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Convolution of second order linear recursive sequences II.

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Abstract. We continue the investigation of convolutions of second order linear recursive sequences (see the first part in [1]). In this paper, we focus on the case when the characteristic polynomials of the sequences have common root.

1 Introduction

The second order linear recursive sequence $\{G_n\}_{n=0}^{\infty}$ is defined by the recursion

$$G_n = AG_{n-1} + BG_{n-2} \quad (n \geq 2),$$

where the initial terms G_0, G_1 and the weights A, B are fixed real numbers with $|G_0| + |G_1| \neq 0$ and $AB \neq 0$. Sometimes the following notation $G_n(G_0, G_1, A, B)$ is used, too. The polynomial

$$p(x) = x^2 - Ax - B \tag{1}$$

is known as the characteristic polynomial of the sequence $\{G_n\}_{n=0}^{\infty}$. If its discriminant $D = A^2 + 4B \neq 0$ then the Binet formula of $\{G_n\}_{n=0}^{\infty}$ is

$$G_n = \frac{G_1 - \beta G_0}{\alpha - \beta}\alpha^n - \frac{G_1 - \alpha G_0}{\alpha - \beta}\beta^n,$$

where α, β are the distinct roots of $p(x)$.

If $G_0 = 0$ and $G_1 = 1$ then $\{G_n\}_{n=0}^{\infty}$ is known as Lucas sequence $\{R_n\}_{n=0}^{\infty}$ with its Binet formula

$$R_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{2}$$

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Name	$G_n(G_0, G_1, A, B)$	Characteristic polynomial	Gen. function
Fibonacci	$F_n(0,1,1,1)$	$p(x) = x^2 - x - 1$	$g(x) = \frac{x}{1-x-x^2}$
Pell	$P_n(0,1,2,1$	$p(x) = x^2 - 2x - 1$	$g(x) = \frac{x}{1-2x-x^2}$
Jacobsthal	$J_n(0,1,1,2)$	$p(x) = x^2 - x - 2$	$g(x) = \frac{x}{1-x-2x^2}$
Mersenne	$M_n(0,1,3,-2)$	$p(x) = x^2 - 3x + 2$	$g(x) = \frac{x}{1-3x+2x^2}$
Lucas	$L_n(2,1,1,1)$	$p(x) = x^2 - x - 1$	$g(x) = \frac{2-x}{1-x-x^2}$
P-Lucas	$p_n(2,2,2,1)$	$p(x) = x^2 - 2x - 1$	$g(x) = \frac{2-2x}{1-2x-x^2}$
J-Lucas	$j_n(2,1,1,2)$	$p(x) = x^2 - x - 2$	$g(x) = \frac{2-x}{1-x-2x^2}$
M-Lucas	$m_n(2,3,3,-2)$	$p(x) = x^2 - 3x + 2$	$g(x) = \frac{2-3x}{1-3x+2x^2}$

Table 1: Some famous sequences

while if $G_0 = 2$ and $G_1 = A$ then the sequence is known as associated Lucas sequence $\{V_n\}_{n=0}^{\infty}$ with its Binet formula

$$V_n = \alpha^n + \beta^n. \tag{3}$$

It is known that the generating function of $\{G_n\}_{n=0}^{\infty}$ is

$$g(x) = \frac{G_0 + (G_1 - AG_0)x}{1 - Ax - Bx^2}. \tag{4}$$

There are some well-known sequences, such as Fibonacci, Pell, Jacobsthal, Mersenne, and their associate sequences. The following table contains the initial terms, characteristic polynomials and generating functions of these sequences.

In this paper, we consider the sequence $\{c(n)\}_{n=0}^{\infty}$ given by the convolution of two second order linear recursive sequences $\{G_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=0}^{\infty}$:

$$c(n) = \sum_{k=0}^{n} G_k H_{n-k},$$

where $\{G_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=0}^{\infty}$ are Lucas or associated Lucas sequences (see (2) and (3)) and the characteristic polynomials of the sequences have common root. We mention that in [1], we have considered that case when the characteristic polynomials have no common root. The applied methods for proofs require the separation of the two cases when the characteristic polynomials have or do not have common root. We open the door for generalization in case of arbitrary second or k-order linear recursive sequences.

2 Results

At first, we consider the convolution of those sequences where the characteristic polynomials have exactly one common root, after that, we deal with the case when the characteristic polynomials have two common roots, that is, the characteristic polynomials are the same ones. After each theorem, we give the exact forms for the sequences contained in Table 1. In the following, we will use the notations:

$$\begin{aligned} b &= (A_1 - A_2)\beta + B_1 - B_2, \\ d &= (A_2 - A_1)\delta + B_2 - B_1, \end{aligned} \tag{5}$$

where $bd \neq 0$, $p(x) = x^2 - A_1x - B_1$ and $q(x) = x^2 - A_2x - B_2$ are the characteristic polynomials of $G_n(G_0, G_1, A_1, B_1)$ and $H_n(H_0, H_1, A_2, B_2)$, respectively. We suppose that $p(\alpha) = q(\alpha) = 0$, $p(\beta) = 0$, $q(\beta) \neq 0$, while $q(\delta) = 0$, $p(\delta) \neq 0$, that is, β and δ are distinct roots, while α is the common root.

2.1 The characteristic polynomials have exactly one common root

In the following theorem, we deal with the convolution of two different Lucas sequences, that is, when the initial terms are $0, 1$.

Theorem 1. *The convolution of $G_n(0, 1, A_1, B_1)$ and $H_n(0, 1, A_2, B_2)$ is*

$$c(n) = \sum_{k=0}^{n} G_k H_{n-k} = \frac{\alpha^n(n+1) + \alpha^n \frac{B_1+B_2-2\alpha^2}{(\alpha-\beta)(\alpha-\delta)} - \beta^{n+1}\frac{\alpha-\delta}{b} - \delta^{n+1}\frac{\alpha-\beta}{d}}{(\alpha-\beta)(\alpha-\delta)}.$$

Corollary 1. *Using Theorem 1 the convolution of Jacobsthal and Mersenne numbers is:*

$$c(n) = \sum_{k=0}^{n} J_k M_{n-k} = \frac{2n + (2n-3)M_n - J_n}{6}.$$

In the following theorem, we deal with the convolution of a Lucas sequence and an associated Lucas sequence, that is, when the initial terms are $0, 1$ and $2, A_2$.

Theorem 2. *The convolution of $G_n(0, 1, A_1, B_1)$ and $H_n(2, A_2, A_2, B_2)$ is*

$$c(n) = \sum_{k=0}^{n} G_k H_{n-k}$$
$$= \frac{\alpha^n(n+1)(2\alpha - A_2) + \alpha^n \frac{B_1-B_2}{\alpha-\beta} - \beta^{n+1}\frac{(\alpha-\delta)(2\beta-A_2)}{b} - \delta^{n+1}\frac{(\alpha-\beta)(2\delta-A_2)}{d}}{(\alpha-\beta)(\alpha-\delta)}.$$

Corollary 2. *Using Theorem 2 the convolution of Jacobsthal and M-Lucas numbers is:*

$$c(n) = \sum_{k=0}^{n} J_k m_{n-k} = \frac{2n + (2n+3)M_n + 5J_n}{6}.$$

Corollary 3. *Using Theorem 2 the convolution of Mersenne and J-Lucas numbers is:*

$$c(n) = \sum_{k=0}^{n} M_k j_{n-k} = \frac{2n + (2n-1)M_n + J_n}{2}.$$

In the following theorem, we deal with the convolution of two different associated Lucas sequences, that is, when the initial terms are $2, A_1$ and $2, A_2$.

Theorem 3. *The convolution of $G_n(2, A_1, A_1, B_1)$ and $H_n(2, A_2, A_2, B_2)$ is*

$$c(n) = \sum_{k=0}^{n} G_k H_{n-k} = \alpha^n(n+1) + \alpha^n \frac{B_1 + B_2 + 2\alpha^2}{(\alpha-\beta)(\alpha-\delta)}$$
$$- \beta^{n+1}\frac{2\beta - A_2}{(\alpha-\beta)(A_1-A_2)} + \delta^{n+1}\frac{2\delta - A_1}{(\alpha-\delta)(A_1-A_2)}.$$

Corollary 4. *Using Theorem 3 the convolution of J-Lucas and M-Lucas numbers is:*

$$c(n) = \sum_{k=0}^{n} j_k m_{n-k} = \frac{n + 1 + (n + 2)M_{n+1} + 5J_{n+1}}{2}.$$

2.2 The characteristic polynomials have two common roots

That is, $p(x) = q(x)$ and so $p(\alpha) = q(\alpha) = 0$, $p(\beta) = q(\beta) = 0$. In the following theorem, we deal with the convolution of a Lucas sequence with itself, that is, the initial terms are $0, 1$. Zhang in [5] has generalized this type of problem, now we give different formulas.

Theorem 4. *The convolution of $R_n(0, 1, A_1, B_1)$ with itself is*

$$c(n) = \sum_{k=0}^{n} R_k R_{n-k} = \frac{1}{(\alpha - \beta)^2}\Big((n + 1)V_n - 2R_{n+1}\Big),$$

where V_n is the associate sequence of R_n.

Corollary 5. *Using Theorem 4 the convolution of Fibonacci numbers with themself:*

$$c(n) = \sum_{k=0}^{n} F_k F_{n-k} = \frac{1}{5}\Big((n + 1)L_n - 2F_{n+1}\Big).$$

Remark 1. The formula given by Zhang in [5] was the following.

$$\sum_{a+b=n} F_a F_b = \frac{1}{5}\Big((n - 1)F_n + 2nF_{n-1}\Big), \quad n \geq 1.$$

It can be easily verifyed that the two formulas are the same ones using some well known identities between the Fibonacci and Lucas numbers: $2F_{n+1} = F_n + L_n$ and $L_n = F_{n-1} + F_{n+1}$. S. Vajda in [6] on page 183 gave the same formula for the convolution of Fibonacci numbers like us in Corollary 5.

In the following theorem, we deal with the convolution of a Lucas sequence and its associated sequence, that is, the initial terms are $0, 1$ and $2, A_1$.

Theorem 5. *The convolution of $R_n(0, 1, A_1, B_1)$ and $V_n(2, A_1, A_1, B_1)$ is*

$$c(n) = \sum_{k=0}^{n} R_k V_{n-k} = (n + 1)R_n.$$

Corollary 6. *Using Theorem 5 the convolution of Fibonacci and Lucas numbers is:*

$$c(n) = \sum_{k=0}^{n} F_k L_{n-k} = (n + 1)F_n.$$

Remark 2. The OEIS [3] contains the above sequence with id A099920.

Corollary 7. *Using Theorem 5 the convolution of Pell and P-Lucas numbers is:*

$$c(n) = \sum_{k=0}^{n} P_k p_{n-k} = (n+1)P_n.$$

Corollary 8. *Using Theorem 5 the convolution of Jacobsthal and J-Lucas numbers is:*

$$c(n) = \sum_{k=0}^{n} J_k j_{n-k} = (n+1)J_n.$$

Corollary 9. *Using Theorem 5 the convolution of Mersenne and M-Lucas numbers is:*

$$c(n) = \sum_{k=0}^{n} M_k m_{n-k} = (n+1)M_n.$$

Remark 3. The OEIS [3] contains the above sequence with id A058877.

In the following theorem, we deal with the convolution of an associated Lucas sequence with itself, that is, the initial terms are $2, A_1$.

Theorem 6. *The convolution of $V_n(2, A_1, A_1, B_1)$ with itself is*

$$c(n) = \sum_{k=0}^{n} V_k V_{n-k} = (n+1)V_n + 2R_{n+1},$$

where V_n is the associate sequence of R_n.

Corollary 10. *Using Theorem 6 the convolution of Lucas numbers with themself is:*

$$c(n) = \sum_{k=0}^{n} L_k L_{n-k} = (n+1)L_n + 2F_{n+1}.$$

Remark 4. In the paper of Zhang and He [4] Corollary 1 contains another formula for the convolution of Lucas numbers with themself:

$$\sum_{a+b=n} L_a L_b = \frac{1}{5}\Big(2L_{n+1} + (5n+9)L_n\Big).$$

It can be easily verifyed that the two formulas are the same ones using some well known identities between the Fibonacci and Lucas numbers: $L_{n+2} + L_n = 5F_{n+1}$ and $L_{n+2} = L_{n+1} + L_n$. S. Vajda in [6] on page 183 gave the same formula for the convolution of Lucas numbers like us in Corollary 10. The OEIS [3] contains this sequence with id A099924.

3 Proofs

In the following proofs, we use the method of partial-fraction decomposition, the generating functions of second order linear recursive sequences and the idea used by Griffiths and Bramham in [2], that is $c(n)$ is the coefficient of x^n in

$$g(x)h(x) = \sum_{n=0}^{\infty} G_n x^n \cdot \sum_{n=0}^{\infty} H_n x^n = \sum_{n=0}^{\infty} c(n) x^n,$$

where $g(x)$, $h(x)$ are the generating functions of sequences $\{G_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=0}^{\infty}$, respectively. Furthermore the following well-known identity will be used throughout the proofs.

$$\frac{1}{1-\alpha x} = \sum_{n=0}^{\infty} (\alpha x)^n, \quad (0 < |\alpha x| < 1). \tag{6}$$

We will use the identity

$$\frac{1}{(1-\alpha x)^2} = \sum_{n=0}^{\infty} (n+1)(\alpha x)^n, \quad (0 < |\alpha x| < 1). \tag{7}$$

too, which can be verified in the following way:

$$\begin{aligned}\frac{1}{(1-\alpha x)^2} &= \left(\frac{1}{\alpha(1-\alpha x)}\right)' = \left(\frac{1}{\alpha}\sum_{n=0}^{\infty}(\alpha x)^n\right)' \\ &= \frac{1}{\alpha}\sum_{n=1}^{\infty} n\alpha^n x^{n-1} = \frac{1}{\alpha}\sum_{n=0}^{\infty}(n+1)\alpha^{n+1}x^n = \sum_{n=0}^{\infty}(n+1)(\alpha x)^n.\end{aligned}$$

Proof of Theorem 1. The generating functions of the sequences $G_n(0,1,A_1,B_1)$ and $H_n(0,1,A_2,B_2)$ follow from (4),

$$g(x) = \frac{x}{1 - A_1 x - B_1 x^2} = \frac{x}{(1-\alpha x)(1-\beta x)}$$

and

$$h(x) = \frac{x}{1 - A_2 x - B_2 x^2} = \frac{x}{(1-\alpha x)(1-\delta x)},$$

where α, β and α, δ are the roots of the characteristic polynomial of $\{G_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=0}^{\infty}$, respectively. The generating functions can be written as (by the method of partial-fraction decomposition)

$$g(x) = \frac{1}{\alpha-\beta}\left(\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right)$$

and

$$h(x) = \frac{1}{\alpha-\delta}\left(\frac{1}{1-\alpha x} - \frac{1}{1-\delta x}\right).$$

From this it follows that

$$\begin{aligned}
&g(x)h(x)(\alpha-\beta)(\alpha-\delta)\\
&=\left(\frac{1}{1-\alpha x}-\frac{1}{1-\beta x}\right)\left(\frac{1}{1-\alpha x}-\frac{1}{1-\delta x}\right)\\
&=\frac{1}{(1-\alpha x)^2}-\frac{1}{(1-\alpha x)(1-\delta x)}-\frac{1}{(1-\beta x)(1-\alpha x)}+\frac{1}{(1-\beta x)(1-\delta x)}\\
&=\frac{1}{(1-\alpha x)^2}-\frac{\frac{\alpha}{\alpha-\delta}}{1-\alpha x}+\frac{\frac{\delta}{\alpha-\delta}}{1-\delta x}-\frac{\frac{\beta}{\beta-\alpha}}{1-\beta x}+\frac{\frac{\alpha}{\beta-\alpha}}{1-\alpha x}+\frac{\frac{\beta}{\beta-\delta}}{1-\beta x}-\frac{\frac{\delta}{\beta-\delta}}{1-\delta x}\\
&=\frac{1}{(1-\alpha x)^2}+\frac{\frac{B_1+B_2-2\alpha^2}{(\alpha-\delta)(\alpha-\beta)}}{1-\alpha x}-\frac{\frac{\beta(\alpha-\delta)}{(A_1-A_2)\beta+B_1-B_2}}{1-\beta x}-\frac{\frac{\delta(\alpha-\beta)}{(A_2-A_1)\delta+B_2-B_1}}{1-\delta x}.
\end{aligned}$$

Now using (6), (7) and the idea that $c(n)$ is the coefficient of x^n in $g(x)h(x)$, we get

$$c(n)=\frac{\alpha^n(n+1)+\alpha^n\frac{B_1+B_2-2\alpha^2}{(\alpha-\beta)(\alpha-\delta)}-\beta^{n+1}\frac{\alpha-\delta}{b}-\delta^{n+1}\frac{\alpha-\beta}{d}}{(\alpha-\beta)(\alpha-\delta)}.$$

□

The corollaries can be reached from Table 1 if we use the values of A_1, B_1, A_2, B_2 and the Binet formula (2), e.g., the proof of Corollary 1:

Proof of Corollary 1. In this special case the sequences are $G_n = J_n(0,1,1,2)$ and $H_n = M_n(0,1,3,-2)$ and we have

$$\alpha=2,\beta=-1,\quad \alpha=2,\delta=1.$$

By (5), we get that

$$\begin{aligned} b&=6,\\ d&=-2.\end{aligned}$$

Applying Theorem 1 and (2), we get the result by a simple calculation. □

Proof of Theorem 2. The generating functions of the sequences $G_n(0,1,A_1,B_1)$ and $H_n(2,A_2,A_2,B_2)$ follow from (4),

$$g(x)=\frac{x}{1-A_1x-B_1x^2}=\frac{x}{(1-\alpha x)(1-\beta x)}$$

and

$$h(x)=\frac{2-A_2x}{1-A_2x-B_2x^2}=\frac{2-A_2x}{(1-\alpha x)(1-\delta x)},$$

where α,β and α,δ are the roots of the characteristic polynomial of $\{G_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=0}^{\infty}$, respectively. The generating functions could be written as (by the method of partial-fraction decomposition)

$$g(x)=\frac{1}{\alpha-\beta}\left(\frac{1}{1-\alpha x}-\frac{1}{1-\beta x}\right)$$

and

$$h(x) = \frac{1}{\alpha - \delta}\left(\frac{2\alpha - A_2}{1-\alpha x} - \frac{2\delta - A_2}{1-\delta x}\right).$$

From this it follows that

$$\begin{aligned}
&g(x)h(x)(\alpha-\beta)(\alpha-\delta) = \\
&= \left(\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right)\left(\frac{2\alpha - A_2}{1-\alpha x} - \frac{2\delta - A_2}{1-\delta x}\right) \\
&= \frac{2\alpha - A_2}{(1-\alpha x)^2} - \frac{2\delta - A_2}{(1-\alpha x)(1-\delta x)} - \frac{2\alpha - A_2}{(1-\beta x)(1-\alpha x)} + \frac{2\delta - A_2}{(1-\beta x)(1-\delta x)} \\
&= \frac{2\alpha - A_2}{(1-\alpha x)^2} - \frac{\frac{\alpha(2\delta - A_2)}{\alpha-\delta}}{1-\alpha x} + \frac{\frac{\delta(2\delta - A_2)}{\alpha-\delta}}{1-\delta x} - \frac{\frac{\beta(2\alpha - A_2)}{\beta-\alpha}}{1-\beta x} + \frac{\frac{\alpha(2\alpha - A_2)}{\beta-\alpha}}{1-\alpha x} + \frac{\frac{\beta(2\delta - A_2)}{\beta-\delta}}{1-\beta x} \\
&- \frac{\frac{\delta(2\delta - A_2)}{\beta-\delta}}{1-\delta x} \\
&= \frac{2\alpha - A_2}{(1-\alpha x)^2} + \frac{\frac{B_1 - B_2}{\alpha-\beta}}{1-\alpha x} - \frac{\frac{\beta(\alpha-\delta)(2\beta - A_2)}{(A_1-A_2)\beta + B_1 - B_2}}{1-\beta x} - \frac{\frac{\delta(\alpha-\beta)(2\delta - A_2)}{(A_2-A_1)\delta + B_2 - B_1}}{1-\delta x}.
\end{aligned}$$

Now using (6), (7) and the idea that $c(n)$ is the coefficient of x^n in $g(x)h(x)$, we get

$$c(n) = \frac{\alpha^n(n+1)(2\alpha - A_2) + \alpha^n\frac{B_1-B_2}{\alpha-\beta} - \beta^{n+1}\frac{(\alpha-\delta)(2\beta-A_2)}{b} - \delta^{n+1}\frac{(\alpha-\beta)(2\delta-A_2)}{d}}{(\alpha-\beta)(\alpha-\delta)}.$$

□

The corollaries can be reached from Table 1 if we use the values of A_1, B_1, A_2, B_2 and the Binet formula (2), e.g., the proof of Corollary 2:

Proof of Corollary 2. In this special case the sequences are $G_n = J_n(0,1,1,2)$ and $H_n = m_n(2,3,3,-2)$ and we have

$$\alpha = 2, \beta = -1, \quad \alpha = 2, \delta = 1.$$

By (5), we get that

$$\begin{aligned} b &= 6, \\ d &= -2. \end{aligned}$$

Applying Theorem 2 and (2), the statement easily follows. □

Proof of Theorem 3. The generating functions of the sequences $G_n(2, A_1, A_1, B_1)$ and $H_n(2, A_2, A_2, B_2)$ follow from (4)

$$g(x) = \frac{2 - A_1 x}{1 - A_1 x - B_1 x^2} = \frac{2 - A_1 x}{(1-\alpha x)(1-\beta x)}$$

and

$$h(x)=\frac{2-A_2x}{1-A_2x-B_2x^2}=\frac{2-A_2x}{(1-\alpha x)(1-\delta x)},$$

where α,β and α,δ are the roots of the characteristic polynomial of $\{G_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=0}^{\infty}$, respectively. The generating functions could be written as (by the method of partial-fraction decomposition)

$$g(x)=\frac{1}{\alpha-\beta}\left(\frac{2\alpha-A_1}{1-\alpha x}-\frac{2\beta-A_1}{1-\beta x}\right)$$

and

$$h(x)=\frac{1}{\alpha-\delta}\left(\frac{2\alpha-A_2}{1-\alpha x}-\frac{2\delta-A_2}{1-\delta x}\right).$$

From this it follows that

$$\begin{aligned}
g(x)h(x)(\alpha-\beta)(\alpha-\delta)&=\left(\frac{2\alpha-A_1}{1-\alpha x}-\frac{2\beta-A_1}{1-\beta x}\right)\left(\frac{2\alpha-A_2}{1-\alpha x}-\frac{2\delta-A_2}{1-\delta x}\right)\\
&=\frac{(2\alpha-A_1)(2\alpha-A_2)}{(1-\alpha x)^2}-\frac{(2\alpha-A_1)(2\delta-A_2)}{(1-\alpha x)(1-\delta x)}\\
&\quad-\frac{(2\beta-A_1)(2\alpha-A_2)}{(1-\beta x)(1-\alpha x)}+\frac{(2\beta-A_1)(2\delta-A_2)}{(1-\beta x)(1-\delta x)}\\
&=\frac{(2\alpha-A_1)(2\alpha-A_2)}{(1-\alpha x)^2}-\frac{\frac{\alpha(2\alpha-A_1)(2\delta-A_2)}{\alpha-\delta}}{1-\alpha x}+\frac{\frac{\delta(2\alpha-A_1)(2\delta-A_2)}{\alpha-\delta}}{1-\delta x}\\
&\quad-\frac{\frac{\beta(2\beta-A_1)(2\alpha-A_2)}{\beta-\alpha}}{1-\beta x}+\frac{\frac{\alpha(2\beta-A_1)(2\alpha-A_2)}{\beta-\alpha}}{1-\alpha x}+\frac{\frac{\beta(2\beta-A_1)(2\delta-A_2)}{\beta-\delta}}{1-\beta x}-\frac{\frac{\delta(2\beta-A_1)(2\delta-A_2)}{\beta-\delta}}{1-\delta x}\\
&=\frac{(2\alpha-A_1)(2\alpha-A_2)}{(1-\alpha x)^2}+\frac{B_1+B_2+2\alpha^2}{1-\alpha x}\\
&\quad-\frac{\beta(\alpha-\delta)(2\beta-A_2)}{(1-\beta x)(A_1-A_2)}+\frac{\delta(\alpha-\beta)(2\delta-A_1)}{(1-\delta x)(A_1-A_2)}.
\end{aligned}$$

Now using (6), (7) and the idea that $c(n)$ is the coefficient of x^n in $g(x)h(x)$, we get

$$\begin{aligned}
c(n)&=\alpha^n(n+1)+\alpha^n\frac{B_1+B_2+2\alpha^2}{(\alpha-\beta)(\alpha-\delta)}\\
&\quad-\beta^{n+1}\frac{2\beta-A_2}{(\alpha-\beta)(A_1-A_2)}+\delta^{n+1}\frac{2\delta-A_1}{(\alpha-\delta)(A_1-A_2)}.
\end{aligned}$$

□

We give the proof of Corollary 4.

Proof of Corollary 4. In this special case the sequences are $G_n=j_n(2,1,1,2)$ and $H_n=m_n(2,3,3,-2)$ and

$$\alpha=2,\beta=-1,\quad \alpha=2,\delta=1.$$

Applying Theorem 3 and (2), we get the result. □

Proof of Theorem 4. By the help of generating function of $R_n(0,1,A_1,B_1)$ and the method of partial-fraction decomposition, we get the following.

$$\begin{aligned}
(g(x))^2 &= \left(\frac{x}{1-A_1x-B_1x^2}\right)^2 \\
&= \left(\frac{x}{(1-\alpha x)(1-\beta x)}\right)^2 = \frac{1}{(\alpha-\beta)^2}\left(\frac{1}{1-\alpha x}-\frac{1}{1-\beta x}\right)^2 \\
&= \frac{1}{(\alpha-\beta)^2}\left(\frac{1}{(1-\alpha x)^2}-\frac{2}{(1-\alpha x)(1-\beta x)}+\frac{1}{(1-\beta x)^2}\right) \\
&= \frac{1}{(\alpha-\beta)^2}\left(\frac{1}{(1-\alpha x)^2}-\frac{\frac{2\alpha}{\alpha-\beta}}{1-\alpha x}+\frac{\frac{2\beta}{\alpha-\beta}}{1-\beta x}+\frac{1}{(1-\beta x)^2}\right).
\end{aligned}$$

Now using (6), (7), (2), (3) and the idea that $c(n)$ is the coefficient of x^n in $(g(x))^2$, we get

$$\begin{aligned}
c(n) &= \frac{1}{(\alpha-\beta)^2}\left(\alpha^n(n+1)-\alpha^n\frac{2\alpha}{\alpha-\beta}+\beta^n\frac{2\beta}{\alpha-\beta}+\beta^n(n+1)\right) \\
&= \frac{1}{(\alpha-\beta)^2}\Big((n+1)V_n-2R_{n+1}\Big).
\end{aligned}$$

□

Proof of Theorem 5. By the help of the generating function of $R_n(0,1,A_1,B_1)$ and $V_n(2,A_1,A_1,B_1)$, the method of partial-fraction decomposition and the Vieta's formula $\alpha+\beta=A_1$, we get the following.

$$\begin{aligned}
g(x)h(x) &= \frac{x}{1-A_1x-B_1x^2}\cdot\frac{2-A_1x}{1-A_1x-B_1x^2} \\
&= \frac{1}{(\alpha-\beta)^2}\left(\frac{1}{1-\alpha x}-\frac{1}{1-\beta x}\right)\left(\frac{2\alpha-A_1}{1-\alpha x}-\frac{2\beta-A_1}{1-\beta x}\right) \\
&= \frac{1}{(\alpha-\beta)^2}\left(\frac{2\alpha-A_1}{(1-\alpha x)^2}-\frac{2\alpha-A_1+2\beta-A_1}{(1-\alpha x)(1-\beta x)}+\frac{2\beta-A_1}{(1-\beta x)^2}\right) \\
&= \frac{1}{(\alpha-\beta)^2}\left(\frac{\alpha-\beta}{(1-\alpha x)^2}-\frac{\alpha-\beta}{(1-\beta x)^2}\right) \\
&= \frac{1}{\alpha-\beta}\left(\frac{1}{(1-\alpha x)^2}-\frac{1}{(1-\beta x)^2}\right).
\end{aligned}$$

Now using (7), (2) and the idea that $c(n)$ is the coefficient of x^n in $g(x)h(x)$, we get

$$c(n) = \frac{1}{\alpha-\beta}\Big(\alpha^n(n+1)-\beta^n(n+1)\Big) = (n+1)R_n.$$

□

Proof of Theorem 6. By the help of the generating function of $V_n(0,1,A_1,B_1)$, the method of partial-fraction decomposition and the Vieta's formula $\alpha+\beta=A_1$, we get the following.

$$\begin{aligned}(h(x))^2 &= \left(\frac{2-A_1x}{1-A_1x-B_1x^2}\right)^2 = \left(\frac{2-A_1x}{(1-\alpha x)(1-\beta x)}\right)^2\\ &= \frac{1}{(\alpha-\beta)^2}\left(\frac{2\alpha-A_1}{1-\alpha x}-\frac{2\beta-A_1}{1-\beta x}\right)^2\\ &= \frac{1}{(\alpha-\beta)^2}\left(\frac{(\alpha-\beta)^2}{(1-\alpha x)^2}-\frac{2(\alpha-\beta)(\beta-\alpha)}{(1-\alpha x)(1-\beta x)}+\frac{(\beta-\alpha)^2}{(1-\beta x)^2}\right)\\ &= \frac{1}{(1-\alpha x)^2}+\frac{2}{(1-\alpha x)(1-\beta x)}+\frac{1}{(1-\beta x)^2}\\ &= \frac{1}{(1-\alpha x)^2}+\frac{\frac{2\alpha}{\alpha-\beta}}{1-\alpha x}-\frac{\frac{2\beta}{\alpha-\beta}}{1-\beta x}+\frac{1}{(1-\beta x)^2}.\end{aligned}$$

Now using (6), (7), (2), (3) and the idea that $c(n)$ is the coefficient of x^n in $(h(x))^2$, we get

$$c(n)=\alpha^n(n+1)+\alpha^n\frac{2\alpha}{\alpha-\beta}-\beta^n\frac{2\beta}{\alpha-\beta}+\beta^n(n+1)=(n+1)V_n-2R_{n+1}.$$

□

All the Corollaries in Subsection 2.2 are special cases of the theorems and can easily be proved by simple substitution using the sequences in Table 1. As an example, we give the proof of Corollary 5.

Proof of Corollary 5. We substitute in this special case the following values

$$R_n=F_n(0,1,1,1),\quad \alpha=\frac{1+\sqrt{5}}{2}\quad\text{and}\quad \beta=\frac{1-\sqrt{5}}{2}$$

into the formula of Theorem 4, where α and β are the roots of the characteristic polynomial of the Fibonacci sequence. We get

$$c(n)=\frac{1}{(\alpha-\beta)^2}\Big((n+1)V_n-2R_{n+1}\Big)=\frac{1}{5}\Big((n+1)L_n-2F_{n+1}\Big),$$

where L_n is the associate sequence of F_n. □

4 Concluding remarks

With this paper, we have completed the investigation of the convolution of two second order linear recursive sequences. We have dealt the cases, when there are no, one or two common root(s) of the characteristic polynomials of the sequences and we used only Lucas and associated Lucas sequences.

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