CONTRIBUTIONS TO THE THEORY OF TWO DIMENSIONAL SYSTEMS OF PHYSICS

Doktori (Ph.D.) értekezés

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1

Introduction

1 List of the topics of the thesis

Two dimensional systems occupies a quite special place in mathematical physics. In many ways, they are often simpler than the higher dimensional system.Sometimes this allows not only a mathematically rigorous treatment, but a complete solution of the problem. Moreover, they serve as simple models of more complicated systems. In this thesis, we present several separate examples of two dimensional physics. We list here the topics of our dissertation:

- We study the deformations of the commutative algebra of functions on a cylinder. The nontrivial topology of the cylinder allows us to derive an interesting modification of von Neumann's formula for the deformed product.

- The deformed product of functions on a surface is used to describe the motion of an oriented membrane in M-theory. We extend this method to nonorientable surfaces. The nonsymplectic nature of nonorientable surfaces is circumvented by the use of Jordan algebras instead of associative ones.

- The simplest solvable two dimensional lattice model of statistical physics is two dimensional lattice gauge theory. Solvable two dimensional models often provide solutions of the Quantum Yang-Baxter Equation. However, the solution corresponding to gauge theory is not invertible. We present a modification of the model circumventing this problem.

- In lattice gauge theory the gauge group can be replaced by a semigroup if the semigroup possesses an involution. In two dimension, the solvability of the model is usually lost by this replacement. We study the conditions which makes these models solvable.

- Here we turn our attention to an other type of solvable models, to a complex variant of the Kortaweg-de Vries two dimensional partial differential equation. The KdV equation can be written as an isospectral deformation of a Schroedinger operator L. We present a scheme when L is replaced by its absolute value $|L| = \sqrt{LL^*}$.

- We study the structure of the coadjoint orbits of some Lie algebras, which emerge in the description of the motion of an quantum mechanical particle in two dimension under the influence of a magnetic field and a periodic cosine potential. We basically present some examples of wild (Type II) groups in solid state physics.

- Conformal field theories with nonconformal boundary conditions might possess an extra central extension term in the algebra of the stress tensor at tha boundary. This extension was discovered by Goncharova. We present some argument for the absence of this term in the model of a massless scalar field with a semitransparent boundary condition.

2 Overview of the thesis

- To describe the motion of a single particle in one dimension two variable (position and velocity) are needed. In quantum mechanics, the are replaced by two noncommuting operators. This replacement can be interpreted as a deformation of the algebra of functions of the two classical phase space variable. An explicit formula for the deformed noncommutative (but associative product was calculated by von Neumann. He discovered that the kernel generating the product on $\mathbb{R}^2$ can be written as

$$f *_h g(r) = (f *_h g)(\mathbf{r}) =$$

$$\frac{1}{h^2\pi^2}\int d^2\mathbf{r}' d^r 2'' f(\mathbf{r}')g(\mathbf{r}'') \exp\frac{-4i}{h} A(\mathbf{r}, \mathbf{r}', \mathbf{r}''),$$

where A is the (symplectic) area of the triangle spanned by $\mathbf{r}, \mathbf{r}'$ and $\mathbf{r}''$. This expression can be used to calculate the deformed product on a cylinder and torus, too. On the plane A can be calculated as either the integral of the symplectic form over the triangle, or as the integral of a suitable one-form over the straight line segments of the boundary of the triangle. On a cylinder three line segments between three point do not necessarily bound a triangle, so it is reasonable to ask what happens if this topologically nontrivial situation is included in a modified form of von Neumann's formula. We study the effect of these nontrivial paths, and conclude that in this case the product remains associative, but the new product is not an algebraic deformation of the commutative one.

- An interesting application of the deformation of algebras on an orientable surface is the regularization of the equations of membrane motion, developed by Goldstone and Hoppe. It was developed as a two dimensional analogue of the dynamics of strings. Recently the supersymmetric version of the theory received quite a lot of attention as a possible nonperturbative formulation of superstring theory.

  In the method of Goldstone and Hoppe algebraic deformation theory requires orientable (symplectic) surfaces. But the equations of motion looks the same for nonorientable surfaces, too. So we looked for an algebraic structure on the functions on the surface which made no reference to the sign of the symplectic form (i.e to the orientation of the surface). We found that the Jordan algebraic deformation satisfies this criteria, and we were able to formulate the regularized version of the equation of motion in term of the Jordan product. We determined what sort of Jordan algebra could be used for the description of a membrane with the topology of $\mathbb{RP}^2$.

- The popularity of group theory in physics enormously increased by the invention of quantum mechanics. The behavior of quantum systems are quite counterintuitive, but their axiomatic mathematical description is quite elegant. The state of a system is described by an unit vector of a Hilbert space, while the symmetries and the dynamics are given by the action of unitary operators. Since very often the symmetry operations are the same in the classical and the corresponding quantum systems, the understanding of quantum theory is greatly facilitated by the knowledge of the unitary (projective) representations of the symmetry group. This group theoretical approach is able to describe the kinematics of a single quantum particle as the representation of the Galileo or the Poincare groups.

This approach was initiated by Dirac, Wigner and Weyl. We should also mention as a major success the theorem of Stone-von Neumann, which roughly states the unicity of the irreducible selfadjoint representation of the Heisenberg commutation relation $[P, Q] = i$. This is basically the most simple nilpotent Lie-algebra.

The Stone-von Neumann theorem is the most simple example of an interesting link between unitary representation theory and quantum mechanics. The "Orbit Method" of Kirillov (and Kostant and Souriau) [45] is able to relate the irreducible representations of a nilpotent group to the structure of its coadjoint orbit space. On a coadjoint orbit there is a natural symplectic two-form, which is the basic structure of the Hamiltonian formulation of classical mechanics. An irreducible unitary representation can be described as a "quantization" of this system. This philosophy was applied for the case of solvable Lie groups by Auslander and Kostant. However, in this case the success was only partial, since these groups are able to exhibit a quite pathological behavior. It might happen that their unitary representations are decomposable as a direct integral of unitary ones several distinct ways. Groups with this sort of behavior are called wild (or Type II) as opposed to "tame" or Type I. One is tempted to think that they belong to the zoo of counterexamples, but they actually occur in the solid-state physics of quasi-periodic systems. We explain this in the case of the Mautner group.

This is a five dimensional solvable Lie-group, whose Lie-algebra can be represented by the following operators:

$$\cos(x),\ \sin(x),\ \cos(\alpha x + \phi),\ \sin(\alpha x + \phi),\ \partial_x,$$

where $\phi \in \mathbb{R}$ and α are irrational. These operators can be assembled into the Hamiltonian of an electron moving in a quasiperiodic potential:

$$H = -1/2\partial_x^2 + \cos(x) + \cos(\alpha x + \phi).$$

The spectral properties of this operator is quite different compared to a periodic one. The spectrum has nontrivial singular part. This behavior is foreshadowed by the wild nature of the group. It is argued that in this case the reason of the wildness is the fact that it is impossible to separate inequivalent representations by any reasonable topology. Indeed, two realization are equivalent if $\phi_1 - \phi_2 = (\alpha k + l)2\pi)$ for some $k, l \in \mathbb{Z}$. As equivalent ϕs are dense in $\mathbb{R}$, the cannot be reasonably separated from other ones.

Similar groups occur in the description of the motion of an electron in a periodic crystal if magnetic field is applied. They can often proved to be wild by the Auslander-Kosant theorem. We regard this as an indication that the system's nature is similar to the phenomenas of the quasiperiodic systems.

- An other example of the influence of physics in group representation theory is the case of the braid group [**20, 21**]. The braid group B_n is described by the following relations of Artin:

  $$\sigma_k\sigma_{k+1}\sigma_k = \sigma_{k+1}\sigma_k\sigma_{k+1}, \quad \sigma_k\sigma_l = \sigma_l\sigma_k \text{ ha } |k-l| \geq 1.$$

  A very similar relation occurs in the theory of solvable (integrable) two dimensional lattice statistical systems:

  $$R_{ij}^{xz}(\lambda)R_{zk}^{yn}(\lambda+\mu)R_{xy}^{ln}(\mu) = R_{jk}^{yx}(\mu)R_{iy}^{lz}(\lambda+\mu)R_{zx}^{mn}(\lambda)$$

  Two dimensional lattice gauge theory is an almost trivial example of solvable systems, so it is a natural idea to check if one can derive from it "R" matrices satisfying this relation. This can be easily checked by the application of some standard character identity from group theory. However, these matrices do not produce representations of B_n, due to the lack of invertibility. Nevertheless, this problem can be eliminated by a small modification of the system. We checked the question of solvability in the case when the gauge group is replaced by certain semigroups. Here solvability is not automatic, but we managed to establish it at certain special cases.

- The existence of symmetries can simplify the solution of a time-evolution equation. According to a fundamental theorem of Emmy Neother, a symmetry of a variational problem implies an existence of a conserved quantity. In certain cases there are so many conserved quantity, that the dynamics is almost trivial in some sense. This happens in the case of completely integrable systems.

A $2n$-dimensional system is completely integrable if there are n conserved quantity I_j in involution, i.e. their Poisson brackets $\{I_j, I_k\}$ are zero. Then typically they generate commutative translations on an n dimensional torus. If the Hamiltonian of the system is a linear combination of them, then it generate a simple quasiperiodic motion.

Most of these (often infinite dimensional, PDE) systems can be written in a Lax form:

$$\dot{L} = PL - LP.$$

As this is the infinitesimal form of a similarity transformation, the spectrum of L remains invariant, which signals the existence of conserved quantities. The most famous example of this scheme is the Kortaweg-de Vries equation

$$u(x,t)_t = u_{xxx} + uu_x,$$

which can be written as an isospectral deformation of the $L = \partial_x^2 + u$ Sturm-Liouville or Schroedinger operator. We describe a variant of this procedure, where we preserve the spectrum of LL^* instead of L. The general form of these equation is

$$\dot{L} = i(PL - LQ), \quad P = P^*, \ Q = Q^*.$$

If this is applied to a Sturm-Liouville operator, then one obtains a complex equation

$$u_t = u_{xxx} - 3\bar{u}_{xxx} - 6uu_x + 6u_x\bar{u} + 12u\bar{u}_x,$$

which is basically a variant of the Hirota-Satsuma equation. As an other application, we derived a complex analog of the Kadmontsev-Petviasvilli hierarchy.

- The existence of a large number of symmetries is very successfully exploited in Conformal Field Theory. In a restricted sense, this theory might be considered as a representation theory of the Virasoro algebra, which is a central extension of $diff(S^1)$ by the Gelfand-Fuks cocycle. The case when the CFT system has a conformal boundary condition was intensively studied recently. Hoverer, the case of nonconformal boundary conditions received much less attention. In this case the usual Virasoro type algebra of the stress tensor potentially acquires an extra central extension term, discovered by Goncharova in her description of the cohomology ring $H^*(L_k(1))$, where $L_k(1)$ is the Lie algebra of the vector fields of the form

$$\nu_k x^{k+1}\partial_x + \nu_{k+1}x^{k+2}\partial_x + \nu_{k+2}x^{k+3}\partial_x + \ldots .$$

The boundary condition $\phi(0)+\phi'(0) = 0$ is preserved by only $L_1(1)$, so Goncharova's central extension might be present when the boundary condition contains the values and the derivatives of a scalar field. Despite of this possibility, we provide some heuristic arguments that, at least in the case of a scalar field on an interval, Goncharova's term does not occur.

2

Nonperturbative effects in deformation quantization

1 Introduction

In this introductory section we briefly review the connection between deformation quantization and the theory of quantum mechanics in the simplest case of the two dimensional $r = (x, p)$ phase space of a point-like particle moving in one dimension. In classical mechanics, the Hamiltonian formalism assign a vector field X_f to every function $f(x, p)$ by the rule

$$f \longrightarrow X_f = f_p \partial_x - f_x \partial_p,$$

which generates the 'time' evolution corresponding to the Hamiltonian f. In quantum mechanics, the function $f(x, p)$ is replaced by an operator

$$\Phi(f) = f(\hat{x}, \hat{p})$$

, where $\hat{x} = x$ and $\hat{p} = -ih\partial_x$. Since $\hat{x}$ and $\hat{p}$ does not commutes with each other, one can define the map Φ for example by the requirement that if f is a two variable polynom, then $f(\hat{x}, \hat{p})$ is obtained from f by averaging all the possible ordering of $\hat{x}$ and $\hat{p}$. This prescription is called the Weyl ordering. Then the map Φ defines a noncommutaive, deformed product of the functions of the (x, p) phase space by the rule

$$f *_h g = \Phi^{-1}(\Phi(f)\Phi(g)).$$

We do not go into any further details (see for example [1]), but recall only the famous formula of von Neumann, which gives an explicit integral representation for $f *_h g$:

$$f *_h g(r) = (f *_h g)(\mathbf{r}) = \frac{1}{h^2\pi^2}\int d^2\mathbf{r}' d^2\mathbf{r}'' f(\mathbf{r}')g(\mathbf{r}'')\exp\frac{-4i}{h}A(\mathbf{r},\mathbf{r}',\mathbf{r}''),$$

where A is the symplectic area of the triangle spanned by three points. This formula works for periodic functions, too, so it describes a deformation of the functions of a cylinder or a torus. A is the integral of the symplectic two form ω so if $\omega = d\beta$, then by Stokes theorem, it can be written as a line integral of β over the boundary of the triangle. On a cylinder one can draw the three straight line segments between the three point, in a manner so that they do not bound a triangle, so one is forced to use the line integral definition of A in von Neumann's formula. By mere curiosity, one can check if this product remains associative. We perform this task in the next section. We also try to provide some more reasonable motivation by recalling the recent results [2] on the path integral representation of the deformed product on Poisson manifolds.

2 A quantization on a cylinder

Kontsevich's solution [2] of the problem of deformation quantization of the algebra of smooth functions on a Poisson manifold $\mathcal{M}$ was interpreted by Cattaneo and Felder [3] with the help of a path integral construction. They gave the following formula for the star product of two functions f and g:

$$(f *_h g)(x) = \int DX\, D\eta \;\; f(X(-1))g(X(i))\exp\frac{i}{h}\int_{D^2} <\eta, dX> + \frac{1}{2}(\alpha\circ X)(\eta,\eta) \tag{2.1}$$

where α is the Poisson bivector field, X is a map from the unit disc D^2 of the complex plane fulfilling $X(1) = x$ and η is also a map from D^2 to the cotangent space of $\mathcal{M}$, mapping $z \in D^2$ to $T^*_{X(z)}$.

On a symplectic manifold this formula simplifies further:

$$(f *_h g)(x) = \int DX\, f(X(-1))g(X(i))\exp\frac{i}{h}A[X] \tag{2.2}$$

where the path integral is over such maps from the boundary $\partial D^2 = S^1$ of the unit disk that $X(1) = x$, and the phase factor A is the integral of the symplectic form ω over the image of an extension of the map X from $S^1 = \partial D^2$ to D^2. Since ω is closed, so at least locally $\omega = d\beta$, A can be written as $\int_{X(S^1)} \beta$ in the perturbative expansion of (3.2).

The perturbative expansion of these integrals provides local solution to the problem of deformation quantization. It was suggested [4] that in the expression (3.2) for certain physical applications one should study nonperturbative effects, i.e. the path integral over the maps X should contain such maps X_1 and X_2, that the union of their images represent a nontrivial element of $\pi_2(\mathcal{M})$ (considered as a map from $D^2 \cup D^2 = S^2$) to $\mathcal{M}$.

The full nonperturbative evaluation of (3.2) might not give an associative product, but it might be necessary in some circumstances, like in the noncommutative geometric description of D-brane physics [5].

If we regard the factor $A[X]$ in (3.2) not as the integral of ω over $X(D^2)$, but as the integral of β over $X(\partial D^2) = X(S^1)$, then one can investigate the effect of the inclusion of nontrivial paths in the path integral, i.e. the case when $X(S^1)$ represent a nontrivial element of $\pi_1(\mathcal{M})$. We perform this program in probably the simplest case, when the symplectic manifold is the cylinder $C = S^1 \times R$ with coordinates $x = 0..2\pi$, $p \in \mathbb{R}$ and symplectic form is $\omega = dx \wedge dp$. Note that $\omega = d(-p dx) = d\beta$ where β is globally defined on C.

Let us recall that on $\mathbb{R}^2$ there is an explicit formula for the deformed product [**6**, **7**]

$$(2.3) \qquad (f *_h g)(\mathbf{r}) = \frac{1}{h^2\pi^2} \int d^2\mathbf{r}' d^2\mathbf{r}'' f(\mathbf{r}') g(\mathbf{r}'') \exp \frac{-4i}{h} A(\mathbf{r}, \mathbf{r}', \mathbf{r}''),$$

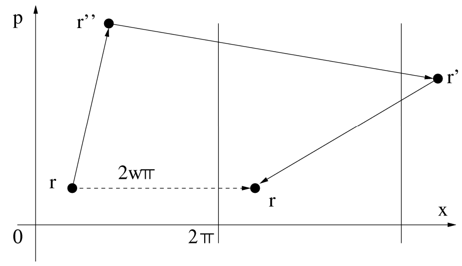
where $\mathbf{r} = (x, p)$ and A is the symplectic area of the triangle $T = \Delta(\mathbf{r}, \mathbf{r}', \mathbf{r}'')$.

The expression (3.3) makes sense if f and g are periodic functions with period vector $(2\pi, 0)$. As these functions can be identified with the functions of the cylinder C, von Neumman's formula solves the deformation problem on C, too.

By the comparison of (3.3) and (3.4) we see that if the values $X(-1) = \mathbf{r}', X(i) = \mathbf{r}'', X(1) = \mathbf{r}$ are fixed, then the integral over the remaining degrees of freedom of the map $S^1 \to R$ is equivalent to an expression involving only straight line segments between the vertices $\mathbf{r}, \mathbf{r}', \mathbf{r}''$ of the triangle T. We make the (hopefully reasonable) assumption that this remains true if the target space of the map from S^1 is not R^2 but our cylinder C.

On C the three line segments between $\mathbf{r}, \mathbf{r}'$,and $\mathbf{r}''$ do not necessarily bound an image of a triangle, so in the expression (3.4) we substitute

$A(\mathbf{r}, \mathbf{r}', \mathbf{r}'')$ by $\int_\Delta -p \;\, dx$ where the integral is over $\Delta = \overleftarrow{\mathbf{r}\mathbf{r}'} \cup \overleftarrow{\mathbf{r}'\mathbf{r}''} \cup \overleftarrow{\mathbf{r}''\mathbf{r}}$. Now let us take $\mathbb{R}^2$ as the universal covering space of C.



Then the path $\mathbf{r}, \overleftarrow{\mathbf{r}'}\overleftarrow{\mathbf{r}''}, \mathbf{r}$, can be uniquely lifted to R^2 once an image of r is chosen. The result is denoted by $\bar{\mathbf{r}}, \overleftarrow{\mathbf{r}'\mathbf{r}},'' \mathbf{r}$, On R^2 the difference between the starting and ending images of $\mathbf{r}$ must has the form $w \cdot (2/pi, 0)$, $w \in \mathbb{Z}$. In the following we fix the value of w and compute this modified deformed product for the basis functions $e_{n,r} = \exp i(nx + rp)$, $n \in \mathbb{Z}, r \in \mathbb{R}$.

To compute the value of $e_{n,r} \cdot_h e_{\tilde{n},\tilde{r}}$ at $\mathbf{r}$ we lift both functions to $\mathbb{R}^2$, choose a representative of $\mathbf{r}$ (denoted also by $\mathbf{r}$), perform the integration over $\mathbf{r}'$ and $\mathbf{r}''$ in (3.3) with the sole difference that the phase factor A is replaced by the line integral of $-p dx$ over the path of straight line segments $\overleftarrow{\bar{\mathbf{r}}\mathbf{r}'} \cup \overleftarrow{\mathbf{r}'\mathbf{r}''} \cup \overleftarrow{\mathbf{r}''\mathbf{r}}$ where $\bar{\mathbf{r}}$ is the other lift of $\mathbf{r}$ with $\bar{\mathbf{r}} - \mathbf{r} = w \cdot (2\pi, 0)$. This sort of modified evaluation of the product of basis functions gives

(2.4)
$$(e_{n,r} *_{h,w} e_{\tilde{n},\tilde{r}})(x,p) = \frac{1}{h^2\pi^2} \int dx' dp' dx'' dp'' \exp[i(nx' + rp' + \tilde{n}x'' + \tilde{r}p'') + \frac{4i}{h} J],$$

where

$$\text{(2.5)} \quad J = I((x,p),(x'',p'')) + I((x'',p''),(x',p')) + I((x',p'),(x + 2w\pi, p)),$$

$$\text{(2.6)} \qquad I((a,b),(c,d)) = \int_{(a,b)}^{(c,d)} -p dx.$$

Since the integral is Gaussian, its evaluation is straightforward:

$$\text{(2.7)} \qquad e_{n,r} *_{h,w} e_{\tilde{n},\tilde{r}} = \exp[\frac{ih}{2}(r\tilde{n} - \tilde{r}n) + 2ih\pi w] e_{n+\tilde{n}, r+\tilde{r}+4w\pi/h}.$$

Surprisingly, this multiplication rule is associative. In fact, by a redefinition of the of the basis

$$e_{n,r} \longrightarrow f_{n,r} = e_{n,r-4w\pi/h} \exp(-2i\pi h w) \tag{2.8}$$

we regain the original $w = 0$ multiplication rule of the $e_{n,r}$ basis:

$$f_{n,r} *_{h,w} f_{\tilde{n},\tilde{r}} = f_{n+\tilde{n},r+\tilde{r}} \cdot \exp\{\frac{ih}{2}[\tilde{n}(r - 4w\pi/h) - n(\tilde{r} - 4w\pi/h)]\} \tag{2.9}$$

$$= f_{n+\tilde{n},r+\tilde{r}} \cdot \exp(\frac{ih}{2}(\tilde{n}r - n\tilde{r})). \tag{2.10}$$

However, the path integral representation (3.2) of the product suggest the inclusion of paths with different winding numbers, possibly weighted by some system of coefficients. So we do not assume anymore that w is fixed, and for the sequence of coefficients $c = \{\dots, c_{-1}, c_0, c_1, c_2, \dots\}$ we define the multiplication rule by

$$f *_c g = \sum_{w \in Z} c_w f *_{h,w} g. \tag{2.11}$$

Usually the linear combination of different associative products is no longer associative, but in our case it is, as both

$$(e_{n,r} *_c e_{\tilde{n},\tilde{r}}) *_c e_{\bar{n},\bar{r}} \qquad \text{and} \qquad e_{n,r} *_c (e_{\tilde{n},\tilde{r}} *_c e_{\bar{n},\bar{r}}) \tag{2.12}$$

evaluates to
(2.13)

$$\sum_{u,w} c_u c_w e_{n+\tilde{n}+\bar{n},r+\tilde{r}+\bar{r}+4\pi(u+w)/h} \cdot \exp\{2\pi i h(u+w) + \frac{ih}{2}[\tilde{n}r - n\tilde{r} + \bar{n}r - n\bar{r} + \bar{n}\tilde{r} - \tilde{n}\bar{n}]\}.$$

So we conclude that on a cylinder the full nonperturbative evaluation of the path integral representation of our variant of von Neumann's expression (3.3) gives an associative product.

Now one might suspect that after a suitable redefinition of the basis functions (like in (2.8)) the $*_c$ product turns out to be the same as the $*_h$ product. However, we show that this is not quite the case. We demonstrate that when the only nonzero elements of the sequence c is $c_0 = c_1 = 1$, then the $*_c$ product algebra of the smooth functions on C does not possess an unit element.

Let us suppose (ad absurdum) that the unit u can be written as

$$u = \sum_{n,r} u_{n,r} e_{n,r}. \tag{2.14}$$

Then

$$e_{\tilde{n},\tilde{r}} = u *_c e_{\tilde{n},\tilde{r}} = \sum_{n,r} e^{\frac{ih}{2}(r\tilde{n}-\tilde{r}n)} e_{n+\tilde{n},r+\tilde{r}} + e^{\frac{ih}{2}(r\tilde{n}-\tilde{r}n)+2ih\pi} e_{n+\tilde{n},r+\tilde{r}+4\pi/h}. \tag{2.15}$$

This would imply that

$$\sum_r u_{0,r} e^{\frac{ih}{2}r\tilde{n}} + u_{0,r-4\pi/h} e^{\frac{ih}{2}r\tilde{n}+2ih\pi} = \begin{cases} 1, & \text{if} \quad r = 0, \\ 0, & \text{if} \quad r \neq 0. \end{cases} \tag{2.16}$$

So

$$\frac{u_{0,(k+1)\cdot 4\pi/h}}{u_{0,k\cdot 4\pi/h}} = -e^{2ih\pi} \tag{2.17}$$

for $k = \ldots, -3, -2, 0, 1, \ldots$. Consequently for $k = 1, 2, 3 \ldots$

$$u_{0,-k\cdot 4\pi/h} = u_{0,-4\pi/h}(-e^{-2ih\pi})^{k-1} \qquad \text{and} \qquad u_{0,k\cdot 4\pi/h} = u_{0,0}(-e^{2ih\pi})^k. \tag{2.18}$$

Since $u_{0,0} + u_{0,-4\pi/h}e^{2ih\pi} = 1$, either the $\{u_{0,-k\cdot 4\pi/h}, k = 1, 2, \ldots\}$ or the $\{u_{0,k\cdot 4\pi/h}, k = 1, 2, \ldots\}$ geometrical sequence has nonzero elements with constant absolute values. As the Fourier series

$$\sum_{n\geq 0} e^{inx}\alpha^n = \frac{1}{1-\alpha e^{ix}}, \tag{2.19}$$

where $|\alpha| = 1$ does not represent a smooth function of x, we conclude that the unit u can not be represented by a smooth function on C. Let us note that as an unital algebra can not be deformed to a nonunital one (see for example [8]), this phenomena is a true nonperturbative effect.

3

Matrix theory of unoriented membranes and Jordan algebras

1 Introduction

The proposal of Matrix theory as a fundamental theory of physics [**9**] and its connection with supermembrane theory [**10**] gave a strong impetus to the study of noncommutative space-time models. The starting point of these investigations is often the correspondence between the Poisson algebra of functions on the surface of the membrane and the associative algebra of the regularized matrix coordinates. Nevertheless, there are models, where even the assumption of the associativity of the space-time coordinates is dropped. We demonstrate that the nonassociative Jordan algebras can be used to describe the motion of the bosonic membrane. In contrast to the case of Poisson algebras, our construction does not require orientable surfaces, so it can describe nonorientable surfaces, too. Let us note that Jordan algebras were used for example in string theory [**12**] and matrix string theory [**11**].

In the second section we briefly review the Matrix theory - membrane correspondence, while the third one contains the Jordan algebraic reformulation of this theory.

2 A short review of the oriented membrane

The classical equation of motion of a membrane is derived from the action

$$S = -T \int d^3\sigma \sqrt{-\det \partial_\alpha X^\mu \partial_\beta X_\mu}, \tag{3.1}$$

where the $X^{mu}(\sigma_0, \sigma_1, \sigma_2)$ coordinates describe the embedding of the membrane's world-volume into the ambient Minkowski space. The Hamiltonian equations of motion in the light-front coordinates are generated by the Hamiltonian

$$H = \frac{\nu T}{4} \int d^2\sigma \left(\dot{X}^i \dot{X}^i + \frac{2}{\nu^2} \{X^i, X^j\}\{X^i, X^j\} \right) \tag{3.2}$$

and the supplementary constraint

$$\{\dot{X}^i, X^i\} = 0 \tag{3.3}$$

The equation of motion is

$$\ddot{X}^i = \frac{4}{\nu^2} \{\{X^i, X^j\}, X^j\}. \tag{3.4}$$

(For references, explanation of notation and further review of this topic we refer to the article [**13**].)

In the regularization procedure of Goldstone and Hoppe [**14**] the X^i coordinate functions are replaced by finite size matrices, Poisson brackets by matrix commutators, and the integration over the membrane's surface by suitably normalized traces. The regularized Hamiltonian and the equations of motion are:

$$H = \frac{1}{2\pi l_p^3} \operatorname{Tr} \left(\frac{1}{2} \dot{\mathbb{X}}^i \dot{\mathbb{X}}^i - \frac{1}{4} [\mathbb{X}^i, \mathbb{X}^j][\mathbb{X}^i, \mathbb{X}^j] \right), \tag{3.5}$$
$$\ddot{\mathbb{X}}^i + [[\dot{\mathbb{X}}^i, \dot{\mathbb{X}}^j], \dot{\mathbb{X}}^j] = 0, \qquad [\dot{\mathbb{X}}^i, \mathbb{X}^i] = 0,$$

where the $\mathbb{X}^i$ coordinates are now Hermitian matrices.

This reformulation requires orientable surfaces. Nevertheless, this procedure can be extended to nonorientable surfaces, too [**15**]. Indeed a nonorientable surface has an orientable double cover (the orientation bundle). Inside the Poisson algebra of the functions of the double cover, one can identify the sub-Lie-algebra of those functions, which generate the area-preserving transformations of the nonorientable surface [**16, 17**]. On $\mathbb{RP}^2$ one obtains the $USp(N)$ Lie-algebra.

3 The unoriented membrane

The surface of the membrane has no intrinsic orientation. Indeed, the Hamiltonian (3.2) is invariant against the flip of the sign of the symplectic form ω of the Poisson bracket. This fact suggest that it might be possible to rewrite the expressions (3.2) and (3.3) with no reference to the sign of ω. Since the correspondence between (3.2) and (3.3) uses the replacement of the commutative algebra of the X^i functions by the noncommutative $\mathbb{X}^i$ matrices via deformation quantization, it is quite reasonable to search for a deformed algebraic structure which does not depend on the sign of ω. Let us recall that on $\mathbb{R}^2$ with $\omega = dx \wedge dy$ the deformed Moyal product look like

(3.6)
$$f *_h g = fg + \frac{ih}{2}(f_x g_y - f_y g_x) + \frac{h^2}{8}\left[f_{xy}g_{xy} - \frac{1}{2}(f_{xx}g_{yy} + f_{yy}g_{xx})\right] + \dots .$$

The second order term is symmetric with respect the exchange of the x and y variables, so it makes no reference to the orientation of the surface. Exactly this term is the first nontrivial term in the Jordan product of f and g

$$(3.7)\quad f \circ_h g = \frac{1}{2}(f *_h g - g *_h f) = fg + \frac{h^2}{8}\left[f_{xy}g_{xy} - \frac{1}{2}(f_{xx}g_{yy} + f_{yy}g_{xx})\right] + \dots ,$$

so we try to rewrite (3.3), (3.4) with the help of anticommutators (Jordan products) instead of commutators.

The Hamiltonian can be rewritten using the following calculation:

$$(3.8)\qquad \begin{aligned} \mathrm{Tr}\,([\mathbb{X},\mathbb{Y}][\mathbb{X},\mathbb{Y}]) &= 2\,\mathrm{Tr}(\mathbb{X}\mathbb{Y})^2 - 2\,\mathrm{Tr}(\mathbb{X}^2\mathbb{Y}^2), \\ \mathrm{Tr}(\mathbb{X}\circ\mathbb{X})\circ(\mathbb{Y}\circ\mathbb{Y}) &= \mathrm{Tr}(\mathbb{X}^2\mathbb{Y}^2), \\ \mathrm{Tr}\,(\mathbb{X}\circ(\mathbb{Y}\circ(\mathbb{X}\circ\mathbb{Y}))) &= \frac{1}{2}\mathrm{Tr}(\mathbb{X}\mathbb{Y})^2 + \frac{1}{2}\mathrm{Tr}(\mathbb{X}^2\mathbb{Y}^2) \end{aligned}$$

(here $\circ$ is the usual Jordan product of matrices $\mathbb{X}\circ\mathbb{Y} = (\mathbb{X}\mathbb{Y} + \mathbb{Y}\mathbb{X})/2$).

Consequently

(3.9)
$$H = \frac{1}{2\pi l_p^3}\,\mathrm{Tr}\,\left(\frac{1}{2}\dot{\mathbb{X}}^i \circ \dot{\mathbb{X}}^i - \mathbb{X}^i \circ (\mathbb{X}^j \circ (\mathbb{X}^i \circ \mathbb{X}^j)) + (\mathbb{X}^i \circ \mathbb{X}^i) \circ (\mathbb{X}^j \circ \mathbb{X}^j)\right).$$

The accelerations $\ddot{\mathbb{X}}^i$ of the matrix coordinates are given by double commutators (3.4), so it can be expressed by with the help of the associator of the Jordan algebra:

$$(\mathbb{X}, \mathbb{Y}, \mathbb{Z}) = (\mathbb{X} \circ \mathbb{Y}) \circ \mathbb{Z} - \mathbb{X} \circ (\mathbb{Y} \circ \mathbb{Z}) = \frac{1}{4}[\mathbb{Y}, [\mathbb{X}, \mathbb{Z}]], \tag{3.10}$$

which identity gives the following equation of motion

$$\ddot{\mathbb{X}}^i = 4(\mathbb{X}^i, \mathbb{X}^j, \mathbb{X}^j) = [\mathbb{X}^j, [\mathbb{X}^i, \mathbb{X}^j]]. \tag{3.11}$$

The substitution of double commutators by associators occurs in almost all papers on the Jordan algebraic reformulation of quantum mechanics. Our next task is to express the constraints $[\dot{\mathbb{X}}^i, \mathbb{X}^i] = 0$ with the Jordan product of matrices. This is obviously impossible directly. The best we can do is to require that

$$4(\dot{\mathbb{X}}^i, \mathbb{U}, \mathbb{X}^i) = [\mathbb{U}, [\dot{\mathbb{X}}^i, \mathbb{X}^i]] = 0 \tag{3.12}$$

for any matrix $\mathbb{U}$. Since only the multiples of the identity matrix commute with all the other matrices, this equation implies that $[\dot{\mathbb{X}}^i, \mathbb{X}^i] = c \cdot 1$, but it is well known that this is impossible for finite size matrices (by taking the trace of both sides).

So we managed to rewrite the equations (3.3), (3.4) in a Jordan algebraic language. Since these equations are only finite dimensional approximations of the continuous equation (3.2), we expect that a similar procedure can be repeated for (3.2), at least up to the leading orders of a deformation parameter. For this purpose, we would like to express $\int d\sigma^2 \{f, g\}^2$ with the help of the Jordan multiplication $\circ_h$. This is possible in the sense that the following identity holds on $\mathbb{R}^2$

$$\begin{gathered}\{f, g\}^2 = (f_x g_y - f_y g_x)^2 = \\ -\frac{4}{h^2}\left[f \circ_h (g \circ_h (f \circ_h g)) - (f \circ_h f) \circ_h (g \circ_h g)\right] + O(h^2) + \alpha_x + \beta_y\end{gathered} \tag{3.13}$$

where the explicit forms of α and β are

$$\begin{aligned}\alpha &= f_x(f g_y^2 - 2 f_y g g_y) + g_x(2 f_y^2 g - f f_y g_y), \\ \beta &= f_y(f g_x^2 - 2 f_x g g_x) + g_x(2 f_x^2 g - f f_x g_x).\end{aligned} \tag{3.14}$$

The verification of these formulas consist of a fairly direct but quite long calculation which can be easily performed by a symbolic algebra package. It is amusing to compare this to the simplicity of the matrix version (3.8), (3.9) of this identities. Since the integral of the $\alpha_x + \beta_y$ term is zero if f and g has compact support (or the integration is done over a compact closed surface), (3.2) can be expressed with the help of the deformed Jordan multiplication up to terms of order $O(h^2)$.

Next we would like to see what sort of Jordan algebra can be associated to a nonorientable surface. We treat here only the simplest case of the real projective plane $\mathbb{RP}^2$, but we believe that the same conclusion would be true for the other nonorientable surfaces, too. (It was demonstrated in **[18]** that the Goldstone-Hoppe regularization procedure is applicable for higher genus orientable surfaces.)

Let us recall that the Goldstone-Hoppe construction on the unit sphere S^2 uses the correspondence

$$x_i \longleftrightarrow \frac{2}{N} J_i, \qquad \{x_i, x_j\} = \epsilon_{ijk} x_k, \qquad -i[J_i, J_j] = \epsilon_{ijk} J_k, \tag{3.15}$$

where the $x_{1,2,3}$ are the three dimensional coordinates of S^2 satisfying $x_1^2 + x_2^2 + x_3^2 = 1$, while $J_{1,2,3}$ are the Hermitian generators of the N dimensional irreducible representation of $SU(2)$. As an algebra, these matrices generate $Mat_N(\mathbb{C})$, while as a (unital) Jordan algebra they generate the selfadjoint part of $Mat_N(\mathbb{C})$ which is denoted by $H_N(\mathbb{C})$. This is probably well known for Jordan algebrists, but with the help of the standard representation of $SU(2)$

$$\begin{aligned} N = 2j + 1, \qquad & J_3|m\rangle = m|m\rangle, \\ J_+|m\rangle &= \sqrt{j(j+1) - m(m+1)}|m\rangle, \\ J_-|m\rangle &= \sqrt{j(j+1) - m(m-1)}|m\rangle, \\ J_1 = \frac{J_+ + J_-}{2}, \qquad & j_2 = \frac{J_+ - J_-}{2i}. \end{aligned} \tag{3.16}$$

this can be demonstrated quite explicitly. Let us denote by e_{ij} the matrix with zero entries except at $(e_{ij})_{ij} = 1$. Then e_{ii} can be written as a suitable polynomial of J_3, since J_3 is diagonal with different diagonal entries. Furthermore

$$\begin{aligned} (e_{ii} \circ J_1) \circ e_{i+1,i+1} &= \frac{1}{4}(J_1)_{i,i+1}(e_{i+1,i} + e_{i,i+1}), \\ (e_{ii} \circ J_2) \circ e_{i+1,i+1} &= \frac{1}{4}(J_1)_{i,i+1}(-ie_{i+1,i} + ie_{i,i+1}). \end{aligned} \tag{3.17}$$

The successive Jordan products of these types of matrices generate the whole of $H_N(\mathbb{C})$. (For example

$$(e_{i+1,i} + e_{i,i+1}) \circ (-ie_{i+1,i+2} + ie_{i+2,i+1}) = \frac{1}{2}(-ie_{i,i+2} + ie_{i+2,i}), \tag{3.18}$$

etc.)

Now let us turn our attention to the case of $\mathbb{RP}^2$. This projective plane is the quotient of S^2 by the antipodal map $(x_1, x_2, x_3) \to (-x_1, -x_2, -x_3)$, so we would like to keep those elements of the Jordan algebra generated by $J_{1,2,3}$ which are invariant with respect to the substitution $J_{1,2,3} \to -J_{1,2,3}$. This part can be generated by the matrices $\{J_i \circ J_k,\ i,k = 1,2,3\}$. We restrict ourself to the very simple case when N is odd, so J_i represent $SO(3)$. In this case J_i can be chosen to be purely imaginary and antisymmetric, so $J_i \circ J_k$ is a symmetric real matrix. The Jordan algebra generated by these matrices is formally real, i.e. $\sum_i a_i^2 = 0$ implies $a_i = 0$. Such unital and finite dimensional Jordan algebras are direct sums of simple ones **[19]** (p.72). If this direct sum were nontrivial, or the generated algebra were realized as a diagonal embedding of for example $H_{N/2}(\mathbb{R})$ into $H_N(\mathbb{R})$, that would implicate the existence of a nontrivial decomposition of the vector space $\mathbb{C}^N$ into the direct sum of vector spaces, with invariant factors with respect the action of $SO(3)$ (since the linear span of the generating set $\{J_i \circ J_k,\ i,k = 1,2,3\}$ is also invariant against $SO(3)$), but this would contradict to the irreducibility of the representation. So we conclude that the Matrix theoretic description of a membrane with topology of the real projective space requires the use of the Jordan algebra of real symmetric matrices. This result is in sharp contrast compared to the construction of **[15]**, where the Lie algebra of $USp(N)$ was used, since the closest Jordan algebraic relative of $USp(N)$ is $H_N(\mathbb{H})$, i.e. the set of selfadjoint quaternionic matrices.

4

Finite groups, semigroups and the Quantum Yang-Baxter Equation

1 Introduction

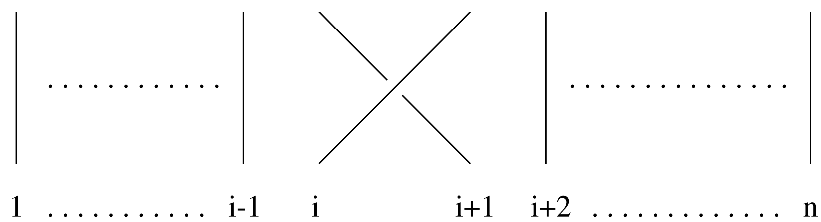
An important development of the mathematics of the last two decades was the discovery of the connections between several fields of mathematics and physics, like the areas of operator algebras [21], Hopf algebras (quantum groups) [1], three dimensional quantum field theory [22], and two dimensional statistical physics [20]. In this chapter we study some aspects of the relationship between statistical physics and the representation theory of the braid group. Braid theory provides provides an algebraic description of knots. The solvable models of two dimensional statistical physics often provide in a natural way representations of the braid group. In the next section we briefly describe this connection.

A very simple solvable system is the two dimensional lattice gauge theory. Since there are no propagating local degrees of freedom, it is almost trivial. Nevertheless, there are quite a few interesting results were obtained in this topic in recent years [24, 25]. At any rate, one can derive some simple representations of the braid relations of Artin with the help of these systems. Unfortunately, these representations are noninvertible, so they cannot represent the whole braid group. However, with a slight modification of the theory this problem can be resolved [30]. We describe this in the third section.

Two dimensional lattice gauge theory can be built on semigroups, too, if the semigroup has an involution wich agrees with the invers on invertible elements. These systems, unlike their group theory counterparts, are not automatically solvable. In the fourth section we determine the conditions of solvability in some simple cases [**30**].

2 On the representation theory of the braid groups

The Braid group B_n consisting of n strings can be described in the following manner [**21**]: Let us fix n points in a plane of $\mathbb{R}^3 = \{(x, y, z)\}$ eg. $\{(i, 0, 0),\ i = 1 \dots n\}$ and connect them with upward moving, nonintersecting curves with the points $\{(i, 0, 1)\}$. The elements of B_n are the equivalence calsses of these string configurations, where two configurations are equivalent if the are continously deformable into each other while preservindg the conditions on the curves. Multiplication is defined by the concatenation of the configurations. B_n is genarated by the $\{\sigma_i,\ i = 1 \dots$
$n - 1\}$ generators, where σ_i is the following braid:

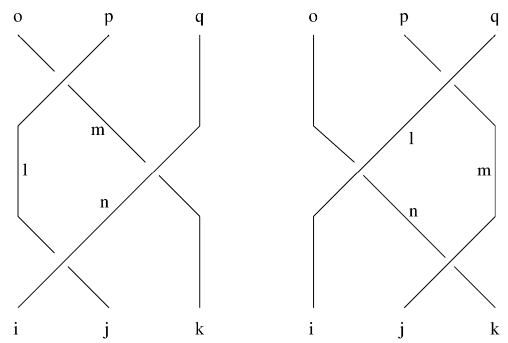


resolved.

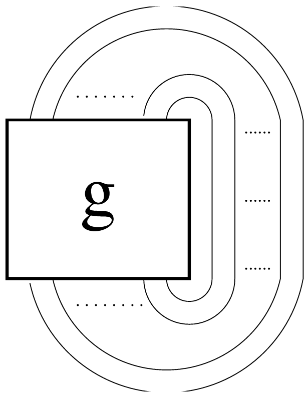
These generators satisfy the braid reletions of Artin:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \text{ ha} |i - j| > 1,$$

as the following figure clearly shows:



Artin proved that every knot can be realised as a closure of a $g \in B_n$ barid.



In principle the construction of the representations of B_n is a distinct problem for each n, but there are two uniform method which works independently of n.

As an example for the first class, consider the Burau representation [**21**]:

$$\sigma_i \longrightarrow \begin{pmatrix} 1 & & & & & & 0 \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & 1-t & 1 & & \\ & & & t & 0 & & \\ & & & & & \cdot & \\ & & & & & & \cdot \\ 0 & & & & & & 1 \end{pmatrix}$$

where the $1-t$ element occupies the (i,i) position. The representation acts on the direct sum $W = \mathbb{R} \oplus \ldots \oplus \mathbb{R}$. The verification of the braid reletions is reduced in this case to a 3×3 matrix calculation.

Instead the direct sum of vector spaces one can try to use direct product, too. So one can try to represent B_n on the vector space

$$W = V \otimes V \otimes \ldots \otimes V = V^{\otimes n}.$$

We require that the structure of σ_i is

$$\sigma_i = 1 \otimes \ldots \otimes 1 \otimes R \otimes 1 \otimes \ldots \otimes 1,$$

$$R : V \otimes V \to V \otimes V.$$

In this case the braid relations are reduced to an identity on operators acting on $V \otimes V \otimes V$:

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23},$$

where R_{12} acts nontrivially on the first two factors of $V \otimes V \otimes V$, etc. In the component notation

$$R(e_i \otimes e_j) = R_{ij}^{kl} e_k e_l$$

this equation reads as

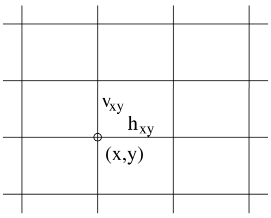
$$R_{ij}^{ln} R_{nk}^{mq} R_{lm}^{op} = R_{jk}^{nm} R_{in}^{ol} R_{lm}^{pq}.$$

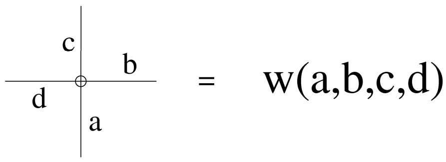
Yang and Baxter obtained a similar, but more general equation during their study of integrable two dimensional statistical physical systems:

$$R_{12}(\mu) R_{23}(\mu + \lambda) R_{12}(\lambda) = R_{23}(\lambda) R_{12}(\mu + \lambda) R_{23}(\mu).$$

This equation is the Quantum Yang-Baxter Equation (QYBE). Here the R matrix depends on a spectral parameter, but in many cases in the limit $\lambda = \mu = \infty$ it reduces to the equation without spectral paramater. This equation emerges in the sudy of solvable vertex models. The definition of theses models is the following:

Let us take a two dimensional square lattice.





To describe a configuration of the system we assign an element to each edge form a prescribed finite set $\{1 \ldots l\}$. Then we assign to each vertex (x, y) of the lettice a number $w_{xy}(\lambda) = w(v_{x,y-1}, h_{x,y}, v_{x,y}, h_{x-1,y} | \lambda)$, where w is a function (which might depend on the spectral parameter λ) $w : \{1 \ldots l\}^4 \to \mathbb{C}$. If we intend to stay within the boundaries of statistical physics, then w should take its values in $\mathbb{R}^+$. Finally we assign to the configuration the weight $\prod_{xy} w_{xy}$. In statistical physics, this describes the probability of the configuration. The basic problem of this theory is the computation of the partiton sum

$$Z(\lambda) = \sum_{\text{confs.}} \prod_{xy} w(v_{x,y-1}, h_{x,y}, v_{x,y}, h_{x-1,y} | \lambda)$$

Yang and Baxter recognised that the computation of Z is possible when the matrix

$$R_{ab}^{cd}(\lambda) = w(a, b, c, d | \lambda)$$

satisfies the QYBE. The complete classification of its solution is achieved only in the $dim\, V = 2$ case [**23**]. In the next sections we present some special solutions based on the two dimensional lattice group and semigroup gauge theories.

3 Finite groups and solutions of the Quantum Yang-Baxter Equations

The solvability of many 2d lattice statistical models is closely connected to the Quantum Yang-Baxter equation (QYBE) [**26**, **27**]. Solutions of the QYBE provide the weight functions of vertex models.

Probably the most simple 2d integrable system is (lattice) gauge theory. The weights of the field configurations around a plaquette (Fig.1a) satisfy the QYBE. (The gauge group is assumed to be finite.)

$$w(a,b,c,d) = R^{d^{-1},c^{-1}}_{a,b}(\lambda) = \sum_{r\in R(G)} \lambda^r \chi_r(abcd) \tag{4.1}$$

$$\sum_{g,h,i\in G} R^{g^{-1},i}_{a,b}(\lambda) R^{h,d}_{i^{-1},c}(\mu) R^{f,e}_{g,h^{-1}}(\nu) = \sum_{g,h,i\in G} R^{h^{-1},g}_{b,c}(\nu) R^{f,i^{-1}}_{a,h}(\mu) R^{e,d}_{i,g^{-1}}(\lambda). \tag{4.2}$$

($a, b, c, \ldots$ are elements of the group G, χ_r is the character of irreducible representation $r \in R(G)$. [**28**])

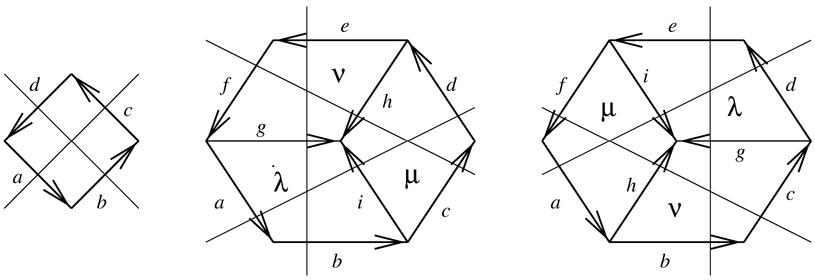


Fig. 1a **Fig. 1b** **Fig. 1c**

Equation (4.2) is satisfied since since both side depend only on the holonomy $abcdef$ around the three plaquettes, which is the same on Fig.1b and Fig.1c. A more direct proof is based on the character identity

$$\sum_{x\in G} \chi_r(ax)\chi_s(x^{-1}b) = \frac{|G|}{d_r}\delta_{rs}\chi_r(ab), \tag{4.3}$$

where d_r is the dimension of the representation r. Using (4.3) three times, both sides of (4.2) evaluate to

$$\sum_{r\in G} \lambda^r \mu^r \nu^r \frac{|G|^3}{d_r^2}. \tag{4.4}$$

A slight modification of (4.2) also satisfies QYBE:

(4.5)

$$w(a,b,c,d) = R_{a,b}^{d^{-1},c^{-1}}(\lambda) =$$

$$\sum_{r\in R(G)} \left(\lambda_0^r \chi_r(abcd) + \lambda_1^r \big(\chi_r(bd) + \chi_r(ac)\big) \right) + \sum_{t,u\in R(G)} \lambda_2^{tu} \chi_t(bd)\chi_u(ac),$$

where $\lambda_2^{tu} = \lambda_2^{ut}$. We omit the proof, since it is a lengthy, but straightforward application of (4.3).

If G is a compact Lie-group, (4.5) can be obtained as weights of the lattice version of the continuous 2d Lie-algebra valued vector field model with action

$$S = \sum_{a=1}^{\dim G} \left\{ \frac{1}{2} {F_{xt}^{a}}^{2} + \alpha\left[(\partial_x A_t^a)^2 + (\partial_t A_x^a)^2\right] + \beta(\partial_x A_t^a)^2(\partial_t A_x^a)^2 \right\}, \tag{4.6}$$

where A_x^a, A_t^a are the spatial end the temporal components of the vector field. (The Killing-metric on G is assumed to be $\delta_{a,b}$.) Since (4.6) is a continuous version of an integrable quantum system, it is reasonable to believe that the classical Euler-Lagrange equations are also integrable ones. The transfer matrices of the lattice system commute for different α and β parameters. However, the classical system is constrained since A_t is not dynamical. The allowable initial conditions live on different constraint surfaces for different parameters, so it is not quite clear to us that the integrability of the lattice version really implies the integrability of its classical version.

After this short digression we return to the lattice world and present another modification of lattice gauge theory. The weight of a field configuration around a plaquette is given by

$$w(a,b,c,d) = R_{a,b}^{d,c} = \sum_{\{\sigma_x=\pm 1\}} \sum_{r\in R(G)} \lambda^r \chi_r(a^{\sigma_a} b^{\sigma_b} c^{\sigma_c} d^{\sigma_d}). \tag{4.7}$$

As it does not matter if the variable assigned to a link is g or g^{-1} the set of link variables are the equivalence classes $\widetilde{G} = G/\{g \sim g^{-1}\}$. For this weight system QYBE does not hold automatically. Summation over the variables g, h, i generates terms like (on Figure 1b):

$$\sum_{g\in G} \chi_r(a^{\sigma_a} b^{\sigma_b} i^{\sigma_i} g)\chi_s(g h^{\sigma_h} e^{\sigma_e} f^{\sigma_f}) = \frac{|G|}{d_r} \delta_{r\bar{s}} \chi_r(a^{\sigma_a} b^{\sigma_b} c^{\sigma_c} f^{-\sigma_f} e^{-\sigma_e} h^{-\sigma_h}). \tag{4.8}$$

($\bar{s}$ is the complex conjugate of the representation s.) Since the cyclic order of the variables a, b, f, e would change a different way on Fig.1c, QYBE is not necessarily satisfied. There are two ways to avoid this problem. The first is to require that if $\lambda_r \neq 0$ then $\lambda_{\bar{r}} = 0$. Unfortunately, in this case weights of variable configurations are not real, so they are not weights of a statistical mechanical system. The second method is to use abelian groups, so the order of group elements is irrelevant.

4 Phase structures

In this section we investigate the ground state structure of these models. Two dimensional lattice theory reduces to a collection of one-dimensional systems in the gauge where all the vertical links are equal to e, so its phase structure is trivial. Although the ground state degeneracy is somewhat lifted in model (4.5) by the extra λ_1, λ_2 terms, the number of ground states is still infinite, since if $a_i, b_i \in G_A$, then configuration on Fig.2a has the same weight as the configuration where all the link variables are equal to e. We assume that $w(a, b, c, d)$ in (4.5) is real and attains its maximum iff

$$abcd = ac = bd = e, \tag{4.9}$$

so configurations with unit holonomy around plaquettes and opposite sides are preferred. We examine the structure of the Gibbs distribution associated to the weights in the low-temperature limit (i.e. when $w(a, b, c, d) \ll w(e, e, e, e)$ if (4.9) does not hold). Let us imagine that the boundary links of a very large rectangular region of area $\mathcal{A}$ are set as on Fig.2a.

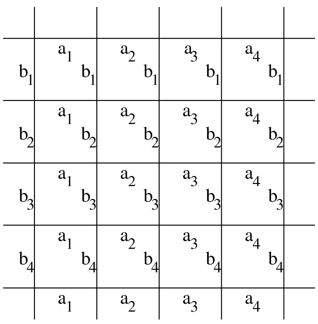


Figure 2.a

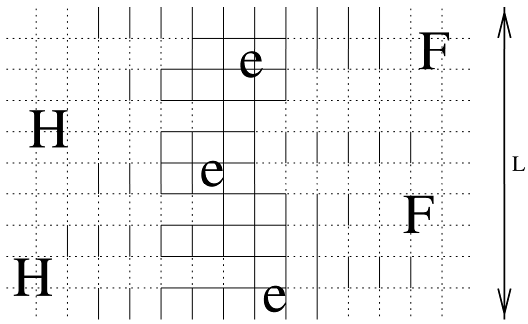


Figure 2.b

We should not expect even at quite low temperature that the horizontal links in a vertical column (or vice versa) are almost all the same, since if they are set to another group element $a \in G_A$ in a column of l consecutive plaquettes, then the weights associated to the plaquettes are nonmaximal only at the top and the bottom of the column, so the 'energy-penalty' is independent of l. So the horizontal links of a column (or vertical links of a row) are typically equal if they are closer to each other then a certain temperature dependent length scale l_0. At this scale the energetic suppression is balanced by the entropic factor l_0. Since the links has $|G_A|$ possible values in such configurations, the entropy per plaquette is approximately

$$\frac{1}{\mathcal{A}} \log \left(|G_A|^{2\mathcal{A}/l_0} \right) . \tag{4.10}$$

Consequently $|G_A|$ should be maximal, otherwise the state is at most a metastable one. So the non-uniqueness of low-temperature Gibbs-states is expected only if G contains several maximal abelian subgroups.

Let suppose that H and F two maximal abelian subgroups of G. For systems similar to the Ising-model the proof of the existence of multiple pure Gibbs-states requires several ingredients [**29**]:

- The system must possess several ground states. This condition is true in our case (H and F).
- The extra energy of a contour separating two ground states should be proportional to the length of the contour. This condition is also satisfied, since the number of plaquettes with non-maximal weights on Fig.2b is proportional to L.
- The number of contours of length L should be less then $e^{c_1 L}$ for some c_1 constant.

Unfortunately, the last condition fails. The reason of this failure is that the unit e belongs to both F and H. The number of contours separating the F and H phases is approximately $e^{c_2 l_e L}$, where l_e is the typical width of the padding region separating the F and H phases, where the links are equal to e (solid links on Fig.2b). Nevertheless, it is still reasonable to conjecture that such configurations are entropically suppressed. In the padding region the links can take only one value, while if they were in F or H then they could take $|G_A|$ values, which would give an extra $|G_A|^{c_3 L}$ factor in the partition sum (which is much larger then the factor 2 corresponding to the choice between F and H). If these speculations are true, then the system must exhibit a phase transition at the temperature where the Gibbs-state becomes nonunique.

In contrast to the previous cases, a system with weights (4.7) has unique ground state if and only if there is no involution $p = p^{-1}$ in G, where we assume that G is abelian and $\tilde{w}(g) = \sum_r \lambda^r \chi_r(g)$ is minimal at $g = e$. If there is an involution $p \in G$, then the configurations where a_i and b_i are either e or p have the same weight as the configuration where all the link variables are equal to e. The absence of involutions in an abelian group implies to that $|G|$ is odd. If $\tilde{w}(g)$ has minimum at some $g \neq e$, then the ground state is infinitely degenerate. Indeed, let us mark a subset of the links so that each plaquette has exactly one marked side, and set the marked link variables to g (or g^{-1}) and set to e the rest of the links. Such configuration is ground state. Since the marking of the links can be done in many ways, the ground state is highly degenerate. If $\tilde{w}(g)$ has two minimums at g_1 and g_2, then the marked links can be set either to g_1 or g_2 in a completely random manner, so the Gibbs-state remains unique even in this case. Consequently the phase structure is trivial.

5 On the solvability of semigroup gauge theories

Let us recall again that the link elimination method of Migdal is based only on the single character identity

$$\int dx\, \chi_r(ax)\chi_s(x^{-1}b) = \delta_{rs}\frac{\chi_r(ab)}{\dim r}, \tag{4.11}$$

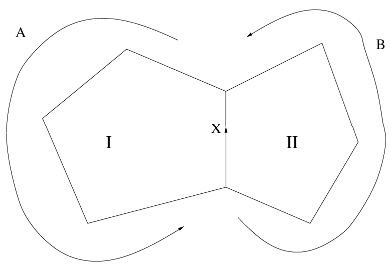
where a, x, b are elements of a compact gauge group G, dx is the normalised Haar measure on G, while r and s are irreducible representations of dimensions $\dim r$ and $\dim s$ with characters χ_r and χ_s. For a finite group this identity can be written as

$$\sum_{x\in G} \chi_r(ax)\chi_s(x^{-1}b) = \frac{|G|}{\dim r}\delta_{rs}\chi_r(ab). \tag{4.12}$$

With the help of this identity, in the partition function the summation over a link variable joining two plaquettes can be performed. Let the weights corresponding to the configuration on the figure be

$$\exp -\beta(S_I(ax) + S_{II}(x^{-1}b)) = w_I(ax)w_{II}(x^{-1}b), \tag{4.13}$$

where w_I, w_{II} are conjugation invariant functions on G.



Then the summation over x gives

$$\begin{aligned}\sum_{x\in G} w_I(ax)w_{II}(x^{-1}b) &= \left(\sum_{r\in\hat{G}} \lambda_r^I\chi_r(ax)\right)\cdot\left(\sum_{s\in\hat{G}} \lambda_s^{II}\chi_s(x^{-1}b)\right)\\ &= |G|\sum_{r\in\hat{G}} \frac{\lambda_r^I\lambda_r^{II}}{\dim r}\chi_r(ab) = \tilde{w}(ab),\end{aligned} \tag{4.14}$$

so the end result depends only on the holonomy around the union of the I and II plaquettes. By the recursive application of this calculation the model can be solved on any planar graph drawn on a surface.

Lattice gauge theory can be generalised to the class of semigroups $\tilde{G}$ possessing an involution i with $i(g) = g^{-1}$ for the invertible elements and $i(rs) = i(s)i(r)$ for $\forall r, s \in \tilde{G}$. In these models, if s is associated to the oriented link $\overrightarrow{l}$, then to $\overleftarrow{l}$ $i(s)$ is associated. The concept of holonomy around a plaquette can be defined just as in ordinary lattice gauge theory by the replacement of inverses by involutions. The Boltzmann weight of a configuration around a plaquette is required to be invariant against gauge transformations by the group of invertible elements of $\tilde{G}$. Several aspects of semigroup gauge theories were extensively studied, see for example **[31]** and its references.

Are these models solvable? The answer is no in general, since the argument at the beginning of the paper (the gauging out of the variables attached to vertical links) does not work in this case as only the invertible elements can be transformed to the identity. Nevertheless, the formula (5) might be true in some cases, at least for suitably chosen weight systems. This would ensure the solvability of the model.

In the next section we demonstrate that this expectation is fulfilled for the fairly simple type of semigroups G_0, where G_0 is obtained from a finite group G by adjoining a zero element 0 with the multiplication rules $0g = g0 = 00 = 0$ for any $g \in G$. The involution i leaves the zero unchanged.

After these studies we investigate the same problem in the case of the semigroup of partial permutations of two elements with the help of symbolic algebra calculations.

6 Semigroups with a zero element

Now let us check if (5) is satisfied for G_0. Let $w : G_0 \to R$ be a conjugacy invariant function, i.e. $w(c) = w(gcg^{-1})$ for $\forall g \in G$. Then w can be expanded as a linear combination of the characters $\chi_s,\ \ s \in \hat{G}$ of the irreducible representations of G plus an additional term χ_0 for the zero element

$$w(c) = \sum_{s \in \hat{G}} \lambda_s \chi_s(c) \ + \ \lambda_0 \chi_0(c) \tag{4.15}$$

where $\chi_s(0) = 0$, and $\chi_0(0) = 1,\ \chi_0(g) = 0$ for $\forall g \in G$.

As the next step, we calculate

$$\sum_{x\in G_0} w(ax)\tilde{w}(i(x)b) = \tag{4.16}$$

$$\sum_{x\in G_0}\left[\sum_{s\in\hat{G}}\lambda_s\chi_s(ax) \ + \ \lambda_0\chi_0(ax)\right]\left[\sum_{r\in\hat{G}}\tilde{\lambda}_r\chi_r(i(x)b) \ + \ \tilde{\lambda}_0\chi_0(i(x)b)\right].$$

For the calculation of the sum we distinguish three cases:

$$1)\quad a,b\in G,\qquad 2)\quad a=0, b\in G\qquad 3)\quad a=b=0. \tag{4.17}$$

1) The expression (4.16) evaluates to

$$|G|\sum_{s\in\hat{G}}\frac{\lambda_s\tilde{\lambda}_s}{\dim s}\chi_s(ab) \ + \ \lambda_0\tilde{\lambda}_0, \tag{4.18}$$

where the first term calculates the sum over the group elements, while the second is responsible for the 0.

2) Now $\chi_s(ax) = 0$, $\chi_0(ax) = 1$ independently of x, so we obtain

$$\left[\sum_{x\in G}\lambda_0\cdot(\sum_{s\in\hat{G}}\hat{\lambda}\chi_s(x^{-1}b)\right] + \lambda_0\tilde{\lambda}_0 = \lambda_0\tilde{\lambda}_1|G| + \lambda_0\tilde{\lambda}_0, \tag{4.19}$$

since

$$\sum_{x\in G}\chi_s(x^{-1}b) = \sum_{x\in G}\chi_1(x)\chi_s(x^{-1}b) = |G|\delta_{1s}\frac{\chi_1(b)}{1} = |G|\delta_{1s}, \tag{4.20}$$

where χ_1 is the identically 1 character of the identity representation of G.

3) This is the easiest case, since now only the $\chi_0(0x)$ and the $\chi_0(i(x)0)$ terms are nonzero for any $x\in G$, so the result is

$$(|G|+1)\lambda_0\tilde{\lambda}_0. \tag{4.21}$$

Our end result is that

$$\sum_{x\in G_0} w(ax)\tilde{w}(i(x)b) = \begin{cases} |G|\sum_{s\in\hat{G}}\frac{\lambda_s\tilde{\lambda}_s}{\dim s}\chi_s(ab) \ + \ \lambda_0\tilde{\lambda}_0, & \text{if } a,b\in G,\\ \lambda_0\tilde{\lambda}_1|G| + \lambda_0\tilde{\lambda}_0, & \text{if } a=0, b\in G,\\ (|G|+1)\lambda_0\hat{\lambda}_0. & \text{if } a=b=0.\end{cases} \tag{4.22}$$

We intend to write this expression in the form $\bar{w}(ab)$ for a conjugacy invariant function $\bar{w}$. Since in the second and the third cases $ab = 0$, this is possible only if $\tilde{\lambda}_1 = \tilde{\lambda}_0$ (and $\lambda_1 = \lambda_0$ from the $a \in G, \; b = 0$ case). Under this condition (4.22) is equal to

(4.23)

$$\hat{w}(ab) = \begin{cases} \sum_{s\in\hat{G}, s\neq 1} |G| \frac{\lambda_s \tilde{\lambda}_s}{\dim s} \chi_s(ab) \; + \; (\lambda_1\tilde{\lambda}_1 + \lambda_0\tilde{\lambda}_0)\chi_1(ab) & \text{if} \quad a, b \in G, \\ (|G|+1)\lambda_0\tilde{\lambda}_0 & \text{if} \quad ab = 0. \end{cases}$$

$$(4.24) \quad = \sum_{s\in\hat{G}, s\neq 1} |G| \frac{\lambda_s \tilde{\lambda}_s}{\dim s} \chi_s(ab) \; + \; (\lambda_1\tilde{\lambda}_1 + \lambda_0\tilde{\lambda}_0)\chi_1(ab) + (|G|+1)\lambda_0\tilde{\lambda}_0.$$

So we see that if the statistical summation is performed over a link joining two plaquettes, then the coefficients of the character expansions of the effective weights of the joint plaquette is obtained by the rule:

(4.25)

$$(\lambda_0, \lambda_1, \lambda_s), \quad (\tilde{\lambda}_0, \tilde{\lambda}_1, \tilde{\lambda}_s) \longrightarrow \left((|G|+1)\lambda_0\tilde{\lambda}_0, (|G|+1)\lambda_1\tilde{\lambda}_1, |G| \frac{\lambda_s \tilde{\lambda}_s}{\dim s} \right)$$

if $\lambda_0 = \lambda_1$ and $\tilde{\lambda}_0 = \tilde{\lambda}_1$. Note that this condition remains true for the coefficients of the joint plaquettes, too. This ensures that the recursive elimination of the link variables can be continued indefinitely. So we conclude that the 2d lattice semigroup gauge theory of G_0 is solvable under the derived restriction on the weight system.

As a byproduct of this investigation, we obtain some simple solution of the Quantum Yang-Baxter Equation. In a previous paper **[30]** we associated some solutions of the QYBE to every finite group. Since the corresponding calculations were based on the character identity (5), the arguments presented in that paper can be entirely carry over to the case of G_0, so the following $(|G|+1)^2) \times (|G|+1)^2$ matrix solves the QYBE:

(4.26)

$$R^{i(d),i(c)}_{a,b} = \sum_{s\in\hat{G}} (\lambda_s\chi_s(abcd) + \lambda'(\chi_s(bd) + \chi_s(ac))) + \lambda_0(abcd) + \lambda'_0(\chi_0(bd) + \chi_0(ac))$$

$a, b, c, d \in G_0$, provided that $\lambda_0 = \lambda_1$ and $\lambda'_0 = \lambda'_1$.

7 The semigroup $\mathbf{PP_2}$

Our result on G_0 shows that for a semigroup the character identity (5) holds only for a restricted set of weight system w. However, we obtained only a single $\lambda_0 = \lambda_1$ condition on w, so one might suspect that semigroup gauge theories are solvable under quite mild restrictions. we study this problem in the case of the semigroup of partial permutations of two elements, which we denote by PP_2. This semigroup consists of the following matrices:

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad g_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{4.27}$$

$$g_5 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad g_6 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_7 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The multiplication is the standard matrix multiplication, while the involution is the matrix transposition. The elements g_1 and g_2 form the group $\mathbb{Z}_2$, while the rest of PP_2 is not invertible. The conjugacy classes are

$$\{g_1\}, \quad \{g_2\}, \quad \{g_3, g_4\}, \quad \{g_5, g_6\}, \quad \{g_7\}. \tag{4.28}$$

So our task is to find those weight systems

$$w_1 = w(g_1), \quad w_2 = w(g_3), \quad w_3 = w(g_3), \quad w_4 = w(g_4), \tag{4.29}$$

$$w_5 = w(g_5), \quad w_6 = w(g_6), \quad w_7 = w(g_7),$$

such that $w_3 = w_4$, $w_5 = w_6$ and

$$\sum_{c \in PP_2} w^{I}(ax) w^{II}(i(x)b) = w^{III}(ab). \tag{4.30}$$

We solved these equations with the help of a symbolic algebra package (MuPad). In order to simplify the calculation, we assumed first that $w^{I} = w^{II}$. Then the only nontrivial (not identically zero on the noninvertible elements or on the group elements) solution was

$$\{w^{I}_1, \ldots, w^{I}_7\} = \{w_1, w_2, \frac{w_1 + w_2}{2}, \ldots, \frac{w_1 + w_2}{2}\}. \tag{4.31}$$

Next we verified that if w^{II} has the same form with different w_1 and w_2 values, then (4.30) still held.

So we managed to construct a solvable lattice model for the semigroup PP_2. However, this solution is only a slightly disguised form of the construction of the previous section in the case of $G = \mathbb{Z}_2$. Indeed, PP_2 has a homomorphism ϕ onto $\mathbb{Z}_2 \cup 0$ with $\phi(x) = 0$ if x is not invertible. So when we check (4.30) for a weight system which is constant on the nonivertible elements, we basically substitute PP_2 with a semigroup consisting of $\mathbb{Z}_2$ plus five copies of the zero element. Since w is constant on these copies, their multiplication rule is irrelevant. The effect of the four extra copies in the computation of (4.22) is that

$$\sum_{x \in G_0 \cup \{0_1, 0_2, 0_3, 0_4, 0_5\}} w(ax)\tilde{w}(i(x)b) \tag{4.32}$$

$$= \begin{cases} |G| \sum_{s \in \hat{G}} \frac{\lambda_s \tilde{\lambda}_s}{\dim s} \chi_s(ab) \ + \ 5\lambda_0\tilde{\lambda}_0, & \text{if } a, b \in G, \\ \lambda_0\tilde{\lambda}_1|G| + 5\lambda_0\tilde{\lambda}_0, & \text{if } a = 0, b \in G, \\ (|G| + 5)\lambda_0\hat{\lambda}_0. & \text{if } a = b = 0. \end{cases}$$

This expression can be rewritten in the form (4.23) if $\lambda_0 = \lambda_1$, just as in the case of G_0. As $(w_1 + w_2)/2$ is just the coefficient of the character of the identity representation of $\mathbb{Z}_2$, the weight system (4.31) is the same than the weight system of $G_0 = \mathbb{Z}_2 \cup 0$. Presumably too general conclusions should not be drawn from a single example, but we think it is reasonable to believe that the solvable weight systems for the semigroups PP_n are derivable from the weight systems of the symmetric groups S_n in a similar fashion.

This example suggest that the solvability of semigroup lattice gauge theories is less widespread than one might think. However, if we delete the group elements g_1 and g_2 from PP_2, then the result is somewhat different. We still require that the conditions $w_3 = w_4$ and $w_5 = w_6$, so the theory still possess a $\mathbb{Z}_2$ gauge group. Then a symbolic algebra calculation shows that (4.30) holds (with the modification $x \in PP_2 - \{g_1, g_2\}$) if the weight systems has one of the following forms:

$$\text{a)} \qquad w_3 = w_4, \quad w_5 = w_6, \quad w_7 = 0, \tag{4.33}$$

$$\text{b)} \qquad w_3 = w + 4 = 2w_7 - w_5, \quad w_5 = w_6. \tag{4.34}$$

Moreover, if w^{I} and w^{II} has either the form a) or b), then w_{III} has the corresponding form, too. This ensures the complete sovability of the model. In this case there is a considerable freedom in the choice of the weight system: one can choose two out of three numbers arbitrarily.

5

A modification of the Lax equation

1 Introduction

It is quite rare that an equation of classical point or continuum mechanics is solvable. The most important class of expections are the linear equations, which are the primary examples of the completely integrable systems. In the next section we present a variation of a very famous integrable system, the KdV (Kortaweg-de Vries) equation, or more precisely, the quite similar Hirota-Satsuma equations. In calssical mechanics, solvability is closely connected with the existence of conserved quantities. The extreme case is when there are n conserve quantity $I_1, \dots, I_n$ in a system with n degrees of freedom, and the Poisson brackets of the conserved quantities are zero. Some sporadic examples of this situation were found quite long ago, like the motion of a rigid body or the geodetic motion an an ellipsoid. Liouville maneged to solve the nonlinear partial differential equation

$$\phi_{tt} - \phi_{xx} = e^{\phi}.$$

These scattered results were quite ad hoc, but in 1967, Gardner, Green, Kruskal and Miura were able to solve the

$$4u_t = u_{xxx} + 6uu_x$$

KdV equation [**36**]. The discovered the connection between this equation and the Sturm-Liouville operator

$$L(t) = \partial_x^2 + u(t, x).$$

They observed that if the time development of u is governed by the KdV equation, then the 'spectrum' of L remains unchanged. This phenomena was further calrified by Peter Lax, who discovered that the KdV equation can be written as the L operator isospectral deformation:

$$u_t = \dot{L} = [L, P], \quad \text{ahol } P = \partial_x^3 + \frac{3}{4}(\partial_x u + u\partial_x).$$

This equation is the infinitesimal form of the similarity transformation

$$L \longrightarrow L \to e^{-tP} L e^{tP}.$$

If L were a finite dimensional matrix, then $I_k = \operatorname{Tr} L^k$ would be conserved, since

$$\dot{I}_k = \sum_{i=0}^{k-1} L^i (LP - PL) L^{k-i-1} = \sum_{i=0}^{k-1} L^{k-1}(LP - PL) = 0,$$

by the cyclicity of the trace operation. Adler and Manin were able to define a similar trace on the space of Volterra operators, so with the help of this scheme the conserved quantities of the KdV equations are computable [**33**].

The paper of Lax left open the question that how could one find those P operators that the $[L, P]$ commutator does not contain terms with ∂_x, so the $\dot{L}$ is a partial differential equation on u. An elegant solution for this problem was discovered by Gelfand and Dickey [**37**]. L should commute with its half integer powers, as for example

$$[L, L^{1/2}] = (L^{1/2})^2 L^{1/2} - L^{1/2}(L^{1/2})^2 = 0.$$

Now $L^{k/2}$ can be split to parts containing nonnegative and negative povers of ∂_x.

$$\begin{gathered} L^{k/2} = \\ = \partial^k + l_{k-2}^{[k/2]} \partial^{k-2} + l_{k-3}^{[k/2]} \partial^{k-3} + \ldots + l_0^{[k/2]} + l_{-1}^{[1/2]} \partial^{-1} + l_{-2}^{[1/2]} \partial^{-2} + \ldots \\ = (L^{k/2})_+ + (L^{k/2})_-. \end{gathered}$$

Then

$$0 = [L, L^{k/2}] = [L, (L^{k/2})_+ + (L^{k/2})_-] \quad \Rightarrow \quad [L, (L^{k/2})_+] = -[L, (L^{k/2})_-].$$

As $(L^{k/2})_+$ contains only the nonnegative powers of ∂, this commutator can not contain any negative powers of ∂. Moreover

$$\begin{gathered} [L, (L^{k/2})_-] = [(\partial^2 + u), l_{-1}^{[k/2]} \partial^{-1} + l_{-2}^{[k/2]} \partial^{-2} + \ldots] = \\ 2(l_{-1}^{[k/2]})_x + (\text{valami})\partial^{-1} + \ldots, \end{gathered}$$

so $[L, (L^{k/2})_+]$ can contain only the zeroth power of ∂. This means that $\dot{L} = [L, (L^{k/2})_+]$ is a partial differential equation on u. For a more detailed and precise presentation of this material see for example [**33**]. In the next section we present a slight modification of this scheme.

2 Unitary deformations and complex soliton equations

Since the Lax-equation

$$\dot{L} = PL - LP$$

is the infinitesimal form of the similarity transformation $L \to e^{tP} L e^{-tP}$, such equations leave invariant the 'spectrum' of L [**32**], ensuring the existence of nontrivial conserved quantities. In many cases this scheme leads to the complete integrability of the corresponding equation (see [**33**] for a textbook account). In the theory of integrable systems on lattices a generalization of the Lax-equation emerges:

$$\dot{L} = PL - LQ,$$

where $Q = \tau(P)$, and τ is an automorphism which commutes with the 'R-matrix' of the problem (see [**34**] for details). Another example of this modification of the Lax equation was observed by Drinfeld and Sokolov [**39**]. They noted, that the spectrum of $L_1 L_2$ is preserved by the following transformations:

$$L_1 \to e^{tP} L_1 e^{-tQ}, \qquad L_2 \to e^{tQ} L_2 e^{-tP}.$$

The infinitesimal forms or these transformations are

$$\dot{L}_1 = PL_1 - L_1 Q, \qquad \dot{L}_2 = QL_2 - L_2 P.$$

With the choice of $L_1 = \partial_x^2 + u + \phi$, $L_2 = \partial_x^2 + u - \phi$ one obtains the Hirota-Satsuma equations [**40**], [**41**]. Another example of this sort of scheme is the constrained KP (cKP) hierarchy [**42**]. Here the spectrum of $L_1^{-1} L_2$ is preserved by the

$$L_1 \to e^{tP} L_1 e^{-tQ}, \qquad L_2 \to e^{tP} L_2 e^{-tQ}$$

transformations.

The generalized Lax-equation also arises if L can be regarded as a linear operator on a Hilbert-space $\mathfrak{H}$. In this case one can try to preserve the spectrums of L^*L and LL^* instead of L's one. In fact, it is fairly natural to associate the operator L^*L to L, since every $L \in \mathbf{B}(\mathfrak{H})$ can be uniquely written as $L = U|L|$, where $|L| = (L^*L)^{1/2}$ is a positive operator, while U is a partial isometry [**35**].

The spectrum of L^*L is not preserved by general similarity transformations of L. However, the transformation

$$L \longrightarrow e^{itP} L e^{-itQ}$$

does leave it invariant if e^{itP} and e^{-itQ} are unitary, i.e. if P and Q are self-adjoint operators. Indeed

$$LL^* \longrightarrow \left(e^{itP} L e^{-itQ}\right)\left(\left(e^{-itQ}\right)^* L^* \left(e^{itP}\right)^*\right) = e^{itP} LL^* e^{-itP}.$$

These considerations suggest that it might be possible to obtain integrable equations in the following form:

$$\dot{L} = i(PL - LQ), \quad P = P^*, \quad Q = Q^*. \tag{5.1}$$

As an illustration of this method, we derive a hierarchy of complex evolution equations of Kortaweg-de Vries type [**36**].

First, let us recall the Gelfand-Dickey [**37**, **33**] construction of the KdV hierarchy. Their starting point is the Schroedinger operator

$$L = \partial^2 + u(x)$$

($\partial = \partial_x$). Its square root $L^{1/2}$ is a pseudo-differential operator

$$L^{1/2} = \partial + l_{-1}^{[1/2]} \partial^{-1} + l_{-2}^{[1/2]} \partial^{-2} + l_{-3}^{[1/2]} \partial^{-3} + \ldots,$$

where the $l_i^{[1/2]}$s are polynomials of u and its derivatives. They are recursively determined by the condition $(L^{1/2})^2 = L$. The crucial property of $L^{1/2}$ is that

$$[L, L^{1/2}] = 0$$

Then

$$0 = [L, L^{k/2}] = [L, (L^{k/2})_+ + (L^{k/2})_-] \quad \Rightarrow \quad [L, (L^{k/2})_+] = -[L, (L^{k/2})_-],$$

where $(L^{k/2})_+$ is the differential operator containing the nonnegative powers of of ∂, while $(L^{k/2})_-$ consists of terms with negative powers of ∂. Since $[L,(L^{k/2})_+]$ is a differential operator, and $[L,(L^{k/2})_-]$ cannot contain positive powers of ∂, both expressions must be polynomials of u and its derivatives, so

$$\partial_{t_k} L = \partial_{t_k} u = [L,(L^{k/2})_+] = -[L,(L^{k/2})_-]$$

is a partial differential equation for $u(x,t)$.

With some minor modification, this scheme works for the generalized Lax-equation, too. As the self-adjointness of operators has an important role, we use the self-adjoint derivation $D = i\partial$ instead of ∂. Let

$$L = D^2 + v(x)D + u(x),$$

where u and v are complex functions of x. Instead of $L^{1/2}$ (which is not self-adjoint), we would like to obtain self-adjoint pseudo-differential operators

$$A = D + a_0 + a_{-1}D^{-1} + a_{-2}D^{-2} + \dots,$$
$$B = D + b_0 + b_{-1}D^{-1} + b_{-2}D^{-2} + \dots,$$

satisfying

$$i(AL - LB) = 0.$$

So

$$AL = LB \quad \Rightarrow \quad L^{-1}AL = B = B^* = L^*AL^{-1*} \quad \Rightarrow \quad A(LL^*) = (LL^*)A,$$

which implies that

$$A = (LL^*)^{1/4} \quad \text{and} \quad B = (L^*L)^{1/4}.$$

The operator equations

$$\begin{aligned} \partial_{t_k} L \;&=\; \partial_{t_k} v D + \partial_{t_k} u \;=\; i\left\{(A^k)_+ L - L(B^k)_+\right\} \\ &= -\,i\left\{(A^k)_- L - L(B^k)_-\right\} \end{aligned} \tag{5.2}$$

generate integrable equations for $u(x,t)$ and $v(x,t)$. Note that the splitting $X = X_+ + X_-$ respects self-adjointness, i.e. $X = X^* \Rightarrow (X_+ = X_+^* \wedge X_- = X_-^*)$. In the KdV hierarchy v can be set to zero, since the term containing ∂ drops out from

$$\begin{aligned} \partial_{t_k} L \;&=\; \partial_{t_k}(\partial^2 + u) \;=\; -[(\partial^2 + u),(L^{k/2})_-] \\ &=\; -[\partial^2 + u, l_{-1}^{[1/2]}\partial_{-1} + l_{-2}^{[1/2]}\partial_{-2} + \dots]. \end{aligned}$$

This reduction does not necessarily works in our case, since

$$
\begin{aligned}
& -i\left((A^k)_-(D^2+u)-(D^2+u)(B^k)_-\right) \\
&= \Big((a_{-1}^{[k]}D^{-1}+\dots)(D^2+u)-(D^2+u)(b_{-1}^{[k]}D^{-1}+\dots)\Big) = -i(a_{-1}^{[k]} - b_{-1}^{[k]})D,
\end{aligned}
$$

so the constraints $\nu(x)=0$ can be violated by the evolution equation. Since the evolution of LL^* and L^*L have the usual Lax form

$$
\partial_{t_k}(LL^*) = i[(A^k)_+, LL^*], \quad \partial_{t_k}(L^*L) = i[(B^k)_+, L^*L],
$$

the standard machinery of the KdV equations **[33]** can be applied. For example, the integrals of motion are

$$
\int res\, A^k\, dx, \quad \int res\, B^k\, dx, \quad k = 1, 2, 3\dots,
$$

where $res\, X$ is the coefficient of D^{-1} in the pseudo-differential operator X. The fact that the equations (5.2) can be embedded into the KdV hierarchy generated by the Lax-operator

$$
L^{(4)} = D^4 + u_3 D^3 + u_2 D^2 + u_1 D1 + u_0
$$

ensures their integrability. The embedding is determined by the constraints

$$
L^{(4)} = LL^* = (D^2+\nu D+u)(D^2+D\bar{\nu}+\bar{w}) = D^4 + (\nu+\bar{\nu})D^3 + \dots.
$$

Since in the KdV hierarchy of $L^{(4)}$ the constraint $u_3=0$ is preserved by the evolution equations, the constraint $\nu+\bar{\nu}=0 \rightarrow \Re\nu=0$ can be imposed on the equations (5.2), too. However, it seems that even $\nu=0$ is compatible with (5.2) for odd k. This was verified by explicit computations for $k = 1, 3, 5$.

Now we[1] compute the explicit forms of the first few of the hierarchy (5.2). For the odd flows, we present only their constrained, $\nu=0$ form, while for the even flows, ν's value is constrained to be pure imaginary ($\nu = iw$).

(1) flow:

$$
\partial_{t_1} u = u'
$$

[1] Me and a symbolic algebra package.

(2) flow:

$$\partial_{t_2} w = 2\Im u'$$

$$\partial_{t_2} u = \frac{i}{2}\Big(w''' - w'(2w' + 2w^2 - 4u) - w(u' - \bar{u}' - w'')\Big)$$

(3) flow:

$$\partial_{t_3} u = \frac{1}{8}\Big(u''' - 3\bar{u}''' - 6uu' + 6u'\bar{u} + 12u\bar{u}'\Big)$$

(4) flow:

$$\partial_{t_4} u = \partial_{t_4} w = 0$$

(5) flow:

$$\begin{aligned}\partial_{t_5} u = \frac{1}{32}\Big\{ & -3u^{(5)} + 5\bar{u}^{(5)} + 5u'''(3u - \bar{u}) + 5\bar{u}'''(-5u + \bar{u}) \\ & + 5u'\left(-3u^2 + 6u\bar{u} + \bar{u}^2 - 3u'' - 5\bar{u}''\right) \\ & + 5\bar{u}'\left(4u^2 + 4u\bar{u} - 3u'' - 3\bar{u}''\right)\Big\}\end{aligned}$$

The adjoint operator L^* was computed with respect to the standard inner product $(\psi_1, \psi_2) = \int \bar{\psi}_1 \psi_2 \, dx$. However, it is possible to use more general inner products (not necessarily positive definite ones), like $(\psi_1, \psi_2) = \int \bar{\psi}_1 F(\psi_2) \, dx$, where F is some pseudodifferential operator (with constant coefficients). In this way, further generalizations of the KdV hierarchies can be obtained.

The operators $P = (A^k)_+$ and $Q = (B^k)_+$ in (5.1) are quite similar to each other. In fact, one is the same as the other, if (u, v) and $(\bar{u}, \bar{v})$ are swapped. This raises the possibility that the presented scheme is a subcase of the twisted Lax equation of Semenov-Tian-Shansky [**34**]. However, in his group theoretical formulation the conserved quantities were functions invariant under twisted conjugations $h \to g h^\tau g^{-1}$ (where τ is a certain fixed automorphism of a group G). If our results can be formulated in a similar framework, it is likely that the conserved quantities are such functions on a group G, which are invariant under $h \to u_1 h u_2^{-1}$ where u_1, u_2 are members of a compact subgroup of G, (so u_1 and u_2 can be represented by unitary operators). Nevertheless, there could be close connections between these frameworks.

The presented construction can be applied to other integrable systems, too. For example, a Kadomtsev-Petviashvilli type hierarchy [**38, 33**] can be derived without any difficulty. Let

$$L = D + u_0 + u_{-1}D^{-1} + u_{-2}D^{-2} + \dots,$$

$$\partial_m L = i\left(B_m L - L\bar{B}_m\right), \quad B_m = (LL^*)^m_+ \ \ \bar{B}_m = (L^*L)^m_+.$$

Then the zero-curvature condition

$$\partial_m B_n - \partial_n B_m - i[B_m, B_n] = 0$$

provides the equations of a KP type hierarchy. The proof of the zero-curvature condition is almost the same as for the standard KP equations:

$$\begin{aligned} \partial_m B_n - \partial_n B_m - i[B_m, B_n] = \\ = i\Big(\sum_{j=0}^{n-1} (LL^*)^{n-1-j}\,(\ [B_m L - L\bar{B}_m]L^* \\ + L[\bar{B}_m L^* - L^* B_m]\ )\,(LL^*)^j - (n \leftrightarrow m)\Big)_+ - [B_m, B_n] \\ = i\Big([B_m, (LL^*)^n]_+ - [B_n, (LL^*)^m]_+ - [B_m, B_n]\Big) \\ = i\left([B_n - (LL^*)^n, B_m - (LL^*)^m]_+ = [(LL^*)^m_-, (LL^*)^n_-]\right) = 0 \end{aligned}$$

It seems reasonable to believe that this equation holds for the half-integer n and/or m, too. The verification of the commutativity of the flows ∂_m is standard.

6

Coadjoint orbits of wild groups in solid-state physics

1 Introduction

Lie-groups are divided into two classes (Types I and II) according to the behaviour of their representations [43]. The unitary representations of Type I (tame) groups have essentially unique decompositions into irreducible representations, while in the case of Type II (wild) groups such decomposition can be highly nonuniqe. Finite groups, semisimple and nilpotent Lie-groups are tame, while infinite discrete groups (except those which contain an Abelian subgroup of finite index) are wild. The type of a solvable Lie-group is determined by the behaviour of its coadjoint orbits. According to a theorem of Auslander and Kostant [44], a solvable Lie-group is tame if and only if the set of its coadjoint orbits are separable and the their standard symplectic two-forms are exact. This theorem provides a fairly convenient method to prove the wildness of some solvable groups. The notation of Type I and II representations comes from the theory of von Neumann algebras. This operator algebraic aspect might be especially relevant in physical applications, where one is interested in the properties of the representations of the enveloping algebra. However, we have little to say about this topics in the present paper.

In Kirillov's book [45] two simple examples of wild solvable groups are given. These examples are not just mathematical curiosities, but they emerge naturally in the description of some quasi-periodic systems in solid-state physics. Kirillov's first example has the following physical realization:

The functions $a\cos x$, $a\sin x$, $b\cos\alpha x$, $b\sin\alpha x$, and the derivation ∂_x form a five dimensional Lie-algebra. If α is irrational, then its Lie-group is wild. These operators are the building blocks of the Hamiltonian of an electron moving a quasi-periodic cosine potential.

The Lie-algebra of the second example can be represented by operators which are necessary for the description of the motion of an electron in two dimension under the influence of periodic cosine potentials and uniform magnetic field. The corresponding group contains the magnetic translation group [**46**, **47**].

The physics of quasi-periodic systems has many characteristic features like the unusual band structure, various types of (de)localisations, etc. [**48**]. The wildness of the groups in these examples foreshadows the appearance of such features, so the theorem of Auslander and Kostant can be used to predict the qualitative nature of physical systems connected with solvable Lie-groups.

We study what happens if the magnetic translation group is extended by generators generating fluctuations of the magnetic field. In this case the conditions of tameness in the Auslander-Kostant theorem is violated only by a single exeptional coadjoint orbit. As all the other orbits satisfy the conditions of the theorem, we expect that this physical system does not exhibit the unusual phenomenas of the quasiperiodic disordered systems.

In the next section we re-present the examples of [**45**] and give physical realizations of the wild solvable groups. We also determine how the characters of the systems changes if some parameters like the magnitude of the potential and magnetic field is allowed to fluctuate. This paper is basically an extra exercise for the last section of [**45**].

2 Solvable Lie-groups in solid-state physics.

Let us first recall the notation of coadjoint orbits. Let G be a Lie-group, $\mathfrak{g}$ its Lie-algebra, and $\mathfrak{g}^*$ its dual. The coadjoint action of G on $\mathfrak{g}^*$ is defined by

$$\langle Ad_g^*\xi, Ad_g X\rangle = \langle \xi, X\rangle, \quad \xi\in\mathfrak{g}^*, X\in\mathfrak{g}, g\in G. \tag{6.1}$$

By differentiating (6.1) we obtain

$$\langle ad_X^*\xi, Y\rangle = -\langle \xi, [X,Y]\rangle. \tag{6.2}$$

On the orbits $\Omega_{\xi_0} = \left\{ Ad_g^* \xi_0, g \in G \right\}$ ad_X^* is represented by a vector field $f_{\Omega_{\xi_0}}(X)$. The symplectic two-form B_Ω on Ω is given by

$$B_{\Omega_\xi}\left(f_{\Omega_\xi}(X), f_{\Omega_\xi}(Y)\right)(\xi) = \langle \xi, [X, Y] \rangle. \tag{6.3}$$

A theorem of Auslander and Kostant characterizes the simply connected solvable Type I Lie-groups:

THEOREM 1. *Let* G *be a simply connected solvable Lie-group. Then G is Type I (tame) if and only if*

(1) *all coadjoint orbits of* G *are* G_δ *sets (i.e. they are countable intersections of open sets) in the usual topology on* $\mathfrak{g}$ *.*

(2) *The symplectic forms* B_{Ω_ξ} *are exact for all* $\xi \in \mathfrak{g}^*$*.*

We use this theorem for the study of some Lie-groups connected with the theory of quasi-periodic systems in solid-state physics.

The simplest example of wild groups is the five dimensional Mautner group [45] consisting of certain 3×3 complex matrices:

$$g(t, w, z) = \begin{pmatrix} e^{it} & 0 & z \\ 0 & e^{i\alpha t} & w \\ 0 & 0 & 1 \end{pmatrix}, \quad t \in \mathbb{R},\ z, w \in \mathbb{C}, \tag{6.4}$$

where α is a fixed irrational number. The non-zero commutators of the Lie-algebra of this group are

$$\begin{aligned} [P, S_1] &= C_1, & [P, S_\alpha] &= \alpha C_\alpha, \\ [P, C_1] &= -S_1, & [P, C_\alpha] &= -\alpha S_\alpha. \end{aligned} \tag{6.5}$$

Operators satisfying the same algebra occur in the theory of one-dimensional quasi-periodic systems. A representation of (6.5) is provided by the following operators acting on $L^2(\mathbb{R}, dt)$:

$$P = \partial_t, \quad \begin{aligned} S_1 &= a \sin(t + \phi_1), & S_\alpha &= a_\alpha \sin(\alpha t + \phi_\alpha), \\ C_1 &= a \cos(t + \phi_1), & C_\alpha &= a_\alpha \cos(\alpha t + \phi_\alpha). \end{aligned} \tag{6.6}$$

A representation with different $a_1', a_\alpha', \phi_1', \phi_\alpha'$ parameters is isomorphic to (6.6) iff $a_1 = a_1', a_\alpha = a_\alpha'$ and $\phi_1 - \phi/\alpha = \phi_1' - \phi'/\alpha + 2m\pi + 2n\pi/\alpha$ for some $m, n \in \mathbb{Z}$. One can build the Hamiltonian of an electron moving in a quasi-periodic cosine potential out of these operators:

$$H = -\frac{1}{2}\partial_x^2 + a_1 \cos(t + \phi_1) + a_\alpha \cos(\alpha t + \phi_\alpha) = -\frac{1}{2}P^2 + a_1 C_1 + a_\alpha C_\alpha. \tag{6.7}$$

In [45] two inequivalent decompositions of the regular representation of (6.4) into irreducible ones are presented. Inequivalent decompositions of a representation of (6.5) occurred in the physics literature, too. It was noted in [**49**, **50**, **51**, **52**] that although (6.7) has no translational symmetry, it is not completely random either. By adding an extra dimension, translations by 2π and $2\pi/\alpha$ can be executed in separate dimensions. For that purpose, we consider the following representation of (6.5) on $L^2(\mathbb{R}^2, dx dy)$:

$$P = \partial_x + \partial_y \qquad \begin{array}{ll} S_1 = a_1 \sin x, & S_\alpha = a_\alpha \sin \alpha y, \\ C_1 = a_1 \cos x, & C_\alpha = a_\alpha \cos \alpha y. \end{array} \tag{6.8}$$

Since P is the generator of translations only along the lines $l_c : y = x + c$, the representation (6.10) is decomposable into irreducible representations acting on the Hilbert-spaces $L^2(l_c)$. These representations are isomorphic to (6.6) with parameters $\phi_1 = 0$, $\phi_\alpha = c\alpha$. A different decomposition of $L^2(\mathbb{R}^2, dx dy)$ is based on the periodicity of (6.8) on the xy-plane. The operator $H = -1/2P^2 + a_1 C_1 + a_\alpha C_\alpha$ is indeed invariant against the translations $(x, y) \to (x + 2\pi, y)$ and $(x, y) \to (x, y + 2\pi/\alpha)$. The translational symmetry entails the existence of Bloch wave-functions

$$\psi(x + 2\pi, y) = e^{is}\psi(x, y), \qquad \psi(x, y + \frac{2\pi}{\alpha}) = e^{it}\psi(x, y). \tag{6.9}$$

The operators acting on such wave-functions for fixed s and t provide exactly the infinitesimal form of the irreducible representation occurring in the second decomposition of the regular representation in [45]. Indeed, if we introduce the periodic functions

$$\tilde{\psi}(x, y) = e^{-i(sx + t\alpha y)}\psi(x, y), \tag{6.10}$$

then the operators (6.8) act on $\tilde{\psi}$ as

$$\tilde{P} = \partial_x + \partial_y + i(s + \alpha t), \qquad (S_1, C_1, S_\alpha, C_\alpha \text{ are unchanged}). \tag{6.11}$$

Since $\tilde{\psi}$ is periodic, we can regard it as a function defined on the torus $S^1 \times S^1 = [0, 2\pi) \times [0, 2\pi/\alpha)$. The action of the operators (6.11) on $L^2(S^1 \times$

S^1) is irreducible. The existence of a representations with Bloch wave-functionals does not *a priori* implies the occurance of extended states in the physical representation (6.8). Indeed, as it was stressed in [**53**], inequivalent representations of the same algebra might have very different spectral and localizational properties. Nevertheless, the existence of extended states in this system was established in [**54, 55, 56**].

Next we study the effect of the fluctuation of the magnitude of the potential. For this purpose we add the generator $M = \partial_a$ to the operators of (6.6). M changes the amplitudes of the potentials S_1 and C_1. To keep the algebra closed we need to add the operators $S_0 = sin(t + \phi_1)$ and $C_0 = cos(t + \phi_1)$ to (6.6), too. The extra non-zero commutators (compared to (6.5)) of the extended Lie-algebra $\mathfrak{g}$ are

$$(6.12)\qquad [P, S_0] = C_0, \quad [P, C_0] = -S_0, \quad [M, S_1] = S_0, \quad [M, C_1] = C_0.$$

The Lie-group G of $\mathfrak{g}$ has a representation by 4×4 matrices

$$(6.13)\qquad g(t, a, u, w, z) = \begin{pmatrix} e^{it} & a & 0 & u \\ 0 & e^{it} & 0 & z \\ 0 & 0 & e^{i\alpha t} & w \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad a, t \in \mathbb{R},\ u, w, z \in \mathbb{C}.$$

If $\mathfrak{g}^*$ is represented by matrices of the following form

$$(6.14)\qquad \xi(\tau, p, l, m,) = \begin{pmatrix} i\tau & 0 & 0 & 0 \\ p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ l & m & n & 0 \end{pmatrix}, \qquad \tau, t \in \mathbb{R},\ l, m, n \in \mathbb{C},$$

so the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ is

$$(6.15)\qquad \langle \xi, h \rangle = \Re(\,\mathrm{Tr}(\xi h))\,, \quad \xi \in \mathfrak{g}^*, h \in \mathfrak{g},$$

then the coadjoint action is

$$(6.16)\quad Ad^*_{g(t,a,u,w,z)}\xi\,(\tau, p, l, m, n) = \\ \xi\left(\tau + \Im(lu + zm + \alpha n w), p - \Re(lz), le^{-it}, me^{-it} - \Re(lz), ne^{-i\alpha t}\right).$$

The four dimensional orbits are given by the parametric equations

$$(6.17)\qquad l = l_0 e^{it}, \qquad n = n_0 e^{i\alpha t}.$$

Since the orbits are dense subsets of the sets

$$|l| = l_0, \qquad |n| = n_0 \tag{6.18}$$

the first criteria of the Auslander-Kostant theorem fails, so the group remains wild despite the fluctuation of the potential.

In the following we turn our attention to Kirillov's second example of wild groups. This group is closely related to the magnetic translation group, whose Type II nature at irrational magnetic flux was pointed out by [57] This is a seven dimensional Lie-algebra whose nonzero commutators are

$$[P_x, P_y] = 2B, \qquad \begin{array}{ll} [P_x, S_x] = C_x, & [P_y, S_y] = C_y, \\ [P_x, C_x] = -S_x, & [P_y, C_y] = -S_y. \end{array} \tag{6.19}$$

This algebra is represented by the operators

$$\begin{array}{lll} \hat{P_x} = i\partial_x - by, & \hat{C_x} = \cos x, & \hat{C_y} = \cos y, \\ \hat{P_y} = i\partial_y + bx, & \hat{S_x} = \sin x, & \hat{S_y} = \sin y, \end{array} \qquad \hat{B} = b, \tag{6.20}$$

on $L^2(\mathbb{R}^3, dx\,dy\,dz)$. The Hamiltonian of an electron moving in constant magnetic field in a periodic crystal can be formed out of these operators:

$$\hat{H} = \hat{P_x}^2 + \hat{P_y}^2 + \hat{C_x} + \hat{C_y}. \tag{6.21}$$

If we regard the generators as linear functions on $\mathfrak{g}^*$, then the coadjoint orbits are

$$C_x^2 + S_x^2 = r_1^2, \qquad C_y^2 + S_y^2 = r_2^2, \qquad B = r_3. \tag{6.22}$$

If the orbits are parametrized as

$$C_x = r_1 \cos\phi, \qquad S_x = r_1 \sin\phi, \qquad C_y = r_2 \cos\psi, \qquad S_y = r_2 \sin\psi, \tag{6.23}$$

then the symplectic two-form B_Ω is

$$B_\Omega = d\phi \wedge dP_x + d\psi \wedge dP_y + 2r_3 d\phi \wedge d\psi. \tag{6.24}$$

Since

$$\int_{\{P_x = P_y = 0,\, B = r_3\}} B_\Omega = 8\pi^2 r_3, \tag{6.25}$$

B_Ω is not exact, so the second criteria of the Auslander-Kostant theorem fails, consequently the group of magnetic translations is wild.

Now let us see what happens if the external magnetic field is dynamical, too. To describe the fluctuation of b we extend the set of generators (6.22) by $\hat{E} = i\partial_b$. In order to keep the commutators closed, we need to adjoin the operators $\hat{Y} = -i[\hat{E}, \hat{P_x}]$ and $\hat{X} = i[\hat{E}, \hat{P_y}]$, too. So the following eleven-dimensional Lie-algebra is necessary to describe the coupled system of an electron and the fluctuating external magnetic field:

$$\begin{array}{lll} [P_x, S_x] = C_x, & [P_y, S_y] = C_y, & [P_x, X] = I, \\ [P_x, C_x] = -S_x, & [P_y, C_y] = -S, & [P_y, Y] = I, \end{array} \tag{6.26}$$

$$\begin{array}{ll} [E, P_x] = -Y, & [P_x, P_y] = 2B, \\ [E, P_y] = X, & [E, B] = I. \end{array}$$

If we use the generators of the Lie-algebra as linear functions on $\mathfrak{g}^*$ then the coadjoint action corresponds to the following vector fields:

$$\begin{aligned} V_{P_x} &= -C_x\partial_{S_x} + S_x\partial_{C_x} + 2B\partial_{P_y} + Y\partial_E + I\partial_Y, \\ V_{P_y} &= -C_y\partial_{S_y} + S_y\partial_{C_y} - 2B\partial_{P_x} - X\partial_E + I\partial_X, \end{aligned} \tag{6.27}$$

$$\begin{array}{ll} V_{S_x} = C_x\partial_{P_x}, & V_{S_y} = C_y\partial_{P_y}, \\ V_{C_x} = -S_x\partial_{P_x}, & V_{C_y} = -S_y\partial_{P_y}, \\ V_B = -I\partial_E, & V_E = I\partial_B - Y\partial_{P_x} + X\partial_{P_y}, \\ V_X = -I\partial_{P_x}, & V_Y = -I\partial_{P_y}, \end{array} \tag{6.28}$$

$$V_I = 0. \tag{6.29}$$

Note that ∂_I does not occur in these expressions, so $I = I_0 = const.$ on each orbit. The form of V_{P_x} and V_{P_y} implies that

$$C_x^2 + S_x^2 = r_x^2, \qquad C_y^2 + S_y^2 = r_y^2. \tag{6.30}$$

If $B \neq 0$, then (6.28) entails

$$\mathcal{L}\left(\{V_X, V_Y, V_E, V_B\}\right) = \mathcal{L}\left(\{\partial_{P_x}, \partial_{P_y}, \partial_E, \partial_B\}\right). \tag{6.31}$$

So the orbits are generated by the vectors $\partial_{P_x}, \partial_{P_y}, \partial_E, \partial_B$ and by

$$\tilde{V}_{P_x} = -C_x\partial_{S_x} + S_x\partial_{C_x} + I_0\partial_Y, \qquad \tilde{V}_{P_y} = -C_y\partial_{S_y} + S_y\partial_{C_y} + I_0\partial_x. \tag{6.32}$$

The integral manifolds of these vectors are

$$
\begin{aligned}
&C_x = r_x \cos\phi, \qquad S_x = r_x \sin\phi, \qquad Y = I_0(\phi + \phi_0) \\
&C_y = r_y \cos\psi, \qquad S_y = r_y \sin\psi, \qquad X = I_0(\psi + \psi_0),
\end{aligned}
\tag{6.33}
$$

while E, B, P_x, P_y are arbitrary. So the maximal dimensional orbits are homeomorph to $\mathbb{R}^6$. Since $H^2(\mathbb{R}^6) = 0$, the symplectic two-form B_Ω is necessarily exact. However, if $I_0 = 0$, then the vector fields $\tilde{V}_{P_x}$ and $\tilde{V}_{P_y}$ generates the product of two circles (a torus) instead of the product of two spirals, and the nonzero integral of the symplectic form violates the conditions of the Auslander-Kostant theorem. This violation happens only at a single vaule of the I_0 coordinate,[1] so the fact that the bulk of the orbits satisfy the conditions of the theorem might imply that this system behaves like the periodic crystalls of solid-state physics.

[1]This case was overlooked in the published version of this chapter. The error was pointed out by Zoltan Magyar in the Mathematical Reviews.

7

On the stress tensor near a nonconformal boundary

1 Introduction

Boundary conformal field theory has received quite a lot of attention recently. It occurs in D-brane physics, in the description of the critical behavior of two dimensional statistical mechanical systems, or in one dimensional quantum systems with ends or with a point-like defect. It has grown into a separate subfield of mathematical physics, too. A simplest example of these systems is a single real scalar field ϕ on an interval. In this case, however, one can prescribe nonconformal boundary conditions for ϕ, so it is a reasonable project to study what happens in this case to the usual ingredients of conformal field theory.

In [59] the authors considered the case of a massless scalar field on the figure eight as the simplest example of quantum field theory on a network. They realized that the system has a nontrivial analogue of conformal symmetry only in special cases. In this paper we study only the "half" of this network, i.e. the quantum field theory of a massless scalar on a circle with a point-like impurity.

Before performing any calculation, we try to guess what sort of modification can be expected for the usual Virasoro algebra of the stress tensor. Let us imagine that ϕ satisfies the nonconformal boundary condition at $x = 0$:

$$\phi'(0) + \alpha\phi(0) = 0. \tag{7.1}$$

(Actually this is not a very sensible boundary condition for a closed physical system, so we consider only it as an illustration.) This condition is invariant against the transformation $\phi(x) \to \phi(f(x))$ only if $f(x) = x + f_1 x^2 + \dots$. The Lie algebra of such transformation is contained in $L_1(1)$, i.e. in the Lie algebra of formal vector fields on a line of the form

$$v = v_1 x^2 \partial + v_2 x^3 + \partial + \dots \tag{7.2}$$

In general, $L_k(1)$ is the formal vector fields on the one dimensional line with the following form:

$$v = v_k x^{k+1} \partial + v_{k+1} x^{k+2} + \partial + \dots \tag{7.3}$$

The (nontrivial) central extensions of $L_1(1)$ was computed by Goncaharova [60]. Since for a free scalar field ϕ the commutator $[\phi, \phi] \sim \mathit{const}$, the schematic form of the Lie algebra of bilinear stress tensor T must be

$$[T, T] \sim [\phi\phi, \phi\phi] \sim \phi\phi + \mathit{const}. \tag{7.4}$$

(In this condensed notation for example $[\phi, \phi] \sim \mathit{const}$ stands for $[\phi(x), \dot{\phi}(y)] = \delta(x - y)$.) The $\phi\phi$ term in the result has the same structure as in the classical system, while the const term of the central extension must be some sum of the Virasoro (Gelfand-Fuks) and the Goncharova cocycles. We note here that Goncharova's cocycle is local in the sense that it can be calculated from the first few derivatives of the vector field v at $x = 0$. Of course we cannot guess in advance the coefficients of the various cocycles in the const. central extensions. They might be even 0 or ∞ in a QFT calculation. In the rest of the paper we try to calculate them for our massless case on an interval with semitransparent (permeable) boundary conditions. We argue that in this case the central extension is absent.

2 Massless field on an interval

Let $\phi(x, t)$ be a massless real scalar field on the interval $[0, \pi]$ with action

$$S = \frac{1}{2} \int_0^{\pi} \dot{\phi}^2 - \phi_x^2 \, dx \, dt \tag{7.5}$$

and equation of motion

$$\phi_{xx} - \phi_{tt} = 0. \tag{7.6}$$

There are several possible treatment of the boundaries at $x = 0$ and $x = \pi$. One can, for example require energy conservation:
(7.7)

$$0 = \frac{d}{dt}\int_0^\pi \dot{\phi}^2 + \phi_x^2 \, dx = \int_0^\pi \dot{\phi}^2 + \phi_{xx} + \dot{\phi}^2 + \phi_x \, dx = \dot{\phi}(\pi)\phi_x(\pi) - \dot{\phi}(0)\phi_x(0).$$

This is satisfied if (but not "only if")

$$\begin{pmatrix} \dot{\phi}(\pi) \\ \phi_x(\pi) \end{pmatrix} = M \begin{pmatrix} \dot{\phi}(0) \\ \phi_x(0) \end{pmatrix}, \qquad M \in O(1,1). \tag{7.8}$$

A simple example of this relation is $\dot{\phi}(\pi) = \lambda\dot{\phi}(0), \quad \phi_x(\pi) = \lambda^{-1}\phi_x(0)$. These conditions describe a scale invariant defect. This system was studied in [58] under the name of "permeable brane".

Another possibility to obtain acceptable boundary conditions is to require that the single particle Hamiltonian

$$H_s = -\partial_{xx} \tag{7.9}$$

is self adjoint on the real Hilbert space $L^2([0,\pi])$, so the system can be decomposed into an infinite collection of harmonic oscillators with real frequencies. This path was followed in [59]. In this case

$$\int_0^\pi \psi\partial_{xx}\phi \, dx = \int_0^\pi \partial_{xx}\psi\phi \, dx, \tag{7.10}$$

which is equivalent to

$$(\psi\phi_x - \psi\phi_x)|_0^\pi = 0. \tag{7.11}$$

This condition is fulfilled if $\phi_x = 0$ or $\phi = 0$ at the endpoints. A more interesting possibility is

$$\begin{pmatrix} \phi(\pi) \\ \phi_x(\pi) \end{pmatrix} = R \begin{pmatrix} \phi(0) \\ \phi_x(0) \end{pmatrix}, \qquad R \in SL(2.\mathbb{R}). \tag{7.12}$$

This can be interpreted as a semitransparent boundary condition corresponding to an impurity at $x = 0 \equiv \pi$. In order to get some intuition about the nature of this system, first we study the solution of the boundary conditions in the case when $R \in SO(2)$.

$$R = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \in SO(2). \tag{7.13}$$

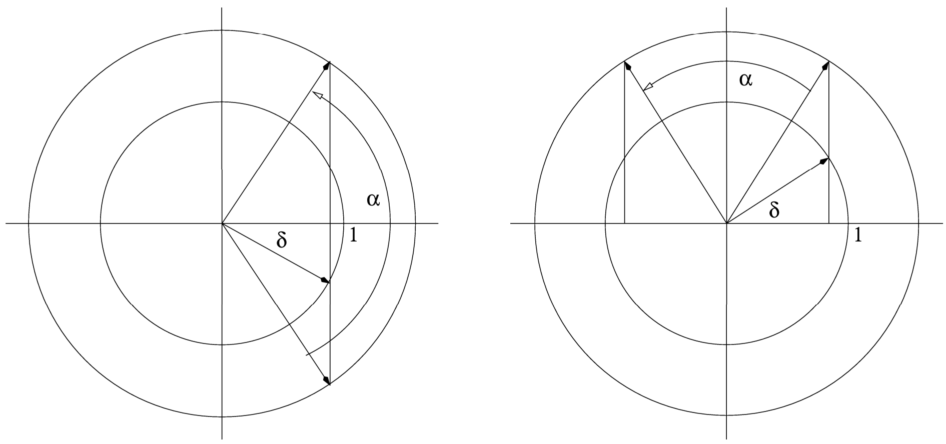
In order to construct the quantum field theory of ϕ, we must find the eigenfunctions of H_s. They can be written in the form

$$\phi_n(x) \sim \cos(k_n x + \delta_n). \tag{7.14}$$

The boundary conditions give

$$\begin{pmatrix} \cos(k_n x + \delta_n) \\ k_n \sin(k_n x + \delta_n) \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos(\delta_n) \\ k_n \sin(\delta_n). \end{pmatrix} \tag{7.15}$$

There are two types of solutions, roughly corresponding to the modes of a vibrating open string with integer and half-integer wave numbers. We hope that the structure of these solutions are evident from the following figure.



(Here the vectors on the outer large circles represent the (ϕ, ϕ') quantities.)

The first possibility is

$$k_n^1 \pi + \delta_n^1 = -\delta_n^1 + 2n\pi, \qquad -\frac{k_n^1 \sin\delta_n^1}{\cos\delta_n^1} = \tan\frac{\alpha}{2} = a. \tag{7.16}$$

With the help of the approximation

$$\begin{pmatrix} \cos\delta_n^1 \\ \sin\delta_n^1 \end{pmatrix} \approx \begin{pmatrix} 1 \\ \delta_n^1 \end{pmatrix} \tag{7.17}$$

we obtain the following approximative solution

$$\phi_n^1(x) \approx \sqrt{\frac{2}{\pi}} \cos\left(\left(2n + \frac{a}{2n\pi}\right)x - \frac{a}{2n}\right). \tag{7.18}$$

The second possibility is

$$k_n^2\pi + \delta_n^2 = (\pi - \delta_n^2) + 2n\pi, \qquad \frac{k_n^2 \sin \delta_n^2}{\cos \delta_n^2} = \mathrm{ctan}\frac{\alpha}{2} = a^{-1}. \tag{7.19}$$

Then

$$\phi_n^2(x) \approx \sqrt{\frac{2}{\pi}} \cos\left(\left(2n+1-\frac{a^{-1}}{(2n+1)\pi}\right)x + \frac{a^{-1}}{(2n+1)}\right). \tag{7.20}$$

The quantum field operator $\phi(x,t)$ is

$$\begin{gathered}\phi(x,t) = \sum_{n=0,1,2,\dots \;\; A=1,2} \frac{\phi_n^A(x)}{\sqrt{2k_n^A}}\left(a_n^A e^{-ik_n^A t} + a_n^{A\,+} e^{ik_n^A t}\right) \\ [a_n^A, a_m^{B\,+}] = \delta_{AB}\delta_{mn}.\end{gathered} \tag{7.21}$$

The zero time fields $\phi(x) = \phi(x,0)$ and $\pi(x) = \dot{\phi}(x,0)$ satisfies the commutation relation

$$[\phi(x), \pi(y)] = i\delta(x-y). \tag{7.22}$$

In principle, one can try to commute the commutators of the various components of the stress tensor with the help of the approximative single particle wave functions, since they correctly reproduce the ultraviolet behavior of ϕ, but this computation becomes very cumbersome,so it is easier to calculate the Green functions of the Euclidean version of the theory, end extract the energy-momentum algebra with the help of the techniques of conformal field theory.

To close this section we recall a partial case of a theorem of Goncharova on the cohomology of $L_1(1)$. (She actually computed $H^*(L_n(1))$ for any n **[60, 61]**.) According to this theorem, $H^2(L_1(1))$ (which classifies the central extensions of $L_1(1)$) is two dimensional with generators α and β whose nonzero values on pairs of the basis vectors

$$e_1 = x^2\partial_x, \quad e_2 = x^3\partial_x, \quad e_3 = x^4\partial_x, \quad \dots \tag{7.23}$$

are

$$\begin{aligned} &\alpha(e_1,e_4) = -\alpha(e_4,e_1) = 1, && \beta(e_2,e_5) = -\beta(e_2,e_5) = 1, \\ &\alpha(e_2,e_2) = -\alpha(e_3,e_2) = -3, && \beta(e_3,e_4) = -\beta(e_4,e_3) = -3. \end{aligned} \tag{7.24}$$

These cocycles are localized at $x = 0$ in the sense that they can be written as
(7.25)

$$\alpha(f(x)\partial_x, g(x)\partial_x) =$$

$$\frac{1}{2!5!}(f''(0)g^{(5)}(0) - f^{(5)}(0)g''(0)) - \frac{3}{3!4!}(f'''(0)g^{(4)}(0) - f^{(4)}(0)g'''(0)),$$

$$\beta(f(x)\partial_x, g(x)\partial_x) =$$

$$\frac{1}{3!6!}(f'''(0)g^{(6)}(0) - f^{(6)}(0)g'''(0)) - \frac{3}{4!5!}(f^{(4)}(0)g^{(5)}(0) - f^{(5)}(0)g^{(4)}(0)),$$

so they might emerge in a local quantum field theory. As our boundary conditions are invariant against vector fields from two copies of $L_1(1)$ (one at $x = 0$ and a second one at $x = \pi$), we might obtain central extensions even from $H^2(L_1(1) \oplus L_1(1))$. Generally,

$$H^2(\mathfrak{g} \oplus \mathfrak{g}) = H^2(\mathfrak{g}) \oplus H^2(\mathfrak{g}) \oplus ((\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]) \otimes (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]))^* . \tag{7.26}$$

To see this, let us write down the Jacobi identity for three elements a_1, b_2, c_2, where a_1 is from the first summand of $\mathfrak{g} \oplus \mathfrak{g}$, while b_2, c_2 are from the second one:

$$[a_1, [b_2, c_2]] + [c_2, [a_1, b_2]] + [b_2, [c_2, a_1]] = 0. \tag{7.27}$$

Now let α be a cocycle from $H^2(\mathfrak{g} \oplus \mathfrak{g})$ which is zero when restricted to the firs or the second summand of $\mathfrak{g} \oplus \mathfrak{g}$. If the central extension is generated by α, i.e. $[a_1, b_2] = \alpha(a_1, b_2)$, etc., then the Jacobi identity gives $\alpha(a_1, [b_2, c_2]) = 0$ (and by symmetry, $\alpha(a_2, [b_1, c_1]) = 0$, too), since for example

$$[c_2, [a_1, b_2]] = [c_2, \alpha(a_1, b_2)] = 0. \tag{7.28}$$

In our case $[L_1(1), L_1(1)] = L_3(1)$, since

$$[f_1 x^2 \partial_x + f_2 x^3 \partial_x + f_3 x^4 \partial_x + \dots, g_1 x^2 \partial_x + g_2 x^3 \partial_x + g_3 x^4 \partial_x + \dots] = \tag{7.29}$$

$$3(f_1 g_2 - f_2 g_1) x^4 \partial_x + \dots. \tag{7.30}$$

Since $L_1(1)/L_3(1) = \mathbb{R}^2$,

$$H^2(L_1(1) \oplus L_1(1)) = \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^4 = \mathbb{R}^8. \tag{7.31}$$

Note that the terms corresponding to $\mathbb{R}^4$ are local, since they can be expressed with the help of the first two components of the vector fields.

In the next section we argue that only the zero cocycle is represented by the behavior of our scalar field ϕ at the boundaries.

3 A defect on the line

Although in the previous section we determined the eigenmodes on an interval fairly accurately (at least in special cases), it is more convenient to work an an infinite line with the boundary conditions located at $x = 0$. In two dimension the massless Green function is

$$G_{plane}((x,t),(y,s)) = \frac{1}{2\pi}\int \frac{e^{-ip(x-y)}e^{-ip_0(t-s)}}{p^2+p_0^2}\,dp\,dp_0. \tag{7.32}$$

In the presence of the defect this is modified to

$$G((x,t),(y,s)) = \frac{1}{2\pi}\int \frac{\sum_{A=1,2}\phi_p^A(x)\bar{\phi}_p^A(y)e^{-ip_0(t-s)}}{p^2+p_0^2}\,dp\,dp_0, \tag{7.33}$$

where ϕ_p^A are plane waves scattered on the defect:

$$\phi_p^1 = \begin{cases} e^{ipx} + R_p^1 e^{-ipx}, & \text{if } x < 0,\\ T_p^1 e^{ipx}, & \text{if } x > 0,\end{cases} \tag{7.34}$$

and

$$\phi_p^1 = \begin{cases} T_p^2 e^{ipx}, & \text{if } x < 0,\\ e^{-ipx} + R_p^2 e^{ipx}, & \text{if } x > 0.\end{cases} \tag{7.35}$$

These waves are orthogonal to each other and has the same normalization as e^{-ipx} has. If we denote the R matrix describing the boundary conditions by $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$, then the scattering coefficients are approximately (for large p)

$$T_p^1 \approx \frac{2i}{bp},\quad R_p^1 \approx 1 - \frac{2id}{bp},\quad T_p^2 \approx \frac{2i}{bp},\quad R_p^2 \approx 1 - \frac{2ia}{bp}. \tag{7.36}$$

Note that $R_p^1 = R_p^2 = 1$ correspond to two noninteracting half space with Neumann (i.e. $\phi' = 0$ on the boundary) boundary conditions. If we write down the x dependent term in the Green function G with the help of these approximations, we obtain that (when $x, y > 0$)

$$\sum_{A=1,2}\phi_p^A(x)\bar{\phi}_p^A(y) \approx \tag{7.37}$$

$$(e^{-ip(x-y)} + e^{ip(x-y)}) + (e^{-ip(x+y)} + e^{ip(x+y)}) + \frac{2ia}{bp}((e^{-ip(x+y)} - e^{ip(x+y)}).$$

The terms in the first two parentheses are responsible for the Green function on a half plane $G_{half}(z, z') \sim \ln|z - z'| + \ln|\bar{z} - z'|$. The third part has softer large p behavior, so it should has only a $|\bar{z} - z'| \ln|\bar{z} - z'|$ type singularity. (Note that $(x \ln x - x)' = \ln x$, and the $\frac{2ia}{bp}$ term can be canceled in (4.30) by a derivation with respect to x or y.) So we see that the correction term in the Green function is less singular than the the original $G_{half}(z, z')$. In conformal field theory, the form of the Gelfand-Fuks cocycle

$$\alpha_{GF}(f(x)\partial_x, g(x)\partial_x) = \int f'(x) g''(x)\, dx \tag{7.38}$$

in the Virasoro algebra can be derived from the short distance behavior of the stress tensor expectation values

$$\langle T(z)T(z')\rangle \sim (\partial_z \partial_{z'} G_{plane}(z, z') \sim \frac{1}{(z - z')^4}. \tag{7.39}$$

The rule of thumb is that exponent 4 is one higher than the number of derivatives in α_{GF}. Now Goncharova's cocycles contains at least 7 derivatives (and they are localized at $x = 0$), so they cannot be the result of a less singular correction to the G_{half} Green function. Of course, we are ready to admit that this argument is not quite rigorous.

8

Summary

Since the Introduction chapter provided an overview and motivation of the results of this thesis, in this chapter we stress only those results and constructions which are original up to our best knowledge. We present them for Chapter 2-7 as separate items.

- *Chapter 2. Nonperturbative effects in deformation quantization.*

  First we describe the basic geometric idea. On $C^\infty(\mathbb{R}^2)$ there is a deformed product

  $$(f *_h g)(\mathbf{r}) = \frac{1}{h^2\pi^2}\int d^2\mathbf{r}' d^2\mathbf{r}'' f(\mathbf{r}')g(\mathbf{r}'')\exp\frac{-4i}{h}A(\mathbf{r},\mathbf{r}',\mathbf{r}''),$$

  where $\mathbf{r} = (x, p)$ and A is the symplectic area of the triangle $T = \Delta(\mathbf{r},\mathbf{r}',\mathbf{r}'')$. This formula works for periodic functions, so it describes a deformed algebra on a cylinder instead of $\mathbb{R}^2$. The factor $A(\mathbf{r},\mathbf{r}',\mathbf{r}'')$ can be written either as the integral of a closed two-form over the triangle $\Delta(\mathbf{r},\mathbf{r}',\mathbf{r}'')$, or as the integral of a one-form over the boundary of the triangle. Now although on a cylinder the straight edges between the points $\mathbf{r},\mathbf{r}',\mathbf{r}''$ does not necessarily bound a triangle, the expression for A as the integral of a one-form over the edges still makes sense. So it is reasonable to study the proprieties of the expression of the deformed product when these topologically nontrivial edge configurations are included. Our result is that if only edge configurations with a fixed winding number w are included, then the new deformed product looks as

  $$e_{n,r} *_{h,w} e_{\tilde{n},\tilde{r}} = \exp[\frac{ih}{2}(r\tilde{n} - \tilde{r}n) + 2ih\pi w]e_{n+\tilde{n},r+\tilde{r}+4w\pi/h}$$

where $e_{n,r} = e^{i(nx+rp)}$. The usual product correspond to $w = 0$. In fact, the $w \neq 0$ product can be obtained from the $w = 0$ one by a redefinition of the basis. However, this is no longer true if one allows the inclusion of edge configurations with various winding numbers

$$f *_c g = \sum_{w \in Z} c_w f *_{h,w} g,$$

where the c_w-s are arbitrary coefficients. We proved that when the nonzero coefficients are $c_0 = c_1 = 1$, then the algebra with the $*_c$ product does not possess an unit element. This phenomena would be impossible if the deformation were algebraic, i.e. when the product depends polynomially on the deformation parameter h. Moreover, the product $*_c$ is associative, which is surprising as the linear combination of associative products are usually not associative (primary examples are the Lie and Jordan algebras for a nonassociative product).

- *Chapter 3. Matrix theory of unoriented membranes and Jordan algebras.*

  This chapter is based on a simple observation. It is well known that the equations of motion of an oriented surface with the standard world-volume action can be regularized with the help of the noncommutative algebra of large Hermitian matrices. This method uses the

  Poisson algebra on the surface $\Longleftrightarrow$ *Matrix commutators* correspondence. The existence of a suitable Poisson algebra on the surface requires orientability. However, the equation of motion of a membrane looks the same for nonorientable ones, so there must be some substitute of the above correspondence for nonorientable surfaces. We found that this aim can be achieved by the

  Jordan algebra of the deformed product on the surface

$$\Longleftrightarrow$$

Jordanian matrix products

correspondence. The applicability of this method for the description of membrane motion is based on the simple identity

$$\mathrm{Tr}\left(\frac{1}{4}[\mathbb{X}^i,\mathbb{X}^j][\mathbb{X}^i,\mathbb{X}^j]\right) =$$

$$\mathrm{Tr}\left(\mathbb{X}^i \circ (\mathbb{X}^j \circ (\mathbb{X}^i \circ \mathbb{X}^j)) - (\mathbb{X}^i \circ \mathbb{X}^i) \circ (\mathbb{X}^j \circ \mathbb{X}^j)\right),$$

which expresses the potential energy of the regularized membrane with the help of the Jordan product. We studied what types of Jordan algebras can be used for the description of membranes with S^2 or $\mathbb{RP}^2$ topologies. It turned out that the latter requires $H_N(\mathbb{R})$, the Jordan algebra of real symmetric matrices.

- *Chapter 4. Finite groups, semigroups and the Quantum Yang-Baxter Equation.*

  It is well known since the work of Migdal that two dimensional lattice gauge theory is solvable. To a solvable two dimensional system almost by default corresponds a solution of the Quantum Yang-Baxter Equation, which is a close relative of Artin's braid relations. We wrote down this relation, and found that the resulting R matrix is not invertible. However, this problem can be cured by a small modification of the action of lattice gauge theory. We obtained the following solution for the QYBE:

$$R_{a,b}^{d^{-1},c^{-1}}(\lambda) =$$

$$\sum_{r\in R(G)}\left(\lambda_0^r\chi_r(abcd) + \lambda_1^r\left(\chi_r(bd)+\chi_r(ac)\right)\right) + \sum_{t,u\in R(G)}\lambda_2^{tu}\chi_t(bd)\chi_u(ac).$$

  (Here the group elements a, b, c, d labels the basis of underlying vector spaces on which R acts, and χ is a character of some representation of the gauge group.) These results were based on the repeated use of a character identity from group representation theory:

$$\left(\sum_{r\in\hat{G}}\lambda_r^I\chi_r(ax)\right)\cdot\left(\sum_{s\in\hat{G}}\lambda_s^{II}\chi_s(x^{-1}b)\right)$$

$$= |G|\sum_{r\in\hat{G}}\frac{\lambda_r^I\lambda_r^{II}}{\dim r}\chi_r(ab)$$

Now if we can prove this for a semigroup G_0 possessing an involution (to replace the inverse in the formula), then we can duplicate our previous results in the semigroup setting. Unfortunately, the character identity is not satisfied unconditionally for semigroups. However, for the case when the semigroup is obtained from a group by the adjunction of a zero element, we were able to determine the condition which ensures the satisfaction of this formula. This condition is $\lambda_0 = \lambda_1$, i.e. the coefficient of the unit representation of the group must be the same as the coefficient of the character which is nonzero only on the zero element. We studied the case of a few small semigroups, too, partially with the help of symbolic computer algebra.

- *Chapter 5. A modification of the Lax equation.*

This chapter consist of some application to the theory of integrable systems of a simple observation. If the linear operator L is replaced by $e^{itP}Le^{-itQ}$ where P and Q are Hermitian, then the spectrum of L^*L remains invariant. The infinitesimal form of this transformation is $\dot{L} = i(PL - LQ)$. Since many solvable PDE in two dimension can be written as an isospectral deformation of some differential operator L (Lax equation), it is reasonable to expect that we can derive equations with many conserved quantities with the help of this scheme. We applied this to the case of

$$L = (i\partial_x)^2 + \nu(x,t)i\partial_x + u(x,t).$$

We managed to emulate the Gelfand-Dickey method of the construction ofintegrable equation for this operator. An example of the obtained PDEs is

$$\partial_{t_2} w = 2\Im u'$$

$$\partial_{t_2} u = \frac{i}{2}\Big(w''' - w'(2w' + 2w^2 - 4u) - w(u' - \bar{u}' - w'')\Big).$$

Very similar equations occurred previously in the literature (Hirota-Satsuma of Fordy-Dodd equations), since the idea of the preservation of the spectrum of the product of operators was rediscovered several times. So our equation is basically a reduction of the complex Hirota-Satsuma equation. We applied this scheme to the Kadmontsev-Petviashvilli hierarchy, too.

- *Chapter 6. Coadjoint orbits of wild groups in solid-state physics.*

  The cosine functions and the partial derivation operators form closed solvable Lie algebras. Since one can build Hamiltonian operators describing the motion of a quantum particle in periodic or quasiperiodic potentials, it is reasonable to study the relation of the 'type' of the Lie algebra (or group) and the nature of the spectrum of the Hamiltonian. For example, the coincidence of the strange nature of the spectrum of a two dimensional lattice in constant magnetic field and the Type II nature of the 'magnetic translation group' was pointed out by Zak.

  For solvable Lie groups, there is a theorem of Auslander and Kostant which decide the nature of solvable Lie groups. We check the conditions of this theorem for several Lie groups connected to solid-state physics. This requires the computation of the coadjoint orbit spaces of these Lie algebras.

- *Chapter 7. On the stress tensor near a nonconformal boundary.*

  In this chapter we attempt to compute the operator product expansion of a two dimensional conformal massless scalar field around a nonconformal boundary. These sort of problems are much studied in the case of conformal boundary conditions. When the boundary condition is nonconformal, potentially a new phenomena might occur. This is related to the fact that a $\phi(0) + \phi'(0) = 0$ (here the 0 refers to the space position of the boundary) type boundary condition of a scalar field ϕ is left invariant by the transformation $\phi(x) \to \phi(f(x))$ only if not only $f(0)$, but $f'(0)$ is zero, too. The (formal) Lie algebra of these transformations is called $L_1(1)$. This Lie algebra has a two dimensional $H^2(L_1(1))$ (this was computed by Goncahrova), so the corresponding central extensions in the Lie algebra of the stress tensor might occur in addition to the Virasoro one. We present some nonrigorous, heuristic argument that this does not occurs in our case.

These results are contained in the following papers, preprints or manuscripts:

- 1. Nonperturbative effects in deformation quantization on a cylinder.

  Submitted to the Journal of Physics A.

- 2. Unoriented membranes and Jordan algebras.

  Journal of Mathematical Physics, Vol. 46, no. 3.

- 3. Finite groups, gauge theories and the Quantum Yang-Baxter Equation.

  Lett. Math. Phys. 43 (1998), no. 4, 295–298.

- 4. On the solvability of two dimensional semigroup gauge theories.

  Submitted to the Journal of Statistical Physics.

- 5. Unitary deformations and complex soliton equations.

  J. Math. Phys. 40 (1999), no. 7, 3404–3408.

- 6. Taming of the wild group of magnetic translations.

  Quantum problems in condensed matter physics. J. Math. Phys. 38 (1997), no. 4, 1864–1869.

- 7. On the stress tensor near a nonconformal boundary.

  Unfinished manuscript.

9

Összefoglalás

Mivel a bevezető fejezetben már áttekintettük dolgozatunk eredményeit, itt csak azokat az eredményeket emeljük ki, amelyek a legjobb tudomásunk szerint eredetiek. Ezeket a következőekben fejezetenként tárgyaljuk.

- *2.Fejezet. A deformácios kvantálás nemperturbative effektusai.*

  Ez a fejezet a következő geometriai képen alapul. A $C^\infty(\mathbb{R}^2)$-beli sima függvények szorzásának egy deformációját Neumann formulája adja meg

  $$(f *_h g)(\mathbf{r}) = \frac{1}{h^2\pi^2}\int d^2\mathbf{r}' d^2\mathbf{r}'' f(\mathbf{r}')g(\mathbf{r}'')\exp\frac{-4i}{h}A(\mathbf{r},\mathbf{r}',\mathbf{r}''),$$

  ahol $\mathbf{r} = (x,p)$ és A a szimplektikus területe a $T = \Delta(\mathbf{r},\mathbf{r}',\mathbf{r}'')$ háromszögnek. Mivel ez a formula értelemmel bír periodikus függvények esetében is, egyszersmind egy henger függvényei algebrájanak valamilyen deformációját is. Az $A(\mathbf{r},\mathbf{r}',\mathbf{r}'')$ tényezőt felírhatjuk egyrészt a szimplektikus formának a $\Delta(\mathbf{r},\mathbf{r}',\mathbf{r}'')$ háromszög feletti integráljaként, de úgy is mint egy alkalmas egy-formának a háromszög pereme fölötti integrálját. Bár a geodetikus ívek három pont között egy hengeren nem feltétlenül határolnak egy háromszöget, az A tényező második megadása ebben az esetben is működik. Így megvizsgálhatjuk a hatását ezeknek a topologikus szempontból nemtriviális konfigurációknak Neumann képletében. Ha ezek közül csak azokat vesszük figyelembe, amelyek pontosan w-szer csavarodnak a hengerre, akkor a szorzasi szabály a következő lesz

  $$e_{n,r} *_{h,w} e_{\tilde{n},\tilde{r}} = \exp[\frac{ih}{2}(r\tilde{n} - \tilde{r}n) + 2ih\pi w]e_{n+\tilde{n},r+\tilde{r}+4w\pi/h},$$

ahol $e_{n,r} = e^{i(nx+rp)}$. Az eredeti szorzatot a $w = 0$ esetben kapjuk vissza. Tulajdonképpen a $w \neq 0$ szorzat ekvivalens a régivel egy megfelelő bázistranszformáció után. Ez azonban már nem lesz így, ha különböző csavarodási számokat is megengedünk. Ekkor a szorzási szabály

$$f *_c g = \sum_{w \in Z} c_w f *_{h,w} g,$$

ahol a c_w számok tetszőleges együtthatók. Megmutattuk, hogy ha a nem nulla együtthatók $c_0 = c_1 = 1$, akkor a $*_c$ szorzás algebrájának nics egysegeleme. Ez a viselkedes nem fordulhatna elő algebrai deformaciók esetében, így ez egy valódi nemperturbatív effektus. Mi több, a $*_c$ szorzás asszociatív marad, ami igazi meglepetés, mivel általában asszociatív szorzatok lineáris szuperpozíciója nem asszociatív (erre a jelenségre a legismertebb példák a Lie és Jordan algebrák).

- *3. Fejezet. Jordan algebrák és a nemorientálható membránok mátrix elmélete.*

  Ez a fejezet egy egyszerű megfigyelésen alapul. Jól ismert, hogy a egy orientálható felület relativisztikus mozgásegyenlete közelíthető a nagyméretű ermitikus matrixok nemkommutatív algebrájának a segítségével. Ez a módszer a

  Poisson algebra a felületen $\longleftrightarrow$ *Mátrix kommutátorok*

  összefüggést használja ki. A megfeleő Poisson algebra létezése kikényszeríti a felület orientálhatóságát. Viszont a membrán lokális mozgásegyenlete nem tartalmaz semmiféle utalást a membrán orientációjára, így szinte biztos, hogy valahogy meg lehet taláni a fenti kapcsolat megfelelőjét a nemorientálható esetben is. Mi ezt a célt a

  A felület deformált algebrájának a Jordan szorzata

  $$\longleftrightarrow$$

  Jordán mátrix szorzat

  megfeleltetéssel értük el. Mindez a membrán mozgásának Jordan algebrai leírására a következő egyszerű összefüggésen keresztül használható fel

  $$\mathrm{Tr}\left(\frac{1}{4}[\mathbb{X}^i, \mathbb{X}^j][\mathbb{X}^i, \mathbb{X}^j]\right) =$$

$$\mathsf{Tr}\left(\mathbb{X}^i \circ (\mathbb{X}^j \circ (\mathbb{X}^i \circ \mathbb{X}^j)) - (\mathbb{X}^i \circ \mathbb{X}^i) \circ (\mathbb{X}^j \circ \mathbb{X}^j)\right).$$

Így ki lehet fejezni a membrán potenciális energiáját a Jordan féle szorzás segítségével. Meghatároztuk, hogy milyen Jordán algebrák szükségesek a gömb, illetve a projektív sík topológiájú membránok leírására. Ez utóbbihoz a valós szimmetrikus mátrixok $\mathsf{H}_N(\mathbb{R})$ algebrája kellett.

- *4. Fejezet. Véges csoportok, félcsoportok és a kvantum Yang-Baxter egyenlet.*

Migdal munkássága nyomán jól ismert, hogy a kétdimenziós rács mértékelmélet egzaktul megoldható. A kétdimenziós megoldható modellek igen gyakran megadják néhány megoldását a Kvantum Yang-Baxter Egyenlet néhány megoldását. Ez utóbbi egyenlet szoros rokonságban van az Artin-féle fonatrelációkkal. Sajnos a kétdimenziós rács mértékelméletnek megfelelő megoldás nem invertálható. Ezt a csorbát sikerült kiküszöbölni a modell egy apró módosításával

$$R_{a,b}^{d^{-1},c^{-1}}(\lambda) =$$

$$\sum_{r\in R(G)} \left(\lambda_0^r\chi_r(abcd) + \lambda_1^r\big(\chi_r(bd) + \chi_r(ac)\big)\right) + \sum_{t,u\in R(G)} \lambda_2^{tu}\chi_t(bd)\chi_u(ac).$$

(Itt az a, b, c, d csoportelemek a a címkéi annak a vektortérnek amelyen R hat, míg χ a mertékcsoport karaktere.) Ez az eredmény egy, a csoportok reprezentációelméletéből jól ismert azonosságon alapul:

$$\left(\sum_{r\in\hat{G}} \lambda_r^I\chi_r(ax)\right) \cdot \left(\sum_{s\in\hat{G}} \lambda_s^{II}\chi_s(x^{-1}b)\right)$$
$$= |G| \sum_{r\in\hat{G}} \frac{\lambda_r^I\lambda_r^{II}}{\dim r}\chi_r(ab)$$

Ha be tudnánk bizonyítani ezt az összefüggést félcsoportokra is, akkor az előzőek már nem csak a csoportok esetén teljesülnének. Abban az igen egyszerű esetben, amikor a félcsoport egy véges csoport kibővítve egy zéruselemmel, a fenti reláció megfelelője igaz marad, ha $\lambda_0 = \lambda_1$. Ez a feltétel azt jelenti, hogy a csoport triviális reprezentációjának és a nulla elemhez tartozó karakternek az együtthatója megegyezik. Ezenfelül megvizsgáltuk ezt a kérdést néhány kis méretű félcsoport esetében is egy szimbolikus algebra program segítségével.

- *5. Fejezet. A Lax egyenlet egy módosítása.*

Ebben a fejezetben a következő egyszerű észrevételt tesszük az integrálható parciális differeenciálegyenletek elméletében: az L és az $e^{itP}Le^{-itQ}$ operátorok spektruma megeggyezik, ha P és Q ermitikusak.

Az ilyen transzformációknak infinitezimális formája $\dot{L} = i(PL - LQ)$. Mivel sok megoldható kétdimenziós PDE felírható úgy, mint egy L differenciáloperátor izospektrális deformációja, várható, hogy ezzel a sémával le lehet származtatni olyan differenciálegyenleteket, amelyek sok megmaradó mennyiséggel rendelkeznek. Mi mindezt az

$$L = (i\partial_x)^2 + v(x,t)i\partial_x + u(x,t).$$

operátor esetében vizsgáltuk meg. Sikerült reprodukaálnunk az integrálható egyenletek generálásának Gelfand és Dickey által kifejlesztett módszerét. Az illusztráció kedvéért leírjuk az egyik egyenletünket:

$$\partial_{t_2} w = 2\Im u'$$

$$\partial_{t_2} u = \frac{i}{2}\Big(w''' - w'(2w' + 2w^2 - 4u) - w(u' - \bar{u}' - w'')\Big).$$

Hasonló típusú egyenletek már korábban is megjenetek a szakirodalomban, mivel az operátorok szorzata megőrzésének módszerét többen is felfedezték. A mi egyenleteink alapvetően a komplex Hirota-Satsuma egyenletek redukciójai. Egy másik alkalmazásként kidolgoztuk a Kadmontsev-Petviashvili egyenlet egy variánsát is.

- *6. Fejezet. Vad csoportok koadjungált obitjai a szilárdtest-fizikában*

  A síkon a koszinusz függvények és a parciális deriválások operátorai egy feloldható Lie algebrát alkotnak. Mivel ezekből az operátorokból felépíthető egy kvantummechanikai részecske periodikus vagy kváziperiodikus potenciálban történő mozgását leíró Hamilton operátor, ésszerű feltételezni valamilyen kapcsolatot a Lie algebra típusa és a Hamilton operátor spektruma között. Erre a kapcsolatra valószínuleg először J.Zak mutatott rá egy állandó mágneses térben elhelyezkedő rács esetében.

  A feloldható Lie csoportok típusát meg lehet határozni Auslander és Kostant egyik tétele segíségével. Mi ellenőriztük ezen tétel feltételeinek teljesülését több, a szilárdtest fizikában előforduló csoport esetében. Ehhez a Lie algebrák koadjungált orbitjait kellett megkeresnünk.

- *7. Fejezet. Az energia-momentum algebra viselkedése nemkonformális határfeltételek mellett.*

Ebben a fejezetben megkíséreltük kiszámítani a kétdimenziós nulla tömegü skaláris tér viselkedését nemkonformális határfeltételek mellett. Ezt a problémát igen sokat tanulmányozták abban az esetben, amikor a határfeltételek konformálisak voltak. A nemkonformális esetben elvileg új jelenség is felléphetne. Ennek az a potenciális oka, hogy a $\phi(0) + \phi'(0) = 0$ típusu határfeltételeket (itt a 0 a határ térkoordinátáját jelöli) a $\phi(x) \to \phi(f(x))$ transzformáció csak akkor hagyja invariánsan, ha nem csak $f(0)$, de $f'(0)$ is eltűnik. Ezeknek a transzformációknak a (formális) Lie algebráját $L_1(1)$-nek nevezik. Mivel $H^2(L_1(1))$ kétdimenziós, így az ennek megfelelő centrális kiterjesztés elvben előfordulhatna az energia-momentum algebra szokásos Virasoro típusú kiterjesztése mellett. Mi felsorakoztatunk néhány érvet amellett, hogy esetünkben ez nem fordulhat elő.

Mindezekeket az eredményeket a következő dolgozatok, preprintek és kéziratok tartalmazzák:

- 1. Nonperturbative effects in deformation quantization on a cylinder.

  Submitted to the Journal of Physics A.

- 2. Unoriented membranes and Jordan algebras.

  Journal of Mathematical Physics, Vol. 46 (2005), no. 3.

- 3. Finite groups, semigroups and the Quantum Yang-Baxter Equation.

  Lett. Math. Phys. 43 (1998), no. 4, 295–298.

- 4. On the solvability of two dimensional semigroup gauge theories.

  Submitted to the Journal of Statistical Physics.

- 5. Unitary deformations and complex soliton equations.

  J. Math. Phys. 40 (1999), no. 7, 3404–3408.

- 6. Taming of the wild group of magnetic translations.

  Quantum problems in condensed matter physics. J. Math. Phys. 38 (1997), no. 4, 1864–1869.

- 7. On the stress tensor near a nonconformal boundary.
  Publikálásra való előkészítés alatt álló kézirat.

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Megjelent vagy elfogadott dolgozatok listája

(1) Unoriented membranes and Jordan algebras.
Journal of Mathematical Physics, 46 (2005), no. 3.

(2) Relative entropy in the Sherrington-Kirkpatrick spin glass model.
Journal of Physics, A 35 (2002), no. 36, 7773–7778.

(3) Unitary deformations and complex soliton equations.
Journal of Mathematical Physics, 40 (1999), no. 7, 3404–3408.

(4) Finite groups, semigroups and the Quantum Yang-Baxter Equation.
Letters of Mathematical Physics, 43 (1998), no. 4, 295–298.

(5) Minimax games, spin glasses, and the polynomial-time hierarchy of complexity classes.
Physical Reviews, E (3) 57 (1998), no. 6, 6487–6492.

(6) Taming of the wild group of magnetic translations.
(In: Quantum problems in condensed matter physics.) Journal of Mathematical Physics, 38 (1997), no. 4, 1864–1869.

Benyújtott dolgozatok listája:

(1) Nonperturbative effects in deformation quantization on a cylinder.
Submitted to the Journal of Physics A.

(2) On the solvability of two dimensional semigroup gauge theories.
Submitted to the Journal of Statistical Physics.

CONTRIBUTIONS TO THE THEORY OF TWO DIMENSIONAL SYSTEMS OF PHYSICS

értekezés a doktori (Ph.D.) fokozat megszerzése érdekében
a matematika tudományágban

Írta Varga Péter, okleveles fizikus

Készült a Debreceni Egyetem Matematika doktori iskolája
Differenciálgeometria programja keretében

Témavezető: Dr. Nagy Péter egyetemi tanár

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