

# Some consequences of the generalized uncertainty principle induced ultraviolet wave-vector cutoff in one-dimensional quantum mechanics

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A projection method is proposed to treat the one-dimensional Schrödinger equation for a single particle when the generalized uncertainty principle (GUP) generates an ultraviolet (UV) wave-vector cutoff. The existence of a unique coordinate representation called the naive one is derived from the one-parameter family of discrete coordinate representations. In this bandlimited quantum mechanics a continuous potential is reconstructed from discrete sampled values observed by means of a particle in maximally localized states. It is shown that bandlimitation modifies the speed of the center and the spreading time of a Gaussian wave packet moving in free space. Indication is found that GUP accompanied by bandlimitation may cause departures of the low-lying energy levels of a particle in a box from those in ordinary quantum mechanics to be much less suppressed than commonly thought when GUP without bandlimitation is at work.

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## I. INTRODUCTION

There are several theoretical indications that quantum gravity may have consequences in the behavior of low-energy quantum systems [1–4]. The corresponding effective quantum mechanics is believed to be based on the generalized uncertainty principle (GUP) [5–11], various modifications of Heisenberg’s uncertainty principle (HUP). Recently a proposal has even been put forward to probe experimentally the departure from HUP in a direct manner [12]. Among the various realizations of GUP there is a class in which the deformation of the commutator relation for the operators of the coordinate  $\hat{x}$  and the canonical momentum  $\hat{p}_x$  depends only on the canonical momentum,

$$[\hat{x}, \hat{p}_x] = i\hbar f(\alpha|\hat{p}_x|), \quad (1)$$

with the deformation function  $f(|u|)$ , where  $u = \alpha p_x$  and  $\alpha = \mathcal{O}(\ell_P/\hbar)$  with the Planck length  $\ell_P \approx 1.616 \times 10^{-35}$  m as a small parameter. In the present paper we shall restrict ourselves to deformation functions for which there exists a minimal wavelength, i.e., a maximal magnitude  $K$  of the wave vector, but the canonical momentum can take arbitrarily large values (as opposed to the cases discussed, e.g., in Ref. [13]). Then physical states are restricted to those of finite bandwidth; i.e., the wave-vector operator  $\hat{k}_x$  can only take values in the interval  $[-K, K]$ .

The mathematical structure of quantum mechanics with finite bandwidth is rather delicate [14], even in the one-dimensional case. Namely, while both operators of the wave vector  $\hat{k}_x$  and the canonical momentum  $\hat{p}_x$  are self-adjoint, the coordinate operator  $\hat{x}$  cannot be self-adjoint but only Hermitian symmetric in order to satisfy the deformed commutator relation (1). It has, however, a one-parameter

family of self-adjoint extensions with eigenvalues determining equidistant grids on the coordinate axis, defining a minimal length scale  $a$ , while the grids belonging to the various extensions are continuously shifted to one another. This is a sign that positions can only be observed with a maximal precision  $\Delta x_{\min} \approx a$ , while momenta can be measured with arbitrary accuracy [9, 15–21].

Although there exist formal coordinate eigenstates even in that case, they cannot be approximated now—as opposed to ordinary quantum mechanics—by a sequence of physical states with uncertainty in position decreasing to zero [15]. A so-called naive coordinate representation can be built up representing the operator algebra of  $[\hat{x}, \hat{k}_x] = i\hbar$  as  $\hat{x} \Rightarrow x$  and  $\hat{k}_x \Rightarrow -i\partial_x$  on square-integrable functions of the coordinate  $x \in \mathbb{R}$ , referred to below as coordinate wave functions. Then the canonical momentum satisfying the relation in Eq. (1) can be represented as  $\hat{p}_x \Rightarrow \alpha^{-1}F^{-1}(-i\alpha\hbar\partial_x)$ , where  $F^{-1}$  stands for the inverse of the function  $F(u) = \int_0^u \frac{du'}{f(|u'|)}$  and relates the operators of the wave vector and the canonical momentum via  $\hat{k}_x = (\alpha\hbar)^{-1}F(\alpha\hat{p}_x)$ . Let us note that the commutator in the left-hand side of Eq. (1) is invariant under the reflection  $x \rightarrow -x$ ,  $p_x \rightarrow -p_x$  so that the deformation function should be an even function of  $p_x$ , which implies that both functions  $F(u)$  and its inverse are even, as well. Furthermore,  $f(0) = 1$  has to be required in order to recover HUP in the limit  $\alpha \rightarrow 0$ . Here we shall restrict ourselves to particular choices of the deformation function which are monotonically increasing with rising  $|u|$  and for which the limits  $F(\pm\infty) = \pm\alpha\hbar K$  remain finite. A few explicit cases of such deformation functions are known, e.g.,  $f = 1 + \alpha^2 p_x^2$  with  $F(u) = \arctan u$  [15, 22],  $f = \exp(\alpha^2 p_x^2)$  with  $F(u) = (\sqrt{\pi}/2)\text{erf}u$  [20], and  $f = \exp(\alpha|p_x|)$  with  $F(u) = (1 - e^{-|u|})\text{sign}u$  used after

Taylor expansion in Refs. [23,24] and all implying  $\hbar K \approx 1/\alpha$  and  $a \approx \alpha \hbar \approx \ell_p$ .

In the present paper we shall take the point of view that the minimal length scale is of the order of the Planck length. Nevertheless, one has to be aware of the lack of an unquestionable theoretical proof of such a conjecture (see, e.g., Ref. [11]). Therefore, all the formulas derived in the present paper can also be considered from another point of view put forward in a series of papers [25–33], namely, that the scale  $\hbar\alpha \approx \Delta x_{\min} \approx a$  denoted here by  $\ell_p$  is an unknown parameter for which an upper bound should be laid either by the discrepancies among data on some observables measured with very high accuracy and their best theoretical estimates, or by the error of some very accurately measured observable. In the former case the best theoretical estimate means one obtained by incorporating all our knowledge on physics affecting the given observable, and the tiny discrepancy between theory and data is ascribed to the GUP effect, while in the latter case the error of the measurement is considered like an upper bound on the GUP effect. Making use of the deformation function  $f = 1 + \alpha^2 p_x^2$ , several upper bounds on  $\Delta x_{\min}$  have been obtained [25–33]. The smallest upper bounds are found, for example, from the investigation of the shift of energy levels of an electron in a Penning trap,  $\Delta x_{\min} \lesssim 2 \times 10^{-16}$  m (1 GeV<sup>-1</sup> in  $\hbar = c = 1$  units) [25] and from that of the 1S Lamb shift of the hydrogen atom  $\Delta x_{\min} \lesssim 3 \times 10^{-17}$  m (1/7 GeV<sup>-1</sup>) [30,31]. In that respect it is interesting to note that even a length scale as small as  $\Delta x_{\min} \sim 2 \times 10^{-19}$  m (1 TeV<sup>-1</sup>) [34] is not expected to influence the cosmic microwave background in any observable way [26], although there occurs a suppression of the high-frequency modes in black-body radiation.

In bandlimited quantum mechanics the coordinate space turns out to exhibit features of discreteness [35] and continuity at the same time, similar to how information does [36–42]. The main idea is that space can be thought of as a differentiable manifold, but the physical degrees of freedom cannot fill it in an arbitrarily dense manner. It has also been conjectured that degrees of freedom corresponding to structures smaller than the resolvable Planck scale turn into internal degrees of freedom [43–46]. Although the coordinate wave functions introduced in the manner described above do not have a simple probabilistic meaning, they provide useful tools to characterize the quantum states of the particle which can then be analyzed, e.g., in terms of maximally localized states [16–19]. Below we shall discuss the justification of that naive coordinate representation for one-dimensional bandlimited quantum mechanics in more detail.

In the present paper we shall concentrate on the solution of the Schrödinger equation for the coordinate wave functions in the case when GUP implies finite bandwidth. It is well known that GUP directly affects the Hamiltonian through the modification of the canonical momentum and

that of the kinetic energy operator  $(\hat{p}_x^2/2m) - (\hbar^2 \hat{k}_x^2/2m)$  which can be expanded—when low-energy states are considered—in powers of the small parameter  $\alpha$ . This pure GUP effect has been treated in the framework of the perturbation expansion using the naive coordinate representation and discussed in detail for various quantum systems (see Refs. [22,24,47–59] without the quest of completeness), among others for the particle in a box [23,24,52,53,56,60,61]. Treatments in the Bargmann-Fock representation [62–64], and various path-integral formulations [65–67] have been worked out. Here we shall take the viewpoint that the bandlimited quantum mechanics is an effective theory in the framework of which no quantum fluctuations of wavelength smaller than those of the order of the minimal length are possible; i.e., the coordinate wave functions should not contain Fourier components with wave vectors outside of the finite band with  $k_x \in [-K, K]$ . In order to build this restriction into the Schrödinger equation we propose to use Hamiltonians operating on the subspace of the square-integrable coordinate wave functions with finite bandwidth. To ensure this, we introduce the projector  $\hat{\Pi}$  onto that subspace and restrict the solutions of the Schrödinger equation to the bandlimited subspace  $\mathcal{H}$  of the Hilbert space. We shall also show that the above-mentioned naive coordinate representation of the bandlimited Hilbert space  $\mathcal{H}$  uniquely exists and solutions of the bandlimited Schrödinger equation automatically reflect the symmetry that the formulations of the theory on any of the equidistant spatial grids of spacing  $a$  exhibit the same physical content. Moreover, we shall discuss the reconstruction of a unique continuous bandlimited potential from sampled values taken on such grids by means of maximally localized states. The band-limitation will be shown to broaden the peaks and smear out the sudden jumps of the microscopic potential over a region of the Planck scale. Below we shall apply our projection method to determine the free motion of a Gaussian wave packet, as well as the energy shifts of the low-lying stationary states of a particle in a box. The stationary problem shall be treated in the framework of first-order perturbation theory.

It is appropriate to make the following remarks: (i) As to our viewpoint of removing the UV components of the wave functions, it is rather a naive approach to estimate the additional effects due to the existence of the finite UV cutoff. Our viewpoint would be exact when the coordinate space were discrete, but it is not, although the self-adjoint extensions of the coordinate operator have discrete eigenvalues forming a grid on the coordinate axis. However, there exists a one-parameter family of such extensions and that of the corresponding grids which can be transformed into each other by continuous shifts. This indicates that quantum fluctuations of wavelength smaller than the minimal length scale  $a$  probably cannot be excluded completely, but rather should have been treated by more

sophisticated methods, like, e.g., renormalization group methods. Fortunately, in the one-dimensional case the solutions of the bandlimited Schrödinger equation in the naive coordinate representation used by us reflect inherently the physical equivalence of any of those grids. Therefore, our projection technique can give a reliable order-of-magnitude estimate of the importance of the additional effect of the UV cutoff as compared to the pure GUP effect. (ii) As to our choice of model, the particle in a box, it is rather a toy model. The pure GUP effect on the low-lying stationary states has already been discussed and it has been noticed that the model with a precisely given box size  $L$  is ill defined in the sense that a change of box size of the order  $\ell_p$ , i.e., that of the maximal accuracy  $\Delta x_{\min}$  of the position determination, causes an energy shift of the order  $\mathcal{O}(\ell_p/L)$ , as compared to the pure GUP effect of the order  $\mathcal{O}((\ell_p/L)^2)$  [68]. In our approach, the Hamiltonian operates on the subspace of states with finite bandwidth and effectively transforms the originally local potential into a nonlocal one. Projecting out the UV components of the potential results in a kind of smearing out of the edges of the box in regions of the size of  $\mathcal{O}(\Delta x_{\min})$ . Our purpose is to determine the additional shift of the low-lying energy levels caused by the existence of the finite UV cutoff. We shall see that this turns out to be of the order  $\mathcal{O}(\ell_p/L)$ , which is much more significant than the pure GUP effect, so that in some sense the result of Pedram [68] will be recovered. Nevertheless, our approach may be a hint that this kind of energy shift might be the true effect, when the physically realistic box with smeared-out walls is considered and modeled by performing the projection which determines the operation of the Hamiltonian on the band-limited Hilbert space  $\mathcal{H}$ .

Our paper is constructed as follows. In Sec. II the projection method is introduced and the integral kernels for the various projected operators determined. For one-dimensional bandlimited quantum mechanics a justification of the naive coordinate representation is given in Sec. III. A method is given in Sec. IV which enables one to reconstruct a bandlimited continuous potential from sampled values obtained by means of a particle in a maximally localized state. The free motion of a Gaussian wave packet is then discussed in Sec. V in the framework of bandlimited quantum mechanics. In Sec. VI the determination of the shifts of the low-lying energy levels of a particle in a potential is formulated in the framework of the first-order perturbation theory. The problem of a particle in a box is considered in Sec. VII as the limiting case of a particle in a square-well potential taking the limit of infinite depth. After setting some notations in Sec. VII A and recovering the well-known result for the pure GUP effect in Sec. VII B, in Sec. VII C the additional energy shifts of the low-lying levels caused by the existence of the finite UV cutoff are shown to be dominant. It is also argued that the latter might influence the estimates on the upper

bound of the minimal length scale as well. Finally, the results are summarized in Sec. VIII. Several technical details are given in the Appendix. Appendix A reminds the reader of some mathematics relevant for the self-adjoint extension of the Hermitian symmetric coordinate operator. In Appendix B the wave functions of the unperturbed system, i.e., those for a particle in the square-well potential in the framework of usual quantum mechanics are derived. The operation of the GUP modified kinetic energy operator on exponential functions is determined in Appendix C. The details of the evaluation of the additional energy shifts of the low-lying energy levels for a particle in a box are presented in Appendixes D and E. In Appendix F maximally localized states of a particle are constructed. Finally, in Appendix G the bandlimited potential reconstructed from sampled values of a Dirac-delta-like potential is presented.

## II. PROJECTORS ONTO THE SUBSPACE OF WAVE FUNCTIONS WITH FINITE BANDWIDTH

Let  $\psi(x)$ ,  $\tilde{\psi}(p_x)$ , and  $\tilde{\psi}(k_x)$  be the wave functions of the state  $|\psi\rangle$  in the naive coordinate, canonical momentum, and wave-vector representations, respectively. The scalar product of arbitrary states  $|\psi\rangle$  and  $|\phi\rangle$  can be written as

$$\begin{aligned} \langle \phi | \psi \rangle &= \int_{-\infty}^{\infty} dx \phi^*(x) \psi(x) \\ &= \int_{-K}^K \frac{dk_x}{2\pi} \tilde{\phi}^*(k_x) \tilde{\psi}(k_x) \\ &= \int_{-\infty}^{\infty} \frac{dp_x}{2\pi \hbar f(\alpha|p_x|)} \tilde{\phi}^*(p_x) \tilde{\psi}(p_x) \end{aligned} \quad (2)$$

and is kept invariant under the transformation

$$\begin{aligned} \psi(x) &= \int_{-K}^K \frac{dk_x}{2\pi} e^{ik_x x} \tilde{\psi}(k_x), \\ \tilde{\psi}(k_x) &= \int_{-\infty}^{\infty} dx e^{-ik_x x} \psi(x). \end{aligned} \quad (3)$$

An arbitrary square-integrable function  $\psi(x) \in L^2(-\infty, \infty)$  contains ultraviolet (UV) Fourier components with  $|k_x| > K$  as well. In order to ensure that the solutions of quantum mechanical eigenvalue equations, as well as that of the Schrödinger equation, belong to the subspace  $L_K^2(-\infty, \infty)$  of wave functions with finite bandwidth, any operator  $\hat{O}$  of an observable should be projected onto that subspace by an appropriate projector  $\hat{\Pi}$ ,  $\hat{O} \Rightarrow \hat{\Pi} \hat{O} \hat{\Pi}$ . The projector should cut off the UV components of any square-integrable function  $f(x) \in L^2(-\infty, \infty)$ ; i.e., for its kernel  $\Pi(x, y)$  the relation

$$\begin{aligned} (\Pi f)(x) &= \int_{-\infty}^{\infty} dy \Pi(x, y) f(y) \\ &= \int_{-K}^K \frac{dk_x}{2\pi} e^{ik_x x} \tilde{f}(k_x) \in L_K^2(-\infty, \infty) \end{aligned} \quad (4)$$

should hold that implies

$$\Pi(x, y) = \int_{-K}^K \frac{dk_x}{2\pi} e^{ik_x(x-y)} = \frac{\sin[K(x-y)]}{\pi(x-y)}. \quad (5)$$

It is straightforward to show that  $\hat{\Pi}$  is a projector satisfying  $\hat{\Pi}^2 = \hat{\Pi}$ . When  $K \rightarrow \infty$  the operation of any Hermitian symmetric operator  $\hat{O}$  on any state  $|\psi\rangle$  can be represented as

$$\int_{-\infty}^{\infty} dy O(x, y) \psi(y) = \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \tilde{\psi}(k_x) \tilde{O}_{k_x} e^{ik_x x}, \quad (6)$$

where the kernel  $O(x, y)$  and the formal differential operators  $O_x$  and  $\tilde{O}_{k_x}$  are related as

$$O(x, y) = O_x \delta(x - y) = \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} e^{-ik_x y} \tilde{O}_{k_x} e^{ik_x x}. \quad (7)$$

Hermitian symmetry implies  $O(x, y) = [O(y, x)]^*$  and  $\tilde{O}_{k_x} = [\tilde{O}_{-k_x}]^*$ . A few examples are summarized in the table:

| $O_x$  | $\tilde{O}_{k_x}$  |
|--|--|
| $x^n (n \in \mathbb{N})$                     | $\frac{1}{2} [(-i\partial_{k_x}^-)^n + (i\partial_{k_x}^+)^n]$ |
| $(-i\partial_x)^n (n \in \mathbb{N})$        | $k_x^n$  |
| $  - i\partial_x   = \sqrt{(i\partial_x)^2}$ | $ k_x $  |

It is straightforward to show that the symmetrized form of  $\tilde{O}_{k_x}$  for  $O_x = x^n$  with partial derivatives  $\partial_{k_x}^+$  and  $\partial_{k_x}^-$  acting to the right and left, respectively, is in agreement with Heisenberg's commutation relation  $[\hat{x}, \hat{k}_x] = i$ .

In quantum mechanics with finite bandwidth  $K$  the kernel  $O(x, y)$  should be projected as

$$\begin{aligned} (\Pi O \Pi)(x, y) &= \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} du \Pi(x, z) O(z, u) \Pi(u, y) \\ &= \int_{-K}^K \frac{dk_x}{2\pi} e^{-ik_x y} \tilde{O}_{k_x} e^{ik_x x}, \end{aligned} \quad (8)$$

while Hermitian symmetry is preserved,  $[(\Pi O \Pi) \times (y, x)]^* = (\Pi O \Pi)(x, y)$ . Now, one easily finds the kernel of  $\hat{\Pi} \hat{k}_x \hat{\Pi}$ ,

$$(\Pi k_x^n \Pi)(x, y) = \int_{-K}^K \frac{dk_x}{2\pi} e^{ik_x(x-y)} k_x^n = (-i\partial_x)^n \Pi(x, y), \quad (9)$$

and that of  $\hat{\Pi} \hat{x}^n \hat{\Pi}$ ,

$$\begin{aligned} (\Pi x^n \Pi)(x, y) &= \int_{-K}^K \frac{dk_x}{2\pi} e^{-ik_x y} \frac{1}{2} [(-i\partial_{k_x}^-)^n + (i\partial_{k_x}^+)^n] e^{ik_x x} \\ &= \frac{1}{2} (x^n + y^n) \Pi(x, y), \end{aligned} \quad (10)$$

which imply for the kernels of the functions  $f(\hat{k}_x)$  and  $V(\hat{x})$  of operators  $\hat{k}_x$  and  $\hat{x}$ , respectively,

$$\begin{aligned} (\Pi f(k_x) \Pi)(x, y) &= f(-i\partial_x) \Pi(x, y), \\ (\Pi V(x) \Pi)(x, y) &= \frac{1}{2} [V(x) + V(y)] \Pi(x, y). \end{aligned} \quad (11)$$

Making use of the projector  $\hat{\Pi}$  introduced above, the time-dependent and the stationary Schrödinger equations can be written as

$$i\hbar \partial_t |\psi\rangle = \hat{\Pi} \hat{H} \hat{\Pi} |\psi\rangle, \quad (12)$$

$$\hat{\Pi} \hat{H} \hat{\Pi} |\psi\rangle = E |\psi\rangle, \quad (13)$$

respectively, in terms of the projected Hamiltonian  $\hat{\Pi} \hat{H} \hat{\Pi}$  when quantum mechanics with finite bandwidth is considered. The initial condition for the time-dependent Schrödinger equation (12) should be a physical state, i.e., also bandlimited. The usage of the projected Hamiltonian obviously ensures that the state will not contain the UV Fourier components. The rules in Eq. (11) for projecting functions of operators enable one to make both the kinetic energy and the potential energy pieces of the projected Hamiltonian explicit in the naive coordinate representation.

### III. COORDINATE REPRESENTATIONS

In this section we show that the naive coordinate representation for one-dimensional bandlimited quantum mechanics formulated by means of the projection technique proposed in Sec. II is equivalent to any of the discrete coordinate representations based on the complete orthonormal sets of eigenvectors of the various self-adjoint extensions of the coordinate operator. Moreover, the time-dependent Schrödinger equation (12) and the stationary one, Eq. (13) in the naive coordinate representation, as well as their solutions, reflect inherently the  $U(1)$  symmetry which reveals itself in the unique physical content of the formulations of the theory in terms of the various discrete coordinate representations. Therefore, the so-called naive coordinate representation is correct for one-dimensional bandlimited quantum mechanics, although the coordinate wave function loses its direct probability meaning, as opposed to ordinary quantum mechanics.

One might accept the usage of the naive coordinate representation with some reservation because the coordinate operator  $\hat{x}$  is not self-adjoint (see Appendix A). Therefore, no coordinate eigenstates exist in the physical domain and the introduction of the coordinate representation becomes questionable. There exists, however, a one-parameter family of the self-adjoint extensions  $\hat{x}_\theta$  of the coordinate operator, labeled by the parameter  $\theta \in [0, a)$ . Any of these extensions for fixed  $\theta$  exhibits an orthonormal complete set of eigenvectors  $|x_n^\theta\rangle$  in the bandlimited Hilbert space  $\mathcal{H}$ . In the wave-vector representation, the corresponding eigenfunctions  $\langle k_x | x_n^\theta \rangle = \tilde{\psi}_{x_n}^\theta(k_x)$  with  $k_x \in [-K, K]$  are given by Eq. A10 in Appendix A, whereas



the corresponding discrete nondegenerate eigenvalues  $x_n^\theta = an + \theta$  form a grid with spacing  $a$  in the one-dimensional space. Therefore, any vector  $|\psi\rangle \in \mathcal{H}$  can be represented as a linear superposition of the base vectors  $|x_n^\theta\rangle \in \mathcal{H}$  for any given  $\theta$ ,  $|\psi\rangle = \sqrt{a} \sum_{n=-\infty}^{\infty} \langle x_n^\theta | \psi \rangle |x_n^\theta\rangle$ , so that discrete coordinate representations  $\mathcal{R}_\theta$  of the normalized vectors  $|\psi\rangle \in \mathcal{H}$  via the vectors  $\{\psi_n^\theta = \langle x_n^\theta | \psi \rangle\} \in \ell^2$  arise (the normalization implies  $a \sum_{n=-\infty}^{\infty} |\psi_n^\theta|^2 = 1$ ). Then the operators  $\hat{O}$  over the bandlimited Hilbert space  $\mathcal{H}$  should be represented by the countably infinite dimensional matrices  $O_{nn'}^\theta = \langle x_n^\theta | \hat{O} | x_{n'}^\theta \rangle$ . Thus, a one-parameter family of discrete coordinate representations  $\mathcal{R}_\theta$  is available.

Using the wave-vector representation, one realizes immediately that the transformation from a discrete coordinate representation  $\mathcal{R}_\theta$  to another one  $\mathcal{R}_{\theta'}$  belongs to the  $U(1)$  group because  $\tilde{\psi}_{x_n}^{\theta'}(k_x) = e^{ik_x(\theta' - \theta)} \tilde{\psi}_{x_n}^\theta(k_x)$ . Such a transformation means a shift of the spatial grid from  $\{x_n^\theta = na + \theta\}$  to  $\{x_n^{\theta'} = na + \theta'\}$  on which one describes one-dimensional bandlimited quantum mechanics. Nevertheless, there should be a distinction of quantum mechanics discretized on a grid and the case discussed here. In the latter case, space reveals discrete and continuous features at the same time, similar to how information does [36–46]. Namely, the physical content of any of the discrete coordinate representations  $\mathcal{R}_\theta$  should be identical; i.e., the  $U(1)$  group of the transformations among them should be a symmetry. Thus, physics contained in the scalar products

$$\langle \phi | \psi \rangle = a \sum_{n=-\infty}^{\infty} \phi_n^{\theta*} \psi_n^\theta \quad (14)$$

of arbitrary vectors  $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ , should be independent of the particular choice of the representation  $\mathcal{R}_\theta$ . In that respect it is important to underline that the Schrödinger equations obtained by using the projection method are given in their abstract forms in Eqs. (12) and (13) without referring to any representation, and their solutions  $|\psi\rangle$  are automatically in the bandlimited Hilbert space,  $|\psi\rangle \in \mathcal{H}$ .

Now we show that to each vector  $|\psi\rangle \in \mathcal{H}$ , i.e., to each vector  $\{\psi_n^\theta = \langle x_n^\theta | \psi \rangle\}$  of the representation  $\mathcal{R}_\theta$ , one can associate a single coordinate wave function  $\psi(x)$ , and the latter is independent of the choice of the representation  $\mathcal{R}_\theta$  used for its construction.

- (1) To any position  $x \in \mathbb{R}$  belongs exactly a single eigenvalue  $x_{n'}^{\theta'} = an' + \theta' = x$  of a particular self-adjoint extension  $\hat{x}^{\theta'}$  of the coordinate operator (see the discussion at the end of Appendix A). This enables one to construct a wave function of continuous variable,

$$\psi(x) = \langle x_{n'}^{\theta'} = x | \psi \rangle = \sum_{n=-\infty}^{\infty} \langle x_{n'}^{\theta'} = x | x_n^\theta \rangle \psi_n^\theta \quad (15)$$

in a reliable manner, where  $\psi(x_n^\theta) = \psi_n^\theta$  are sampled values of the coordinate wave function on the arbitrarily chosen grid  $\{x_n^\theta\}$ . Here figures the matrix element  $\langle x_{n'}^{\theta'} = x | x_n^\theta \rangle = a \Pi(x, x_n^\theta)$  of the unitary transformation from the discrete coordinate representation characterized by  $\theta$  to the one characterized by  $\theta'$ , which is directly related to the projector  $\hat{\Pi}$  onto the bandlimited Hilbert space. Thus Eq. (15), recast into the form

$$\psi(x) = a \Pi(x, x_n) \psi_n^\theta, \quad (16)$$

can be interpreted as the particular case of the sampling theorem in the bandlimited Hilbert space [38], a generalization of Shannon's sampling theorem [69]. With a similar logic identifying  $x = x_{n'}^{\theta'}$  and  $y = x_{n''}^{\theta''}$  as the  $n'$ th and  $n''$ th eigenvalues of the appropriate self-adjoint extensions  $\hat{x}_{\theta'}$  and  $\hat{x}_{\theta''}$ , respectively, one is enabled to reexpress any matrix element  $(\Pi \hat{O} \Pi)(x, y) = \langle x | \hat{\Pi} \hat{O} \hat{\Pi} | y \rangle$  of an arbitrary operator  $\hat{O}$  in terms of the matrix  $O_{nm}^\theta = \langle x_n^\theta | \hat{\Pi} \hat{O} \hat{\Pi} | x_m^\theta \rangle = \langle x_n^\theta | \hat{O} | x_m^\theta \rangle$  as

$$O(x, y) = a^2 \sum_{n,m=-\infty}^{\infty} \Pi(x, x_n^\theta) O_{nm}^\theta \Pi(x_m^\theta, y), \quad (17)$$

implying  $O(x_n^\theta, x_m^\theta) = O_{nm}^\theta$ . Therefore, to any vector of the bandlimited Hilbert space represented by the vector  $\{\psi_n^\theta\} \in \ell^2$  in the discrete coordinate representation  $\mathcal{R}_\theta$ , there corresponds a continuous coordinate wave function  $\psi(x)$ , and to any operator  $\hat{\Pi} \hat{O} \hat{\Pi}$  mapping the bandlimited Hilbert space into itself, i.e., to any matrix of the discrete coordinate representation  $\mathcal{R}_\theta$ , corresponds the kernel  $O(x, y)$  of the continuous coordinate representation.

- (2) Being aware of the construction of the coordinate wave function  $\psi(x) = \sum_{n=-\infty}^{\infty} \langle x = x_n^{\theta'} | \hat{\Pi} | \psi \rangle$  via Eq. (15), one can insert any of the decompositions of the unit operator  $\hat{\mathbb{I}} = \sum_{n=-\infty}^{\infty} |x_n^\theta\rangle \langle x_n^\theta|$  over  $\mathcal{H}$  associated with any of the representations  $\mathcal{R}_\theta$ . This means, on one hand, that the resulting wave function  $\psi(x)$  associated in the naive coordinate representation to the bandlimited vector  $|\psi\rangle \in \mathcal{H}$  is unique, independent of the representation  $\mathcal{R}_\theta$  used for its construction. On the other hand, it also means that taking sampled values on various spatially shifted grids  $\{x_n^\theta\}$  and inserting those into Eq. (16), one finally obtains the same wave function of the continuous variable  $x$ . Similar arguments lead to the unique kernel  $O(x, y)$  associated to the operator  $\hat{O}$ , mapping  $\mathcal{H}$  into itself.

Thus one can conclude that the solutions of the bandlimited Schrödinger equations for one spatial dimension, being bandlimited themselves, satisfy the generalized sampling theorem expressed in Eq. (16) automatically and therefore reflect inherently the  $U(1)$  symmetry under the continuous

shifts of the spatial grid determined by the discrete eigenvalues of the various self-adjoint extensions of the coordinate operator.

#### IV. BANDLIMITED POTENTIALS

Let us here illustrate the manner in which one could observe the formally local potential  $V(x)$  in the framework of quantum mechanics with finite bandwidth. The best one can do is to construct maximally localized states  $\varphi_{\bar{x}}(x)$  centered at arbitrary positions  $\bar{x}$  [16–19] and detect the potential exerted on it. One should, however, be aware of the fact that there exists only a countable set of physically distinguishable positions  $x_n^\theta$ , those of the eigenvalues of an arbitrarily chosen self-adjoint extension  $\hat{x}^\theta$  of the coordinate operator, which form a grid of spacing  $a = \pi/K$ . Thus, we can sample the potential only at the grid points. Let  $\{|\varphi_n^\theta\rangle\}$  be the sequence of the maximally localized states centered on the grid points. Thus, potential values

$$\bar{V}_n^\theta = \langle \varphi_n^\theta | \hat{\Pi} \hat{V} \hat{\Pi} | \varphi_n^\theta \rangle \quad (18)$$

can only be observed at a discrete set of points of a grid with spacing  $a$ . The maximally localized states  $|\varphi_n^\theta\rangle$  belong to the subspace of bandlimited wave functions,  $\hat{\Pi}|\varphi_n^\theta\rangle = |\varphi_n^\theta\rangle$ , so that Eq. (18) reduces to

$$\bar{V}_n^\theta = \langle \varphi_n^\theta | \hat{V} | \varphi_n^\theta \rangle. \quad (19)$$

According to Shannon's basic sampling theorem on bandlimited real functions [69], a continuous potential  $\bar{V}(\bar{x})$  with finite bandwidth  $K$  ( $k_x \in [-K, K]$ ) can be perfectly reconstructed from its values  $\bar{V}_n^\theta = \bar{V}(x_n^\theta)$  taken on the set of equidistant points  $\{x_n^\theta\}$  spaced  $\frac{2\pi}{2K} = a$  apart:

$$\bar{V}(\bar{x}) = a \sum_{n=-\infty}^{\infty} \bar{V}_n^\theta \Pi(\bar{x} - x_n^\theta). \quad (20)$$

Now one has to show that the reconstructed potential  $\bar{V}(\bar{x})$  is bandlimited and does not depend on the particular choice of the grid, i.e., that of the self-adjoint extension of the coordinate operator. One can choose the sampled values  $\bar{V}_n^\theta$  on the grid  $\{x_n^\theta = na + \theta\}$  shifted by any constant  $\theta \in [0, a)$ . Let  $\mathcal{V}(\bar{x})$  be the function which takes the values  $\mathcal{V}(x_n^\theta) = \bar{V}_n^\theta$  for any  $n \in \mathbb{Z}$  and  $\theta \in [0, a)$ . It is generally not bandlimited, implying

$$\mathcal{V}(x) = \int_{-\infty}^{\infty} \frac{dl}{2\pi} e^{ilx} \tilde{\mathcal{V}}(l). \quad (21)$$

Let us now reconstruct a potential  $\bar{V}(\bar{x})$  from the sampled values  $\bar{V}_n^\theta$  for given  $\theta$  and ask how far the resulting function depends on the particular choice of  $\theta$ , i.e., that of the particular choice of the self-adjoint extension of the coordinate operator  $\hat{x}$ . According to the reconstruction formula in Eq. (20) we get

$$\bar{V}(\bar{x}) = a \sum_{n=-\infty}^{\infty} \mathcal{V}(x_n^\theta) \Pi(\bar{x} - x_n^\theta), \quad (22)$$

which implies

$$\begin{aligned} \bar{V}(\bar{x} + \theta) &= a \sum_{n=-\infty}^{\infty} \mathcal{V}(na + \theta) \Pi(\bar{x} - na) \\ &= a \int_{-\infty}^{\infty} \frac{dl}{2\pi} \tilde{\mathcal{V}}(l) \int_{-K}^K \frac{dq}{2\pi} e^{il\theta + iq\bar{x}} \sum_{n=-\infty}^{\infty} e^{i(l-q)na}. \end{aligned} \quad (23)$$

Making use of the sum (G2), one obtains

$$\bar{V}(\bar{x} + \theta) = \int_{-K}^K \frac{dl}{2\pi} \tilde{\mathcal{V}}(l) e^{il(\bar{x} + \theta)} = \mathcal{V}_K(\bar{x} + \theta), \quad (24)$$

i.e., the reconstructed function  $\bar{V}(\bar{x}) = \mathcal{V}_K(\bar{x})$ , which is independent of the choice of the particular self-adjoint representation of  $\hat{x}$  and is bandlimited,  $\mathcal{V}_K(\bar{x}) = (\hat{\Pi} \mathcal{V})(\bar{x})$ . One cannot, of course, reconstruct the function  $\mathcal{V}(\bar{x})$  which contains modes outside of the band  $[-K, K]$ . Below we shall take the sampled potential values on the grid  $x_n = na$  (for  $\theta = 0$ ) and, for the sake of simplicity, suppress the upper indices  $\theta$ .

In Appendix F we have determined the maximally localized state  $\tilde{\varphi}_{\bar{x}}(k_x)$  in the wave-vector representation by making use of the method of Detournay *et al.* [18]; see Eq. (F7). Rewriting it into the coordinate representation, one finds

$$\begin{aligned} \varphi_{\bar{x}}(x) &= \int_{-K}^K \frac{dk_x}{2\pi} e^{ik_x x} \tilde{\varphi}_{\bar{x}}(k_x) \\ &= \sqrt{\frac{a}{2}} \left[ \Pi\left(x - \bar{x} + \frac{1}{2}a\right) + \Pi\left(x - \bar{x} - \frac{1}{2}a\right) \right], \end{aligned} \quad (25)$$

implying a position inaccuracy of  $\langle \varphi_{\bar{x}} | \hat{x}^2 - \bar{x}^2 | \varphi_{\bar{x}} \rangle = a^2/4$ . The wave function (25) is real, has a maximum at  $x = \bar{x}$ , varies slightly in the small interval  $x \in I_{\bar{x}} \equiv [\bar{x} - \frac{1}{2}a, \bar{x} + \frac{1}{2}a]$  centered at the point  $\bar{x}$ , and falls off rapidly in an oscillatory manner outside of the interval  $I_{\bar{x}}$ . Obviously such a state enables one to detect a bandlimited potential  $\bar{V}(\bar{x})$  smeared out as compared to  $V(x)$ . For example, the Dirac-delta-like potential,  $V(x) = V_0 a \delta(x)$ , is observed as a broadened one with finite height  $V_0$  and finite width,

$$\begin{aligned} \bar{V}_n &= V_0 a |\varphi_{x_n}(0)|^2 \\ &= \frac{V_0 a^2}{2} \left[ \Pi\left(x_n - \frac{1}{2}a\right) + \Pi\left(x_n + \frac{1}{2}a\right) \right]^2. \end{aligned} \quad (26)$$

The reconstructed bandlimited continuous potential is then given by Eq. G6 in Appendix G and is shown in Fig. 1. It is peaked, taking values of the order  $V_0$  in the interval  $\bar{x} \in [-\frac{1}{2}a, \frac{1}{2}a]$ , and falls off rapidly in an oscillatory manner outside of that interval. The characteristic wavelength of the oscillations is  $2a$ .

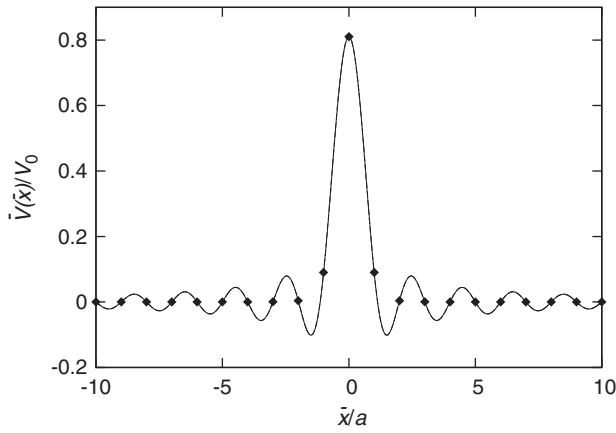


FIG. 1. Sampled values (boxes) and the reconstructed bandlimited continuous potential (solid line) for  $V(x) = V_0 a \delta(x)$ . The numerical result obtained by terminating the sum in Eq. (20) at  $n = \pm 50$  and the analytic one given by Eq. (G4) cannot be distinguished in the figure.

Another example is that the finite jump at  $x = 0$  of the potential step  $V(x) = V_0 \Theta(-x)$  becomes smeared out. The sampled values taken at  $x_n = na$  are

$$\bar{V}_n = V_0 \int_{-\infty}^0 dx |\varphi_{na}(x)|^2. \quad (27)$$

The values  $\bar{V}_n$  are monotonically decreasing with increasing  $n$ . Moreover, the relation  $|\varphi_{na}(x)|^2 = |\varphi_0(x - na)|^2$  holds for the integrand, implying

$$\bar{V}_n = V_0 \int_{-\infty}^{-na} dx |\varphi_0(x)|^2. \quad (28)$$

Because  $\varphi_0(x)$  is even and normalized to 1, one gets  $\bar{V}_0 = \frac{1}{2} V_0$ , and  $\bar{V}_{\pm n} = V_0(\frac{1}{2} \pm r_n)$  with  $r_n = \int_0^{na} dx |\varphi_0(x)|^2$  for  $n > 0$ , where trivially  $r_n$  increases strictly monotonically from  $r_0 = 0$  to 1, with  $n$  going to infinity so that  $\bar{V}_n$  takes the value  $V_0$  for large negative index  $n$ , and decreases strictly monotonically through the value  $\bar{V}_0 = \frac{1}{2} V_0$  at  $n = 0$  to zero when  $n$  increases to infinitely large integer values. So long as the localized state  $\varphi_{na}(x)$  almost entirely extends over a region where the potential is  $V_0$  or zero, it detects its original value, but when it “feels” the sudden jump of the potential step it detects monotonically decreasing values when the increasing sequence  $x_n$  runs through  $x = 0$ . The sampled values already smear out the sudden jump over a region of width  $a$ . Reconstruction of the bandlimited potential  $\bar{V}(\bar{x})$  from the sample shall make the fall of the potential around  $x = 0$  oscillatory. Instead of evaluating analytically the reconstructed potential itself, we shall illustrate the oscillations introduced by the reconstruction with a simpler example. For that purpose, let us choose the sample  $\bar{V}_n = \bar{V}(x_n = na)$  taken of the simple step function  $\bar{V}(x) = V_0 \Theta(-x)$ , for which the well-known integral representation

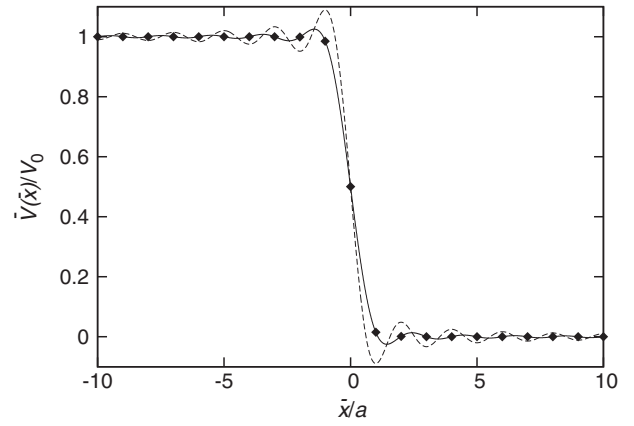


FIG. 2. Sampled values (boxes), the reconstructed bandlimited continuous potential (solid line), and its analytic approximation (dashed line) for the potential step  $V(x) = V_0 \Theta(-x)$ . The numerical result (solid line) has been obtained by terminating the sum in Eq. (20) at  $n = \pm 50$ .

$$\bar{V}(x) = \frac{V_0}{2\pi i} \int_{-\infty}^{\infty} dl \frac{e^{-ilx}}{l - i\epsilon} \quad (29)$$

holds. The bandlimited potential reconstructed from this sample  $\bar{V}_n$  is then given as

$$\bar{V}(x) = \frac{V_0}{2\pi i} \int_{-K}^K dl \frac{e^{-ilx}}{l - i\epsilon}. \quad (30)$$

Let us evaluate this integral by closing the straight-line section  $[-K, K]$  on the real axis via a half circle  $\mathcal{C}_-$  of radius  $K$  on the lower half of the complex  $l$  plane, along which one has  $l = Ke^{-i\alpha}$ , with  $\alpha$  running from zero to  $\pi$ ,

$$\bar{V}(x) = \frac{V_0}{2\pi i} \int_{\mathcal{C}_-} dl \frac{e^{-ilx}}{l} = \frac{V_0}{2\pi} \int_0^\pi d\alpha e^{-iKx \cos \alpha - Kx \sin \alpha}. \quad (31)$$

Changing the integration variable from  $\alpha$  to  $\beta = \alpha - \frac{1}{2}\pi$ , one can recast the integral in the form

$$\begin{aligned} \bar{V}(x) &= \frac{V_0}{\pi} \int_0^{\pi/2} d\beta \cos(Kx \sin \beta) e^{-Kx \cos \beta} \\ &= V_0 \left( \frac{1}{2} - \frac{\text{Si}(Kx)}{\pi} \right). \end{aligned} \quad (32)$$

The reconstructed potential tends to  $V_0$  for  $x \rightarrow -\infty$ , zero for  $x \rightarrow +\infty$ , taking the value  $\frac{1}{2} V_0$  at  $x = 0$ , and it falls in an oscillatory manner from the value  $V_0$  to zero. The reconstructed potential and its analytic approximation in Eq. (32) are demonstrated in Fig. 2.

## V. MOTION OF A WAVE PACKET IN FREE SPACE

The Schrödinger equation for the free motion of a particle in one-dimensional space (say, the  $x$  axis) is

$$i\hbar\partial_t\psi(x,t) = \int_{-\infty}^{\infty} dy(\Pi H^{\text{free}}\Pi)(x,y)\psi(y,t), \quad (33)$$

with the Hamiltonian

$$H^{\text{free}}(x,y) = \frac{1}{2m\alpha^2}[F^{-1}(-i\alpha\hbar\partial_x)]^2\delta(x-y). \quad (34)$$

The Hamiltonian  $\hat{H}^{\text{free}}$ , the operators of the wave vector  $\hat{k}_x$ , and the canonical momentum  $\hat{p}_x = F^{-1}(\alpha\hbar\hat{k}_x)/\alpha$  are pairwise commuting, and their common set of eigenfunctions is the plane waves  $e^{ik_x x}$ . The projector acts on plane waves as

$$\int_{-\infty}^{\infty} dy\Pi(x,y)e^{ik_x y} = \Theta(K - |k_x|)e^{ik_x x}, \quad (35)$$

so that the subset of plane waves with wave vector  $k_x \in [-K, K]$ , canonical momentum  $p_x(k_x) = F^{-1}(\alpha\hbar k_x)/\alpha \in (-\infty, \infty)$ , and energy  $E_{k_x} = [F^{-1}(\alpha\hbar k_x)]^2/(2m\alpha^2) \in [0, \infty)$  eigenvalues, respectively, form the set of common eigenfunctions of the operators  $\hat{\Pi}\hat{H}^{\text{free}}\hat{\Pi}$ ,  $\hat{\Pi}\hat{k}_x\hat{\Pi}$ , and  $\hat{\Pi}\hat{p}_x\hat{\Pi}$  in bandlimited quantum mechanics. For canonical momenta much less than the Planck momentum, i.e., for  $|p_x| \ll (1/\alpha)$ , one can expand the deformation function in powers of the magnitude of  $u = \alpha p_x$  as  $f(u) = 1 + f_1|u| + f_2u^2 + \dots$ , which implies the expansions  $F(u) = u - \frac{1}{2}f_1u|u| + \frac{1}{3}(f_1^2 - f_2)u^3 + \dots$ ,  $F^{-1}(v) = v + \frac{1}{2}f_1v|v| + \frac{1}{6}(f_1^2 + 2f_2)v^3 + \dots$ , with  $v = \alpha\hbar k_x$  and the well-known modification of the energy eigenvalue due to the GUP

$$E_{k_x} \approx \frac{\hbar^2 k_x^2}{2m} \left( 1 + f_1 \alpha \hbar |k_x| + \frac{1}{12}(7f_1^2 + 8f_2)\alpha^2 \hbar^2 k_x^2 + \dots \right) \quad (36)$$

for  $|k_x| \ll K$ . As to the plane waves the only effect due to the finite bandwidth is the lack of plane waves with wave vectors from the UV region.

Now let us seek the solution of the time-dependent Schrödinger equation (33) in the form of a wave packet, i.e., that of a superposition from plane waves belonging to the finite band of wave vectors,

$$\psi(x,t) = \int_{-K}^K \frac{dk_x}{2\pi} a(k_x, t) e^{ik_x x}, \quad (37)$$

$$i\hbar\partial_t a(k_x, t) = \frac{[p_x(k_x)]^2}{2m} a(k_x, t). \quad (38)$$

In order to choose an initial condition  $a_0(k_x) = a(k_x, t=0)$  consistent with the generalized uncertainty relation implied by Eq. (1), one has to be rather cautious. In the wave-vector representation, the coordinate operator  $\hat{x}$  is represented by the formal differential operator  $i\partial_{k_x}$  defined in the dense domain  $\mathcal{D}_x \subset L^2[-K, K]$  of square-integrable wave functions  $\tilde{\psi}(k_x)$ , consisting of absolutely continuous functions whose derivatives also belong to  $L^2[-K, K]$ . The physical domain  $\mathcal{D}_{\text{phys}}$  consisting of the

wave functions satisfying the generalized uncertainty relation should be a subset of the domain  $\mathcal{D}_x$ ,  $\mathcal{D}_{\text{phys}} \subseteq \mathcal{D}_x$ . We should choose the initial condition from that physical domain,  $a_0(k_x) \in \mathcal{D}_{\text{phys}}$ . For  $\hat{x}$  being a symmetric operator, only two types of boundary conditions are allowed [70]. In the set of functions satisfying  $\tilde{\psi}(K) = C\tilde{\psi}(-K)$ , with  $|C| = 1$ , the coordinate operator is self-adjoint, but in our case this should be excluded because in that case the coordinate operators were diagonal in the coordinate representation and the GUP in Eq. (1) could not be satisfied. Thus, the wave functions  $\tilde{\psi}(k_x) \in \mathcal{D}_{\text{phys}}$  should satisfy the other type of boundary conditions, namely, Dirichlet's boundary conditions  $\tilde{\psi}(-K) = \tilde{\psi}(K) = 0$  for which  $\hat{x}$  is not self-adjoint. Therefore, we choose an initial condition  $a_0(k_x)$  satisfying the boundary conditions  $a_0(\pm K) = 0$ ,

$$a_0(k_x) = \mathcal{N} \exp\left(-\frac{[p_x(k_x) - \bar{p}]^2}{4\sigma_p^2}\right), \quad (39)$$

corresponding to a Gaussian wave packet with the mean wave vector  $\bar{k}$ , the mean canonical momentum  $\bar{p} = p_x(\bar{k})$ , the variance  $\sigma_p$ , and the normalization factor  $\mathcal{N}$ . Without loss of generality, one can assume  $\bar{k} > 0$  (i.e.,  $\bar{p} > 0$ ) and  $\sigma_p \ll K\hbar$ . The solution of Eq. (38) with the initial condition (39) is

$$a(k_x, t) = a_0(k_x) \exp\left(-\frac{i}{\hbar} \frac{[p_x(k_x)]^2}{2m} t\right), \quad (40)$$

and the evolution of the wave packet is given by the time-dependent wave function

$$\psi(x,t) = \int_{-K}^K \frac{dk_x}{2\pi} a_0(k_x) \exp\left(-\frac{i}{\hbar} \frac{[p_x(k_x)]^2}{2m} t + ik_x x\right). \quad (41)$$

For the sake of simplicity let us assume that the deformation function  $f(u)$  is analytic at  $u = 0$ . For sufficiently sharp distribution,  $\sigma_p \ll K\hbar$ , one can expand the exponent at the mean  $\bar{k}$  as

$$\text{Exp} = -\frac{i}{\hbar} \frac{\bar{p}^2}{2m} t + i\bar{k}x - A(t)s^2 + iB(x,t)s, \quad (42)$$

where

$$A(t) = \frac{\hbar^2 \bar{f}^2}{4\sigma_p^2} + \frac{i\beta_2 t}{2m\hbar}, \quad B(x,t) = x - \frac{\beta_1 t}{2m\hbar}, \quad (43)$$

$$\beta_1 = 2\hbar\bar{p}\bar{f}, \quad \beta_2 = \hbar^2\bar{f}(\bar{f} + 2\alpha^2\bar{p}^2\bar{f}'),$$

with  $s = k_x - \bar{k}$ ,  $\bar{p} = p_x(\bar{k})$ ,  $\bar{f} = f(\alpha^2\bar{p}^2) > 1$ ,  $\bar{f}' = f'(\alpha^2\bar{p}^2)$ ,  $f'(u) = df(u)/du$ . The time-dependent wave function (41) can then be rewritten as



$$\psi(x, t) = \mathcal{N} \exp\left(-\frac{i}{\hbar} \frac{\bar{p}^2}{2m} t + i\bar{k}x\right) \times \int_{-K+\bar{k}}^{K+\bar{k}} \frac{ds}{2\pi} e^{-A(t)s^2 + iB(x,t)s}. \quad (44)$$

Let us now make use of  $\sigma_k \ll K$  and consider the limiting cases (i)  $\bar{k} = 0$  and (ii)  $\bar{k} = K$ . As a good approximation we can replace the definite integrals by improper ones for case (i),  $\int_{-K}^K dk_x \Rightarrow \int_{-\infty}^{\infty} dk_x$ , and for case (ii),  $\int_{-2K}^0 dk_x \Rightarrow \int_{-\infty}^0 dk_x$ . Because of the inequality  $\text{Re}A(t) > 0$ , the integral remains convergent and one ends up with

$$\psi(x, t) \propto A^{-1/2}(t) \exp\left(-\frac{i}{\hbar} \frac{\bar{p}^2}{2m} t + i\bar{k}x - \frac{B^2(x, t)}{4A(t)}\right), \quad (45)$$

implying the Gaussian spatial distribution

$$|\psi(x, t)|^2 \propto A^{-1}(t) \exp\left(-\frac{(x - \bar{v}t)^2}{2\sigma_x^2(t)}\right), \quad (46)$$

centered at the position  $\bar{x} = \bar{v}t$  at time  $t$  and moving with the speed  $\bar{v} = \bar{f} \bar{p} / m > \bar{p} / m$ . Therefore, the center of the wave packet moves with a larger speed in the bandlimited case than in ordinary quantum mechanics. The variance of the position distribution is given by

$$\sigma_x^2(t) = \frac{|A(t)|^2}{\text{Re}A(t)} = \frac{\hbar^2 \bar{f}^2}{4\sigma_p^2} \left[1 + \frac{t^2}{\tau^2}\right], \quad (47)$$

where the characteristic time for the spreading of the wave packet is given as

$$\tau = \tau_0 \frac{\bar{f}}{\bar{f} + 2\alpha^2 \bar{p}^2 \bar{f}'} < \tau_0 \quad (48)$$

in terms of the characteristic spreading time  $\tau_0 = m\hbar/(2\sigma_p^2)$  in ordinary quantum mechanics. We see that the finite bandwidth can cause a much faster spread of the wave packet when its mean wave vector approaches the limiting value  $K$ .

## VI. STATIONARY STATES OF A PARTICLE IN A POTENTIAL

Let us discuss now the problem of stationary states of a particle in a potential  $V(x)$ . Even if GUP does not imply a finite wave-vector cutoff, it results in the modified kinetic energy operator  $\hat{H}^{\text{free}}$ , introduced in the previous section, so that the Hamiltonian can be written as  $\hat{H} = \hat{H}^{\text{free}} + \hat{V}$ . In order to determine the low-energy states, the nondegenerate stationary perturbation expansion has been widely applied with an expansion in powers of the small parameter  $\alpha$  (see, e.g., Refs. [22–24, 47–61]). When additionally even a finite bandwidth is enforced by GUP, the projected Hamiltonian  $\hat{\Pi}(\hat{H}^{\text{free}} + \hat{V})\hat{\Pi}$  figures in the stationary Schrödinger equation (13). Now we shall use the perturbation expansion for the low-energy states in another manner, without expanding in the small parameter  $\alpha$ . Namely, we

shall simply say that the whole GUP effect, including the effect of projection to states with finite bandwidth, is a perturbation and account for it in the first order. Therefore, we split the projected Hamiltonian as

$$\hat{\Pi} \hat{H} \hat{\Pi} = \hat{H}_0 + \hat{H}', \quad (49)$$

where  $\hat{H}_0 = \hat{T}_0 + \hat{V}$  is the Hamiltonian in ordinary quantum mechanics with the usual kinetic energy operator  $\hat{T}_0 = \hbar^2 \hat{k}_x^2 / (2m)$ , and

$$\hat{H}' = \hat{h} + \hat{v} + \hat{t} \quad (50)$$

represents the perturbation caused by GUP and the restriction to finite bandwidth. The latter consists of the pure GUP effect  $\hat{h} = \hat{H} - \hat{H}_0 = \hat{H}^{\text{free}} - \hat{T}_0$  discussed widely in the literature, the pieces  $\hat{v} = \hat{\Pi} \hat{V} \hat{\Pi} - \hat{V}$  and  $\hat{t} = \hat{\Pi} \hat{H}^{\text{free}} \hat{\Pi} - \hat{T}_0$  responsible for the additional modification of the potential, and the kinetic energy operator due to the restriction of the states to those with finite bandwidth. Obviously, the projection alters the local potential and kinetic energy into nonlocal quantities.

Let  $\{\varphi_n\}$  be the complete set of eigenstates of the unperturbed Hamiltonian  $\hat{H}_0$ ,

$$\hat{H}_0 |\varphi_n\rangle = \epsilon_n |\varphi_n\rangle; \quad (51)$$

then the energy levels  $E_n = \epsilon_n + \Delta\epsilon_n$  of the perturbed system are shifted by

$$\Delta\epsilon_n = \langle \varphi_n | \hat{H}' | \varphi_n \rangle = h_{nn} + v_{nn} + t_{nn}, \quad (52)$$

where  $h_{nn} = \langle \varphi_n | \hat{h} | \varphi_n \rangle$ ,  $v_{nn} = \langle \varphi_n | \hat{v} | \varphi_n \rangle$ , and  $t_{nn} = \langle \varphi_n | \hat{t} | \varphi_n \rangle$  represent the energy shifts caused by the pure GUP effect and by the projection of the potential and that of the kinetic energy, respectively.

## VII. TOY MODEL: PARTICLE IN A BOX

### A. Energy shifts of stationary states

We shall apply the method described in the previous section to a toy model, a particle bounded in a square-well potential

$$V(x) = V_0 [\Theta(-x) + \Theta(x - L)] \quad (53)$$

of width  $L$ , and let the depth of the potential well  $V_0$  go finally to infinity. Although a sudden jump of the potential is unrealistic in bandlimited quantum mechanics as emphasized in Ref. [68], in our treatment that problem shall be cured by a projection that makes potential edges effectively smeared out over a range of the order  $a$ , as argued in Sec. IV previously.

When GUP effects (including finite bandwidth) are neglected, the solutions  $\varphi_n(x)$  corresponding to the unperturbed bound states with energy  $\epsilon_n < V_0$  in the square-well potential are given in Appendix B. The expressions for asymptotically large depth  $V_0$  of the potential, i.e., for states with  $\epsilon_n/V_0 \ll 1$ , are also given. The matrix

elements  $h_{nn}$ ,  $v_{nn}$ , and  $t_{nn}$  contributing additively to the energy shift can be expressed in terms of the various pieces  $\varphi_i(x)$  ( $i = I, II, III$ ) of the wave function defined in the intervals  $I_i$ , respectively (cf. Appendix B). Since the operator  $\hat{h}$  is local, while the operators  $\hat{v}$  and  $\hat{t}$  are nonlocal due to the projection operator, we can write their matrix elements in the form

$$\begin{aligned} h_{nn} &= \sum_{i=I}^{III} h_i, & h_i &= \int_{I_i} dx \varphi_i^*(x) h_x \varphi_i(x), \\ v_{nn} &= \sum_{i,j=I}^{III} v_{i,j}, & v_{i,j} &= \int_{I_i} dx \int_{I_j} dy \varphi_i^*(x) v(x,y) \varphi_j(y), \\ t_{nn} &= \sum_{i,j=I}^{III} t_{i,j}, & t_{i,j} &= \int_{I_i} dx \int_{I_j} dy \varphi_i^*(x) t(x,y) \varphi_j(y), \end{aligned} \quad (54)$$

where

$$\begin{aligned} h_x &= \frac{1}{2m} \left( \frac{1}{\alpha^2} [F^{-1}(-i\alpha\hbar\partial_x)]^2 - (-\hbar^2\partial_x^2) \right), \\ v(x,y) &= \frac{1}{2} [V(x) + V(y)] [\Pi(x,y) - \delta(x-y)], \\ t(x,y) &= \frac{1}{2m\alpha^2} [F^{-1}(-i\alpha\hbar\partial_x)]^2 [\Pi(x,y) - \delta(x-y)] \end{aligned} \quad (55)$$

are the appropriate kernels. Hermitian symmetry of the operators  $\hat{h}$ ,  $\hat{v}$ , and  $\hat{t}$ ; reflection symmetry of the potential to  $x = L/2$ ; and the operator  $[F^{-1}(\alpha\hbar\hat{k}_x)]^2$  being even lead to the symmetry relations

$$\begin{aligned} h_I &= h_{III}, \\ v_{j,i} &= (v_{i,j})^*, & v_{III,III} &= v_{I,I}, & v_{III,II} &= \pm v_{I,II}, \\ t_{j,i} &= (t_{i,j})^*, & t_{III,III} &= t_{I,I}, & t_{III,II} &= \pm t_{I,II}. \end{aligned} \quad (56)$$

Here the  $\pm$  signs correspond to eigenstates characterized by the wave vectors  $k_{\pm}$ . Furthermore,  $v_{II,II} = 0$  trivially because  $V(x) = 0$  for  $x \in I_{II}$ .

### B. Shift due to pure GUP effect

We call the pure GUP effect the energy shift  $h_{nn}$  of stationary states because of the modification of the canonical momentum and that of the kinetic energy from  $\hbar\hat{k}_x$  and  $(\hbar\hat{k}_x)^2/(2m)$  in ordinary quantum mechanics to  $\alpha^{-1}F^{-1}(\alpha\hbar\hat{k}_x)$  and  $(2m\alpha^2)^{-1}[F^{-1}(\alpha\hbar\hat{k}_x)]^2$  when GUP is at work. Making use of the results of Appendix C, one finds that the functions  $\varphi_i(x)$  ( $i = I, II, III$ ) are eigenfunctions of the kinetic energy operator. Then one gets

$$\begin{aligned} h_{II} &= \frac{1}{2m} \left( \frac{1}{\alpha^2} [F^{-1}(\alpha\hbar k_{\pm})]^2 - \hbar^2 k_{\pm}^2 \right) \int_{I_{II}} dx |\varphi_{II}(x)|^2, \\ h_I &= \frac{1}{2m} \left( \frac{1}{\alpha^2} [F^{-1}(-i\alpha\hbar\kappa)]^2 + (\hbar\kappa_{\pm})^2 \right) \int_{I_I} dx |\varphi_I(x)|^2 \\ &= h_{III}, \end{aligned} \quad (57)$$

and the energy shift due to the pure GUP effect

$$\begin{aligned} h_{nn} &= \left[ \left( 1 + \frac{\sin k_{\pm} L}{k_{\pm} L} \right) ([F^{-1}(\alpha\hbar k_{\pm})]^2 - (\alpha\hbar k_{\pm})^2) \right. \\ &\quad \left. + \frac{|\rho_{\pm}|^2}{2\kappa_{\pm} L} ([F^{-1}(-i\alpha\hbar\kappa_{\pm})]^2 + (\alpha\hbar\kappa_{\pm})^2) \right] \\ &\quad \times \left[ 2m\alpha^2 \left( 1 + \frac{\sin k_{\pm} L}{k_{\pm} L} + \frac{|\rho_{\pm}|^2}{2\kappa_{\pm} L} \right) \right]^{-1}. \end{aligned} \quad (58)$$

Let us consider now the limit  $\kappa_{\pm} \rightarrow \infty$ , the case of a particle in a box. In that limit  $|\rho_{\pm}|^2 \sim k_{\pm}^2/\kappa_{\pm}^2$  and, consequently, one obtains a finite energy shift if and only if the limit  $\lim_{\kappa \rightarrow \infty} [F^{-1}(-i\alpha\hbar\kappa)]^2 \kappa^{-3} = C_{\infty}$  remains finite. We shall assume that only such deformation functions for which  $C_{\infty} = 0$  are physically reasonable, which means that the tails of the wave function in the outer regions  $I_I$  and  $I_{III}$  of the square-well potential give vanishing contributions to the kinetic energy when the depth of the potential becomes infinite. The deformation function  $f = 1 + \alpha^2 p_x^2$ , with  $F^{-1}(u) = \tan u$ , satisfies that condition because the limit  $\lim_{u \rightarrow \pm\infty} \tanh u = \pm 1$  is finite. Finally, we end up with the pure GUP energy shift  $h_{nn} = R_h \epsilon_n$ , with

$$R_h = \left( \frac{F^{-1}(\alpha\hbar k_{\pm})}{\alpha\hbar k_{\pm}} \right)^2 - 1, \quad (59)$$

for any deformation function that is reasonable in the above discussed sense. For highly excited states characterized by wave vectors  $k_{\pm} \approx K$  close to the UV cutoff, the ratio  $R_h$  explodes, which signals simply that the perturbation expansion ceases to work. For low-lying states to which our approach is applicable, the expansion in the small parameter  $\alpha\hbar k_{\pm} = n\alpha\hbar\pi/L$  yields

$$R_h \approx \frac{\alpha\hbar\pi}{L} \left[ f_1 n + \frac{1}{12} (7f_1^2 + 8f_2) \frac{\alpha\hbar\pi}{L} n^2 + \dots \right]. \quad (60)$$

For  $h\alpha = \ell_P$  and  $f = 1 + \alpha^2 p_x^2$ , one gets  $f_1 = 0$  and  $f_2 = 1$  and the ratio

$$R_h \approx \frac{2}{3} \left( \frac{\ell_P}{2L} \right)^2 n^2 + \mathcal{O}((n\ell_P/L)^2), \quad (61)$$

which rises quadratically with increasing  $n$  and is independent of the mass of the particle in the box. Thus, we recovered the result obtained in Refs. [61,71] [given after Eq. (14) for  $j = 1$ ]. An order-of-magnitude estimate gives  $(\ell_P/L)^2 \approx 10^{-40}$ ,  $10^{-50}$ , and  $10^{-58}$  for boxes of the size  $L = 10^{-15}$  m (size of a nucleon),  $10^{-10}$  m (size of a

hydrogen atom), and  $10^{-6}$  m (the wavelength of infrared radiation), respectively. So even for the first few thousands of energy levels the pure GUP correction remains a tiny correction.

### C. Shift due to finite bandwidth

Finite bandwidth, i.e., the existence of the finite UV wave-vector cutoff  $K$ , results in the absence of quantum fluctuations with UV wave vectors  $|k_x| > K$  and is expressed in our approach by the projection of the operators of potential and kinetic energies. The energy shift  $v_{nn}$  of the  $n$ th energy level caused by the replacement of the potential by its projected counterpart can be expressed in terms of the independent integrals  $v_{I,I}$ ,  $v_{I,II}$ , and  $v_{I,III}$  when the symmetry relations discussed above are accounted for. Here  $v_{I,I}$  and  $v_{I,III}$  are real because  $\varphi_I(x)$  and  $\varphi_{III}(x)$  are real functions. As discussed in Appendix D the leading order contribution comes from the integral  $v_{I,II}$  in the limit of infinite potential depth, while the other independent integrals  $v_{i,j}$  are suppressed like powers of  $1/\kappa_{\pm}$  as compared to it. One finds [cf. Eq. (D12)] that  $v_{nn}$  vanishes for the energy levels  $n$  even and for the energy levels  $n$  odd it is given as [cf. Eqs. (D10)–(D12)]

$$v_{nn} \approx \frac{\hbar^2 k}{2mL} \left( \frac{2k}{K\pi} [4\sin^2(\nu\pi) + (n\pi)^2 \cos(2\nu\pi) + \mathcal{O}(n^4)] + \mathcal{O}((k/K)^2) \right). \quad (62)$$

The vanishing of  $v_{nn}$  for even  $n$  is a consequence of the particular form of the wave function in a square-well potential, namely, the alternating sign of the tail of the wave function in the region *III* with the alteration of even and odd  $n$  values in the numeration of the stationary states with increasing energy. The ratio of the energy shift  $v_{nn}$  for odd  $n$  to the unperturbed energy  $\epsilon_n$  of the stationary state  $n$ ,

$$R_v = \frac{v_{nn}}{\epsilon_n} \approx \frac{2}{KL\pi} [4\sin^2(\nu\pi) + (n\pi)^2 \cos(2\nu\pi) + \mathcal{O}(n^4)] + \mathcal{O}((\ell_p/L)^2), \quad (63)$$

turns out to take values of the order  $(\ell_p/L)$ .

Thus, the potential energy shift due to the UV cutoff seems to be many orders of magnitude larger—at least for the lowest energy levels—as compared to the energy shift caused by pure GUP, because it holds  $R_h/R_v \approx \mathcal{O}(\ell_p/L)$ . It is remarkable that  $R_v$  oscillates strongly with the fine-tuning of the length of the box  $L$  confining the particle. The variation of  $\nu$  in the interval  $[0, 1]$  corresponds to the tiny change of the box size  $L$  in a range of  $\mathcal{O}(\ell_p) = \mathcal{O}(a)$ , the size of the grid constant, as well as that of the maximal accuracy  $\Delta x_{\min}$  of position determination. Therefore, an averaging over  $\nu$  might be more reliable when one wants to

incorporate the indefiniteness of the size of the box, a direct consequence of the impossibility to determine positions more precisely than the distance  $\Delta x_{\min}$ . This yields then

$$\int_0^1 d\nu R_v \approx \frac{4}{KL\pi} [1 + \mathcal{O}(n^4)] + \mathcal{O}((KL)^{-2}). \quad (64)$$

An order-of-magnitude estimate gives  $(KL)^{-1} \sim (\ell_p/L) \approx 10^{-20}$ ,  $10^{-25}$ , and  $10^{-29}$  for boxes of the size  $L = 10^{-15}$  m (size of a nucleon),  $10^{-10}$  m (size of a hydrogen atom), and  $10^{-6}$  m (the wavelength of infrared radiation), respectively. These are still small effects but they are 20 to 30 orders of magnitude larger than the energy shift due to pure GUP without wave-vector cutoff.

Another contribution  $t_{nn}$  arises due to finite bandwidth, which represents the difference of the expectation values of the projected and unprojected kinetic energy operators. According to the symmetry relations, the only independent integrals contributing to the kinetic energy shift  $t_{nn}$  are  $t_{I,I}$ ,  $t_{II,I}$ ,  $t_{III,I}$ ,  $t_{II,II}$  as given in Eqs. E1 in Appendix D. Among them  $t_{II,II}$  is the only one surviving the limit  $\kappa \rightarrow \infty$ , that of taking the square-well potential with infinite depth. According to Eq. (E10) and the estimate in Eq. (E12) one obtains

$$t_{II,II} \approx R_t(1 + R_h)\epsilon_n \approx R_t\epsilon_n \quad (65)$$

with

$$R_t(\nu) \approx \frac{t_{II,II}}{\epsilon_n} \approx I_K(L/2) - 1. \quad (66)$$

Here  $R_t(\nu)$  is the ratio defined as the additional shift of the expectation value of the kinetic energy due to the finite bandwidth divided by the unperturbed energy of the stationary state  $n$ . While the additional energy shift  $v_{nn}$  appeared to be vanishing for states with even  $n$ , the shift  $t_{nn}$  is nonvanishing for all  $n$ . As seen in Eq. (E12) this ratio oscillates with the fine-tuning of the length  $L$  of the potential box again. Referring to the impossibility of determining positions and distances more precisely than  $\Delta x_{\min}$  in bandlimited quantum mechanics, one can perform averaging over  $\nu \in [0, 1]$ , similarly to the case for the shift of the expectation value of the potential operator due to the finite bandwidth. The averaging over the interval  $\nu \in [0, 1]$  results in

$$\int_0^1 d\nu R_t(\nu) \approx \frac{4}{KL\pi} + \mathcal{O}((KL)^{-2}), \quad (67)$$

which is the same as the relative contribution of the shift of the potential energy in Eq. (64).

It is worthwhile to come back now to the viewpoint of considering the length scale  $\Delta x_{\min}$  ( $\ell_p$ ) as an unknown parameter and looking for its experimental upper bound [25–33], as mentioned in the introduction. The additional effect of the existence of the UV wave-vector cutoff  $K \sim 1/\ell_p$ , taken into account by us by means of the projection method, may have influence on the estimates

of such upper bounds. This can be illustrated—although in a rather crude manner—by the following argument. Let us consider, for example, the 1S Lamb shift  $\Delta f_L$  of the hydrogen atom and ascribe the tiny discrepancy of the best available experimental and theoretical values  $\delta f = \Delta f_L^{\text{exp}} - \Delta f_L^{\text{th}} \approx 1.06 \times 10^5 \text{ s}^{-1}$  to the GUP effect like the authors did in Refs. [30,31]. The lesson learned from our treatment of the particle in a one-dimensional box is that the additional effect of the existence of the UV wave-vector cutoff being of the order  $(\ell_p/L)$  is much larger than the direct GUP effect caused by the modification of the free particle's Hamiltonian of the order  $(\ell_p/L)^2$ . An energy shift of the same order  $(\ell_p/L)^2$  has also been found in Refs. [30,31] for the energy shift of the ground state of the hydrogen atom because the modification of the Hamiltonian due to the direct GUP effect is of that order of magnitude. Let us assume that the indirect effect of GUP via the existence of the UV wave-vector cutoff is of the order  $\ell_p/R$  for the hydrogen-atom ground state, where the Bohr radius  $R = 5.29 \times 10^{-11} \text{ m}$  stands as the box size. Then we expect the relative energy shift of the ground state to be of the order  $\delta f/f_{1S} \sim \ell_p/R$ , where  $f_{1S} \approx 2.447 \times 10^{14} \text{ s}^{-1}$  is related to the energy  $E = -hf_{1S} = -13.6 \text{ eV}$  of the ground state. This would yield an upper bound  $\ell_p \lesssim R\delta f/f_{1S} \approx 2.3 \times 10^{-20} \text{ m}$  ( $1.2 \times 10^{-4} \text{ GeV}^{-1}$ ), which is considerably smaller than the one obtained in Refs. [30,31]. Without taking it word for word, we believe that our very crude estimate should rather be taken as a motivation for the redetermination of the possible bounds on the characteristic length scale while taking the additional effect due to the existence of the UV cutoff into account.

### VIII. SUMMARY

The free motion of a wave packet and the energy levels of a particle in a box have been discussed in one-dimensional quantum mechanics when the existence of a finite bandwidth, i.e., that of an UV wave-vector cutoff  $K$  is present as a consequence of the GUP. The latter is implemented by generalizing Heisenberg's commutation relations for quantization with the particular choice of the deformation function  $f(|u|)$  ( $u = \alpha p_x$ ) occurring in the commutator relation for the coordinate  $\hat{x}$  and the canonical momentum  $\hat{p}_x$ . Deformation functions  $f(|u|)$ , which strictly monotonically increase with  $|u|$ , for which  $F(u) = \int_0^u \frac{du'}{f(|u'|)}$  remains finite in the limits  $u \rightarrow \infty$ , provide such an UV cutoff,  $F(\pm\infty) = \pm\alpha\hbar K$ . We took the point of view that the existence of the UV wave number cutoff, corresponding to an infinite canonical momentum eigenvalue, excludes the UV components of the wave function. In order to enforce this in the naive coordinate representation, the Hamiltonian and all operators of observables should be sandwiched by a projector restricting wave functions to the subspace of bandlimited wave functions. Such a projector

$\hat{\Pi}$  has been constructed and the rules for the operators acting on the bandlimited subspace have been established. It has also been shown that the proposed projection method justifies the usage of the naive coordinate representation through a generalization of Shannon's basic sampling theorem taken from information theory to one-dimensional bandlimited quantum mechanics. We have discussed the relation of the naive coordinate representation using coordinate wave functions of continuous variables to discrete coordinate representations based on the self-adjoint extensions of the coordinate operator.

A method is proposed to observe potential values exerted on a particle by means of preparing it in a state of maximal localization. Although any self-adjoint extension of the coordinate operator enables one to take such sampled values at the discrete points of equidistant spacing  $a = \pi/K$ , the reconstruction of a bandlimited continuous potential is possible according to Shannon's basic sampling theorem on bandlimited real functions. Applying that reconstruction procedure, the broadening of the Dirac-delta-like potential and the smearing out of the potential step over a region of the order of the spacing  $a$  have been shown, both accompanied with oscillations of wavelength of  $\sim 2a$ , as well.

It has been shown that the free motion of the wave packet is modified as a consequence of the finite bandwidth. The center of a Gaussian wave packet with mean canonical momentum  $\bar{p}$  moves with a speed  $\bar{V}$  larger than  $\bar{v} = \bar{p}/m$  [cf. Eq. (46)] and the spreading time  $\tau$  of the wave packet gets smaller than the corresponding characteristic time  $\tau_0$  in usual quantum mechanics [cf. Eq. (48)]. The ratios  $\bar{V}/\bar{v}$  and  $\tau/\tau_0$  are strictly monotonically increasing and decreasing functions, respectively, of the mean momentum  $\bar{p}$ .

The shifts of the low-lying energy levels of a particle in a box have been determined considering the effect of GUP and the additional effect caused by the UV cutoff in first-order perturbation theory. For the pure GUP effect the well-known result has been recovered and is of the order  $(\ell_p/L)^2$  for the box size  $L$  (in terms of the Planck length  $\ell_p$ ) and the deformation function  $f = 1 + \alpha p_x^2$ . The additional effect caused by the UV cutoff, i.e., by the projection to the subspace of states with finite bandwidth, has occurred as the shift of the expectation values of both the potential and kinetic energy operators and turned out to be of the order  $\ell_p/L$ . This result indicates that the effect of GUP on low-energy quantum mechanics may be much more significant indirectly, through the existence of the finite UV cutoff, than directly by providing small correction terms to the Hamiltonian. It is also remarkable that the additional effect caused by the UV cutoff appeared to have an oscillatory dependence on the variation of the box size  $L$  in a range of the minimal accuracy of position determination. We have suggested that any observation of the box size should average over such a range principally;



therefore, the true correction should be averaged over that range as well. We also argued that the existence of the UV wave-vector cutoff might affect the determination of the upper bounds on the length scale  $\ell_P$  when it is not taken *a priori* for the Planck length but instead constrained either by the comparison of high-accuracy experimental data with theoretical calculations or by ascribing the tiny error of experimental data entirely to the GUP effect.

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### APPENDIX A: SELF-ADJOINT EXTENSIONS OF THE COORDINATE OPERATOR

Here we give a short summary of some mathematical results [70] relevant for the self-adjoint extension of the coordinate operator (cf. also Refs. [14,36–42]). In the wave-vector representation, the states are represented by the wave-vector wave functions  $\tilde{\psi}(k_x)$ , and the coordinate operator by the formal differential operator  $i\partial_{k_x}$  (it is called formal because the definition of an operator should also include the boundary conditions set on the functions on which it operates). The coordinate operator  $\hat{x}$  is defined in the dense domain  $\mathcal{D}(\hat{x}) \subset L^2[-K, K]$  of those square-integrable functions, which are absolutely continuous (implying infinite differentiability) inside the interval  $[-K, K]$ , and whose derivatives also belong to  $L^2[-K, K]$ . The physical domain  $\mathcal{D}_{\text{phys}}$  consisting of the wave functions satisfying the generalized uncertainty relation implied by GUP in Eq. (1) should be a subset of the domain  $\mathcal{D}(\hat{x})$ ,  $\mathcal{D}_{\text{phys}} \subseteq \mathcal{D}(\hat{x})$ . In order to have real expectation values, the coordinate operator  $\hat{x}$  should be symmetric. Therefore, only two types of boundary conditions are allowed to be set at  $k_x = \pm K$  for the wave functions  $\tilde{\psi}(k_x)$ . It is well known that the operator  $i\partial_{k_x}$  is self-adjoint for the boundary conditions  $\tilde{\psi}(K) = C\tilde{\psi}(-K)$  with  $|C| = 1$ , i.e.,  $C = e^{i\alpha}$  with  $\alpha \in [0, 2\pi)$ , and it is not self-adjoint, although Hermitian symmetric for Dirichlet's boundary conditions  $\tilde{\psi}(-K) = \tilde{\psi}(K) = 0$ . If  $\hat{x}$  were self-adjoint in the physical domain  $\mathcal{D}_{\text{phys}}$ , its eigenstates  $e^{-ik_x x}$  would belong to that domain. This cannot, however, be the case because the formal eigenfunctions of the operator  $i\partial_{k_x}$  cannot satisfy the boundary condition  $e^{-iKx} = e^{i\alpha + iKx}$  for arbitrary eigenvalue  $x$ . One can also argue that if  $\hat{x}$  were self-adjoint in the physical domain, the coordinate eigenstates with zero position uncertainty would be physical states and that would contradict GUP by implying a wave-vector cutoff, i.e., a nonvanishing minimal position uncertainty. Therefore, the only remaining possibility is that  $\hat{x}$  is symmetric on functions with Dirichlet's boundary condition.

In this case the adjoint operator  $\hat{x}^\dagger$ , represented formally also by  $i\partial_{k_x}$ , has the domain  $\mathcal{D}(\hat{x}^\dagger)$  consisting of all differentiable functions of  $L^2[-K, K]$ , whose derivatives also belong to  $L^2[-K, K]$ , and which are not restricted by any kind of boundary condition. Thus, clearly one has  $\mathcal{D}(\hat{x}) \subset \mathcal{D}(\hat{x}^\dagger)$ . Symmetric operators can be characterized by their deficiency indices  $\nu_\pm$  and the dimensions of the null spaces of the operators  $\hat{x}^\dagger - (\pm i)\xi$ , where  $\xi \in \mathbb{R}$ . The solutions of the equations

$$0 = (i\partial_{k_x} \mp i\xi)\tilde{\varphi}_\pm, \quad \xi \in \mathbb{R} \quad (\text{A1})$$

are the square-integrable functions  $\tilde{\varphi}_\pm(x) = e^{\pm\xi k_x}$ , which span one-dimensional null spaces, so that the deficiency indices of  $\hat{x}$  are equal,  $\nu_+ = \nu_- \equiv \nu = 1$ . Then there exist self-adjoint extensions  $\hat{x}_e = \hat{x}_e^\dagger$  of  $\hat{x}$ . According to the general theory, these can be constructed by means of the boundary conditions prescribed for the functions of the domain  $\mathcal{D}(\hat{x}_e)$ . In the case with  $\nu = 1$  a single boundary condition is needed to specify the domains  $\mathcal{D}(\hat{x}_e)$  of the self-adjoint extension  $\hat{x}_e$  and that can be done in terms of a single function  $f_1(k_x) \in \mathcal{D}(\hat{x}^\dagger)$  being linearly independent relative to  $\mathcal{D}(\hat{x})$  (because  $\hat{x}$  is a closed operator):

$$\begin{aligned} \mathcal{D}(\hat{x}_e) &= \{f | f \in \mathcal{D}(\hat{x}^\dagger), [f(k_x)f_1^*(k_x)]_{-K}^K = 0, \\ &[f_1(k_x)f_1^*(k_x)]_{-K}^K = 0, f_1 \in \mathcal{D}(\hat{x}^\dagger)\}, \end{aligned} \quad (\text{A2})$$

with  $[fg^*]_a^b = f(b)g^*(b) - f(a)g^*(a)$ . The function  $f_1$  linearly independent relative to  $\mathcal{D}(\hat{x})$  can be constructed as the linear combination  $f_1(k_x) = \beta h_1(k_x) + \gamma h_{-1}(k_x)$  ( $\beta, \gamma \in \mathbb{C}$ ) of a number of  $2\nu = 2$  functions,  $h_s(k_x)$  ( $s = 1, -1$ ), which are linearly independent relative to  $\mathcal{D}(\hat{x})$ , i.e., for which the relations

$$[h_r h_s]_{-K}^K = r(2K)^2 \delta_{rs}, \quad r, s = 1, -1 \quad (\text{A3})$$

hold. These are the functions  $h_1 = K + k_x$  and  $h_{-1} = K - k_x$ . Then one finds  $f_1(k_x) = \beta(K + k_x) + \gamma(K - k_x)$  and

$$\begin{aligned} 0 &= [f_1 f_1]_{-K}^K = \beta\beta^* h_1^2(K) - \gamma\gamma^* h_{-1}^2(-K) \\ &= (2K)^2(\beta\beta^* - \gamma\gamma^*), \end{aligned} \quad (\text{A4})$$

implying

$$\begin{aligned} \beta/\gamma &= \frac{1}{(\beta/\gamma)^*}, \Rightarrow |\beta/\gamma|^2 = 1, \\ \beta/\gamma &= e^{i\alpha + 2\pi ni}, \quad \alpha \in [0, 2\pi), \quad n \in \mathbb{Z} \end{aligned} \quad (\text{A5})$$

and

$$[ff_1]_{-K}^K = f(K)f_1^*(K) - f(-K)f_1^*(-K) = 0, \quad (\text{A6})$$

restricting the domain of the self-adjoint extension to the functions with boundary conditions

$$f(K) = f(-K) \frac{f_1^*(-K)}{f_1^*(K)} = f(-K)e^{-i\alpha}, \quad (\text{A7})$$

given through a particular choice of the parameter  $\alpha \in [0, 2\pi)$  [when  $f(\pm K) \neq 0$ ] and to those with Dirichlet's boundary conditions. This means that the various self-adjoint extensions  $\hat{x}_e$  of the operator  $\hat{x}$  are obtained when the domain of the formal differential operator  $i\partial_{k_x}$  is defined as

$$\mathcal{D}(\hat{x}_e) = \mathcal{D}(\hat{x}) \cup \{f | f \in \mathcal{D}(\hat{x}^\dagger), f(K) = f(-K)e^{-i\alpha}, \alpha \in [0, 2\pi)\}. \quad (\text{A8})$$

Thus, the various self-adjoint extensions are identical on the domain  $\mathcal{D}(\hat{x})$  and can be parametrized by the real number  $\alpha \in [0, 2\pi)$  so that we can write for them  $\hat{x}_e = \hat{x}_\alpha$ . The eigenfunctions  $e^{-ik_x x} \in \mathcal{D}(\hat{x}_\alpha)$  [but  $\notin \mathcal{D}(\hat{x})$ ] of the particular self-adjoint extension  $\hat{x}_\alpha$  are those satisfying the boundary conditions in Eq. (A7),

$$e^{-iKx} = e^{iKx - i\alpha}, \quad (\text{A9})$$

belonging to the eigenvalues  $x = x_n^\theta = na + \theta$  with  $a = \pi/K$  and  $\theta = \alpha/2K \in [0, a)$ . These eigenvalues determine a grid of equidistant points on the coordinate axis with spacing  $a$ . We can change the notation of the particular self-adjoint extensions from  $\hat{x}_\alpha$  to  $\hat{x}_\theta$ . Since the eigenfunctions

$$\tilde{\psi}^\theta_{x_n}(k_x) = \sqrt{a}e^{ik_x x_n^\theta}, \quad k_x \in [-K, K] \quad (\text{A10})$$

of any particular self-adjoint extension  $\hat{x}_\theta$  form a complete orthonormal set, satisfying the orthonormality conditions (the upper index  $\theta$  of the eigenvalues is suppressed)

$$\begin{aligned} \delta_{nn'} &= \int_{-K}^K \frac{dk_x}{2\pi} \tilde{\psi}^{\theta*}_{x_n}(k_x) \tilde{\psi}^\theta_{x_{n'}}(k_x) \\ &= \int_{-\infty}^{\infty} \frac{dp_x}{2\pi f(\alpha|p_x|)} \tilde{\psi}^{\theta*}_{x_n}(p_x) \tilde{\psi}^\theta_{x_{n'}}(p_x), \end{aligned} \quad (\text{A11})$$

there exists a one-parameter family of such orthonormal bases in the Hilbert space  $\mathcal{H}$  of bandlimited wave functions. Moreover, the one-parameter family of all eigenvalues, i.e., the union  $\cup_{\theta \in [0, a)} \{x_n^\theta\}$  of all sets of eigenvalues of the various self-adjoint extensions, can be mapped trivially in a one-to-one way on the real line  $\mathbb{R}$ . Namely, each real number  $x \in \mathbb{R}$  occurs as a single eigenvalue of a single self-adjoint extension  $\hat{x}_\theta$ .

## APPENDIX B: BOUND STATES IN A SQUARE-WELL POTENTIAL

The solution  $\varphi_n(x)$  with energy  $\epsilon_n < V_0$  of the stationary Schrödinger equation (51) for the square-well potential in Eq. (53) should be constructed from the functions  $\varphi_i(x)$  ( $i = I, II, III$ ) defined in the intervals  $I_I: x \in (-\infty, 0]$ ,  $I_{II}: x \in [0, L]$ , and  $I_{III}: x \in [L, \infty)$ , respectively, satisfying the equations

$$\begin{aligned} -\frac{\hbar^2}{2m} \varphi_{II}''(x) &= \epsilon \varphi_{II}(x), \\ -\frac{\hbar^2}{2m} \varphi_i''(x) &= -(V_0 - \epsilon) \varphi_i(x), \quad i = I, III. \end{aligned} \quad (\text{B1})$$

(We shall suppress the index  $n$  numerating the energy levels.) The boundary conditions ensure continuous differentiability of the solution at the boundaries  $x = 0$  and  $x = L$  of the various intervals, as well as exponential fall-off at infinities  $|x| \rightarrow \infty$  for square integrability. Looking for the solutions of Eqs. (B1) in the form

$$\varphi_{II}(x) = B e^{ikx} + C e^{-ikx}, \quad \varphi_I(x) = A e^{\kappa x}, \quad \varphi_{III}(x) = D e^{-\kappa x}, \quad (\text{B2})$$

with the real parameters  $\kappa = \sqrt{2m(V_0 - \epsilon)}/\hbar$  and  $\kappa_n = \sqrt{2m(V_0 - \epsilon_n)}/\hbar$ , the boundary conditions at  $|x| \rightarrow \infty$  are automatically satisfied. The boundary conditions at  $x = 0$  and  $x = L$  result in the set of homogeneous linear equations,

$$A = B + C, \quad (\text{B3})$$

$$A\kappa = ik(B - C), \quad (\text{B4})$$

$$B e^{ikL} + C e^{-ikL} = D e^{-\kappa L}, \quad (\text{B5})$$

$$ik(B e^{ikL} - C e^{-ikL}) = -\kappa D e^{-\kappa L}. \quad (\text{B6})$$

There exists a nontrivial solution for the coefficients  $A, B, C$ , and  $D$  if and only if the determinant of the set of linear equations vanishes, yielding the implicit equation

$$e^{2ikL} = \left( \frac{\kappa - ik}{\kappa + ik} \right)^2, \quad (\text{B7})$$

for the energy eigenvalues  $\epsilon_n = \hbar^2 k_n^2 / (2m) < V_0$  of bound states with discrete values  $k_n$  and  $\kappa_n = \sqrt{2m(V_0 - \epsilon_n)}/\hbar$ . The numeration of the states by the integer  $n = 1, 2, \dots$  can be established in the limit  $V_0 \rightarrow \infty$  when  $\kappa \rightarrow \infty$  and  $e^{2ikL} \rightarrow 1$ , which yields the wave vectors  $k_n = n\pi/L$  with  $n \in \mathbb{N}$ . It is straightforward to realize that the wave vectors  $k_\pm$  satisfying

$$e^{ik_\pm L} = \mp \frac{\kappa_\pm - ik_\pm}{\kappa_\pm + ik_\pm}, \quad (\text{B8})$$

with  $\kappa_\pm = \sqrt{2mV_0 - k_\pm^2} \hbar / \hbar$  in the limit  $V_0 \rightarrow \infty$  behave as  $k_- = 2n'\pi/L$  and  $k_+ = (2n' - 1)\pi/L$  for  $n' = 1, 2, \dots$

Equations (B3) and (B5) can be used to express  $A$  and  $D$  via  $B$  and  $C$ , whereas taking the ratio of the appropriate sides of Eqs. (B3) and (B4), one finds after trivial manipulations

$$C = -\frac{\kappa_\pm - ik_\pm}{\kappa_\pm + ik_\pm} B = \pm e^{ik_\pm L} B, \quad (\text{B9})$$

implying  $A = B\rho_{\pm}$  and  $D = \pm B e^{\kappa_{\pm} L} \rho_{\pm}$  with  $\rho_{\pm} = 1 \pm e^{ik_{\pm} L}$ . The normalization condition  $\int_{-\infty}^{\infty} dx |\varphi_n(x)|^2 = \sum_{i=I}^{III} \int_{I_i} dx |\varphi_i(x)|^2 = 1$ , with

$$\int_{I_{II}} dx |\varphi_{II}(x)|^2 = |B|^2 2L \left( 1 + \frac{\sin k_{\pm} L}{k_{\pm} L} \right), \quad (B10)$$

$$\int_{I_i} dx |\varphi_i(x)|^2 = \frac{|\pm B \rho_{\pm}|^2}{\kappa_{\pm}}, \quad i = I, III,$$

yields

$$|B|^{-2} = 2L \left( 1 + \frac{\sin k_{\pm} L}{k_{\pm} L} + \frac{|\rho_{\pm}|^2}{2\kappa_{\pm} L} \right) \approx 2L \left( 1 + \frac{2}{\kappa_{\pm} L} + \mathcal{O}((k/\kappa)^3) \right). \quad (B11)$$

For later use we shall need the coefficients for asymptotically large values of  $V_0$ , i.e., those of  $\kappa \gg k$  which means  $\kappa \gg 1/L$  for sufficiently low-lying states. Keeping the terms up to the order  $\mathcal{O}(\kappa^{-2})$ , one obtains

$$\rho_{\pm} \approx \frac{2ik_{\pm}}{\kappa_{\pm}} + \frac{2\kappa_{\pm}^2}{k_{\pm}^2} + \mathcal{O}((k/\kappa)^3),$$

$$|\rho_{\pm}|^2 \approx \frac{4\kappa_{\pm}^2}{k_{\pm}^2} + \mathcal{O}((k/\kappa)^3), \quad (B12)$$

$$\sin k_{\pm} L \approx \pm \frac{2k_{\pm}}{\kappa_{\pm}} + \mathcal{O}((k/\kappa)^3),$$

and

$$B \approx \frac{1}{\sqrt{2L}} \left( 1 + \frac{1}{\kappa_{\pm} L} + \frac{3}{2\kappa_{\pm}^2 L^2} + \mathcal{O}((k/\kappa)^3) \right),$$

$$B\rho_{\pm} \approx \frac{1}{\sqrt{2L}} \left( \frac{2ik_{\pm}}{\kappa_{\pm}} + \frac{2k_{\pm}^2}{\kappa_{\pm}^2} + \frac{2ik_{\pm}}{\kappa_{\pm}^2 L} + \mathcal{O}((k/\kappa)^3) \right),$$

$$|B\rho_{\pm}|^2 \approx \frac{1}{2L} \left( \frac{4k_{\pm}^2}{\kappa_{\pm}^2} + \mathcal{O}((k/\kappa)^3) \right). \quad (B13)$$

### APPENDIX C: KINETIC ENERGY OPERATOR ON EXPONENTIAL FUNCTIONS

In general an arbitrary operator function  $g(\hat{k}_x)$  operates on a function  $f(x)$  as

$$g(\hat{k}_x)f(x) = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} e^{-ik_x y} g(k_x) e^{ik_x x} f(y) = \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} g(k_x) \tilde{f}(k_x) e^{ik_x x} \quad (C1)$$

with the Fourier transform of the function  $f(x)$ ,

$$\tilde{f}(k_x) = \int_{-\infty}^{\infty} dx e^{-ik_x x} f(x). \quad (C2)$$

In the case of exponential functions, however, the Fourier transform is not a well-behaved function, but rather a distribution. Therefore, one has to be careful when using

the integral representation of various operators on exponential functions. In order to be more definite, one can consider the Fourier transforms of exponential functions as limits of Gaussian integrals. For  $f(x) = e^{isx}$  ( $s \in \mathbb{R}$ ), one can write

$$\tilde{f}(k_x) = \lim_{\sigma \rightarrow 0^+} \int_{-\infty}^{\infty} dx e^{i(s-k_x)x - \frac{1}{2}\sigma^2 x^2} = \lim_{\sigma \rightarrow 0^+} \sqrt{\frac{2\pi}{\sigma^2}} e^{-\frac{(s-k_x)^2}{2\sigma^2}}, \quad (C3)$$

and similarly for  $f(x) = e^{sx}$ , ( $s \in \mathbb{R}$ )

$$\tilde{f}(k_x) = \lim_{\sigma \rightarrow 0^+} \int_{-\infty}^{\infty} dx e^{-ik_x x + sx - \frac{1}{2}\sigma^2 x^2} = \lim_{\sigma \rightarrow 0^+} \sqrt{\frac{2\pi}{\sigma^2}} e^{\frac{(s-ik_x)^2}{2\sigma^2}}, \quad (C4)$$

and the trivial operation of the powers of  $\hat{k}_x$  on exponential functions,  $(-i\partial_x)^n e^{isx} = s^n e^{isx}$  and  $(-i\partial_x)^n e^{sx} = (-is)^n e^{sx}$ , can also be recovered by saddle point integration:

$$\begin{aligned} (-i\partial_x)^n e^{isx} &= \lim_{\sigma \rightarrow 0^+} \sqrt{\frac{2\pi}{\sigma^2}} \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} e^{-\frac{(s-k_x)^2}{2\sigma^2}} k_x^n e^{ik_x x} \\ &= \lim_{\sigma \rightarrow 0^+} \frac{1}{\sqrt{2\pi}\sigma} (s + i\sigma^2 x)^n e^{isx - \frac{1}{2}\sigma^2 x^2} \int_{-\infty}^{\infty} d\eta e^{-\frac{\eta^2}{2\sigma^2}} \\ &= s^n e^{isx} \end{aligned} \quad (C5)$$

and

$$\begin{aligned} (-i\partial_x)^n e^{sx} &= \lim_{\sigma \rightarrow 0^+} \sqrt{\frac{2\pi}{\sigma^2}} \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} e^{\frac{(s-ik_x)^2}{2\sigma^2}} k_x^n e^{ik_x x} \\ &= \lim_{\sigma \rightarrow 0^+} \frac{1}{\sqrt{2\pi}\sigma} (-is + i\sigma^2 x)^n e^{-\frac{1}{2}\sigma^2 x^2 + sx} \int_{-\infty}^{\infty} d\eta e^{-\frac{\eta^2}{2\sigma^2}} \\ &= (-is)^n e^{sx}. \end{aligned} \quad (C6)$$

Then the less trivial action of powers of the operator  $|\hat{k}_x|$  can be obtained in a similar manner:

$$|-i\partial_x|^n e^{isx} = \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} |k_x|^n e^{ik_x x} 2\pi \delta(s - k_x) = |s|^n e^{isx}, \quad (C7)$$

or otherwise

$$\begin{aligned} |-i\partial_x|^n e^{isx} &= \lim_{\sigma \rightarrow 0^+} \sqrt{\frac{2\pi}{\sigma^2}} \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} |k_x|^n e^{ik_x x} e^{-\frac{(s-k_x)^2}{2\sigma^2}} \\ &= \lim_{\sigma \rightarrow 0^+} \frac{1}{\sqrt{2\pi}\sigma} |s + i\sigma^2 x|^n e^{isx - \frac{1}{2}\sigma^2 x^2} \int_{-\infty}^{\infty} d\eta e^{-\frac{\eta^2}{2\sigma^2}} \\ &= |s|^n e^{isx} \end{aligned} \quad (C8)$$

and

$$\begin{aligned}
& | -i\partial_x |^n e^{sx} \\
&= \lim_{\sigma \rightarrow 0^+} \sqrt{\frac{2\pi}{\sigma^2}} \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} |k_x|^n e^{ik_x x + \frac{(s-ik_x)^2}{2\sigma^2}} \\
&= \lim_{\sigma \rightarrow 0^+} \frac{1}{\sqrt{2\pi}\sigma} | -is + i\sigma^2 x |^n e^{sx - \frac{1}{2}\sigma^2 x^2} \int_{-\infty}^{\infty} d\eta e^{-\frac{\eta^2}{2\sigma^2}} \\
&= |s|^n e^{sx}. \tag{C9}
\end{aligned}$$

The lesson we have learned is the following. Let  $\mathcal{T}(v^2, |w|)$  be given as a double Taylor expansion in powers of  $v^2$  and  $|w|$ ; then the operator obtained by inserting  $v = w = -i\hbar\alpha\partial_x$  acts on exponential functions as

$$\begin{aligned}
\mathcal{T}(-\hbar^2\alpha^2\partial_x^2, | -i\hbar\alpha\partial_x |) e^{isx} &= \mathcal{T}(\hbar^2\alpha^2s^2, \hbar\alpha|s|) e^{isx}, \\
\mathcal{T}(-\hbar^2\alpha^2\partial_x^2, | -i\hbar\alpha\partial_x |) e^{sx} &= \mathcal{T}(-\hbar^2\alpha^2s^2, \hbar\alpha|s|) e^{sx} \tag{C10}
\end{aligned}$$

for  $s \in \mathbb{R}$ , or otherwise exponential functions are eigenfunctions of the operator  $\mathcal{T}(-\hbar^2\alpha^2\partial_x^2, | -i\hbar\alpha\partial_x |) = [F^{-1}(\hbar\alpha\hat{k}_x)]^2$ , i.e., those of the kinetic energy operator. Moreover, the function  $[F^{-1}(u)]^2$  is even, so that the eigenvalues of the kinetic energy operator when acting on exponential functions are independent of the sign of  $s$ .

#### APPENDIX D: EVALUATION OF $v_{nn}$

The independent integrals  $v_{i,j}$  contributing to the matrix element  $v_{nn}$  are

$$\begin{aligned}
v_{I,I} &= |B\rho_{\pm}|^2 V_0 \left[ \int_{-\infty}^0 dx e^{\kappa_{\pm}x} \right. \\
&\quad \times \left. \int_{-\infty}^0 dy e^{\kappa_{\pm}y} \Pi(x-y) - \frac{1}{2\kappa_{\pm}} \right], \\
v_{I,III} &= \pm |B\rho_{\pm}|^2 V_0 \int_{-\infty}^0 dx e^{\kappa_{\pm}x} \int_L^{\infty} dy e^{\kappa_{\pm}(L-y)} \Pi(x-y), \\
v_{I,II} &= \frac{1}{2} |B|^2 \rho_{\pm}^* V_0 \int_{-\infty}^0 dx e^{\kappa_{\pm}x} \\
&\quad \times \int_0^L dy (e^{ik_{\pm}y} \pm e^{ik_{\pm}(L-y)}) \Pi(x-y). \tag{D1}
\end{aligned}$$

We shall determine these integrals in the asymptotic limit  $V_0 \rightarrow \infty$ , implying  $\kappa \rightarrow \infty$ . (For the sake of simplicity let us suppress the lower index  $\pm$ .) Then the main contribution to the  $x$  integral comes in each case from  $x = 0$  due to the extremely rapidly falling off factors  $e^{\kappa x}$  in the integrands.

Therefore, we can expand the slowly varying  $x$ -dependent factor  $\Pi(x-y)$  of the integrand at  $x = 0$  and recast the integral over  $x$ :

$$\begin{aligned}
& \int_{-\infty}^0 dx e^{\kappa x} \Pi(x-y) \\
&\approx \int_{-\infty}^0 dx e^{\kappa x} [\Pi(-y) + x\Pi'(-y) + \mathcal{O}(x^2)] \\
&\approx [\Pi(-y) + \Pi'(-y)\partial_{\kappa} + \mathcal{O}(\partial_{\kappa}^2)] \int_{-\infty}^0 dx e^{\kappa x} \\
&\approx [\Pi(-y) + \Pi'(-y)\partial_{\kappa} + \mathcal{O}(\partial_{\kappa}^2)] \kappa_{\pm}^{-1} \\
&\approx \kappa^{-1} \Pi(-y) + \kappa^{-2} \Pi'(-y) + \mathcal{O}(\kappa^{-3}), \tag{D2}
\end{aligned}$$

with the notations  $\Pi'(u) = d\Pi(u)/du$ ,  $\Pi''(u) = d^2\Pi(u)/du^2$ . Thus we get an expansion of the integral in powers of  $1/\kappa$ . In the limit  $V_0 \rightarrow \infty$ , i.e.,  $k/\kappa \rightarrow 0$ , one is only interested in the leading order terms of  $v_{i,j}$ 's.

With similar logic, one can expand the integrals over  $y$  in the expressions of  $v_{I,I}$  and  $v_{I,III}$  and also in powers of  $1/\kappa$ . Since the even and odd derivatives of  $\Pi(u)$  at  $u = 0$  are finite and zero, respectively [e.g.,  $\Pi(0) = K/\pi$ ,  $\Pi'(0) = 0$ ,  $\Pi''(0) = -K^3/\pi$ ] the following well-defined expansion occurs in the expression of  $v_{I,I}$ :

$$\begin{aligned}
& \int_{-\infty}^0 dy e^{\kappa y} (\kappa^{-1} \Pi(-y) + \kappa^{-2} \Pi'(-y) + \mathcal{O}(\kappa^{-3})) \\
&\approx (\kappa^{-1} \Pi(0) + \kappa^{-2} \Pi'(0) + \mathcal{O}(\kappa^{-3})) \kappa^{-1} \\
&\approx \kappa^{-2} \Pi(0) + \mathcal{O}(\kappa^{-3}), \tag{D3}
\end{aligned}$$

which implies

$$\begin{aligned}
v_{I,I} &\approx |B\rho|^2 V_0 \left( \frac{\Pi(0)}{\kappa^2} - \frac{1}{2\kappa} + \mathcal{O}(\kappa^{-3}) \right) \\
&\approx |B\rho|^2 V_0 \left( -\frac{1}{2\kappa} + \mathcal{O}(\kappa^{-2}) \right). \tag{D4}
\end{aligned}$$

Similarly, expanding the  $y$ -dependent factor of the integrand of  $v_{I,III}$  at  $y = L$ , the leading order term provides

$$v_{I,III} = \pm |B\rho|^2 V_0 \left( \frac{\Pi(-L)}{\kappa^2} + \mathcal{O}(\kappa^{-3}) \right) \tag{D5}$$

in the limit  $V_0 \rightarrow \infty$  (with the sign  $\pm$  when the lower index  $\pm$  is everywhere restored). Furthermore, the integral  $v_{I,II}$  can be recast as

$$\begin{aligned}
v_{I,II} &\approx \frac{1}{2\kappa} |B|^2 V_0 \rho^* \int_0^L dy (e^{iky} \pm e^{ik(L-y)}) \times [\Pi(-y) + \mathcal{O}(\kappa^{-1})] \\
&\approx \frac{1}{2\kappa i} |B|^2 V_0 \rho^* \int_{-K}^K \frac{dq}{2\pi} \left( \frac{e^{i(k-q)L} - 1}{k-q} \mp e^{ikL} \frac{e^{-i(k+q)L} - 1}{k+q} \right) \\
&\approx \frac{1}{2\kappa i 2\pi} |B|^2 V_0 \rho^* \left( \int_{-KL-kL}^{KL-kL} du \frac{1 - e^{-iu}}{u} \pm e^{ikL} \int_{-KL+kL}^{KL+kL} du \frac{1 - e^{-iu}}{u} \right) \tag{D6}
\end{aligned}$$



for asymptotically large values of  $\kappa$ . Here the expression in the bracket can be recast into the sum

$$\begin{aligned} (\dots) &= I_{c-} + iI_{s-} \pm e^{ikL}(I_{c+} + iI_{s+}) \\ &= [\pm e^{ikL} - 1]I_{c+} + i[1 \pm e^{ikL}]I_{s+}, \end{aligned} \quad (\text{D7})$$

with

$$\begin{aligned} I_{c\pm} &= \int_{-KL \pm kL}^{KL \pm kL} du \frac{1 - \cos u}{u}, \\ I_{s\pm} &= \int_{-KL \pm kL}^{KL \pm kL} du \frac{\sin u}{u}, \end{aligned} \quad (\text{D8})$$

and  $I_{c-} = -I_{c+}$ ,  $I_{s-} = I_{s+}$ . For the low-lying excited states we can Taylor-expand these integrals in the small parameter  $k/K \ll 1$ ,

$$\begin{aligned} I_{c+} &\approx \frac{k}{K} \left( 4\sin^2 \frac{KL}{2} + (kL)^2 \cos(KL) + \mathcal{O}((kL)^4) \right) \\ &\quad + \mathcal{O}((k/K)^2), \\ I_{s+} &\approx \int_{-KL}^{KL} du \frac{\sin u}{u} + \mathcal{O}((k/K)^2). \end{aligned} \quad (\text{D9})$$

Making use of the asymptotic relations  $\kappa_{\pm} \approx \sqrt{2mV_0}/\hbar$ ,  $\pm e^{ik_{\pm}L} \rightarrow -1$  and those in Eqs. (B12) and (B13), one easily establishes the order-of-magnitude relations  $v_{I,I} \sim \mathcal{O}(\kappa^{-1})$ ,  $v_{I,III} \sim \mathcal{O}(\kappa^{-2})$ , and the leading order contribution comes from

$$v_{I,II} \approx \frac{\hbar^2 k}{4mL\pi} I_{c+}. \quad (\text{D10})$$

One can write  $KL = 2\pi(N + \nu)$  where  $N$  and  $0 \leq \nu < 1$  stand for the integer and fractional parts of the ratio  $KL/(2\pi)$ , respectively. Consequently, one finds  $\sin^2(KL/2) = \sin^2(\nu\pi)$  and

$$\begin{aligned} I_{c+} &= \frac{k}{K} [4\sin^2(\nu\pi) + (n\pi)^2 \cos(2\nu\pi) + \mathcal{O}(n^4)] \\ &\quad + \mathcal{O}((k/K)^2), \end{aligned} \quad (\text{D11})$$

which has the order of magnitude  $k/K \sim n(\ell_p/L)$ .

Summing the contributions of leading order in  $1/\kappa$ , and making use of  $v_{III,II} = \pm v_{I,II}$ , finally one obtains for the matrix element  $v_{nn}$ ,

$$\begin{aligned} v_{nn} &\approx v_{I,II} + v_{II,I} + v_{III,II} + v_{II,III} \\ &\approx 2(v_{I,II} + v_{III,II}) \\ &\approx \begin{cases} 4v_{I,II} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}. \end{aligned} \quad (\text{D12})$$

The vanishing of the potential energy shift for  $n$  even is a consequence of the sign difference of the wave function in the outer regions  $I_I$  and  $I_{III}$ , i.e., a consequence of the sign difference of the coefficients  $A$  and  $D$ , implying  $v_{III,II} = -v_{I,II}$  in that case.

## APPENDIX E: EVALUATION OF $t_{nn}$

The independent integrals contributing to the kinetic energy shift  $t_{nn}$  arising due to the finite bandwidth are as follows:

$$\begin{aligned} t_{I,I} &= \frac{|B\rho|^2}{2m\alpha^2} \int_{-\infty}^0 dx e^{\kappa x} [F^{-1}(-i\alpha\hbar\partial_x)]^2 \left( \int_{-\infty}^0 dy \Pi(x-y) e^{\kappa y} - e^{\kappa x} \right), \\ t_{II,I} &= \frac{|B|^2 \rho}{2m\alpha^2} \int_0^L dx (e^{-ikx} \pm e^{-ik(L-x)}) [F^{-1}(-i\alpha\hbar\partial_x)]^2 \int_{-\infty}^0 dy \Pi(x-y) e^{\kappa y}, \\ t_{III,I} &= \pm \frac{|B\rho|^2}{2m\alpha^2} \int_L^\infty dx e^{\kappa(L-x)} [F^{-1}(-i\alpha\hbar\partial_x)]^2 \int_{-\infty}^0 dy \Pi(x-y) e^{\kappa y}, \\ t_{II,II} &= \frac{|B|^2}{2m\alpha^2} \int_0^L dx (e^{-ikx} \pm e^{-ik(L-x)}) [F^{-1}(-i\alpha\hbar\partial_x)]^2 \left( \int_0^L dy \Pi(x-y) (e^{iky} \pm e^{ik(L-y)}) - (e^{ikx} \pm e^{ik(L-x)}) \right). \end{aligned} \quad (\text{E1})$$

Let us expand the integrals over  $y$  in  $t_{I,I}$ ,  $t_{II,I}$ , and  $t_{III,I}$  in powers of  $\kappa^{-1}$ , similar to what we did in Appendix D for the  $x$  integrals,

$$\int_{-\infty}^0 dy \Pi(x-y) e^{\kappa y} \approx \int_{-\infty}^0 dy [\Pi(x) - y\Pi'(x) + \mathcal{O}(y^2)] e^{\kappa y} \approx [\Pi(x) - \Pi'(x)\partial_\kappa + \mathcal{O}(\partial_\kappa^2)] \kappa^{-1} \approx \kappa^{-1} \Pi(x) + \mathcal{O}(\kappa^{-2}). \quad (\text{E2})$$

Then a similar  $(1/\kappa)$  expansion of the integrals over  $x$  occurring in  $t_{I,I}$  and  $t_{III,I}$ , respectively, is possible:

$$\kappa^{-1} \int_{-\infty}^0 dx e^{\kappa x} G_{\Pi}(x) \approx \kappa^{-2} G_{\Pi}(0) + \mathcal{O}(\kappa^{-3}), \quad \kappa^{-1} \int_L^\infty dx e^{\kappa(L-x)} G_{\Pi}(x) \approx \kappa^{-2} G_{\Pi}(L) + \mathcal{O}(\kappa^{-3}), \quad (\text{E3})$$

with  $G_{\Pi}(x) = [F^{-1}(-i\alpha\hbar\partial_x)]^2 \Pi(x)$ . Then one finds in the leading order of  $1/\kappa$ ,

$$\begin{aligned}
t_{I,I} &\approx \frac{|B\rho|^2}{2m\alpha^2} \left( \kappa^{-2} G_{\Pi}(0) - \frac{[F^{-1}(-i\alpha\hbar\kappa)]^2}{2\kappa} + \mathcal{O}(\kappa^{-3}) \right) \approx \mathcal{O}(\kappa^{-3}), \\
t_{III,I} &\approx \pm \frac{|B\rho|^2}{2m\alpha^2} [\kappa^{-2} G_{\Pi}(L) + \mathcal{O}(\kappa^{-3})] \approx \mathcal{O}(\kappa^{-4}),
\end{aligned} \tag{E4}$$

where we made use of our particular choice of the deformation function  $f(u)$ . The leading order terms of  $t_{II,I}$  are given as

$$t_{II,I} \approx \frac{|B|^2 \rho}{2m\alpha^2} \kappa^{-1} \int_0^L dx (e^{-ikx} \pm e^{-ik(L-x)}) G_{\Pi}(x) \approx \mathcal{O}(\kappa^{-2}). \tag{E5}$$

Therefore, the only independent integral contributing to  $t_{nn}$  in the limit  $\kappa \rightarrow \infty$  is  $t_{II,II}$ .

As for the next, we try to estimate the integral  $t_{II,II}$  in the limit  $\kappa \rightarrow \infty$ . Let  $\chi_{[0,L]}(x)$  be the characteristic function of the interval  $x \in [0, L]$ . In order to perform the integral over  $y$ , let us first rewrite the trivial integral  $\int_0^L dy \delta(x-y) e^{iky} = e^{ikx} \chi_{[0,L]}(x)$  as a limit,

$$\begin{aligned}
\int_0^L dy \delta(x-y) e^{iky} &= \lim_{\Lambda \rightarrow \infty} \int_{-\Lambda}^{\Lambda} \frac{dq}{2\pi} e^{iqx} \int_0^L dy e^{-i(q-k)y} \\
&= e^{ikx} \lim_{\Lambda \rightarrow \infty} \int_{-\Lambda}^{\Lambda} \frac{dq}{2\pi} e^{i(q-k)x} i \frac{e^{-i(q-k)L} - 1}{q-k} \\
&= e^{ikx} \lim_{\Lambda \rightarrow \infty} \int_{-\Lambda}^{\Lambda} \frac{dp}{2\pi} e^{ipx} i \frac{e^{-ipL} - 1}{p} \\
&= e^{ikx} \lim_{\Lambda \rightarrow \infty} \int_{-\Lambda}^{\Lambda} \frac{dp}{2\pi} \left[ \frac{\sin pL}{p} \cos px - [\cos(pL) - 1] \frac{\sin px}{p} \right] \\
&= e^{ikx} \lim_{\Lambda \rightarrow \infty} \frac{1}{2} \int_{-1}^1 ds \left[ \frac{\sin[s\Lambda(L-x)]}{s\pi} + \frac{\sin(s\Lambda x)}{s\pi} \right] \\
&= e^{ikx} \frac{1}{2} \int_{-1}^1 ds [\chi_{[0,L]}(x)(\delta(s) + \delta(s)) + \Theta(L-x)(-\delta(s) + \delta(s)) + \Theta(-x)(\delta(x) - \delta(x))] \\
&= e^{ikx} \chi_{[0,L]}(x) \frac{1}{2} \int_{-1}^1 ds (\delta(s) + \delta(s)) \\
&= e^{ikx} \chi_{[0,L]}(x).
\end{aligned} \tag{E6}$$

Now let us evaluate  $\int_0^L dy \Pi(x-y) e^{iky}$  in a similar manner, where the limit  $\Lambda \rightarrow \infty$  is removed and  $\Lambda$  is replaced by the finite cutoff  $K$ ,

$$\begin{aligned}
\int_0^L dy \Pi(x-y) e^{iky} &= \int_{-K}^K \frac{dq}{2\pi} e^{iqx} \int_0^L dy e^{-i(q-k)y} \\
&= e^{ikx} \frac{1}{2} \int_{-1}^1 ds \left[ \frac{\sin[sK(L-x)]}{s\pi} + \frac{\sin(sKx)}{s\pi} \right] \\
&= e^{ikx} I_K(x).
\end{aligned} \tag{E7}$$

Making use of this and the limit  $\pm e^{ik\pm L} \rightarrow -1$  for  $\kappa \rightarrow \infty$ , one can recast the integral  $t_{II,II}$  in the form

$$\begin{aligned}
t_{II,II} &= -\frac{|B|^2}{2m\alpha^2} \int_0^L dx (e^{ikx} - e^{-ikx}) [I_K(x) - 1] \\
&\quad \times [F^{-1}(-i\alpha\hbar\partial_x)]^2 (e^{ikx} - e^{-ikx}).
\end{aligned} \tag{E8}$$

In order to obtain an order-of-magnitude estimate for  $t_{II,II}$ , let us note that the integral  $I_K(x)$  for sufficiently large cutoff  $K$  should be a rather smooth function of  $x$  because it

is independent of  $x$  for  $K \rightarrow \infty$ ,  $I_{K \rightarrow \infty}(x) \rightarrow 1$ . Then the following approximations seem to be justified: (i) the replacement of the kinetic energy operator by its eigenvalue when acting on the functions  $e^{\pm ikx}$  (cf. Appendix C), (ii) the replacement of  $I_K(x)$  by  $I_K(L/2)$ . The integral over  $x$  reduces then to

$$\int_0^L dx \sin^2(kx) = \frac{1}{2} L \tag{E9}$$

for  $kL = n\pi$  with any  $n = 1, 2, \dots$ , and one finds

$$t_{II,II} \approx \frac{[F^{-1}(\alpha\hbar k)]^2}{2m\alpha^2} [I_K(L/2) - 1], \tag{E10}$$

with

$$I_K(L/2) - 1 = \frac{2}{\pi} \int_0^{KL/2} du \frac{\sin u}{u} - 1, \tag{E11}$$

where the new integration variable  $u = sKL/2$  has been introduced.

Let  $N'$  and  $\nu' \in [0, 1)$  be the integer and fractional parts of  $KL/(4\pi)$ , respectively. The integral (E11) can be rewritten as

$$\begin{aligned}
I_K(L/2) - 1 &= -\frac{2}{\pi} \int_{KL/2}^{\infty} du \frac{\sin u}{u} \\
&\approx -\frac{2}{\pi} \frac{2}{KL} \int_{\nu/2\pi}^{2\pi} du \sin u \\
&\approx \frac{4}{KL\pi} [1 - \cos(2\nu\pi)]. \quad (\text{E12})
\end{aligned}$$

Here one has split the interval  $u \in [KL/2, \infty)$  into subintervals  $[(N' + \nu')2\pi, (N' + 1)2\pi]$ ,  $[2(N' + j)\pi, 2(N' + j + 1)\pi]$ , with  $j = 1, 2, \dots$ , and replaced the factor  $1/u$  in the integrand by  $[(N' + \nu')2\pi]^{-1}$ ,  $[(N' + j)2\pi]^{-1}$ , with  $j = 1, 2, \dots$ , respectively, in the subsequent intervals.

## APPENDIX F: MAXIMALLY LOCALIZED STATES

Following the method of Detournay *et al.* [18] we construct the maximally localized state centered at position  $\bar{x}$  for an arbitrary deformation function. One should look for the state  $|\phi\rangle$  with a given undetermined position uncertainty  $\mu$ , i.e., the state satisfying

$$\mu^2 = \langle \phi | \hat{x}^2 - \bar{x}^2 | \phi \rangle \quad (\text{F1})$$

and the subsidiary conditions

$$\langle \phi | \hat{x} | \phi \rangle = \bar{x} \quad (\text{F2})$$

and  $\langle \phi | \phi \rangle = 1$ . Then one selects out the state  $|\varphi_{\bar{x}}\rangle$  with the minimal value of  $\mu$ . In the wave-vector representation, the variational problem is equivalent with the solution of the differential equation

$$0 = [(i\partial_{k_x})^2 - \bar{x}^2 - \mu^2 - 2\lambda(i\partial_{k_x} - \bar{x})]\tilde{\phi}(k_x), \quad (\text{F3})$$

where the wave function  $\tilde{\phi}(k_x)$  should satisfy Dirichlet's boundary conditions  $\tilde{\phi}(\pm K) = 0$  and be normalized,  $\int_{-K}^K \frac{dk_x}{2\pi} |\tilde{\phi}(k_x)|^2 = 1$ ;  $\lambda \in \mathbb{R}$  is a Lagrange multiplier that should be determined from the subsidiary condition (F2). Looking for the solutions in exponential form, one finds two independent solutions,  $\tilde{\phi}_{\pm} = e^{ik_x \sigma_{\pm}}$  with  $\sigma_{\pm} = -\lambda \pm \sqrt{(\lambda - \bar{x})^2 + \mu^2}$ . Then the general solution of Eq. (F3) is given as  $\tilde{\phi}(k_x) = Ae^{ik_x \sigma_+} + Be^{ik_x \sigma_-}$  and one gets from Dirichlet's boundary conditions,  $\tilde{\phi}(\pm K) = 0$ , that

$$\begin{aligned}
K(\sigma_+ - \sigma_-) &= 2\sqrt{(\lambda - \bar{x})^2 + \mu^2} = N\pi, \\
N \in \mathbb{N} \quad B &= -Ae^{iN\pi}. \quad (\text{F4})
\end{aligned}$$

Thus, one finds the sets of solutions:

$$\begin{aligned}
\tilde{\phi}_{N=2n}(k_x) &= 2iAe^{-ik_x \lambda} \sin\left(\frac{n\pi}{K}k_x\right), \\
\tilde{\phi}_{N=2n-1}(k_x) &= 2Ae^{-ik_x \lambda} \cos\left(\frac{(2n-1)\pi}{2K}k_x\right), \quad (\text{F5})
\end{aligned}$$

with  $n \in \mathbb{N}$ , again. For both sets the subsidiary condition (F2) yields  $\lambda = \bar{x}$ , which implies

$$\mu^2 = \left(\frac{N\pi}{2K}\right)^2. \quad (\text{F6})$$

Therefore, the state centered at  $x = \bar{x}$  with minimal position uncertainty is the one with  $N = 1$ ,

$$\tilde{\phi}_{\bar{x}}(k_x) = \tilde{\phi}_{N=1}(k_x) = \sqrt{2a}e^{-ik_x \bar{x}} \cos\left(\frac{k_x a}{2}\right), \quad (\text{F7})$$

with  $a = \pi/K$  after normalization. What one has to check yet is that this state is of finite energy. For a particle with the usual kinetic energy  $\frac{\hat{p}_x^2}{2m}$ , the integral

$$\int_{-\infty}^{\infty} \frac{dp_x}{2\pi f(\alpha|p_x|)} p_x^2 \cos^2\left(\frac{k_x(p_x)a}{2}\right) \quad (\text{F8})$$

should converge, which happens if the deformation function increases for  $|p_x| \rightarrow \infty$  faster than  $p_x^2$ , a condition satisfied by the deformation functions  $f = \exp(\alpha^2 p_x^2)$  and  $f = \exp(\alpha|p_x|)$  cited in Sec. I, except for the case with  $f = 1 + \alpha^2 p_x^2$ , although the latter might be a good low-momentum approximation of some realistic deformation function. The function (F7) and its derivatives with respect to  $k_x$  are bounded functions of  $k_x$ . Therefore, any potential energy which can be approximated with a sequence of polynomials will have a finite expectation value in the state given by Eq. (F7). Therefore, we can indeed consider the wave function in Eq. (F7) for the particular class of the deformation function as that of the physical state of a particle centered at  $\bar{x}$  with the minimal position uncertainty. It is an advantage of the wave-vector representation as compared to the canonical momentum representation that the wave-vector wave functions of the states maximally localized at various positions do not depend on the explicit form of the deformation function.

## APPENDIX G: RECONSTRUCTION OF A CONTINUOUS BANDLIMITED POTENTIAL FROM SAMPLED VALUES OF DIRAC-DELTA POTENTIAL

First, we give a unique definition of the sum  $\sum_{n=-\infty}^{\infty} e^{iqna}$  which turns out to be useful for the determination of reconstructed bandlimited potentials. We settle the ordering of the terms in the sum via

$$\sum_{n=-\infty}^{\infty} e^{iqna} = \lim_{N \rightarrow \infty} \left( \sum_{n=0}^N e^{-iqna} + \sum_{n=0}^N e^{inqa} - 1 \right), \quad (\text{G1})$$

i.e., in the manner that ensures the completeness of the eigenstates  $\tilde{\psi}_{x_n}(k_x) = \sqrt{a}e^{-ik_x x_n}$  ( $x_n = na + \theta$ ,  $\theta \in [0, a)$ ,  $n \in \mathbb{Z}$ ) of an arbitrarily chosen self-adjoint extension  $\hat{x}_{\theta}$  of the coordinate operator  $-i\partial_{k_x}$  in the bandlimited Hilbert space  $\mathcal{H}$ . Then one finds

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} e^{iqna+iq\theta} &= \lim_{N \rightarrow \infty} e^{iq\theta} \left( \frac{e^{-iq(N+1)a} - 1}{e^{-iq a} - 1} + \frac{e^{iq(N+1)a} - 1}{e^{iq a} - 1} - 1 \right) \\
&= \lim_{N \rightarrow \infty} e^{iq\theta} \frac{\cos(Nqa) - \cos[(N+1)qa]}{1 - \cos(qa)} \\
&= \pi qa e^{iq\theta} \lim_{N \rightarrow \infty} \left( \frac{\cos(Nqa)}{\pi qa} + \frac{\sin(qa)}{1 - \cos(qa)} \frac{\sin(Nqa)}{\pi qa} \right) \\
&= \pi qa \left( -1 + \frac{\sin(qa)}{1 - \cos(qa)} \right) \delta(qa) \\
&= \frac{2\pi}{a} \delta(q).
\end{aligned} \tag{G2}$$

Let us emphasize that the definition of the sum does not depend on the choice of  $\theta$ , i.e., that of the self-adjoint extension of the coordinate operator. Hence, the completeness relation of the coordinate eigenstates  $|x_n^\theta\rangle$  in the bandlimited Hilbert space  $\mathcal{H}$  takes the form

$$\sum_{n=-\infty}^{\infty} \tilde{\psi}_{x_n}^{\theta*}(k_x) \tilde{\psi}_{x_n}^{\theta}(k'_x) = a \left( \sum_{n=-\infty}^{\infty} e^{i(k_x - k'_x)(na + \theta)} \right) = 2\pi \delta(k_x - k'_x) \tag{G3}$$

in the wave-vector representation.

As for the next, we determine the bandlimited continuous potential  $\bar{V}(\bar{x})$  reconstructed from discrete sampled values  $\bar{V}_n$  (with the notations of Sec. IV) of the Dirac-delta-like potential  $V(x) = V_0 a \delta(x)$ . Making use of the reconstruction formula in Eq. (20), the sample (26), and the rule to evaluate the sum like in Eq. (G2), one reconstructs the following continuous bandlimited potential:

$$\begin{aligned}
\bar{V}(\bar{x}) &= \frac{V_0 a^3}{2} \sum_{n=-\infty}^{\infty} \left[ \Pi\left(x_n - \frac{1}{2}a\right) + \Pi\left(x_n + \frac{1}{2}a\right) \right]^2 \Pi(\bar{x} - x_n) \\
&= \frac{V_0 a^3}{2} \sum_{n=-\infty}^{\infty} \int_{-K}^K \frac{dk_1}{2\pi} [e^{ik_1(x_n - \frac{1}{2}a)} + e^{ik_1(x_n + \frac{1}{2}a)}] \int_{-K}^K \frac{dk_2}{2\pi} [e^{ik_2(x_n - \frac{1}{2}a)} + e^{ik_2(x_n + \frac{1}{2}a)}] \int_{-K}^K \frac{dq}{2\pi} e^{iq(\bar{x} - x_n)} \\
&= \frac{V_0 a^3}{2} \int_{-K}^K \frac{dk_1}{2\pi} \int_{-K}^K \frac{dk_2}{2\pi} \int_{-K}^K \frac{dq}{2\pi} [e^{-ik_1 \frac{1}{2}a} + e^{ik_1 \frac{1}{2}a}] [e^{-ik_2 \frac{1}{2}a} + e^{ik_2 \frac{1}{2}a}] e^{iq\bar{x}} \sum_{n=-\infty}^{\infty} e^{i(k_1 + k_2 - q)x_n} \\
&= \frac{V_0 a^2}{2(2\pi)^2} \int_{-K}^K dk_1 \int_{-K}^K dk_2 [e^{-ik_1 \frac{1}{2}a} + e^{ik_1 \frac{1}{2}a}] [e^{-ik_2 \frac{1}{2}a} + e^{ik_2 \frac{1}{2}a}] \int_{-K}^K dq e^{iq\bar{x}} \delta(k_1 + k_2 - q) \\
&= \frac{V_0 a^2}{2(2\pi)^2} \int_{-K}^K dk_1 \int_{-K}^K dk_2 [e^{-ik_1 \frac{1}{2}a} + e^{ik_1 \frac{1}{2}a}] [e^{-ik_2 \frac{1}{2}a} + e^{ik_2 \frac{1}{2}a}] e^{i(k_1 + k_2)\bar{x}} (\Theta(k_1 + k_2 + K) - \Theta(K - k_1 - k_2)) \\
&= \frac{V_0 a^2}{2(2\pi)^2} [I(\bar{x}) + J(\bar{x})],
\end{aligned} \tag{G4}$$

with

$$\begin{aligned}
I(\bar{x}) &= \int_{-K}^0 dk_1 \int_{-K-k_1}^K dk_2 [e^{-ik_1 \frac{1}{2}a} + e^{ik_1 \frac{1}{2}a}] [e^{-ik_2 \frac{1}{2}a} + e^{ik_2 \frac{1}{2}a}] e^{i(k_1 + k_2)\bar{x}}, \\
J(\bar{x}) &= \int_0^K dk_1 \int_{-K}^{K-k_1} dk_2 [e^{-ik_1 \frac{1}{2}a} + e^{ik_1 \frac{1}{2}a}] [e^{-ik_2 \frac{1}{2}a} + e^{ik_2 \frac{1}{2}a}] e^{i(k_1 + k_2)\bar{x}} = I(-\bar{x}).
\end{aligned} \tag{G5}$$

Then a somewhat lengthy but straightforward calculation yields

$$\bar{V}(\bar{x}) = V_0 \frac{a^2}{(2\pi)^2 [\bar{x}^2 - (a/2)^2]} \left( \frac{a^2}{\bar{x}^2 - (a/2)^2} - \frac{2a\bar{x} \sin(K\bar{x})}{\bar{x}^2 - (a/2)^2} + \frac{4\bar{x}}{a} \sin K\bar{x} - \pi \cos K\bar{x} \right). \tag{G6}$$

This function turns out to be a unique even function of  $\bar{x}$  in all of the various  $\theta$  sectors. It takes the typical values  $\bar{V}(0) = V_0 \frac{4+\pi}{\pi^2} \approx 0.72V_0$ ,  $\bar{V}(\pm \frac{1}{2}a) = V_0 \frac{1+5(\pi/4)^2}{\pi^2} \approx 0.41V_0$  and falls off rapidly outside the interval  $\bar{x} \in [-\frac{1}{2}a, \frac{1}{2}a]$  in an oscillatory manner.



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