q(x) \le p(y)-p(x)+d(x,y).$$

Assume that (2.1.1) is fulfilled for $t_1 < s_1, \dots, t_n < s_n, u_1 < v_1, \dots, u_m < v_m$. Then, using Lemma 2.1.1 and the above inequality, we get

$$\sum_{i=1}^{n} \big(p(s_i) - p(t_i) - d(t_i, s_i)\big)^+ \le \sum_{i=1}^{n} \big(q(s_i) - q(t_i)\big)$$
$$= \sum_{j=1}^{m} \big(q(v_j) - q(u_j)\big) \le \sum_{j=1}^{m} \big(p(v_j) - p(u_j) + d(u_j, v_j)\big).$$

Therefore (2.2.1) holds true.

Finally, we prove the sufficiency of the condition: Let $\mathcal{P} = \{]x, y] \mid x, y \in I, x \le y\}$, and define the set functions $\Phi, \Psi : \mathcal{P} \to \mathbb{R}$ by

$$\Phi\big(]x, y]\big) = \big(p(y) - p(x) - d(x, y)\big)^+, \qquad \Psi\big(]x, y]\big) = p(y) - p(x) + d(x, y).$$

When, for $t_1 < s_1, \dots, t_n < s_n, u_1 < v_1, \dots, u_m < v_m$, equation (2.1.1) holds, then, in view of (2.2.1), we have

$$\sum_{i=1}^{n} \Phi(]t_i, s_i]) \le \sum_{j=1}^{m} \Psi(]u_j, v_j]).$$

Due to the theorem of Kindler, there exists a set function $\Gamma : \mathcal{P} \to \mathbb{R}$ so that Γ satisfies $\Gamma \preceq \Gamma$ and $\Phi \le \Gamma \le \Psi$. Therefore Γ is additive. Now let $x_0 \in I$ be fixed and define the function $q : I \to \mathbb{R}$ by

$$q(y) := \begin{cases} \Gamma\big(]x_0, y]\big) & \text{if} \quad y > x_0, \\ 0 & \text{if} \quad x_0 = y, \\ -\Gamma\big(]y, x_0]\big) & \text{if} \quad y < x_0. \end{cases}$$

We prove that $\Gamma\big(]x, y]\big) = q(y) - q(x)$ for $x \le y$. In the proof we distinguish the five cases:

Case 1: When $x_0 < x < y$ then $q(y) = \Gamma\big(]x_0, y]\big)$ and $q(x) = \Gamma\big(]x_0, x]\big)$, that is,

$$q(y) - q(x) = \Gamma\big(]x_0, y]\big) - \Gamma\big(]x_0, x]\big).$$

Since $]x_0, y] =]x_0, x] \cup]x, y]$ and Γ is additive, hence

$$\Gamma\big(]x_0, y]\big) = \Gamma\big(]x_0, x]\big) + \Gamma\big(]x, y]\big).$$

Thus,

$$q(y) - q(x) = \Gamma\big(]x_0, x]\big) + \Gamma\big(]x, y]\big) - \Gamma\big(]x_0, x]\big) = \Gamma\big(]x, y]\big).$$

Case 2: When $x_0 = x < y$, then the statement is trivial.

Case 3: When $x < x_0 < y$, then $q(y) = \Gamma\big(]x_0, y]\big)$ and $q(x) = -\Gamma\big(]x, x_0]\big)$ that is

$$q(y) - q(x) = \Gamma\big(]x_0, y]\big) + \Gamma\big(]x, x_0]\big).$$

Since $]x, x_0] \cup]x_0, y] =]x, y]$ and Γ is additive, hence

$$\Gamma\big(]x, x_0]\big) + \Gamma\big(]x_0, y]\big) = \Gamma\big(]x, y]\big).$$

Thus,

$$q(y) - q(x) = \Gamma\big(]x, y]\big).$$

Case 4: When $x < y = x_0$ the statement is trivial.

Case 5: When $x < y < x_0$ then $q(y) = -\Gamma\big(]y, x_0]\big)$ and $q(x) = -\Gamma\big(]x, x_0]\big)$, that is

$$q(y) - q(x) = -\Gamma\big(]y, x_0]\big) + \Gamma\big(]x, x_0]\big).$$

Since $]x, x_0] =]x, y] \cup]y, x_0]$ and Γ is additive, hence

$$\Gamma\big(]x, x_0]\big) = \Gamma\big(]x, y]\big) + \Gamma\big(]y, x_0]\big).$$

Thus,

$$q(y) - q(x) = -\Gamma\big(]y, x_0]\big) + \Gamma\big(]x, y]\big) + \Gamma\big(]y, x_0]\big) = \Gamma\big(]x, y]\big).$$

Using the inequalities $\Phi \le \Gamma \le \Psi$ we get

$$\big(p(y) - p(x) - d(x, y)\big)^+ \le q(y) - q(x) \le p(y) - p(x) + d(x, y),$$

for $x < y$. The left hand side inequality yields that $0 \le q(y) - q(x)$, hence q is monotone increasing. It also follows that

$$p(y) - p(x) - d(x, y) \le q(y) - q(x) \le p(y) - p(x) + d(x, y),$$

hence,

$$-d(x, y) \le (p - q)(y) - (p - q)(x) \le d(x, y).$$

Thus $\ell := p - q$ is d-Lipschitz, that is, p has the desired decomposition $p = q + \ell$. □

In what follows, we deduce an equivalent form of the condition offered by Theorem 2.2.1.

LEMMA 2.2.2. *Let $t_1 < s_1, \ldots, t_n < s_n$ and $u_1 < v_1, \ldots, u_m < v_m$ in I satisfying* (2.1.1). *Then* (2.2.1) *holds for a function $p : I \to \mathbb{R}$ if and only if*

$$0 \le \sum_{i=1}^{n} \min\big(d(t_i, s_i), p(s_i) - p(t_i)\big) + \sum_{j=1}^{m} d(u_j, v_j). \tag{2.2.2}$$

PROOF. Using Lemma 2.1.1, we can see that (2.2.1) is equivalent to the inequality

$$\sum_{i=1}^{n} \big(p(s_i) - p(t_i) - d(t_i, s_i)\big)^+ \le \sum_{i=1}^{n} \big(p(s_i) - p(t_i)\big) + \sum_{j=1}^{m} d(u_j, v_j),$$

which can be rewritten as

$$\sum_{i=1}^{n} \Big(\max\big(p(s_i) - p(t_i) - d(t_i, s_i), 0\big) - \big(p(s_i) - p(t_i)\big)\Big) \le \sum_{j=1}^{m} d(u_j, v_j),$$

that is, as

$$\sum_{i=1}^{n} -\min\Big(d(t_i,s_i),\big(p(s_i)-p(t_i)\big)\Big) \le \sum_{j=1}^{m} d(u_j,v_j).$$

This latter inequality is clearly equivalent to condition (2.2.2). $\square$

Using Lemma 2.2.2, we obtain another characterization of the decomposability $p = q + \ell$. Here, instead of requiring (2.1.1) we need only inequality (2.2.4) for the intervals $]t_1, s_1], \ldots,]t_n, s_n]$ and $]u_1, v_1], \ldots,]u_m, v_m]$.

THEOREM 2.2.3. *A function $p : I \to \mathbb{R}$ can be written in the form $p = q + \ell$, where q is increasing and ℓ is d-Lipschitz if and only if*

$$0 \le \sum_{i=1}^{n} d(t_i,s_i) + \sum_{j=1}^{m} d(u_j,v_j) + \mathcal{I}_p\Big(\sum_{j=1}^{m} 1_{]u_j,v_j]} - \sum_{i=1}^{n} 1_{]t_i,s_i]}\Big) \tag{2.2.3}$$

for all real numbers $t_1 < s_1, \ldots, t_n < s_n$ and $u_1 < v_1, \ldots, u_m, < v_m$ in I satisfying

$$\sum_{i=1}^{n} 1_{]t_i,s_i]} \le \sum_{j=1}^{m} 1_{]u_j,v_j]}. \tag{2.2.4}$$

PROOF. Assume that p is of the form $q + \ell$, where q is increasing and ℓ is d-Lipschitz. Let $t_1 < s_1, \ldots, t_n < s_n$ and $u_1 < v_1, \ldots, u_m, < v_m$ in I satisfying (2.2.4). Then there exist $t_{n+1} < s_{n+1}, \ldots, t_N < s_N$ such that

$$\sum_{i=1}^{N} 1_{]t_i,s_i]} = \sum_{j=1}^{m} 1_{]u_j,v_j]}. \tag{2.2.5}$$

In view of Theorem 2.2.1, we have that

$$\sum_{i=1}^{N} \big(p(s_i) - p(t_i) - d(t_i,s_i)\big)^{+} \le \sum_{j=1}^{m} \big(p(v_j) - p(u_j) + d(u_j,v_j)\big).$$

Using Lemma 2.2.2, it follows from this inequality and from (2.2.5) that

$$\begin{aligned}
0 &\leq \sum_{i=1}^{N} \min\big(d(t_i,s_i), p(s_i)-p(t_i)\big) + \sum_{j=1}^{m} d(u_j,v_j) \\
&\leq \sum_{i=1}^{n} d(t_i,s_i) + \sum_{i=n+1}^{N} \big(p(s_i)-p(t_i)\big) + \sum_{j=1}^{m} d(u_j,v_j) \\
&= \sum_{i=1}^{n} d(t_i,s_i) + \mathfrak{I}_p\Big(\sum_{i=n+1}^{N} 1_{]t_i,s_i]}\Big) + \sum_{j=1}^{m} d(u_j,v_j) \\
&= \sum_{i=1}^{n} d(t_i,s_i) + \sum_{j=1}^{m} d(u_j,v_j) + \mathfrak{I}_p\Big(\sum_{j=1}^{m} 1_{]u_j,v_j]} - \sum_{i=1}^{n} 1_{]t_i,s_i]}\Big),
\end{aligned}$$

which yields (2.2.3).

Conversely, assume that (2.2.3) holds for all $t_1 < s_1, \dots, t_n < s_n$ and $u_1 < v_1, \dots, u_m, < v_m$ in I whenever (2.2.4) is satisfied. We intend to prove that condition (2.2.1) of Theorem 2.2.1 is also fulfilled for all $t_1 < s_1, \dots, t_n < s_n$ and $u_1 < v_1, \dots, u_m, < v_m$ in I satisfying (2.1.1). This, by Theorem 2.2.1, will imply that p has a desired decomposition.

Let $t_1 < s_1, \dots, t_n < s_n$ and $u_1 < v_1, \dots, u_m < v_m$ in I with (2.1.1). Denote by Γ the set of those indices $i \in \{1, \dots, n\}$ such that

$$d(t_i,s_i) \leq p(s_i) - p(t_i).$$

Then, in view of (2.1.1), we have that

$$\sum_{i\in\Gamma} 1_{]t_i,s_i]} \leq \sum_{j=1}^{m} 1_{]u_j,v_j]}.$$

Thus, conditions (2.2.3) and (2.1.1) give

$$\begin{aligned}
0 &\leq \sum_{i\in\Gamma} d(t_i,s_i) + \sum_{j=1}^{m} d(u_j,v_j) + \mathfrak{I}_p\Big(\sum_{j=1}^{m} 1_{]u_j,v_j]} - \sum_{i\in\Gamma} 1_{]t_i,s_i]}\Big) \\
&= \sum_{i\in\Gamma} \min\big(d(t_i,s_i), p(s_i)-p(t_i)\big) + \sum_{j=1}^{m} d(u_j,v_j) + \mathfrak{I}_p\Big(\sum_{i\notin\Gamma} 1_{]t_i,s_i]}\Big) \\
&= \sum_{i\in\Gamma} \min\big(d(t_i,s_i), p(s_i)-p(t_i)\big) + \sum_{j=1}^{m} d(u_j,v_j) + \sum_{i\notin\Gamma} \big(p(s_i)-p(t_i)\big)
\end{aligned}$$

$$= \sum_{i\in\Gamma} \min\big(d(t_i,s_i), p(s_i)-p(t_i)\big) + \sum_{j=1}^{m} d(u_j,v_j)$$
$$+ \sum_{i\notin\Gamma} \min\big(d(t_i,s_i), p(s_i)-p(t_i)\big)$$
$$= \sum_{i=1}^{n} \min\big(d(t_i,s_i), p(s_i)-p(t_i)\big) + \sum_{j=1}^{m} d(u_j,v_j),$$

which yields (2.2.2). □

2.3. The case of concave semimetrics

In this section, provided that the semimetric d possesses further properties, we are going to obtain simpler necessary and sufficient conditions in order that a function p could be decomposed as $q+\ell$ where q is monotone increasing and ℓ is d-Lipschitz.

DEFINITION 2.3.1. A system of intervals $\{]a_i,b_i] : i=1,\dots,k\}$ is said to be *nested* if, for all $i,j\in\{1,\dots,k\}$,

either $[a_i,b_i]\cap[a_j,b_j]=\emptyset$ or $]a_i,b_i]\subset]a_j,b_j]$ or $]a_j,b_j]\subset]a_i,b_i]$.

DEFINITION 2.3.2. A semimetric $d: I\times I\to\mathbb{R}$ is called *concave* if, for all $x\le y\le z\le w$ in I, it satisfies

$$d(x,w)+d(y,z)\le d(x,z)+d(y,w). \tag{2.3.1}$$

Observe that, with $y=z$, (2.3.1) reduces to the triangle inequality. The reason why semimetrics satisfying (2.3.1) are called concave is explained by the following result.

LEMMA 2.3.3. *Let $\varphi : \big(\mathbb{R}_+\cap(I-I)\big)\to\mathbb{R}$ be a monotone increasing function with $\varphi(0)=0$. Then the function $d_\varphi : I\times I\to\mathbb{R}$ defined by*

$$d_\varphi(x,y) := \varphi(|x-y|) \tag{2.3.2}$$

is a concave semimetric on I if and only φ is concave on $\big(\mathbb{R}_+\cap(I-I)\big)$.

PROOF. Assume that d_φ is a concave semimetric on I. Then, it follows from (2.3.1) that

$$\varphi(w-x)+\varphi(z-y)\le\varphi(z-x)+\varphi(w-y) \tag{2.3.3}$$

for all $x\le y\le z\le w$ in I. Let $\alpha,\beta\in\mathbb{R}_+\cap(I-I)$ with $0\le\alpha\le\beta$ be arbitrary. Then there exist $x\le w$ in I such that $\beta=w-x$. Let

$$y := w-\frac{\alpha+\beta}{2} \qquad\text{and}\qquad z := x+\frac{\alpha+\beta}{2}.$$

Then, one can see that $x\le y\le z\le w$ holds. Thus, by (2.3.3), it follows that

$$\varphi(\beta)+\varphi(\alpha)\le 2\varphi\Big(\frac{\alpha+\beta}{2}\Big)$$

for all $\alpha, \beta \in \mathbb{R}_+ \cap (I - I)$ with $0 \leq \alpha \leq \beta$. Thus, φ is Jensen-concave on $\mathbb{R}_+ \cap (I - I)$.

Assume that φ is concave on $\big(\mathbb{R}_+ \cap (I - I)\big)$. Obviously, the function d_φ is symmetric, nonnegative and $d_\varphi(x, x) = 0$. In order to prove (2.3.1), let $x \leq y \leq z \leq w$ in I be arbitrary. If $x = y$ or $z = w$ then (2.3.1) is obvious. Thus, we may assume that $x < y$ and $z < w$. Denote $\alpha = y - x$, $\beta = z - y$, and $\gamma = w - z$. Then $\alpha, \gamma > 0$,

$$\alpha + \beta = \frac{\gamma}{\alpha + \gamma}\beta + \frac{\alpha}{\alpha + \gamma}(\alpha + \beta + \gamma),$$

and

$$\beta + \gamma = \frac{\alpha}{\alpha + \gamma}\beta + \frac{\gamma}{\alpha + \gamma}(\alpha + \beta + \gamma).$$

Since φ is concave, we get that

$$\varphi(\alpha + \beta) \geq \frac{\gamma}{\alpha + \gamma}\varphi(\beta) + \frac{\alpha}{\alpha + \gamma}\varphi(\alpha + \beta + \gamma),$$

and

$$\varphi(\beta + \gamma) \geq \frac{\alpha}{\alpha + \gamma}\varphi(\beta) + \frac{\gamma}{\alpha + \gamma}\varphi(\alpha + \beta + \gamma).$$

Adding these inequalities, it follows that

$$\varphi(\alpha + \beta) + \varphi(\beta + \gamma) \geq \varphi(\beta) + \varphi(\alpha + \beta + \gamma).$$

which is equivalent to (2.3.3) and also to (2.3.1). This property yields that d_φ satisfies the triangle inequality, too. Thus d_φ is concave semimetric. $\square$

REMARK. If $0 < p \leq 1$ then $\varphi(t) := t^p$ is a concave function. Thus, if $0 < p_1 < \cdots < p_k \leq 1$ and $c_1, \ldots, c_k > 0$, then the formula

$$d(x, y) = \sum_{i=1}^{k} c_i |x - y|^{p_i} \qquad (x, y \in I)$$

yields a concave metric on I. This way, a large class of concave metrics can be obtained.

The next lemma describes a connection of concave semimetrics and nested systems of intervals.

LEMMA 2.3.4. *If d is a concave semimetric and $u_1 < v_1, \ldots, u_m < v_m$ are in I, then there exist a nested system of intervals $\{]a_i, b_i] : i = 1, \ldots, k\}$ such that*

$$\sum_{j=1}^{m} 1_{]u_j, v_j]} = \sum_{i=1}^{k} 1_{]a_i, b_i]}, \tag{2.3.4}$$

$$\sum_{i=1}^{k} d(a_i, b_i) \leq \sum_{j=1}^{m} d(u_j, v_j) \tag{2.3.5}$$

and

$$\{a_1, \dots, a_k, b_1, \dots, b_k\} \subset \{u_1, \dots, u_m, v_1, \dots, v_m\}. \tag{2.3.6}$$

PROOF. If the system of intervals $\{]u_j, v_j] : j = 1, \dots, m\}$ is nested, then there is nothing to prove. If this is not the case, then there exist $l, n \in \{1, \dots, m\}$ such that

$$[u_l, v_l] \cap [u_n, v_n] \neq \emptyset, \qquad]u_l, v_l] \not\subseteq]u_n, v_n] \qquad \text{and} \qquad]u_n, v_n] \not\subseteq]u_l, v_l]. \tag{2.3.7}$$

Due to the symmetry, we may assume that $u_l \le u_n$. Then, it follows from (2.3.7) that $u_l < u_n \le v_l < v_n$. Obviously,

$$1_{]u_l,v_l]} + 1_{]u_n,v_n]} = 1_{]u_l,v_n]} + 1_{]u_n,v_l]},$$

therefore, using that d is a concave semimetric, we get that

$$d(u_l, v_n) + d(u_n, v_l) \le d(u_l, v_l) + d(u_n, v_n).$$

Then, replacing $]u_l, v_l]$ and $]u_n, v_n]$ of the system of intervals $\{]u_j, v_j] : j = 1, \dots, m\}$ by $]u_l, v_n]$ and by (the possibly empty) $]u_n, v_l]$, the sum of the characteristic functions of the intervals remains unchanged, the sum of the d-length of these intervals does not increase and the set of endpoints does not increase as well. Observe that, due to the strict inequalities $u_l < u_n$ and $v_l < v_n$,

$$(v_n - u_l)^2 + (v_l - u_n)^2 > (v_n - u_n)^2 + (v_l - u_l)^2$$

holds. Therefore, by the above replacement, the sum of the squares of the (ordinary) length of the intervals strictly increases.

When repeating the above replacement step, we cannot return to a previous system of intervals because the sum of the squares of the lengths strictly increases. There are only finitely many systems of intervals with the same sum of the corresponding characteristic functions such that the set of endpoints of the intervals is included in $\{u_1, \dots, u_m, v_1, \dots, v_m\}$, therefore, there are only finitely many replacement steps and in each step the sum of the d-length of the intervals does not increase. When the procedure terminates, the resulting system of intervals has to be nested. $\square$

The main result of this section is contained in the following theorem which characterizes the decomposability $p = q + \ell$ in case of concave semimetrics. The sufficient and necessary condition of Theorem 2.2.3 dramatically simplifies in this setting.

THEOREM 2.3.5. *A function $p : I \to \mathbb{R}$ can be written in the form $p = q + \ell$, where q is increasing and ℓ is d-Lipschitz if and only if*

$$0 \le \sum_{k=1}^{n} d(x_{2k-1}, x_{2k}) + d(x_0, x_{2n+1}) + \sum_{k=0}^{n} \big(p(x_{2k+1}) - p(x_{2k})\big) \tag{2.3.8}$$

holds for all $x_0 \le x_1 < \cdots < x_{2n} \le x_{2n+1}$ in I.

PROOF. *The necessity of the condition* (2.3.8). If $p : I \to \mathbb{R}$ can be written in the form $p = q + \ell$, where q is increasing and ℓ is d-Lipschitz, then in view of Theorem 2.2.3, inequality (2.2.3) holds whenever $t_1 < s_1, \dots, t_n < s_n, u_1 < v_1, \dots, u_m < v_m$ in I satisfies (2.2.4).

Let $x_0 \le x_1 < \cdots < x_{2n} \le x_{2n+1}$ in I be arbitrary. Then (2.2.3) applied to the system of intervals

$$]t_k, s_k] :=]x_{2k-1}, x_{2k}] \qquad (k = 1, \dots, n), \qquad]u_1, v_1] :=]x_0, x_{2n+2}],$$

yields

$$\sum_{k=1}^{n} d(x_{2k-1}, x_{2k}) + d(x_0, x_{2n+1}) + \sum_{k=0}^{n} \big(p(x_{2k+1}) - p(x_{2k})\big) \ge 0$$

because

$$\begin{aligned}\mathcal{I}_p\Big(1_{]x_0,x_{2n+1}]} - \sum_{k=1}^{n} 1_{]x_{2k-1},x_{2k}]}\Big) &= \mathcal{I}_p\Big(\sum_{k=0}^{n} 1_{]x_{2k+1},x_{2k}]}\Big) \\ &= \sum_{k=0}^{n} \big(p(x_{2k+1}) - p(x_{2k})\big).\end{aligned}$$

The sufficiency of the condition (2.3.8). Again by Theorem 2.2.3 it is enough to show that (2.2.3) holds for all $t_1 < s_1, \dots, t_n < s_n$ and $u_1 < v_1, \dots, u_m < v_m$ in I satisfying (2.2.4). In view of Lemma 2.2.2, we may assume that the systems of intervals

$$\mathcal{S} := \big\{]t_i, s_i] : i = 1, \dots, n\big\} \qquad \text{and} \qquad \mathcal{V} := \big\{]u_j, v_j] : j = 1, \dots, m\big\}$$

are nested. We prove by induction on m.

If $m = 1$ then (2.2.4) shows that the intervals $]t_i, s_i]$ are disjoint subintervals of $]u_1, v_1]$. Without loss of generality, we may assume that $u_1 \le t_1 < s_1 < \cdots < t_n < s_n \le v_1$. Set

$$x_0 := u_1,\ x_1 := t_1,\ x_2 := s_1,\ \dots,\ x_{2n-1} := t_n,\ x_{2n} := s_n,\ x_{2n+1} := v_1.$$

The right hand side of (2.2.3) can be written as

$$\begin{aligned}&\sum_{i=1}^{n} d(t_i, s_i) + \sum_{j=1}^{m} d(u_j, v_j) + \mathcal{I}_p\Big(\sum_{j=1}^{m} 1_{]u_j,v_j]} - \sum_{i=1}^{n} 1_{]t_i,s_i]}\Big) \\ &\qquad= \sum_{i=1}^{n} d(t_i, s_i) + d(u_1, v_1) + \mathcal{I}_p\Big(1_{]u_1,v_1]} - \sum_{i=1}^{n} 1_{]t_i,s_i]}\Big) \\ &\qquad= \sum_{i=1}^{n} d(t_i, s_i) + d(u_1, v_1) + \mathcal{I}_p\Big(1_{]u_1,t_1]} + \sum_{i=1}^{n-1} 1_{]s_i,t_{i+1}]} + 1_{]s_n,v_1]}\Big) \\ &\qquad= \sum_{i=1}^{n} d(x_{2i-1}, x_{2i}) + d(x_0, x_{2n+1}) + \sum_{i=0}^{n} \big(p(x_{2i+1}) - p(x_{2i})\big),\end{aligned}$$

which is nonnegative by (2.3.8).

Now assume that we have proved the condition of Theorem 2.2.3 in the case when $m = k - 1$ where $k \geq 2$. We shall prove that this condition is also true in the case $m = k$.

Let $t_1 < s_1, \dots, t_n < s_n, u_1 < v_1, \dots, u_m < v_m$ are arbitrary elements in I with (2.2.4). Let j_0 be an index so that the interval $]u_{j_0}, v_{j_0}]$ is maximal in the system $\mathcal{V}$. Furthermore let Γ_{j_0} be a subset of $\{1, \dots, n\}$ so that the system $T_{j_0} = \big\{]t_i, s_i] : i \in \Gamma_{j_0}\big\}$ is a maximal disjoint subsystem of $\mathcal{S}$ such that $]t_i, s_i] \subseteq]u_{j_0}, v_{j_0}]$ for all $i \in \Gamma_{j_0}$.

The intervals in T_{j_0} are pairwise disjoint and are contained in $]u_{j_0}, v_{j_0}]$, therefore we get

$$\sum_{i \in \Gamma_{j_0}} 1_{]t_i, s_i]} \leq 1_{]u_{j_0}, v_{j_0}]}.$$

What we have proved for the case $m = 1$ yields that

$$0 \leq \sum_{i \in \Gamma_{j_0}} d(t_i, s_i) + d(u_{j_0}, v_{j_0}) + \mathcal{I}_p\Big(1_{]u_{j_0}, v_{j_0}]} - \sum_{i \in \Gamma_{j_0}} 1_{]t_i, s_i]}\Big). \tag{2.3.9}$$

Now we show that the inequality

$$\sum_{i \notin \Gamma_{j_0}} 1_{]t_i, s_i]} \leq \sum_{j \neq j_0} 1_{]u_j, v_j]} \tag{2.3.10}$$

is fulfilled, too. We prove this statement by contradiction. Assume that there exists $x \in I$ such that

$$\sum_{i \notin \Gamma_{j_0}} 1_{]t_i, s_i]}(x) > \sum_{j \neq j_0} 1_{]u_j, v_j]}(x).$$

Then, we also have that

$$\sum_{j=1}^{m} 1_{]u_j, v_j]}(x) \geq \sum_{i=1}^{n} 1_{]t_i, s_i]}(x) \geq \sum_{i \notin \Gamma_{j_0}} 1_{]t_i, s_i]}(x) > \sum_{j \neq j_0} 1_{]u_j, v_j]}(x).$$

It follows from these inequalities that $x \in]u_{j_0}, v_{j_0}]$. The left hand side is bigger than the right hand side by one, therefore, the first two inequalities in the above chain are equalities, in fact. Thus, $x \notin]t_i, s_i]$ if $i \in \Gamma_{j_0}$. On the other hand, due to the strict inequality above, there exists $i_0 \notin \Gamma_{j_0}$ such that $x \in]t_{i_0}, s_{i_0}]$ and the interval $]t_{i_0}, s_{i_0}]$ is maximal in $\mathcal{S}$. The maximal intervals in $\mathcal{V}$ cover the interval $]t_{i_0}, s_{i_0}]$. These maximal intervals are nested and thus pairwise disjoint, therefore one of them covers also the interval $]t_{i_0}, s_{i_0}]$. Being x a common element of $]t_{i_0}, s_{i_0}]$ and $]u_{j_0}, v_{j_0}]$, we get that $]u_{j_0}, v_{j_0}]$ covers $]t_{i_0}, s_{i_0}]$. We also have that $]t_{i_0}, s_{i_0}] \cap]t_i, s_i] \neq \emptyset$ if $i \in \Gamma_{j_0}$ because the system of intervals $\mathcal{S}$ is nested. Thus $]t_{i_0}, s_{i_0}]$ is a maximal subinterval of $]u_{j_0}, v_{j_0}]$ which is disjoint from $]t_i, s_i]$ for all $i \in \Gamma_{j_0}$. This contradicts the construction of Γ_{j_0}. This contradiction validates (2.3.10).

By the inductive assumption, it follows from (2.3.10) that

$$0 \le \sum_{i \notin \Gamma_{j_0}} d(t_i, s_i) + \sum_{j \neq j_0} d(u_j, v_j) + \mathcal{I}_p\Big(\sum_{j \neq j_0} 1_{]u_j, v_j]} - \sum_{i \notin \Gamma_{j_0}} 1_{]t_i, s_i]} \Big). \tag{2.3.11}$$

Then, using the additivity of $\mathcal{I}_p$, and the inequalities (2.3.9), (2.3.11), we get

$$\begin{aligned}
\sum_{i=1}^{n} d(t_i, s_i) &+ \sum_{j=1}^{m} d(u_j, v_j) + \mathcal{I}_p\Big(\sum_{j=1}^{m} 1_{]u_j, v_j]} - \sum_{i=1}^{n} 1_{]t_i, s_i]} \Big) \\
&= \sum_{i \in \Gamma_{j_0}} d(t_i, s_i) + d(u_{j_0}, v_{j_0}) + \mathcal{I}_p\Big(1_{]u_{j_0}, v_{j_0}]} - \sum_{i \in \Gamma_{j_0}} 1_{]t_i, s_i]} \Big) \\
&\quad + \sum_{i \notin \Gamma_{j_0}} d(t_i, s_i) + \sum_{j \neq j_0} d(u_j, v_j) + \mathcal{I}_p\Big(\sum_{j \neq j_0} 1_{]u_j, v_j]} - \sum_{i \notin \Gamma_{j_0}} 1_{]t_i, s_i]} \Big) \ge 0.
\end{aligned}$$

Thus, the proof of the theorem is complete. □

In the case of when the semimetric coincides with the ordinary distance function, the condition (2.3.8) simplifies to a two variable inequality only.

THEOREM 2.3.6. *If the metric d is given by $d(x, y) = |y - x|$ $(x, y \in I)$, then the condition* (2.3.8) *holds for all $x_0 \le x_1 < \cdots < x_{2n} \le x_{2n+1}$ in I if and only if*

$$p(x) \le p(y) + d(x, y) \tag{2.3.12}$$

for all $x < y$ in I.

PROOF. *The necessity of the condition* (2.3.12). When the condition (2.3.8) holds for all $x_0 \le x_1 < \cdots < x_{2n} \le x_{2n+1}$ in I then the condition (2.3.12) is fulfilled trivially.

The sufficiency of the condition (2.3.12). We use that, for $x < y$ in I, $p(x) \le p(y) + d(x, y)$ and $d(x, y) = y - x$, that is,

$$x - y \le p(y) - p(x).$$

Thus

$$\begin{aligned}
\sum_{k=1}^{n} d(x_{2k-1}, x_{2k}) + d(x_0, x_{2n+1}) + \sum_{k=0}^{n} \big(p(x_{2k+1}) - p(x_{2k}) \big) \\
&= d(x_1, x_2) + d(x_3, x_4) + \cdots + d(x_{2n-1}, x_{2n}) + d(x_0, x_{2n}) \\
&\quad + p(x_1) - p(x_0) + \cdots + p(x_{2n+1}) - p(x_{2n}) \\
&\ge x_2 - x_1 + x_4 - x_3 + \cdots + x_{2n} - x_{2n-1} + x_{2n+1} - x_0 \\
&\quad + x_0 - x_1 + x_2 - x_3 + \cdots + x_{2n} - x_{2n+1} \\
&= 2(x_2 - x_1) + 2(x_4 - x_3) + \cdots + 2(x_{2n} - x_{2n-1}) \\
&\ge 0,
\end{aligned}$$

which was to be proved. □

Summary

1. A. On the equality for two variable means

One of the aims of the dissertation is to investigate the equality and invariance problem of generalized quasi-arithmetic means. We define the generalized quasi-arithmetic mean as follows.

DEFINITION. Given a continuous strictly monotone function $\varphi : I \to \mathbb{R}$ and a probability measure μ on the Borel subsets of $[0,1]$, the two variable mean $\mathcal{M}_{\varphi,\mu} : I^2 \to I$ is defined by

$$\mathcal{M}_{\varphi,\mu}(x,y) := \varphi^{-1}\Big(\int_0^1 \varphi\big(tx+(1-t)y\big)d\mu(t)\Big) \qquad (x,y\in I).$$

If $\mu = \dfrac{\delta_0+\delta_1}{2}$, then $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_\varphi$, where δ_t is the Dirac measure concentrated at the point $t\in[0,1]$. If $\mu =$ Lebesgue measure on $[0,1]$, then $\mathcal{M}_{\varphi,\mu} = \mathcal{L}_\varphi$.

In the first part of the first chapter contains the basic notations and lemmas, which we need to present our results.

Given a Borel probability measure μ on the interval $[0,1]$, we define the *kth moment* and the *kth centralized moment* of μ by

$$\widehat{\mu}_k := \int_0^1 t^k d\mu(t) \qquad \text{and} \qquad \mu_k := \int_0^1 (t-\widehat{\mu}_1)^k d\mu(t) \qquad (k\in\mathbb{N}\cup\{0\}).$$

The reflection of the measure μ with respect to the point $1/2$ is defined by

$$\widetilde{\mu}(A) = \mu(\widetilde{A}),$$

where A is an arbitrary Borel subset of $[0,1]$ and $\widetilde{A} := 1-A := \{1-x \mid x\in A\}$.

To formulate the main results of this chapter, we consider the cases when the first n moments of the measures μ and ν in the equality problem are identical. For $n\in\mathbb{N}\cup\{0,\infty\}$, we say that the *nth-order moment condition* $\mathcal{M}_n$ *holds* if μ,ν are Borel probability measures on $[0,1]$, furthermore,

$$\widehat{\mu}_k = \widehat{\nu}_k \quad \text{for all} \quad 1\le k\le n.$$

Thus the $\mathcal{M}_\infty$ condition means that all the moments of μ and ν are equal, whence, by well-known results of measure and approximation theory, the equality of the two measure μ and ν follows. On the other hand, the condition $\mathcal{M}_0$ simply means that μ,ν are probability measures on the Borel subsets of $[0,1]$. For $n\in\mathbb{N}\cup\{0\}$, we say that the *exact nth-order moment condition* $\mathcal{M}_n^*$ *holds* if $\mathcal{M}_n$ is valid but $\mathcal{M}_{n+1}$ fails, i.e.,

$$\widehat{\mu}_k = \widehat{\nu}_k \quad \text{for all} \quad 1 \le k \le n \qquad \text{and} \qquad \widehat{\mu}_{n+1} \neq \widehat{\nu}_{n+1}.$$

In order to describe the various regularity conditions on the two unknown functions φ and ψ, for $\in \mathbb{N} \cup \{\infty\}$, we say that the *nth-order regularity condition* $\mathcal{C}_n$ *holds* if $\varphi, \psi : I \to \mathbb{R}$ are n-times continuously differentiable functions with non-vanishing first-order derivatives. For convenience, we also say that $\mathcal{C}_0$ holds if $\varphi, \psi : I \to \mathbb{R}$ are just continuous strictly monotone functions.

In our first result, we compute the first partial derivatives of the mean $\mathcal{M}_{\varphi,\mu}$ at a point of the diagonal of $I \times I$ under a weak regularity assumption.

LEMMA. *Let μ be a Borel probability measure, let $\varphi : I \to \mathbb{R}$ be a continuous strictly monotone function and assume that φ is differentiable at a point $p \in I$ and $\varphi'(p) \neq 0$. Then $\partial_1 \mathcal{M}_{\varphi,\mu}(p,p) = \widehat{\mu}_1$.*

In the second part of the first chapter we characterize those pairs (φ, μ) and (ψ, ν) such that

$$\mathcal{M}_{\varphi,\mu}(x,y) = \mathcal{M}_{\psi,\nu}(x,y) \quad (x, y \in I)$$

holds.

In the following result we obtain the first necessary condition for the equality of the generalized quasi-arithmetic means. This shows that, under weak regularity assumptions, there is no solution of the equality problem if the exact moment condition $\mathcal{M}_0^*$ holds.

COROLLARY. *Assume $\mathcal{C}_0$ and $\mathcal{M}_0$. Suppose that there exists a point $p \in I$ such that φ and ψ are differentiable at p and $\varphi'(p)\psi'(p) \neq 0$. Then, in order that $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ be valid, it is necessary that*

$$\widehat{\mu}_1 = \widehat{\nu}_1,$$

i.e., $\mathcal{M}_1$ be satisfied.

In our next result, assuming $\mathcal{C}_1$, we obtain a characterization of the equality problem that does not involve the inverses of the unknown functions φ and ψ.

THEOREM. *Assume $\mathcal{C}_1$ and $\mathcal{M}_1$. Then $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ holds for all $x, y \in I$ if and only if*

$$\int_0^1 \int_0^1 (t-s)\varphi'\big(tx + (1-t)y\big)\psi'\big(sx + (1-s)y\big)d\mu(t)d\nu(s) = 0.$$

Assuming $\mathcal{C}_{n+1}$, we now deduce further conditions that are necessary for the equality problem.

THEOREM. *Assume $\mathcal{C}_{n+1}$ for some $n \in \mathbb{N}$ and $\mathcal{M}_1$. Then, in order that $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ be valid, it is necessary that*

$$(1) \qquad \sum_{i=0}^{n} \binom{n}{i} \big(\mu_{i+1}\nu_{n-i} - \mu_i \nu_{n+1-i}\big) \frac{\varphi^{(i+1)}}{\varphi'} \cdot \frac{\psi^{(n+1-i)}}{\psi'} = 0.$$

Conversely, if φ, ψ are analytic functions and (1) *holds for all $n \in \mathbb{N}$, then $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ is satisfied.*

In this section we solve the equality problem, if the two measures μ and ν coincide.

THEOREM. *Assume $\mathcal{C}_0$ and $\mathcal{M}_\infty$. Then $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ holds if and only if*

(i) either $\mu = \nu = \delta_\tau$ for some $\tau \in [0,1]$ and φ, ψ are arbitrary,

(ii) or $\mu = \nu$ is not a Dirac measure and there exist constants $a \neq 0$ and b such that

$$\psi = a\varphi + b.$$

If at least the first two moments of the measures μ and ν are the same but the measures are not identical. The investigation of this case requires twice continuous differentiability of the unknown functions φ and ψ.

THEOREM. *Assume $\mathcal{C}_2$ and $\mathcal{M}_n^*$ for some $2 \leq n < \infty$. Then $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ holds if and only if there exist constants $a \neq 0$ and b such that*

$$\psi = a\varphi + b$$

and φ is a polynomial with $\deg \varphi \leq n$.

In the investigation of this case we consider two subcases according as $\mu_2\nu_2 = 0$, respectively $\mu_2\nu_2 \neq 0$.

THEOREM. *Assume $\mathcal{C}_2$ and $\mathcal{M}_1^*$ with $\mu_2\nu_2 = 0$. Then $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ holds if and only if*

(i) either μ and ψ are arbitrary, $\nu = \delta_{\widehat{\mu}_1}$, and there exist constants $a \neq 0$ and b such that

$$\varphi(x) = ax + b \qquad (x \in I).$$

(ii) or ν and φ are arbitrary, $\mu = \delta_{\widehat{\nu}_1}$, and there exist constants $c \neq 0$ and d such that

$$\psi(x) = cx + d \qquad (x \in I).$$

In the following result we derive further necessary conditions for the equality problem.

THEOREM. *Assume $\mathcal{C}_2$ and $\mathcal{M}_1$ with $\mu_2\nu_2 \neq 0$ and assume that $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ holds. Then*

$$\nu_2 \frac{\psi''}{\psi'} = \mu_2 \frac{\varphi''}{\varphi'} =: \Phi.$$

If $\mathcal{C}_3$ is valid then the function $\Phi : I \to \mathbb{R}$ introduced above satisfies the differential equation

$$\left(\frac{\mu_3}{\mu_2} - \frac{\nu_3}{\nu_2}\right)\Phi' + \left(\frac{\mu_3}{\mu_2^2} - \frac{\nu_3}{\nu_2^2}\right)\Phi^2 = 0.$$

If $\mathcal{C}_4$ is also valid, then φ and ψ are analytic functions and Φ satisfies the differential equations

$$\left(\frac{\mu_4}{\mu_2}-\frac{\nu_4}{\nu_2}\right)\Phi''+\left(\frac{3\mu_4}{\mu_2^2}-\frac{3\nu_4}{\nu_2^2}\right)\Phi\Phi'+\left(\frac{\mu_4-3\mu_2^2}{\mu_2^3}-\frac{\nu_4-3\nu_2^2}{\nu_2^3}\right)\Phi^3=0.$$

If $\mathcal{M}_1$ holds then the three coefficients in this equation do not vanish simultaneously.

In the main result of this part, we obtain a necessary and sufficient condition for the equality problem under the additional assumption that Φ satisfies a first-order polynomial differential equation.

THEOREM. *Assume $\mathcal{C}_3$ and $\mathcal{M}_1$ with $\mu_2\nu_2 \neq 0$. Suppose that $\nu_2\frac{\psi''}{\psi'} = \mu_2\frac{\varphi''}{\varphi'} =: \Phi$ holds and that there exists integer numbers $0 \leq 2n \leq k$ and a constant vector $(c_0,\dots,c_n) \neq (0,\dots,0)$ such that the function $\Phi : I \to \mathbb{R}$ satisfies the following first-order polynomial differential equation*

$$\sum_{i=0}^{n} c_i\Phi^{k-2i}\big(\Phi'\big)^i=0.$$

Then $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ holds if and only if

(i) either there exist real constants a,b,c,d with $ac \neq 0$ such that

$$\varphi(x)=ax+b \qquad \textit{and} \qquad \psi(x)=cx+d \qquad (x\in I);$$

(ii) or there exist real constants a,b,c,d,p,q with $ac(p-q)\neq 0$, $pq>0$ such that

$$\varphi(x)=ae^{px}+b \qquad \textit{and} \qquad \psi(x)=ce^{qx}+d \qquad (x\in I)$$

and, for $n \in \mathbb{N}$

$$\sum_{i=0}^{n}\binom{n}{i}p^iq^{n-i}\big(\mu_{i+1}\nu_{n-i}-\mu_i\nu_{n+1-i}\big)=0;$$

(iii) or there exist real constants a,b,c,d,p,q with $ac(p-q)\neq 0$, $(p-1)(q-1)>0$ and $x_0 \notin I$ such that, for $x \in I$

$$\varphi(x)=\begin{cases} a|x-x_0|^p+b, & \textit{if } p\neq 0\\ a\ln|x-x_0|+b, & \textit{if } p=0,\end{cases}$$

$$\psi(x)=\begin{cases} c|x-x_0|^q+d, & \textit{if } q\neq 0\\ c\ln|x-x_0|+d, & \textit{if } q=0\end{cases}$$

and, for $n \in \mathbb{N}$

$$\sum_{i=0}^{n}\binom{p-1}{i}\binom{q-1}{n-i}\big(\mu_{i+1}\nu_{n-i}-\mu_i\nu_{n+1-i}\big)=0.$$

By solving of the following examples we apply our main result.

EXAMPLE. Consider the functional equation

$$\varphi^{-1}\left(\frac{\varphi\left(\frac{2x+y}{3}\right)+\varphi\left(\frac{x+2y}{3}\right)}{2}\right)=\psi^{-1}\left(\frac{\psi(x)+16\psi\left(\frac{x+y}{2}\right)+\psi(y)}{18}\right),$$

where $\varphi,\psi : I \to \mathbb{R}$ are continuous strictly monotone functions.

If $\mathcal{C}_4$ is assumed, the generating functions $\varphi,\psi : I \to \mathbb{R}$ satisfy this functional equation if and only if there exist constants $a \neq 0$ and b such that $\psi = a\varphi + b$ and φ is an arbitrary strictly monotone polynomial with $\deg \varphi \leq 3$.

It remains an open problem to find the solutions of this functional equation under the regularity assumption $\mathcal{C}_0$ only.

EXAMPLE. Consider the functional equation

$$\varphi^{-1}\left(\frac{2\varphi(x)+\varphi(y)}{3}\right)=\psi^{-1}\left(\int_0^1 2t\psi(tx+(1-t)y)dt\right),$$

where $\varphi,\psi : I \to \mathbb{R}$ are continuous strictly monotone functions.

If $\mathcal{C}_3$ is assumed, the generating functions $\varphi,\psi : I \to \mathbb{R}$ satisfy this functional equation if and only if there exist real constants a,b,c,d with $ac \neq 0$ such that

(i) either $\varphi(x) = ax + b$ and $\psi(x) = cx + d$, i.e., they are affine functions,
(ii) or $\varphi(x) = a\ln x + b$ and $\psi(x) = cx^{-3} + d$.

EXAMPLE. Consider the functional equation

$$\varphi^{-1}\left(\frac{2\varphi(x)+\varphi(y)}{3}\right)=\psi^{-1}\left(\frac{4\psi(x)+4\psi\left(\frac{x+y}{2}\right)+\psi(y)}{9}\right),$$

where $\varphi,\psi : I \to \mathbb{R}$ are continuous strictly monotone functions.

If $\mathcal{C}_3$ is assumed, the generating functions $\varphi,\psi : I \to \mathbb{R}$ satisfy this functional equation if and only if

(i) either there exist real constants a,b,c,d with $ac \neq 0$ such that $\varphi(x) = ax+b$ and $\psi(x) = cx + d$, i.e., they are affine functions,
(ii) or there exist real constants a,b,c,d,p with $acp \neq 0$ such that $\varphi(x) = ae^{px}+b$ and $\psi(x) = ce^{2px} + d$.

1. B. The invariance problem for two variable means

In the third part of the first chapter of the dissertation we characterize the continuous strictly monotone functions φ,ψ and Borel probability measures μ,ν such that

$$\mathcal{M}_{\varphi,\mu}(x,y)+\mathcal{M}_{\psi,\nu}(x,y)=x+y \quad (x,y\in I)$$

holds.

The first result present a necessary condition of first-order.

COROLLARY. *Let μ and ν be a Borel probability measures. Assume $\mathcal{C}_0$. Suppose that there exists a point $p \in I$ such that φ and ψ are differentiable at p and $\varphi'(p)\psi'(p) \neq 0$. Then, in order that the invariance equation be valid, it is necessary that*

$$\widehat{\mu}_1 + \widehat{\nu}_1 = 1.$$

In the solution of the invariance equation, we consider two subcases according as $\mu_2\nu_2 = 0$, respectively $\mu_2\nu_2 \neq 0$.

THEOREM. *Let μ and ν be a Borel probability measures with $\mu_2\nu_2 = 0$. Assume $\mathcal{C}_2$. Then the invariance equation holds if and only if*

(i) either $\mu = \delta_\tau$, $\nu = \delta_{1-\tau}$ for some $\tau \in [0,1]$ and φ, ψ are arbitrary,
(ii) or $\mu = \delta_\tau$ for some $\tau \in [0,1]$, $\nu_2 \neq 0$, $\widehat{\nu}_1 = 1-\tau$, φ is arbitrary and there exist constants $a \neq 0$ and b such that

$$\psi(x) = ax + b \qquad (x \in I),$$

(iii) or $\nu = \delta_{1-\tau}$ for some $\tau \in [0,1]$, $\mu_2 \neq 0$, $\widehat{\mu}_1 = \tau$, ψ is arbitrary and there exist constants $a \neq 0$ and b such that

$$\varphi(x) = ax + b \qquad (x \in I).$$

Our first main result in this part offers a necessary condition for the validity of the invariance equation in terms of two differential equations for the second-order partial derivative $\partial_1^2\mathcal{M}_{\varphi,\mu}$ of the mean $\mathcal{M}_{\varphi,\mu}$.

THEOREM. *Let μ and ν be a Borel probability measures with $\mu_2\nu_2 \neq 0$ and assume that the invariance equation is satisfied. If $\mathcal{C}_3$ holds then the function $\Phi : I \to \mathbb{R}$ defined by*

$$\Phi(x) := \partial_1^2\mathcal{M}_{\varphi,\mu}(x,x)$$

satisfies the differential equation

$$\left(\frac{3\widehat{\mu}_1\mu_2 + \mu_3}{\mu_2} - \frac{3\widehat{\nu}_1\nu_2 + \nu_3}{\nu_2}\right)\Phi' + \left(\frac{\mu_3}{\mu_2^2} + \frac{\nu_3}{\nu_2^2}\right)\Phi^2 = 0$$

and if

$$(\mu_3, \nu_3) \neq \frac{3(\widehat{\mu}_1 - \widehat{\nu}_1)}{\mu_2 + \nu_2}(-\mu_2^2, \nu_2^2),$$

then the coefficients in this equation do not vanish simultaneously.
If, in addition, $\mathcal{C}_4$ holds then Φ also satisfies the differential equation

$$\begin{aligned}&\Big(\frac{6\widehat{\mu}_1^2\mu_2 + 4\widehat{\mu}_1\mu_3 + \mu_4}{\mu_2} - \frac{6\widehat{\nu}_1^2\nu_2 + 4\widehat{\nu}_1\nu_3 + \nu_4}{\nu_2}\Big)\Phi'' \\ &+ \Big(\frac{8\widehat{\mu}_1\mu_3 + 3\mu_4}{\mu_2^2} + \frac{8\widehat{\nu}_1\nu_3 + 3\nu_4}{\nu_2^2}\Big)\Phi\Phi' + \Big(\frac{\mu_4 - 3\mu_2^2}{\mu_2^3} - \frac{\nu_4 - 3\nu_2^2}{\nu_2^3}\Big)\Phi^3 = 0.\end{aligned}$$

By our second main result, under three times continuous differentiability assumptions and certain non-degeneracy conditions on the second and third centralized moments of the two measures, the solutions of the invariance equation fall into three different classes. The unknown generator functions φ and ψ are either linear, or exponential or power functions.

THEOREM. *Let μ and ν be a Borel probability measures with $\mu_2\nu_2 \neq 0$ and $(\mu_3, \nu_3) \neq \frac{3(\widehat{\mu}_1-\widehat{\nu}_1)}{\mu_2+\nu_2}(-\mu_2^2, \nu_2^2)$. Assume also $\mathcal{C}_3$. Then the invariance equation holds if and only if $\widehat{\mu}_1 + \widehat{\nu}_1 = 1$ and*

(i) either there exist real constants a, b, c, d with $ac \neq 0$ such that

$$\varphi(x) = ax + b \qquad \textit{and} \qquad \psi(x) = cx + d \qquad (x \in I);$$

(ii) or there exist real constants a, b, c, d, p, q with $ac \neq 0$, $pq < 0$ such that

$$\varphi(x) = ae^{px} + b \qquad \textit{and} \qquad \psi(x) = ce^{qx} + d \qquad (x \in I)$$

and, for $n \in \mathbb{N}$

$$\sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} \big(\mu_{i+1}\nu_{n-i} + \mu_i \nu_{n+1-i}\big) = 0;$$

(iii) or there exist real constants a, b, c, d, p, q with $ac \neq 0$, $(p-1)(q-1) < 0$, and $x_0 \notin I$ such that, for $x \in I$,

$$\varphi(x) = \begin{cases} a|x - x_0|^p + b, & \textit{if } p \neq 0 \\ a \ln |x - x_0| + b, & \textit{if } p = 0, \end{cases}$$

$$\psi(x) = \begin{cases} c|x - x_0|^q + d, & \textit{if } q \neq 0 \\ c \ln |x - x_0| + d, & \textit{if } q = 0 \end{cases}$$

and, with the notation

$$F_{p,\mu}(z) := \begin{cases} \Big(\int_0^1 (1 + tz)^p d\mu(t) \Big)^{\frac{1}{p}}, & \textit{if } p \neq 0 \\ \exp \Big(\int_0^1 \ln(1 + tz) d\mu(t) \Big), & \textit{if } p = 0 \end{cases} \qquad (z > -1),$$

the identity

$$F_{p,\mu}(z) + F_{q,\nu}(z) = 2 + z \qquad (z > -1)$$

holds.

THEOREM G. *Let μ, ν be a Borel probability measures with $\widehat{\mu}_1 + \widehat{\nu}_1 = 1$, $\mu_2 = \nu_2 \neq 0$, $\mu_3 = -\nu_3$, such that*

$$\mu_3 \neq 3\Big(\frac{1}{2} - \widehat{\mu}_1\Big)\mu_2.$$

Assume also $\mathcal{C}_3$. Then the invariance equation is satisfied if and only if *(i) either there exist real constants* a, b, c, d *with* $ac \neq 0$ *such that*

$$\varphi(x) = ax + b \qquad \textit{and} \qquad \psi(x) = cx + d \qquad (x \in I);$$

(ii) or there exist real constants a, b, c, d, p *with* $acp \neq 0$*, such that*

$$\varphi(x) = ae^{px} + b \qquad \textit{and} \qquad \psi(x) = ce^{-px} + d \qquad (x \in I)$$

and ν *is the reflection of* μ *with respect to the point* $1/2$*.*

THEOREM. *Let* μ, ν *be a Borel probability measures with* $\widehat{\mu}_1 = \widehat{\nu}_1 = \frac{1}{2}$*,* $\mu_2 = \nu_2 \neq 0$*,* $\mu_3 = -\nu_3$*,* $\mu_4 = \nu_4$*. Assume also* $\mathcal{C}_4$*. Then the invariance equation is satisfied if and only if one of the alternatives of Theorem G holds.*

In the next result we consider the particular case of the previous theorem when $\mu = \nu$ is a symmetric measure.

COROLLARY. *Let* μ *be a Borel probability measure with* $\mu_2 \neq 0$ *which is symmetric with respect to the point* $1/2$*. Assume also* $\mathcal{C}_4$*. Then the invariance equation*

$$\mathcal{M}_{\varphi,\mu}(x, y) + \mathcal{M}_{\psi,\mu}(x, y) = x + y \qquad (x, y \in I)$$

is satisfied if and only if

(i) either there exist real constants a, b, c, d *with* $ac \neq 0$ *such that*

$$\varphi(x) = ax + b \qquad \textit{and} \qquad \psi(x) = cx + d \qquad (x \in I);$$

(ii) or there exist real constants a, b, c, d, p *with* $acp \neq 0$*, such that*

$$\varphi(x) = ae^{px} + b \qquad \textit{and} \qquad \psi(x) = ce^{-px} + d \qquad (x \in I).$$

In the subsequent examples we demonstrate how some known results of the literature follow from ours.

EXAMPLE. Consider the functional equation

$$\varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right) + \psi^{-1}\left(\frac{\psi(x) + \psi(y)}{2}\right) = x + y,$$

where $\varphi, \psi : I \to \mathbb{R}$ are continuous strictly monotone functions and $x, y \in I$.

If $\mathcal{C}_4$ is assumed, the generating functions φ and ψ satisfy this functional equation if and only if

(i) either there exist real constants a, b, c, d with $ac \neq 0$ such that $\varphi(x) = ax + b$ and $\psi(x) = cx + d$ $(x \in I)$;

(ii) or there exist real constants a, b, c, d, p with $acp \neq 0$, such that $\varphi(x) = ae^{px} + b$ and $\psi(x) = ce^{-px} + d$ $(x \in I)$.

This statement was first proved by Sutô [**71**], [**72**] assuming analyticity and by Matkowski [**57**] who supposed twice continuous differentiability. After some preliminary regularity improving steps [**22**], [**27**], the main goal of the paper [**28**] was to show that the same conclusion can be obtained without any superflouos differentiability assumptions.

EXAMPLE. Consider the functional equation

$$\varphi^{-1}\big(\lambda\varphi(x) + (1-\lambda)\varphi(y)\big) + \psi^{-1}\big((1-\lambda)\psi(x) + \lambda\psi(y)\big) = x + y,$$

where $\varphi, \psi : I \to \mathbb{R}$ are continuous strictly monotone functions, $\lambda \in [0,1] \setminus \big\{0, \frac{1}{2}, 1\big\}$ and $x, y \in I$.

If $\mathcal{C}_3$ is assumed, the generating functions φ and ψ satisfy this functional equation if and only if

(i) either there exist real constants a, b, c, d with $ac \neq 0$ such that

$$\varphi(x) = ax + b \qquad \text{and} \qquad \psi(x) = cx + d \qquad (x \in I);$$

(ii) or there exist real constants a, b, c, d, p with $acp \neq 0$, such that

$$\varphi(x) = ae^{px} + b \qquad \text{and} \qquad \psi(x) = ce^{-px} + d \qquad (x \in I).$$

and ν is the reflection of μ with respect to the point $1/2$.

The result so obtained has been discovered by Jarczyk and Matkowski [**44**] and has recently been proved without any continuous differentiability assumptions by Jarczyk [**43**].

EXAMPLE. Consider the functional equation

$$\varphi^{-1}\left(\frac{1}{y-x}\int_x^y \varphi(t)dt\right) + \psi^{-1}\left(\frac{1}{y-x}\int_x^y \psi(t)dt\right) = x + y,$$

where $\varphi, \psi : I \to \mathbb{R}$ are continuous strictly monotone functions and $x, y \in I,\ x \neq y$.

If $\mathcal{C}_4$ is assumed, the generating functions φ and ψ satisfy this functional equation if and only if

(i) either there exist real constants a, b, c, d with $ac \neq 0$ such that $\varphi(x) = ax + b$ and $\psi(x) = cx + d$ $(x \in I)$;

(ii) or there exist real constants a, b, c, d, p with $acp \neq 0$, such that $\varphi(x) = ae^{px} + b$ and $\psi(x) = ce^{-px} + d$ $(x \in I)$.

This result has been discovered with stronger regularity assumptions by Matkowski [**61**].

EXAMPLE. Consider the functional equation

$$\varphi^{-1}\left(\frac{2\varphi(x) + \varphi(y)}{3}\right) + \psi^{-1}\left(\frac{\psi(x) + 4\psi\big(\frac{x+y}{2}\big) + 4\psi(y)}{9}\right) = x + y,$$

where $\varphi, \psi : I \to \mathbb{R}$ are continuous strictly monotone functions and $x, y \in I$.

If $\mathcal{C}_3$ is assumed, the generating functions φ and ψ satisfy this functional equation if and only if

(i) either there exist real constants a, b, c, d with $ac \neq 0$ such that $\varphi(x) = ax + b$ and $\psi(x) = cx + d$,

(ii) or there exist real constants a, b, c, d, p with $acp \neq 0$ such that $\varphi(x) = ae^{px} + b$ and $\psi(x) = ce^{-2px} + d$.

2. Lipschitz perturbation of monotone functions

In the second chapter of the dissertation we investigate when a function $p : I \to \mathbb{R}$ can be decomposed in the form $p = q + \ell$, where q is increasing and ℓ is d-Lipschitz function.

The stability theory of functional inequalities started with the paper of Hyers and Ulam [**42**] (cf. also [**38**]). They discovered that the so-called δ-convex functions can be decomposed as the sum of a convex and a bounded function if the underlying space is of finite dimension. A more general form of this stability theorem has recently been obtained in [**69**], where the stability of convex functions was investigated under Lipschitz perturbations. A useful auxiliary concept introduced in [**69**] was the notion of ϵ-monotonicity which leaded to the stability properties of monotonic functions. A function $p : I \to \mathbb{R}$ is called ϵ-increasing if

$$p(x) \leq p(y) + \epsilon$$

holds for all $x \leq y$. It turned out in [**69**] that ϵ-increasing functions are closely related to increasing functions, more precisely, p is ϵ-increasing if and only if $p = q + h$, where q is an increasing function and h is a bounded function with $||h|| \leq \epsilon/2$.

Motivated by the above theorem, we investigate when a function p can be written in the form $p = q + \ell$, where q is increasing and ℓ is d-Lipschitz (i.e., it satisfies

$$|\ell(x) - \ell(y)| \leq d(x, y)$$

for $x, y \in I$.) Here $d : I^2 \to \mathbb{R}$ is assumed to be a semimetric on I. Our main results offer necessary and sufficient conditions for the above decomposability in the cases of general semimetrics and concave semimetrics.

LEMMA. *Let I be an interval of $\mathbb{R}$, $t_1 < s_1, \dots, t_n < s_n$ and $u_1 < v_1, \dots, u_m < v_m$ be real numbers in I such that*

$$\sum_{i=1}^{n} 1_{]t_i, s_i]} = \sum_{i=1}^{m} 1_{]u_i, v_i]}$$

be fulfilled. Then the following equality is true:

$$\sum_{i=1}^{n} \big(q(s_i) - q(t_i)\big) = \sum_{i=1}^{m} \big(q(v_i) - q(u_i)\big).$$

Let $d : I^2 \to I$ be a semimetric throughout the rest of the paper (i.e., d is a nonnegative symmetric two variable function satisfying also the triangle inequality). A function $\ell : I \to \mathbb{R}$ is said to be *d-Lipschitz* if $|\ell(x) - \ell(y)| \leq d(x, y)$ for $x, y \in I$. The notation x^+ will stand for the positive part of $x \in \mathbb{R}$, i.e., $x^+ := \max(0, x)$.

The next theorem contains our main result that gives the first characterization for Lipschitz perturbations of increasing functions. The proof of the sufficiency will directly utilize the theorem of Kindler quoted in the previous section.

THEOREM. *A function $p : I \to \mathbb{R}$ can be written in the form $p = q + \ell$, where q is increasing and ℓ is d-Lipschitz if and only if*

$$\sum_{i=1}^{n} \big(p(s_i) - p(t_i) - d(t_i, s_i)\big)^+ \le \sum_{j=1}^{m} \big(p(v_j) - p(u_j) + d(u_j, v_j)\big)$$

is fulfilled for all real numbers $t_1 < s_1, \dots, t_n < s_n$ and $u_1 < v_1, \dots, u_m < v_m$ in I satisfying $\sum_{i=1}^{n} 1_{]t_i,s_i]} = \sum_{i=1}^{m} 1_{]u_i,v_i]}$.

Using this lemma, we obtain another characterization of the decomposability $p = q + \ell$.

LEMMA. *Let $t_1 < s_1, \dots, t_n < s_n$ and $u_1 < v_1, \dots, u_m < v_m$ in I satisfying $\sum_{i=1}^{n} 1_{]t_i,s_i]} = \sum_{i=1}^{m} 1_{]u_i,v_i]}$. Then*

$$\sum_{i=1}^{n} \big(p(s_i) - p(t_i) - d(t_i, s_i)\big)^+ \le \sum_{j=1}^{m} \big(p(v_j) - p(u_j) + d(u_j, v_j)\big)$$

holds for a function $p : I \to \mathbb{R}$ if and only if

$$0 \le \sum_{i=1}^{n} \min\big(d(t_i, s_i), p(s_i) - p(t_i)\big) + \sum_{j=1}^{m} d(u_j, v_j).$$

Denote by $\mathcal{F}(I)$ the class of those functions that are of the following form

$$f = \sum_{i=1}^{n} 1_{]t_i,s_i]} - \sum_{j=1}^{m} 1_{]u_j,v_j]}, \tag{2}$$

where $t_1 < s_1, \dots, t_n < s_n, u_1 < v_1, \dots, u_m < v_m$ are in I. Then $\mathcal{F}(I)$ is closed under the usual pointwise addition.

Given an arbitrary function $q : I \to \mathbb{R}$, define now a functional $\mathcal{I}_q(f) : \mathcal{F}(I) \to \mathbb{R}$ by

$$\mathcal{I}_q(f) = \sum_{i=1}^{n} \big(q(s_i) - q(t_i)\big) - \sum_{j=1}^{m} \big(q(v_j) - q(u_j)\big),$$

where f is given by (2).

THEOREM. *A function $p : I \to \mathbb{R}$ can be written in the form $p = q + \ell$, where q is increasing and ℓ is d-Lipschitz if and only if*

$$0 \le \sum_{i=1}^{n} d(t_i, s_i) + \sum_{j=1}^{m} d(u_j, v_j) + \mathcal{I}_p\Big(\sum_{j=1}^{m} 1_{]u_j,v_j]} - \sum_{i=1}^{n} 1_{]t_i,s_i]}\Big)$$

for all real numbers $t_1 < s_1, \dots, t_n < s_n$ *and* $u_1 < v_1, \dots, u_m, < v_m \in I$ *satisfying*

$$\sum_{i=1}^{n} 1_{]t_i,s_i]} \le \sum_{j=1}^{m} 1_{]u_j,v_j]}.$$

If the semimetric d possesses further properties, we are going to obtain simpler necessary and sufficient conditions in order that a function p could be decomposed as $q + \ell$ where q is monotone increasing and ℓ is d-Lipschitz.

DEFINITION. A semimetric $d : I \times I \to \mathbb{R}$ is called *concave* if, for all $x \le y \le z \le w \in I$, it satisfies

$$d(x, w) + d(y, z) \le d(x, z) + d(y, w).$$

The following theorem characterizes the decomposability $p = q + \ell$ in case of concave semimetrics. In the case of when the semimetric coincides with the ordinary distance function, our condition simplifies to a two variable inequality only.

THEOREM. *A function* $p : I \to \mathbb{R}$ *can be written in the form* $p = q + \ell$*, where* q *is increasing and* ℓ *is* d*-Lipschitz if and only if*

$$0 \le \sum_{k=1}^{n} d(x_{2k-1}, x_{2k}) + d(x_0, x_{2n+1}) + \sum_{k=0}^{n} \big(p(x_{2k+1}) - p(x_{2k})\big)$$

holds for all $x_0 \le x_1 < \cdots < x_{2n} \le x_{2n+1} \in I$.

THEOREM. *If the metric* d *is given by* $d(x, y) = |y - x|$ $(x, y \in I)$*, then*

$$0 \le \sum_{k=1}^{n} d(x_{2k-1}, x_{2k}) + d(x_0, x_{2n+1}) + \sum_{k=0}^{n} \big(p(x_{2k+1}) - p(x_{2k})\big)$$

holds for all $x_0 \le x_1 < \cdots < x_{2n} \le x_{2n+1} \in I$ *if and only if*

$$p(x) \le p(y) + d(x, y)$$

for all $x < y \in I$.

Összefoglaló

1.A Általánosított kvázi-aritmetikai közepek egyenlőségi problémájáról

A klasszikus kváziaritmetikai közepek fogalmát súlyfüggvények és paraméterek hozzáadása révén többféleképpen is általánosíthatjuk. A disszertáció egyik célja általánosított kvázi-aritmetikai közepek egyenlőségi és invariancia problémájának vizsgálata. A disszertációban vizsgált általánosított kvázi-aritmetikai közepeket a következőképpen definiáljuk.

DEFINÍCIÓ. Legyen $\varphi : I \to \mathbb{R}$ egy folytonos, szigorúan monoton függvény, μ egy, a $[0,1]$ intervallum Borel halmazain értelmezett valószínűségi mérték. Ekkor az $\mathcal{M}_{\varphi,\mu} : I^2 \to I$ általánosított kvázi-aritmetikai középet a következő képlettel értelmezzük:

$$\mathcal{M}_{\varphi,\mu}(x,y) := \varphi^{-1}\Big(\int_0^1 \varphi\big(tx+(1-t)y\big)d\mu(t)\Big) \qquad (x,y \in I).$$

Ha $\mu = \dfrac{\delta_0+\delta_1}{2}$, ahol δ_t a $t \in [0,1]$ pontra koncentrált Dirac mérték, akkor $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\varphi}$. Ha pedig μ Lebesgue mérték a $[0,1]$ intervallumon, akkor $\mathcal{M}_{\varphi,\mu} = \mathcal{L}_{\varphi}$.

Az első fejezet első részében összefoglaljuk az eredmények bemutatásához szükséges jelöléseket és alapvető eredményeket. Definiáljuk egy μ Borel valószínűségi mérték *k-adik momentumát* és *k-adik centrális momentumát*:

$$\widehat{\mu}_k := \int_0^1 t^k d\mu(t) \qquad \text{és} \qquad \mu_k := \int_0^1 (t-\widehat{\mu}_1)^k d\mu(t) \qquad (k \in \mathbb{N} \cup \{0\}),$$

és az $\frac{1}{2}$ pontra vonatkozó tükörképét:

$$\widetilde{\mu}(A) = \mu(\widetilde{A}),$$

ahol A a $[0,1]$ intervallum Borel részhalmaza és $\widetilde{A} := 1 - A := \{1-x \mid x \in A\}$. Az általánosított kvázi-aritmetikai közepek egyenlőségi problémájára vonatkozó eredményeket a különböző rendű momentum feltételek teljesülése szerint mutatjuk be, ehhez szükségünk van a következő definíciókra:

Azt mondjuk, hogy az *n-ed rendű momentum feltétel* $\mathcal{M}_n$ teljesül valamely $n \in \mathbb{N} \cup \{0,\infty\}$ esetén, ha μ, ν a $[0,1]$ intervallumon definiált Borel valószínűségi mértékek, és

$$\widehat{\mu}_k = \widehat{\nu}_k \quad \text{minden} \quad 1 \le k \le n,$$

illetve, hogy az *egzakt n-ed rendű momentum feltétel* $\mathcal{M}_n^*$ teljesül, ha $\mathcal{M}_n$ teljesül, de $\mathcal{M}_{n+1}$ nem teljesül, azaz

$$\widehat{\mu}_k = \widehat{\nu}_k \quad \text{minden} \quad 1 \le k \le n \qquad \text{és} \qquad \widehat{\mu}_{n+1} \ne \widehat{\nu}_{n+1}.$$

A kvázi-aritmetikai közepek generátor függvényeire vonatkozó különböző regularitási feltételek megadásához bevezetjük az *n-ed rendű regularitási feltétel* definícióját. A $\varphi, \psi : I \to \mathbb{R}$ függvények teljesítik az n-ed rendű regularitási feltételt, ha n-szer folytonosan differenciálhatóak, és az első rendű deriváltjuk seholsem tűnik el.

LEMMA. *Legyen μ Borel valószínűségi mérték, $\varphi : I \to \mathbb{R}$ folytonos, szigorúan monoton függvény, és tegyük fel, hogy φ differenciálható a $p \in I$ pontban és $\varphi'(p) \ne 0$. Ekkor $\partial_1 M_{\varphi,\mu}(p,p) = \widehat{\mu}_1$.*

A fejezet további részeiben megadjuk azoknak a (φ, μ) és (ψ, ν) pároknak a jellemzését, amelyek megoldásai az általánosított kvázi-aritmetikai közepek egyenlőségi, illetve invariancia problémájának. Az első fejezet második részében az egyenlőségi problémát, azaz a következő egyenletet vizsgáljuk:

$$\mathcal{M}_{\varphi,\mu}(x,y) = \mathcal{M}_{\psi,\nu}(x,y) \quad (x, y \in I).$$

Az alábbi szükséges feltétel azt mutatja, hogy gyenge regularitási feltételek mellett, ha az $\mathcal{M}_0^*$ egzakt momentum feltétel teljesül, akkor az egyenlőségi problémának nincsen megoldása.

KÖVETKEZMÉNY. *Tegyük fel, hogy $\mathcal{C}_0$ és $\mathcal{M}_0$ teljesül, és hogy létezik egy $p \in I$ pont úgy, hogy φ és ψ differenciálható a p pontban és $\varphi'(p)\psi'(p) \ne 0$. Ekkor az $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ egyenlet teljesülésének szükséges feltétele*

$$\widehat{\mu}_1 = \widehat{\nu}_1,$$

azaz, $\mathcal{M}_1$ teljesül.

A következő eredményben az egyenlőségi probléma olyan jellemzését kapjuk, melyhez nincs szükség az ismeretlen függvények inverzeire.

TÉTEL. *Tegyük fel, hogy $\mathcal{C}_1$ és $\mathcal{M}_1$ teljesül. Ekkor az $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ egyenlet akkor és csak akkor teljesül bármely $x, y \in I$ esetén, ha*

$$\int_0^1 \int_0^1 (t-s)\varphi'\big(tx + (1-t)y\big)\psi'\big(sx + (1-s)y\big)d\mu(t)d\nu(s) = 0.$$

Feltételezve, hogy az $n+1$-ed rendű regularitási feltétel teljesül, egy újabb szükséges feltételt kapunk az egyenlőségi probléma teljesülésére.

TÉTEL. *Tegyük fel, hogy $\mathcal{C}_{n+1}$ $(n \in \mathbb{N})$ és $\mathcal{M}_1$ teljesül. Ekkor az $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ egyenlet teljesülésének szükséges feltétele, hogy*

$$\text{(1)} \qquad \sum_{i=0}^{n} \binom{n}{i} \big(\mu_{i+1}\nu_{n-i} - \mu_i \nu_{n+1-i}\big) \frac{\varphi^{(i+1)}}{\varphi'} \cdot \frac{\psi^{(n+1-i)}}{\psi'} = 0.$$

Megfordítva, ha φ, ψ analitikus függvények, és (1) *teljesül minden $n \in \mathbb{N}$ esetén, akkor $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ teljesül.*

Ha a két mérték egyenlő, az egyenlőségi problémára a következő megoldást kapjuk.

TÉTEL. *Tegyük fel, hogy $\mathcal{C}_0$ és $\mathcal{M}_\infty$ teljesül. Ekkor $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ akkor és csak akkor igaz, ha*

(i) vagy $\mu = \nu = \delta_\tau$ valamely $\tau \in [0,1]$ esetén és φ, ψ tetszőleges függvények,
(ii) vagy $\mu = \nu$ nem Dirac mérték és léteznek $a \neq 0$ és b konstansok úgy, hogy

$$\psi = a\varphi + b.$$

Ha a két mérték nem egyenlő, de legalább az első két momentumuk megegyezik, akkor az egyenlőségi problémára az alábbi jellemzést kapjuk.

TÉTEL. *Tegyük fel, hogy $\mathcal{C}_2$ és $\mathcal{M}_n^*$ teljesül valamely $2 \leq n < \infty$ esetén. Ekkor $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ akkor és csak akkor áll fenn, ha léteznek $a \neq 0$ és b konstansok úgy, hogy*

$$\psi = a\varphi + b$$

és φ n-től nem nagyobb fokszámú polinom.

Ha $\mathcal{M}_1^*$ teljesül, akkor két esetet különböztetünk meg aszerint, hogy $\mu_2\nu_2 = 0$, illetve $\mu_2\nu_2 \neq 0$.

TÉTEL. *Tegyük fel, hogy $\mathcal{C}_2$, $\mathcal{M}_1^*$ és $\mu_2\nu_2 = 0$ teljesül. Ekkor $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ akkor és csak akkor igaz, ha*

(i) vagy μ és ψ tetszőleges, $\nu = \delta_{\widehat{\mu}_1}$, és léteznek $a \neq 0$ és b konstansok úgy, hogy

$$\varphi(x) = ax + b \qquad (x \in I),$$

(ii) vagy ν és φ tetszőleges, $\mu = \delta_{\widehat{\nu}_1}$, és léteznek $c \neq 0$ és d konstansok úgy, hogy

$$\psi(x) = cx + d \qquad (x \in I).$$

Abban az esetben, amikor $\mu_2\nu_2 \neq 0$, további szükséges feltételeket kapunk az egyenlőségi probléma teljesülésére.

TÉTEL. *Tegyük fel, hogy $\mathcal{C}_2$, $\mathcal{M}_1$, $\mu_2\nu_2 \neq 0$ teljesül, és tegyük fel, hogy fennáll az $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ egyenlet. Ekkor*

$$\nu_2 \frac{\psi''}{\psi'} = \mu_2 \frac{\varphi''}{\varphi'} =: \Phi.$$

Ha teljesül a 3-ad rendű regularitási feltétel is, akkor a fenti $\Phi : I \to \mathbb{R}$ függvény kielégíti a következő differenciálegyenletet:

$$\left(\frac{\mu_3}{\mu_2} - \frac{\nu_3}{\nu_2}\right)\Phi' + \left(\frac{\mu_3}{\mu_2^2} - \frac{\nu_3}{\nu_2^2}\right)\Phi^2 = 0.$$

Ha még a 4-ed rendű regularitási feltétel is teljesül, akkor φ és ψ analitikus függvények és Φ kielégíti az alábbi differenciálegyenletet.

$$\left(\frac{\mu_4}{\mu_2}-\frac{\nu_4}{\nu_2}\right)\Phi''+\left(\frac{3\mu_4}{\mu_2^2}-\frac{3\nu_4}{\nu_2^2}\right)\Phi\Phi'+\left(\frac{\mu_4-3\mu_2^2}{\mu_2^3}-\frac{\nu_4-3\nu_2^2}{\nu_2^3}\right)\Phi^3=0.$$

Ha teljesül az első rendű momentum feltétel, akkor az utóbbi differenciálegyenlet együtthatói egyszerre nem tűnnek el.

A következő eredmény szükséges és elégséges feltételt ad az egyenlőségi problémára, azzal a feltétellel, hogy a Φ függvény kielégít egy elsőrendű differenciálegyenletet.

TÉTEL. *Tegyük fel, hogy $\mathcal{C}_3$, $\mathcal{M}_1$ és $\mu_2\nu_2\neq 0$, továbbá*

$$\nu_2\frac{\psi''}{\psi'}=\mu_2\frac{\varphi''}{\varphi'}=:\Phi$$

teljesül, léteznek $0\leq 2n\leq k$ egész számok és egy $(c_0,\dots,c_n)\neq(0,\dots,0)$ konstans vektor úgy, hogy a $\Phi: I\to\mathbb{R}$ függvény teljesíti a következő polinomiális differenciálegyenletet:

$$\sum_{i=0}^{n}c_i\Phi^{k-2i}\big(\Phi'\big)^i=0.$$

Ekkor $\mathcal{M}_{\varphi,\mu}=\mathcal{M}_{\psi,\nu}$ akkor és csak akkor teljesül, ha

(i) vagy léteznek valós konstansok a,b,c,d, $ac\neq 0$ úgy, hogy

$$\varphi(x)=ax+b,\qquad\text{és}\qquad\psi(x)=cx+d\qquad(x\in I),$$

(ii) vagy léteznek valós konstansok a,b,c,d,p,q, $ac(p-q)\neq 0$, $pq>0$ úgy, hogy

$$\varphi(x)=ae^{px}+b\qquad\text{és}\qquad\psi(x)=ce^{qx}+d\qquad(x\in I),\tag{2}$$

és $n\in\mathbb{N}$ esetén

$$\sum_{i=0}^{n}\binom{n}{i}p^iq^{n-i}\big(\mu_{i+1}\nu_{n-i}-\mu_i\nu_{n+1-i}\big)=0;$$

(iii) vagy léteznek valós konstansok a,b,c,d,p,q, $ac(p-q)\neq 0$, $(p-1)(q-1)>0$, és $x_0\notin I$ úgy, hogy $x\in I$ esetén

$$\varphi(x)=\begin{cases}a|x-x_0|^p+b, & \text{ha } p\neq 0\\ a\ln|x-x_0|+b, & \text{ha } p=0,\end{cases}$$

$$\psi(x)=\begin{cases}c|x-x_0|^q+d, & \text{ha } q\neq 0\\ c\ln|x-x_0|+d, & \text{ha } q=0\end{cases}$$

és $n\in\mathbb{N}$ esetén

$$\sum_{i=0}^{n}\binom{p-1}{i}\binom{q-1}{n-i}\big(\mu_{i+1}\nu_{n-i}-\mu_i\nu_{n+1-i}\big)=0.$$

Ha $\mu_2\nu_2 \neq 0$ és $(\mu_3,\nu_3) \neq (0,0)$, vagy $\mu_2\nu_2 \neq 0$, $(\mu_3,\nu_3) = (0,0)$ és $\mu_2\nu_4 = \nu_2\mu_4$ vagy $\mu_2\nu_2 \neq 0$, $(\mu_3,\nu_3) = (0,0)$ és $\mu_2\nu_4 \neq \nu_2\mu_4$ és $(\mu_5,\nu_5) \neq (0,0)$, akkor a megfelelő regularitási és momentum feltételek teljesülése mellett azt kapjuk, hogy az $\mathcal{M}_{\varphi,\mu} = \mathcal{M}_{\psi,\nu}$ egyenlet akkor és csak akkor igaz, ha az előző tétel állításai közül valamelyik teljesül, azaz a (φ,ψ) megoldás pár vagy lineáris, vagy exponenciális, vagy hatványfüggvény. Ezen esetek mindegyikében megmutatható, hogy Φ kielégít egy polinomiális differenciálegyenletet.

Az utolsó alfejezetben példákat mutatunk be arra vonatkozóan, hogy eredményeink segítségével hogyan kaphatjuk meg különböző függvényegyenletek megoldásait.

PÉLDA. Tekintsük a

$$\varphi^{-1}\left(\frac{\varphi\left(\frac{2x+y}{3}\right)+\varphi\left(\frac{x+2y}{3}\right)}{2}\right)=\psi^{-1}\left(\frac{\psi(x)+16\psi\left(\frac{x+y}{2}\right)+\psi(y)}{18}\right)$$

függvényegyenletet, ahol $\varphi,\psi : I \to \mathbb{R}$ folytonos, szigorúan monoton függvények.

Ha $\mathcal{C}_4$ teljesül, a $\varphi,\psi : I \to \mathbb{R}$ függvények akkor és csak akkor megoldásai a fenti függvényegyenletnek, ha léteznek $a \neq 0$ és b számok úgy, hogy $\psi = a\varphi + b$ és φ egy tetszőleges 3-nál nem nagyobb fokszámú polinom.

PÉLDA. Tekintsük a

$$\varphi^{-1}\left(\frac{2\varphi(x)+\varphi(y)}{3}\right)=\psi^{-1}\left(\int_0^1 2t\psi(tx+(1-t)y)dt\right)$$

függvényegyenletet, ahol $\varphi,\psi : I \to \mathbb{R}$ folytonos, szigorúan monoton függvények.

Ha $\mathcal{C}_3$ teljesül, a φ és ψ generátorfüggvények akkor és csak akkor megoldásai a fenti függvényegyenletnek, ha léteznek olyan a,b,c,d valós számok, melyekre $ac \neq 0$ úgy, hogy

(i) $\varphi(x) = ax + b$ és $\psi(x) = cx + d$, azaz φ és ψ affin függvények,
(ii) vagy $\varphi(x) = a\ln x + b$ és $\psi(x) = cx^{-3} + d$.

PÉLDA. Tekintsük a

$$\varphi^{-1}\left(\frac{2\varphi(x)+\varphi(y)}{3}\right)=\psi^{-1}\left(\frac{4\psi(x)+4\psi\left(\frac{x+y}{2}\right)+\psi(y)}{9}\right)$$

függvényegyenletet, ahol $\varphi,\psi : I \to \mathbb{R}$ folytonos, szigorúan monoton függvények.

Ha $\mathcal{C}_3$ teljesül, a φ és ψ generátorfüggvények akkor és csak akkor megoldásai a fenti függvényegyenletnek, ha

(i) léteznek olyan a,b,c,d valós számok, melyekre $ac \neq 0$ úgy, hogy $\varphi(x) = ax + b$ és $\psi(x) = cx + d$,

(ii) vagy léteznek a, b, c, d, p, olyan valós számok, melyekre $acp \neq 0$ úgy, hogy $\varphi(x) = ae^{px} + b$ és $\psi(x) = ce^{2px} + d$.

1.B Az invariancia egyenlet általánosított kvázi-aritmetikai közepekre

Az első fejezet harmadik részében a Matkowski-Sutô problémát, azaz a következő egyenletet vizsgáljuk:

$$\mathcal{M}_{\varphi,\mu}(x, y) + \mathcal{M}_{\psi,\nu}(x, y) = x + y \qquad (x, y \in I).$$

Első eredményünk az első szükséges feltétel az invariancia egyenlet teljesülésére.

KÖVETKEZMÉNY. *Legyen μ és ν Borel valószínűségi mérték. Tegyük fel, hogy $\mathcal{C}_0$ teljesül, és létezik egy $p \in I$ pont úgy, hogy a φ és ψ függvények differenciálhatók p-ben és $\varphi'(p)\psi'(p) \neq 0$. Ekkor az $\mathcal{M}_{\varphi,\mu}(x, y)+\mathcal{M}_{\psi,\nu}(x, y) = x+y$ egyenlet teljesülésének szükséges feltétele, hogy*

$$\widehat{\mu}_1 + \widehat{\nu}_1 = 1.$$

Az $\mathcal{M}_{\varphi,\mu}(x, y)+\mathcal{M}_{\psi,\nu}(x, y) = x+y$ egyenlet megoldásánál szintén két esetet különböztetünk meg aszerint, hogy $\mu_2\nu_2 = 0$, illetve $\mu_2\nu_2 \neq 0$.

TÉTEL. *Legyen μ és ν Borel valószínűségi mérték úgy, hogy $\mu_2\nu_2 = 0$. Tegyük fel, hogy $\mathcal{C}_2$ teljesül. Ekkor az invariancia egyenlet akkor és csak akkor igaz, ha*

(i) vagy $\mu = \delta_\tau$, $\nu = \delta_{1-\tau}$ valamely $\tau \in [0, 1]$-ra, és φ, ψ tetszőleges függvények,

(ii) vagy $\mu = \delta_\tau$ valamely $\tau \in [0, 1]$-ra, $\nu_2 \neq 0$, $\widehat{\nu}_1 = 1 - \tau$, φ tetszőleges és létezik $a \neq 0$ és b konstans úgy, hogy

$$\psi(x) = ax + b \qquad (x \in I),$$

(iii) vagy $\nu = \delta_{1-\tau}$ valamely $\tau \in [0, 1]$-ra, $\mu_2 \neq 0$, $\widehat{\mu}_1 = \tau$, ψ tetszőleges, és létezik $a \neq 0$ és b konstans úgy, hogy

$$\varphi(x) = ax + b \qquad (x \in I).$$

A $\mu_2\nu_2 \neq 0$ esetben a következő eredményünk szükséges feltételt ad az invariancia egyenlet teljesülésére az $\mathcal{M}_{\varphi,\mu}$ közép másodrendű parciális deriváltjaira vonatkozó differenciálegyenletek segítségével.

TÉTEL. *Legyen μ és ν Borel valószínűségi mérték úgy, hogy $\mu_2\nu_2 \neq 0$ és tegyük fel, hogy az invariancia egyenletünk teljesül. Ha $\mathcal{C}_3$ fennáll, akkor a*

$$\Phi(x) := \partial_1^2\mathcal{M}_{\varphi,\mu}(x, x)$$

függvény kielégíti az alábbi differenciálegyenletet:

$$\left(\frac{3\widehat{\mu}_1\mu_2 + \mu_3}{\mu_2} - \frac{3\widehat{\nu}_1\nu_2 + \nu_3}{\nu_2}\right)\Phi' + \left(\frac{\mu_3}{\mu_2^2} + \frac{\nu_3}{\nu_2^2}\right)\Phi^2 = 0$$

és ha

$$(\mu_3, \nu_3) \neq \frac{3(\widehat{\mu}_1 - \widehat{\nu}_1)}{\mu_2 + \nu_2}(-\mu_2^2, \nu_2^2),$$

akkor az előző differenciálegyenletben az együtthatók egyszerre nem tűnnek el. Ha $\mathcal{C}_4$ fennáll, akkor Φ az alábbi differenciálegyenletnek megoldása:

$$\Big(\frac{6\widehat{\mu}_1^2\mu_2+4\widehat{\mu}_1\mu_3+\mu_4}{\mu_2}-\frac{6\widehat{\nu}_1^2\nu_2+4\widehat{\nu}_1\nu_3+\nu_4}{\nu_2}\Big)\Phi''$$
$$+\Big(\frac{8\widehat{\mu}_1\mu_3+3\mu_4}{\mu_2^2}+\frac{8\widehat{\nu}_1\nu_3+3\nu_4}{\nu_2^2}\Big)\Phi\Phi'+\Big(\frac{\mu_4-3\mu_2^2}{\mu_2^3}-\frac{\nu_4-3\nu_2^2}{\nu_2^3}\Big)\Phi^3=0.$$

Az invariancia egyenletre vonatkozó fő eredményünkben megkapjuk, hogy az

$$\mathcal{M}_{\varphi,\mu}(x,y)+\mathcal{M}_{\psi,\nu}(x,y)=x+y \qquad (x,y\in I)$$

egyenlet megoldásai 3 különböző osztályba sorolhatók. A φ és ψ generátor függvények lineáris, exponenciális vagy hatvány függvények.

TÉTEL. *Legyen μ és ν Borel valószínűségi mérték úgy, hogy $\mu_2\nu_2\neq 0$ és $(\mu_3,\nu_3)\neq\frac{3(\widehat{\mu}_1-\widehat{\nu}_1)}{\mu_2+\nu_2}(-\mu_2^2,\nu_2^2)$. Tegyük fel, hogy $\mathcal{C}_3$ teljesül. Ekkor az invariancia egyenlet akkor és csak akkor teljesül, ha $\widehat{\mu}_1+\widehat{\nu}_1=1$ és*

(i) vagy léteznek a,b,c,d konstansok, melyekre $ac\neq 0$ úgy, hogy

$$\varphi(x)=ax+b \qquad \text{és} \qquad \psi(x)=cx+d \qquad (x\in I);$$

(ii) vagy léteznek a,b,c,d,p,q konstansok, melyekre $ac\neq 0$, $pq<0$ úgy, hogy

$$\varphi(x)=ae^{px}+b \qquad \text{és} \qquad \psi(x)=ce^{qx}+d \qquad (x\in I)$$

és minden $n\in\mathbb{N}$ esetén

$$\sum_{i=0}^{n}\binom{n}{i}p^iq^{n-i}\big(\mu_{i+1}\nu_{n-i}+\mu_i\nu_{n+1-i}\big)=0;$$

(iii) vagy léteznek a,b,c,d,p,q konstansok, melyekre $ac\neq 0$, $(p-1)(q-1)<0$, és $x_0\notin I$ úgy, hogy minden $x\in I$ esetén

$$\varphi(x)=\begin{cases}a|x-x_0|^p+b, & \text{ha } p\neq 0\\ a\ln|x-x_0|+b, & \text{ha } p=0,\end{cases}$$

$$\psi(x)=\begin{cases}c|x-x_0|^q+d, & \text{ha } q\neq 0\\ c\ln|x-x_0|+d, & \text{ha } q=0\end{cases}$$

és az alábbi jelöléssel

$$F_{p,\mu}(z):=\begin{cases}\Big(\int_0^1(1+tz)^p d\mu(t)\Big)^{\frac{1}{p}}, & \text{ha } p\neq 0\\ \exp\Big(\int_0^1\ln(1+tz)d\mu(t)\Big), & \text{ha } p=0 \qquad (z>-1),\end{cases}$$

a következő azonosság teljesül:

$$F_{p,\mu}(z)+F_{q,\nu}(z)=2+z \qquad (z>-1).$$

Az alábbi két eredményben a μ és a ν mértékek első néhány momentumára különböző kikötéseket teszünk.

TÉTEL. *Legyen* μ, ν *Borel valószínűségi mérték úgy, hogy* $\widehat{\mu}_1 + \widehat{\nu}_1 = 1$, $\mu_2 = \nu_2 \neq 0$, $\mu_3 = -\nu_3$, *és*

$$\mu_3 \neq 3\Big(\frac{1}{2} - \widehat{\mu}_1\Big)\mu_2.$$

Tegyük fel, hogy $\mathcal{C}_3$ *teljesül. Ekkor az invariancia egyenletünk akkor és csak akkor teljesül, ha*

(i) vagy léteznek a, b, c, d *valós konstansok, melyekre* $ac \neq 0$ *úgy, hogy*

$$\varphi(x) = ax + b \qquad \textit{and} \qquad \psi(x) = cx + d \qquad (x \in I);$$

(ii) vagy léteznek a, b, c, d, p *valós konstansok, melyekre* $acp \neq 0$ *úgy, hogy*

$$\varphi(x) = ae^{px} + b \qquad \textit{and} \qquad \psi(x) = ce^{-px} + d \qquad (x \in I)$$

és a ν *mérték a* μ *mérték tükörképe az* $1/2$ *pontra nézve.*

TÉTEL. *Legyen* μ, ν *Borel valószínűségi mérték úgy, hogy* $\widehat{\mu}_1 = \widehat{\nu}_1 = \frac{1}{2}$, $\mu_2 = \nu_2 \neq 0$, $\mu_3 = -\nu_3$, $\mu_4 = \nu_4$. *Tegyük fel, hogy* $\mathcal{C}_4$ *teljesül. Ekkor az invariancia egyenletünk akkor és csak akkor teljesül, ha az előző tétel valamelyik állítása teljesül.*

A következő eredményben az a speciális esetet tekintjük, amikor $\mu = \nu$ szimmetrikus mérték.

KÖVETKEZMÉNY. *Legyen* μ *Borel valószínűségi mérték,* $\mu_2 \neq 0$ *és* μ *szimmetrikus az* $1/2$ *pontra nézve. Tegyük fel,* $\mathcal{C}_4$ *teljesül. Ekkor az invariancia egyenletünk akkor és csak akkor teljesül, ha*

(i) vagy léteznek a, b, c, d *valós konstansok, melyekre* $ac \neq 0$ *úgy, hogy*

$$\varphi(x) = ax + b \qquad \textit{and} \qquad \psi(x) = cx + d \qquad (x \in I);$$

(ii) vagy léteznek a, b, c, d, p *valós konstansok, melyekre* $acp \neq 0$ *úgy, hogy*

$$\varphi(x) = ae^{px} + b \qquad \textit{and} \qquad \psi(x) = ce^{-px} + d \qquad (x \in I).$$

Az alábbi példákban bemutatjuk, hogy eredményeink hogyan alkalmazhatók függvényegyenletek megoldására.

PÉLDA. Tekintsük a

$$\varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right) + \psi^{-1}\left(\frac{\psi(x) + \psi(y)}{2}\right) = x + y \qquad (x, y \in I),$$

függvényegyenletet, ahol $\varphi, \psi : I \to \mathbb{R}$ folytonos, szigorúan monoton függvények.

Ha $\mathcal{C}_4$ teljesül, a fenti függvényegyenlet megoldásai lineáris vagy exponenciális alakú függvények, melyet először, 1914-ben Sutô [**71**],[**72**] bizonyított, aki megadta az analitikus megoldásokat. 1999-ben Matkowski [**57**] a kétszer folytonosan differenciálható megoldásait adta meg ennek a függvényegyenletnek. A generátor függvényekre vonatkozó regularitási feltételeket Daróczy, Maksa és

Páles [**22**],[**27**] fokozatosan gyengítették, majd végül 2002-ben Daróczy és Páles [**28**] minden regularitási feltétel nélkül megoldották a problémát.

PÉLDA. Tekintsük a

$$\varphi^{-1}\big(\lambda\varphi(x)+(1-\lambda)\varphi(y)\big)+\psi^{-1}\big((1-\lambda)\psi(x)+\lambda\psi(y)\big)=x+y,$$

függvényegyenletet, ahol $\varphi,\psi:I\to\mathbb{R}$ folytonos szigorúan monoton függvények és $\lambda\in[0,1]\setminus\left\{0,\frac{1}{2},1\right\}$, és $x,y\in I$.

Ha $\mathcal{C}_3$ teljesül, a vizsgált függvényegyenlet megoldásai lineáris vagy exponenciális alakú függvények. Ezt az eredményt Jarczyk and Matkowski fedezték fel 2006-ban [**44**], és Jarczyk folytonos differenciálhatósági feltételek nélkül bizonyította 2007-ben [**43**].

PÉLDA. Tekintsük a

$$\varphi^{-1}\left(\int_0^1\varphi\big(tx+(1-t)y\big)dt\right)+\psi^{-1}\left(\int_0^1\psi\big(tx+(1-t)y\big)dt\right)=x+y,$$

függvényegyenletet, ahol $\varphi,\psi:I\to\mathbb{R}$ folytonos szigorúan monoton függvények, és $x,y\in I$, $x\neq y$.

Ha $\mathcal{C}_4$ teljesül, a függvényegyenlet megoldásai lineáris vagy exponenciális alakú függvények, amelyet Matkowski, erősebb regularitási feltételek mellett, bizonyított 2005-ben [**61**].

PÉLDA. Tekintsük a

$$\varphi^{-1}\left(\frac{2\varphi(x)+\varphi(y)}{3}\right)+\psi^{-1}\left(\frac{\psi(x)+4\psi\big(\frac{x+y}{2}\big)+4\psi(y)}{9}\right)=x+y,$$

függvényegyenletet, ahol $\varphi,\psi:I\to\mathbb{R}$ folytonos szigorúan monoton függvények.

Ha $\mathcal{C}_3$ teljesül, a φ és ψ generátorfüggvények akkor és csak akkor megoldásai a fenti függvényegyenletnek, ha

(i) léteznek olyan a,b,c,d valós számok, melyekre $ac\neq 0$ úgy, hogy $\varphi(x)=ax+b$ és $\psi(x)=cx+d$,
(ii) vagy léteznek a,b,c,d,p, olyan valós számok, melyekre $acp\neq 0$ úgy, hogy $\varphi(x)=ae^{px}+b$ és $\psi(x)=ce^{-2px}+d$.

2. Monoton függvények Lipschitz perturbációjáról

A függvényegyenletek stabilitás vizsgálatának elmélete 1952-ben indult Hyers és Ulam [**42**] cikkével. Hyers és Ulam felfedezték, hogy az ún. δ-konvex függvények felbonthatók egy konvex és egy korlátos függvény összegére véges dimenziós terek fölött. E stabilitási tétel még általánosabb formáját Páles adta meg 2003-ban [**69**]. Bevezette az ϵ-monotonitás fogalmát, amely elvezetett a monoton függvények stabilitási tulajdonságaihoz. A $p : I \to \mathbb{R}$ függvényt ϵ-növekvőnek nevezzük, ha

$$p(x) \leq p(y) + \epsilon$$

minden $x \leq y$ esetén. Páles ebben a cikkében megmutatta, hogy egy függvény akkor és csak akkor ϵ-növekvő, ha felbontható egy növekvő és egy korlátos függvény összegére. A disszertáció másik célja annak vizsgálata, hogy egy függvény mikor bontható fel egy növekvő és egy d-Lipschitz függvény összegére. A $d : I^2 \to I$ függvény *szemimetrika*, ha nemnegatív, szimmetrikus és teljesíti a háromszögegyenlőtlenséget.

DEFINÍCIÓ. Az $\ell : I \to \mathbb{R}$ függvényt *d-Lipschitznek* nevezzük, ha

$$|\ell(x) - \ell(y)| \leq d(x, y)$$

minden $x, y \in I$ esetén teljesül.

A fő eredményeink szükséges és elégséges feltételeket adnak a fenti felbontásra tetszőleges szemimetrika és konkáv szemimetrika esetén.

Fő eredményünk bizonyításához felhasználjuk a következő lemmát.

LEMMA. *Legyen $I \subseteq \mathbb{R}$, $t_1 < s_1, \dots, t_n < s_n$ és $u_1 < v_1, \dots, u_m < v_m$ olyan I-beli valós számok, melyekre*

$$\sum_{i=1}^{n} 1_{]t_i,s_i]} = \sum_{i=1}^{m} 1_{]u_i,v_i]}.$$

Ekkor az alábbi egyenlőség teljesül.

$$\sum_{i=1}^{n} \big(q(s_i) - q(t_i)\big) = \sum_{i=1}^{m} \big(q(v_i) - q(u_i)\big).$$

Ha d tetszőleges szemimetrika, a következő tétel megadja növekvő függvények Lipschitz perturbációjának egyik jellemzését. Jelölje x^+ az $x \in \mathbb{R}$ pozitív részét, azaz $x^+ := \max(0, x)$.

TÉTEL. *A $p : I \to \mathbb{R}$ függvény akkor és csak akkor írható $p = q + \ell$ alakban, ahol q növekvő és ℓ d-Lipschitz, ha*

$$\sum_{i=1}^{n} \big(p(s_i) - p(t_i) - d(t_i, s_i)\big)^+ \leq \sum_{j=1}^{m} \big(p(v_j) - p(u_j) + d(u_j, v_j)\big)$$

teljesül minden $t_1 < s_1, \ldots, t_n < s_n$ és $u_1 < v_1, \ldots, u_m < v_m$ I-beli valós számra, melyekre teljesül, hogy

$$\sum_{i=1}^{n} 1_{]t_i,s_i]} = \sum_{i=1}^{m} 1_{]u_i,v_i]}.$$

A következő lemma felhasználásával a $p = q + \ell$ felbontás egy másik jellemzését kapjuk.

LEMMA. *Legyenek $t_1 < s_1, \ldots, t_n < s_n$ és $u_1 < v_1, \ldots, u_m < v_m$ I-beli valós számok úgy, hogy $\sum_{i=1}^{n} 1_{]t_i,s_i]} = \sum_{i=1}^{m} 1_{]u_i,v_i]}$ teljesül. Ekkor*

$$\sum_{i=1}^{n} \big(p(s_i) - p(t_i) - d(t_i, s_i)\big)^+ \le \sum_{j=1}^{m} \big(p(v_j) - p(u_j) + d(u_j, v_j)\big)$$

akkor és csak akkor teljesül a $p : I \to \mathbb{R}$ függvényre, ha

$$0 \le \sum_{i=1}^{n} \min\big(d(t_i, s_i), p(s_i) - p(t_i)\big) + \sum_{j=1}^{m} d(u_j, v_j).$$

Jelölje $\mathcal{F}(I)$ azon függvények osztályát, melyek felírhatók

$$f = \sum_{i=1}^{n} 1_{]t_i,s_i]} - \sum_{j=1}^{m} 1_{]u_j,v_j]}$$

alakban, ahol $t_1 < s_1, \ldots, t_n < s_n, u_1 < v_1, \ldots, u_m < v_m \in I$.

Adott egy tetszőleges $q : I \to \mathbb{R}$ függvény, az $\mathcal{I}_q(f) : \mathcal{F}(I) \to \mathbb{R}$ funkcionált a következő módon definiáljuk:

$$\mathcal{I}_q(f) = \sum_{i=1}^{n} \big(q(s_i) - q(t_i)\big) - \sum_{j=1}^{m} \big(q(v_j) - q(u_j)\big),$$

ahol $f \in \mathcal{F}(I)$.

TÉTEL. *A $p : I \to \mathbb{R}$ függvény akkor és csak akkor írható fel $p = q + \ell$ alakban, ahol q növekvő, ℓ pedig d-Lipschitz, ha*

$$0 \le \sum_{i=1}^{n} d(t_i, s_i) + \sum_{j=1}^{m} d(u_j, v_j) + \mathcal{I}_p\Big(\sum_{j=1}^{m} 1_{]u_j,v_j]} - \sum_{i=1}^{n} 1_{]t_i,s_i]}\Big)$$

minden $t_1 < s_1, \ldots, t_n < s_n$ és $u_1 < v_1, \ldots, u_m, < v_m$ I-beli valós számra, melyre

$$\sum_{i=1}^{n} 1_{]t_i,s_i]} \le \sum_{j=1}^{m} 1_{]u_j,v_j]}.$$

Ha a d függvény konkáv szemimetrika, akkor egyszerűbb szükséges és elégséges feltételeket kapunk arra, hogy egy p függvényt mikor tudunk felbontani egy monoton növekvő és egy d-Lipschitz függvény összegére.

DEFINÍCIÓ. A $d : I \times I \to \mathbb{R}$ szemimetrikát *konkáv*nak nevezzük, ha minden $x \le y \le z \le w$ I-beli valós szám esetén

$$d(x,w) + d(y,z) \le d(x,z) + d(y,w)$$

teljesül.

A következő fő eredmény abban az esetben ad szükséges és elégséges feltételt a vizsgált felbontásra, amikor a d konkáv szemimetrika. Ha a d metrika a szokásos távolság függvénnyel egyenlő, akkor a feltételünk egy kétváltozós egyenlőtlenséggé egyszerűsödik.

TÉTEL. *A $p : I \to \mathbb{R}$ függvény akkor és csak akkor írható fel $p = q + \ell$ alakban, ahol q növekvő, ℓ pedig d-Lipschitz, ha*

$$0 \le \sum_{k=1}^{n} d(x_{2k-1}, x_{2k}) + d(x_0, x_{2n+1}) + \sum_{k=0}^{n} \big(p(x_{2k+1}) - p(x_{2k})\big)$$

teljesül minden $x_0 \le x_1 < \cdots < x_{2n} \le x_{2n+1}$ I-beli valós szám esetén.

TÉTEL. *Ha $d(x,y) = |y - x|$ $(x, y \in I)$, akkor*

$$0 \le \sum_{k=1}^{n} d(x_{2k-1}, x_{2k}) + d(x_0, x_{2n+1}) + \sum_{k=0}^{n} \big(p(x_{2k+1}) - p(x_{2k})\big)$$

akkor és csak akkor teljesül minden $x_0 \le x_1 < \cdots < x_{2n} \le x_{2n+1}$ I-beli valós számra, ha

$$p(x) \le p(y) + d(x,y).$$

fennáll bármely $x < y$ I-beli valós számok esetén.

List of publications

[1] Z. Makó and Zs. Páles, *On Lipschitz perturbation of monotonic functions*, Acta Math. Hungar. **113** (2006), no. 1-2, 1–18. MR 2007f:26010

[2] Z. Makó and Zs. Páles, *On the equality of generalized quasi-arithmetic means*, Publ.Math. Debrecen **72** (2008), no. 3-4, 407–440.

[3] Z. Makó and Zs. Páles, *The invariance of the arithmetic mean with respect to generalized quasi-arithmetic means*, J. Math. Anal. Appl. **353** (2009), 8–23.

List of talks

[1] *On the Lipschitz perturbation of monotone functions*, The 5^{th} Joint Conference on Mathematics and Computer Science, Debrecen (Hungary), 2004.

[2] *Monoton függvények Lipschitz perturbációjáról*, Síkfőkút (Magyarország), 2004. (in Hungarian)

[3] *On the Lipschitz perturbation of monotone functions*, The 5^{th} Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities, Bedlewo (Poland), 2005.

[4] *On the Lipschitz perturbation of monotone functions*, The 1^{st} International Student's Conference on Analysis, Szczyrk (Poland), 2005.

[5] *On the equality of generalized quasi-arithmetic means*, Conference on Inequalities and Applications '07, Noszvaj (Hungary), 2007.

[6] *The invariance of the arithmetic mean with respect to generalized quasi-arithmetic means*, Numbers, Functions, Equations '08, Noszvaj (Hungary), 2008.

Bibliography

[1] J. Aczél, *The notion of mean values*, Norske Vid. Selsk. Forh., Trondhjem **19** (1947), no. 23, 83–86.

[2] ———, *On mean values*, Bull. Amer. Math. Soc. **54** (1948), 392–400.

[3] ———, *On mean values and operations defined for two variables*, Norske Vid. Selsk. Forh., Trondhjem **20** (1948), no. 10, 37–40.

[4] ———, *On the theory of means*, Colloq. Math. **4** (1956), 33–55.

[5] L. R. Berrone, *The mean value theorem: functional equations and Lagrangian means*, Epsilon **14** (1998), no. 1(40), 131–151.

[6] L. R. Berrone and J. Moro, *Lagrangian means*, Aequationes Math. **55** (1998), no. 3, 217–226.

[7] J. Błasińska-Lesk, D. Głazowska, and J. Matkowski, *An invariance of the geometric mean with respect to Stolarsky mean-type mappings*, Results Math. **43** (2003), no. 1-2, 42–55.

[8] J. M. Borwein and P. B. Borwein, *Pi and the AGM*, John Wiley & Sons Inc., New York, 1987, A study in analytic number theory and computational complexity.

[9] P. Burai, *Extension theorem for a functional equation*, J. Appl. Anal. **12** (2006), no. 2, 293–299.

[10] ———, *A Matkowski–Sutô type equation*, Publ. Math. Debrecen **70** (2007), no. 1-2, 233–247.

[11] E. Casini and P. L. Papini, *A counterexample to the infinity version of the Hyers-Ulam stability theorem*, Proc. Amer. Math. Soc. **118** (1993), 885–890.

[12] P. W. Cholewa, *The stability problem for a generalized Cauchy type functional equation*, Rev. Roumaine Math. Pures Appl. **29** (1984), no. 6, 457–460.

[13] Z. Daróczy, *On a class of means of two variables*, Publ. Math. Debrecen **55** (1999), no. 1-2, 177–197.

[14] ———, *Matkowski-Sutô type problem for conjugate arithmetic means*, Rocznik Nauk.-Dydakt. Prace Mat. **17** (2000), 89–100, Dedicated to Professor Zenon Moszner on the occasion of his seventieth birthday.

[15] ———, *Gaussian iteration of mean values and the existence of* $\sqrt{2}$, Teaching Math. Comp. Sci. **1** (2003), no. 1, 35–42.

[16] ———, *Quasi-arithmetic elements of a given class of means*, Publ. Math. Debrecen **65** (2004), no. 3-4, 317–322, Dedicated to the memory of Béla Brindza and Jenő Erdős.

[17] ———, *Functional equations involving means and Gauss compositions of means*, Nonlinear Anal. **63** (2005), no. 5-7, e417–e425.

[18] Z. Daróczy and G. Hajdu, *On linear combinations of weighted quasi-arithmetic means*, Aequationes Math. **69** (2005), no. 1-2, 58–67.

[19] Z. Daróczy, G. Hajdu, and C. T. Ng, *An extension theorem for a Matkowski-Sutô problem*, Colloq. Math. **95** (2003), no. 2, 153–161.

[20] Z. Daróczy, K. Lajkó, R. L. Lovas, Gy. Maksa, and Zs. Páles, *Functional equations involving means*, Acta Math. Hungar. **116** (2007), no. 1-2, 79–87.

[21] Z. Daróczy and Gy. Maksa, *On a problem of Matkowski*, Colloq. Math. **82** (1999), no. 1, 117–123.

[22] Z. Daróczy, Gy. Maksa, and Zs. Páles, *Extension theorems for the Matkowski-Sutô problem*, Demonstratio Math. **33** (2000), no. 3, 547–556.

[23] ______, *On two-variable means with variable weights*, Aequationes Math. **67** (2004), no. 1-2, 154–159.
[24] ______, *Functional equations involving means and their Gauss composition*, Proc. Amer. Math. Soc. **134** (2006), no. 2, 521–530.
[25] Z. Daróczy and C. T. Ng, *A functional equation on complementary means*, Acta Sci. Math. (Szeged) **66** (2000), 603–611.
[26] Z. Daróczy and Zs. Páles, *Középértékek Gauss-féle kompozíciója és a Matkowski–Sutô probléma megoldása*, Mat. Lapok (1998-99 (2003)), no. 3-4, 1–53.
[27] ______, *On means that are both quasi-arithmetic and conjugate arithmetic*, Acta Math. Hungar. **90** (2001), no. 4, 271–282.
[28] ______, *Gauss-composition of means and the solution of the Matkowski–Sutô problem*, Publ. Math. Debrecen **61** (2002), no. 1-2, 157–218.
[29] ______, *A Matkowski-Sutô type problem for quasi-arithmetic means of order α*, Functional Equations — Results and Advances (Z. Daróczy and Zs. Páles, eds.), Adv. Math. (Dordr.), vol. 3, Kluwer Acad. Publ., Dordrecht, 2002, pp. 189–200.
[30] ______, *The Matkowski–Sutô problem for weighted quasi-arithmetic means*, Acta Math. Hungar. **100** (2003), no. 3, 237–243.
[31] ______, *A Matkowski–Sutô-type problem for weighted quasi-arithmetic means*, Ann. Univ. Sci. Budapest. Sect. Comput. **22** (2003), 69–81, Dedicated to the 60th birthday of Professor Karl-Heinz Indlekofer.
[32] ______, *On functional equations involving means*, Publ. Math. Debrecen **62** (2003), no. 3-4, 363–377, Dedicated to Professor Lajos Tamássy on the occasion of his 80th birthday.
[33] B. De Finetti, *Sul concetto di media*, Giornale dell' Instituto, Italiano degli Attuarii **2** (1931), 369–396.
[34] J. Domsta and J. Matkowski, *Invariance of the arithmetic mean with respect to special mean-type mappings*, Aequationes Math. **71** (2006), no. 1-2, 70–85.
[35] C. F. Gauss, *Bestimmung der Anziehung eines elliptischen Ringes*, Akademische Verlagsgesellschaft M. B. H., Leipzig, 1927, Nachlass zur Theorie des arithmetisch-geometrischen Mittels und der Modulfunktion.
[36] R. Ger, *Stability aspects of delta-convexity*, Stability of mappings of Hyers-Ulam type, Hadronic Press, Palm Harbor, FL, 1994, pp. 99–109.
[37] D. Głazowska, W. Jarczyk, and J. Matkowski, *Arithmetic mean as a linear combination of two quasi-arithmetic means*, Publ. Math. Debrecen **61** (2002), no. 3-4, 455–467.
[38] J. W. Green, *Approximately convex functions*, Duke Math. J. **19** (1952), 499–504.
[39] G. Hajdu, *An extension theorem for the Matkowski–Sutô problem for conjugate arithmetic means*, Functional Equations — Results and Advances (Z. Daróczy and Zs. Páles, eds.), Advances in Mathematics, vol. 3, Kluwer Acad. Publ., Dordrecht, 2002, pp. 201–208.
[40] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1934, (first edition), 1952 (second edition).
[41] H. Haruki and Th. M. Rassias, *New characterizations of some mean-values*, J. Math. Anal. Appl. **202** (1996), no. 1, 333–348.
[42] D. H. Hyers and S. M. Ulam, *Approximately convex functions*, Proc. Amer. Math. Soc. **3** (1952), 821–828.
[43] J. Jarczyk, *Invariance of weighted quasi-arithmetic means with continuous generators*, Publ. Math. Debrecen **71** (2007), no. 3-4, 279–294.
[44] J. Jarczyk and J. Matkowski, *Invariance in the class of weighted quasi-arithmetic means*, Ann. Polon. Math. **88** (2006), no. 1, 39–51.
[45] J. Kindler, *Sandwich theorems for set functions*, J. Math. Anal. Appl. **133** (1988), no. 2, 529–542.
[46] A. Kolmogorov, *Sur la notion de la moyenne*, Rend. Accad. dei Lincei (6) **12** (1930), 388–391.

[47] M. Laczkovich, *The local stability of convexity, affinity and of the Jensen equation*, Aequationes Math. **58** (1999), 135–142.
[48] L. Losonczi, *Equality of two variable weighted means: reduction to differential equations*, Aequationes Math. **58** (1999), no. 3, 223–241.
[49] ______, *Equality of Cauchy mean values*, Publ. Math. Debrecen **57** (2000), no. 1-2, 217–230.
[50] ______, *Homogeneous Cauchy mean values*, Functional Equations — Results and Advances (Z. Daróczy and Zs. Páles, eds.), Advances in Mathematics, vol. 3, Kluwer Acad. Publ., Dordrecht, 2002, pp. 209–218.
[51] ______, *Equality of two variable Cauchy mean values*, Aequationes Math. **65** (2003), no. 1-2, 61–81.
[52] ______, *Equality of two variable means revisited*, Aequationes Math. **71** (2006), no. 3, 228–245.
[53] Z. Makó and Zs. Páles, *On Lipschitz perturbation of monotonic functions*, Acta Math. Hungar. **113** (2006), no. 1-2, 1–18.
[54] ______, *On the equality of generalized quasiarithmetic means*, Publ. Math. Debrecen **72** (2008), no. 3-4, 407–440.
[55] ______, *The invariance of the arithmetic mean with respect to generalized quasi-arithmetic means*, J. Math. Anal. Appl. **353** (2009), 8–23.
[56] J. Matkowski, *Invariant and complementary quasi-arithmetic means*, Aequationes Math. **57** (1999), no. 1, 87–107.
[57] ______, *Iterations of mean-type mappings and invariant means*, Ann. Math. Sil. (1999), no. 13, 211–226, European Conference on Iteration Theory (Muszyna-Złockie, 1998).
[58] ______, *On invariant generalized Beckenbach-Gini means*, Functional Equations — Results and Advances (Z. Daróczy and Zs. Páles, eds.), Advances in Mathematics, vol. 3, Kluwer Acad. Publ., Dordrecht, 2002, pp. 219–230.
[59] ______, *On iteration semigroups of mean-type mappings and invariant means*, Aequationes Math. **64** (2002), no. 3, 297–303.
[60] ______, *Convex functions with respect to a mean and a characterization of quasi-arithmetic means*, Real Anal. Exchange **29** (2003/2004), no. 1, 229–246.
[61] ______, *Lagrangian mean-type mappings for which the arithmetic mean is invariant*, J. Math. Anal. Appl. **309** (2005), no. 1, 15–24.
[62] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives*, Mathematics and its Applications (East European Series), vol. 53, Kluwer Academic Publishers Group, Dordrecht, 1991.
[63] ______, *Classical and New Inequalities in Analysis*, Mathematics and its Applications (East European Series), vol. 61, Kluwer Academic Publishers Group, Dordrecht, 1993.
[64] Á. Münnich, Gy. Maksa, and R. J. Mokken, *Collective judgement: combining individual value judgements*, Math. Social Sci. **37** (1999), no. 3, 211–233.
[65] M. Nagumo, *Über eine Klasse der Mittelwerte*, Jap. Jour. of Math. **7** (1930), 71–79.
[66] C. P. Niculescu and L.-E. Persson, *Convex Functions and Their Applications*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 23, Springer-Verlag, New York, 2006, A contemporary approach.
[67] Zs. Páles, *On the equality of quasi-arithmetic means and Lagrangian means*, manuscript.
[68] ______, *Nonconvex functions and separation by power means*, Math. Inequal. Appl. **3** (2000), no. 2, 169–176.
[69] ______, *On approximately convex functions*, Proc. Amer. Math. Soc. **131** (2003), no. 1, 243–252.
[70] ______, *On functional equations characterizing polynomials*, Acta Sci. Math. (Szeged) **74** (2008), 581–592.
[71] O. Sutô, *Studies on some functional equations I*, Tôhoku Math. J. **6** (1914), 1–15.

[72] ———, *Studies on some functional equations II*, Tôhoku Math. J. **6** (1914), 82–101.