Merging to semistable processes[1]

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Abstract

A functional merge theorem is obtained for distributions being in the domain of geometric partial attraction of a semistable law.

Key words and phrases: distances of probability measures, merging, tightness, functional limit theorem, semistable law, domain of partial attraction, slowly varying function.

2000 Mathematics Subject Classification: 60F17, 60E07.

1 Introduction

The concept of semistable distributions was introduced by Paul Lévy in 1937 (see [17]). The characteristic function of a semistable distribution was obtained by Kruglov [16]. The domain of geometric partial attraction of a semistable law was described by Grinevich and Khokhlov [13]. In Csörgő and Megyesi [7] the theory of semistable laws was studied in the framework of the 'probabilistic' approach of Csörgő [5] and Csörgő, Haeusler, and Mason [6]. In the case of distributions being in the domain of geometric partial attraction of a semistable law ordinary convergence in distribution takes place only along some subsequences. However, a merge theorem is valid (Csörgő and Megyesi [7]). We say that two sequences $\{\mu_n\}$ and $\{\nu_n\}$ of probability measures are merging if $\lim_{n\to\infty} \varrho(\mu_n, \nu_n) = 0$ where ϱ is a distance of probability measures (the Lévy distance or the Prokhorov distance, say).

A considerable part of probability theory is devoted to functional versions of ordinary limit theorems (the standard references are [19], [21], [2]). The classical result is Donsker's theorem. Let $\xi_1, \xi_2, \dots$ be independent identically distributed random variables with $\mathbb{E}\xi_1 = 0$, $\mathbb{D}^2\xi_1 = 1$. Denote by $X_n(t)$ the usual step function constructed from the partial sums. Then $X_n \xrightarrow{d} W$ in $\mathbb{D}[0,1]$ where W is the standard Wiener process. However, if we assume that $\xi_1, \xi_2, \dots$ belong to the domain of geometric partial attraction of a semistable law, then the step functions constructed from $\xi_1, \xi_2, \dots$ will not converge (as the finite dimensional distributions will not converge). Therefore the following question arises. How can we describe the asymptotic behaviour of the step functions?

In this paper we prove a functional merge theorem for laws being in the domain of geometric partial attraction of a semistable law (Theorem 2.1). It is the functional

[1]Supported by the Hungarian Foundation of Scientific Researches under Grant No. OTKA T-079128/2009.

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version of the merge theorem by Csörgő and Megyesi [7]. The accompanying laws in our theorem are distributions of semistable Lévy processes (random processes with stationary independent increments). As we want to prove the precise functional version of (a part of) Theorem 2 of [7], in the following sections we shall use the same centering and norming constants as the ones in [7].

To handle the concept of merging distributions of stochastic processes, we need some facts about merging probabilities on metric spaces (see Davydov and Rotar [8]). It turns out that in our case both sequences of probabilities are tight.

In Section 2 first some known facts on semistable distributions and their domain of geometric partial attraction are listed. Then the main result is given (Theorem 2.1). Section 3 describes the approximation of sums of independent random variables being in the domain of geometric partial attraction of a semistable law. In Section 4 some general facts on merging probability measures are listed, moreover, tightness of our sequences are proved.

We shall use the following notation. $\mathbb{N}$ is the set of positive integers, $\mathbb{R}$ denotes the set of real numbers. $\mathbb{D}[0,1]$ is the Skorokhod space of functions without discontinuities of second kind (see [2]). The distribution of a random element ξ will be denoted by $\mathcal{L}_\xi$. Sign $\xrightarrow{d}$ denotes the convergence in distribution.

Finally, we mention that in Berkes, Csáki, Csörgő, and Megyesi [1] an almost sure limit theorem was proved for laws being in the domain of geometric partial attraction of a semistable law. The almost sure limit theorem is valid with the usual weights. In [10] a functional analogue of the a.s. limit theorem of [1] was obtained.

2 The merge theorem

Consider the probabilistic approach to the theory of infinitely divisible distributions presented in Csörgő, Haeusler and Mason [6] and Csörgő [5]. We shall use it to handle p-semistable distributions, $0 < p < 2$, and the domain of the geometric partial attraction of a p-semistable distribution, see Megyesi [18], Csörgő and Megyesi [7].

Let Ψ be the class of all non-positive, non-decreasing, right-continuous functions $g(\cdot)$ defined on the positive half-line $(0,\infty)$ such that $\int_\varepsilon^\infty g^2(s)ds < \infty$ for all $\varepsilon > 0$. Let N_j, $j = 1, 2$, be independent standard left-continuous Poisson processes. Let

$$V_j(g) = \int_1^\infty [N_j(s) - s]dg(s) + \int_0^1 N_j(s)dg(s) - g(1), \quad j = 1, 2. \tag{2.1}$$

Let U be a standard normal random variable such that $N_1(\cdot)$, U, and $N_2(\cdot)$ are independent. Let $g_1, g_2 \in \Psi$ and let $\sigma \geq 0$. Introduce the random variable

$$V(g_1, g_2, \sigma) = -V_1(g_1) + \sigma U + V_2(g_2). \tag{2.2}$$

Up to an additive constant, any infinitely divisible distribution can be described in this way.

Infinitely divisible distributions which arise as limiting distributions of suitable centered and normalized sums $S_{k_n} = \frac{1}{B_{k_n}} \sum_{i=1}^{k_n} X_i - A_{k_n}$ of independent identically distributed random variables $X_1, X_2, \ldots$ along subsequences $\{k_n\}$ satisfying condition $\lim_{n\to\infty} k_{n+1}/k_n = c \geq 1$ are called semistable laws. It is known that a semistable law is either normal or p-semistable with $0 < p < 2$.

We consider the case of the p-semistable distribution, $0 < p < 2$. Suppose that

$$g_j(s) = -M_j(s)s^{-1/p}, \quad s > 0, \quad j = 1, 2, \tag{2.3}$$

are non-decreasing functions, where M_1, M_2 are non-negative, right-continuous functions on $(0, \infty)$, either identically zero or bounded away from both zero and infinity, such that $M_1 + M_2$ is not identically zero, moreover $M_j(cs) = M_j(s)$ for all $s > 0$, $j = 1, 2$, for some constant $c > 1$. Let $g_j(s)$ be defined by (2.3) and let

$$W_j(M_j) = W_j(M_j, p) = V_j(g_j), \quad j = 1, 2,$$

$$W(M_1, M_2) = W_2(M_2) - W_1(M_1). \tag{2.4}$$

A random variable W is a p-semistable random variable with $0 < p < 2$ if and only if $W \stackrel{d}{=} W(M_1, M_2) + b$ for some M_1, M_2 and $b \in \mathbb{R}$. We will denote by

$$\psi(x) = \psi(x, M_1, M_2) = \mathbb{E}e^{ixW(M_1,M_2)} \tag{2.5}$$

the characteristic function of $W(M_1, M_2)$.

Let X, X_i, $i \in \mathbb{N}$, be independent identically distributed random variables. Denote by $F(x) = \mathbb{P}(X \leq x)$ the distribution function of X. Let $\{k_n\}$ be a sequence of positive integers with the property

$$\lim_{n\to\infty} k_{n+1}/k_n = c > 1. \tag{2.6}$$

Recall that F (or X) is said to belong to the domain of geometric partial attraction of a semistable law, if for some sequence $\{k_n\}$ of positive integers with property (2.6), for some norming numbers B_{k_n} and some centering numbers A_{k_n}, the sequence $S_{k_n} = \frac{1}{B_{k_n}} \sum_{i=1}^{k_n} X_i - A_{k_n} \xrightarrow{d} W$, as $n \to \infty$, where W is a p-semistable random variable ($W \stackrel{d}{=} W(M_1, M_2)$, say). In this case we will write $F \in \mathbb{D}_{gp}(M_1, M_2, p)$, see [18]. Denote by Q the quantile function of X, i.e.

$$Q(s) = \inf\{x \in \mathbb{R} : F(x) \geq s\}, \quad 0 < s < 1.$$

Denote by Q_+ the right-continuous version of the quantile function Q.

Consider a subsequence $\{k_n\}_{n=1}^{\infty} \subset \mathbb{N}$ satisfying (2.6). As $c > 1$, the sequence $\{k_n\}$ is eventually strictly increasing. So for all $s \in (0, s_0)$, with $s_0 \in (0, 1]$ small enough, there exists a unique $k_{n^*(s)}$ such that $k_{n^*(s)}^{-1} \leq s < k_{n^*(s)-1}^{-1}$. We define $\gamma(s) = sk_{n^*(s)}$ for $s \in (0, s_0)$ and $\gamma(s) = 1$ for $s \in [s_0, 1)$. So $1 \leq \gamma(s) < c + \varepsilon$ for any fixed $\varepsilon > 0$ and all $s \in (0,1)$ small enough for the limiting $c > 1$ from (2.6). Then $F \in \mathbb{D}_{gp}(M_1, M_2, p)$ along a subsequence $\{k_n\}$ satisfying (2.6), $p \in (0,2)$, if and only if for all $s \in (0,1)$ small enough

$$Q_+(s) = -s^{-1/p} l(s)[M_1(\gamma(s)) + h_1(s)], \tag{2.7}$$

$$Q(1-s) = s^{-1/p} l(s)[M_2(\gamma(s) + h_2(s)], \tag{2.8}$$

where $l(\cdot)$ is a right-continuous function, slowly varying at zero, and the error functions h_1 and h_2 are right-continuous such that $\lim_{n\to\infty} h_j(t/k_n) = 0$, for every continuity point $t > 0$ of M_j, $j = 1, 2$ ([13], [18]).

We will use the norming and centering constants (see [7])

$$B_n = n^{1/p} l\Big(\frac{1}{n}\Big), \quad A_{nk} = \frac{k}{B_n} \int_{1/n}^{1-1/n} Q(s)ds, \quad k = 1, \dots, n, \ n = 1, 2, \dots. \tag{2.9}$$

Let

$$S_{nk} = \frac{1}{B_n} \sum\nolimits_{i=1}^{k} X_i - A_{nk}, \quad k = 1, \dots, n, \ n = 1, 2, \dots. \tag{2.10}$$

We will denote S_{nn} and A_{nn} by S_n and A_n, respectively.

The aim of this paper is to obtain a functional version of the following result.

Theorem A. (Merge theorem, i.e. Theorem 2 in [7].) *Let $F \in \mathbb{D}_{gp}(M_1, M_2, p)$ along a subsequence $\{k_n\}$ satisfying* (2.6), $0 < p < 2$. *Then, as $n \to \infty$, we have*

$$\sup_{x\in\mathbb{R}} \Big| \mathbb{P}\{S_n \le x\} - \mathbb{P}\{W(M_1(\gamma(1/n)y), M_2(\gamma(1/n)y)) \le x\} \Big| \to 0. \quad \square$$

Here and in what follows $W(M_1(\gamma(1/n)y), M_2(\gamma(1/n)y))$ denotes $W(\widetilde{M}_1, \widetilde{M}_2)$ with $\widetilde{M}_1(y) = M_1(\gamma(1/n)y)$ and $\widetilde{M}_1(y) = M_1(\gamma(1/n)y)$.

To obtain the functional version of Theorem A, consider the random processes

$$Z_n(t) = S_{n[nt]}, \quad t \in [0,1], \tag{2.11}$$

with sample paths in the Skorokhod space $\mathbb{D}[0,1]$.

Also we will consider the Lévy processes $Y_n(t)$, $t \in [0,1]$, such that

$$\text{the distribution of } Y_n(1) \text{ is the same as that of } W(M_1(\gamma(1/n)y), M_2(\gamma(1/n)y)). \tag{2.12}$$

(For the definition of a Lévy process see [20], Definition 1.6.) By Corollary 11.6 of [20], there exists a Lévy process with property (2.12). In particular, the trajectories of Y_n belong to $\mathbb{D}[0,1]$.

Theorem 2.1. *Let $F \in \mathbb{D}_{gp}(M_1, M_2, p)$ along a subsequence $\{k_n\}$ satisfying* (2.6), $0 < p < 2$. *Let Z_n be defined by* (2.10)–(2.11) *and let Y_n be the Lévy process given by* (2.12). *Then, as $n \to \infty$,*

$$\varrho(\mathcal{L}_{Z_n}, \mathcal{L}_{Y_n}) \to 0 \quad \textit{in} \quad \mathbb{D}[0,1]. \tag{2.13}$$

Proof. By Lemma 4.1, we have to show, that $\{\mathcal{L}_{Z_n}, n = 1, 2, \dots\}$ and $\{\mathcal{L}_{Y_n}, n = 1, 2, \dots\}$ are tight, moreover, the distances of the finite dimensional distributions converge to 0. These facts are proved in Lemmas 4.3, 4.2, and 3.1. □

3 Approximation with semistable distributions

Let the sequences of processes Z_n and Y_n be defined by (2.11) and (2.12), respectively. In this section we prove the convergence of the finite dimensional distributions. Moreover, we present the distribution of Y_n in terms of Lévy's description of infinitely divisible laws.

First we collect some known facts on characteristic functions and infinitely divisible laws.

Remark 3.1. (See [4], Ch. 12.) 1. Let $f(t)$ be a continuous, non-vanishing, complex valued function of the interval $[-T, T]$ with $f(0) = 1$. Then there exists a unique (single-valued) continuous complex valued function $\lambda(t)$ defined on $[-T, T]$ with $\lambda(0) = 0$ such that $f(t) = e^{\lambda(t)}$ for $t \in [-T, T]$. Moreover, $[-T, T]$ is replaceable by $(-\infty, \infty)$.

2. The function $\lambda(t)$ defined above is called the distinguished logarithm of $f(t)$ and is denoted by $\mathrm{Log} f(t)$. Also, for $v \in \mathbb{R}$, $\exp(v\lambda(t))$ is called the distinguished vth power of $f(t)$ and is denoted by $f^v(t)$.

3. Let f, f_k, $k = 1, 2, \dots$, be continuous, non-vanishing, complex valued functions of the interval $[-T, T]$ with $f(0) = 1$, $f_k(0) = 1$, $k = 1, 2, \dots$. If $f_k \to f$ uniformly in $[-T, T]$, then $\mathrm{Log} f_k \to \mathrm{Log} f$ uniformly in $[-T, T]$.

4. For an infinitely divisible characteristic function φ we have $\varphi(t) \neq 0$ for $t \in \mathbb{R}$. The product of infinitely divisible characteristic functions is infinitely divisible. (A characteristic functions is called infinitely divisible if it belongs to an infinitely divisible law.)

5. The limit of infinitely divisible laws is infinitely divisible.

Remark 3.2. (See [15], Theorem 3.1.) Let $\{\mu_n : n = 1, 2, \dots\}$ and $\{\nu_n : n = 1, 2, \dots\}$ be families of probability measures on the real line with characteristic functions $\{\varphi_n : n = 1, 2, \dots\}$ and $\{\psi_n : n = 1, 2, \dots\}$, respectively. Let $\{\mu_n : n = 1, 2, \dots\}$ be relatively compact. Then $\varrho(\mu_n, \nu_n) \to 0$ if and only if $\varphi_n(t) - \psi_n(t) \to 0$ for every $t \in \mathbb{R}$. (Here ϱ denotes the Lévy distance.)

Lemma 3.1. *The Lévy distances of the finite dimensional distributions of the sequence $Z_n(\cdot)$ and those of $Y_n(\cdot)$ converge to 0, as $n \to \infty$.*

Proof. We shall write $W(M(\gamma(1/n)y))$ instead of $W(M_1(\gamma(1/n)y), M_2(\gamma(1/n)y))$.

Both of the processes $Z_n(\cdot)$ and $Y_n(\cdot)$ have independent increments and their values at zero are zero. So it is enough to prove that the distances of the distributions of the increments converge to 0.

Let $0 \le t_0 < t_1 \le 1$ and let $\Delta t = t_1 - t_0$. We shall use Remark 3.2. So first we shall prove that the set of distributions of the increments on the interval $[t_0, t_1]$ of the family of processes $Y_n(\cdot)$ is a relatively compact set of probabilities.

Let $\varphi_n = \varphi_{W\{M[\gamma(1/n)y]\}}$ be the characteristic function of $W\{M[\gamma(1/n)y]\}$. Then the characteristic function of the increment $Y_n(t_1) - Y_n(t_0)$ is $\varphi_n^{\Delta t}$. We have to prove that every subsequence $\varphi_{n'}^{\Delta t}$ contains a convergent subsequence. By Theorem 1 of [7] the distributions of the sequence $\{S_n\}_{n=1}^{\infty}$ is relatively compact. So, by Theorem A, the sequence $W\{M[\gamma(1/n)y]\}$, $n = 1, 2, \dots$, is also relatively compact. Therefore, the subsequence $\{n'\}$ contains a further subsequence $\{n''\}$ such that $\varphi_{n''} \to \varphi_\infty$ where φ_∞ is a characteristic function. Here the convergence is uniform in any bounded interval, and the characteristic functions involved are infinitely divisible. We have to prove that $\varphi_{n''}^{\Delta t}(x_0) \to \varphi_\infty^{\Delta t}(x_0)$ for each fixed x_0. Let T be large enough so that $x_0 \in [-T, T]$. By Remark 3.1, $\mathrm{Log}\varphi_{n''}(x_0) \to \mathrm{Log}\varphi_\infty(x_0)$ for $x_0 \in [-T, T]$. Therefore,

$$\varphi_{n''}^{\Delta t}(x_0) = e^{\Delta t \mathrm{Log}\varphi_{n''}(x_0)} \to e^{\Delta t \mathrm{Log}\varphi_\infty(x_0)} = \varphi_\infty^{\Delta t}(x_0).$$

Therefore, the set of the distributions of the increments of the processes $Y_n(\cdot)$ on the interval $[t_0, t_1]$ is relatively compact.

Now turn to the process Z_n. The increment of Z_n on the interval $[t_0, t_1]$ is

$$S_{n[nt_1]} - S_{n[nt_0]} = \frac{1}{B_n} \sum_{i=[nt_0]+1}^{[nt_1]} X_i - l_n A_{n1}$$

where $l_n = [nt_1] - [nt_0]$. We have $l_n/n \to \Delta t$. The characteristic function of the increment is

$$(\psi_n(x))^{l_n} = \left(\varphi_X\left(x/B_n\right) e^{ixA_{n1}}\right)^{l_n}$$

where φ_X is the characteristic function of X. By Remark 3.2, we have to prove that

$$\psi_n^{l_n}(x) - \varphi_n^{\Delta t}(x) \to 0 \tag{3.1}$$

for every $x \in \mathbb{R}$. Assume that it is not true, therefore there exists a fixed $x_0 \in \mathbb{R}$, a positive ε and a subsequence $\{n'\}$ such that

$$|\psi_{n'}^{l_{n'}}(x_0) - \varphi_{n'}^{\Delta t}(x_0)| > \varepsilon \tag{3.2}$$

for each n'. As the sequence $W\{M[\gamma(1/n)y]\}$, $n = 1, 2, \dots$, is relatively compact, we have a further subsequence $\{n''\}$ of $\{n'\}$ such that

$$\varphi_{n''}(x) \to \varphi_\infty(x). \tag{3.3}$$

As in (3.3) the characteristic functions are infinitely divisible, that is they are non-vanishing, we have

$$\varphi_{n''}^{\Delta t}(x_0) \to \varphi_\infty^{\Delta t}(x_0). \tag{3.4}$$

From Theorem A and (3.3), we have $\psi_{n''}^{n''}(x) \to \varphi_\infty(x)$. Let T be so large that $x_0 \in [-T,T]$. The previous convergence is uniform in the interval $[-T,T]$ and the limit is infinitely divisible, so it is non-vanishing. Therefore $\psi_{n''}^{n''}(x)$ is also non-vanishing if n'' is large enough and $x \in [-T,T]$. By Remark 3.1, we can take the distinguished logarithm and we have $n''\mathrm{Log}\psi_{n''}(x) = \mathrm{Log}\psi_{n''}^{n''}(x) \to \mathrm{Log}\varphi_\infty(x)$. Therefore $(l_{n''}/n'')n''\mathrm{Log}\psi_{n''}(x) \to \Delta t\,\mathrm{Log}\varphi_\infty(x)$. It implies that

$$\psi_{n''}^{l_{n''}}(x) = e^{l_{n''}\mathrm{Log}\psi_{n''}(x)} \to e^{\Delta t \mathrm{Log}\varphi_\infty(x)} = \varphi_\infty^{\Delta t}(x). \tag{3.5}$$

As (3.5) is valid for any $x \in [-T,T]$ and we have $x_0 \in [-T,T]$, therefore (3.5) combined with (3.4), contradicts to (3.2). So (3.1) is valid.

By Remark 3.2, the proof is complete. □

Now we turn to the description of $Y_n(\cdot)$ in terms of Lévy's original approach to infinitely divisible distributions. Consider the random variable $V(g_1, g_2, \sigma) = -V_1(g_1) + \sigma U + V_2(g_2)$ from equation (2.2). For any $g \in \Psi$ let

$$\theta(g) = \int_0^1 \frac{g(s)}{1+g^2(s)}ds - \int_1^\infty \frac{g^3(s)}{1+g^2(s)}ds.$$

Then, by Theorem 3 of [6], the characteristic function of

$$V_0(g_1, g_2, \sigma) = V(g_1, g_2, \sigma) + \theta(g_2) - \theta(g_1) \tag{3.6}$$

is

$$\mathbb{E}\left(e^{ixV_0(g_1,g_2,\sigma)}\right) = \tag{3.7}$$

$$= \exp\left\{-\frac{\sigma^2x^2}{2} + \int_{-\infty}^0 \left(e^{ixu} - 1 - \frac{ixu}{1+u^2}\right)dL(u) + \int_0^\infty \left(e^{ixu} - 1 - \frac{ixu}{1+u^2}\right)dR(u)\right\}$$

for all real x, where $L(u) = \inf\{s > 0 : g_1(s) \ge u\}$, $u < 0$, and $R(u) = -\inf\{s > 0 : g_2(s) \ge -u\}$, $u > 0$. Here $L(\cdot)$ is left-continuous and non-decreasing on $(-\infty, 0)$ with $L(-\infty) = 0$ and $R(\cdot)$ is right-continuous and non-decreasing on $(0,\infty)$ with $R(\infty) = 0$, moreover $\int_{-\varepsilon}^0 u^2 dL(u) + \int_0^\varepsilon u^2 dR(u) < \infty$ for every $\varepsilon > 0$. Conversely, consider two functions $L(\cdot)$ and $R(\cdot)$ with the properties just listed, and choose $g_1(s) = \inf\{u < 0 : L(u) > s\}$, $s > 0$, and $g_2(s) = \inf\{u < 0 : -R(-u) > s\}$, $s > 0$, then $g_1, g_2 \in \Psi$. Using these g_1, g_2, the characteristic function of $V_0(g_1, g_2, \sigma)$ satisfies (3.7).

Now fix the functions M_1 and M_2 in Theorem 2.1 and $g_j(s) = -M_j(s)s^{-1/p}$, $j = 1, 2$. Let L and R be the functions in Lévy's approach corresponding to g_1 and g_2, respectively. Let $a_n = k_{n^*(1/n)}/n$. Then $1 \le a_n \le c_0 < \infty$. The L-function corresponding to the g_1-function $-M_1(a_n s)\frac{1}{s^{1/p}} = -M_1\left(\frac{k_{n^*(1/n)}}{n}s\right)\frac{1}{s^{1/p}}$ is $\frac{1}{a_n}L\left(u\left(\frac{1}{a_n}\right)^{1/p}\right)$. The connection between M_2 and R is similar.

We shall write $\int \cdot\, dJ(u)$ instead of $\int_{-\infty}^0 \cdot\, dL(u) + \int_0^\infty \cdot\, dR(u)$.

So the characteristic function of $Y_n(1)$ is (compare with (2.12))

$$\phi_n^1(x) = \exp\left\{\int \left(e^{ixu} - 1 - \frac{ixu}{1+u^2}\right) d\left[\frac{1}{a_n} J\left(u\left(\frac{1}{a_n}\right)^{1/p}\right)\right]\right\} \times$$

$$\times \exp\left\{ix\theta(a_n)\right\}$$

where

$$a_n = k_{n^*(1/n)}/n, \quad \theta(a_n) = \theta_1(a_n) - \theta_2(a_n),$$

$$\theta_j(a_n) = \int_0^1 \frac{-M_j(sa_n)s^{-1/p}}{1 + \left(M_j(sa_n)s^{-1/p}\right)^2} ds - \int_1^\infty \frac{\left(-M_j(sa_n)s^{-1/p}\right)^3}{1 + \left(M_j(sa_n)s^{-1/p}\right)^2} ds =$$

$$= \frac{1}{a_n}\left[\int_0^{a_n} \frac{-M_j(v)(v/a_n)^{-1/p}}{1 + \left(M_j(v)(v/a_n)^{-1/p}\right)^2} dv - \int_{a_n}^\infty \frac{\left(-M_j(v)(v/a_n)^{-1/p}\right)^3}{1 + \left(M_j(v)(v/a_n)^{-1/p}\right)^2} dv\right],$$

$j = 1, 2$.

Finally, we obtain another version of Lévy's description. Let $H : \mathbb{R} \to \mathbb{R}$ be a truncation function (i.e. H is bounded, $H(u) = u$ in a neighbourhood of the origin and the support of H is compact). Then

$$\phi_n^1(x) = \exp\left\{\int \left(e^{ixv} - 1 - ixH(v)\right) d\left[\frac{1}{a_n} J\left(v\left(\frac{1}{a_n}\right)^{1/p}\right)\right]\right\} \times \tag{3.8}$$

$$\times \exp\left\{ix\left(\theta(a_n) + \kappa(a_n)\right)\right\}.$$

Here

$$\kappa(a_n) = \int \left(H(v) - \frac{v}{1+v^2}\right) d\left[\frac{1}{a_n} J\left(v\left(\frac{1}{a_n}\right)^{1/p}\right)\right].$$

We see that the sequences $\theta(a_n)$ and $\kappa(a_n)$, $n = 1, 2, \ldots$, are bounded.

4 Approximation and tightness in space $\mathbb{D}[0,1]$

In this section some general facts on merging probability measures are listed, moreover, tightness of our sequences of probabilities are proved.

Let $\mathbb{H}$ be a complete separable metric space. We shall consider probability measures on the Borel sets of $\mathbb{H}$. Let ϱ denote the Lévy-Prokhorov metric of probability measures (see [9]).

Definition 4.1. We say that two sequences $\{\mu_n\}$ and $\{\nu_n\}$ of probability measures on $\mathbb{H}$ are merging if $\lim_{n\to\infty} \varrho(\mu_n, \nu_n) = 0$.

Remark 4.1. (Theorem 1 of Davydov and Rotar [8].) $\lim_{n\to\infty} \varrho(\mu_n, \nu_n) = 0$ if and only if

$$\int f d\mu_n - \int f d\nu_n \to 0 \tag{4.1}$$

for each bounded and uniformly continuous functions $f : \mathbb{H} \to \mathbb{R}$. □

If at least one of the two sequences of measures are relatively compact, then the situation is quite simple.

Remark 4.2. (See [8], p. 87.) Let $\{\mu_n\}$ be relatively compact and assume that $\lim_{n\to\infty} \varrho(\mu_n, \nu_n) = 0$. Then $\{\nu_n\}$ is also relatively compact.

To see it choose a subsequence $\{\nu_{n'}\}$. Then we can choose a further subsequence $\{n''\}$ of $\{n'\}$ such that $\{\mu_{n''}\}$ is convergent. Then $\{\nu_{n''}\}$ converges to the same limit. That is $\{\nu_n\}$ is relatively compact. □

Remark 4.3. (See [8], p. 87.) Let $\{\mu_n\}$ be relatively compact. Then $\lim_{n\to\infty} \varrho(\mu_n, \nu_n) = 0$ if and only if (4.1) is satisfied for each bounded and continuous functions $f : \mathbb{H} \to \mathbb{R}$.

To see it first observe that the proposition is true if $\mu_n \equiv \mu$ (see the usual theory in [2]). Now both directions can be obtained using indirect proofs, and convergent subsequences. □

Lemma 4.1. *Assume that* $\{\mu_n\}$ *and* $\{\nu_n\}$ *are tight sequences of probability measures on* $\mathbb{D}[0,1]$*, moreover*

the distances of the finite dimensional distributions of μ_n *and* ν_n *converge to* 0. (4.2)

Then $\lim_{n\to\infty} \varrho(\mu_n, \nu_n) = 0$.

Proof. Assume that $\lim_{n\to\infty} \varrho(\mu_n, \nu_n) = 0$ is not satisfied. Then there exist an $\varepsilon > 0$ and a subsequence $\{m_n\}$ such that $\varrho(\mu_{m_n}, \nu_{m_n}) \geq \varepsilon$ for every n. As $\{\mu_n\}$ and $\{\nu_n\}$ are relative compact, therefore we can choose a further subsequence $\{m'_n\}$ of $\{m_n\}$ such that $\mu_{m'_n} \xrightarrow{d} \mu$ and $\nu_{m'_n} \xrightarrow{d} \nu$, say. Then $\varrho(\mu, \nu) \geq \varepsilon$.

Now consider the finite dimensional distributions. First we list some known facts. Let $t_1, \ldots, t_k$ be points in $[0,1]$. The projection

$$\pi_{t_1,\ldots,t_k} : \mathbb{D}[0,1] \to \mathbb{R}^k$$

is defined by $\pi_{t_1,\ldots,t_k}(x) = (x(t_1), \ldots, x(t_k))$, $x \in \mathbb{D}[0,1]$.

Let T be a subset of $[0,1]$. Let $\mathcal{F}_T$ denote the class of sets $\pi^{-1}_{t_1,\ldots,t_k} H$, where k is arbitrary, the t_i are arbitrary points of T, and H is arbitrary k-dimensional Borel set. By Theorem 14.5 of [2], if T contains 1 and is dense in$[0,1]$, then $\mathcal{F}_T$ generates $\mathcal{D}$ (the class of the Borel sets of $\mathbb{D}[0,1]$).

For any probability measure P on $\mathbb{D}[0,1]$, let T_P consists of those t in $[0,1]$ for which the projection π_t is continuous except at points forming a set of P-measure

0. It is known (see [2], Section 15) that T_P contains 0 and 1 and its complement in $[0,1]$ is at most countable.

If P_n and P are probability measures on $\mathbb{D}[0,1]$ and $P_n \xrightarrow{d} P$, then for the finite dimensional distributions we have that $P_n\pi^{-1}_{t_1,\dots,t_k} \xrightarrow{d} P\pi^{-1}_{t_1,\dots,t_k}$ holds if all the t_i lie in T_P (see [2], Section 15).

By Theorem 15.1 of [2], if the family $\{P_n\}$ is tight and $P_n\pi^{-1}_{t_1,\dots,t_k} \xrightarrow{d} P\pi^{-1}_{t_1,\dots,t_k}$ holds whenever $t_1,\dots,t_k$ all lie in T_P, then $P_n \xrightarrow{d} P$.

Now we can finish the proof in the following way. $\mu_{m'_n} \xrightarrow{d} \mu$ implies that $\mu_{m'_n}\pi^{-1}_{t_1,\dots,t_k} \xrightarrow{d} \mu\pi^{-1}_{t_1,\dots,t_k}$ whenever $t_1,\dots,t_k$ all lie in T_μ. This fact and

$$\varrho(\mu_n\pi^{-1}_{t_1,\dots,t_k}, \nu_n\pi^{-1}_{t_1,\dots,t_k}) \to 0$$

implies that

$$\nu_{m'_n}\pi^{-1}_{t_1,\dots,t_k} \xrightarrow{d} \mu\pi^{-1}_{t_1,\dots,t_k} \tag{4.3}$$

whenever $t_1,\dots,t_k$ all lie in T_μ. However, $\nu_{m'_n} \xrightarrow{d} \nu$ implies

$$\nu_{m'_n}\pi^{-1}_{t_1,\dots,t_k} \xrightarrow{d} \nu\pi^{-1}_{t_1,\dots,t_k} \tag{4.4}$$

whenever $t_1,\dots,t_k$ all lie in T_ν. Now, by (4.3) and (4.4),

$$\mu\pi^{-1}_{t_1,\dots,t_k} = \nu\pi^{-1}_{t_1,\dots,t_k}$$

if $t_1,\dots,t_k \in T_\nu \cap T_\mu = T$. Now, $\mathcal{F}_T$ is an algebra of sets and, by the above equality, μ and ν coincide on $\mathcal{F}_T$. As T is dense in $[0,1]$ and it contains 1, therefore $\mathcal{F}_T$ generates $\mathcal{D}$. So μ and ν coincide on $\mathcal{D}$. It contradicts to $\varrho(\mu,\nu) \ge \varepsilon$. □

Remark 4.4. Let $\{\mu_n\}$ and $\{\nu_n\}$ be sequences of probability measures on $\mathbb{C}[0,1]$. Assume that $\{\mu_n\}$ is tight. Then $\lim_{n\to\infty}\varrho(\mu_n,\nu_n) = 0$ if and only if $\{\nu_n\}$ is tight and (4.2) is satisfied.

To see it first we remark that $\pi_{t_1,\dots,t_k}$ is continuous for every $t_1,\dots,t_k \in [0,1]$ (see [2], Section 3). So, for any probability measures $P, P_1, P_2, \dots$ the convergence $P_n \xrightarrow{d} P$ implies for the finite dimensional distributions that $P_n\pi^{-1}_{t_1,\dots,t_k} \xrightarrow{d} P\pi^{-1}_{t_1,\dots,t_k}$ for every $t_1,\dots,t_k \in [0,1]$. Therefore the proof of Lemma 4.1 implies one direction.

To obtain the other direction let $\lim_{n\to\infty}\varrho(\mu_n,\nu_n) = 0$. Then Remark 4.2 implies tightness of $\{\nu_n\}$. Now assume that (4.2) is not satisfied. Then there exists $t_1,\dots,t_k \in [0,1]$ and a subsequence n' such that

$$\varrho(\mu_{n'}\pi^{-1}_{t_1,\dots,t_k}, \nu_{n'}\pi^{-1}_{t_1,\dots,t_k}) \ge \varepsilon > 0.$$

However, there exist a further subsequence $\{n''\}$ of $\{n'\}$ such that $\mu_{n''} \to \mu$ and $\nu_{n''} \to \mu$. Their finite dimensional distributions also converge. It is a contradiction. □

Now we turn to the proof of tightness of $Y_n(t)$. We shall use the notation of [14]. A process $Y(t)$, $t \geq 0$, is called a process with stationary independent increments, if it is adapted to the filtration $\mathcal{F}_t$, it is càdlàg, $Y(0) = 0$, $Y(t) - Y(s)$ is independent from $\mathcal{F}_s$ ($0 \leq s \leq t$), and the distribution of $Y(t) - Y(s)$ depends only on the difference $t - s$. (We shall consider $\mathcal{F}_t$ as the σ-field generated by $\{Y(s) : s \leq t\}$.)

Let $Y_n(t)$, $t \geq 0$, $n = 0, 1, 2, \ldots$, be a sequence of processes with stationary independent increments with characteristic functions

$$\mathbb{E}\left(e^{ixY_n(t)}\right) = \exp\left\{t\left[ixb_n - \frac{\sigma_n^2 x^2}{2} + \int \left(e^{ixv} - 1 - ixH(v)\right) dK_n(v)\right]\right\}.$$

Here H is a truncation function that we choose to be continuous. Moreover, for each n, $b_n \in \mathbb{R}$, $\sigma_n \geq 0$, K_n is a positive measure on $\mathbb{R}$ that integrates $\min\{x^2, 1\}$ and satisfies $K_n(0) = 0$.

By Corollary 3.6 in Chapter VII of [14], we have the following criterion of convergence. Let $Y_n(t)$, $t \geq 0$, $n = 0, 1, 2, \ldots$, be a sequence of processes with stationary independent increments with characteristics $b_n t$, $\sigma_n^2 t$, $dtK_n(dx)$. Then there is equivalence between the following three statements.

(a) $Y_n \xrightarrow{d} Y_0$;

(b) $Y_n(1) \xrightarrow{d} Y_0(1)$;

(c) conditions $b_n \to b_0$, $\tilde{c}_n = \sigma_n^2 + \int H^2(x) dK_n(x) \to \sigma_0^2 + \int H^2(x) dK_0(x) = \tilde{c}_0$, $\int g(x) dK_n(x) \to \int g(x) dK_0(x)$ for each continuous bounded function $g : \mathbb{R} \to \mathbb{R}$ which is 0 around 0 and has a limit at infinity.

Lemma 4.2. *Let the sequence of processes with stationary independent increments Y_n, $n = 1, 2, \ldots$, be defined by* (2.12). *Then the sequence of measures $\mathcal{L}_{Y_n}$, $n = 1, 2, \ldots$ is tight in $\mathbb{D}[0,1]$.*

Proof. We shall apply the above facts from the theory of processes with stationary independent increments. Let Y_n be defined by (2.12). Then, by (3.8), $b_n = \theta(a_n) + \kappa(a_n)$, $\sigma_n = 0$, $K_n(\cdot) = \frac{1}{a_n} J\left(\left(\frac{1}{a_n}\right)^{1/p} \cdot\right)$.

We have to prove that $\mathcal{L}_{Y_n}$, $n = 1, 2, \ldots$ is relatively compact. Consider an arbitrary subsequence. We have $1 \leq a_n \leq c_0 < \infty$. Therefore our subsequence contains a further subsequence n' such that $a_{n'} \to a_0$ where $1 \leq a_0 \leq c_0 < \infty$. It is easy to see that the sequences $b_{n'}$ is convergent. Moreover, using the properties of H and g, we obtain

$$\tilde{c}_{n'} = \int H^2(x)\, d\left[\frac{1}{a'_n} J\left(\left(\frac{1}{a_{n'}}\right)^{1/p} x\right)\right] \to \int H^2(x)\, d\left[\frac{1}{a_0} J\left(\left(\frac{1}{a_0}\right)^{1/p} x\right)\right] = \tilde{c}_0,$$

$$\int g(x)\, d\left[\frac{1}{a'_n} J\left(\left(\frac{1}{a_{n'}}\right)^{1/p} x\right)\right] \to \int g(x)\, d\left[\frac{1}{a_0} J\left(\left(\frac{1}{a_0}\right)^{1/p} x\right)\right].$$

Therefore the subsequence $\mathcal{L}_{Y_{n'}}$ is convergent. □

Lemma 4.3. *Let the sequence of processes $Z_n(t)$ be defined by* (2.11). *Then the sequence of measures $\mathcal{L}_{Z_n}$, $n = 1, 2, \dots,$ is tight in $\mathbb{D}[0,1]$.*

Proof. Introduce the auxiliary sequence $S'_n = \frac{1}{B_n}\sum_{i=1}^n X_i - A'_n$ where

$$A'_{n1} = (1/B_n)\mathbb{E}XI_{\{|X|\le B_n\}}, \quad A'_{nk} = kA'_{n1}, \ k = 1, 2, \dots, n, \quad A'_n = A'_{nn},$$

for $n = 1, 2, \dots$.

Consider the random processes

$$Z'_n(t) = \frac{1}{B_n}\sum_{i=1}^{[nt]} X_i - [nt]A'_{n1} = Z_n(t) + [nt](A_{n1} - A'_{n1}), \quad t \in [0,1].$$

We shall prove the tightness of the distribution family $\{\mathcal{L}_{Z'_n} : n \in \mathbb{N}\}$.

We denote by L^0 the space of all random variables endowed with the topology of convergence in probability. By Theorem 1 of [7], the sequence $\{S_n\}_{n=1}^\infty$ is bounded in L^0.

By Megyesi [18] (see the proof of Theorem 3)

$$\max\{|Q_+(s)|, Q(1-s)\} \le Cs^{-1/p}l(s), \tag{4.5}$$

for $s > 0$ small enough, where $C < \infty$.

We need the following facts. Let $a > 0$.

$$\text{If} \quad a > Q(1-s), \quad \text{then} \quad 1 - F(a) \le s. \tag{4.6}$$

$$\text{If} \quad -a < Q_+(s), \quad \text{then} \quad F_-(-a) \le s, \tag{4.7}$$

where F_- is the left-continuous version of F.

Let $\overline{F}(x) = \mathbb{P}(|X| > x) = 1 - F(x) + F_-(-x)$, for $x > 0$. Then, using basic properties of slowly varying functions (see Ch. 1 of [3]), (4.5) and (4.6)–(4.7) give

$$\sup_{n\in\mathbb{N}} n\mathbb{P}\{|X/B_n| > 1\} = \sup_{n\in\mathbb{N}} n\overline{F}(B_n) = C_1 < \infty. \tag{4.8}$$

Now let $U_n = \frac{1}{B_n}\sum_{i=1}^n |X_i| I_{\{|X_i|>B_n\}}$, $n = 1, 2, \dots$. Integrating by parts, we obtain

$$\mathbb{P}(|U_n| > K) \le \frac{n\overline{F}(B_n)}{K} + \int_{1/K}^1 n\overline{F}(KB_n x)dx. \tag{4.9}$$

Then, using known facts on regularly varying functions (see Potter's theorem in Ch. 1 of [3]), (4.5), (4.6)–(4.7), we can see that $\int_{1/K}^1 n\overline{F}(KB_n x)dx < \varepsilon$ if $K > K_\varepsilon$. This fact, (4.9) and (4.8) imply that U_n is bounded in probability. Therefore $U'_n = \frac{1}{B_n}\sum_{i=1}^n X_i I_{\{|X_i|>B_n\}}$, $n = 1, 2, \dots,$ is also bounded in probability.

Let $V_n = \frac{1}{B_n}\sum_{i=1}^n X_i I_{\{|X_i|\le B_n\}}$, $n = 1, 2, \dots$. Then $V_n - A_n = S_n - U'_n$, $n = 1, 2, \dots,$ is also bounded in probability. From this fact and using the same method (symmetrization, Lévy's inequality, and stopping times) as in the proof of Theorem 10.1.1 in [4], we can prove that

$$\sup_{n\in\mathbb{N}} n\mathbb{D}^2\left((X/B_n)I_{\{|X|\le B_n\}}\right) = C_2 < \infty. \tag{4.10}$$

Using this fact and Tchebychev's inequality, we can see that the sequence $\{S'_n\}$ is bounded in probability. As $\{S_n\}$ and $\{S'_n\}$ are bounded in probability, the sequence $\{A_n - A'_n\}$ is bounded.

Now, turn to the tightness of the distribution family $\{\mathcal{L}_{Z'_n} : n \in \mathbb{N}\}$. To this end we apply Theorem 15.3 of Billingsley [2]. Let $0 < \varepsilon < 1$ and $0 < u < 1$. We have

$$\mathbb{P}\{\sup_{0\le t\le u} |Z'_n(t)| > \varepsilon\} \le \tag{4.11}$$

$$\le \mathbb{P}\Big\{\sup_{0\le t\le u}\Big|\frac{1}{B_n}\sum_{i=1}^{[nt]}\left(X_iI_{\{|X_i|\le B_n\}} - \mathbb{E}X_iI_{\{|X_i|\le B_n\}}\right)\Big| > \frac{\varepsilon}{2}\Big\} + [nu]\mathbb{P}\{|X| > B_n\}.$$

Now, applying Kolmogorov's inequality, (4.11), (4.8) and (4.10), we obtain

$$\lim_{u\to 0}\limsup_{n\to\infty}\left(\mathbb{P}\{\sup_{0\le t\le u}|Z'_n(t)| > \varepsilon\} + \mathbb{P}\{\sup_{1-u\le t\le 1}|Z'_n(1) - Z'_n(t)| > \varepsilon\}\right) = 0 \tag{4.12}$$

Using the inequality in Sect. 6 Ch. 9 of [11], (4.11), Kolmogorov's inequality, (4.8) and (4.10), we obtain

$$\lim_{m\to\infty}\limsup_{n\to\infty}\sum_{k=1}^{m}\mathbb{P}\left\{\sup_{\frac{k-1}{m}\le t_1\le t_2\le t_3<\frac{k}{m}}\min\{|Z'_n(t_2) - Z'_n(t_1)|, |Z'_n(t_3) - Z'_n(t_2)|\} > \varepsilon\right\} \le$$

$$\le \lim_{m\to\infty}\limsup_{n\to\infty}\sum_{k=1}^{m}\left(\mathbb{P}\left\{\sup_{\frac{k-1}{m}\le t<\frac{k}{m}}\left|Z'_n(\frac{k}{m}) - Z'_n(t)\right| > \frac{\varepsilon}{4}\right\}\right)^2 \le$$

$$\le \lim_{m\to\infty}\limsup_{n\to\infty}\max_{1\le k\le m}\left\{\frac{C([\frac{k}{m}n] - [\frac{k-1}{m}n])}{\varepsilon^2}\Big\{\mathbb{E}\Big\{\frac{X}{B_n}I_{\{|X|\le B_n\}} - A'_{n1}\Big\}^2 + \mathbb{P}\{|X| > B_n\}\Big\}\right\}\times$$

$$\times\sum_{k=1}^{m}\frac{C([\frac{k}{m}n] - [\frac{k-1}{m}n])}{\varepsilon^2}\Big\{\mathbb{E}\Big\{\frac{X}{B_n}I_{\{|X|\le B_n\}} - A'_{n1}\Big\}^2 + \mathbb{P}\{|X| > B_n\}\Big\} = 0. \tag{4.13}$$

Let $Z'_n(t) = Z'_{1n}(t) + Z'_{2n}(t)$, $t \in [0, 1]$, where

$$Z'_{1n}(t) = \frac{1}{B_n}\sum_{i=1}^{[nt]}X_iI_{\{|X_i|\le B_n\}} - [nt]A'_{n1} \quad\text{and}\quad Z'_{2n}(t) = \frac{1}{B_n}\sum_{i=1}^{[nt]}X_iI_{\{|X_i|>B_n\}},$$

for $t \in [0, 1]$. By Kolmogorov's inequality, and (4.10), we have

$$\sup_{n\in\mathbb{N}}\mathbb{P}\left\{\|Z'_{1n}\|_\infty > K\right\} \le \sup_{n\in\mathbb{N}}\frac{n\mathbb{E}\left(\frac{1}{B_n}XI_{\{|X|\le B_n\}} - A'_{n1}\right)^2}{K^2} \le \frac{C_2}{K^2} \to 0, \tag{4.14}$$

as $K \to \infty$. We have already proved, that the family of random variables $\{U_n\}$ is bounded in L^0. Therefore

$$\sup_{n\in\mathbb{N}} \mathbb{P}\{\|Z'_{2n}\|_\infty > K\} \leq \sup_{n\in\mathbb{N}} \mathbb{P}\{|U_n| > K\} \to 0, \text{ as } K \to \infty. \tag{4.15}$$

From (4.14) and (4.15) we obtain

$$\sup_{n\in\mathbb{N}} \mathbb{P}\{\|Z'_n\|_\infty > K\} \to 0, \text{ as } K \to \infty. \tag{4.16}$$

Now, by Theorem 15.3 in Billingsley [2], relations (4.12), (4.13), and (4.16) imply the tightness of the distribution family $\{\mathcal{L}_{Z'_n} : n \in \mathbb{N}\}$. Therefore for any $\varepsilon > 0$ there exists a compact set $K'_\varepsilon \subset \mathbb{D}[0,1]$ such that $\mathcal{L}_{Z'_n}(K'_\varepsilon) > 1 - \varepsilon$ for all $n \in \mathbb{N}$.

Now, we prove the tightness of $\{\mathcal{L}_{Z_n} : n \in \mathbb{N}\}$. Let $c_n = A'_n - A_n$. We have already proved that $\{c_n\}$ is bounded. Consider the sequence of functions

$$f_n(t) = [nt](A'_{n1} - A_{n1}) = c_n[nt]/n, \quad t \in [0,1].$$

Let $K_1 = \{f_n : n = 1, 2, \dots\}$. Let Λ denote the class of strictly increasing, continuous mappings of $[0,1]$ onto itself (i.e. Λ is the set of functions used to define the topology of $\mathbb{D}[0,1]$, see [2]). As $t - (1/n) \leq [nt]/n \leq t$, we have

$$|f_n(\lambda(t)) - c_n\lambda(t)| \leq |c_n|/n.$$

From this inequality and from the boundedness of $\{c_n\}$ we obtain that from any infinite sequence from K_1 we can select a subsequence $\{f_{k_n}\}$ such that

$$\lim_{n\to\infty} f_{k_n}(\lambda_n(t)) = ct \tag{4.17}$$

uniformly in t for an arbitrary sequence $\{\lambda_n(.)\}$ from Λ with $\lim_{n\to\infty} \lambda_n(t) = t$ uniformly in t.

Now we can prove that $K_\varepsilon = K'_\varepsilon + K_1$ is a compact set in $\mathbb{D}[0,1]$. Consider a sequence $g_n + h_n$ from $K'_\varepsilon + K_1$. As K'_ε is compact, we can choose a convergent subsequence g_{k_n}. That is (see [2], section 14) there exists functions λ_{k_n} in Λ such that $\lim_{n\to\infty} g_{k_n}(\lambda_{k_n}(t)) = g(t)$ and $\lim_{n\to\infty} \lambda_{k_n}(t) = t$ uniformly in t. From here, by (4.17), there exists a further subsequence such that $\lim_{n\to\infty}[g_{k'_n}(\lambda_{k'_n}(t)) + h_{k'_n}(\lambda_{k'_n}(t))] = g(t) + ct$ uniformly in t. It means that $g_{k'_n}(t) + h_{k'_n}(t) \to g(t) + ct$ in Skorokhod's topology.

Finally, for all $n \in \mathbb{N}$ we have

$$\mathcal{L}_{Z_n}(\mathbb{D}[0,1] \setminus K_\varepsilon) \leq \mathcal{L}_{Z'_n}(\mathbb{D}[0,1] \setminus K'_\varepsilon) < \varepsilon. \qquad \square$$

ACKNOWLEDGMENTS. I am grateful to Prof. I. Berkes for proposing the topic. I thank to Prof. A. Rackauskas for reference [8].

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