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The Mixed Partition Dimension: A New Resolvability Parameter in Graph Theory

SITI NORZIAHIDAYU AMZEE ZAMRI,[1,2], SIKANDER ALI[3], MUHAMMAD AZEEM[3,4], HUSAM A. NEAMAH[5], BANDAR ALMOHSEN[6],

[1]Science and Medicine Foundation Centre, Universiti Sultan Zainal Abidin, Gong Badak Campus, 21300 Kuala Nerus, Terengganu
[2]East Coast Environmental Research Institute (ESERI), Universiti Sultan Zainal Abidin, Gong Badak Campus, 21300, Kuala Terengganu, Malaysia sitinamzee@unisza.edu.my
[3]Department of Mathematics, Riphah International University Lahore, Pakistan;(e-mail: sikandralicui@gmail.com azeemali7009@gmail.com).
[4]Department of Solids and Structures, School of Engineering, The University of Manchester, Oxford Road, Manchester, M13 9PL, UK, muhammad.azeem-3@postgrad.manchester.ac.uk
[5]Department of Electrical Engineering and Mechatronics, Faculty of Engineering, University of Debrecen, Òtemetö Utca 2-4, 4028 Debrecen, Hungary; husam@eng.unideb.hu
[6]Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia.balmohsen@ksu.edu.sa

Corresponding author:Husam A. Neamah(e-mail:husam@eng.unideb.hu)

ABSTRACT In this article, we introduce a novel graph-theoretical parameter called the mixed partition dimension and apply it to the path graph and the hexagonal network. This parameter builds on the concept of resolvability in graphs, integrating vertex-based partition dimensions with edge-oriented strategies to characterize the complexity of graph structures. It is the extension of the mixed metric dimension and partition dimension. Suppose Let $R = \{W_1, W_2, \ldots, W_k\}$ be a partition of the vertex set $V(G)$ of a graph $G = (V, E)$, where $W_1 \cup W_2 \cup \cdots \cup W_k = V(G)$ and $W_i \cap W_j = \emptyset$ for $i \neq j$. Each subset W_i is non-empty, mutually disjoint, and collectively covers all vertices. The partition set $R_{pm})$ is called mixed resolving partition set if it satisfies: For any two distinct vertices $x, y \in V$, there exists $W_i \in R$ such that:$d(u, W_i) \neq d(v, W_i)$, for any two distinct edges $e_1, e_2 \in E$, there exists $W_i \in R$ such that:$d(e_1, W_i) \neq d(e_2, W_i)$ and for any vertex $u \in V$ and edge $e \in E$, there exists $W_i \in R$ such that: $d(u, W_i) \neq d(e, W_i)$. The mixed partition dimension of G is the minimum number of subsets in a mixed resolving partition set $R_{pm})$. This parameter provides a unified measure of a graph's complexity by accounting for both vertex and edge distinguishability, offering new insights into the structure of complex networks.

INDEX TERMS partition resolving set; edge partition dimension; partition dimension; mixed partition dimension; hexagonal network

I. INTRODUCTION

THE Chemical graph theory helps scientists understand complicated compounds' structure, making it possible to build new materials logically with the necessary features in mind. By bridging the fields of chemistry and mathematics, this interdisciplinary approach promotes creativity in creating innovative materials with many uses. Recent developments in the mathematical modeling of chemical processes and the use of mathematical ideas in chemistry have led to a substantial evolution in mathematical chemistry. These developments are assisting modern drug development tactics by facilitating interactions with drug models [1]. Chemical graph theory helps to make sense of and simplify massive, complex chemical structures, leading to insightful discoveries that would be hard to obtain in any other way.

Determining a node's exact location within a network is known as network localization, and it is essential for many applications. When a computer sends a printing command within a facility, accurate localization aids in locating the closest printer, identifying unauthorized connections, finding a broken device, detecting a malfunctioning node, and tracking the location of a moving robot. Nevertheless, network localization is frequently difficult, expensive, and time-consuming to do [2].

Slater referred to the idea of the resolving set as having a graph locating number [3]. He illustrated the importance of this concept by employing Loran stations and sonar. Loran stations were considered to be obsolete as soon as GPS became widely accessible on a commercial level. Later, Harary and Melter conducted a study on this subject as well, albeit they used the term metric dimension [4] rather than location number to describe it. According to Chartrand et

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al., [5], this concept was referred to as the metric basis, and the resolving set was the smallest set of the metric basis. Blumenthal had long before characterized the rotating set and the metric dimension [6].

In geometry or topology, the term mixed metric dimension usually refers to the concept of space equipped with different measurements or measurements simultaneously. This concept is used in many areas of mathematics (see metric geometry, geometric group theory, and geometric topology). In geometric group theory, we can also talk about mixed metric dimensions of space, that is, the behavior of space groups with different metrics. To do this, it is necessary to understand how measurements work on the geometric shapes of space and how they interact with the behavior of the group. Thus, in geometric topology, places with distinct considerations in terms of local distance or size will be associated with places with mixed metric dimensions. For example, a space may have an Euclidean metric in some areas and a hyperbolic metric in others [2].

Partition dimension is an extension of the concept of metric dimension in graph theory, applied to take a look at molecular systems in chemical graph concept. While the metric dimension specializes in figuring out a fixed number of vertices that uniquely decide the location of each different vertex based totally on distances, the partition dimension takes this idea in addition to thinking about the walls of the vertex set. The mixed partition dimension integrates the perspectives of metric and edge metric dimensions by analyzing both vertices and edges through a partition-based. This unified framework captures the interplay among vertices and edges in hexagonal networks, which is crucial for applications like molecular graphs and conversation networks. It is the extension of metric, edge metric dimension, partition, and edge partition. In which we take a partition set of the vertices of the graph and check the distances from both vertices as well as edges. It uses the concept of mixed metric dimension [7], [8]

Metric dimensions have many applications in daily life, inspire scientists, and have been studied extensively. Specifically, identify similar patterns across multiple drugs using a longitudinal index [9]. Some applications of the dimensionality scale include combinatorial optimization [10], [11], robot navigation [12], chemical chemistry [13], computer networks [14], canonical labeled8 location problem, sonar . and coast guard Loran [3], imaging facilities, weight problem [15], coding and decoding of the genius game discussed in [16]. Use of resolving sets in development of a city [17]. To learn more about this dimension's chemical and physical components, see [18]–[21]. I learned a lot. For example, [22] studied the measurement of cell length.

The literature on resolvability parameters is, the metric dimension of the cellulose network bonds determined in [23], and the metric dimension of the resulting images of the base material [24] determining the crystals is Metric. Double resolving sets are given in [25] and discussion of convex polyhedral graphs is shown in [26], a novel resolving parameter is discussed in [27], double resolving set of anti-malaria drug is mentioned in [28] and edge metric dimensions for the centroid subdivision of the Cayley diagram are given in [29], as many difficult problems due to their diversity It is solved using the concept of metric dimension. We refer to [30], [31] for the solution of many chemical formulas.

The literature of graphs in the field of improvements in engineering, laptop technological know-how, and graph idea, addressing real-world challenges [32]–[35]. Key contributions include cost-efficient carrier feature chain orchestration in NFV networks, dynamic provisioning for business applications, and fashions for temporal know-how graphs [36]–[39]. Applications increase to 3-D reconstruction of lunar craters, fatigue popularity systems for miners, and stable transmission in wi-fi networks. Innovations in robotics, sensible class systems, and UAV-enabled aspect computing demonstrate large development in healthcare, infrastructure, and automation [40]–[43]. These research together emphasize the significance of pass-disciplinary approaches in fixing complex problems throughout various domains [44]–[48]. For more details about machine learning see [49]–[51].

Definition I.1. *Let* $G = (V, E)$ *be a simple, connected graph with vertex set* V *and edge set* E*. A set* $R = \{w_1, w_2, \ldots, w_k\}$ *is a collection of vertices of* $V(G)$*. For any vertex* $x \in V$*, edge* $e = (x, y) \in E$*, and a vertex in* $w_i \in R$*, the distance between* $d(x, w)$ *is the shortest path between vertices* x *and* w *and* $d(e, w)$ *is the shortest path between vertex* e *and any vertex in* w*. A set* R *is called a mixed resolving set if for every pair of distinct vertices* $x, y \in V$*, distinct edges* $e_1, e_2 \in E$*, and for vertices and edges, there exists at least one vertex* $w_i \in R$ *such that:*$d(x, w_i) \neq d(y, w_i)$ *or* $d(e_1, w_i) \neq d(e_2, w_i)$ *or* $d(x, w_i) \neq d(e, w_i)$*. The mixed metric dimension of the graph* G *is the minimum number of vertices in the mixed resolving set of* G *[52].*

We move on to the different combinatorial properties of mixed-metric generators. The first thing to note is that every mixed metric generator is both a metric generator and an edge metric generator. This implies the immediate relationship that follows. For any graph G,

$$dim_m(G) \geq max\{dim(G), dim_e(G)\}$$

[52].

Definition I.2. ***Mixed Partition Dimension***

Let $G = (V, E)$ *be a simple, connected graph with vertex set* V *and edge set* E*. A partition* $R = \{W_1, W_2, \ldots, W_k\}$ *of* V *is a collection of non-empty, disjoint subsets of* $V(G)$ *such that* $W_1 \cup W_2 \cup \cdots \cup W_k = V(G)$*. For any vertex* $x \in V$*, edge* $e = (x, y) \in E$*, and a subset* $W_i \in R$*, the distance between:*

- *A vertex* x *and a subset* W_i *is:*

$$d(x, W_i) = \min\{d(x, w) : w \in W_i\},$$

where $d(x, w)$ is the shortest path between vertices x and w.

- *An edge $e = (x, y)$ and a subset W_i is:*

  $$d(e, W_i) = \min\{d(x, W_i), d(y, W_i)\},$$

  where $d(x, W_i)$ is the shortest path between vertex x and any vertex in W_i.

A partition set R is called a ***mixed resolving partition set*** *if for every pair of distinct vertices $u, v \in V$, distinct edges $e_1, e_2 \in E$, and for distinct vertices and edges there exists at least one subset $W_i \in R$ such that: $d(x, W_i) \neq d(y, W_i)$ or $d(e_1, W_i) \neq d(e_2, W_i)$ or $d(x, W_i) \neq d(e, W_i)$.*

The ***mixed partition dimension*** *of the graph G, denoted by $pd_m(G)$, is the minimum number of subsets in a mixed resolving partition set of V:*

$$pd_m(G) = \min\{|R| : R \textit{ is a mixed resolving partition set of } G\}.$$

Example I.3. *Here we have an example that illustrates the concept of mixed partition dimension. $R_{pm} = \{R_{p1}, R_{p2}, R_{p3}, R_{p4}\}$ where $R_{p1} = \{v_1\}$, $R_{p2} = \{v_3\}, R_{p3} = \{v_6\}, R_{p4} = \{v_2, v_5, v_5\}$*

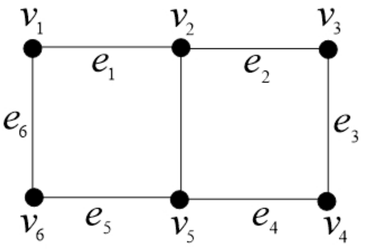

FIGURE 1. The graph $P_2 \square P_1 \cong L_2$.

$R_{pm} = \{R_{p1}, R_{p2}, R_{p3}, R_{p4}\}$

$Vertices$	v_1	v_2	v_3
$r(. \mid R_{pm})$	(0,2,1,1)	(1,1,2,0)	(2,0,3,1)
$Vertices$	v_4	v_5	v_6
$r(. \mid R_{pm})$	(3,1,2,0)	(2,2,1,0)	(1,3,0,1)
$Edes$	e_1	e_2	e_3
$r(. \mid R_{pm})$	(0,1,1,0)	(1,0,2,0)	(2,0,2,0)
$Edes$	e_4	e_5	e_6
$r(. \mid R_{pm})$	(2,1,1,0)	(1,2,0,0)	(0,2,0,1)
$Edes$	e_7	—	—
$r(. \mid R_{pm})$	(1,1,1,0)	—	—

TABLE 1. Representation of edges w.r.t R_{pm} of Figure 1

From Table 1 it is clear that the representation of all vertices and edges are unique so Figure 1 has 4 mixed partition dimension.

II. MAIN RESULTS

In this section, we discussed novel resolvability parameters, the mixed partition dimension of the path graph, and the hexagonal network.

FIGURE 2. path graph

Theorem II.1. *The mixed partition dimension of a graph G is 3 and it is denoted by $pd_m(P_n) = 3$.*

Proof. To prove our claim that the mixed partition dimension of the path graph is 3, we want to show that the representation of $R_{pm})$ with cardinality 3 is unique. We will now use the mixed partition resolving set definition to demonstrate this assertion. Let $R_{pm})\{R_{p1}, R_{p2}, R_{p3}\}$ where $R_{p1} = \{a_1\}$, $R_{p2} = \{a_n\}$, and $R_{p3} = V(P_n)\backslash\{a_1, a_n\}$. Given below are the unique distances of all vertices and edges of the path graph that show that the mixed partition dimension of P_n is 3 for $n \geq 2$. We make generalized formulas for distances of all vertices and edges to $R_{pm})$ according to Figure2.

$$d(a_i, R_{p1}) = d(a_i, a_1) = \{ \; i-1 \quad \text{for } 1 \leq i \leq n,$$

$$d(e_i, R_{p1}) = d(e_i, a_1 = \{ \; i-1 \quad \text{for } 1 \leq i \leq n-1,$$

$$d(a_i, R_{p2}) = d(a_i, a_n = \{ \; n-i \quad \text{for } 1 \leq i \leq n,$$

$$d(e_i, R_{p2}) = d(e_i, a_n = \{ \; n-i-1 \quad \text{for } 1 \leq i \leq n-1,$$

$$d(a_i, R_{p3}) = \begin{cases} 1 & \text{for } a_1, a_n \\ 0 & \text{otherwise} \end{cases}$$

$$d(e_i, R_{p3}) = \{ \; 0 \quad \text{for } 1 \leq i \leq n-1,$$

This formulation clearly shows that the path graph's mixed partition dimension is 3 because all distances are unique. Hence prove that the mixed partition dimension of path is 3 and it is represented by $pd_m(P_n) = 3$. □

Relationship between existing resolvability parameters and Mixed partition dimension:

Theorem II.2. *Let G be a connected graph. Then, the mixed partition dimension $pd_m(G)$ of G satisfies the relation:*

$$pd_m(G) = \dim_m(G) + 1,$$

where $\dim_m(G)$ is the mixed metric dimension of G.

Proof.

1) The mixed metric dimension $\dim_m(G)$ is the minimum size of a resolving set $S \subseteq V(G) \cup E(G)$ such that every element (vertex or edge) of G is uniquely determined by its distances to the elements of S.
2) The mixed partition dimension $pd_m(G)$ is the minimum number of partitions $\Pi = \{P_1, P_2, \ldots, P_k\}$ of $V(G)$ such that every vertex $v \in V(G)$ and edge $e \in E(G)$ is uniquely identified by its distances to all partitions in Π.
3) In any graph, the process of creating resolving partitions inherently adds one additional structural element over simply selecting a mixed resolving set. The inclusion of the partitioning structure accounts for this extra element.

4) Therefore, $pd_m(G)$, which requires both partitioning and resolving, is exactly one more than $\dim_m(G)$, which only resolves the elements.

Hence, the relationship holds:

$$pd_m(G) = \dim_m(G) + 1.$$

□

Corollary: This result can be extended to specific graph families, such as paths, cycles, and trees, where the mixed metric dimension and partition structures are more explicitly computable.

III. MIXED PARTITION DIMENSION OF HEXAGONAL NETWORK

This section defines and explains the hexagonal network's mixed partition dimension. The new mixed partition dimension provides a more thorough framework for assessing network complexity, which combines edge- and vertex-based resolving partition. We develop a technique to uniquely identify each node in the network by splitting its vertices and edges into different partition-resolving sets. The hexagonal network is a perfect model to demonstrate the potential of this new dimension because of its rich structural features and numerous applications in chemistry and nanotechnology.

A. CONSTRUCTION OF HEXAGONAL NETWORK

In Figure 3, the red color represents edges with a degree of end vertices 2 and 4.. The blue edges ended with vertices of degree two, and the black color is for all edges with degree endpoints 4. The green color is used for two-degree corners. The two-colored corners are the points of the evaluation process. $a_{1,1}, a_{1,2h+1}, a_{1,v+1}$ Since the 2 level is green and red, these points are the trap of the owner of the analysis. Let v and h denote the vertical and horizontal number of C_6 and $v, h \geq 2$, $v, h \in \mathbb{Z}^+$. The count of nodes of degree two are $2h + v + 3$, and the node of degree 4 are $5h + 4v - 3 - (2h + v)$. The total number of vertices of $HN_{h,v}$ is $|V(HN_{h,v})| = (2h + 1)(v + 1)$, and number of edges $|E(HN_{h,v})| = (2h + 1)(v + 1) + 1$. h and v are two parameters are used and two index ζ, ξ are used for drawing; where $1 \leq \zeta \leq h$ and ξ vary twice with v. Figure 3 shows our main results' log on vertex and edge clusters. The edge and vertex set of the network are

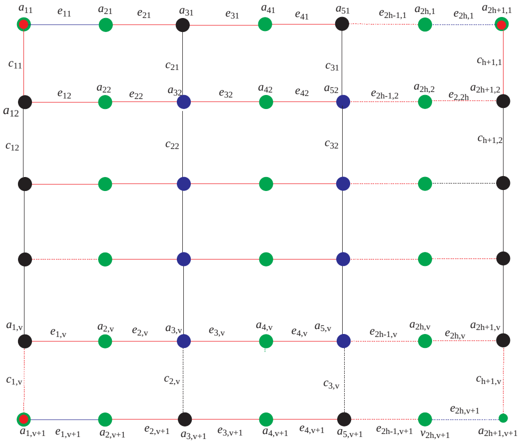

FIGURE 3. Generalize network derived from Hexagon

B. EDGE AND VERTEX SET

$$E(HN) = \{a_{i,j}a_{i,j+1}, a_{i,j\,i+1j}; 1 \leq i \leq v,\ 1 \leq j \leq 2h+1\}$$
$$V(HN) = \{a_{i,j}; 1 \leq i \leq v,\ 1 \leq j \leq 2h+1\}$$

Theorem III.1. *Let $HN_{h,v}$ be a hexagonal network for $h, v \geq 1$, then the mixed partition dimension is 4 and denoted by $pd_m(HN_{h,v}) = 4$.*

Proof. To prove our claim that the mixed partition dimension is 4, we want to show that the hexagonal network has partition and edge partition dimension 4 at the same partition resolving set R_{pm}). We will now use the mixed partition resolving set definition to demonstrate this assertion. Let $R_{pm})\{R_{p1}, R_{p2}, R_{p3}, R_{p4}\}$ where $R_{p1} = \{a_{1,1}\}$, $R_{p2} = \{a_{2h+1,1}\}$, $R_{p3} = \{a_{1,v+1}\}$ and $R_{p4} = V(HN)\backslash\{a_{1,1}, a_{2h+1,1}, a_{1,v+1}\}$. Given below are the unique representation of all vertices and edges of $HN_{h,v}$ for $h, v \geq 1$.

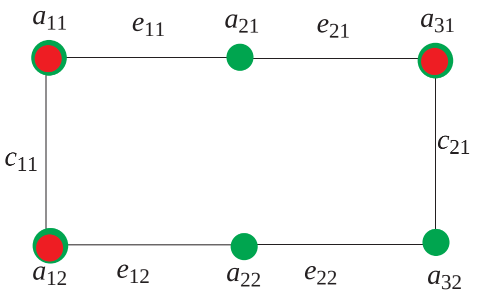

FIGURE 4. Hexagon

$R_{pm})\{R_{p1}, R_{p2}, R_{p3}, R_{p4}\}$

vertex	$a_{1,1}$	$a_{2,1}$	$a_{3,1}$
$r(.\mid R_{pm}))$	(0 ,2, 1,1)	(1,1,2,0)	(2,0,3,1)
vertex	$a_{1,2}$	$a_{2,2}$	$a_{3,2}$
$r(.\mid R_{pm}))$	(2,3,0,1)	(2,2,1,0)	(3,1,2,0
Edges	$e_{1,1}$	$e_{2,1}$	$c_{1,1}$
$r(.\mid R_{pm}))$	(0,1,1,0)	(1,0,2,0)	(0,2,0,1)
Edges	$e_{1,2}$	$e_{2,2}$	$c_{2,1}$
$r(.\mid R_{pm}))$	(1,2,0,0)	(2,1,1,0)	(2,0,2,0)

TABLE 2. Codes of nodes and edges of Figure 4

Table 2 shows the unique representation of all vertices and edges of the hexagonal network. So R is mixed partition resolving set of cardinality 4.

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Now we want to show the representation for $h = 2 = v$

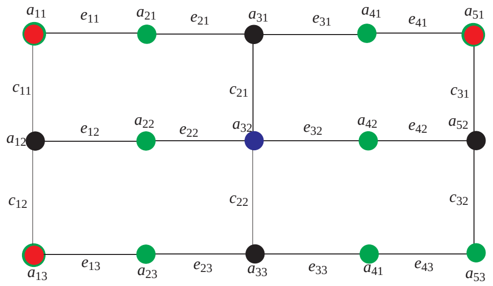

FIGURE 5. 2 by 2 Hexagone

The revolving set is $R_{pm})\{R_{p1}, R_{p2}, R_{p3}, R_{p4}\}$

$vertex$	$a_{1,1}$	$a_{2,1}$	$a_{3,1}$	$a_{4,1}$	$a_{5,1}$
$r(. \mid R_{pm}))$	(0,4,2,1)	(1,3,3,0)	(2,2,4,0)	(3,1,5,0)	(4,0,6,1)
$vertex$	$a_{1,2}$	$a_{2,2}$	$a_{3,2}$	$a_{4,2}$	$a_{5,2}$
$r(. \mid R_{pm}))$	(1,4,1,0)	(2,4,3,0)	(3,3,3,0)	(4,2,4,0)	(5,1,5,0)
$vertex$	$a_{1,3}$	$a_{2,3}$	$a_{3,3}$	$a_{4,3}$	$a_{5,3}$
$r(. \mid R_{pm})))$	(2,6,0,1)	(3,5,1,0)	(4,4,2,0)	(5,3,3,0)	(6,2,4,0)
$Edges$	$e_{1,1}$	$e_{2,1}$	$e_{3,1}$	$e_{4,1}$	$c_{1,1}$
$r(. \mid R_{pm}))$	(0,3,2,0)	(1,2,3,0)	(2,1,4,0)	(3,0,5,0)	(0,4,1,0)
$Edges$	$e_{1,2}$	$e_{2,2}$	$e_{3,2}$	$e_{4,2}$	$c_{2,1}$
$r(. \mid R_{pm}))$	(1,4,1,0)	(2,3,2,0)	(3,2,3,0)	(4,1,4,0)	(2,2,3,0)
$Edges$	$e_{1,3}$	$e_{2,3}$	$e_{3,3}$	$e_{4,3}$	$c_{3,1}$
$r(. \mid R_{pm}))$	(2,5,0,0)	(3,4,1,0)	(4,3,2,0)	(5,2,3,0)	(4,0,5,0)
$Edges$	$c_{1,2}$	$c_{2,2}$	$c_{3,2}$	—	—
$r(. \mid R_{pm}))$	(1,5,0,0)	(3,3,2,0)	(5,1,4,0)	—	—

TABLE 3. all codes w.r.t nodes and links of Figure 5

Table 3 shows the hexagonal network's unique representation of vertices and edges. So $R_{pm})$ is a partition resolving a set of cardinality 4. Now we want to make generalized formulas of distances that are unique.

GENERALIZED FORMULAS OF DISTANCE TO CHECK UNIQUENESS

A general expression for the distance for all vertices and edges of a hexagonal network shows that the mixed partition dimension is 4 since each distance is different. Let $d(a_{i,j}, R_{p1}) = W_1, d(a_{i,j}, R_{p2}) = W_2, d(a_{i,j}, R_{p3}) = W_3, d(a_{i,j}, R_{p4}) = W_4$ and $r(a_{i,j} \mid R) = (w_1, w_2, w_3, w_4)$

$$w_1 = \{ \ i + j - 2 \quad \text{for } 1 \leq i \leq h, \text{ and } 1 \leq j \leq v.$$
$$w_2 = \{ \ 2h + i - j \quad \text{for } 1 \leq i \leq h, \text{ and } 1 \leq j \leq v.$$
$$w_3 = \{ \ h + i - j \quad \text{for } 1 \leq i \leq h, \text{ and } 1 \leq j \leq v.$$
$$w_4 = \begin{cases} 1 & \text{for } a_{1,1}, a_{1,2h+1}, a_{1,v+1} \\ 0 & \text{otherwise} \end{cases}$$

Let $d(e_{i,j}, R_{p1}) = w'_1, d(e_{i,j}, R_{p2}) = w'_2, d(e_{i,j}, R_{p3}) = w'_3, d(e_{i,j}, R_{p4}) = w'_4$ and $r(e_{i,j} \mid R) = (w'_1, w'_2, w'_3, w'_4)$

$$w'_1 = \{ \ i + j - 2 \quad \text{for } 1 \leq i \leq h, \text{ and } 1 \leq j \leq v,$$
$$w'_2 = \{ \ 2h + i - j - 1 \quad \text{for } 1 \leq i \leq h, \text{ and } 1 \leq j \leq v.$$
$$w'_3 = \{ \ h + i - j \quad \text{for } 1 \leq i \leq h, \text{ and } 1 \leq j \leq v,$$
$$w'_4 = \{ \ 0 \quad \text{for all}$$

Let $d(c_{i,j}, R_{p1}) = w''_1, d(c_{i,j}, R_{p2}) = w''_2, d(c_{i,j}, R_{p3}) = w''_3, d(c_{i,j}, R_{p4}) = w''_4$ and $r(c_{i,j} \mid R) = (w''_1, w''_3, w''_3, w''_4)$

$$w''_1 = \{ \ 2i + j - 3 \quad \text{for } 1 \leq i \leq h, \text{ and } 1 \leq j \leq v,$$
$$w''_2 = \{ \ 2h + i - 2j + 1 \quad \text{for } 1 \leq i \leq h, \text{ and } 1 \leq j \leq v.$$
$$w''_3 = \{ \ h + 2i - j - 2 \quad \text{for } 1 \leq i \leq h, \text{ and } 1 \leq j \leq v.$$
$$w''_4 = \{ \ 0 \quad \text{for all}$$

Let λ_1 and λ_2 be any two random vertices on hexagonal network $HN_{h,v}$. Let $R_{pm})\{a_{1,1}, a_{1,2h+1}, a_{1,v+1}\}$ and $WLOG$ denotes without loss of generality and this would imply that represented by $TWIT$.
Case I: When $\lambda_1 = a_{i,n}$ and $\lambda_2 = a_{i',j'}$ then further 4 cases accore.
Case 1: if $i = i^{'}$, $j \neq j^{'}$ then WLOG we say that $j < j^{'}$ $TWIT$ $d(\lambda_1, a_{1,1}) \neq d(\lambda_2, a_{1,1})$, because $d(\lambda_1, a_{1,1}) = d(\lambda_2, a_{1,1}) + t$ where $t = i^{'} - i$ so $r(\lambda_1 \mid R) \neq r(\lambda_2 \mid R)$.
Case 2: if $i \neq i^{'}$, $j = j^{'}$ then WLOG we say that $i < i^{'}$ $TWIT$ $d(\lambda_1, a_{1,1}) \neq d(\lambda_2, a_{1,1})$, because $d(\lambda_1, a_{1,1}) = d(\lambda_2, a_{1,1}) + s$ where $s = 2(i^{'} - i)$ so $r(\lambda_1 \mid R) \neq r(\lambda_2 \mid R)$.
Case 3: if $i \neq i^{'}$, $j \neq j^{'}$ then WLOG we say that $j < j^{'}$, $i < i^{'}$ $TWIT$ $d(\lambda_1, R) \neq d(\lambda_2, R)$, because $d(\lambda_1, R) = d(\lambda_2, R) + (s + t)$ so $r(\lambda_1 \mid R) \neq r(\lambda_2 \mid R)$.
When $i \neq i^{'}$, $j \neq j^{'}$ the positions of λ_1 and λ_2 where $d(\lambda_1, a_{1,1}) = d(\lambda_2, a_{1,1})$,then $d(\lambda_1, a_{1,2h+}) \neq d(\lambda_2, a_{1,2h+1})$.
One can note from the Figure 3 when $d(\lambda_1, a_{1,1}) = d(\lambda_2, a_{1,1})$ $TWIT$ $d(\lambda_2, a_{1,2h+1}) \neq d(\lambda_1, a_{1,2h+1})$ also $d(\lambda_2, a_{1,2h+1}) = d(\lambda_1, a_{1,2h+1})$ $TWIT$ $d(\lambda_1, a_{1,1}) \neq d(\lambda_2, a_{1,1})$.
we discuss all cases where $d(\lambda_1, a_{1,1}) \neq d(\lambda_2, a_{1,1})$ while $d(\lambda_1, a_{1,2h+1}) = d(\lambda_2, a_{1,2h+1})$
or $d(\lambda_1, a_{1,1}) = d(\lambda_2, a_{1,1})$, $d(\lambda_1, a_{1,2h+1}) \neq d(\lambda_2, a_{1,2h+1})$.
Case 4: if $i = i^{'}$, $j \neq j^{'}$ and WLOG we can say that $j < j^{'}$ then $d(\lambda_1, a_{1,1}) \neq d(\lambda_2, a_{1,1})$ Because $d(\lambda_1, a_{1,1}) = d(\lambda_2, a_{1,1}) + t$ where $t = i^{'} - m$ so $r(\lambda_1 \mid R) \neq r(\lambda_2 \mid R)$.
Case 5: if $i \neq i^{'}$, $j = j^{'}$ and WLOG we can say that $i < i^{'}$ then $d(\lambda_1, a_{1,1}) \neq d(\lambda_2, a_{1,1})$ Because $d(\lambda_1, a_{1,1}) = d(\lambda_2, a_{1,1}) + s$ where $s = 2(i^{'} - m)$ so $r(\lambda_1 \mid R) \neq r(\lambda_2 \mid R)$.
Case 6: if $i \neq i^{'}$, $j \neq j^{'}$ and WLOG we can say that $i < i^{'}$, $j < j^{'}$ then $d(\lambda_1, R) \neq d(\lambda_2, R)$ Because $d(\lambda_1, R) = d(\lambda_2, R) + (s + t)$ so $r(\lambda_1 \mid R) \neq r(\lambda_2 \mid R)$.
From all the above cases, there is no possibility that the two

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representations are identical. This shows that $pd_m(G) \leq 4$.

Contrary Case

For $pd_m(HN_{h,v}) \geq 3$

$\sim pd_m(HN_{h,v}) < 3$

$\Rightarrow pd_m(HN_{h,v}) = 1, 2$. The mixed partition dimension is not 1 because one is not possible for a path graph so 1 is not for this structure. The mixed partition dimension is not 2 we discussed some reasons.

Case 1:

Let$R'_1 \subseteq \{a_i, j, a_{i,j+1} : 1 \leq i \leq h\ , 1 \leq j \leq 2v\}$ with the cardinality of 2, the analogous descriptions are provided by $r(a_{1,j}|R'_1) = r(c_{1,j}|R'_1)$.

Case 2:

Let$R'_1 \subseteq \{a_i, j, a_{i+1,j} : 1 \leq i \leq h\ , 1 \leq j \leq 2v\}$ with the cardinality of 2, the analogous descriptions are provided by $r(a_{1,j}|R'_1) = r(e_{1,j}|R'_1)$.

Case 3:

Let$R'_1 \subseteq \{a_i, j, a_{i+1,j+1} : 1 \leq i \leq h\ , 1 \leq j \leq 2v\}$ with the cardinality of 2, the analogous descriptions are provided by $r(a_{1+2,j}|R'_1) = r(e_{i+2,j}|R'_1)$.

Hence the mixed partition dimension is not 2 so the mixed partition dimension of the hexagonal network is 4. Hence prove that the $pd_m(HN_{h,v}) = 4$. □

IV. APPLICATION

Introducing the Mixed partition dimension, which considers both vertex and edge partitions to offer a greater comprehensive analysis of graph systems, opens up a new avenue for applications throughout numerous fields. Here are several ability applications. These applications are based on all other resolvability parameters.

Advanced Network Optimization: In telecommunications and records networks, the Mixed partition dimension can optimize routing algorithms by thinking about both nodes and connections. This dual attention reduces latency, improves facts throughput, and enhances common community robustness. It also aids in the early detection of network failures or inefficiencies.

Integrated Urban Planning: Urban planners can use the Mixed partition dimension for designing and optimizing integrated application networks (water, electricity, sewage) and transportation structures. By studying infrastructure nodes and connections, planners can make certain efficient useful resource distribution and minimize provider disruptions, mainly to smarter cities.

Enhanced Cybersecurity: In cybersecurity, the Mixed partition dimension can improve the detection and prevention of cyber threats. By tracking endpoints (nodes) and statistics pathways (edges), safety structures can perceive anomalous behaviors, including the unauthorized right of entry or records breaches, thereby offering a greater holistic protection approach.

Biological Network Analysis: In systems biology, the Mixed partition dimension facilitates understanding complicated biological networks consisting of metabolic pathways, protein-protein interactions, and neural networks. By thinking about both interactions and additives, researchers can benefit from insights into ailment mechanisms, drug goals, and the overall functionality of organic structures.

Smart grid management: Mixed partition dimension for power grid can enhance the power distribution management by simultaneously considering power stations (nodes) and transmission lines (edges) These two analyses improve load balancing, prevent blackouts, have better integration of renewable energy, and are a contributor.

Robotic swarm coordination: Mixed partition dimension in swarm robotics can improve robot communication and task allocation. Looking at robots (nodes) and their communication channels (edges) ensures better coordination, division of labor, and fault tolerance, and enhances the overall productivity of the robot.

Detailed social network analysis: In social network analysis, the mixed partition dimension provides a deeper understanding of social networks by analyzing individuals (nodes) and their relationships (streams) This method helps to identify key influences, understand community structures, and model the expansion exactly information or actions.

Integrated transportation design: Mixed partition dimension for optimizing transport networks considers both roads (edges) and stations or junctions (nodes) This integrated approach improves traffic management, reduces accidents, and improves the public transport system works well.

Complex Systems Engineering: In engineering complex systems, such as aerospace or car systems, mixed partition dimension helps to analyze components (nodes) and their connections (edges) This detailed analysis helps to improve system reliability, find potential failure areas, optimizing system design.

Environmental Monitoring and Management: In environmental sciences, the hybrid separation scale can be used to manage and manage ecosystems through species (nodes) and their interactions (water) to be analyzed This approach helps to understand ecological networks, evaluate conservation efforts, and predict environmental impact changes.

By introducing the mixed partition dimension, researchers and practitioners can gain a more nuanced and comprehensive understanding of networks and systems, leading to increased performance, reliability, and productivity across various applications.

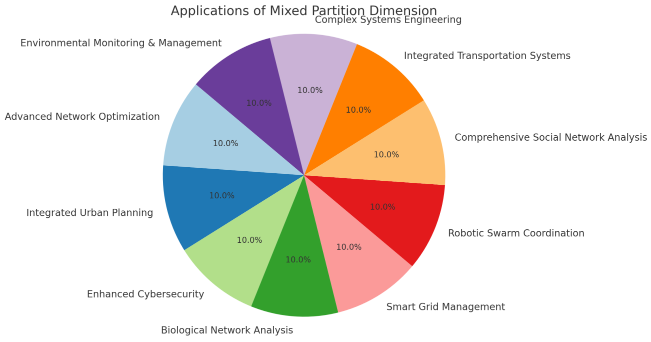

FIGURE 6. Pi chart

Here is the pi 6 chart representing the applications of the blended partition dimension throughout diverse fields. Each pie chart slice corresponds to a utility location, and they are proven as similarly allotted for simplicity.

Advantages of mixed partition dimension over traditional measures:

The mixed partition dimension (MPD) offers a good-sized development inside the area of graph resolvability, addressing obstacles of conventional measures like the metric dimension (MD) and edge metric dimension (EMD). By incorporating a partition-primarily based technique, MPD extends the scope of graph theoretical analysis to a broader range of real-world applications.

From optimizing sensor placement in community layout and ensuring efficient site visitors go with the flow in city systems to enhancing telecommunications and reading organic networks, MPD offers a robust framework for uniquely resolving nodes and edges. Its capability to partition graph factors helps hierarchical and aid-green solutions, making it a flexible tool for tackling complex troubles in logistics, emergency reactions, and secure communications.

The concrete use cases discussed show that MPD is not simply a theoretical extension but a practical metric with tangible blessings in various fields. This positions MPD as a precious addition to the suite of graph invariants, paving the manner for revolutionary solutions to real-world challenges. Future studies can explore its implementation in greater specialized domains, similarly solidifying its relevance and software.

COMPARISON BETWEEN MIXED PARTITION DIMENSION AND EXISTING RESOLVABILITY PARAMETERS

Graphs	MD	PD	MMD	MPD
Path graph	1	2	2	3
Cycle graph	2	3	3	4
Hexagonal sheet	2	3	3	4
Ladder	2	3	3	4

TABLE 4. Comparison MD, PD, MMD, and MPD

Here MD is the metric dimension, PD partition dimension, MMD is the mixed metric dimension and MPD is the mixed partition dimension. Table 4 shows the comparative analysis with existing resolvability parameters and mixed partition dimensions. For the results see [8], [53]

CONCLUSION

In this work, we introduce a novel resolvability parameter named the Mixed Partition Dimension and confirm its utility on unique graph lessons. The mixed partition dimension of a path graph is 3, and a hexagonal network has a mixed partition dimension of 4. These effects monitor that a graph's structural complexity and geometric arrangement substantially affect its mixed partition dimension. The parameter affords a strong framework for reading and evaluating distinct graph structures, highlighting its capability for further.

FUTURE WORK

One can work on mixed partition dimensions and exchange property for any structure.

CONJECTURE

Let G be any graph then

$$pd_m(G) \geq \max\{pd(G), epd(G)\}$$

OPEN PROBLEM

There is a graph showing that inequality is true ?.

$$pd_m(G) < pd(G) \ \ or \ \ epd(G)$$

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DATA AVAILABILITY STATEMENT

All the data supporting the results are included in the manuscript.

CONFLICTS OF INTEREST

The authors declare that they have no conflicts of interest.

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SITI NORZIAHIDAYU AMZEE ZAMRI She is currently a Senior Lecturer, at UniSZA Science and Medicine Foundation Centre, Universiti Sultan Zainal Abidin. DOCTOR OF PHILOSOPHY (MATHEMATICS), Universiti Teknologi Malaysia, Skudai. Her research interests include graph theory, group theory, and soliton theory. He has published more than 36 research articles in different high-reputed journals. He is a reviewer for international journals.

SIKANDER ALI is a research associate in the department of mathematics at Riphah International University Lahore Campus. He received his Bachelor's degree from GOVT Post Graduate College Bahawal Nagar affiliated to Islamia University Bahawalpur, in 2017 under the supervision of Prof. Rana Muhammad Aslam. He did his Master of Mathematics (2018-2020) from COMSATS University Islamabad, Sahiwal Campus under the supervision of Prof. Dr. Muhammad Asad Meraj in cryptography. He pursues his Master of Science in Mathematics at the same university under the supervision of Prof. Dr. Manzoor Ahmad Zahid in graph theory. His research interests include Cryptography, geometry, metric graph theory, graph labeling, spectral graph theory, and chemical graph theory. He is a reviewer for different international journals. He was awarded by**best researcher in graph theory** by Science Father on 22/10/2024 and he was also awarded by **Young Scientist in Chemistry** by Science Father on 29/10/2024

MUHAMMAD AZEEM He received his BS degree from COMSATS University Islamabad, Lahore Campus in 2018. He has published research articles in reputed international journals of mathematics and informatics. His research interests include control theory, metric graph theory, graph labeling, spectral graph theory, fuzzy set theory, fuzzy graph theory, and soliton theory. He has published more than 60 research articles in different high-reputed journals. He is a reviewer for more than 30 international journals.

HUSAM A. NEAMAH He is currently a Lecturer at the University of Debrecen Hungary. He did his Ph.D. in informatics and engineering technologies (Robotics). He has published research articles in reputed international journals of mathematics and informatics. His research interests include Mechatronics, Automation, Robotics, Exoskeleton, Robotics, Biomechatronics, Biomechanical, Engineering, Control Theory, Modeling and Simulation, Cognitive, Systems Assistive Robotics, and Social Robotics. He has published more than 34 research articles in different high-reputed journals. He is a reviewer for more than 7 international journals.

BANDAR ALMOHSEN He is the Saudi Association for Mathematical Science (SAMS) president. He was the head of the Mathematics Department from 2014-2016. He obtained a Ph.D. in applied mathematics from the University of Leeds, Leeds, U.K., in 2013. He is a Mathematics Department Professor at King Saud University, Saudi Arabia. He has published scientific papers in high-quality international mathematics and engineering sciences journals. His research interests include numerical analysis, inverse problems, PDEs, optimization, and numerical computational methods.

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