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THE EQUALITY CASE IN SOME RECENT CONVEXITY INEQUALITIES

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Abstract. In this paper, we investigate a functional equation related to some recently introduced and investigated convexity type inequalities.

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1. INTRODUCTION

In a recent paper [24] by Varošanec, a common generalization of convex and s-convex functions, Godunova-Levin functions, and $\mathcal{P}$-functions is introduced in the following way: Let I be a nonvoid subinterval of $\mathbb{R}$ (the set of all real numbers), $h : [0, 1] \to \mathbb{R}$ and $f : I \to \mathbb{R}$ be real-valued functions satisfying the inequality

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) \tag{1.1}$$

for all $x, y \in I$ and $t \in]0, 1[$. An even more general notion, the so-called (T, h)-convexity, can be found in Házy [11]: Let X be a real or complex normed space, $D \subset X$ be a nonempty convex set, $\emptyset \neq T \subset [0, 1]$, and $h : T \to \mathbb{R}$ be a function. A function $f : D \to \mathbb{R}$ is (T, h)-convex if (1.1) holds for all $x, y \in D$ and $t \in T$. It is clear that this generalizes the concepts of convexity ($h(t) = t$, $t \in [0, 1]$, [24], [21]), the Breckner-convexity ($h(t) = t^s$, $t \in]0, 1[$, for some $s \in \mathbb{R}$, [5], [6]), the Godunova-Levin functions ($h(t) = t^{-1}$, $t \in]0, 1[$, [10]), the $\mathcal{P}$-functions ($h(t) = 1$, $t \in [0, 1]$, [18]), and the t-convexity ($T = \{t, 1-t\}$, $h(t) = t$, $h(1-t) = 1-t$, where $0 < t < 1$ is a fixed number, Kuhn [14]). For further related results see Burai-Házy [1, 2] and Burai-Házy-Juhász [3, 4].

In this note, we focus on the functional equation related to these convexity properties and give the solutions of the following problem. Let X be a real or complex topological vector space, $D \subset X$ be a nonempty open set, T be a nonempty set, and $\alpha, \beta, a, b : T \to \mathbb{R}$ be given functions. The problem is to find all the solutions $f : D \to \mathbb{R}$ of the functional equation

$$f(\alpha(t)x + \beta(t)y) = a(t)f(x) + b(t)f(y) \qquad (x, y \in D,\, t \in T) \tag{1.2}$$

provided that D is (α, β)-convex, that is, $\alpha(t)x + \beta(t)y \in D$ whenever $x, y \in D$ and $t \in T$. To avoid the trivialities and the unimportant cases, we suppose that there exists an element $t_0 \in T$ such that

$$\alpha(t_0)\beta(t_0)a(t_0)b(t_0) \neq 0. \tag{1.3}$$

We refer to the solutions of (1.2) as (α, β, a, b)-affine functions and the solutions f of the corresponding inequality

$$f(\alpha(t)x + \beta(t)y) \leq a(t)f(x) + b(t)f(y) \qquad (x, y \in D,\, t \in T)$$

will be called (α, β, a, b)-convex functions. Besides those convexity notions we listed above this is a generalization of (t, q)-convexity ($T = \{t\}, \alpha(t) = t$, $\beta(t) = 1-t$, $a(t) = q$, $b(t) = 1-q$, where $t, q \in]0, 1[$ are fixed numbers, Kuhn [15], Matkowski-Pycia [16]), and Orlicz s-convexity ($T = [0, 1]$, $\alpha(t) = t^s$, $\beta(t) = (1-t)^s$, $a(t) = t$, $b(t) = 1-t$ for all $t \in T$ and for some $s \geq 1$, Orlicz [17], Hudzik-Maligranda [12]).

Our purpose is to describe the (α, β, a, b)-affine functions. Throughout this paper X denotes a real or complex topological vector space. A function $A : X \to \mathbb{R}$ is called additive if it satisfies the Cauchy functional equation

$$A(x + y) = A(x) + A(y) \qquad (x, y \in X).$$

Given a subfield $S \subseteq \mathbb{R}$, a function $\varphi : S \to \mathbb{R}$ is said to be a field-homomorphism if φ is additive and multplicative on S, i.e.,

$$\varphi(s + t) = \varphi(s) + \varphi(t) \qquad \text{and} \qquad \varphi(st) = \varphi(s)\varphi(t) \qquad (s, t \in S).$$

2. THE RESULTS

Our investigations are based on the following extension theorem which is an immediate consequence of Theorem 1 in Radó-Baker [19].

Theorem 2.1. *Let U be a nonempty, open, connected subset of $X \times X$ and define the following sets*

$$\begin{aligned} U_0 &:= \{x + y \mid (x, y) \in U\}, \\ U_1 &:= \{x \mid \exists\, y \in X : (x, y) \in U\}, \quad \textit{and} \\ U_2 &:= \{y \mid \exists\, x \in X : (x, y) \in U\}. \end{aligned}$$

Suppose that the functions $f_i : U_i \to \mathbb{R}$, $(i = 0, 1, 2)$ satisfy the functional equation

$$f_0(x + y) = f_1(x) + f_2(y)$$

for all $(x, y) \in U$. *Then there exist a unique additive function* $A : X \to \mathbb{R}$ *and a unique pair* $(c_1, c_2) \in \mathbb{R}^2$ *such that*

$$\begin{aligned} f_0(x) &= A(x) + c_1 + c_2 && (x \in U_0), \\ f_1(x) &= A(x) + c_1 && (x \in U_1), \ \textit{and} \\ f_2(x) &= A(x) + c_2 && (x \in U_2). \end{aligned}$$

An important consequence of the above theorem is the following result.

Theorem 2.2. *Let* $\gamma, \delta, p, q \in \mathbb{R}$ *and* $\emptyset \neq D \subset X$ *be an open and connected set such that* $\gamma\delta pq \neq 0$ *and* $\gamma x + \delta y \in D$ *for all* $x, y \in D$. *Then the function* $f : D \to \mathbb{R}$ *satisfies the functional equation*

$$f(\gamma x + \delta y) = pf(x) + qf(y) \qquad (x, y \in D) \tag{2.1}$$

if, and only if, there exist an additive function $A : X \to \mathbb{R}$ *and a constant* $c \in \mathbb{R}$ *such that*

$$\begin{aligned} A(\gamma x) &= pA(x) && (x \in X), \\ A(\delta x) &= qA(x) && (x \in X), \\ c(p + q - 1) &= 0, && \textit{and} \\ f(x) &= A(x) + c && (x \in D). \end{aligned} \tag{2.2}$$

Proof. Equation (2.1) implies that

$$f(x + y) = pf\Big(\frac{1}{\gamma}x\Big) + qf\Big(\frac{1}{\delta}y\Big) \qquad (x \in \gamma D,\ y \in \delta D).$$

Applying Theorem 2.1 for the open and connected set $U := (\gamma D) \times (\delta D)$ and the triplet of functions

$$\begin{aligned} f_0(x) &:= f(x),\ x \in \gamma D + \delta D \subset D, \\ f_1(x) &:= pf\Big(\frac{1}{\gamma}x\Big),\ x \in \gamma D, \\ f_2(x) &:= qf\Big(\frac{1}{\delta}x\Big),\ x \in \delta D, \end{aligned}$$

we obtain that

$$pf\Big(\frac{1}{\gamma}x\Big) = A_0(x) + c_0 \qquad (x \in \gamma D)$$

with some additive function $A_0 : X \to \mathbb{R}$ and $c_0 \in \mathbb{R}$. Thus

$$f(x) = \frac{1}{p}A_0(\gamma x) + \frac{c_0}{p} \qquad (x \in D),$$

whence, with the definitions $A(x) := \frac{1}{p}A_0(\gamma x)$, $x \in X$ and $c := \frac{c_0}{p}$,

$$f(x) = A(x) + c \qquad (x \in D)$$

follows.

Obviously, $A : X \to \mathbb{R}$ is additive. Replacing this form of f into (2.1), we find that

$$A(\gamma x) - pA(x) + A(\delta y) - qA(y) - c(p+q-1) = 0 \qquad (x, y \in D).$$

This shows that, for all fixed $y \in D$, the polynomial function

$$x \mapsto A(\gamma x) - pA(x) + A(\delta y) - qA(y) - c(p+q-1) \qquad (x \in X)$$

vanishes on D, therefore it vanishes everywhere on X (see Székelyhidi [23]). This implies the other equalities of (2.2), as well. The converse is straightforward. □

In the result below we investigate homogeneity properties of additive functions. Given an additive function $A : \mathbb{R} \to \mathbb{R}$, we introduce its set of homogeneity pairs H_A as follows:

$$H_A := \{(s,t) \in \mathbb{R}^2 \mid A(sx) = tA(x) \text{ for all } x \in \mathbb{R}\}.$$

Theorem 2.3. *Let $A : \mathbb{R} \to \mathbb{R}$ be a nonzero additive function. Then there exist a subfield $S_A \subseteq \mathbb{R}$ (called the homogeneity field of A) and an injective field-homomorphism $\varphi_A : S_A \to \mathbb{R}$ (called the homogeneity field-homomorphism of A) such that H_A is equal to the graph of φ_A, i.e.,*

$$H_A = \{(s, \varphi_A(s)) \mid s \in S_A\}. \tag{2.3}$$

Conversely, for every subfield $S \subseteq \mathbb{R}$ and injective field-homomorphism $\varphi : S \to \mathbb{R}$, there exists a nonzero additive function $A : X \to \mathbb{R}$ such that $S \subseteq S_A$ and $\varphi_A|_S = \varphi$.

Proof. Denote by S_A the domain of the relation H_A. We show that, H_A is in fact a function. Assume that $(s, t_1), (s, t_2) \in H_A$. Then, for all $x \in X$,

$$(t_1 - t_2)A(x) = t_1 A(x) - t_2 A(x) = A(sx) - A(sx) = 0,$$

which, by the nontriviality of A, yields that $t_1 = t_2$ proving that the relation H_A is a function. This means that there exists a function $\varphi_A : S_A \to \mathbb{R}$ such that (2.3) holds. It remains to show that S_A is a subfield of $\mathbb{R}$ and φ_A is an injective field-homomorphism.

To prove the injectivity, let $(s_1, t), (s_2, t) \in H_A$. Then, for all $x \in X$,

$$A((s_1 - s_2)x) = A(s_1 x) - A(s_2 x) = tA(x) - tA(x) = 0,$$

which, by the nontriviality of A, yields that $s_1 = s_2$. By the injectivity, $\varphi_A(s)$ is nonzero whenever s is different from zero.

Let $s, t \in S$. Then, using (2.3), for all $x \in X$, we get that

$$A((s-t)x) = A(sx) - A(tx) = \varphi_A(s)A(x) - \varphi_A(t)A(x) = (\varphi_A(s) - \varphi_A(t))A(x).$$

Hence, $(s - t, \varphi_A(s) - \varphi_A(t)) \in H_A$, which yields that $s - t \in S$ and $\varphi_A(s - t) = \varphi_A(s) - \varphi_A(t)$. Thus S is a group with respect to the addition and φ_A is additive.

Similarly, for all $s \in S$, $t \in S \setminus \{0\}$, and $x \in X$, we obtain that

$$\varphi_A(t) A\Big(\frac{s}{t}x\Big) = A(sx) = \varphi_A(s)A(x).$$

Hence $\left(\frac{s}{t}, \frac{\varphi_A(s)}{\varphi_A(t)}\right) \in H_A$, which yields that $\frac{s}{t} \in S$ and $\varphi_A\left(\frac{s}{t}\right) = \frac{\varphi_A(s)}{\varphi_A(t)}$. This proves that S is a semigroup under the multiplication whose nonzero elements form a group and φ_A is also multiplicative.

To prove the reversed statement, let $S \subseteq \mathbb{R}$ be a subfield and $\varphi : S \to \mathbb{R}$ be an injective field-homomorphism. Consider X as a vector space over S and let $\{x_\gamma \mid \gamma \in \Gamma\}$ be a Hamel base of X over S. In addition, let $\{a_\gamma \mid \gamma \in \Gamma\}$ be an arbitrary family of real numbers such that at least one of these elements is different from zero. Given an element $x \in X$, it can uniquely be written in the form

$$x = s_1 x_{\gamma_1} + \ldots + s_m x_{\gamma_m}, \tag{2.4}$$

where $m \in \mathbb{N} \cup \{0\}$, $s_1, \ldots, s_m \in S$, and $\gamma_1, \ldots, \gamma_m$ are pairwise distinct elements of the index set Γ. Now define $A(x)$ by

$$A(x) := \varphi(s_1) a_{\gamma_1} + \ldots + \varphi(s_m) a_{\gamma_m}.$$

Using the additivity of φ, it is immediate to see that A is a nonzero additive function. It remains to show that, for all $s \in S$, $(s, \varphi(s)) \in H_A$, i.e.,

$$A(sx) = \varphi(s) A(x) \qquad (x \in X). \tag{2.5}$$

If x is of the form (2.4), then $sx = (ss_1)x_{\gamma_1} + \ldots + (ss_m)x_{\gamma_m}$ and hence, by the multiplicativity of φ, we get

$$\begin{aligned} A(sx) &= \varphi(ss_1)a_{\gamma_1} + \ldots + \varphi(ss_m)a_{\gamma_m} = \varphi(s)\big(\varphi(s_1)a_{\gamma_1} + \ldots + \varphi(s_m)a_{\gamma_m}\big) = \\ &= \varphi(s)A(x), \end{aligned}$$

which completes the proof of (2.5). $\square$

Remark 2.4. The equality stated in (2.3) can be rewritten as the following identity:

$$A(sx) = \varphi_A(s) A(x) \qquad (s \in S_A,\ x \in X). \tag{2.6}$$

The additive and multiplicative properties of φ_A imply that if $s \in S$ is an algebraic number over a subfield of $\mathbb{R}$ then $\varphi_A(s)$ must be one of its algebraic conjugates. In particular, if s is a rational number then, $\varphi_A(s) = s$. On the other hand, if $s \in S$ is transcendent, then $\varphi_A(s)$ can be any transcendental number. For an account of such results see the paper [8] by Z. Daróczy. Those real numbers s such that $(s, s) \in H_A$ also form a subfield of $\mathbb{R}$ (cf. Rätz [20]). This easily follows from the fact that they are characterized by the fixed point equation $\varphi_A(s) = s$.

An easy consequence of Theorem 2.2 and Theorem 2.3 is the following result.

Theorem 2.5. *Let T be a nonempty set, and $\alpha, \beta, a, b : T \to \mathbb{R}$ be given functions satisfying property* (1.3) *for some $t_0 \in T$. Let furthermore, $\emptyset \neq D \subset X$ be an open connected and (α, β)-convex set. Then $f : D \to \mathbb{R}$ is a nonconstant (α, β, a, b)-affine function if, and only if, there exist a nonzero additive function $A : X \to \mathbb{R}$ and a* *constant $c \in \mathbb{R}$ such that $\alpha(T) \cup \beta(T)$ is contained by the homogeneity field S_A of A and*

$$\begin{aligned} a(t) &= \varphi_A(\alpha(t)) && (t \in T), \\ b(t) &= \varphi_A(\beta(t)) && (t \in T), \\ c(a(t)+b(t)-1) &= 0 && (t \in T), \quad \text{and} \\ f(x) &= A(x) + c && (x \in D) \end{aligned} \tag{2.7}$$

where $\varphi_A : S_A \to \mathbb{R}$ is the homogeneity field-homomorphism of A.

Proof. Applying Theorem 2.2 with $\gamma := \alpha(t_0)$, $\delta := \beta(t_0)$, $p := a(t_0)$, and $q := b(t_0)$, it follows that there exist an additive function $A : X \to \mathbb{R}$ and a constant $c \in \mathbb{R}$ such that $f(x) = A(x) + c$ for all $x \in D$.

To see that the first three equations in (2.7) are valid, we substitute this form of f into (1.2) and get that, for all $x, y \in D$ and $t \in T$,

$$A(\alpha(t)x) - a(t)A(x) + A(\beta(t)y) - b(t)A(y) - c(a(t)+b(t)-1) = 0. \tag{2.8}$$

In other words, for all fixed $y \in D$ and $t \in T$, the polynomial function

$$x \mapsto A(\alpha(t)x) - a(t)A(x) + A(\beta(t)y) - b(t)A(y) - c(a(t)+b(t)-1) \qquad (x \in X)$$

vanishes on the open set D, therefore it vanishes everywhere on X. (See Székelyhidi [23].) Analogously, for all fixed $x \in X$ and $t \in T$, the polynomial function

$$y \mapsto A(\alpha(t)x) - a(t)A(x) + A(\beta(t)y) - b(t)A(y) - c(a(t)+b(t)-1) \qquad (y \in X)$$

vanishes on D, therefore it vanishes everywhere on X. Therefore, (2.8) holds for all $x, y \in X$ and $t \in T$.

Thus, with simple substitutions, for all $t \in T$ and $x \in X$, we obtain that

$$A(\alpha(t)x) = a(t)A(x), \qquad A(\beta(t)x) = b(t)A(x), \qquad c(a(t)+b(t)-1) = 0.$$

The first two equalities yield that $(\alpha(t), a(t))$ and $(\beta(t), b(t))$ belong to H_A for all $t \in T$. Therefore, $\alpha(T) \cup \beta(T) \subseteq S_A$ and the first two equations in (2.7) are also satisfied. □

3. REMARKS AND EASY CONSEQUENCES OF THEOREM 2.5

Remark 3.1. Suppose that $\alpha, \beta, a, b : T \to \mathbb{R}$ are given functions, $\emptyset \neq D \subset X$ such that, for some $t \in T$,

$$\alpha(t) + \beta(t) = a(t) + b(t) = 1,\ a(t) > 0,\ b(t) > 0,\ \text{and}\ \alpha(t)x + \beta(t)y,\ \frac{x+y}{2} \in D$$

whenever $x, y \in D$. Then every (α, β, a, b)-convex function $f : D \to \mathbb{R}$ is Jensen convex, i.e.

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} \qquad (x, y \in D),$$

and every (α, β, a, b)-affine function $f : D \to \mathbb{R}$ satisfies the Jensen equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \qquad (x, y \in D).$$

In Kuczma [13, p. 315], there is an extension theorem for the Jensen equation. There D is a subset of $\mathbb{R}^n$ with nonempty interior. Our statements follow easily from the identity (see Daróczy-Páles [9], and also Matkowski-Pycia [16])

$$\frac{x+y}{2} = \alpha(t)\left[\alpha(t)\frac{x+y}{2} + \beta(t)y\right] + \beta(t)\left[\alpha(t)x + \beta(t)\frac{x+y}{2}\right] \qquad (x, y \in D).$$

Finally, we list some easy consequences of Theorem 2.5.

Corollary 3.2. *If $\alpha(T) \cup \beta(T)$ contains a set of positive Lebesgue measure then the additive function A in Theorem 2.5 is a linear functional on X and $a = \alpha, b = \beta$.*

Proof. In this case, by a well-known theorem of Steinhaus [22], the homogeneity field S_A must contain an interval of positive length. Therefore $S_A = \mathbb{R}$. Thus, by the classical theorem of Darboux [7] and taking into consideration (1.3) to hold for some $t_0 \in T$, we have that $\varphi_A(t) = t$ for all $t \in \mathbb{R}$. The remaining statements are obvious. □

The following corollary is a trivial consequence of Corollary 3.2.

Corollary 3.3. *Suppose that, for $f : D \to \mathbb{R}$ and for all $x, y \in D$, the equality holds in the defining inequality of Breckner-convexity or Orlicz-convexity. Then f must be the constant function except the case $s = 1$.*

Taking into consideration Remark 2.4 (see also Daróczy [8]), we have

Corollary 3.4. *If $t, q \in]0, 1[$ are fixed, $T = \{t\}, \alpha(t) = t, \beta(t) = 1 - t, a(t) = q, b(t) = 1 - q$ then there exists nonconstant (α, β, a, b)-affine function if, and only if, t and q are conjugate, i.e., they are both transcendental or they are both algebraic and have the same minimal polynomial with rational coefficients.*

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