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REMARKS TO A THEOREM OF SINCLAIR AND VAALER

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Abstract. Sinclair and Vaaler in [6] Theorem 1.2 found sufficient conditions, nonlinear in the coefficients depending on a parameter $p \geq 1$, for self-inversive polynomials to have all their zeros on the unit circle. Here we discuss the dependence of the conditions on the parameter and through it we show that applying Theorem 1 of Lakatos and Losonczi [4] their result can be strengthened by giving the locations of the zeros.

1. Introduction

Let $P_m(z) = \sum_{k=0}^{m} A_k z^k = A_m \prod_{k=0}^{m} (z - z_k) \in \mathbb{C}[z]$ be a polynomial of degree m with zeros $z_1, \dots, z_m$. Further let P_m^* be the polynomial defined by

$$P_m^*(z) := z^m \bar{P}(1/z) = \sum_{k=0}^{m} \bar{A}_k z^{n-k} = \bar{A}_0 \prod_{k=0}^{m} (z - z_k^*)$$

whose zeros are $z_k^* = 1/\bar{z}_k, k = 0, \dots, m$ (the inverses of z_k with respect to the unit circle).

DEFINITION 1. A polynomial $P_m(z)$ of degree m is said to be *self-inversive* if there exists an $\varepsilon \in \mathbb{C}, |\varepsilon| = 1$ such that $P_m^*(z) = \varepsilon P_m(z)$.

There are several equivalent definitions of self-inversive polynomials. It is well-known (see e.g. [5]) that *for a polynomial* $P_m(z) = \sum_{k=0}^{m} A_k z^k$ *of degree* m *the following statement are equivalent*

1. P_m *is self-inversive,*
2. $\bar{A}_k = \varepsilon A_{m-k}, k = 0, \dots, m,$ *where* $|\varepsilon| = 1$,
3. *for the zeros* z_k *of* P_m *we have* $\{z_1, z_2, \dots, z_m\} = \{1/\bar{z}_1, 1/\bar{z}_2, \dots, 1/\bar{z}_m\}$.

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If a polynomial with real coefficients is self-inversive then ε is necessarily real hence either $\varepsilon = 1$ our polynomial is called reciprocal, or $\varepsilon = -1$ and our polynomial is called antireciprocal.

C. D. Sinclair and J. D. Vaaler in [6] Theorem 1.2 found sufficient conditions, nonlinear in the coefficients, for self-inversive polynomials to have all their zeros on the unit circle.

Their results reads as follows (the notations and formulation are slightly changed).

THEOREM 1. (Sinclair and Vaaler) If $P_m(z) = \sum_{k=0}^{m} A_k z^k$ is a monic self-inversive polynomial of degree m with $L \geq 3$ non-zero coefficients such that for some $p \geq 1$

$$|P_m|_p^p \leq 2 + \frac{2^p}{(L-2)^{p-1}}, \tag{1}$$

then P_m has all of its zeros on the unit circle.

Here the p norm $|P_m|_p$ is defined by

$$|P_m|_p := \left(|A_m|^p + |A_{m-1}|^p + \cdots + |A_0|^p\right)^{1/p} \qquad (p \geq 1).$$

Since here P_m is monic $L \geq 2$, in case of $L = 2$ clearly all zeros of P_m are on the unit circle. thus we may assume that $L \geq 3$.

The authors remark that their result is similar in spirit to recent results of Schinzel [7] and Lakatos and Losonczi [3].

Here we show that condition (1) is the strongest (gives the largest set of polynomials) if $p = 1$ and for this value (1) is identical to the sufficient condition (ii)-1 in Theorem 1 of Lakatos and Losonczi [4]. Applying this theorem the result of Sinclair and Vaaler can be strengthened by giving the location of zeros.

2. Results

To find the dependence of (1) on p first we rewrite it in an equivalent form as

$$\left(\sum_{k=1}^{m-1} |A_k|^p/(L-2)\right)^{1/p} \leq 2/(L-2). \tag{2}$$

For positive p let

$$\mathscr{M}_p(x_1, \ldots, x_n) := \left(\sum_{i=1}^{n} x_i^p/n\right)^{1/p}$$

be the pth power mean of the (nonnegative) numbers $x_1, \ldots, x_n$.

It is well known (see e.g. [1] p.16) that $\mathscr{M}_p(x_1, \ldots, x_n)$ is an nondecreasing function of p, strictly increasing unless $x_1 = \cdots = x_n$ and $\lim_{p\to\infty} \mathscr{M}_p(x_1, \ldots, x_n) = \max_{1\leq i\leq n} x_i$.

It is easy to recognize that the left hand side of (2) is exactly $\mathscr{M}_p(|A_1|,\dots,|A_{m-1}|)$ thus (1) has now the form

$$\mathscr{M}_p(|A_1|,\dots,|A_{m-1}|) \le \frac{2}{L-2}. \tag{3}$$

Suppose now that (3) holds for some $p \ge 1$. Then we have

$$\frac{|A_1|+\cdots+|A_{m-1}|}{L-2} = \mathscr{M}_1(|A_1|,\dots,|A_{m-1}|) \le \mathscr{M}_p(|A_1|,\dots,|A_{m-1}|) \le \frac{2}{L-2}$$

hence

$$|A_1|+\cdots+|A_{m-1}| \le 2(=2|A_m|). \tag{4}$$

Here *strict inequality* holds either if (1) holds with strict inequality or if (1) holds with equality, $p>1$ and not all absolute values of the nonzero coefficients (of P_m) are equal.

Equality holds in (4) either if (1) holds with equality and $p=1$ or $p>1$ and the absolute values of all nonzero coefficients (of P_m) are equal.

In [4] we proved the following

THEOREM 2. (Lakatos and Losonczi) (i) If all zeros of the polynomial $P_m(z) = \sum_{k=0}^{m} A_k z^k \in \mathbb{C}[z]$ of degree $m \ge 1$ are on the unit circle then P_m is self-inversive.

(ii)-1 If P_m is self-inversive and

$$|A_m| \ge \frac{1}{2}\sum_{k=1}^{m-1} |A_k| \tag{5}$$

holds then all zeros of P_m are on the unit circle.

Let

$$\beta_{m-l} = \arg A_{m-l}\left(\frac{\bar{A}_0}{A_m}\right)^{\frac{1}{2}} \left(l=0,\dots,\left[\frac{m}{2}\right]\right), \quad \varphi_l = \frac{2(l\pi-\beta_m)}{m}\ (l=0,\dots,m) \tag{6}$$

where $\left[\frac{m}{2}\right]$ denotes the integer part of $\frac{m}{2}$.

(ii)-2 If the inequality (5) is strict then the zeros e^{iu_l} $(l=1,\dots,m)$ of P_m are simple and can be arranged such that

$$\varphi_{l-1} < u_l < \varphi_l \quad (l=1,\dots,m). \tag{7}$$

(ii)-3 If (5) holds with equality then double zeros may arise. If (5) holds with equality then $e^{i\varphi_l}$ $(1 \le l \le m)$ is a zero of P_m if and only if the coefficients of P_m satisfy the conditions

$$\cos\left(\beta_{m-k} + \left(\frac{m}{2}-k\right)\varphi_l\right) = (-1)^{l+1} \text{ for all } k=1,\dots,\left[\tfrac{m}{2}\right] \text{ for which } A_k \ne 0. \tag{8}$$

If (8) holds then $e^{i\varphi_l}$ is necessarily a double zero of P_m.

One can easily see that (5) is identical to (4) hence the assertions of Theorem 2 can be applied. In this way Theorem 1 of Sinclair and Vaaler can be strengthened to

THEOREM 3. (j)-1 Let $P_m(z) = \sum_{k=0}^{m} A_k z^k$ be a monic self-inversive polynomial of degree m with $L \geq 3$ non-zero coefficients such that for some $p \geq 1$

$$\sum_{k=0}^{m} |A_k|^p \leq 2 + \frac{2^p}{(L-2)^{p-1}}, \tag{9}$$

(or $\max_{1 \leq k \leq m-1} |A_k| \leq 2/(L-2)$ obtained from (9) by $p \to \infty$) then P_m has all of its zeros on the unit circle.

Let β_{m-l}, φ_l be defined by (6) with the coefficients of our monic polynomial.

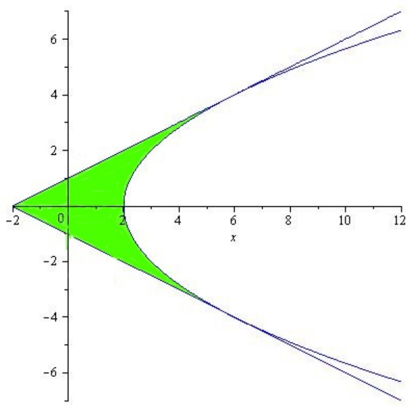
(jj)-2 If (9) holds with strict inequality or if (9) holds with equality, $p > 1$ and not all absolute values of the nonzero coefficients (of P_m) are equal then the zeros e^{iu_l} $(l = 1, \ldots, m)$ of P_m are simple and can be arranged such that

$$\varphi_{l-1} < u_l < \varphi_l \quad (l = 1, \ldots, m) \tag{10}$$

(jj)-3 If (9) holds with equality and $p = 1$ or $p > 1$ and the absolute values of all nonzero coefficients (of P_m) are equal then double zeros may arise. In this case $e^{i\varphi_l}$ $(1 \leq l \leq m)$ is a zero of P_m if and only if the coefficients of the monic P_m satisfy the conditions (8) and if this holds then $e^{i\varphi_l}$ is necessarily a double zero of P_m.

3. The case of degree four reciprocal polynomials

Let us consider a degree four monic reciprocal polynomial $f_4(z) = z^4 + c_1 z^3 + c_2 z^2 + c_1 z + 1$ with real coefficients. Its zeros are on the unit circle if and only if (see Lakatos [2] p. 659-660) $2\sqrt{\max\{c_2 - 2, 0\}} \leq |c_1| \leq \min\{4, (c_2+2)/2\}$. The figure below shows the closed region D in the (c_1, c_2) plane, satisfying these inequalities colored in gray (green in the pdf). Here and in the next figure the horizontal axis is c_2 the vertical axis is c_1.

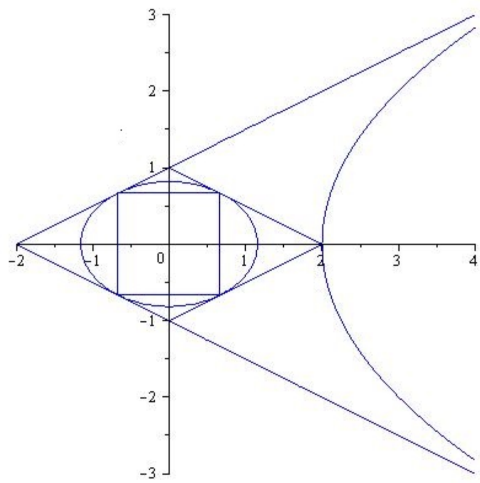


The closed region D

f_4 satisfies inequality (1) or (9)

$$\begin{array}{lll} \text{for } p=1 & \text{if and only if} & 2|c_1|+|c_2|\le 2, \\ \text{for } p=2 & \text{if and only if} & 2|c_1|^2+|c_2|^2\le 4/3, \\ \text{for } p\to\infty & \text{if and only if} & \max|c_1|,|c_2|\le 2/3. \end{array}$$

Denote by D_1, D_2, D_∞ the closed regions corresponding to $p=1$ (the closed rhombus with vertices $(-2,0),(0,1),(2,0),(0,-1)$), to $p=2$ (the interior and boundary of the ellipse with horizontal half-axis $\sqrt{4/3}$ vertical half-axis $\sqrt{2/3}$), to $p\to\infty$ (the closed square of sides $2/3$) respectively. The next figure shows the closed regions $D_\infty\subset D_2\subset D_1\subset D$ together.



The closed regions $D_\infty\subset D_2\subset D_1\subset D$

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