Some questions of probability theory on special topological groups

Ph.D. Thesis

Mátyás Barczy

University of Debrecen
Faculty of Informatics

Debrecen, 2006

Ezen értekezést a Debreceni Egyetem Matematika- és Számítástudományok Doktori Iskola Valószínűségelmélet, matematikai statisztika és alkalmazott matematika programja keretében készítettem 2001–2005 között és ezúton benyújtom a Debreceni Egyetem doktori Ph.D. fokozatának elnyerése céljából. Kijelentem, hogy ezt a doktori értekezést magam készítettem és abban csak a megadott forrásokat használtam fel.

Debrecen, 2006. február 24.

....
Barczy Mátyás
jelölt

Tanúsítom, hogy Barczy Mátyás doktorjelölt 2001–2005 között a fent megnevezett doktori iskola Valószínűségelmélet, matematikai statisztika és alkalmazott matematika programja keretében irányításommal végezte munkáját. Az értekezésben foglaltak a jelölt önálló munkáján alapulnak, az eredményekhez önálló alkotó tevékenységével meghatározóan hozzájárult. Az értekezés elfogadását javaslom.

Debrecen, 2006. február 24.

....
Dr. Pap Gyula
témavezető

Contents

Chapter 1

Introduction

1.1 Motivation and historical background

Five years ago I chose probability theory on locally compact groups as the topic of my Ph.D. thesis, since I was always interested in probability theory and functional analysis, especially the theoretical part of them. I thought that working on this field I would learn many new things from mathematics, not just from probability theory. Now I think it was a good choice.

The idea of studying probability measures on spheres in Euclidean space $\mathbb{R}^d$ rather than on the Euclidean space itself as old as the beginnings of probability theory. In 1734 Daniel Bernoulli looked at the orbital planes of the planets known at his time as random points on the surface of a sphere and asserted their uniform distribution. In 1940 Itô and Kawada in their paper [32] established the fundamentals of a probability theory on general compact groups. Bochner, in his basic works [11] and [12], studied for the first time probability mesures on locally compact Abelian groups. Then in 1963 Grenander, in his book [25], summarized all the available knowledge at his time about probability measures on locally compact groups. In 1965 Hannan, in his book [26], dealt with the relationship between the theory of probability measures on groups and the theory of group representations. In 1967 Parthasarathy, in his book [46], summarized and improved the general theory of probability measures on second countable locally compact Abelian groups (LCA2 groups). The content of this paragraph comes from the book of Heyer [30].

In 1977 Heyer's very famous book entitled *Probability measures on locally* *compact groups* [30] appeared. The goal of his book is to give a fairly complete treatment of the central limit problem for probability measures on a locally compact group. In analogy to the classical theory his discussion is centered around infinitely divisible probability measures on a locally compact group and their relationship to convergence of infinitesimal triangular arrays. In 1988 Diaconis, in his book [17], showed how the mathematical theory of group representations can be used to solve very concrete problems in probability and statistics. It is mainly concerned with noncommutative finite groups. In 1988 Ruzsa and Székely, in their book [48], considered a number of problems in probability theory from an algebraic viewpoint by studying the semigroup of distributions on a locally compact group, endowed with the operation of convolution and the weak topology. In 2000 Woess, in his book [61], dealt with random walks on infinite graphs and groups. In 2001 Hazod and Siebert, in their detailed and comprehensive monograph [28], treated stability properties of probability measures on locally compact groups.

Besides the above mentioned authors we have to refer to other active researchers who are working on this field and with whom we have real contacts: D. Applebaum, A. Bendikov, M. Bingham, Ph. Feinsilver, M. McCrudden, D. Neuenschwander, R. Schott and M. Voit.

The present dissertation is based on two more or less independent topics and we deal with probability theory on special topological groups. First we investigate questions concerning Gauss measures on special noncommutative Lie groups, such as on the Heisenberg group and on the affine group. We describe the distribution of the convolution of two Gauss measures on the 3-dimensional Heisenberg group. We show that a Gauss measure on the affine group can be embedded only in a uniquely determined Gauss semigroup. Then we deal with proving (central) limit theorems for infinitesimal triangular arrays of random elements with values in special LCA2 groups, such as in the torus group, in the group of p-adic integers and in the p-adic solenoid. We also consider the problem of representation of weakly infinitely divisible probability measures on these groups. In the next section we give a detailed presentation overview of our results.

1.2 Presentation overview and our results

The present work consists of two main topics, these topics lead into three more or less independent directions. Namely, we deal with calculating the Fourier transform of a Gauss measure on the Heisenberg group, proving uniqueness of embedding of a Gauss measure on the affine group into a Gauss semigroup and proving limit theorems on LCA2 groups.

More precisely, this dissertation consists of the following parts. The introduction (first chapter) contains our motivation, the historical background, the presentation overview and our main results.

In the second and third chapters we deal with some analytic properties of Gauss measures on two special Lie groups, on the 3-dimensional Heisenberg group and on the affine group.

In the second chapter we consider the case of the 3-dimensional Heisenberg group. We derive an explicit formula for the Fourier transform of a Gauss measure on this group at the Schrödinger representation (see Theorem 2.3.1). Using this explicit formula necessary and sufficient conditions are given for the convolution of two Gauss measures to be a Gauss measure (see Theorem 2.2.1). It turns out that a convolution of Gauss measures on the Heisenberg group is almost never a Gauss measure. We also give the Fourier transform of the convolution of two Gauss measures on the Heisenberg group including the case when the convolution is not a Gauss measure (see Theorem 2.6.1).

The third chapter is devoted to Gauss measures on the affine group. We show that a Gauss measure on this group can be embedded only in a uniquely determined Gauss semigroup (see Theorem 3.3.1). The proof is based on the fact that a Gauss Lévy process in the affine group satisfies a certain stochastic differential equation (SDE). Theorem 3.2.1 contains the solution of this SDE. Moreover, we give a complete description of supports of Gauss measures on the affine group using Siebert's support formula (see Theorem 3.4.1).

The fourth chapter deals with proving (central) limit theorems on locally compact Abelian groups. We also consider the question of giving a construction of an arbitrary weakly infinitely divisible measure on special LCA2 groups using only real valued random variables. First we collect all the necessary information about measures on LCA2 groups and about their properties. Then we prove limit theorems for row sums of a rowwise independent infinitesimal array of random elements with values in an LCA2 group. We give a proof of Gaiser's theorem on convergence of triangular arrays [23, Satz 1.3.6], since it does not have an easy access and it is not complete (see Theorem 4.3.1). This theorem gives sufficient conditions for convergence of the row sums of a rowwise independent infinitesimal array of random elements with values in an LCA2 group, but the limit measure can not have a nondegenerate idempotent factor, i.e., a nondegenerate Haar measure on some compact subgroup as its factor.

As new results we prove necessary and sufficient conditions for convergence of the row sums of symmetric arrays and Bernoulli arrays, where the limit measure can also be a nondegenerate normalized Haar measure on a compact subgroup (see Theorems 4.4.2 and 4.5.1). Then we investigate special LCA2 groups: the torus group (see Section 4.6), the group of p-adic integers (see Section 4.7) and the p-adic solenoid (see Section 4.8).

Besides proving limit theorems, we give a construction of an arbitrary weakly infinitely divisible probability measure on the torus group and the group of p-adic integers (see Theorems 4.6.4 and 4.7.4). On the p-adic solenoid we give a construction of weakly infinitely divisible probability measures without nondegenerate idempotent factors (see Theorem 4.8.4). In our constructions we only use real valued random variables. We note that, as a special case of our results, we have a new construction of the normalized Haar measure on the group of p-adic integers and the p-adic solenoid.

In the fifth chapter we prove an analogue of the portmanteau theorem on weak convergence of probability measures allowing measures which are finite on the complement of any Borel neighbourhood of a fixed element of an underlying metric space. We use this result in proving Gaiser's limit theorem (Theorem 4.3.1). We present this separately, because it can be formulated in a more general setting than it is needed in proving Gaiser's limit theorem.

In terms of notations, we try to avoid using non-standard terminology. The basic notations are given at the beginning of each chapter. In all chapters $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denotes the set of positive integers, the set of integers, the set of real numbers and the set of complex numbers, respectively. The expression "a measure on a topological space" means a measure on the σ-algebra of Borel subsets of the topological space in question. By a Borel neighbourhood U of an element x of a topological space G we mean a Borel subset of G for which there exists an open subset $\widetilde{U}$ of G such that $x \in \widetilde{U} \subset U$. The weak convergence of bounded measures on a topological space is denoted by $\xrightarrow{\mathsf{w}}$.

1.3 Credits

All the proofs of this dissertation are joint work with my supervisor, Gyula Pap.

The proofs of the chapter *Gauss measures on the Heisenberg group* are based on

M. Barczy and G. Pap, Fourier transform of a Gaussian measure on the Heisenberg group, to appear in *Annales de L'Institut Henri Poincaré Probabilités et Statistiques.*

The proofs of the chapter *Gauss measures on the affine group* are based on

M. Barczy and G. Pap, Gaussian measures on the affine group: uniqueness of embedding and supports. *Publ. Math. Debrecen* 63(1-2) (2003), 221-234.

The proofs of the chapter *Limit theorems on LCA2 groups* are based on

M. Barczy, A. Bendikov and G. Pap, Limit theorems on locally compact Abelian groups, submitted to *Mathematische Nachrichten*,

M. Barczy and G. Pap, Weakly infinitely divisible measures on some locally compact Abelian groups, submitted to *Bulletin of Australian Mathematical Society.*

The proof of Gaiser's theorem (see Theorem 4.3.1) is a correction of Gaiser's original proof ([23, Satz 1.3.6]). We clarify and complete some questionable parts of the original proof.

The proofs of the chapter *Portmanteau theorem for unbounded measures* are based on

M. Barczy and G. Pap, Portmanteau theorem for unbounded measures, submitted to *Statistics & Probability Letters.*

Chapter 2

Gauss measures on the Heisenberg group

Fourier transform of a probability measure on a locally compact group plays an important role in several problems concerning convolution and weak convergence of probability measures. In case of a locally compact Abelian group, an explicit formula is available for the Fourier transform of an arbitrary infinitely divisible probability measure (see Parthasarathy [46]). The case of non-Abelian groups is much more complicated. For Lie groups, Tomé [58] proposed a method how to calculate Fourier transforms based on Feynman's path integrals and discussed the physical motivation, but explicit expressions have been derived only in very special cases.

In this chapter we examine some properties of Gauss measures on the 3-dimensional Heisenberg group. An explicit formula is derived for the Fourier transform of a Gauss measure on the 3-dimensional Heisenberg group at the Schrödinger representation (see Theorem 2.3.1). Using this explicit formula, we give necessary and sufficient conditions for the convolution of two Gauss measures to be a Gauss measure (see Theorem 2.2.1). It turns out that a convolution of Gauss measures on the Heisenberg group is almost never a Gauss measure. We also give the Fourier transform of the convolution of two Gauss measures on the Heisenberg group including the case when the convolution is not a Gauss measure (see Theorem 2.6.1).

The structure of the present chapter is similar to Pap [45]. Theorems 2.2.1 and 2.3.1 of the present chapter are generalizations of the corresponding results for symmetric Gauss measures on the Heisenberg group due to Pap [45]. We summarize briefly the new ingredients. Comparing Lemma 6.1 in Pap [45] and Proposition 2.5.3 of the present chapter, one can realize that now we have to calculate a much more complicated (Euclidean) Fourier transform (see (2.5.6)). For this reason we generalized a result due to Chaleyat-Maurel [13] (see Lemma 2.5.2). We note that using Lemma 2.6.3 one can easily derive Theorem 1.1 in Pap [45] from Theorem 2.2.1 of the present chapter.

The results of this chapter are contained in our accepted paper [6].

2.1 Preliminaries

In what follows $\mathbb{H}$ will denote the 3-dimensional Heisenberg group which can be obtained by furnishing $\mathbb{R}^3$ with its natural topology and with the product

$$(g_1, g_2, g_3)(h_1, h_2, h_3) = \Big(g_1 + h_1, g_2 + h_2, g_3 + h_3 + \frac{1}{2}(g_1 h_2 - g_2 h_1)\Big).$$

Then $\mathbb{H}$ is a connected nilpotent Lie group. The Schrödinger representations $\{\pi_{\pm\lambda} : \lambda > 0\}$ of $\mathbb{H}$ are representations in the group of unitary operators of the complex Hilbert space $L^2(\mathbb{R})$ given by

$$[\pi_{\pm\lambda}(g)u](x) := \mathsf{e}^{\pm i(\lambda g_3 + \sqrt{\lambda} g_2 x + \lambda g_1 g_2/2)} u(x + \sqrt{\lambda} g_1) \tag{2.1.1}$$

for $g = (g_1, g_2, g_3) \in \mathbb{H}$, $u \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$ (see Taylor [56, p. 46, Theorem 2.1]). The value of the Fourier transform of a probability measure μ on $\mathbb{H}$ at the Schrödinger representation $\pi_{\pm\lambda}$ is the bounded linear operator $\widehat{\mu}(\pi_{\pm\lambda}) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given by

$$\widehat{\mu}(\pi_{\pm\lambda})u := \int_{\mathbb{H}} \pi_{\pm\lambda}(g)u \, \mu(\mathsf{d}g), \qquad u \in L^2(\mathbb{R}),$$

interpreted as a Bochner integral.

The Lie algebra $\mathcal{H}$ of $\mathbb{H}$ can be realized as the vector space $\mathbb{R}^3$ furnished with multiplication

$$[(p_1, p_2, p_3), (q_1, q_2, q_3)] = (0, 0, p_1 q_2 - p_2 q_1).$$

To an element $X \in \mathcal{H}$ one can correspond a left-invariant differential operator on $\mathbb{H}$, namely, for continuously differentiable functions $f : \mathbb{H} \to \mathbb{R}$ we put

$$\widetilde{X} f(g) := \lim_{t \to 0} \frac{1}{t} \Big(f(g \exp(tX)) - f(g) \Big), \qquad g \in \mathbb{H},$$

where the exponential mapping $\exp : \mathcal{H} \to \mathbb{H}$ is now the identity mapping. We note that the mapping $X \in \mathcal{H} \mapsto \widetilde{X}$ is injective and linear (see, e.g., Corwin–Greenleaf [15, p. 110]).

A family $(\mu_t)_{t\geqslant 0}$ of probability measures on $\mathbb{H}$ is said to be a *continuous convolution semigroup* if we have $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \geqslant 0$, and $\mu_t \xrightarrow{\mathsf{w}} \mu_0 = \delta_e$ as $t \downarrow 0$, where δ_e denotes the Dirac measure concentrated on the unit element $e = (0, 0, 0)$ of $\mathbb{H}$. Its *infinitesimal generator* is defined by

$$(\widetilde{N} f)(g) := \lim_{t\downarrow 0} \frac{1}{t} \int_{\mathbb{H}} \big(f(gh) - f(g)\big)\mu_t(\mathsf{d}h), \qquad g \in \mathbb{H},$$

for suitable functions $f : \mathbb{H} \to \mathbb{R}$. (The infinitesimal generator is always defined for infinitely differentiable functions $f : \mathbb{H} \to \mathbb{R}$ with compact support.) A convolution semigroup $(\mu_t)_{t\geqslant 0}$ is called a *Gauss semigroup* if

$$\lim_{t\downarrow 0} \frac{1}{t} \mu_t(\mathbb{H} \setminus U) = 0$$

for all Borel neighbourhoods U of e. We note that the definition of a Gauss semigroup slightly differs from the Definition 6.2.1 in Heyer [30], since in our definition, given a Gauss semigroup $(\mu_t)_{t\geqslant 0}$, the measure μ_t can be a Dirac measure for any $t > 0$ (see Remark 3.1.1 in Chapter 3).

Let $\{X_1, X_2, X_3\}$ denote the natural basis in $\mathcal{H}$ (that is, $X_1 = (1, 0, 0)$, $X_2 = (0, 1, 0)$ and $X_3 = (0, 0, 1)$). It is known that a convolution semigroup $(\mu_t)_{t\geqslant 0}$ is a Gauss semigroup if and only if its infinitesimal generator has the form

$$\widetilde{N} = \sum_{k=1}^{3} a_k \widetilde{X}_k + \frac{1}{2} \sum_{j=1}^{3} \sum_{k=1}^{3} b_{j,k} \widetilde{X}_j \widetilde{X}_k, \tag{2.1.2}$$

where $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ and $B = (b_{j,k})_{1\leqslant j,k\leqslant 3}$ is a real, symmetric, positive semidefinite matrix. This easily follows from Theorem 4.2.4 and Lemma 6.2.6 in Heyer [30] and from the fact that given a Gauss semigroup $(\mu_t)_{t\geqslant 0}$ such that μ_{t_0} is a Dirac measure on $\mathbb{H}$ for some $t_0 > 0$, there exist a_1, a_2, $a_3 \in \mathbb{R}$ such that $\mu_t = \delta_{\exp(ta_1X_1+ta_2X_2+ta_3X_3)}$ for all $t \geqslant 0$. A probability measure μ on $\mathbb{H}$ is called a *Gauss measure* if there exists a Gauss semigroup $(\mu_t)_{t\geqslant 0}$ such that $\mu = \mu_1$. A Gauss measure on $\mathbb{H}$ can be embedded only in a uniquely determined Gauss semigroup (see Baldi [4], Pap [44]). (Neuenschwander [40] showed that a Gauss measure on $\mathbb{H}$ can not be embedded in a non-Gauss convolution semigroup. We note that in Chapter 3 we show that a Gauss measure on the affine group can be embedded only in a uniquely determined Gauss semigroup, see Theorem 3.3.1.) Thus for a vector $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ and a real, symmetric, positive semidefinite matrix $B = (b_{j,k})_{1 \leqslant j,k \leqslant 3}$ we can speak about the Gauss measure μ with parameters (a, B) which is by definition $\mu := \mu_1$, where $(\mu_t)_{t \geqslant 0}$ is the Gauss semigroup with infinitesimal generator $\widetilde{N}$ given by (2.1.2). If ν is a Gauss measure with parameters (a, B) and $(\nu_s)_{s \geqslant 0}$ is the Gauss semigroup with infinitesimal generator $\widetilde{N}$ given by (2.1.2) then ν_t is a Gauss measure with parameters (ta, tB) for all $t \geqslant 0$, since $\mu_s := \nu_{st}$, $s \geqslant 0$ defines a Gauss semigroup with infinitesimal generator $t\widetilde{N}$. Hence $\nu_t = \mu_1$, so it will be sufficient to calculate the Fourier transform of μ_1.

Let us consider a Gauss semigroup $(\mu_t)_{t \geqslant 0}$ with parameters (a, B) on $\mathbb{H}$. Its infinitesimal generator $\widetilde{N}$ can also be written in the form

$$\widetilde{N} = \widetilde{Y}_0 + \frac{1}{2} \sum_{j=1}^{d} \widetilde{Y}_j^2, \tag{2.1.3}$$

where $0 \leqslant d \leqslant 3$ and

$$Y_0 = \sum_{k=1}^{3} a_k X_k, \qquad Y_j = \sum_{k=1}^{3} \sigma_{k,j} X_k, \quad 1 \leqslant j \leqslant d,$$

where $\Sigma = (\sigma_{k,j})$ is a $3 \times d$ matrix with $\mathsf{rank}\,(\Sigma) = \mathsf{rank}\,(B) = d$. Moreover, $B = \Sigma \cdot \Sigma^\top$. (We just diagonalise the quadratic form appearing in (2.1.2) and use that the mapping $X \in \mathcal{H} \mapsto \widetilde{X}$ is injective and linear.) Then the measure μ_t can be described as the distribution of the random vector $Z(t) = (Z_1(t), Z_2(t), Z_3(t))$ with values in $\mathbb{R}^3$, where

$$\begin{aligned}
Z_1(t) &= a_1 t + \sum_{k=1}^{d} \sigma_{1,k} W_k(t), \qquad Z_2(t) = a_2 t + \sum_{k=1}^{d} \sigma_{2,k} W_k(t), \\
Z_3(t) &= a_3 t + \sum_{k=1}^{d} \sigma_{3,k} W_k(t) + \frac{1}{2} \int_0^t (Z_1(s)\,\mathsf{d}Z_2(s) - Z_2(s)\,\mathsf{d}Z_1(s)) \\
&= a_3 t + \sum_{k=1}^{d} \sigma_{3,k} W_k(t) + \sum_{1 \leqslant k < \ell \leqslant d} (\sigma_{1,k}\sigma_{2,\ell} - \sigma_{1,\ell}\sigma_{2,k}) W_{k,\ell}(t) \\
&\quad + \sum_{k=1}^{d} (a_2 \sigma_{1,k} - a_1 \sigma_{2,k}) W_k^*(t),
\end{aligned}$$

where $(W_1(t), \dots, W_d(t))_{t\geqslant 0}$ is a standard Wiener process in $\mathbb{R}^d$ and

$$W_k^*(t) := \frac{1}{2}\left(\int_0^t W_k(s)\,\mathsf{d}s - \int_0^t s\,\mathsf{d}W_k(s)\right),$$
$$W_{k,\ell}(t) := \frac{1}{2}\left(\int_0^t W_k(s)\,\mathsf{d}W_\ell(s) - \int_0^t W_\ell(s)\,\mathsf{d}W_k(s)\right).$$

(See, e.g., Roynette [47].) The process $(W_{k,\ell}(t))_{t\geqslant 0}$ is the so-called Lévy's stochastic area swept by the process $(W_k(s), W_\ell(s))_{s\in[0,t]}$ on $\mathbb{R}^2$.

2.2 Main results

Let $(\mu_t)_{t\geqslant 0}$ be a Gauss semigroup of probability measures on $\mathbb{H}$. By a result of Siebert [53, Proposition 3.1, Lemma 3.1], $\big(\widehat{\mu}_t(\pi_{\pm\lambda})\big)_{t\geqslant 0}$ is a strongly continuous semigroup of contractions on $L^2(\mathbb{R})$ with infinitesimal generator

$$N(\pi_{\pm\lambda}) = \alpha_1 I + \alpha_2 x + \alpha_3 D + \alpha_4 x^2 + \alpha_5(xD + Dx) + \alpha_6 D^2,$$

where $\alpha_1, \dots, \alpha_6$ are certain complex numbers (depending on $(\mu_t)_{t\geqslant 0}$, see Remark 2.3.2), I denotes the identity operator on $L^2(\mathbb{R})$, x is the multiplication by the variable x, and $Du(x) = u'(x)$. One of our purposes is to determine the action of the operators

$$\widehat{\mu}_t(\pi_{\pm\lambda}) = \mathsf{e}^{tN(\pi_{\pm\lambda})}, \qquad t \geqslant 0,$$

on $L^2(\mathbb{R})$. (Here the notation $(\mathsf{e}^{tA})_{t\geqslant 0}$ means a semigroup of operators with infinitesimal generator A.) When $N(\pi_{\pm\lambda})$ has the special form $\frac{1}{2}(D^2 - x^2)$, the celebrated Mehler's formula gives us

$$\mathsf{e}^{t(D^2-x^2)/2}u(x) = \frac{1}{\sqrt{2\pi\sinh t}}\int_{\mathbb{R}} \exp\left\{-\frac{(x^2+y^2)\cosh t - 2xy}{2\sinh t}\right\} u(y)\,\mathsf{d}y$$

for all $t > 0$, $u \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$, (see, e.g., Taylor [56], Davies [16]). Our Theorem 2.3.1 in Section 2.3 can be regarded as a generalization of Mehler's formula.

It turns out that $\widehat{\mu}_t(\pi_{\pm\lambda}) = \mathsf{e}^{tN(\pi_{\pm\lambda})}$, $t \geqslant 0$ are again integral operators on $L^2(\mathbb{R})$ if α_6 is a positive real number. One of the main results of this chapter is an explicit formula for the kernel function of these integral operators (see Theorem 2.3.1). We apply a probabilistic method using that the Fourier transform $\widehat{\mu}(\pi_{\pm\lambda})$ of an absolutely continuous probability measure μ on $\mathbb{H}$ can be derived from the Euclidean Fourier transform of μ considering μ as a measure on $\mathbb{R}^3$ (see Proposition 2.4.1).

The second part of this chapter deals with convolutions of Gauss measures on $\mathbb{H}$. The convolution of two Gauss measures on a locally compact Abelian group is again a Gauss measure (it can be proved by the help of Fourier transforms; see Parthasarathy [46]). We prove that a convolution of Gauss measures on $\mathbb{H}$ is almost never a Gauss measure. More exactly, we obtain the following result (using our explicit formula for the Fourier transforms).

2.2.1 Theorem. *Let μ' and μ'' be Gauss measures on $\mathbb{H}$. Then the convolution $\mu' * \mu''$ is a Gauss measure on $\mathbb{H}$ if and only if one of the following conditions holds:*

(C1) *there exist elements Y_0', Y_0'', Y_1, Y_2 in the Lie algebra of $\mathbb{H}$ such that $[Y_1, Y_2] = 0$, and the supports of μ' and μ'' are contained in $\exp\{Y_0' + \mathbb{R}\cdot Y_1 + \mathbb{R}\cdot Y_2\}$ and $\exp\{Y_0'' + \mathbb{R}\cdot Y_1 + \mathbb{R}\cdot Y_2\}$, respectively. (Equivalently, there exists an Abelian subgroup $\mathbb{G}$ of $\mathbb{H}$ such that $\mathsf{supp}\,(\mu')$ and $\mathsf{supp}\,(\mu'')$ are contained in "Eucledian cosets" of $\mathbb{G}$.)*

(C2) *there exist a Gauss semigroup $(\mu_t)_{t\geqslant 0}$ and $t', t'' \geqslant 0$ and a Gauss measure ν such that $\mathsf{supp}\,(\nu)$ is contained in the center of $\mathbb{H}$ and either $\mu' = \mu_{t'}$, $\mu'' = \mu_{t''} * \nu$ or $\mu' = \mu_{t'} * \nu$, $\mu'' = \mu_{t''}$ holds. (Equivalently, μ' and μ'' are sitting on the same Gauss semigroup modulo a Gauss measure with support contained in the center of $\mathbb{H}$.)*

By the support $\mathsf{supp}\,(\mu)$ of a measure μ on $\mathbb{H}$ we mean the complement of the union of all open subsets U of $\mathbb{H}$ on which μ vanishes in the sense that for all continuous real valued functions f on $\mathbb{H}$ with compact support contained in U we have $\int_{\mathbb{H}} f \,\mathsf{d}\mu = 0$.

We note that in case of (C1), μ' and μ'' are Gauss measures also in the "Euclidean sense" (i.e., considering them as measures on $\mathbb{R}^3$). Moreover, Theorem 2.6.1 contains an explicit formula for the Fourier transform of a convolution of arbitrary Gauss measures on $\mathbb{H}$.

2.3 Fourier transform of a Gauss measure

The Schrödinger representations are infinite dimensional, irreducible, unitary representations, and each irreducible, unitary representation is unitarily equivalent with one of the Schrödinger representations or with $\chi_{\alpha,\beta}$ for some $\alpha, \beta \in \mathbb{R}$, where $\chi_{\alpha,\beta}$ is a one-dimensional representation given by

$$\chi_{\alpha,\beta}(g) := \mathsf{e}^{i(\alpha g_1+\beta g_2)}, \qquad g = (g_1, g_2, g_3) \in \mathbb{H},$$

(see Taylor [56, p. 49, Theorem 2.5]). The value of the *Fourier transform* of a probability measure μ on $\mathbb{H}$ at the representation $\chi_{\alpha,\beta}$ is

$$\widehat{\mu}(\chi_{\alpha,\beta}) := \int_{\mathbb{H}} \chi_{\alpha,\beta}(g)\,\mu(\mathsf{d}g) = \int_{\mathbb{H}} \mathsf{e}^{i(\alpha g_1+\beta g_2)}\,\mu(\mathsf{d}g) = \widetilde{\mu}(\alpha,\beta,0),$$

where $\widetilde{\mu} : \mathbb{R}^3 \to \mathbb{C}$ denotes the *Euclidean Fourier transform* of μ,

$$\widetilde{\mu}(\alpha,\beta,\gamma) := \int_{\mathbb{H}} \mathsf{e}^{i(\alpha g_1+\beta g_2+\gamma g_3)}\,\mu(\mathsf{d}g).$$

Let us consider a Gauss semigroup $(\mu_t)_{t\geqslant 0}$ with parameters (a, B) on $\mathbb{H}$. The Fourier transform of $\mu := \mu_1$ at the one-dimensional representations can be calculated easily, since the description of $(\mu_t)_{t\geqslant 0}$ given in Section 2.1 implies that

$$\widehat{\mu}(\chi_{\alpha,\beta}) = \mathsf{E}\exp\left\{ i(\alpha a_1+\beta a_2) + i\left(\alpha\sum_{k=1}^{d}\sigma_{1,k}W_k(1) + \beta\sum_{k=1}^{d}\sigma_{2,k}W_k(1)\right)\right\}$$

for $\alpha, \beta \in \mathbb{R}$. The random variable

$$\left(\sum_{k=1}^{d}\sigma_{1,k}W_k(1),\ \sum_{k=1}^{d}\sigma_{2,k}W_k(1)\right)$$

has a normal distribution with zero mean and covariance matrix

$$\begin{bmatrix}\sigma_{1,1} & \dots & \sigma_{1,d}\\ \sigma_{2,1} & \dots & \sigma_{2,d}\end{bmatrix}\begin{bmatrix}\sigma_{1,1} & \sigma_{2,1}\\ \vdots & \vdots\\ \sigma_{1,d} & \sigma_{2,d}\end{bmatrix} = \begin{bmatrix} b_{1,1} & b_{1,2}\\ b_{2,1} & b_{2,2}\end{bmatrix},$$

since $\Sigma\Sigma^\top = B$. Consequently,

$$\widehat{\mu}(\chi_{\alpha,\beta}) = \exp\left\{ i(\alpha a_1+\beta a_2) - \frac{1}{2}(b_{1,1}\alpha^2 + 2b_{1,2}\alpha\beta + b_{2,2}\beta^2)\right\}.$$

One of the main results of the present chapter is an explicit formula for the Fourier transform of a Gauss measure on the Heisenberg group $\mathbb{H}$ at the Schrödinger representations.

2.3.1 Theorem. *Let* μ *be a Gauss measure on* $\mathbb{H}$ *with parameters* (a, B). *Then*

$$[\widehat{\mu}(\pi_{\pm\lambda})u](x) = \begin{cases} \int_{\mathbb{R}} K_{\pm\lambda}(x,y)u(y)\,\mathsf{d}y & \text{if } b_{1,1} > 0, \\ L_{\pm\lambda}(x)u(x+\sqrt{\lambda}a_1) & \text{if } b_{1,1} = 0, \end{cases}$$

for $u \in L^2(\mathbb{R})$, $x \in \mathbb{R}$, *where*

$$K_{\pm\lambda}(x,y) := C_{\pm\lambda}(B)\exp\left\{-\frac{1}{2}\mathbf{z}^\top D_{\pm\lambda}(a,B)\mathbf{z}\right\}, \qquad \mathbf{z} := (x,y,1)^\top,$$

where, with $\delta := \sqrt{b_{1,1}b_{2,2} - b_{1,2}^2}$, $\delta_1 := b_{1,1}b_{2,3} - b_{1,2}b_{1,3}$, $\delta_2 := a_1b_{1,2} - a_2b_{1,1}$,

$$C_{\pm\lambda}(B) := \begin{cases} \frac{1}{\sqrt{2\pi\lambda b_{1,1}}} & \text{if } \delta = 0, \\ \sqrt{\frac{\delta}{2\pi b_{1,1}\sinh(\lambda\delta)}} & \text{if } \delta > 0, \end{cases}$$

and $D_{\pm\lambda}(a,B) = (d_{j,k}^{\pm\lambda}(a,B))_{1\leqslant j,k\leqslant 3}$ *are symmetric matrices defined for* $b_{1,1} > 0$ *and* $\delta = 0$ *by*

$$d_{1,1}^{\pm\lambda}(a,B) := \frac{\lambda^{-1} \pm ib_{1,2}}{b_{1,1}}, \quad d_{1,2}^{\pm\lambda}(a,B) := -\frac{1}{\lambda b_{1,1}}, \quad d_{2,2}^{\pm\lambda}(a,B) := \frac{\lambda^{-1} \mp ib_{1,2}}{b_{1,1}},$$

$$d_{1,3}^{\pm\lambda}(a,B) := \frac{a_1 \pm i\lambda b_{1,3}}{\sqrt{\lambda}b_{1,1}} \pm i\frac{\sqrt{\lambda}\delta_2}{2b_{1,1}}, \qquad d_{2,3}^{\pm\lambda}(a,B) := -\frac{a_1 \pm i\lambda b_{1,3}}{\sqrt{\lambda}b_{1,1}} \pm i\frac{\sqrt{\lambda}\delta_2}{2b_{1,1}},$$

$$d_{3,3}^{\pm\lambda}(a,B) := \frac{(a_1 \pm i\lambda b_{1,3})^2}{b_{1,1}} + \frac{\lambda^2\delta_2^2}{12b_{1,1}} + \lambda^2 b_{3,3} \mp 2i\lambda a_3,$$

and for $\delta > 0$ *by*

$$d_{1,1}^{\pm\lambda}(a,B) := \frac{\delta\coth(\lambda\delta) \pm ib_{1,2}}{b_{1,1}}, \qquad d_{2,2}^{\pm\lambda}(a,B) := \frac{\delta\coth(\lambda\delta) \mp ib_{1,2}}{b_{1,1}},$$

$$d_{1,2}^{\pm\lambda}(a,B) := -\frac{\delta}{b_{1,1}\sinh(\lambda\delta)}, \quad d_{1,3}^{\pm\lambda}(a,B) := \frac{a_1 \pm i\lambda b_{1,3}}{\sqrt{\lambda}b_{1,1}} + \frac{\lambda\delta_1 \pm i\delta_2}{\sqrt{\lambda}b_{1,1}\delta\coth(\lambda\delta/2)},$$

$$d_{2,3}^{\pm\lambda}(a,B) := -\frac{a_1 \pm i\lambda b_{1,3}}{\sqrt{\lambda}b_{1,1}} + \frac{\lambda\delta_1 \pm i\delta_2}{\sqrt{\lambda}b_{1,1}\delta\coth(\lambda\delta/2)},$$

$$d_{3,3}^{\pm\lambda}(a,B) := \frac{(a_1 \pm i\lambda b_{1,3})^2}{b_{1,1}} - \frac{(\lambda\delta_1 \pm i\delta_2)^2}{\lambda b_{1,1}\delta^3}\Big(\lambda\delta - 2\tanh(\lambda\delta/2)\Big) + \lambda^2 b_{3,3} \mp 2i\lambda a_3,$$

and

$$L_{\pm\lambda}(x) := \exp\bigg\{ \pm \frac{i\sqrt{\lambda}}{2}\Big(\sqrt{\lambda}(2a_3 + a_1a_2) + 2a_2x\Big) - \frac{\lambda^2}{6}(3b_{3,3} + 3a_1b_{2,3} + a_1^2 b_{2,2})$$
$$- \frac{\lambda^{3/2}}{2}(2b_{2,3} + a_1b_{2,2})x - \frac{\lambda}{2}b_{2,2}x^2\bigg\}.$$

We prove this theorem in Section 2.5.

2.3.2 Remark. Consider a Gauss semigroup $(\mu_t)_{t\geqslant 0}$ with infinitesimal generator $\widetilde{N}$ given in (2.1.2). Siebert [53, Proposition 3.1, Lemma 3.1] proved that $\big(\widehat{\mu_t}(\pi_{\pm\lambda})\big)_{t\geqslant 0}$ is a strongly continuous semigroup of contractions on $L^2(\mathbb{R})$ with infinitesimal generator

$$N(\pi_{\pm\lambda}) = \sum_{k=1}^{3} a_k X_k(\pi_{\pm\lambda}) + \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3} b_{j,k} X_j(\pi_{\pm\lambda}) X_k(\pi_{\pm\lambda}),$$

where

$$X(\pi_{\pm\lambda})u := \lim_{t\to 0} t^{-1}\big(\pi_{\pm\lambda}(\exp(tX))u - u\big)$$

for all differentiable vectors $u \in L^2(\mathbb{R})$. Here the infinitesimal generator $N(\pi_{\pm\lambda})$ of $\big(\widehat{\mu_t}(\pi_{\pm\lambda})\big)_{t\geqslant 0}$ is the linear operator defined by

$$N(\pi_{\pm\lambda})u := \lim_{t\downarrow 0} \frac{\widehat{\mu_t}(\pi_{\pm\lambda})u - u}{t} \quad \text{for } u \in D(N(\pi_{\pm\lambda})),$$

where

$$D(N(\pi_{\pm\lambda})) := \Big\{ u \in L^2(\mathbb{R}) : \lim_{t\downarrow 0} \frac{\widehat{\mu_t}(\pi_{\pm\lambda})u - u}{t} \quad \text{exists in } L^2(\mathbb{R}) \Big\}.$$

(Then $N(\pi_{\pm\lambda})$ is always defined for all differentiable vectors $u \in L^2(\mathbb{R})$.) We note that the infinitesimal generator $\widetilde{N}$ of a Gauss semigroup $(\mu_t)_{t\geqslant 0}$ can also be considered as the infinitesimal generator of a suitable one-parameter semigroup of bounded linear operators. Namely, for all $t \geqslant 0$ and for all bounded continuous functions $f : \mathbb{H} \to \mathbb{R}$ vanishing at infinity, let

$$(T_{\mu_t} f)(g) := \int_{\mathbb{H}} f(gh)\,\mu_t(\mathsf{d}h), \quad g \in \mathbb{H}.$$

Then $(T_{\mu_t})_{t\geqslant 0}$ is a one-parameter semigroup of bounded linear operators on the Banach space of all bounded continuous functions $f:\mathbb{H}\to\mathbb{R}$ vanishing at infinity equipped with the supremum norm. Moreover, the infinitesimal generator of $(T_{\mu_t})_{t\geqslant 0}$ coincides with the infinitesimal generator $\widetilde{N}$ of $(\mu_t)_{t\geqslant 0}$.

We get

$$\begin{aligned}
[X_1(\pi_{\pm\lambda})u](x) &= \sqrt{\lambda}u'(x) = \sqrt{\lambda}Du(x),\\
[X_2(\pi_{\pm\lambda})u](x) &= \pm i\sqrt{\lambda}xu(x),\\
[X_3(\pi_{\pm\lambda})u](x) &= \pm i\lambda u(x)
\end{aligned}$$

for all $x\in\mathbb{R}$. Consequently,

$$N(\pi_{\pm\lambda}) = \alpha_1 I + \alpha_2 x + \alpha_3 D + \alpha_4 x^2 + \alpha_5(xD + Dx) + \alpha_6 D^2,$$

where

$$\alpha_1 = -\frac{1}{2}\lambda^2 b_{3,3} \pm i\lambda a_3, \qquad \alpha_2 = -\lambda^{3/2} b_{2,3} \pm i\lambda^{1/2} a_2, \qquad \alpha_3 = \lambda^{1/2} a_1 \pm i\lambda^{3/2} b_{1,3},$$

$$\alpha_4 = -\frac{1}{2}\lambda b_{2,2}, \qquad \alpha_5 = \pm\frac{i}{2}\lambda b_{1,2}, \qquad \alpha_6 = \frac{1}{2}\lambda b_{1,1}.$$

2.4 Absolute continuity and singularity of a Gauss measure

A probability measure μ on $\mathbb{H}$ is said to be absolutely continuous or singular if it is absolutely continuous or singular with respect to a (and then necessarily to any) Haar measure on $\mathbb{H}$. It is known that the class of left Haar measures on $\mathbb{H}$ is the same as the class of right Haar measures on $\mathbb{H}$ and hence we can use the expression "a Haar measure on $\mathbb{H}$". It is also known that a measure ν on $\mathbb{H}$ is a Haar measure if and only if ν is the Lebesgue measure on $\mathbb{R}^3$ multiplied by some positive constant (see Corwin–Greenleaf [15, Theorem 1.2.10] and Hewitt–Ross [29, Remarks 15.8]). The following proposition is the same as Proposition 2.1 in Pap [45]. But the proof given here is simpler, we do not use Weyl calculus.

2.4.1 Proposition. *If μ is an absolutely continuous probability measure on $\mathbb{H}$ with density f then the Fourier transform $\widehat{\mu}(\pi_{\pm\lambda})$ is an integral operator* *on* $L^2(\mathbb{R})$,

$$[\widehat{\mu}(\pi_{\pm\lambda})u](x) = \int_{\mathbb{R}} K_{\pm\lambda}(x,y)u(y)\,\mathsf{d}y, \quad u \in \mathbb{L}^2(\mathbb{R}), \ \ x \in \mathbb{R},$$

with kernel function $K_{\pm\lambda} : \mathbb{R}^2 \to \mathbb{C}$ *given by*

$$K_{\pm\lambda}(x,y) := \frac{1}{\sqrt{\lambda}} \widetilde{f}_{2,3}\left(\frac{y-x}{\sqrt{\lambda}}, \pm\sqrt{\lambda}\left(\frac{y+x}{2}\right), \pm\lambda\right),$$

where

$$\widetilde{f}_{2,3}(s_1, \widetilde{s}_2, \widetilde{s}_3) := \int_{\mathbb{R}^2} \mathsf{e}^{i(s_2\widetilde{s}_2 + s_3\widetilde{s}_3)} f(s_1, s_2, s_3)\,\mathsf{d}s_2\,\mathsf{d}s_3, \qquad (s_1, \widetilde{s}_2, \widetilde{s}_3) \in \mathbb{R}^3,$$

denotes a partial Euclidean Fourier transform of f *(considering* f *as a function on* $\mathbb{R}^3$*).*

Proof. Using the definition of the Schrödinger representation we obtain

$$\begin{aligned}
[\widehat{\mu}(\pi_{\pm\lambda})u](x) &= \int_{\mathbb{R}^3} \mathsf{e}^{\pm i(\lambda s_3 + \sqrt{\lambda}s_2 x + \lambda s_1 s_2/2)} u(x + \sqrt{\lambda}s_1) f(s_1, s_2, s_3)\,\mathsf{d}s_1\,\mathsf{d}s_2\,\mathsf{d}s_3 \\
&= \frac{1}{\sqrt{\lambda}} \int_{\mathbb{R}^3} \mathsf{e}^{\pm i(\lambda s_3 + \sqrt{\lambda}s_2 x + \sqrt{\lambda}(y-x)s_2/2)} u(y) f\left(\frac{y-x}{\sqrt{\lambda}}, s_2, s_3\right) \mathsf{d}y\,\mathsf{d}s_2\,\mathsf{d}s_3 \\
&= \int_{\mathbb{R}} K_{\pm\lambda}(x,y)u(y)\,\mathsf{d}y,
\end{aligned}$$

where

$$\begin{aligned}
K_{\pm\lambda}(x,y) &= \frac{1}{\sqrt{\lambda}} \int_{\mathbb{R}^2} \mathsf{e}^{\pm i(\lambda s_3 + \sqrt{\lambda}(x+y)s_2/2)} f\left(\frac{y-x}{\sqrt{\lambda}}, s_2, s_3\right) \mathsf{d}s_2\,\mathsf{d}s_3 \\
&= \frac{1}{\sqrt{\lambda}} \widetilde{f}_{2,3}\left(\frac{y-x}{\sqrt{\lambda}}, \pm\sqrt{\lambda}\left(\frac{y+x}{2}\right), \pm\lambda\right).
\end{aligned}$$

Hence the assertion. □

The partial Euclidean Fourier transform $\widetilde{f}_{2,3}$ can be obtained by the inverse Euclidean Fourier transform:

$$\widetilde{f}_{2,3}(s_1, \widetilde{s}_2, \widetilde{s}_3) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathsf{e}^{-is_1\widetilde{s}_1} \widetilde{f}(\widetilde{s}_1, \widetilde{s}_2, \widetilde{s}_3)\,\mathsf{d}\widetilde{s}_1, \qquad (s_1, \widetilde{s}_2, \widetilde{s}_3) \in \mathbb{R}^3, \tag{2.4.1}$$

where $\widetilde{f}$ denotes the (full) Euclidean Fourier transform of f:

$$\widetilde{f}(\widetilde{s}_1, \widetilde{s}_2, \widetilde{s}_3) := \int_{\mathbb{R}^3} \mathsf{e}^{i(s_1\widetilde{s}_1 + s_2\widetilde{s}_2 + s_3\widetilde{s}_3)} f(s_1, s_2, s_3)\,\mathsf{d}s_1\,\mathsf{d}s_2\,\mathsf{d}s_3$$

for $(\widetilde{s}_1, \widetilde{s}_2, \widetilde{s}_3) \in \mathbb{R}^3$. Moreover, $\widehat{\mu}(\pi_{\pm\lambda})$ is a compact operator. If the density f of μ belongs to the Schwartz space then $\widehat{\mu}(\pi_{\pm\lambda})$ is a trace class (i.e., nuclear) operator.

In order to apply Proposition 2.4.1 we shall need the description of the set of absolutely continuous Gauss measures on $\mathbb{H}$. Using a general result due to Siebert [54, Theorem 2] one can prove the following lemma as in Pap [45, Lemma 3.3].

2.4.2 Lemma. *A Gauss measure μ on $\mathbb{H}$ with parameters (a, B) is either absolutely continuous or singular. More precisely, μ is absolutely continuous if $b_{1,1}b_{2,2} - b_{1,2}^2 > 0$ and singular if $b_{1,1}b_{2,2} - b_{1,2}^2 = 0$.*

By Siebert [54, Theorem 2], given a Gauss semigroup $(\mu_t)_{t\geqslant 0}$ on $\mathbb{H}$, either the measures μ_t are absolutely continuous with respect to the Haar measures on $\mathbb{H}$ for all $t > 0$, or the measures μ_t are singular with respect to the Haar measures on $\mathbb{H}$ for all $t > 0$. In the first case we say that $(\mu_t)_{t\geqslant 0}$ is an absolutely continuous semigroup on $\mathbb{H}$, otherwise it is called singular. The next lemma describes Gauss semigroups on $\mathbb{H}$ and the support of a Gauss measure on $\mathbb{H}$.

2.4.3 Lemma. *Let $(\mu_t)_{t\geqslant 0}$ be a Gauss semigroup on $\mathbb{H}$ with infinitesimal generator $\widetilde{N}$ given by (2.1.3). According to the structure of $\widetilde{N}$ we can distinguish five different types of Gauss semigroups:*

(i) $\widetilde{N} = \widetilde{Y}_0 + \frac{1}{2}(\widetilde{Y}_1^2 + \widetilde{Y}_2^2 + \widetilde{Y}_3^2)$ *with Y_1, Y_2 and Y_3 linearly independent. Then the semigroup is absolutely continuous and $\mathsf{supp}\,(\mu_t) = \mathbb{H}$ for all $t > 0$. Moreover, $\mathsf{rank}\,(B) = 3$, $b_{1,1}b_{2,2} - b_{1,2}^2 \neq 0$.*

(ii) $\widetilde{N} = \widetilde{Y}_0 + \frac{1}{2}(\widetilde{Y}_1^2 + \widetilde{Y}_2^2)$ *with Y_1 and Y_2 linearly independent and $[Y_1, Y_2] \neq 0$. Then the semigroup is absolutely continuous and $\mathsf{supp}\,(\mu_t) = \mathbb{H}$ for all $t > 0$. Moreover, $\mathsf{rank}\,(B) = 2$, $b_{1,1}b_{2,2} - b_{1,2}^2 \neq 0$.*

(iii) $\widetilde{N} = \widetilde{Y}_0 + \frac{1}{2}(\widetilde{Y}_1^2 + \widetilde{Y}_2^2)$ *with Y_1 and Y_2 linearly independent and $[Y_1, Y_2] = 0$. Then the semigroup is singular, it is a Gauss semigroup on $\mathbb{R}^3$ as well, and it is supported by a 'Euclidean coset' of the same closed normal subgroup, namely,*

$$\mathsf{supp}\,(\mu_t) = \exp(tY_0 + \mathbb{R} \cdot Y_1 + \mathbb{R} \cdot Y_2)$$

for all $t > 0$. Moreover, $\mathsf{rank}\,(B) = 2$, $b_{1,1}b_{2,2} - b_{1,2}^2 = 0$.

(iv) $\widetilde{N} = \widetilde{Y}_0 + \frac{1}{2}\widetilde{Y}_1^2$. *Then the semigroup is singular, it is a Gauss semigroup on* $\mathbb{R}^3$ *as well, and it is supported by a "Euclidean coset" of the same closed normal subgroup, namely,*

$$\mathsf{supp}\,(\mu_t) = \exp(tY_0 + \mathbb{R}\cdot Y_1 + \mathbb{R}\cdot [Y_0, Y_1])$$

for all $t > 0$. *Moreover,* $\mathsf{rank}\,(B) = 1$, $b_{1,1}b_{2,2} - b_{1,2}^2 = 0$.

(v) $\widetilde{N} = \widetilde{Y}_0$. *Then the semigroup is singular and consists of Dirac measures, namely,* $\mu_t = \delta_{\exp(tY_0)}$ *for all* $t \geqslant 0$.

Proof. From general results due to Siebert [54, Theorems 2 and 4], it follows that a Gauss measure μ on $\mathbb{H}$ is absolutely continuous if and only if $\mathcal{G} := \mathcal{L}(Y_i, [Y_j, Y_k] : 1 \leqslant i \leqslant d,\, 0 \leqslant j < k \leqslant d) = \mathbb{R}^3$, where $\mathcal{L}(\cdot)$ denotes the linear hull of the given vectors, and $Y_i \in \mathcal{H}$, $0 \leqslant i \leqslant d$ are described in (2.1.3). Moreover, the support of μ_t is

$$\mathsf{supp}\,(\mu_t) = \overline{\bigcup_{n=1}^{\infty} \left(M \exp\left(\frac{tY_0}{n}\right)\right)^n} \qquad \text{for all} \qquad t > 0,$$

where M is the analytic subgroup of $\mathbb{H}$ corresponding to the Lie subalgebra generated by $\{Y_i : 1 \leqslant i \leqslant d\}$ and the bar denotes the closure in $\mathbb{H}$. Clearly $[Y_i, Y_j] = (\sigma_{1,i}\sigma_{2,j} - \sigma_{1,j}\sigma_{2,i})\, X_3$ for $1 \leqslant i < j \leqslant d$ and $[Y, Z] \in \mathcal{L}(X_3)$ for all $Y, Z \in \mathcal{H}$.

We prove only the cases (iii) and (iv), the other cases can be proved similarly.

In case of (iii) we have $\mathcal{G} = \mathcal{L}(Y_1, Y_2, [Y_0, Y_1], [Y_0, Y_2])$. Since $[Y_1, Y_2] = 0$, we have $\sigma_{1,1}\sigma_{2,2} - \sigma_{1,2}^2 = 0$, so Y_1 and Y_2 are linearly dependent in their first two coordinates, thus their linear independence yields $X_3 \in \mathcal{L}(Y_1, Y_2)$. Moreover, $[Y_0, Y_1], [Y_0, Y_2] \in \mathcal{L}(X_3) \subset \mathcal{L}(Y_1, Y_2)$. So $\mathcal{G} = \mathcal{L}(Y_1, Y_2) \neq \mathbb{R}^3$, i.e., the semigroup $(\mu_t)_{t \geqslant 0}$ is singular.

To obtain the formula for the support of μ_t it is sufficient to prove that

$$\left(M \exp\left(\frac{t}{n} Y_0\right)\right)^n = \exp(tY_0 + \mathbb{R}\cdot Y_1 + \mathbb{R}\cdot Y_2)$$

for all $t > 0$ and $n \in \mathbb{N}$, where now $M = \exp(\mathbb{R}\cdot Y_1 + \mathbb{R}\cdot Y_2)$. The multiplication in $\mathbb{H}$ can be reconstructed by the help of the Campbell–Haussdorf formula

$$\exp(X)\exp(Y) = \exp\Big(X + Y + \frac{1}{2}[X, Y]\Big), \quad X, Y \in \mathcal{H},$$

(see Corwin–Greenleaf [15, Theorem 1.2.1]). Applying induction by n gives the assertion. Indeed, for $n = 1$ we have

$$M \exp(tY_0) = \exp(\mathbb{R} \cdot Y_1 + \mathbb{R} \cdot Y_2) \exp(tY_0) = \exp(tY_0 + \mathbb{R} \cdot Y_1 + \mathbb{R} \cdot Y_2),$$

since $[Y_0, Y_1], [Y_0, Y_2] \in \mathcal{L}(X_3) \subset \mathcal{L}(Y_1, Y_2)$. Suppose that

$$\Big(M \exp\big(\frac{t}{n-1} Y_0\big)\Big)^{n-1} = \exp(tY_0 + \mathbb{R} \cdot Y_1 + \mathbb{R} \cdot Y_2)$$

holds for all $t > 0$. Using the Campbell–Haussdorf formula and the induction hypothesis we get

$$\Big(M \exp\big(\frac{t}{n} Y_0\big)\Big)^{n} = \exp\big(\frac{n-1}{n} tY_0 + \mathbb{R} \cdot Y_1 + \mathbb{R} \cdot Y_2\big) \exp\big(\frac{t}{n} Y_0 + \mathbb{R} \cdot Y_1 + \mathbb{R} \cdot Y_2\big).$$

Since $[Y_0, Y_1]$, $[Y_0, Y_2] \in \mathcal{L}(X_3) \subset \mathcal{L}(Y_1, Y_2)$, another application of the Campbell–Haussdorf formula gives the assertion.

The case (iv) can be obtained similarly. Indeed, we have $\mathcal{G} = \mathcal{L}(Y_1, [Y_0, Y_1]) \neq \mathbb{R}^3$, $M = \exp(\mathbb{R} \cdot Y_1)$, hence

$$\mathsf{supp}\,(\mu_t) = \exp(tY_0 + \mathbb{R} \cdot Y_1 + \mathbb{R} \cdot [Y_1, Y_0]) \quad \text{for all } t > 0.$$

□

2.5 Euclidean Fourier transform of a Gauss measure

Now we investigate the processes $(W_k^*(t))_{t \geqslant 0}$ and $(W_{k,\ell}(t))_{t \geqslant 0}$ (defined in Section 2.1). Let $t > 0$ be fixed. We prove that $W_k^*(t)$ and $W_{k,\ell}(t)$ can be constructed by the help of infinitely many independent identically distributed real valued random variables with standard normal distribution. Because of the self-similarity property of the Wiener process it is sufficient to prove the case $t = 2\pi$. The rigorous proof of the following lemma is due to Endre Iglói.

2.5.1 Lemma. *Let* $(W_1(s), \dots, W_d(s))_{s \in [0, 2\pi]}$ *be a standard Wiener process in* $\mathbb{R}^d$ *on a probability space* $(\Omega, \mathcal{A}, \mathrm{P})$. *Let us consider the orthonormal basis* $f_n(s) = (2\pi)^{-1/2} \mathrm{e}^{ins}$, $s \in [0, 2\pi]$, $n \in \mathbb{Z}$ *in the complex Hilbert space* $L^2([0, 2\pi])$. *If* $(g(s))_{s \in [0, 2\pi]}$ *is an adapted, measurable, complex valued process,* *independent of* $(W_1(s),\dots,W_d(s))_{s\in[0,2\pi]}$ *such that* $\mathsf{E}\left(\int_0^{2\pi}|g(s)|^2\,\mathsf{d}s\right)<\infty$ *then for all* $j=1,\dots,d$,

$$\int_0^{2\pi} g(s)\,\mathsf{d}W_j(s)=\sum_{n\in\mathbb{Z}}\langle g,f_n\rangle\int_0^{2\pi} f_n(s)\,\mathsf{d}W_j(s)\quad \text{a. s.,} \tag{2.5.1}$$

where $\langle\cdot,\cdot\rangle$ *denotes the inner product in* $L^2([0,2\pi])$ *and the convergence of the series on the right-hand side of* (2.5.1) *is meant in* $L^2(\Omega,\mathcal{A},\mathrm{P})$.

Proof. Let $1\leqslant j\leqslant d$ be arbitrary, but fixed. First we prove that the right-hand side of (2.5.1) is convergent in $L^2(\Omega,\mathcal{A},\mathrm{P})$. Using that the processes $(g(s))_{s\in[0,2\pi]}$ and $(W_1(s),\dots,W_d(s))_{s\in[0,2\pi]}$ are independent, for $n,m\in\mathbb{Z}$, $n\neq m$, we get

$$\begin{aligned}
&\mathsf{E}\Bigg(\langle g,f_n\rangle\int_0^{2\pi} f_n(s)\,\mathsf{d}W_j(s)\,\overline{\langle g,f_m\rangle}\int_0^{2\pi}\overline{f_m(s)}\,\mathsf{d}W_j(s)\Bigg)\\
&\qquad=\mathsf{E}\big(\langle g,f_n\rangle\overline{\langle g,f_m\rangle}\big)\mathsf{E}\Bigg(\int_0^{2\pi} f_n(s)\,\mathsf{d}W_j(s)\int_0^{2\pi}\overline{f_m(s)}\,\mathsf{d}W_j(s)\Bigg)\\
&\qquad=\mathsf{E}\big(\langle g,f_n\rangle\overline{\langle g,f_m\rangle}\big)\int_0^{2\pi} f_n(s)\overline{f_m(s)}\,\mathsf{d}s=0.
\end{aligned}$$

Using again the independence of $(g(s))_{s\in[0,2\pi]}$ and $(W_1(s),\dots,W_d(s))_{s\in[0,2\pi]}$, we have

$$\begin{aligned}
\mathsf{E}\left|\langle g,f_n\rangle\int_0^{2\pi} f_n(s)\,\mathsf{d}W_j(s)\right|^2&=\mathsf{E}\big|\langle g,f_n\rangle\big|^2\mathsf{E}\left|\int_0^{2\pi} f_n(s)\,\mathsf{d}W_j(s)\right|^2\\
&=\mathsf{E}\big|\langle g,f_n\rangle\big|^2\int_0^{2\pi}|f_n(s)|^2\,\mathsf{d}s=\mathsf{E}\big|\langle g,f_n\rangle\big|^2.
\end{aligned}$$

Since $\mathsf{E}\left(\int_0^{2\pi}|g(s)|^2\,\mathsf{d}s\right)<\infty$, Parseval's identity in $L^2([0,2\pi])$ gives us that

$$\sum_{n\in\mathbb{Z}}\big|\langle g,f_n\rangle\big|^2=\int_0^{2\pi}|g(s)|^2\,\mathsf{d}s\quad\text{a. s.}$$

This implies that

$$\sum_{n\in\mathbb{Z}}\mathsf{E}\big|\langle g,f_n\rangle\big|^2=\mathsf{E}\left(\int_0^{2\pi}|g(s)|^2\,\mathsf{d}s\right)<\infty.$$

Hence the right-hand side of (2.5.1) is convergent in $L^2(\Omega, \mathcal{A}, \mathrm{P})$.

Now we show that

$$\mathsf{E}\left|\int_0^{2\pi} g(s)\,\mathsf{d}W_j(s) - \sum_{n\in\mathbb{Z}}\langle g, f_n\rangle \int_0^{2\pi} f_n(s)\,\mathsf{d}W_j(s)\right|^2 = 0,$$

which implies (2.5.1). We have

$$\begin{aligned}
&\mathsf{E}\left|\int_0^{2\pi} g(s)\,\mathsf{d}W_j(s) - \sum_{n\in\mathbb{Z}}\langle g, f_n\rangle \int_0^{2\pi} f_n(s)\,\mathsf{d}W_j(s)\right|^2 \\
&= \mathsf{E}\left|\int_0^{2\pi} g(s)\,\mathsf{d}W_j(s)\right|^2 + \mathsf{E}\left|\sum_{n\in\mathbb{Z}}\langle g, f_n\rangle \int_0^{2\pi} f_n(s)\,\mathsf{d}W_j(s)\right|^2 \\
&\quad - 2\mathsf{Re}\,\mathsf{E}\left(\int_0^{2\pi} g(s)\,\mathsf{d}W_j(s) \sum_{n\in\mathbb{Z}}\overline{\langle g, f_n\rangle} \int_0^{2\pi} \overline{f_n(s)}\,\mathsf{d}W_j(s)\right) =: A_1 + A_2 - 2\mathsf{Re}\,A_3.
\end{aligned}$$

Then, using that the inner product in $L^2(\Omega, \mathcal{A}, \mathrm{P})$ is continuous, we get

$$\begin{aligned}
A_1 &= \mathsf{E}\left(\int_0^{2\pi} |g(s)|^2\,\mathsf{d}s\right), \\
A_2 &= \sum_{n\in\mathbb{Z}} \mathsf{E}\left|\langle g, f_n\rangle \int_0^{2\pi} f_n(s)\,\mathsf{d}W_j(s)\right|^2 = \sum_{n\in\mathbb{Z}} \mathsf{E}\big|\langle g, f_n\rangle\big|^2 = \mathsf{E}\left(\int_0^{2\pi} |g(s)|^2\,\mathsf{d}s\right), \\
A_3 &= \sum_{n\in\mathbb{Z}} \mathsf{E}\left(\int_0^{2\pi} g(s)\,\mathsf{d}W_j(s)\,\overline{\langle g, f_n\rangle} \int_0^{2\pi} \overline{f_n(s)}\,\mathsf{d}W_j(s)\right).
\end{aligned}$$

Let us denote the σ-algebra generated by the process $(g(s))_{s\in[0,2\pi]}$ by $\mathcal{F}(g)$. Then we obtain

$$\begin{aligned}
A_3 &= \sum_{n\in\mathbb{Z}} \mathsf{E}\,\mathsf{E}\left(\int_0^{2\pi} g(s)\,\mathsf{d}W_j(s)\,\overline{\langle g, f_n\rangle} \int_0^{2\pi} \overline{f_n(s)}\,\mathsf{d}W_j(s)\,\Big|\,\mathcal{F}(g)\right) \\
&= \sum_{n\in\mathbb{Z}} \mathsf{E}\left(\overline{\langle g, f_n\rangle}\,\mathsf{E}\left(\int_0^{2\pi} g(s)\,\mathsf{d}W_j(s) \int_0^{2\pi} \overline{f_n(s)}\,\mathsf{d}W_j(s)\,\Big|\,\mathcal{F}(g)\right)\right) \\
&= \sum_{n\in\mathbb{Z}} \mathsf{E}\left(\overline{\langle g, f_n\rangle} \int_0^{2\pi} g(s)\overline{f_n(s)}\,\mathsf{d}s\right) = \sum_{n\in\mathbb{Z}} \mathsf{E}\big|\langle g, f_n\rangle\big|^2 = \mathsf{E}\left(\int_0^{2\pi} |g(s)|^2\,\mathsf{d}s\right).
\end{aligned}$$

Hence the assertion. □

The next statement is a generalization of Section 1.2 in Chaleyat-Maurel [13].

2.5.2 Lemma. *Let* $(W_1(s), \ldots, W_d(s))_{s\in[0,2\pi]}$ *be a standard Wiener process in* $\mathbb{R}^d$. *Then there exist random variables* $a_n^{(j)}, b_n^{(j)}, n \in \mathbb{N}, j = 1, \ldots, d$, *with standard normal distribution, independent of each other and of the random variable* $(W_1(2\pi), \ldots, W_d(2\pi))$ *such that the following constructions hold*

$$W_{j,k}(2\pi) = \sum_{n=1}^{\infty} \frac{1}{n}\left[b_n^{(j)}\left(a_n^{(k)} - \frac{1}{\sqrt{\pi}} W_k(2\pi)\right) - b_n^{(k)}\left(a_n^{(j)} - \frac{1}{\sqrt{\pi}} W_j(2\pi)\right)\right] \quad \text{a. s.,} \tag{2.5.2}$$

$$W_\ell^*(2\pi) = -2\sqrt{\pi}\sum_{n=1}^{\infty} \frac{b_n^{(\ell)}}{n} \qquad \text{a. s.} \tag{2.5.3}$$

for all $1 \leqslant j < k \leqslant d$ *and* $\ell = 1, \ldots, d$, *where the series on the right-hand sides of* (2.5.2) *and* (2.5.3) *are convergent almost surely.*

Proof. Retain the notations of Lemma 2.5.1 and let us denote

$$c_n^{(j)} := \int_0^{2\pi} \overline{f_n(s)}\, \mathsf{d}W_j(s), \qquad n \in \mathbb{Z}, \quad j = 1, \ldots, d.$$

Then $c_n^{(j)}$, $n \in \mathbb{Z}, n \neq 0, j = 1, \ldots, d$, are independent identically distributed complex valued random variables with standard normal distribution, i.e., the decompositions $c_n^{(j)} = (a_n^{(j)} + i\, b_n^{(j)})/\sqrt{2}$, $n \in \mathbb{Z}, n \neq 0, j = 1, \ldots, d$, hold with independent identically distributed real valued random variables $a_n^{(j)}, b_n^{(j)}$, $n \in \mathbb{Z}$, $n \neq 0, j = 1, \ldots, d$, having standard normal distribution. Specifying g as the indicator function $\mathbb{1}_{[0,t]}$ of the interval $[0, t]$ $(t \in [0, 2\pi])$ in Lemma 2.5.1, we have for all $t \in [0, 2\pi]$

$$W_\ell(t) = \sum_{n\in\mathbb{Z},\, n\neq 0} c_{-n}^{(\ell)} \frac{i}{n}\big(f_{-n}(t) - f_0(t)\big) + \frac{c_0^{(\ell)} t}{\sqrt{2\pi}} \quad \text{a. s.,} \qquad \ell = 1, \ldots, d. \tag{2.5.4}$$

Moreover, there is a set Ω_0 with $\mathrm{P}(\Omega_0) = 0$ such that (2.5.4) holds for all $\omega \notin \Omega_0$ and for almost every $t \in [0, 2\pi]$ (see, e.g., Ash [2, p. 107, Problem 4]). Applying (2.5.1) for $\int_0^{2\pi} W_j(s)\, \mathsf{d}W_k(s)$ and $\int_0^{2\pi} W_k(s)\, \mathsf{d}W_j(s)$ and using the construction (2.5.4) for W_j and W_k, Chaleyat-Maurel [13] showed that (2.5.2) holds. Choosing $g(s) = s\mathbb{1}_{[0,t]}(s)$ $(t \in [0, 2\pi])$ in Lemma 2.5.1 it can be easily checked that

$$\int_0^t s\,\mathsf{d}W_\ell(s) = \sum_{n\in\mathbb{Z},\,n\neq 0} \frac{c_{-n}^{(\ell)}(i\,nt+1)}{n^2} f_{-n}(t) - \sum_{n\in\mathbb{Z},\,n\neq 0} \frac{c_{-n}^{(\ell)}}{n^2}\, f_0(t) + c_0^{(\ell)} \frac{t^2}{2\sqrt{2\pi}} \quad \text{a. s.}$$

By Itô's formula we get $W_\ell^*(t) = \frac{1}{2} t W_\ell(t) - \int_0^t s\,\mathsf{d}W_\ell(s)$. Using the construction (2.5.4) of $W_\ell(t)$ and the definition of $c_n^{(\ell)}$ a simple computation shows that (2.5.3) holds. By Lemma 2.5.1 the series in the constructions (2.5.2), (2.5.3) and (2.5.4) are convergent in $L^2(\Omega, \mathcal{A}, \mathrm{P})$. Since the summands in the series in (2.5.3) and (2.5.4) are independent, Lévy's theorem implies that they are convergent almost surely as well. Finally we show that the series in (2.5.2) is also convergent almost surely. For this, using that $\sum_{n=1}^\infty b_n^{(\ell)}/n$ is convergent almost surely for all $\ell = 1, \dots, d$, it is enough to prove that the series

$$\sum_{n=1}^\infty \frac{1}{n} \big(b_n^{(j)} a_n^{(k)} - b_n^{(k)} a_n^{(j)} \big) \tag{2.5.5}$$

is convergent almost surely. Here $b_n^{(j)} a_n^{(k)} - b_n^{(k)} a_n^{(j)}$, $n \in \mathbb{N}$, are independent, identically distributed real valued random variables with zero mean and finite second moment. Hence Kolmogorov's One-Series Theorem yields that the series in (2.5.5) is convergent almost surely. □

Taking into account Proposition 2.4.1 and the representation of a Gauss semigroup $(\mu_t)_{t\geqslant 0}$ by the process $(Z(t))_{t\geqslant 0}$ (given in Section 2.1), in order to prove Theorem 2.3.1 we need the joint (Euclidean) Fourier transform of the 9-dimensional random vector

$$\big(W_1(t), W_2(t), W_3(t), W_1^*(t), W_2^*(t), W_3^*(t), W_{1,2}(t), W_{1,3}(t), W_{2,3}(t)\big). \tag{2.5.6}$$

2.5.3 Proposition. *The Fourier transform* $\widetilde{F}_t : \mathbb{R}^9 \to \mathbb{C}$ *of the random vector* (2.5.6) *is*

$$\begin{aligned} &\widetilde{F}_t(\eta_1, \eta_2, \eta_3, \zeta_1, \zeta_2, \zeta_3, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}) \\ &\quad = \frac{1}{\cosh(t\|\widetilde{\xi}\|/2)} \exp\left\{ \frac{\|\widetilde{\xi}\|^2 \|\widetilde{\eta}\|^2 + \kappa \langle \widetilde{\xi}, \widetilde{\eta} \rangle^2 - t\kappa(1+\kappa)\|\zeta\|^2}{2(1+\kappa)\|\widetilde{\xi}\|^2} \right. \\ &\qquad\qquad \left. - \frac{t^3}{4\|\widetilde{\xi}\|^2} \left(\frac{1}{6} - \frac{2\kappa}{t^2\|\widetilde{\xi}\|^2} \right) \langle \widetilde{\xi}, \zeta \rangle^2 \right\} \end{aligned}$$

for $\widetilde{\xi} := (\xi_{2,3}, -\xi_{1,3}, \xi_{1,2})^\top \in \mathbb{R}^3$ *with* $\widetilde{\xi} \neq 0$, *where* $\zeta := (\zeta_1, \zeta_2, \zeta_3)^\top \in \mathbb{R}^3$ *and*

$$\kappa := \frac{t\|\widetilde{\xi}\|}{2} \coth\left(\frac{t\|\widetilde{\xi}\|}{2}\right) - 1, \qquad \widetilde{\eta} := \frac{\sqrt{t}\kappa}{\|\widetilde{\xi}\|^2}\xi\zeta + i\sqrt{t}\eta,$$

with $\eta := (\eta_1, \eta_2, \eta_3)^\top \in \mathbb{R}^3$ *and*

$$\xi := \begin{bmatrix} 0 & \xi_{1,2} & \xi_{1,3} \\ -\xi_{1,2} & 0 & \xi_{2,3} \\ -\xi_{1,3} & -\xi_{2,3} & 0 \end{bmatrix}.$$

(Here $\|\cdot\|$ *and* $\langle \cdot, \cdot \rangle$ *denote the Euclidean norm and scalar product, respectively.)*

To prove Proposition 2.5.3 we will use the constructions of the processes $(W_k^*(t))_{t\geqslant 0}$ and $(W_{k,\ell}(t))_{\geqslant 0}$ (see Lemma 2.5.2) and the following lemma.

2.5.4 Lemma. *Let* X *be a* k*-dimensional real random vector with standard normal distribution. Then we have*

$$\mathsf{E}\exp\left\{\langle \widetilde{\eta}, X\rangle - s\langle BX, X\rangle\right\} = \frac{1}{\sqrt{\det(I + 2sB)}}\exp\left\{\frac{1}{2}\langle \widetilde{\eta}, (I + 2sB)^{-1}\widetilde{\eta}\rangle\right\},$$

for all $\widetilde{\eta} \in \mathbb{C}^k$, *nonnegative real numbers* s *and real symmetric positive semidefinite matrices* B. *(Here* I *denotes the* $k \times k$ *identity matrix.)*

Proof. Consider a decomposition $B = U\Lambda U^\top$, where Λ is the $k \times k$ diagonal matrix containing the eigenvalues of B in its diagonal and U is an orthogonal matrix. Then the random vector $Y := U^\top X$ has also a standard normal distribution. This implies that

$$\begin{aligned} \mathsf{E}\exp\left\{\langle \widetilde{\eta}, X\rangle - s\langle BX, X\rangle\right\} &= \mathsf{E}\exp\left\{\langle \widetilde{\eta}, UY\rangle - s\langle \Lambda Y, Y\rangle\right\} \\ &= \frac{1}{\sqrt{(2\pi)^k}}\int_{\mathbb{R}^k}\exp\left\{\langle \widetilde{\eta}, Uy\rangle - s\langle \Lambda y, y\rangle - \frac{1}{2}\langle y, y\rangle\right\}\mathsf{d}y, \end{aligned}$$

where $y = (y_1, \ldots, y_k)^\top \in \mathbb{R}^k$. Let $\lambda_1, \ldots, \lambda_k$ denote the eigenvalues of the matrix B. A simple computation shows that

$$\begin{aligned}
&\langle \widetilde{\eta}, Uy \rangle - s\langle \Lambda y, y \rangle - \frac{1}{2}\langle y, y \rangle \\
&\quad = -\sum_{j=1}^{k} \left(s\lambda_j + \frac{1}{2} \right) y_j^2 + \sum_{j=1}^{k} (U^\top \mathsf{Re}\,\widetilde{\eta})_j y_j + i \sum_{j=1}^{k} (U^\top \mathsf{Im}\,\widetilde{\eta})_j y_j \\
&\quad = i \sum_{j=1}^{k} (U^\top \mathsf{Im}\,\widetilde{\eta})_j y_j - \sum_{j=1}^{k} \frac{1 + 2s\lambda_j}{2} \left(y_j - \frac{(U^\top \mathsf{Re}\,\widetilde{\eta})_j}{1 + 2s\lambda_j} \right)^2 + \sum_{j=1}^{k} \frac{(U^\top \mathsf{Re}\,\widetilde{\eta})_j^2}{2(1 + 2s\lambda_j)}.
\end{aligned}$$

Using the well-known formula for the Fourier transform of a standard normal distribution

$$\int_{\mathbb{R}} \exp\left\{ ixt - \frac{(x-m)^2}{2\sigma^2} \right\} \mathsf{d}x = \sqrt{2\pi}\sigma \exp\left\{ imt - \frac{1}{2}\sigma^2 t^2 \right\}, \tag{2.5.7}$$

for all $t, m \in \mathbb{R}$ and $\sigma > 0$, we obtain

$$\begin{aligned}
&\mathsf{E} \exp\left\{ \langle \widetilde{\eta}, X \rangle - s\langle BX, X \rangle \right\} \\
&= \frac{1}{\sqrt{\prod_{j=1}^{k} (1 + 2s\lambda_j)}} \exp\left\{ i \sum_{j=1}^{k} \frac{(U^\top \mathsf{Re}\,\widetilde{\eta})_j (U^\top \mathsf{Im}\,\widetilde{\eta})_j}{1 + 2s\lambda_j} - \sum_{j=1}^{k} \frac{(U^\top \mathsf{Im}\,\widetilde{\eta})_j^2}{2(1 + 2s\lambda_j)} \right. \\
&\qquad\qquad\qquad \left. + \sum_{j=1}^{k} \frac{(U^\top \mathsf{Re}\,\widetilde{\eta})_j^2}{2(1 + 2s\lambda_j)} \right\}.
\end{aligned}$$

Hence the assertion. □

Proof of Proposition 2.5.3. Because of the self-similarity property of the Wiener process, the random vectors $\big(W_k(t), W_\ell^*(t), W_{p,q}(t) : 1 \leqslant k, \ell \leqslant d,\ 1 \leqslant p < q \leqslant d\big)$ and $\big(c^{-1/2} W_k(ct), c^{-3/2} W_\ell^*(ct), c^{-1} W_{p,q}(ct) : 1 \leqslant k, \ell \leqslant d,\ 1 \leqslant p < q \leqslant d\big)$ have the same distribution for all $t \geqslant 0$ and $c > 0$. Hence

$$\begin{aligned}
&\widetilde{F}_t(\eta_1, \eta_2, \eta_3, \zeta_1, \zeta_2, \zeta_3, \xi_{1,2}, \xi_{1,3}, \xi_{2,3}) \\
&\quad = \widetilde{F}_{2\pi} \left(\sqrt{\frac{t}{2\pi}} \eta_1, \sqrt{\frac{t}{2\pi}} \eta_2, \sqrt{\frac{t}{2\pi}} \eta_3, \left(\frac{t}{2\pi} \right)^{3/2} \zeta_1, \left(\frac{t}{2\pi} \right)^{3/2} \zeta_2, \left(\frac{t}{2\pi} \right)^{3/2} \zeta_3, \right. \\
&\qquad\qquad \left. \frac{t}{2\pi} \xi_{1,2}, \frac{t}{2\pi} \xi_{1,3}, \frac{t}{2\pi} \xi_{2,3} \right),
\end{aligned}$$

so it is sufficient to determine $\widetilde{F}_{2\pi}$. By the definition of the Fourier transform we get

$$\begin{aligned}
&\widetilde{F}_{2\pi}(\eta_1,\eta_2,\eta_3,\zeta_1,\zeta_2,\zeta_3,\xi_{1,2},\xi_{1,3},\xi_{2,3}) \qquad (2.5.8)\\
&= \mathsf{E}\exp\bigg\{i\bigg(\sum_{j=1}^{3}\eta_j W_j(2\pi)+\sum_{j=1}^{3}\zeta_j W_j^*(2\pi)+\sum_{1\leqslant j<k\leqslant 3}\xi_{j,k}W_{j,k}(2\pi)\bigg)\bigg\}.
\end{aligned}$$

For abbreviation let $\widetilde{F}_{2\pi}$ denote $\widetilde{F}_{2\pi}(\eta_1,\eta_2,\eta_3,\zeta_1,\zeta_2,\zeta_3,\xi_{1,2},\xi_{1,3},\xi_{2,3})$. Define the random vector $\chi := (\chi_1,\chi_2,\chi_3)^\top$ by

$$\begin{aligned}
\chi_1 &:= -\xi_{1,2}\frac{1}{\sqrt{\pi}}W_2(2\pi)-\xi_{1,3}\frac{1}{\sqrt{\pi}}W_3(2\pi)-2\sqrt{\pi}\zeta_1,\\
\chi_2 &:= \xi_{1,2}\frac{1}{\sqrt{\pi}}W_1(2\pi)-\xi_{2,3}\frac{1}{\sqrt{\pi}}W_3(2\pi)-2\sqrt{\pi}\zeta_2,\\
\chi_3 &:= \xi_{1,3}\frac{1}{\sqrt{\pi}}W_1(2\pi)+\xi_{2,3}\frac{1}{\sqrt{\pi}}W_2(2\pi)-2\sqrt{\pi}\zeta_3.
\end{aligned}$$

Substituting the expressions (2.5.2), (2.5.3) for $W_{j,k}(2\pi)$ and $W_\ell^*(2\pi)$ into the formula (2.5.8), taking conditional expectation with respect to the random variables $\{W_j(2\pi),\, a_n^{(j)},\, 1\leqslant j\leqslant 3,\, n\geqslant 1\}$, and using the identity $\mathsf{E}(\mathsf{E}(X|Y))=\mathsf{E}X$ (where X, Y random variables, $\mathsf{E}|X|<\infty$), we obtain

$$\begin{aligned}
\widetilde{F}_{2\pi} = \mathsf{E}\bigg[&\exp\Big\{i\big(\eta_1W_1(2\pi)+\eta_2W_2(2\pi)+\eta_3W_3(2\pi)\big)\Big\}\\
&\times\mathsf{E}\bigg(\exp\bigg\{i\sum_{n=1}^{\infty}\frac{1}{n}\langle\xi\cdot a_n+\chi,b_n\rangle\bigg\}\,\bigg|\,W_j(2\pi),\,a_n^{(j)},\,1\leqslant j\leqslant 3,\,n\geqslant 1\bigg)\bigg],
\end{aligned}$$

where $a_n := (a_n^{(1)},a_n^{(2)},a_n^{(3)})^\top$ and $b_n := (b_n^{(1)},b_n^{(2)},b_n^{(3)})^\top$. Taking into account that $b_n^{(1)}$, $b_n^{(2)}$, $b_n^{(3)}$ are independent of the condition above and of each other for all $n\in\mathbb{N}$, using the dominated convergence theorem and the explicit formula for the Fourier transform of a standard normal distribution we get

$$\begin{aligned}
\widetilde{F}_{2\pi} = \mathsf{E}\bigg[&\exp\Big\{i\big(\eta_1W_1(2\pi)+\eta_2W_2(2\pi)+\eta_3W_3(2\pi)\big)\Big\}\\
&\times\prod_{n=1}^{\infty}\exp\bigg\{-\frac{1}{2n^2}\|\xi\cdot a_n+\chi\|^2\bigg\}\bigg].
\end{aligned}$$

Since ξ is a skew symmetric matrix, there exists an orthogonal matrix $M = (m_{j,k})_{1\leqslant j,k\leqslant 3}$ such that

$$M^\top \xi M = \begin{bmatrix} 0 & p & 0 \\ -p & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} =: P.$$

The orthogonality of M implies $M^{-1} = M^\top$, hence $\xi M = MP$. We have

$$MP = \begin{bmatrix} -pm_{1,2} & pm_{1,1} & 0 \\ -pm_{2,2} & pm_{2,1} & 0 \\ -pm_{3,2} & pm_{3,1} & 0 \end{bmatrix} = [-p\mathbf{m}_2, p\mathbf{m}_1, 0],$$

where $\mathbf{m}_i$, $i = 1,2,3$, denotes the column vectors of M, that is, $M = [\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3]$. Obviously, $\xi M = [\xi\mathbf{m}_1, \xi\mathbf{m}_2, \xi\mathbf{m}_3]$, hence $\xi\mathbf{m}_1 = -p\mathbf{m}_2$, $\xi\mathbf{m}_2 = p\mathbf{m}_1$, $\xi\mathbf{m}_3 = 0$. Taking into account that M is orthogonal, we have $\|\mathbf{m}_3\| = 1$, hence

$$\mathbf{m}_3 = \pm \frac{1}{\sqrt{\xi_{1,2}^2 + \xi_{1,3}^2 + \xi_{2,3}^2}} (\xi_{2,3}, -\xi_{1,3}, \xi_{1,2})^\top.$$

Moreover, $\xi^2\mathbf{m}_1 = \xi(\xi\mathbf{m}_1) = \xi(-p\mathbf{m}_2) = -p^2\mathbf{m}_1$. The only nonzero eigenvalue of ξ^2 is $-(\xi_{1,2}^2 + \xi_{1,3}^2 + \xi_{2,3}^2)$, hence $p = \pm\sqrt{\xi_{1,2}^2 + \xi_{1,3}^2 + \xi_{2,3}^2}$, and M can be chosen such that $\mathbf{m}_3 = \widetilde{\xi}/\|\widetilde{\xi}\|$, $p = \|\widetilde{\xi}\|$, and thus

$$\langle \mathbf{m}_1, u\rangle^2 + \langle \mathbf{m}_2, u\rangle^2 = \|M^\top u\|^2 - \langle \mathbf{m}_3, u\rangle^2 = \|u\|^2 - \frac{1}{\|\widetilde{\xi}\|^2}\langle \widetilde{\xi}, u\rangle^2, \tag{2.5.9}$$

for all $u \in \mathbb{R}^3$. We also get

$$-\xi^2 = M \begin{bmatrix} \|\widetilde{\xi}\|^2 & 0 & 0 \\ 0 & \|\widetilde{\xi}\|^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} M^\top =: M\Lambda M^\top.$$

To continue the calculation of the Fourier transform of (2.5.6) we take conditional expectation with respect to $\{W_1(2\pi), W_2(2\pi), W_3(2\pi)\}$. A special case of Lemma 2.5.4 is that

$$\begin{aligned} \mathsf{E}\exp\Bigg\{ -s\sum_{j=1}^n Y_j^2 \Bigg\} &= \frac{1}{\sqrt{\det(I + 2sD)}} \\ &\quad\times \exp\bigg\{ \Big\langle \big(2s^2 D^{1/2}(I+2sD)^{-1}D^{1/2} - sI\big)m, m \Big\rangle \bigg\} \end{aligned}$$

for all nonnegative real numbers s, where $Y = (Y_1, \dots, Y_k)^\top$ is a k-dimensional random variable with normal distribution such that $\mathsf{E}Y = m$ and $\mathsf{Var}Y = D$. Applying this formula for $Y = \xi \cdot a_n + \chi$ with $s = (2n^2)^{-1}$, $m = \chi$ and $D = \xi \cdot \xi^\top = -\xi^2 = M\Lambda M^\top$ we get

$$
\begin{aligned}
\widetilde{F}_{2\pi} = \mathsf{E}\bigg[&\exp\Big\{ i\big(\eta_1 W_1(2\pi) + \eta_2 W_2(2\pi) + \eta_3 W_3(2\pi)\big)\Big\} \\
&\times \prod_{n=1}^{\infty} \frac{1}{\sqrt{\mathsf{det}\,(I + n^{-2}\Lambda)}} \\
&\qquad \times \exp\left\{ \frac{1}{2}\Big\langle \big(n^{-4}\sqrt{\Lambda}(I + n^{-2}\Lambda)^{-1}\sqrt{\Lambda} - n^{-2}I\big)M^{-1}\chi, M^{-1}\chi \Big\rangle \right\}\bigg].
\end{aligned}
$$

Clearly $\mathsf{det}\,(I + n^{-2}\Lambda) = (1 + n^{-2}\|\widetilde{\xi}\|^2)^2$. Using that

$$
\prod_{k=1}^{\infty} \frac{k^2\pi^2}{k^2\pi^2 + x^2} = \frac{x}{\sinh x}, \qquad x\coth x - 1 = x^2 \sum_{k=1}^{\infty} \frac{2}{k^2\pi^2 + x^2}, \qquad x \in \mathbb{R},
$$

(see Gradshteyn–Ryzhik [24, formulas 1.431 and 1.421]), the identity (2.5.9) and the fact that $\langle \widetilde{\xi}, \chi\rangle^2 = 4\pi\langle \zeta, \widetilde{\xi}\rangle^2$ we obtain

$$
\begin{aligned}
\widetilde{F}_{2\pi} =& \frac{\pi\|\widetilde{\xi}\|}{\sinh(\pi\|\widetilde{\xi}\|)} \exp\left\{ -\frac{\pi^3}{\|\widetilde{\xi}\|^2}\left(\frac{1}{3} - \frac{\kappa}{\pi^2\|\widetilde{\xi}\|^2}\right)\langle \zeta, \widetilde{\xi}\rangle^2 \right\} \\
&\times \mathsf{E}\exp\left\{ i\big(\eta_1 W_1(2\pi) + \eta_2 W_2(2\pi) + \eta_3 W_3(2\pi)\big) - \frac{\kappa}{4\|\widetilde{\xi}\|^2}\|\chi\|^2 \right\},
\end{aligned}
$$

where $\kappa = \pi\|\widetilde{\xi}\|\coth(\pi\|\widetilde{\xi}\|) - 1$. A simple computation shows that

$$
\begin{aligned}
\|\chi\|^2 = & \frac{1}{\pi}\Big((\xi_{1,2}^2 + \xi_{1,3}^2)W_1^2(2\pi) + (\xi_{1,2}^2 + \xi_{2,3}^2)W_2^2(2\pi) + (\xi_{2,3}^2 + \xi_{1,3}^2)W_3^2(2\pi) \Big) \\
& + \frac{2}{\pi}\Big(\xi_{1,3}\xi_{2,3}W_1(2\pi)W_2(2\pi) - \xi_{1,2}\xi_{2,3}W_1(2\pi)W_3(2\pi) \\
& \qquad + \xi_{1,2}\xi_{1,3}W_2(2\pi)W_3(2\pi) \Big) - 4(\xi_{1,2}\zeta_2 + \xi_{1,3}\zeta_3)W_1(2\pi) \\
& + 4(\xi_{1,2}\zeta_1 - \xi_{2,3}\zeta_3)W_2(2\pi) + 4(\xi_{1,3}\zeta_1 + \xi_{2,3}\zeta_2)W_3(2\pi) + 4\pi\|\zeta\|^2.
\end{aligned}
$$

Using Lemma 2.5.4 with $\widetilde{\eta} = \frac{\sqrt{2\pi\kappa}}{\|\xi\|^2}\xi\zeta + i\sqrt{2\pi}\eta$, $B := -2\xi^2$, $s = \frac{\kappa}{4\|\xi\|^2}$ and taking into account that $\sqrt{\mathsf{det}\,(I+2sB)} = 1+\kappa$ we conclude

$$\begin{aligned}\widetilde{F}_{2\pi} =& \frac{\pi\|\widetilde{\xi}\|}{(1+\kappa)\sinh(\pi\|\widetilde{\xi}\|)}\exp\left\{-\frac{\pi^3}{\|\widetilde{\xi}\|^2}\left(\frac{1}{3}-\frac{\kappa}{\pi^2\|\widetilde{\xi}\|^2}\right)\langle\zeta,\widetilde{\xi}\rangle^2\right\}\\ &\times\exp\left\{-\frac{\pi\kappa}{\|\widetilde{\xi}\|^2}\|\zeta\|^2+\frac{1}{2}\left\langle\widetilde{\eta},\left(I-\frac{\kappa}{\|\widetilde{\xi}\|^2}\xi^2\right)^{-1}\widetilde{\eta}\right\rangle\right\}.\end{aligned}$$

Using (2.5.9) we get

$$\left\langle\widetilde{\eta},\left(I-\frac{\kappa}{\|\widetilde{\xi}\|^2}\xi^2\right)^{-1}\widetilde{\eta}\right\rangle = \frac{1}{1+\kappa}\|\widetilde{\eta}\|^2+\frac{\kappa}{1+\kappa}\frac{\langle\widetilde{\xi},\widetilde{\eta}\rangle^2}{\|\widetilde{\xi}\|^2}.$$

Hence the assertion. □

Proof of Theorem 2.3.1. We prove only the case $\mathsf{rank}\,(B)=3$. The cases $\mathsf{rank}\,(B)=1$ and $\mathsf{rank}\,(B)=2$ can be handled in a similar way. In case $\mathsf{rank}\,(B)=3$ the measure μ is absolutely continuous and so Proposition 2.4.1 implies that the partial Euclidean Fourier transform $\widetilde{f}_{2,3}$ of the measure μ has to be calculated in order to obtain the Fourier transform $\widehat{\mu}(\pi_{\pm\lambda})$. Let $(\mu_t)_{t\geqslant 0}$ be a Gauss semigroup such that $\mu_1=\mu$ and let $\rho_1 := \sigma_{1,1}\sigma_{2,2}-\sigma_{1,2}\sigma_{2,1}$, $\rho_2 := \sigma_{1,1}\sigma_{2,3}-\sigma_{1,3}\sigma_{2,1}$, $\rho_3 := \sigma_{1,2}\sigma_{2,3}-\sigma_{1,3}\sigma_{2,2}$ by definition. In case $\mathsf{rank}\,(B)=3$, the representation of $(\mu_t)_{t\geqslant 0}$ by the process $(Z(t))_{t\geqslant 0}$ (see Section 2.1) gives us

$$\begin{aligned}Z_1(1) &= a_1+\sum_{k=1}^{3}\sigma_{1,k}W_k(1), \qquad Z_2(1) = a_2+\sum_{k=1}^{3}\sigma_{2,k}W_k(1),\\ Z_3(1) &= a_3+\sum_{k=1}^{3}\sigma_{3,k}W_k(1)+\sum_{k=1}^{3}(a_2\sigma_{1,k}-a_1\sigma_{2,k})W_k^*(1)\\ &\quad+\rho_1W_{1,2}(1)+\rho_2W_{1,3}(1)+\rho_3W_{2,3}(1).\end{aligned}$$

This implies that the (full) Euclidean Fourier transform of the measure μ is

$$\begin{aligned}
\widetilde{f}(\widetilde{s}_1, \widetilde{s}_2, \widetilde{s}_3) &= \mathsf{E} \exp\Big\{ i\big(\widetilde{s}_1 Z_1(1) + \widetilde{s}_2 Z_2(1) + \widetilde{s}_3 Z_3(1)\big) \Big\} \\
&= \exp\big\{ i(\widetilde{s}_1 a_1 + \widetilde{s}_2 a_2 + \widetilde{s}_3 a_3) \big\} \\
&\quad \times \mathsf{E} \exp\Bigg\{ i\Bigg(\sum_{k=1}^{3} (\sigma_{1,k}\widetilde{s}_1 + \sigma_{2,k}\widetilde{s}_2 + \sigma_{3,k}\widetilde{s}_3) W_k(1) \\
&\qquad\qquad + \widetilde{s}_3 \rho_1 W_{1,2}(1) + \widetilde{s}_3 \rho_2 W_{1,3}(1) + \widetilde{s}_3 \rho_3 W_{2,3}(1) \\
&\qquad\qquad + \sum_{k=1}^{3} (a_2 \sigma_{1,k} - a_1 \sigma_{2,k}) \widetilde{s}_3 W_k^*(1) \Bigg) \Bigg\}.
\end{aligned}$$

Proposition 2.4.1 shows that we may suppose $\widetilde{s}_3 \neq 0$. Using Proposition 2.5.3 and the facts that

$$\begin{aligned}
&\sum_{k=1}^{d} (a_2 \sigma_{1,k} - a_1 \sigma_{2,k})^2 = b_{2,2} a_1^2 - 2 b_{1,2} a_1 a_2 + b_{1,1} a_2^2, \qquad d = 1, 2, 3, \\
&\rho_1 (a_1 \sigma_{2,3} - a_2 \sigma_{1,3}) - \rho_2 (a_1 \sigma_{2,2} - a_2 \sigma_{1,2}) + \rho_3 (a_1 \sigma_{2,1} - a_2 \sigma_{1,1}) = 0, \\
&\delta^2 = \rho_1^2 + \rho_2^2 + \rho_3^2,
\end{aligned} \tag{2.5.10}$$

we get

$$\begin{aligned}
\widetilde{f}(\widetilde{s}_1, \widetilde{s}_2, \widetilde{s}_3) = \frac{1}{\cosh(|\widetilde{s}_3|\delta/2)} \exp\Bigg\{ & i\big(\widetilde{s}_1 a_1 + \widetilde{s}_2 a_2 + \widetilde{s}_3 a_3\big) + \frac{\kappa}{2(1+\kappa)} \frac{\langle \widetilde{\xi}, \widetilde{\eta} \rangle^2}{\delta^2} \\
& - \frac{\kappa}{2\delta^2} \big(b_{2,2} a_1^2 - 2 b_{1,2} a_1 a_2 + b_{1,1} a_2^2 \big) \\
& + \frac{1}{2(1+\kappa)} \|\widetilde{\eta}\|^2 \Bigg\},
\end{aligned}$$

where

$$\kappa := \frac{|\widetilde{s}_3|\delta}{2} \coth\left(\frac{|\widetilde{s}_3|\delta}{2} \right) - 1, \qquad \widetilde{\eta} := -\frac{\kappa}{\delta^2} (v_1, v_2, v_3)^\top + i \Sigma^\top \widetilde{s}$$

with

$$\begin{aligned}
v_1 &:= \rho_1 (a_1 \sigma_{2,2} - a_2 \sigma_{1,2}) + \rho_2 (a_1 \sigma_{2,3} - a_2 \sigma_{1,3}), \\
v_2 &:= -\rho_1 (a_1 \sigma_{2,1} - a_2 \sigma_{1,1}) + \rho_3 (a_1 \sigma_{2,3} - a_2 \sigma_{1,3}), \\
v_3 &:= -\rho_2 (a_1 \sigma_{2,1} - a_2 \sigma_{1,1}) - \rho_3 (a_1 \sigma_{2,2} - a_2 \sigma_{1,2}),
\end{aligned}$$

and $\widetilde{s} := (\widetilde{s}_1, \widetilde{s}_2, \widetilde{s}_3)^\top$, $\widetilde{\xi} := (\rho_3, -\rho_2, \rho_1)^\top$. It can be easily checked that

$$\begin{aligned}
&\langle \widetilde{\xi}, \widetilde{\eta} \rangle^2 = -\widetilde{s}_3^2 \operatorname{det} B, \\
&\|\widetilde{\eta}\|^2 = -\langle B\widetilde{s}, \widetilde{s}\rangle + \frac{\kappa^2}{\delta^4}\langle v, v\rangle - 2i\frac{\kappa}{\delta^2}\big((\widetilde{s}_1 a_1 + \widetilde{s}_2 a_2)\delta^2 + \widetilde{s}_3(a_1\delta_3 + a_2\delta_1)\big), \\
&\widetilde{s}^\top B \widetilde{s} = b_{1,1}\left(\widetilde{s}_1 + \frac{b_{1,2}\widetilde{s}_2 + b_{1,3}\widetilde{s}_3}{b_{1,1}}\right)^2 + \frac{1}{b_{1,1}}\begin{bmatrix}\widetilde{s}_2\\ \widetilde{s}_3\end{bmatrix}^\top \begin{bmatrix}\delta^2 & \delta_1\\ \delta_1 & \delta_4\end{bmatrix}\begin{bmatrix}\widetilde{s}_2\\ \widetilde{s}_3\end{bmatrix},
\end{aligned}$$

where $\delta_3 := b_{1,3}b_{2,2} - b_{1,2}b_{2,3}$ and $\delta_4 := b_{1,1}b_{3,3} - b_{1,3}^2$. Using (2.4.1), the identities above and (2.5.7), the partial Fourier transform $\widetilde{f}_{2,3}$ can be calculated as follows

$$\begin{aligned}
\widetilde{f}_{2,3}(s_1, \widetilde{s}_2, \widetilde{s}_3) = &\sqrt{\frac{|\widetilde{s}_3|\delta}{2\pi b_{1,1}\sinh(|\widetilde{s}_3|\delta)}}\exp\Bigg\{ -\frac{1}{2(1+\kappa)b_{1,1}}\begin{bmatrix}\widetilde{s}_2\\ \widetilde{s}_3\end{bmatrix}^\top \begin{bmatrix}\delta^2 & \delta_1\\ \delta_1 & \delta_4\end{bmatrix}\begin{bmatrix}\widetilde{s}_2\\ \widetilde{s}_3\end{bmatrix} \\
&- \frac{\kappa}{2(1+\kappa)\delta^2}\widetilde{s}_3^2 \operatorname{det} B - \frac{\kappa}{2(1+\kappa)\delta^2}(b_{2,2}a_1^2 - 2b_{1,2}a_1a_2 + b_{1,1}a_2^2) \\
&- \frac{1+\kappa}{2b_{1,1}}\left(\frac{a_1}{1+\kappa} - s_1\right)^2 - \frac{b_{1,2}\widetilde{s}_2 + b_{1,3}\widetilde{s}_3}{b_{1,1}}\left(\frac{a_1}{1+\kappa} - s_1\right) \\
&+ i\bigg(\widetilde{s}_2 a_2 + \widetilde{s}_3 a_3 - \frac{\kappa}{(1+\kappa)\delta^2}\big(\widetilde{s}_2 a_2 \delta^2 + \widetilde{s}_3(a_1\delta_3 + a_2\delta_1)\big)\bigg)\Bigg\}.
\end{aligned}$$

Finally Proposition 2.4.1 implies that the Fourier transform $\widehat{\mu}(\pi_{\pm\lambda})$ is an integral operator on $L^2(\mathbb{R})$,

$$[\widehat{\mu}(\pi_{\pm\lambda})u](x) = \int_{\mathbb{R}} K_{\pm\lambda}(x, y)u(y)\,\mathsf{d}y,$$

where $K_{\pm\lambda}$ has the form given in Theorem 2.3.1. □

2.6 Convolution of Gauss measures

The convolution of two probability measures μ' and μ'' on $\mathbb{H}$ is defined by

$$(\mu' * \mu'')(A) := \int_{\mathbb{H}} \mu''(h^{-1}A)\,\mu'(\mathsf{d}h),$$

for all Borel sets A in $\mathbb{H}$.

First we give an explicit formula for the Fourier transform of a convolution of two Gauss measures on $\mathbb{H}$.

2.6.1 Theorem. *Let* μ' *and* μ'' *be Gauss measures on* $\mathbb{H}$ *with parameters* (a', B') *and* (a'', B''), *respectively. Then we have*

$$
\begin{aligned}
(\mu' * \mu'')\widehat{\ }(\chi_{\alpha,\beta}) = \exp\Big\{ & i\big((a_1' + a_1'')\alpha + (a_2' + a_2'')\beta\big) \\
& - \frac{1}{2}\Big((b_{1,1}' + b_{1,1}'')\alpha^2 + 2(b_{1,2}' + b_{1,2}'')\alpha\beta + (b_{2,2}' + b_{2,2}'')\beta^2\Big)\Big\},
\end{aligned}
$$

$$
\Big[(\mu' * \mu'')\widehat{\ }(\pi_{\pm\lambda})u\Big](x) = \begin{cases} L_{\pm\lambda}(x)u\big(x + \sqrt{\lambda}(a_1' + a_1'')\big) & \text{if } b_{1,1}' = b_{1,1}'' = 0, \\ \displaystyle\int_{\mathbb{R}} K_{\pm\lambda}(x,y)u(y)\,\mathsf{d}y & \text{otherwise,} \end{cases}
$$

where $L_{\pm\lambda}(x)$ *is given by*

$$
\begin{aligned}
\exp\Big\{ & \pm i\Big(\lambda\big(a_3' + a_3'' + (a_1'a_2' + a_1''a_2'')/2\big) + \sqrt{\lambda}(a_2' + a_2'')x + \lambda a_1' a_2''\Big) \\
& - \frac{\lambda}{2}x^2(b_{2,2}' + b_{2,2}'') - \frac{\lambda^{3/2}}{2}x\Big(2b_{2,3}' + 2b_{2,3}'' + a_1'b_{2,2}' + (2a_1' + a_1'')b_{2,2}''\Big) \\
& - \frac{\lambda^2}{2}\Big(b_{3,3}' + b_{3,3}'' + a_1'b_{2,3}' + (2a_1' + a_1'')b_{2,3}'' + \big((a_1')^2 b_{2,2}' + (a_1'')^2 b_{2,2}''\big)/3 \\
& \qquad + a_1'(a_1' + a_1'')b_{2,2}''\Big)\Big\},
\end{aligned}
$$

and $K_{\pm\lambda}(x,y) := C\exp\big\{-\frac{1}{2}\mathbf{z}^\top V\mathbf{z}\big\}$, $\mathbf{z} := (x, y, 1)^\top$, *with*

$$
C := \begin{cases} C_{\pm\lambda}(B') & \text{if } b_{1,1}' > 0 \text{ and } b_{1,1}'' = 0, \\ C_{\pm\lambda}(B'') & \text{if } b_{1,1}' = 0 \text{ and } b_{1,1}'' > 0, \\ C_{\pm\lambda}(B')C_{\pm\lambda}(B'')\sqrt{\dfrac{2\pi}{d_{2,2}' + d_{1,1}''}} & \text{if } b_{1,1}' > 0 \text{ and } b_{1,1}'' > 0, \end{cases}
$$

(taking the square root with positive real part) where $C_{\pm\lambda}(B')$, $C_{\pm\lambda}(B'')$ *are*

defined in Theorem 2.3.1 and

$$V := \begin{cases} D_{\pm\lambda}(a', B') + \begin{bmatrix} 0 & 0 & -\sqrt{\lambda} a_1'' d_{1,2}' \\ 0 & \lambda b_{2,2}'' & p_{2,3} \\ -\sqrt{\lambda} a_1'' d_{1,2}' & p_{3,2} & p_{3,3} \end{bmatrix} & \text{if } b_{1,1}' > 0 \text{ and } b_{1,1}'' = 0, \\ \begin{bmatrix} \lambda b_{2,2}' & 0 & q_{1,3} \\ 0 & 0 & \sqrt{\lambda} a_1' d_{1,2}'' \\ q_{3,1} & \sqrt{\lambda} a_1' d_{1,2}'' & q_{3,3} \end{bmatrix} + D_{\pm\lambda}(a'', B'') & \text{if } b_{1,1}' = 0 \text{ and } b_{1,1}'' > 0, \\ \begin{bmatrix} d_{1,1}' & 0 & d_{1,3}' \\ 0 & d_{2,2}'' & d_{2,3}'' \\ d_{3,1}' & d_{3,2}'' & d_{3,3}' + d_{3,3}'' \end{bmatrix} - \dfrac{UU^\top}{d_{2,2}' + d_{1,1}''} & \text{if } b_{1,1}' > 0 \text{ and } b_{1,1}'' > 0, \end{cases}$$

where $d_{j,k}' := d_{j,k}^{\pm\lambda}(a', B')$, $d_{j,k}'' := d_{j,k}^{\pm\lambda}(a'', B'')$ *for* $1 \leqslant j, k \leqslant 3$ *are defined in Theorem 2.3.1 and*

$$\begin{aligned} U &:= (d_{1,2}', d_{2,1}'', d_{3,2}' + d_{3,1}'')^\top, \\ p_{2,3} &:= p_{3,2} := -\sqrt{\lambda} a_1'' d_{2,2}' + \lambda^{3/2}(2b_{2,3}'' - a_1'' b_{2,2}'')/2 \mp i\sqrt{\lambda} a_2'', \\ p_{3,3} &:= -\sqrt{\lambda} a_1''(d_{2,3}' + d_{3,2}') + \lambda(a_1'')^2 d_{2,2}' + \lambda^2\big(b_{3,3}'' - a_1'' b_{2,3}'' + (a_1'')^2 b_{2,2}''/3\big) \\ &\quad \mp i\lambda(2a_3'' - a_1'' a_2''), \\ q_{1,3} &:= q_{3,1} := \sqrt{\lambda} a_1' d_{1,1}'' + \lambda^{3/2}(a_1' b_{2,2}' + 2b_{2,3}')/2 \mp i\sqrt{\lambda} a_2', \\ q_{3,3} &:= \sqrt{\lambda} a_1'(d_{1,3}'' + d_{3,1}'') + \lambda(a_1')^2 d_{1,1}'' + \lambda^2\big(b_{3,3}' + a_1' b_{2,3}' + (a_1')^2 b_{2,2}'/3\big) \\ &\quad \mp i\lambda(2a_3' + a_1' a_2'). \end{aligned}$$

Proof. If $b_{1,1}' > 0$ and $b_{1,1}'' > 0$ then the assertion can be proved as in Pap [45, Theorem 7.2]. If $b_{1,1}' > 0$ and $b_{1,1}'' = 0$ then by Theorem 2.3.1

$$[\widehat{\mu'}(\pi_{\pm\lambda})u](x) = \int_{\mathbb{R}} K_{\pm\lambda}'(x, y) u(y)\, \mathsf{d}y$$

with

$$K_{\pm\lambda}'(x, y) := C_{\pm\lambda}(B') \exp\left\{-\frac{1}{2}\mathbf{z}^\top D_{\pm\lambda}(a', B')\mathbf{z}\right\}, \qquad \mathbf{z} = (x, y, 1)^\top,$$

and

$$[\widehat{\mu''}(\pi_{\pm\lambda})u](y)$$
$$= \exp\left\{ \pm \frac{i\sqrt{\lambda}}{2}\big(\sqrt{\lambda}(2a_3'' + a_1''a_2'') + 2a_2''y\big) - \frac{\lambda^2}{6}\big(3b_{3,3}'' + 3a_1''b_{2,3}'' + (a_1'')^2 b_{2,2}''\big) - \frac{\lambda^{3/2}}{2}(2b_{2,3}'' + a_1''b_{2,2}'')y - \frac{\lambda}{2}b_{2,2}''y^2 \right\} u(y + \sqrt{\lambda}a_1'').$$

Clearly we have

$$\Big[(\mu' * \mu'')\widehat{\ }(\pi_{\pm\lambda})u\Big](x) = [\widehat{\mu'}(\pi_{\pm\lambda})\widehat{\mu''}(\pi_{\pm\lambda})u](x) = \int_{\mathbb{R}} K_{\pm\lambda}'(x,y)[\widehat{\mu''}(\pi_{\pm\lambda})u](y)\,\mathsf{d}y.$$

Using the formulas for $\widehat{\mu'}(\pi_{\pm\lambda})$ and $\widehat{\mu''}(\pi_{\pm\lambda})$ an easy calculation yields that $K_{\pm\lambda}$ has the form given in the theorem. The other cases $b_{1,1}' = 0,\ b_{1,1}'' > 0$ and $b_{1,1}' = b_{1,1}'' = 0$ can be handled in the same way. □

We need two lemmas concerning the parameters of a Gauss measure on $\mathbb{H}$.

2.6.2 Lemma. *Let us consider a Gauss semigroup $(\mu_t)_{t\geqslant 0}$ such that μ_1 is a Gauss measure on $\mathbb{H}$ with parameters (a, B). Then we have*

$$a_i = \mathsf{E}Z_i,\ i = 1,2,3, \qquad b_{i,j} = \mathsf{Cov}(Z_i, Z_j) \quad \text{if} \quad (i,j) \neq (3,3),$$

and

$$b_{3,3} = \mathsf{Var}Z_3 - \frac{1}{4}\Big(\mathsf{Var}Z_1\mathsf{Var}Z_2 - \mathsf{Cov}(Z_1,Z_2)^2\Big) - \frac{1}{12}\Big(\mathsf{Var}Z_2\,(\mathsf{E}Z_1)^2 - 2\mathsf{Cov}(Z_1,Z_2)\,\mathsf{E}Z_1\mathsf{E}Z_2 + \mathsf{Var}Z_1\,(\mathsf{E}Z_2)^2\Big),$$

where the distribution of the random vector (Z_1, Z_2, Z_3) with values in $\mathbb{R}^3$ is μ_1.

Proof. Let $Z(t) := (Z_1(t), Z_2(t), Z_3(t))$, $t \geqslant 0$ be given as in Section 2.1. Taking the expectation of $Z(1)$ yields that $\mathsf{E}(Z_i(1)) = a_i$, $i = 1,2,3$. Using again the definition of $Z(1)$ and the fact that $B = \Sigma\cdot\Sigma^\top$ we get

$$\mathsf{Var}(Z_1(1)) = \sum_{k=1}^d \sum_{\ell=1}^d \sigma_{1,k}\sigma_{1,\ell}\mathsf{E}(W_k(1)W_\ell(1)) = \sum_{k=1}^d \sigma_{1,k}^2 = b_{1,1}.$$

Similar arguments show $\mathsf{Var}(Z_2(1)) = b_{2,2}$ and $\mathsf{Cov}(Z_1(1), Z_2(1)) = b_{1,2}$. We also obtain

$$\begin{aligned}\mathsf{Cov}(Z_1(1), Z_3(1)) = \mathsf{E}\bigg[&\sum_{i=1}^{d} \sigma_{1,i} W_i(1) \bigg(\sum_{k=1}^{d} \sigma_{3,k} W_k(1) + \sum_{k=1}^{d} (a_2\sigma_{1,k} - a_1\sigma_{2,k}) W_k^*(1) \bigg) \\ &+ \sum_{i=1}^{d} \sigma_{1,i} W_i(1) \sum_{1 \leqslant k < \ell \leqslant d} (\sigma_{1,k}\sigma_{2,\ell} - \sigma_{1,\ell}\sigma_{2,k}) W_{k,\ell}(1) \bigg],\end{aligned}$$

which implies that

$$\begin{aligned}\mathsf{Cov}(Z_1(1), Z_3(1)) &= \sum_{k=1}^{d} \sigma_{1,k}\sigma_{3,k} + \sum_{i=1}^{d}\sum_{k=1}^{d} \sigma_{1,i}(a_2\sigma_{1,k} - a_1\sigma_{2,k}) \mathsf{E}(W_i(1) W_k^*(1)) \\ &\quad + \sum_{i=1}^{d} \sum_{1 \leqslant k < \ell \leqslant d} \sigma_{1,i}(\sigma_{1,k}\sigma_{2,\ell} - \sigma_{1,\ell}\sigma_{2,k}) \mathsf{E}(W_i(1) W_{k,\ell}(1)) \\ &= b_{1,3},\end{aligned}$$

since $W_i(1),\ 1 \leqslant i \leqslant d$ are independent of each other and

$$\mathsf{E}(W_i(1) W_k^*(1)) = \mathsf{E}(W_i(1) W_{k,\ell}(1)) = 0, \quad 1 \leqslant i \leqslant d, \ \ 1 \leqslant k < \ell \leqslant d. \tag{2.6.1}$$

Indeed,

$$\begin{aligned}\mathsf{E}(W_i(1) W_k^*(1)) = \frac{1}{2} \lim_{n\to\infty} \mathsf{E}\bigg[W_i(1) \sum_{j=1}^{n} \Big(&W_k(s_{j-1}^{(n)})(s_j^{(n)} - s_{j-1}^{(n)}) \\ &- s_{j-1}^{(n)}\big(W_k(s_j^{(n)}) - W_k(s_{j-1}^{(n)})\big)\Big)\bigg],\end{aligned}$$

$$\begin{aligned}\mathsf{E}(W_i(1) W_{k,\ell}(1)) = \frac{1}{2} \lim_{n\to\infty} \mathsf{E}\bigg[W_i(1) \sum_{j=1}^{n} \Big(&W_k(s_{j-1}^{(n)})\big(W_\ell(s_j^{(n)}) - W_\ell(s_{j-1}^{(n)})\big) \\ &- W_\ell(s_{j-1}^{(n)})\big(W_k(s_j^{(n)}) - W_k(s_{j-1}^{(n)})\big)\Big)\bigg]\end{aligned}$$

for all $1 \leqslant i \leqslant d$, $1 \leqslant k < \ell \leqslant d$, where $\{s_j^{(n)} : j = 0, \ldots, n\}$ denotes a partition of the interval $[0, 1]$ such that $\max_{1\leqslant j\leqslant n}(s_j^{(n)} - s_{j-1}^{(n)})$ tends to 0

as n goes to infinity. We can obtain $\mathsf{Cov}(Z_2(1), Z_3(1)) = b_{2,3}$ in the same way. Using again the form of $Z(t)$, (2.6.1) and the facts that

$$\begin{aligned}
&\mathsf{Cov}(W_{i,j}(1), W_{k,\ell}(1)) = 0 \quad \text{for all } 1 \leqslant i < j \leqslant d,\ 1 \leqslant k < \ell \leqslant d,\ (i,j) \neq (k,\ell), \\
&\mathsf{Cov}(W_k^*(1), W_\ell^*(1)) = 0 \qquad \text{for all } 1 \leqslant k, \ell \leqslant d,\ k \neq \ell,
\end{aligned}$$

we get

$$\begin{aligned}
\mathsf{Var}(Z_3(1)) = &\sum_{k=1}^d \sigma_{3,k}^2 + \sum_{k=1}^d (a_2\sigma_{1,k} - a_1\sigma_{2,k})^2 \mathsf{Var}(W_k^*(1)) \\
&+ \sum_{1 \leqslant k < \ell \leqslant d} (\sigma_{1,k}\sigma_{2,\ell} - \sigma_{1,\ell}\sigma_{2,k})^2 \mathsf{Var}(W_{k,\ell}(1)).
\end{aligned}$$

Lévy proved that the (Euclidean) Fourier transform of $W_{k,\ell}(1)$, $1 \leqslant k < \ell \leqslant d$ (i.e., the characteristic function of $W_{k,\ell}(1)$) is

$$\mathsf{E}\left(e^{itW_{k,\ell}(1)}\right) = \frac{1}{\cosh(t/2)}, \qquad 1 \leqslant k < \ell \leqslant d,$$

for all $t \in \mathbb{R}$ (this follows also from Proposition 2.5.3), so

$$\mathsf{Var}(W_{k,\ell}(1)) = -\frac{\mathsf{d}^2}{\mathsf{d}\, t^2}\left(\frac{1}{\cosh(t/2)}\right)\bigg|_{t=0} = \frac{1}{4}, \qquad 1 \leqslant k < \ell \leqslant d.$$

Clearly W_k^* has a normal distribution with zero mean and with variance $\mathsf{Var}(W_k^*(1)) = \frac{1}{12}$, $1 \leqslant k \leqslant d$. Using (2.5.10) we have

$$\mathsf{Var}(Z_3(1)) = b_{3,3} + \frac{1}{4}(b_{1,1}b_{2,2} - b_{1,2}^2) + \frac{1}{12}(a_1^2 b_{2,2} - 2a_1a_2b_{1,2} + a_2^2 b_{1,1}).$$

Hence the assertion. □

2.6.3 Lemma. *Let μ' and μ'' be Gauss measures on $\mathbb{H}$ with parameters (a', B') and (a'', B''), respectively. If the convolution $\mu' * \mu''$ is a Gauss* *measure on* $\mathbb{H}$ *with parameters* (a, B) *then we have*

$$a_1 = a_1' + a_1'', \qquad a_2 = a_2' + a_2'', \qquad a_3 = a_3' + a_3'' + \frac{1}{2}(a_1'a_2'' - a_2'a_1''),$$

$$b_{1,1} = b_{1,1}' + b_{1,1}'', \qquad b_{1,2} = b_{1,2}' + b_{1,2}'', \qquad b_{2,2} = b_{2,2}' + b_{2,2}'',$$

$$b_{1,3} = b_{1,3}' + b_{1,3}'' + \frac{1}{2}\big(a_2''b_{1,1}' - a_1''b_{1,2}' + a_1'b_{1,2}'' - a_2'b_{1,1}''\big),$$

$$b_{2,3} = b_{2,3}' + b_{2,3}'' + \frac{1}{2}\big(a_2''b_{1,2}' - a_1''b_{2,2}' + a_1'b_{2,2}'' - a_2'b_{1,2}''\big),$$

$$\begin{aligned} b_{3,3} = b_{3,3}' + b_{3,3}'' + a_2''b_{1,3}' - a_1''b_{2,3}' + a_1'b_{2,3}'' - a_2'b_{1,3}'' \\ + \frac{1}{6}\Big(- a_1'a_1''b_{2,2}' + (a_1'')^2b_{2,2}' + (a_1')^2b_{2,2}'' - a_1'a_2''b_{2,2}'' + a_1'a_2''b_{1,2}' + a_1''a_2'b_{1,2}' \\ - 2a_1''a_2''b_{1,2}' - 2a_1'a_2'b_{1,2}'' + a_1'a_2''b_{1,2}'' + a_1''a_2'b_{1,2}'' - a_2'a_2''b_{1,1}' + (a_2'')^2b_{1,1}' \\ + (a_2')^2b_{1,1}'' - a_2'a_2''b_{1,1}''\Big). \end{aligned}$$

Proof. Let $Z' = (Z_1', Z_2', Z_3')^\top$ and $Z'' = (Z_1'', Z_2'', Z_3'')^\top$ be independent random variables with values in $\mathbb{R}^3$ such that the distribution of Z' is μ' and the distribution of Z'' is μ'', respectively. Then the convolution $\mu' * \mu''$ is the distribution of the random variable

$$\Big(Z_1' + Z_1'', Z_2' + Z_2'', Z_3' + Z_3'' + \frac{1}{2}(Z_1'Z_2'' - Z_1''Z_2')\Big) =: (Z_1, Z_2, Z_3).$$

Using Lemma 2.6.2 we get

$$a_1 = \mathsf{E}Z_1 = \mathsf{E}Z_1' + \mathsf{E}Z_1'' = a_1' + a_1'',$$
$$a_2 = \mathsf{E}Z_2 = \mathsf{E}Z_2' + \mathsf{E}Z_2'' = a_2' + a_2'',$$
$$a_3 = \mathsf{E}Z_3 = \mathsf{E}Z_3' + \mathsf{E}Z_3'' + \frac{1}{2}\big(\mathsf{E}Z_1'\mathsf{E}Z_2'' - \mathsf{E}Z_1''\mathsf{E}Z_2'\big) = a_3' + a_3'' + \frac{1}{2}(a_1'a_2'' - a_2'a_1''),$$

since Z' and Z'' are independent. Similar arguments show that

$$b_{1,1} = \mathsf{Var}Z_1 = \mathsf{Var}Z_1' + \mathsf{Var}Z_1'' = b_{1,1}' + b_{1,1}'',$$
$$b_{2,2} = \mathsf{Var}Z_2 = \mathsf{Var}Z_2' + \mathsf{Var}Z_2'' = b_{2,2}' + b_{2,2}'',$$
$$b_{1,2} = \mathsf{Cov}(Z_1, Z_2) = b_{1,2}' + b_{1,2}''.$$

We also have

$$\begin{aligned} b_{1,3} = \mathsf{Cov}(Z_1, Z_3) = \mathsf{Cov}(Z_1', Z_3') + \mathsf{Cov}(Z_1'', Z_3'') \\ + \frac{1}{2}\Big(\mathsf{Cov}(Z_1', Z_1'Z_2'') - \mathsf{Cov}(Z_1', Z_2'Z_1'') + \mathsf{Cov}(Z_1'', Z_1'Z_2'') - \mathsf{Cov}(Z_1'', Z_1''Z_2')\Big). \end{aligned}$$

Using this and Lemma 2.6.2 the validity of the formula for $b_{1,3}$ can be easily checked. For example, we have

$$\mathsf{Cov}(Z_1', Z_1'Z_2'') = \mathsf{E}\big((Z_1')^2 Z_2''\big) - \mathsf{E}Z_1'\mathsf{E}(Z_1'Z_2'') = \big(b_{1,1}' + (a_1')^2\big)a_2'' - (a_1')^2 a_2'' = a_2'' b_{1,1}'.$$

The validity of the formula for $b_{2,3}$ can be proved in the same way. Lemma 2.6.2 implies that

$$\begin{aligned}
\mathsf{Var} Z_3 &= b_{3,3} + \frac{1}{4}(b_{1,1}b_{2,2} - b_{1,2}^2) + \frac{1}{12}(a_1^2 b_{2,2} - 2a_1a_2b_{1,2} + a_2^2 b_{1,1}) = \mathsf{Cov}(Z_3, Z_3) \\
&= \mathsf{Cov}(Z_3', Z_3') + \mathsf{Cov}(Z_3'', Z_3'') + \mathsf{Cov}(Z_3', Z_1'Z_2'') - \mathsf{Cov}(Z_3', Z_1''Z_2') \\
&\quad + \mathsf{Cov}(Z_3'', Z_1'Z_2'') - \mathsf{Cov}(Z_3'', Z_1''Z_2') + \frac{1}{4}\Big(\mathsf{Cov}(Z_1'Z_2'', Z_1'Z_2'') \\
&\quad - \mathsf{Cov}(Z_1'Z_2'', Z_1''Z_2') - \mathsf{Cov}(Z_1''Z_2', Z_1'Z_2'') + \mathsf{Cov}(Z_1''Z_2', Z_1''Z_2')\Big).
\end{aligned}$$

Using again Lemma 2.6.2 and substituting the formulas for $b_{1,1}$, $b_{1,2}$, $b_{2,2}$, a_1 and a_2 into the formula above, an easy calculation shows the validity of the formula for $b_{3,3}$. □

Our aim is to give necessary and sufficient conditions for a convolution of two Gauss measures to be a Gauss measure. Using the fact that the Fourier transform is injective (i.e., if μ and ν are probability measures on $\mathbb{H}$ such that $\widehat{\mu}(\chi_{\alpha,\beta}) = \widehat{\nu}(\chi_{\alpha,\beta})$ for all $\alpha, \beta \in \mathbb{R}$ and $\widehat{\mu}(\pi_{\pm\lambda}) = \widehat{\nu}(\pi_{\pm\lambda})$ for all $\lambda > 0$ then $\mu = \nu$), our task can be fulfilled in the following way. We take the Fourier transform of the convolution of two Gauss measures μ' and μ'' with parameters (a', B') and (a'', B'') at all one-dimensional and at all Schrödinger representations and then we search for necessary and sufficient conditions under which this Fourier transform has the form given in Theorem 2.3.1. First we sketch our approach to obtain necessary conditions. By Theorem 2.6.1, $(\mu' * \mu'')\widehat{\ }(\pi_{\pm\lambda})$ is an integral operator for $b_{1,1}' + b_{1,1}'' > 0$, and it is a product of certain shift and multiplication operators for $b_{1,1}' + b_{1,1}'' = 0$. If the convolution $\mu' * \mu''$ is a Gauss measure with parameters (a, B) then, by Theorem 2.3.1, $(\mu' * \mu'')\widehat{\ }(\pi_{\pm\lambda})$ is an integral operator for $b_{1,1} > 0$, and it is a product of certain shift and multiplication operators for $b_{1,1} = 0$. By Lemma 2.6.3, we have $b_{1,1} = b_{1,1}' + b_{1,1}''$, hence $b_{1,1} = 0$ if and only if $b_{1,1}' + b_{1,1}'' = 0$. Hence if $b_{1,1} > 0$, the integral operator $(\mu' * \mu'')\widehat{\ }(\pi_{\pm\lambda})$ can be written with the kernel function given in Theorem 2.3.1 and also with the kernel function given in Theorem 2.6.1. In the next lemma we derive some consequences of this observation.

2.6.4 Lemma. *Let μ' and μ'' be Gauss measures on $\mathbb{H}$ with parameters (a', B') and (a'', B''), respectively. Suppose that $\mu' * \mu''$ is a Gauss measure on $\mathbb{H}$ with parameters $a = (a_i)_{1 \leqslant i \leqslant 3}$, $B = (b_{j,k})_{1 \leqslant j,k \leqslant 3}$ such that $b_{1,1} > 0$. Then $d_{j,k}^{\pm\lambda} = v_{j,k}^{\pm\lambda}$ for all $1 \leqslant j,k \leqslant 3$ with $(j,k) \neq (3,3)$ and for all $\lambda > 0$, and*

$$C_{\pm\lambda}(B) \exp\left\{-\frac{1}{2} d_{3,3}^{\pm\lambda}\right\} = C \exp\left\{-\frac{1}{2} v_{3,3}^{\pm\lambda}\right\}, \quad \lambda > 0,$$

where $C_{\pm\lambda}(B)$, $d_{j,k}^{\pm\lambda} := d_{j,k}^{\pm\lambda}(a, B)$, $1 \leqslant j,k \leqslant 3$ and C, $V =: (v_{j,k}^{\pm\lambda})_{1 \leqslant j,k \leqslant 3}$ are defined in Theorems 2.3.1 and 2.6.1, respectively.

Proof. The Fourier transform $(\mu' * \mu'')\widehat{\ }(\pi_{\pm\lambda})$ is a bounded linear operator on $L^2(\mathbb{R})$, and since $b_{1,1} > 0$, Theorem 2.3.1 yields that it is an integral operator on $L^2(\mathbb{R})$,

$$\left[(\mu' * \mu'')\widehat{\ }(\pi_{\pm\lambda}) u\right](x) = \int_{\mathbb{R}} K_{\pm\lambda}(x,y) u(y) \, \mathsf{d}y, \quad u \in L^2(\mathbb{R}), \ x \in \mathbb{R}, \qquad (2.6.2)$$

where

$$K_{\pm\lambda}(x,y) = C_{\pm\lambda}(B) \exp\left\{-\frac{1}{2} \mathbf{z}^\top D_{\pm\lambda}(a,B) \mathbf{z}\right\}, \quad \mathbf{z} = (x,y,1)^\top.$$

Let us write $d'_{j,k} =: d_{j,k}^{\pm\lambda}(a', B')$ and $d''_{j,k} =: d_{j,k}^{\pm\lambda}(a'', B'')$ for $1 \leqslant j,k \leqslant 3$ as in Theorem 2.6.1. By Lemma 2.6.3, we have $b_{1,1} = b'_{1,1} + b''_{1,1}$, hence $b_{1,1} > 0$ implies that $b'_{1,1} > 0$ or $b''_{1,1} > 0$. Using Theorem 2.6.1 we have

$$\left[(\mu' * \mu'')\widehat{\ }(\pi_{\pm\lambda}) u\right](x) = \int_{\mathbb{R}} \widetilde{K}_{\pm\lambda}(x,y) u(y) \, \mathsf{d}y, \quad u \in L^2(\mathbb{R}), \ x \in \mathbb{R}, \qquad (2.6.3)$$

where

$$\widetilde{K}_{\pm\lambda}(x,y) = C \exp\left\{-\frac{1}{2} \mathbf{z}^\top V \mathbf{z}\right\}, \quad \mathbf{z} = (x,y,1)^\top.$$

Using (2.6.2) and (2.6.3), we have

$$0 = \int_{\mathbb{R}} \big(K_{\pm\lambda}(x,y) - \widetilde{K}_{\pm\lambda}(x,y)\big) u(y) \, \mathsf{d}y, \quad u \in L^2(\mathbb{R}), \ x \in \mathbb{R}.$$

We show that if

$$\int_{\mathbb{R}} |K_{\pm\lambda}(x,y)|^2 \, \mathsf{d}y < \infty, \quad \int_{\mathbb{R}} |\widetilde{K}_{\pm\lambda}(x,y)|^2 \, \mathsf{d}y < \infty, \quad x \in \mathbb{R}, \qquad (2.6.4)$$

then $K_{\pm\lambda}(x,y) = \widetilde{K}_{\pm\lambda}(x,y)$, $x,y \in \mathbb{R}$. Indeed, for all $x \in \mathbb{R}$, the function $y \in \mathbb{R} \mapsto K_{\pm\lambda}(x,y) - \widetilde{K}_{\pm\lambda}(x,y)$ is in $L^2(\mathbb{R})$. Hence

$$0 = \int_{\mathbb{R}} |K_{\pm\lambda}(x,y) - \widetilde{K}_{\pm\lambda}(x,y)|^2 \,\mathsf{d}y, \quad x \in \mathbb{R}.$$

Then we get

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |K_{\pm\lambda}(x,y) - \widetilde{K}_{\pm\lambda}(x,y)|^2 \,\mathsf{d}x\,\mathsf{d}y = 0,$$

which implies that $K_{\pm\lambda}(x,y) = \widetilde{K}_{\pm\lambda}(x,y)$ for almost every $x,y \in \mathbb{R}$. Using that $K_{\pm\lambda}$ and $\widetilde{K}_{\pm\lambda}$ are continuous, we get $K_{\pm\lambda}(x,y) = \widetilde{K}_{\pm\lambda}(x,y)$, $x,y \in \mathbb{R}$. Now we check that (2.6.4) is satisfied. Using the forms of $K_{\pm\lambda}$ and $\widetilde{K}_{\pm\lambda}$, it is enough to check that

$$\int_{\mathbb{R}} \exp\left\{-\mathbf{z}^\top \mathsf{Re}\,(D_{\pm\lambda}(a,B))\mathbf{z}\right\} \,\mathsf{d}y < \infty, \quad x \in \mathbb{R}, \tag{2.6.5}$$

$$\int_{\mathbb{R}} \exp\left\{-\mathbf{z}^\top \mathsf{Re}\,(V)\mathbf{z}\right\} \,\mathsf{d}y < \infty, \quad x \in \mathbb{R}, \tag{2.6.6}$$

where $\mathbf{z} = (x,y,1)^\top$. Here $\mathsf{Re}\,(D_{\pm\lambda}(a,B))$ and $\mathsf{Re}\,(V)$ are real, symmetric matrices. Let us consider an arbitrary real, symmetric matrix $M = (m_{i,j})_{1\leqslant i,j\leqslant 3}$ with $m_{2,2} > 0$. Then

$$\begin{aligned}
\mathbf{z}^\top M\mathbf{z} &= m_{1,1}x^2 + 2m_{1,2}xy + m_{2,2}y^2 + 2m_{1,3}x + 2m_{2,3}y + m_{3,3} \\
&= \left(\sqrt{m_{2,2}}\,y + \frac{1}{\sqrt{m_{2,2}}}(m_{1,2}x + m_{2,3})\right)^2 - \frac{1}{m_{2,2}}(m_{1,2}x + m_{2,3})^2 \\
&\quad + m_{1,1}x^2 + 2m_{1,3}x + m_{3,3}.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{\mathbb{R}} \exp\left\{-\mathbf{z}^\top M\mathbf{z}\right\}\, \mathsf{d}y &= \exp\left\{\frac{1}{m_{2,2}}(m_{1,2}x+m_{2,3})^2 - m_{1,1}x^2 - 2m_{1,3}x - m_{3,3}\right\} \\
&\qquad \times \int_{\mathbb{R}} \exp\left\{-\left(\sqrt{m_{2,2}}\,y + \frac{1}{\sqrt{m_{2,2}}}(m_{1,2}x+m_{2,3})\right)^2\right\}\, \mathsf{d}y \\
&= \exp\left\{\frac{1}{m_{2,2}}(m_{1,2}x+m_{2,3})^2 - m_{1,1}x^2 - 2m_{1,3}x - m_{3,3}\right\} \\
&\qquad \times \frac{1}{\sqrt{2m_{2,2}}} \int_{\mathbb{R}} \exp\left\{-\frac{t^2}{2}\right\}\, \mathsf{d}t \\
&= \sqrt{\frac{\pi}{m_{2,2}}} \exp\left\{\frac{1}{m_{2,2}}(m_{1,2}x+m_{2,3})^2 - m_{1,1}x^2 - 2m_{1,3}x - m_{3,3}\right\},
\end{aligned}$$

which yields that

$$\int_{\mathbb{R}} \exp\left\{-\mathbf{z}^\top M\mathbf{z}\right\}\, \mathsf{d}y < \infty, \quad x \in \mathbb{R}.$$

Hence in order to prove that (2.6.5) and (2.6.5) are valid we only have to check that the $(2,2)$-entries of the matrices $\mathsf{Re}\,(D_{\pm\lambda}(a,B))$ and $\mathsf{Re}\,(V)$ are positive. For example, if $b'_{1,1} > 0$ and $b''_{1,1} > 0$, then

$$\big(\mathsf{Re}\,(V)\big)_{2,2} = \mathsf{Re}\,(d''_{2,2}) - \mathsf{Re}\left(\frac{(d''_{2,1})^2}{d'_{2,2} + d''_{1,1}}\right).$$

If $b'_{1,1}b'_{2,2} - (b'_{1,2})^2 = b''_{1,1}b''_{2,2} - (b''_{1,2})^2 = 0$, then

$$\big(\mathsf{Re}\,(V)\big)_{2,2} = \frac{1}{\lambda b''_{1,1}} - \frac{1}{\lambda^2 (b''_{1,1})^2} \frac{\frac{1}{\lambda b'_{1,1}} + \frac{1}{\lambda b''_{1,1}}}{\left(\frac{1}{\lambda b'_{1,1}} + \frac{1}{\lambda b''_{1,1}}\right)^2 + \left(\frac{b''_{1,2}}{b''_{1,1}} - \frac{b'_{1,2}}{b'_{1,1}}\right)^2}.$$

Hence $\big(\mathsf{Re}\,(V)\big)_{2,2} > 0$ if and only if

$$\lambda b''_{1,1}\left[\left(\frac{1}{\lambda b'_{1,1}} + \frac{1}{\lambda b''_{1,1}}\right)^2 + \left(\frac{b''_{1,2}}{b''_{1,1}} - \frac{b'_{1,2}}{b'_{1,1}}\right)^2\right] > \frac{1}{\lambda b'_{1,1}} + \frac{1}{\lambda b''_{1,1}}.$$

A simple calculation shows that the latter inequality is equivalent to

$$\frac{b''_{1,1}}{b'_{1,1}}\left(\frac{1}{\lambda b'_{1,1}} + \frac{1}{\lambda b''_{1,1}}\right) + \lambda b''_{1,1}\left(\frac{b''_{1,2}}{b''_{1,1}} - \frac{b'_{1,2}}{b'_{1,1}}\right)^2 > 0,$$

which holds since $b'_{1,1} > 0$, $b''_{1,1} > 0$ and $\lambda > 0$. The other cases can be handled similarly. Hence (2.6.5) and (2.6.6) are satisfied, and then $K_{\pm\lambda}(x,y) = \widetilde{K}_{\pm\lambda}(x,y)$, $x, y \in \mathbb{R}$.

Using the forms of $K_{\pm\lambda}$ and $\widetilde{K}_{\pm\lambda}$, we get

$$C_{\pm\lambda}(B) \exp\left\{-\frac{1}{2}\mathbf{z}^\top D_{\pm\lambda}(a,B)\mathbf{z}\right\} = C \exp\left\{-\frac{1}{2}\mathbf{z}^\top V\mathbf{z}\right\}, \quad \mathbf{z} = (x,y,1)^\top.$$

Putting $\mathbf{z} = (0,0,1)^\top$ gives

$$C_{\pm\lambda}(B) \exp\left\{-\frac{1}{2}d^{\pm\lambda}_{3,3}\right\} = C \exp\left\{-\frac{1}{2}v^{\pm\lambda}_{3,3}\right\}. \tag{2.6.7}$$

Substituting $\mathbf{z} = (1,0,1)^\top$ implies

$$C_{\pm\lambda}(B) \exp\left\{-\frac{1}{2}\big(d^{\pm\lambda}_{1,1} + 2d^{\pm\lambda}_{1,3} + d^{\pm\lambda}_{3,3}\big)\right\} = C \exp\left\{-\frac{1}{2}\big(v^{\pm\lambda}_{1,1} + 2v^{\pm\lambda}_{1,3} + v^{\pm\lambda}_{3,3}\big)\right\}.$$

Using (2.6.7) we have

$$d^{\pm\lambda}_{1,1} + 2d^{\pm\lambda}_{1,3} = v^{\pm\lambda}_{1,1} + 2v^{\pm\lambda}_{1,3}. \tag{2.6.8}$$

With $\mathbf{z} = (0,1,1)^\top$ a similar argument shows that

$$d^{\pm\lambda}_{2,2} + 2d^{\pm\lambda}_{2,3} = v^{\pm\lambda}_{2,2} + 2v^{\pm\lambda}_{2,3}. \tag{2.6.9}$$

Putting $\mathbf{z} = (1,1,1)^\top$ and using (2.6.7) we obtain

$$\begin{aligned} &d^{\pm\lambda}_{1,1} + 2d^{\pm\lambda}_{1,2} + 2d^{\pm\lambda}_{1,3} + d^{\pm\lambda}_{2,2} + 2d^{\pm\lambda}_{2,3} \\ &\qquad = v^{\pm\lambda}_{1,1} + 2v^{\pm\lambda}_{1,2} + 2v^{\pm\lambda}_{1,3} + v^{\pm\lambda}_{2,2} + 2v^{\pm\lambda}_{2,3}. \end{aligned} \tag{2.6.10}$$

Using (2.6.8),(2.6.9) and (2.6.10), we have $d^{\pm\lambda}_{1,2} = v^{\pm\lambda}_{1,2}$. If $\mathbf{z} = (2,0,1)^\top$ then using (2.6.7) we have

$$d^{\pm\lambda}_{1,1} + d^{\pm\lambda}_{1,3} = v^{\pm\lambda}_{1,1} + v^{\pm\lambda}_{1,3}.$$

Using (2.6.8) we have $d^{\pm\lambda}_{1,3} = v^{\pm\lambda}_{1,3}$. If $\mathbf{z} = (0,2,1)^\top$ then

$$d^{\pm\lambda}_{2,2} + d^{\pm\lambda}_{2,3} = v^{\pm\lambda}_{2,2} + v^{\pm\lambda}_{2,3}.$$

Using (2.6.9) we have $d^{\pm\lambda}_{2,3} = v^{\pm\lambda}_{2,3}$. □

Using Lemma 2.6.4 we derive necessary conditions for a convolution of two Gauss measures to be a Gauss measure and then prove that these conditions are also sufficient. The above train of thoughts will be used in the proof of Proposition 2.6.6 and Theorem 2.6.7.

2.6.5 Remark. By Lemma 2.4.3, it can be easily checked that a Gauss measure μ admits parameters (a, B) with $b_{j,k} = 0$ for $1 \leqslant j, k \leqslant 3$ with $(j, k) \neq (3, 3)$ and $a_1 = a_2 = 0$ if and only if the support of μ is contained in the center of $\mathbb{H}$.

Now we can derive a special case of Theorem 2.6.7 which will be used in the proof of Theorem 2.6.7.

2.6.6 Proposition. *If μ'' is a Gauss measure on $\mathbb{H}$ with parameters (a'', B'') such that the support of μ'' is contained in the center of $\mathbb{H}$ then for all Gauss measures μ' on $\mathbb{H}$ with parameters (a', B'), the convolutions $\mu' * \mu''$ and $\mu'' * \mu'$ are Gauss measures with parameters $(a' + a'', B' + B'')$, and $\mu' * \mu'' = \mu'' * \mu'$.*

Proof. Let μ be a Gauss measure with parameters $(a' + a'', B' + B'')$. By the injectivity of the Fourier transform, in order to prove that $\mu' * \mu'' = \mu$ is valid, it is sufficient to show that $(\mu' * \mu'')\widehat{\ }(\chi_{\alpha,\beta}) = \widehat{\mu}(\chi_{\alpha,\beta})$ for all $\alpha, \beta > 0$ and $(\mu' * \mu'')\widehat{\ }(\pi_{\pm\lambda}) = \widehat{\mu}(\pi_{\pm\lambda})$ for all $\lambda > 0$. Theorem 2.6.1 implies that $(\mu' * \mu'')\widehat{\ }(\chi_{\alpha,\beta}) = \widehat{\mu}(\chi_{\alpha,\beta})$ is valid for all one-dimensional representations $\chi_{\alpha,\beta}$, $\alpha, \beta \in \mathbb{R}$. Suppose that $b'_{1,1} \neq 0$ and $b'_{1,1}b'_{2,2} - (b'_{1,2})^2 \neq 0$. By Theorem 2.6.1, to prove $(\mu' * \mu'')\widehat{\ }(\pi_{\pm\lambda}) = \widehat{\mu}(\pi_{\pm\lambda})$ for all $\lambda > 0$ it is sufficient to show that

$$D_{\pm\lambda}(a', B') + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda^2 b''_{3,3} \mp 2i\lambda a''_3 \end{bmatrix} = D_{\pm\lambda}(a' + a'', B' + B'')$$

for all $\lambda > 0$. Since $b''_{j,k} = 0$ for $1 \leqslant j, k \leqslant 3$ with $(j, k) \neq (3, 3)$, we have $d^{\pm\lambda}_{j,k}(a' + a'', B' + B'') = d^{\pm\lambda}_{j,k}(a', B')$ for $1 \leqslant j, k \leqslant 3$ with $(j, k) \neq (3, 3)$. So we have to check only that

$$d^{\pm\lambda}_{3,3}(a', B') + \lambda^2 b''_{3,3} \mp 2i\lambda a''_3 = d^{\pm\lambda}_{3,3}(a' + a'', B' + B'')$$

for all $\lambda > 0$. Theorem 2.3.1 implies this. The case $b'_{1,1} \neq 0$, $b'_{1,1}b'_{2,2} - (b'_{1,2})^2 = 0$ can be proved similarly. Suppose that $b'_{1,1} = b''_{1,1} = 0$. Using again Theorem

2.3.1, we have

$$\Big[\widehat{\mu''}(\pi_{\pm\lambda})u\Big](x) = \exp\bigg\{\pm i\lambda a_3'' - \frac{\lambda^2}{2} b_{3,3}''\bigg\} u(x),$$

$$\Big[\widehat{\mu'}(\pi_{\pm\lambda})u\Big](x)$$
$$= \exp\bigg\{\pm \frac{i\sqrt{\lambda}}{2}\big(\sqrt{\lambda}(2a_3' + a_1'a_2') + 2a_2'x\big) - \frac{\lambda^2}{6}\big(3b_{3,3}' + 3a_1'b_{2,3}' + (a_1')^2 b_{2,2}'\big)$$
$$- \frac{\lambda^{3/2}}{2}\big(2b_{2,3}' + a_1'b_{2,2}'\big)x - \frac{\lambda}{2} b_{2,2}' x^2\bigg\} u(x + \sqrt{\lambda} a_1').$$

Theorem 2.3.1 implies that $[\widehat{\mu}(\pi_{\pm\lambda})u](x) = \Big[(\mu' * \mu'')\widehat{\ }(\pi_{\pm\lambda})u\Big](x)$ for all $\lambda > 0$, $u \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$. Hence the assertion. $\square$

Now we give necessary and sufficient conditions under which the convolution of two Gauss measures is a Gauss measure.

2.6.7 Theorem. *Let μ' and μ'' be Gauss measures on $\mathbb{H}$ with parameters $a' = (a_i')_{1\leqslant i\leqslant 3}$, $B' = (b_{j,k}')_{1\leqslant j,k\leqslant 3}$ and $a'' = (a_i'')_{1\leqslant i\leqslant 3}$, $B'' = (b_{j,k}'')_{1\leqslant j,k\leqslant 3}$, respectively. Then the convolution $\mu' * \mu''$ is a Gauss measure on $\mathbb{H}$ if and only if one of the following conditions holds:*

($\widetilde{\mathrm{C}}1$) $b_{1,1}' > 0$, $\delta' > 0$, $b_{1,1}'' > 0$, $\delta'' > 0$, *and there exists* $\varrho > 0$ *such that* $b_{j,k}'' = \varrho b_{j,k}'$ *for* $1 \leqslant j,k \leqslant 3$ *with* $(j,k) \neq (3,3)$ *and* $a_i'' = \varrho a_i'$ *for* $i = 1,2$,

($\widetilde{\mathrm{C}}2$) $b_{1,1}' > 0$, $\delta' = 0$, $b_{1,1}'' > 0$, $\delta'' = 0$, *and there exists* $\varrho > 0$ *such that* $b_{j,k}'' = \varrho b_{j,k}'$ *for* $1 \leqslant j,k \leqslant 2$,

($\widetilde{\mathrm{C}}3$) $b_{1,1}' > 0$, $\delta' > 0$, $b_{j,k}'' = 0$ *for* $1 \leqslant j,k \leqslant 3$ *with* $(j,k) \neq (3,3)$ *and* $a_i'' = 0$ *for* $i = 1,2$,

($\widetilde{\mathrm{C}}4$) $b_{1,1}' > 0$, $\delta' = 0$, $b_{j,k}'' = 0$ *for* $1 \leqslant j,k \leqslant 3$ *with* $(j,k) \neq (3,3)$,

($\widetilde{\mathrm{C}}5$) $b_{1,1}'' > 0$, $\delta'' > 0$, $b_{j,k}' = 0$ *for* $1 \leqslant j,k \leqslant 3$ *with* $(j,k) \neq (3,3)$ *and* $a_i' = 0$ *for* $i = 1,2$,

($\widetilde{\mathrm{C}}6$) $b_{1,1}'' > 0$, $\delta'' = 0$, $b_{j,k}' = 0$ *for* $1 \leqslant j,k \leqslant 3$ *with* $(j,k) \neq (3,3)$,

($\widetilde{\mathrm{C}}7$) $b_{1,1}' = 0$ *and* $b_{1,1}'' = 0$, *where* $\delta' := \sqrt{b'_{1,1}b'_{2,2} - (b'_{1,2})^2}$ *and* $\delta'' := \sqrt{b''_{1,1}b''_{2,2} - (b''_{1,2})^2}$. *In cases* $(\widetilde{C}1)$, $(\widetilde{C}3)$, $(\widetilde{C}5)$ *the parameters of the convolution* $\mu' * \mu''$ *are* $(a' + a'', B' + B'')$, *but in the other cases it does not hold necessarily (compare with Lemma 2.6.3).*

Proof. First we show necessity, i.e., if $\mu' * \mu''$ is a Gauss measure then one of the conditions $(\widetilde{\mathrm{C}}1) - (\widetilde{\mathrm{C}}7)$ holds. Let us denote the parameters of the convolution $\mu' * \mu''$ by (a, B) and we write $d_{j,k} := d^{\pm\lambda}_{j,k}(a, B)$, $d'_{j,k} := d^{\pm\lambda}_{j,k}(a', B')$ and $d''_{j,k} := d^{\pm\lambda}_{j,k}(a'', B'')$ for $1 \leqslant j, k \leqslant 3$ as in Theorem 2.6.1. If $b'_{1,1} > 0$ and $b''_{1,1} > 0$, we can easily prove that

$$\frac{b_{1,2}}{b_{1,1}} = \frac{b'_{1,2}}{b'_{1,1}} = \frac{b''_{1,2}}{b''_{1,1}}, \qquad \frac{b_{2,2}}{b_{1,1}} = \frac{b'_{2,2}}{b'_{1,1}} = \frac{b''_{2,2}}{b''_{1,1}},$$

and $d'_{2,2} + d''_{1,1} \in \mathbb{R}$ as in Pap [45, Theorem 7.3]. This implies that there exists $\varrho > 0$ such that $b''_{j,k} = \varrho b'_{j,k}$ for $1 \leqslant j, k \leqslant 2$, i.e., $(\widetilde{\mathrm{C}}2)$ holds.

When $b'_{1,1} > 0$, $\delta' > 0$ and $b''_{1,1} > 0$, $\delta'' > 0$, we show that $(\widetilde{\mathrm{C}}1)$ holds. To derive this it is sufficient to show that $b''_{1,3} = \varrho b'_{1,3}$, $b''_{2,3} = \varrho b'_{2,3}$, $a''_1 = \varrho a'_1$ and $a''_2 = \varrho a'_2$. Using Theorem 2.6.1 we obtain

(i) $(d'_{2,2} + d''_{1,1})(\operatorname{Re} d'_{1,3} - \operatorname{Re} d_{1,3}) = d'_{1,2}(\operatorname{Re} d''_{1,3} + \operatorname{Re} d'_{2,3})$,

(ii) $(d'_{2,2} + d''_{1,1})(\operatorname{Re} d''_{2,3} - \operatorname{Re} d_{2,3}) = d''_{1,2}(\operatorname{Re} d''_{1,3} + \operatorname{Re} d'_{2,3})$,

(iii) $(d'_{2,2} + d''_{1,1})(\operatorname{Im} d'_{1,3} - \operatorname{Im} d_{1,3}) = d'_{1,2}(\operatorname{Im} d''_{1,3} + \operatorname{Im} d'_{2,3})$,

(iv) $(d'_{2,2} + d''_{1,1})(\operatorname{Im} d'_{2,3} - \operatorname{Im} d_{2,3}) = d''_{1,2}(\operatorname{Im} d''_{1,3} + \operatorname{Im} d'_{2,3})$.

Let us denote $\delta'_1 := b'_{1,1}b'_{2,3} - b'_{1,2}b'_{1,3}$, $\delta''_1 := b''_{1,1}b''_{2,3} - b''_{1,2}b''_{1,3}$, $\delta'_2 := a'_1 b'_{1,2} - a'_2 b'_{1,1}$, $\delta''_2 := a''_1 b''_{1,2} - a''_2 b''_{1,1}$. Summing up (iii) and (iv) we have

$$(d'_{2,2} + d''_{1,1})(\operatorname{Im} d'_{1,3} + \operatorname{Im} d''_{2,3} - \operatorname{Im} d_{1,3} - \operatorname{Im} d_{2,3}) = (d'_{1,2} + d''_{1,2})(\operatorname{Im} d''_{1,3} + \operatorname{Im} d'_{2,3}).$$

Using the definition of $d_{j,k}$, $d'_{j,k}$, $d''_{j,k}$ $(1 \leqslant j,k \leqslant 3)$ we get

$$\begin{aligned}\big(\coth(\lambda\delta') + \coth(\lambda\delta'')\big)\bigg(\frac{b'_{1,3}}{b'_{1,1}} + \frac{\delta'_2}{\lambda b'_{1,1}\delta'\coth(\lambda\delta'/2)} - \frac{b''_{1,3}}{b''_{1,1}} + \frac{\delta''_2}{\lambda b''_{1,1}\delta''\coth(\lambda\delta''/2)}\\ - \frac{2\delta_2}{\lambda b_{1,1}\delta\coth(\lambda\delta/2)}\bigg)\\ = -\bigg(\frac{1}{\sinh(\lambda\delta')} + \frac{1}{\sinh(\lambda\delta'')}\bigg)\bigg(\frac{b''_{1,3}}{b''_{1,1}} + \frac{\delta''_2}{\lambda b''_{1,1}\delta''\coth(\lambda\delta''/2)} - \frac{b'_{1,3}}{b'_{1,1}}\\ + \frac{\delta'_2}{\lambda b'_{1,1}\delta'\coth(\lambda\delta'/2)}\bigg).\end{aligned}$$

An easy calculation shows that

$$\begin{aligned}&\bigg(\frac{b'_{1,3}}{b'_{1,1}} - \frac{b''_{1,3}}{b''_{1,1}}\bigg)\lambda\sinh(\lambda\delta'/2)\sinh(\lambda\delta''/2)\\ &= \bigg(\frac{1}{\delta'+\delta''}\Big(a_1\frac{b_{1,2}}{b_{1,1}} - a_2\Big) - \frac{1}{\delta'}\Big(a'_1\frac{b'_{1,2}}{b'_{1,1}} - a'_2\Big)\bigg)\sinh(\lambda\delta'/2)\cosh(\lambda\delta''/2)\\ &\quad + \bigg(\frac{1}{\delta'+\delta''}\Big(a_1\frac{b_{1,2}}{b_{1,1}} - a_2\Big) - \frac{1}{\delta''}\Big(a''_1\frac{b''_{1,2}}{b''_{1,1}} - a''_2\Big)\bigg)\cosh(\lambda\delta'/2)\sinh(\lambda\delta''/2)\end{aligned}$$

for all $\lambda > 0$. We show that the functions

$$\lambda\sinh(\lambda\delta'/2)\sinh(\lambda\delta''/2), \quad \sinh(\lambda\delta'/2)\cosh(\lambda\delta''/2), \quad \cosh(\lambda\delta'/2)\sinh(\lambda\delta''/2),$$

$(\lambda > 0)$ are linearly independent. We have

$$\begin{aligned}&\lambda\sinh(\lambda\delta'/2)\sinh(\lambda\delta''/2)\\ &\qquad = \frac{\lambda}{4}\big(\mathsf{e}^{\lambda(\delta'+\delta'')/2} - \mathsf{e}^{\lambda(\delta''-\delta')/2} - \mathsf{e}^{\lambda(\delta'-\delta'')/2} + \mathsf{e}^{-\lambda(\delta'+\delta'')/2}\big),\\ &\sinh(\lambda\delta'/2)\cosh(\lambda\delta''/2)\\ &\qquad = \frac{1}{4}\big(\mathsf{e}^{\lambda(\delta'+\delta'')/2} + \mathsf{e}^{\lambda(\delta'-\delta'')/2} - \mathsf{e}^{\lambda(\delta''-\delta')/2} - \mathsf{e}^{-\lambda(\delta'+\delta'')/2}\big),\\ &\cosh(\lambda\delta'/2)\sinh(\lambda\delta''/2)\\ &\qquad = \frac{1}{4}\big(\mathsf{e}^{\lambda(\delta'+\delta'')/2} - \mathsf{e}^{\lambda(\delta'-\delta'')/2} + \mathsf{e}^{\lambda(\delta''-\delta')/2} - \mathsf{e}^{-\lambda(\delta'+\delta'')/2}\big).\end{aligned}$$

The linear independence of these functions follows from the following fact: if $c_1, \dots, c_n$ are pairwise different complex numbers and $Q_1, \dots, Q_n$ are complex polynomials such that $\sum_{j=1}^{n} Q_j(\lambda)e^{c_j\lambda} = 0$ for all $\lambda > 0$ then $Q_1 = \cdots = Q_n = 0$. Hence we get

$$\begin{aligned}
&\frac{b'_{1,3}}{b'_{1,1}} - \frac{b''_{1,3}}{b''_{1,1}} = 0, \\
&\frac{1}{\delta' + \delta''}\left(a_1\frac{b_{1,2}}{b_{1,1}} - a_2\right) = \frac{1}{\delta'}\left(a'_1\frac{b'_{1,2}}{b'_{1,1}} - a'_2\right) = \frac{1}{\delta''}\left(a''_1\frac{b''_{1,2}}{b''_{1,1}} - a''_2\right).
\end{aligned} \tag{2.6.11}$$

Subtracting the equation (i) from (ii) we get

$$(d'_{2,2} + d''_{1,1})(\operatorname{Re} d'_{1,3} - \operatorname{Re} d''_{2,3} - \operatorname{Re} d_{1,3} + \operatorname{Re} d_{2,3}) = (d'_{1,2} - d''_{1,2})(\operatorname{Re} d''_{1,3} + \operatorname{Re} d'_{2,3}).$$

Using again the definition of $d_{j,k},\ d'_{j,k},\ d''_{j,k}$ $(1 \leqslant j,k \leqslant 3)$ we obtain

$$\begin{aligned}
&\big(\coth(\lambda\delta') + \coth(\lambda\delta'')\big)\Bigg(\frac{a'_1}{\sqrt{\lambda}b'_{1,1}} + \frac{a''_1}{\sqrt{\lambda}b''_{1,1}} - \frac{2a_1}{\sqrt{\lambda}b_{1,1}} \\
&\qquad\qquad + \frac{\sqrt{\lambda}\delta'_1}{b'_{1,1}\delta'\coth(\lambda\delta'/2)} - \frac{\sqrt{\lambda}\delta''_1}{b''_{1,1}\delta''\coth(\lambda\delta''/2)}\Bigg) \\
&= \left(\frac{1}{\sinh(\lambda\delta'')} - \frac{1}{\sinh(\lambda\delta')}\right)\Bigg(\frac{a''_1}{\sqrt{\lambda}b''_{1,1}} - \frac{a'_1}{\sqrt{\lambda}b'_{1,1}} + \frac{\sqrt{\lambda}\delta'_1}{b'_{1,1}\delta'\coth(\lambda\delta'/2)} \\
&\qquad\qquad + \frac{\sqrt{\lambda}\delta''_1}{b''_{1,1}\delta''\coth(\lambda\delta''/2)}\Bigg).
\end{aligned}$$

A simple calculation shows that

$$\begin{aligned}
&\lambda\big(1 + \tanh(\lambda\delta'/2)\tanh(\lambda\delta''/2)\big)\left(\frac{\delta'_1}{\delta' b'_{1,1}} - \frac{\delta''_1}{\delta'' b''_{1,1}}\right) \\
&= \big(\coth(\lambda\delta') + \coth(\lambda\delta'')\big)\left(2\frac{a_1}{b_{1,1}} - \frac{a'_1}{b'_{1,1}} - \frac{a''_1}{b''_{1,1}}\right) \\
&\qquad\qquad + \left(\frac{1}{\sinh(\lambda\delta')} - \frac{1}{\sinh(\lambda\delta'')}\right)\left(\frac{a'_1}{b'_{1,1}} - \frac{a''_1}{b''_{1,1}}\right).
\end{aligned}$$

It can be easily checked that the functions $\lambda\big(1 + \tanh(\lambda\delta'/2)\tanh(\lambda\delta''/2)\big)$, $\coth(\lambda\delta') + \coth(\lambda\delta'')$ and $(\sinh(\lambda\delta'))^{-1} - (\sinh(\lambda\delta''))^{-1}$ $(\lambda > 0)$ are linearly independent. Hence we have

$$\frac{a'_1}{b'_{1,1}} - \frac{a''_1}{b''_{1,1}} = 0, \qquad 2\frac{a_1}{b_{1,1}} - \frac{a'_1}{b'_{1,1}} - \frac{a''_1}{b''_{1,1}} = 0, \qquad \frac{\delta'_1}{\delta' b'_{1,1}} = \frac{\delta''_1}{\delta'' b''_{1,1}}. \tag{2.6.12}$$

Taking into account (2.6.11) and (2.6.12), we conclude that $(\widetilde{\mathrm{C}}1)$ holds. Using Lemma 2.6.3 it turns out that in this case $a = a' + a''$ and $B = B' + B''$.

If $b'_{1,1} > 0$, $\delta' > 0$ and $b''_{1,1} > 0$, $\delta'' = 0$ we show that $\mu' * \mu''$ can not be a Gauss measure. Our proof goes along the lines of the proof of Theorem 7.3 in Pap [45]. Since the proof given in Pap [45] contains a mistake we write down the details. Suppose that, on the contrary, $\mu' * \mu''$ is a Gauss measure on $\mathbb{H}$ with parameters (a, B). By Lemma 2.6.3, we have $b_{1,1} = b'_{1,1} + b''_{1,1}$, hence $b_{1,1} > 0$. By Theorem 2.3.1, we have $\widehat{(\mu' * \mu'')}(\pi_{\pm\lambda})$ is an integral operator. Using Theorem 2.6.1 we obtain

$$d_{1,1} = d'_{1,1} - \frac{(d'_{1,2})^2}{d'_{2,2} + d''_{1,1}}, \tag{2.6.13}$$

$$d_{2,2} = d''_{2,2} - \frac{(d''_{1,2})^2}{d'_{2,2} + d''_{1,1}}. \tag{2.6.14}$$

We show that $d'_{2,2} + d''_{1,1} \in \mathbb{R}$ and $\frac{b'_{1,2}}{b'_{1,1}} = \frac{b''_{1,2}}{b''_{1,1}}$. (The derivations of these two facts are not correct in the proof of Theorem 7.3 in Pap [45].) By Theorem 2.3.1, we have

$$\mathsf{Im}\,(d'_{2,2} + d''_{1,1}) = \mp \left(\frac{b'_{1,2}}{b'_{1,1}} - \frac{b''_{1,2}}{b''_{1,1}} \right) = -\mathsf{Im}\,(d'_{1,1} + d''_{2,2}).$$

Using that $\mathsf{Im}\,(d_{1,1} + d_{2,2}) = 0$, by (2.6.13) and (2.6.14) we get

$$\begin{aligned} 0 &= \pm \left(\frac{b'_{1,2}}{b'_{1,1}} - \frac{b''_{1,2}}{b''_{1,1}} \right) - \mathsf{Im} \left(\frac{(d'_{1,2})^2 + (d''_{1,2})^2}{d'_{2,2} + d''_{1,1}} \right) \\ &= \pm \left(\frac{b'_{1,2}}{b'_{1,1}} - \frac{b''_{1,2}}{b''_{1,1}} \right) \mp \frac{(d'_{1,2})^2 + (d''_{1,2})^2}{|d'_{2,2} + d''_{1,1}|^2} \left(\frac{b'_{1,2}}{b'_{1,1}} - \frac{b''_{1,2}}{b''_{1,1}} \right). \end{aligned}$$

Hence

$$\Big(|d'_{2,2} + d''_{1,1}|^2 - (d'_{1,2})^2 - (d''_{1,2})^2 \Big) \left(\frac{b'_{1,2}}{b'_{1,1}} - \frac{b''_{1,2}}{b''_{1,1}} \right) = 0.$$

Then

$$
\begin{aligned}
|d'_{2,2} + d''_{1,1}|^2 - (d'_{1,2})^2 - (d''_{1,2})^2 &= \left| \frac{\delta' \coth(\lambda\delta') \mp i b'_{1,2}}{b'_{1,1}} + \frac{\lambda^{-1} \pm i b''_{1,2}}{b''_{1,1}} \right|^2 \\
&\quad - \frac{(\delta')^2}{(b'_{1,1})^2 \sinh^2(\lambda\delta')} - \frac{1}{\lambda^2 (b''_{1,1})^2} \\
&= \frac{(\delta')^2}{(b'_{1,1})^2} + \frac{2\delta' \coth(\lambda\delta')}{\lambda b'_{1,1} b''_{1,1}} + \left(\frac{b'_{1,2}}{b'_{1,1}} - \frac{b''_{1,2}}{b''_{1,1}} \right)^2 > 0.
\end{aligned}
$$

This yields $\frac{b'_{1,2}}{b'_{1,1}} = \frac{b''_{1,2}}{b''_{1,1}}$. Particularly, $d'_{2,2} + d''_{1,1} \in \mathbb{R}$. Rewrite (2.6.13) and (2.6.14) in the form

$$
\begin{aligned}
(d'_{1,1} - d_{1,1})(d'_{2,2} + d''_{1,1}) &= (d'_{1,2})^2, \\
(d''_{2,2} - d_{2,2})(d'_{2,2} + d''_{1,1}) &= (d''_{1,2})^2.
\end{aligned}
$$

It follows that

$$
(d'_{1,1} - d''_{2,2} - d_{1,1} + d_{2,2})(d'_{2,2} + d''_{1,1}) = (d'_{1,2})^2 - (d''_{1,2})^2.
$$

Using that $d'_{2,2} + d''_{1,1} \in \mathbb{R}$ and $\mathsf{Re}\,(d_{1,1} - d_{2,2}) = 0$, taking real parts we get

$$
(\mathsf{Re}\,(d'_{1,1}) - \mathsf{Re}\,(d''_{2,2}))(d'_{2,2} + d''_{1,1}) = (d'_{1,2})^2 - (d''_{1,2})^2.
$$

Thus

$$
\left(\frac{\delta' \coth(\lambda\delta')}{b'_{1,1}} - \frac{1}{\lambda b''_{1,1}} \right)\left(\frac{\delta' \coth(\lambda\delta')}{b'_{1,1}} + \frac{1}{\lambda b''_{1,1}} \right) = \frac{(\delta')^2}{(b'_{1,1})^2 \sinh^2(\lambda\delta')} - \frac{1}{\lambda^2 (b''_{1,1})^2}.
$$

From this we conclude

$$
\frac{(\delta')^2 \coth^2(\lambda\delta')}{(b'_{1,1})^2} - \frac{1}{\lambda^2 (b''_{1,1})^2} = \frac{(\delta')^2}{(b'_{1,1})^2 \sinh^2(\lambda\delta')} - \frac{1}{\lambda^2 (b''_{1,1})^2},
$$

and it follows that $\cosh(\lambda\delta') = 1$. Hence $\delta' = 0$, which leads to a contradiction.

If $b'_{1,1} > 0$, $\delta' > 0$, and $b''_{1,1} = 0$ we show that $(\widetilde{\mathrm{C}}3)$ holds. The symmetry and positive semi-definiteness of the matrix B'' imply $b''_{1,2} = b''_{1,3} = 0$. Lemma 2.6.3 yields that $b_{1,1} = b'_{1,1} + b''_{1,1} > 0$. Hence Theorem 2.3.1 implies that

$(\mu' * \mu'')\widehat{\ }(\pi_{\pm\lambda})$ is an integral operator and $\mathsf{Im}\,(d_{1,1}+d_{2,2})=0$ holds. By Theorems 2.3.1 and 2.6.1 we obtain $\mathsf{Im}\,(d_{1,1}+d_{2,2})=\mathsf{Im}\,(d'_{1,1}+d'_{2,2}+\lambda b''_{2,2})=\mathsf{Im}\,(\lambda b''_{2,2})$. Thus $b''_{2,2}=0$, which implies that $b''_{2,3}=0$ and $\delta=\delta'>0$. Using again Theorem 2.6.1 we get

$$d_{1,3}=d'_{1,3}-\sqrt{\lambda}a''_1 d'_{1,2}, \tag{2.6.15}$$

$$d_{2,3}=d'_{2,3}-\sqrt{\lambda}a''_1 d'_{2,2}\mp i\sqrt{\lambda}a''_2. \tag{2.6.16}$$

Taking the real part of the difference of equations (2.6.15) and (2.6.16) we have

$$2\left(\frac{a_1}{b_{1,1}}-\frac{a'_1}{b'_{1,1}}\right)=\lambda\delta'\frac{a''_1}{b'_{1,1}}\left(\frac{1+\cosh(\lambda\delta')}{\sinh(\lambda\delta')}\right). \tag{2.6.17}$$

Since (2.6.17) is valid for all $\lambda>0$, we have $a''_1=0$. Taking the imaginary part of (2.6.16) and using the fact that $a''_1=0$ we get

$$a''_2\left(1-\frac{1}{\lambda\delta'\coth(\lambda\delta'/2)}\right)=\frac{b_{1,3}}{b_{1,1}}-\frac{b'_{1,3}}{b'_{1,1}}=0. \tag{2.6.18}$$

Since (2.6.18) is valid for all $\lambda>0$, we get $a''_2=0$, so $(\widetilde{\mathrm{C}}3)$ holds. If $b'_{1,1}>0$, $\delta'=0$ and $b''_{1,1}=0$ a similar argument shows that $(\widetilde{\mathrm{C}}4)$ holds.

The aim of the following discussion is to show the converse. Suppose that $(\widetilde{\mathrm{C}}1)$ holds. We prove that the convolution $\mu' * \mu''$ is a Gauss measure on $\mathbb{H}$ with parameters $(a'+a'',B'+B'')$. By Theorem 2.6.1, the Fourier transform $(\mu' * \mu'')\widehat{\ }(\chi_{\alpha,\beta})$ equals the Fourier transform of a Gauss measure with parameters $(a'+a'',B'+B'')$ at the representation $\chi_{\alpha,\beta}$ for all $\alpha,\beta>0$. Since $b'_{1,1}+b''_{1,1}>0$, the Fourier transform $(\mu' * \mu'')\widehat{\ }(\pi_{\pm\lambda})$ is an integral operator on $L^2(\mathbb{R})$ with kernel function $K_{\pm\lambda}$ given in Theorem 2.6.1 for all $\lambda>0$. It is enough to show that $C=C_{\pm\lambda}(B'+B'')$ and $V=D_{\pm\lambda}(a'+a'',B'+B'')=(d^{\pm\lambda}_{j,k}(a'+a'',B'+B''))_{1\leqslant j,k\leqslant 3}$. We have

$$d'_{2,2}+d''_{1,1}=\frac{\delta'\sinh\big(\lambda(1+\varrho)\delta'\big)}{b'_{1,1}\sinh(\lambda\delta')\sinh(\lambda\varrho\delta')},$$

hence using Theorem 2.6.1 we obtain

$$C=\sqrt{\frac{\delta'}{2\pi b'_{1,1}\sinh(\lambda(1+\varrho)\delta')}}=C_{\pm\lambda}(B'+B'').$$

Let $(\mu_t)_{t\geqslant 0}$ be a Gauss semigroup such that μ_1 is a Gauss measure with parameters (a', B'). By the help of the semigroup property we have $\mu_1 * \mu_\varrho = \mu_{1+\varrho}$. Taking into account that a'_3 and $b'_{3,3}$ appear only in $d^{\pm\lambda}_{3,3}(a', B')$ (see Theorem 2.3.1) and the fact that μ_t is a Gauss measure with parameters (ta', tB') for all $t \geqslant 0$, Theorem 2.3.1 and Theorem 2.6.1 give us

$$v_{j,k} = d^{\pm\lambda}_{j,k}(a' + a'', B' + B'').$$

for $1 \leqslant j, k \leqslant 3$ with $(j,k) \neq (3,3)$. So we have to check only that $v_{3,3} = d^{\pm\lambda}_{3,3}(a' + a'', B' + B'')$. By the help of Theorem 2.6.1 we get

$$v_{3,3} = d'_{3,3} + d''_{3,3} - \frac{1}{d'_{2,2} + d''_{1,1}}(d'_{3,2} + d''_{3,1})^2. \tag{2.6.19}$$

Calculating the real and imaginary part of (2.6.19) one can easily check that $v_{3,3} = d^{\pm\lambda}_{3,3}(a' + a'', B' + B'')$ is valid.

Now suppose that $(\widetilde{\mathrm{C}}2)$ holds. Using the parameters of μ' and μ'', define a vector $a = (a_i)_{1\leqslant i\leqslant 3}$ and a matrix $B = (b_{i,j})_{1\leqslant i,j\leqslant 3}$, as in Lemma 2.6.3. We show that the convolution $\mu := \mu' * \mu''$ is a Gauss measure on $\mathbb{H}$ with parameters (a, B). An easy calculation shows that the Fourier transforms of $\mu' * \mu''$ and μ at the one-dimensional representations coincide. Concerning the Fourier transforms at the Schrödinger representations, as in case of $(\widetilde{\mathrm{C}}1)$, it is enough to show that

$$C_{\pm\lambda}(B) = C_{\pm\lambda}(B')C_{\pm\lambda}(B'')\sqrt{\frac{2\pi}{d'_{2,2} + d''_{1,1}}}$$

and $V = D_{\pm\lambda}(a' + a'', B' + B'')$. Using Theorem 2.3.1 we have

$$\begin{aligned}\frac{1}{\sqrt{2\pi\lambda b'_{1,1}}}\frac{1}{\sqrt{2\pi\lambda b''_{1,1}}}\sqrt{\frac{2\pi}{\frac{1}{\lambda b'_{1,1}} + \frac{1}{\lambda b''_{1,1}} \pm i\left(\frac{b''_{1,2}}{b''_{1,1}} - \frac{b'_{1,2}}{b'_{1,1}}\right)}} &= \frac{1}{\sqrt{2\pi\lambda(b'_{1,1} + b''_{1,1})}} \\ &= \frac{1}{\sqrt{2\pi\lambda b_{1,1}}},\end{aligned}$$

since $b''_{1,2}/b''_{1,1} = b'_{1,2}/b'_{1,1} = \varrho$. Using similar arguments one can also easily check that $V = D_{\pm\lambda}(a' + a'', B' + B'')$ holds. We note that in this case the parameters of $\mu' * \mu''$ is not the sum of the parameters of μ' and μ''.

Suppose that $(\widetilde{\mathrm{C}}3)$ holds. Proposition 2.6.6 gives us that the convolution $\mu' * \mu''$ is a Gauss measure on $\mathbb{H}$ with parameters $(a' + a'', B' + B'')$. In cases

$(\widetilde{\text{C}}4)$, $(\widetilde{\text{C}}5)$, $(\widetilde{\text{C}}6)$, $(\widetilde{\text{C}}7)$ we can argue as in cases $(\widetilde{\text{C}}2)$, $(\widetilde{\text{C}}3)$. Consequently, the proof is complete. □

For the proof of Theorem 2.2.1 we need the following lemma about the support of a Gauss measure on $\mathbb{H}$.

2.6.8 Lemma. *Let μ be a Gauss measure on $\mathbb{H}$ with parameters (a, B) such that $b_{1,1}b_{2,2} - b_{1,2}^2 = 0$. Let $Y_0 \in \mathcal{H}$ be defined as in Section 2.1. If* $\mathsf{rank}\,(B) = 2$ *then* $\mathsf{supp}\,(\mu) = \exp\big(Y_0 + \mathbb{R} \cdot U + \mathbb{R} \cdot X_3\big)$, *where*

$$U := \begin{cases} b_{1,1}X_1 + b_{2,1}X_2 & \text{if } b_{1,1} > 0, \\ b_{2,2}X_2 & \text{if } b_{1,1} = 0 \text{ and } b_{2,2} > 0. \end{cases}$$

If $\mathsf{rank}\,(B) = 1$ *then* $\mathsf{supp}\,(\mu) = \exp\big(Y_0 + \mathbb{R} \cdot U + \mathbb{R} \cdot [Y_0, U]\big)$, *where*

$$U := \begin{cases} b_{1,1}X_1 + b_{2,1}X_2 + b_{3,1}X_3 & \text{if } b_{1,1} > 0, \\ b_{2,2}X_2 + b_{3,2}X_3 & \text{if } b_{1,1} = 0 \text{ and } b_{2,2} > 0, \\ b_{3,3}X_3 & \text{if } b_{1,1} = b_{2,2} = 0 \text{ and } b_{3,3} > 0. \end{cases}$$

If $\mathsf{rank}\,(B) = 0$ *then* $\mathsf{supp}\,(\mu) = \exp(Y_0)$.

Proof. We apply (iii) – (v) of Lemma 2.4.3. If $\mathsf{rank}\,(B) = 2$ then one can check that $\mathcal{L}(Y_1, Y_2) = \mathcal{L}(U, X_3)$. If $\mathsf{rank}\,(B) = 1$ then $\mathcal{L}(Y_1) = \mathcal{L}(U)$. □

Proof of Theorem 2.2.1. First we prove that if one of the conditions (C1) and (C2) holds then one of the conditions $(\widetilde{\text{C}}1)$ – $(\widetilde{\text{C}}7)$ in Theorem 2.6.7 is valid, which implies that the convolution $\mu' * \mu''$ is a Gauss measure on $\mathbb{H}$.

Suppose that (C1) holds. Lemma 2.4.3 implies $\delta' = \delta'' = 0$.

If $b'_{1,1} = b''_{1,1} = 0$ then $(\widetilde{\text{C}}7)$ holds.

If $b'_{1,1} > 0$, $\delta' = 0$ and $b''_{1,1} = 0$, $\delta'' = 0$ we show that $(\widetilde{\text{C}}4)$ holds. It is sufficient to show that $b''_{2,2} = 0$. Suppose that, on the contrary, $b''_{2,2} \neq 0$. When $\mathsf{rank}\,(B') = \mathsf{rank}\,(B'') = 2$, by the help of Lemma 2.6.8, we get

$$\mathsf{supp}\,(\mu') = \exp\big(Y'_0 + \mathbb{R} \cdot U' + \mathbb{R} \cdot X_3\big), \qquad \mathsf{supp}\,(\mu'') = \exp\big(Y''_0 + \mathbb{R} \cdot U'' + \mathbb{R} \cdot X_3\big),$$

where $U' = b'_{1,1}X_1 + b'_{2,1}X_2$ and $U'' = b''_{2,2}X_2$. Since in this case $\mathsf{supp}\,(\mu')$ and $\mathsf{supp}\,(\mu'')$ are contained in "Euclidean cosets" of the same 2-dimensional Abelian subgroup of $\mathbb{H}$, we obtain that $\mathcal{L}(U', X_3) = \mathcal{L}(U'', X_3)$. From this we conclude $b'_{1,1} = 0$, which leads to a contradiction. When $\mathsf{rank}\,(B') = 1$, $\mathsf{rank}\,(B'') = 2$ and in other cases one can argue similarly, so $(\widetilde{\mathrm{C}}4)$ holds.

If $b'_{1,1} = 0$, $\delta' = 0$ and $b''_{1,1} > 0$, $\delta'' = 0$ the same argument shows that $(\widetilde{\mathrm{C}}6)$ holds.

If $b'_{1,1} > 0$, $\delta' = 0$ and $b''_{1,1} > 0$, $\delta'' = 0$ we show that $(\widetilde{\mathrm{C}}2)$ holds. When $\mathsf{rank}\,(B') = \mathsf{rank}\,(B'') = 2$, Lemma 2.6.8 implies that

$$\mathsf{supp}\,(\mu') = \exp\bigl(Y_0' + \mathbb{R}\cdot U' + \mathbb{R}\cdot X_3\bigr), \qquad \mathsf{supp}\,(\mu'') = \exp\bigl(Y_0'' + \mathbb{R}\cdot U'' + \mathbb{R}\cdot X_3\bigr),$$

where $U' = b'_{1,1}X_1 + b'_{2,1}X_2$ and $U'' = b''_{1,1}X_1 + b''_{2,1}X_2$. Condition (C1) yields that $\mathcal{L}(U', X_3) = \mathcal{L}(U'', X_3)$, hence we have $b''_{2,1}b'_{1,1} = b'_{2,1}b''_{1,1}$. Since $\delta' = \delta'' = 0$ we get $b''_{2,2}b'_{1,1} = b'_{2,2}b''_{1,1}$. Thus $(\widetilde{\mathrm{C}}2)$ holds with $\varrho := b''_{1,1}/b'_{1,1}$. When $\mathsf{rank}\,(B') = \mathsf{rank}\,(B'') = 1$, Lemma 2.6.8 implies that

$$\begin{aligned}\mathsf{supp}\,(\mu') &= \exp\bigl(Y_0' + \mathbb{R}\cdot U' + \mathbb{R}\cdot [Y_0', U']\bigr),\\ \mathsf{supp}\,(\mu'') &= \exp\bigl(Y_0'' + \mathbb{R}\cdot U'' + \mathbb{R}\cdot [Y_0'', U'']\bigr),\end{aligned}$$

where $U' = b'_{1,1}X_1 + b'_{2,1}X_2 + b'_{3,1}X_3$ and $U'' = b''_{1,1}X_1 + b''_{2,1}X_2 + b''_{3,1}X_3$. Condition (C1) yields $\mathcal{L}(U', [Y_0', U']) = \mathcal{L}(U'', [Y_0'', U''])$, hence $\mathcal{L}(b'_{1,1}X_1 + b'_{2,1}X_2) = \mathcal{L}(b''_{1,1}X_1 + b''_{2,1}X_2)$. It can be easily checked that $(\widetilde{\mathrm{C}}2)$ holds with $\varrho := b''_{1,1}/b'_{1,1}$. When $\mathsf{rank}\,(B') = 1$, $\mathsf{rank}\,(B'') = 2$ or $\mathsf{rank}\,(B') = 2$, $\mathsf{rank}\,(B'') = 1$ we also have $(\widetilde{\mathrm{C}}2)$ holds.

Suppose that (C2) holds (i.e., $\mu' = \mu_{t'}$, $\mu'' = \mu_{t''} * \nu$ or $\mu' = \mu_{t'} * \nu$, $\mu'' = \mu_{t''}$ with appropriate nonnegative real numbers t', t'' and a Gauss measure ν with support contained in the center of $\mathbb{H}$). Then we have

$$\mu' * \mu'' = \mu_{t'} * \mu_{t''} * \nu = \mu_{t'+t''} * \nu \qquad \text{or} \qquad \mu' * \mu'' = \mu_{t'} * \nu * \mu_{t''} = \mu_{t'+t''} * \nu.$$

Remark 2.6.5 and Proposition 2.6.6 yield that $\mu' * \mu''$ is a Gauss measure on $\mathbb{H}$.

Conversely, suppose that $\mu' * \mu''$ is a Gauss measure on $\mathbb{H}$. Then by Theorem 2.6.7, one of the conditions $(\widetilde{\mathrm{C}}1) - (\widetilde{\mathrm{C}}7)$ holds. We show that then one of the conditions (C1) and (C2) is valid.

Suppose that $(\widetilde{\mathrm{C}}1)$ holds. If $b''_{3,3} - \varrho b'_{3,3} \geqslant 0$ then let $(\alpha'_t)_{t\geqslant 0}$ be a Gauss semigroup such that $\alpha'_1 = \mu'$ and let ν be a Gauss measure on $\mathbb{H}$ with parameters (a_ν, B_ν) such that

$$B_\nu := \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & b''_{3,3} - \varrho b'_{3,3}\end{bmatrix}, \qquad a_\nu := \begin{bmatrix} 0\\ 0\\ a''_3 - \varrho a'_3\end{bmatrix}.$$

Remark 2.6.5 and Proposition 2.6.6 imply that $\mu'' = \alpha'_\varrho * \nu$, hence (C2) holds. If $b''_{3,3} - \varrho b'_{3,3} < 0$ then let $(\alpha''_t)_{t\geqslant 0}$ be a Gauss semigroup such that $\alpha''_1 = \mu''$ and let ν be a Gauss measure on $\mathbb{H}$ with parameters (a_ν, B_ν) such that

$$B_\nu := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b'_{3,3} - \varrho^{-1} b''_{3,3} \end{bmatrix}, \qquad a_\nu := \begin{bmatrix} 0 \\ 0 \\ a'_3 - \varrho^{-1} a'_3 \end{bmatrix}.$$

Remark 2.6.5 and Proposition 2.6.6 imply that $\mu' = \alpha''_{1/\varrho} * \nu$, hence (C2) holds.

Suppose that $(\widetilde{\mathrm{C}}2)$ holds. Lemma 2.6.8 implies that

$$\mathsf{supp}\,(\mu') \subset \exp\big(Y'_0 + \mathbb{R}\cdot U' + \mathbb{R}\cdot X_3\big), \qquad \mathsf{supp}\,(\mu'') \subset \exp\big(Y''_0 + \mathbb{R}\cdot U'' + \mathbb{R}\cdot X_3\big),$$

where $U' = b'_{1,1}X_1 + b'_{2,1}X_2$ and $U'' = b''_{1,1}X_1 + b''_{2,1}X_2$. Condition $(\widetilde{\mathrm{C}}2)$ gives us that $\mathcal{L}(U') = \mathcal{L}(U'')$, hence (C1) holds.

Suppose that $(\widetilde{\mathrm{C}}3)$ holds. Let $(\alpha'_t)_{t\geqslant 0}$ be a Gauss semigroup such that $\alpha'_1 = \mu'$ and let ν be a Gauss measure with parameters (a_ν, B_ν) such that

$$B_\nu := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b''_{3,3} \end{bmatrix}, \qquad a_\nu := \begin{bmatrix} 0 \\ 0 \\ a''_3 \end{bmatrix}.$$

Then we have $\mu'' = \nu = \alpha'_0 * \nu$, so (C2) holds.

Suppose that $(\widetilde{\mathrm{C}}4)$ holds. By the help of Lemma 2.6.8, we have

$$\mathsf{supp}\,(\mu') \subset \exp\big(Y'_0 + \mathbb{R}\cdot U' + \mathbb{R}\cdot X_3\big), \qquad \mathsf{supp}\,(\mu'') \subset \exp\big(Y''_0 + \mathbb{R}\cdot U''\big),$$

where $U' = b'_{1,1}X_1 + b'_{2,1}X_2$ and $U'' = b''_{3,3}X_3$. Hence the support of μ' is contained in $\exp\big(Y'_0 + \mathbb{R}\cdot U' + \mathbb{R}\cdot X_3\big)$ and the support of μ'' is contained in $\exp\big(Y''_0 + \mathbb{R}\cdot U' + \mathbb{R}\cdot X_3\big)$, so (C1) holds. Similar arguments show that when $(\widetilde{\mathrm{C}}5)$ holds then (C2) is valid, and when $(\widetilde{\mathrm{C}}6)$ holds then (C1) is valid.

Suppose that $(\widetilde{\mathrm{C}}7)$ holds. Using Lemma 2.6.8, we have

$$\mathsf{supp}\,(\mu') \subset \exp\big(Y'_0 + \mathbb{R}\cdot U' + \mathbb{R}\cdot X_3\big), \qquad \mathsf{supp}\,(\mu'') \subset \exp\big(Y''_0 + \mathbb{R}\cdot U'' + \mathbb{R}\cdot X_3\big),$$

where $U' = b'_{2,2}X_2$ and $U'' = b''_{2,2}X_2$, so (C1) holds. $\square$

2.6.9 Remark. In case of (C1) in Theorem 2.2.1, μ' and μ'' are Gauss measures also in the "Euclidean sense" (i.e., considering them as measures on $\mathbb{R}^3$), but the parameters of the convolution $\mu' * \mu''$ are not necessarily the sum of the parameters of μ' and μ''. In case of (C2) in Theorem 2.2.1, μ' and μ'' are not necessarily Gauss measures in the "Euclidean sense", but the parameters of the convolution $\mu' * \mu''$ are the sum of the parameters of μ' and μ''.

2.6.10 Remark. It is natural to ask whether we can prove our results for non-symmetric Gauss measures using only the results for symmetric Gauss measures. First we recall that a measure ν on $\mathbb{H}$ is called symmetric if $\nu = \nu^\star$, where $\nu^\star(B) := \nu(B^{-1})$ for all Borel subsets B of $\mathbb{H}$. The measure $\nu^\star$ is called the adjoint of ν. We check that a Gauss measure μ on $\mathbb{H}$ with parameters (a, B) is symmetric if and only if $a = 0$. First we suppose that μ is a symmetric Gauss measure on $\mathbb{H}$ with parameters (a, B). Then there exists a unique Gauss semigroup $(\mu_t)_{t\geqslant 0}$ such that $\mu_1 = \mu$ and the canonical representation of the infinitesimal generator of $(\mu_t)_{t\geqslant 0}$ is $(a, B, 0)$ (for the canonical representation, see Heyer [30, Theorem 4.3.1]). Then the canonical representation of the infinitesimal generator of the adjoint semigroup $(\mu_t^\star)_{t\geqslant 0}$ is $(-a, B, 0)$ (see Siebert [53, Section 3]). Moreover, $\mu_1^\star = \mu^\star = \mu$. By Lemma 6.2.6 in Heyer [30], $(\mu_t^\star)_{t\geqslant 0}$ is a Gauss semigroup. Using that a Gauss measure on $\mathbb{H}$ can be embedded only in a uniquely determined Gauss semigroup, we get $\mu_t^\star = \mu_t$ for all $t \geqslant 0$. Hence the canonical representations of the infinitesimal generators of $(\mu_t)_{t\geqslant 0}$ and $(\mu_t^\star)_{t\geqslant 0}$ coincide, which implies $a = 0$.

Conversely, let μ be a Gauss measure on $\mathbb{H}$ with parameters $(0, B)$. Then there exists a unique Gauss semigroup $(\mu_t)_{t\geqslant 0}$ such that $\mu_1 = \mu$ and the canonical representation of the infinitesimal generator of $(\mu_t)_{t\geqslant 0}$ is $(0, B, 0)$. Then the infinitesimal generator of the adjoint semigroup $(\mu_t^\star)_{t\geqslant 0}$ admits canonical representation $(0, B, 0)$. By Theorem 4.2.5 in Heyer [30], we get $\mu_t^\star = \mu_t$ for all $t \geqslant 0$, which implies $\mu_1^\star = \mu_1 = \mu$. Since $\mu_1^\star = \mu^\star$, we get $\mu = \mu^\star$, i.e., μ is symmetric.

The answer to our original question concerning symmetric and non-symmetric Gauss measures on $\mathbb{H}$ is negative. The reason for this is that in case of $\mathbb{H}$ the convolution of a symmetric Gauss measure and a Dirac measure is in general not a Gauss measure. For example, if $a = (1, 0, 0) \in \mathbb{H}$ and $(\mu_t)_{t\geqslant 0}$ is a Gauss semigroup with infinitesimal generator $\widetilde{X}_1^2 + \widetilde{X}_2^2$, then using Theorem 2.2.1 and Lemma 2.4.3, one can easily check that $\mu_1 * \delta_a$ is not a Gauss measure on $\mathbb{H}$.

2.6.11 Remark. We note that if the convolution of two Gauss measures on $\mathbb{H}$ is again a Gauss measure on $\mathbb{H}$, then the corresponding infinitesimal generators not necessarily commute, nor even if the infinitesimal generator corresponding to the convolution is the sum of the original infinitesimal generators. Now we give an illuminating counterexample. Let μ' and μ'' be Gauss measures on $\mathbb{H}$ such that the corresponding Gauss semigroups have infinitesimal generators

$$\widetilde{N}' = \frac{1}{2}(\widetilde{X}_1 + \widetilde{X}_2)^2 \qquad \text{and} \qquad \widetilde{N}'' = \frac{1}{2}(\widetilde{X}_1 + \widetilde{X}_2)^2 + \widetilde{X}_1\widetilde{X}_3, \qquad \text{respectively.}$$

Using Theorem 2.6.7 and Lemma 2.6.3, $\mu' * \mu''$ is a symmetric Gauss measure on $\mathbb{H}$ such that the corresponding Gauss semigroup has infinitesimal generator $\widetilde{N}' + \widetilde{N}''$. But $\widetilde{N}'$ and $\widetilde{N}''$ do not commute. Indeed, $\widetilde{N}'\widetilde{N}'' - \widetilde{N}''\widetilde{N}' = -(\widetilde{X}_1 + \widetilde{X}_2)\widetilde{X}_3^2 \neq 0$.

Chapter 3

Gauss measures on the affine group

In this chapter it is shown that a Gauss measure on the affine group (i.e., the group of affine mappings on $\mathbb{R}$) can be embedded only in a uniquely determined Gauss semigroup (see Theorem 3.3.1). The starting point of the proof is the fact that a Gauss Lévy process in the affine group satisfies a certain stochastic differential equation (SDE). Theorem 3.2.1 contains the solution of this SDE. Moreover, we give a complete description of supports of Gauss measures on the affine group using Siebert's support formula (see Theorem 3.4.1).

The results of this chapter appeared in our paper [5].

3.1 Motivation

A probability measure μ on a locally compact group G is called *continuously embeddable* if there exists a continuous convolution semigroup $(\mu_t)_{t\geqslant 0}$ of probability measures on G (i.e., $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \geqslant 0$, and $\mu_t \xrightarrow{\mathsf{w}} \mu_0 = \delta_e$ as $t \downarrow 0$) satisfying $\mu_1 = \mu$. (Here δ_e denotes the Dirac measure concentrated on the unit element e of G.)

For a general locally compact group G one does not know whether the embedding convolution semigroup of a continuously embeddable probability measure on G is unique. If $(\mu_t)_{t\geqslant 0}$ and $(\nu_t)_{t\geqslant 0}$ are convolution semigroups of probability measures on $(\mathbb{R}^d, +)$ then it is well-known that $\mu_1 = \nu_1$ implies

$\mu_t = \nu_t$ for all $t \geqslant 0$. The same statement holds for locally compact Abelian groups without non-trivial compact subgroups (cf. Heyer [30, Theorem 3.5.15]). But for example in case of the one-dimensional torus group $\{\mathsf{e}^{ix} : -\pi \leqslant x < \pi\}$ (which is compact), the Dirac measure $\delta_{\mathsf{e}^{-\pi}}$ is continuously embeddable into the continuous convolution semigroups $(\delta_{\mathsf{e}^{-t\pi}})_{t\geqslant 0}$ and $(\delta_{\mathsf{e}^{-3t\pi}})_{t\geqslant 0}$, which do not coincide (their infinitesimal generators are different). The question of unicity of embedding into stable and semi-stable semigroups on simply connected nilpotent Lie groups has been studied by Drisch and Gallardo [18], Nobel [43] and see also a detailed discussion by Hazod and Siebert [28, Section 2.6]. Neuenschwander [41] studied Poisson semigroups on simply connected nilpotent Lie groups.

By a *Gauss measure* on a locally compact group G we mean a probability measure μ on G for which there exists a Gauss semigroup $(\mu_t)_{t\geqslant 0}$ (i.e., a continuous convolution semigroup $(\mu_t)_{t\geqslant 0}$ for which $\lim_{t\downarrow 0} t^{-1}\mu_t(G \setminus U) = 0$ for all Borel neighbourhoods U of e) such that $\mu = \mu_1$.

3.1.1 Remark. We note that the definition of a Gauss semigroup slightly differs from the Definition 6.2.1 in Heyer [30], since in our definition, given a Gauss semigroup $(\mu_t)_{t\geqslant 0}$, the measure μ_t can be a Dirac measure for any $t > 0$. More precisely, one can prove the following assertion. Suppose that G is second countable, $(\mu_t)_{t\geqslant 0}$ is a continuous convolution semigroup on G and there exists some $t_0 > 0$ such that μ_{t_0} is a Dirac measure on G. Then there exists a continuous one-parameter subsemigroup $(x_t)_{t\geqslant 0}$ of G such that $\mu_t = \delta_{x_t}$ for all $t \geqslant 0$.

Pap [44] proved that a Gauss measure on a simply connected nilpotent Lie group has a unique embedding semigroup among Gauss semigroups. We prove the same result for the 2-dimensional affine group, i.e., the group of affine mappings on $\mathbb{R}$, which is a Lie group but not nilpotent (see Theorem 3.3.1). Our method, which is related to the idea of Pap [44], consists of recursively calculating the first and second moments. In order to prove the uniqueness of embedding we consider a Gauss Lévy process $(\xi(t))_{t\geqslant 0}$ in the affine group related to a Gauss semigroup, and we show that $(\xi(t))_{t\geqslant 0}$ satisfies a certain stochastic differential equation (SDE). Theorem 3.2.1 contains the solution of this SDE. The question about the existence of a non-Gauss embedding semigroup of a Gauss measure remains still open. In the special case of simply connected step 2-nilpotent Lie groups Neuenschwander [42] showed that a Gauss measure does not admit a non-Gauss embedding semigroup.

We will also investigate the support of μ_t for $t > 0$ where $(\mu_t)_{t\geqslant 0}$ forms a

Gauss semigroup on the affine group. Siebert [54, Theorem 2] showed that given a Gauss semigroup $(\mu_t)_{t\geqslant 0}$ on a connected Lie group G, either the measures μ_t are absolutely continuous with respect to a left or right (and then necessarily to any left or right) Haar measure on G for all $t > 0$, or the measures μ_t are singular with respect to a left or right (and then necessarily to any left or right) Haar measure on G for all $t > 0$. In the first case we say that $(\mu_t)_{t\geqslant 0}$ is an absolutely continuous semigroup on G, otherwise it is called singular. For any absolutely continuous Gauss semigroup $(\mu_t)_{t\geqslant 0}$ on a connected Abelian Lie group G, we have $\mathsf{supp}\,(\mu_t) = G$ for all $t > 0$, where $\mathsf{supp}\,(\mu)$ denotes the support of the measure μ. McCrudden [37] showed that for any absolutely continuous Gauss semigroup $(\mu_t)_{t\geqslant 0}$ on any connected nilpotent Lie group G, we have $\mathsf{supp}\,(\mu_t) = G$ for all $t > 0$. But in the solvable case the situation becomes more complicated. Siebert [54] showed that on the affine group there exists an absolutely continuous Gauss semigroup $(\mu_t)_{t\geqslant 0}$ with $\mathsf{supp}\,(\mu_t) \neq G$ for every $t > 0$. We will give a complete description of supports for Gauss semigroups on the affine group using Siebert's support formula (see Theorem 3.4.1). See further investigations on other Lie groups by McCrudden [36], [37], [38], Kelly-Lyth and McCrudden [35].

3.2 Gauss Lévy processes

Let G be a second countable locally compact T_0-topological group. A stochastic process $(\xi(t))_{t\geqslant 0}$ (on a probability space $(\Omega, \mathcal{A}, \mathrm{P})$) with values in G has stationary independent left-increments if for all $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$, $n \in \mathbb{N}$, the random elements $\xi(t_1)$, $\xi(t_1)^{-1}\xi(t_2), \ldots, \xi(t_{n-1})^{-1}\xi(t_n)$ are independent and the distribution of $\xi(s)^{-1}\xi(t)$ depends only on $t-s$ for all $0 \leqslant s \leqslant t$. Now we recall the notion of stochastic continuity of a stochastic process $(\xi(t))_{t\geqslant 0}$ with values in G. By Hewitt–Ross [29, Theorem 8.3], G admits a left-invariant metric ρ compatible with its topology. We say that $(\xi(t))_{t\geqslant 0}$ is stochastically continuous if for all $t_0 \geqslant 0$ and for all $t_n \geqslant 0$, $n \in \mathbb{N}$, with $\lim_{n\to\infty} t_n = t_0$ we have

$$\lim_{n\to\infty} \mathrm{P}\big(\rho(\xi(t_n), \xi(t_0)) > \varepsilon\big) = 0, \quad \forall\, \varepsilon > 0.$$

If $(\xi(t))_{t\geqslant 0}$ has stationary independent left-increments then the left-invariant property of ρ implies that $(\xi(t))_{t\geqslant 0}$ is stochastically continuous if and only if for all $t_n \geqslant 0$, $n \in \mathbb{N}$, with $\lim_{n\to\infty} t_n = 0$ we have

$$\lim_{n\to\infty} \mathrm{P}\big(\rho(\xi(t_n), e) > \varepsilon\big) = 0, \quad \forall\, \varepsilon > 0.$$

By Vakhania–Tarieladze–Chobanyan [59, p. 91], this latter condition is equivalent to the fact that the sequence $\xi(t_n)$, $n \in \mathbb{N}$, is convergent in distribution to the Dirac measure δ_e. Hence the definition of stochastic continuity of a process with values in G having stationary independent left-increments is independent of the choice of left-invariant metrics on G (compatible with the topology of G). By a *Lévy process* $(\xi(t))_{t\geqslant 0}$ (on a probability space $(\Omega, \mathcal{A}, \mathrm{P})$) with values in G we mean a stochastically continuous process with stationary independent left-increments such that $\xi(0) = e$ and regular in the sense that for almost every $\omega \in \Omega$ the path $t \mapsto \xi(t)(\omega)$ is right continuous on $[0, \infty)$ and has left-hand limits on $(0, \infty)$.

To a Lévy process $(\xi(t))_{t\geqslant 0}$ with values in G one can correspond a unique continuous convolution semigroup $(\mu_t)_{t\geqslant 0}$ such that the distribution of $\xi(t)$ is μ_t for all $t \geqslant 0$. Conversely, for a continuous convolution semigroup $(\mu_t)_{t\geqslant 0}$ there exist a probability space $(\Omega, \mathcal{A}, \mathrm{P})$ and a Lévy process $(\xi(t))_{t\geqslant 0}$ on $(\Omega, \mathcal{A}, \mathrm{P})$ with values in G such that the distribution of $\xi(t)$ is μ_t for all $t \geqslant 0$ (see Heyer [30, p. 334–335]). Moreover, the distribution of $\xi(s)^{-1}\xi(t)$ is μ_{t-s} for all $0 \leqslant s \leqslant t$.

By a *Gauss Lévy process* we mean a Lévy process $(\xi(t))_{t\geqslant 0}$ for which the corresponding continuous convolution semigroup $(\mu_t)_{t\geqslant 0}$ is a Gauss semigroup, i.e.,

$$0 = \lim_{t\downarrow 0} \frac{1}{t}\mu_t(G \setminus U) = \lim_{t\downarrow 0} \frac{1}{t}\mathrm{P}(\xi(t) \notin U)$$

for all Borel neighbourhoods U of e. Corollary 2 of Theorem 2 in Siebert [55] implies that for a Gauss Lévy process $(\xi(t))_{t\geqslant 0}$ the path $t \mapsto \xi(t)(\omega)$ is continuous on $[0, \infty)$ for almost every $\omega \in \Omega$. Moreover, given a continuous convolution semigroup, if each of its associated Lévy processes has continuous paths with probability one then the convolution semigroup in question is a Gauss semigroup. Hence a Gauss Lévy process with values in G can also be called a Brownian motion in G.

By the infinitesimal generator of a Lévy process $(\xi(t))_{t\geqslant 0}$ we mean the infinitesimal generator of the continuous convolution semigroup $(\mu_t)_{t\geqslant 0}$ corresponding to it, i.e.,

$$(\widetilde{N}f)(x) := \lim_{t\downarrow 0} \frac{1}{t}\int_G (f(xy) - f(x))\,\mu_t(\mathsf{d}y) = \lim_{t\downarrow 0} \frac{1}{t}\mathsf{E}(f(x\xi(t)) - f(x)), \quad x \in G,$$

for suitable functions $f : G \to \mathbb{R}$. (The infinitesimal generator is always defined for infinitely differentiable functions $f : G \to \mathbb{R}$ with compact support.)

Roynette [47] gave a recursive formula for constructing Gauss Lévy processes in an arbitrary nilpotent Lie group by the help of a corresponding Gauss Lévy process in the corresponding Lie algebra, that is, by some independent Wiener processes in $\mathbb{R}$. The formula involves Itô integrals and reflects the group law. In Feinsilver and Schott [21], [22] one can find an operator approach (applicable for other Lie groups and based on limit theorems) which can be used to obtain similar explicit formulas. Applebaum and Kunita [1] studied Lévy processes $(\xi(t))_{t\geqslant 0}$ with values in a connected Lie group G. They showed that for all bounded twice continuously differentiable functions $f : G \to \mathbb{R}$ having limit at infinity, the process $(f(\xi(t)))_{t\geqslant 0}$ satisfies a stochastic differential equation connected to the infinitesimal generator of the process $(\xi(t))_{t\geqslant 0}$.

In case of the affine group it turns out that a Gauss Lévy process $(\xi(t))_{t\geqslant 0}$ can be constructed by the help of one standard Wiener process, or two independent standard Wiener processes. The formula involves again Itô integrals and reflects the group law as in the case of nilpotent Lie groups (see, e.g., Roynette [47]).

Concerning Gauss Lévy processes and Gauss measures on the affine group F (the group of affine mappings on $\mathbb{R}$) we can restrict ourselves to the group G of direction preserving affine mappings on $\mathbb{R}$. Indeed, the connected component of the identity e of F coincides with G, hence, for a Gauss semigroup $(\mu_t)_{t\geqslant 0}$ of probability measures on F, the support of μ_t is contained in G for all $t \geqslant 0$ (see Heyer [30, Theorem 6.2.3]). Hence the restriction of a Gauss measure on F onto G is a Gauss measure on G. Similarly, a Gauss Lévy process with values in F can be considered as a Gauss Lévy process with values in G.

The 2-dimensional affine group F can be realized as the matrix group

$$F = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0,\, b \in \mathbb{R} \right\}.$$

Here the notion "a matrix group" means a closed subgroup of the group $\mathrm{GL}_2(\mathbb{R})$ of all invertible, 2×2 real matrices. Endowing $\mathrm{GL}_2(\mathbb{R})$ with the topology induced on it by the natural topology of $\mathbb{R}^4$, it is a Lie group. By Baker [3, Theorem 7.24] each matrix group is a Lie subgroup of $\mathrm{GL}_2(\mathbb{R})$. Hence F is a Lie group and it is not connected, not compact and not nilpotent.

The group G of direction preserving affine mappings on $\mathbb{R}$ can be realized as the matrix group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0,\, b \in \mathbb{R} \right\}.$$

Then G is a connected solvable Lie group which is not nilpotent.

The Lie algebra $\mathcal{G}$ of G can be realized as the matrix algebra

$$\mathcal{G} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}.$$

Moreover, the Lie algebra of F coincides with $\mathcal{G}$. Consider the basis $\{e_1, e_2\}$ of $\mathcal{G}$ defined by

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_2 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then we have the commutation relation $[e_1, e_2] = e_1e_2 - e_2e_1 = e_2$.

Lévy processes with values in a Lie group can be given by their infinitesimal generators containing left-invariant differential operators (see Heyer [30, Theorems 4.2.4 and 4.2.5]). If $f : F \to \mathbb{R}$ is continuously differentiable then, for every $X \in \mathcal{G}$, we can introduce the left-invariant differential operator $\widetilde{X}$ defined by

$$\widetilde{X} f(g) := \lim_{t \to 0} \frac{f(g \exp(tX)) - f(g)}{t}, \qquad g \in F.$$

Here $\exp$ denotes the exponential mapping from $\mathcal{G}$ into F. Note that the mapping $X \in \mathcal{G} \mapsto \widetilde{X}$ is injective and linear (see, e.g., Corwin–Greenleaf [15, p. 110]). It is known that a Lévy process $(\xi(t))_{t \geqslant 0}$ in F is a Gauss Lévy process if and only if its infinitesimal generator admits the form

$$\widetilde{N} = \sum_{i=1}^{2} a_i \widetilde{e}_i + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} b_{i,j} \widetilde{e}_i \widetilde{e}_j, \tag{3.2.1}$$

where $a_1, a_2 \in \mathbb{R}$ and $B = (b_{i,j})_{1 \leqslant i,j \leqslant 2}$ is a real symmetric positive semidefinite matrix. This easily follows from Theorem 4.2.4 and Lemma 6.2.6 in Heyer [30] and from the fact that given a Gauss Lévy process $(\xi(t))_{t \geqslant 0}$ in F such that the distribution of $\xi(t_0)$ is a Dirac measure on F for some $t_0 > 0$ then there exist a_1, $a_2 \in \mathbb{R}$ such that the distribution of $\xi(t)$ is $\delta_{\exp(ta_1e_1 + ta_2e_2)}$ for all $t \geqslant 0$.

The infinitesimal generator $\widetilde{N}$ can be written in the form

$$\widetilde{N} = \widetilde{Y} + \frac{1}{2} \sum_{k=1}^{r} \widetilde{X}_k^2, \tag{3.2.2}$$

where

$$Y = \sum_{i=1}^{2} a_i e_i, \qquad X_j = \sum_{i=1}^{2} \sigma_{i,j} e_i, \quad 1 \leqslant j \leqslant r \leqslant 2,$$

where $\Sigma = (\sigma_{i,j})$ is a $2 \times r$ matrix such that $B = \Sigma \cdot \Sigma^\top$ and $\operatorname{rank} \Sigma = \operatorname{rank} B = r$.

3.2.1 Theorem. *Let $(\xi(t))_{t \geqslant 0}$ be a Gauss Lévy process with values in the affine group F with infinitesimal generator* (3.2.1)*. Then*

$$\xi(t) = \begin{pmatrix} \mathrm{e}^{Z_1(t)} & \int_0^t \mathrm{e}^{Z_1(s)} \, \mathrm{d}(Z_2(s) + b_{1,2}s/2) \\ 0 & 1 \end{pmatrix}, \qquad t \geqslant 0,$$

where

$$Z_i(t) = a_i t + \sum_{j=1}^{r} \sigma_{i,j} W_j(t), \qquad i = 1, 2,$$

and $(W_1(t))_{t \geqslant 0}$ and $(W_2(t))_{t \geqslant 0}$ are independent standard Wiener processes in $\mathbb{R}$.

Proof. If $b_{i,j} = 0$ for all $1 \leqslant i, j \leqslant 2$ then one can check that the process

$$x(t) := \exp(ta_1e_1 + ta_2e_2) = \begin{cases} \begin{pmatrix} \mathrm{e}^{a_1 t} & a_2 \frac{\mathrm{e}^{a_1 t} - 1}{a_1} \\ 0 & 1 \end{pmatrix} & \text{if } a_1 \neq 0, \\ \begin{pmatrix} 1 & a_2 t \\ 0 & 1 \end{pmatrix} & \text{if } a_1 = 0, \end{cases} \qquad t \geqslant 0,$$

is a Gauss Lévy process in F with infinitesimal generator $\widetilde{N} = \sum_{i=1}^2 a_i \widetilde{e}_i$.

If $b_{i,j} \neq 0$ for some $1 \leqslant i, j \leqslant 2$ then, applying Theorem 3.1 in Applebaum and Kunita [1], $(\xi(t))_{t \geqslant 0}$ can be represented as a solution of the SDE

$$\xi(t) = I + \sum_{i=1}^{2} \int_0^t a_i \xi(s) e_i \, \mathrm{d}s + \frac{1}{2} \sum_{i,j=1}^{2} \int_0^t b_{i,j} \xi(s) e_i e_j \, \mathrm{d}s + \sum_{i=1}^{2} \int_0^t \xi(s) e_i \, \mathrm{d}B_i(s),$$

where I is the 2×2 identity matrix, and $B(t) = (B_1(t), B_2(t))$ is a Gauss Lévy process in $\mathbb{R}^2$ with zero mean and covariances $\mathsf{Cov}(B_i(t), B_j(t)) = tb_{i,j}$, $1 \leqslant i, j \leqslant 2$.

Writing $\xi(t)$ in the form

$$\xi(t) = \begin{pmatrix} \xi_1(t) & \xi_2(t) \\ 0 & 1 \end{pmatrix},$$

and using $e_1^2 = e_1$, $e_2^2 = 0$, $e_1 e_2 = e_2$, $e_2 e_1 = 0$ we obtain the SDE

$$\begin{aligned} \mathrm{d}\xi_1(t) &= \left(a_1 + \frac{b_{1,1}}{2}\right)\xi_1(t)\,\mathrm{d}t + \xi_1(t)\,\mathrm{d}B_1(t), \\ \mathrm{d}\xi_2(t) &= \left(a_2 + \frac{b_{1,2}}{2}\right)\xi_1(t)\,\mathrm{d}t + \xi_1(t)\,\mathrm{d}B_2(t). \end{aligned} \tag{3.2.3}$$

Clearly $B_1(t) = \sum_{j=1}^r \sigma_{1,j} W_j(t)$ and $B_2(t) = \sum_{j=1}^r \sigma_{2,j} W_j(t)$, where $(W_1(t))_{t\geqslant 0}$ and $(W_2(t))_{t\geqslant 0}$ are independent standard Wiener processes in $\mathbb{R}$. By a simple application of Itô's formula we obtain

$$\xi_1(t) = \mathrm{e}^{(a_1+b_{1,1}/2)t + \sum_{j=1}^r \sigma_{1,j}W_j(t) - \sum_{j=1}^r \sigma_{1,j}^2 t/2} = \mathrm{e}^{Z_1(t)}.$$

since $B = \Sigma \cdot \Sigma^\top$ implies $\sum_{j=1}^r \sigma_{1,j}^2 = b_{1,1}$. Moreover,

$$\xi_2(t) = \int_0^t \xi_1(s)\,\mathrm{d}\left(\left(a_2+\frac{b_{1,2}}{2}\right)s+\sum_{j=1}^r \sigma_{2,j}W_j(s)\right) = \int_0^t \mathrm{e}^{Z_1(s)}\,\mathrm{d}(Z_2(s)+b_{1,2}s/2).$$

Hence the assertion. □

3.2.2 Remark. The process $(Z_1(t), Z_2(t))_{t\geqslant 0}$ is a Gauss Lévy process in $\mathbb{R}^2$ with infinitesimal generator

$$\sum_{i=1}^2 a_i \partial_i + \frac{1}{2}\sum_{i,j=1}^2 b_{i,j}\partial_i\partial_j,$$

i.e., replacing in (3.2.1) the differential operators $\widetilde{e}_1$ and $\widetilde{e}_2$ by ∂_1 and ∂_2, respectively.

3.3 Uniqueness of embedding

3.3.1 Theorem. *Let $(\mu_t)_{t\geqslant 0}$ and $(\nu_t)_{t\geqslant 0}$ be Gauss semigroups on the affine group F. If $\mu_1 = \nu_1$ then we have $\mu_t = \nu_t$ for all $t \geqslant 0$. In other words, a Gauss measure on the affine group can be embedded only in a uniquely determined Gauss semigroup.*

Proof. It is sufficient to show that by the help of the measure μ_1 we can construct the whole Gauss semigroup $(\mu_t)_{t\geqslant 0}$. A convolution semigroup is uniquely determined by its infinitesimal generator, hence it is sufficient to construct the infinitesimal generator of $(\mu_t)_{t\geqslant 0}$. Consider a Gauss Lévy process $(\xi(t))_{t\geqslant 0}$ which corresponds to $(\mu_t)_{t\geqslant 0}$. We will show that the distribution of $\xi(1)$ uniquely determines the parameters a_1, a_2, $b_{1,1}$, $b_{1,2}$ and $b_{2,2}$ of the infinitesimal generator (3.2.1). It turns out that the knowledge of the expectation vector and covariance matrix of $\xi(1)$ uniquely defines these parameters.

First we calculate the expectation of $\xi(t)$. Taking the expectation of the integrated forms of the stochastic differential equations (3.2.3) we obtain

$$\begin{aligned}\mathsf{E}\xi_1(t) &= 1 + \left(a_1 + \frac{b_{1,1}}{2}\right)\int_0^t \mathsf{E}\xi_1(s)\,\mathsf{d}s,\\ \mathsf{E}\xi_2(t) &= \left(a_2 + \frac{b_{1,2}}{2}\right)\int_0^t \mathsf{E}\xi_1(s)\,\mathsf{d}s.\end{aligned}$$

Indeed, we check that

$$\mathsf{E}\left(\int_0^t \xi_1(s)\,\mathsf{d}B_1(s)\right) = 0, \qquad \mathsf{E}\left(\int_0^t \xi_1(s)\,\mathsf{d}B_2(s)\right) = 0, \quad t \geqslant 0.$$

For this it is enough to show that (see, e.g., Karatzas–Shreve [34, Definition 3.2.9])

$$\mathsf{E}\left(\int_0^t \xi_1^2(s)\,\mathsf{d}s\right) < \infty, \quad t \geqslant 0. \tag{3.3.1}$$

If $\xi_1(t) = \mathsf{e}^{W(t)}$, $t \geqslant 0$, where $(W(t))_{t\geqslant 0}$ is a standard Wiener process in $\mathbb{R}$, then

$$\mathsf{E}\left(\int_0^t \mathsf{e}^{2W(s)}\,\mathsf{d}s\right) = \int_0^t \mathsf{E}(\mathsf{e}^{2W(s)})\,\mathsf{d}s = \int_0^t \exp\left\{\frac{4s}{2}\right\}\,\mathsf{d}s = \frac{1}{2}(\mathsf{e}^{2t} - 1) < \infty.$$

The general case can be handled similarly.

It follows that

$$\mathsf{E}\xi_1(t) = \mathsf{e}^{(a_1+b_{1,1}/2)t}, \tag{3.3.2}$$

$$\mathsf{E}\xi_2(t) = \left(a_2 + \frac{b_{1,2}}{2}\right)\int_0^t \mathsf{e}^{(a_1+b_{1,1}/2)s}\,\mathsf{d}s. \tag{3.3.3}$$

Using Itô's formula we have the following stochastic differential equations

$$\begin{aligned}\mathsf{d}\xi_1^2(t) &= 2\xi_1(t)\,\mathsf{d}\xi_1(t) + \mathsf{d}[\xi_1,\xi_1]_t,\\ \mathsf{d}\xi_2^2(t) &= 2\xi_2(t)\,\mathsf{d}\xi_2(t) + \mathsf{d}[\xi_2,\xi_2]_t,\\ \mathsf{d}(\xi_1(t)\xi_2(t)) &= \xi_2(t)\,\mathsf{d}\xi_1(t) + \xi_1(t)\,\mathsf{d}\xi_2(t) + \mathsf{d}[\xi_1,\xi_2]_t,\end{aligned}$$

where $[\,.\,,\,.\,]_t$ denotes the cross quadratic variation of continuous semimartingales.
Taking into account (3.2.3) and the facts that $B_i(t) = \sum_{j=1}^{r} \sigma_{i,j} W_j(t)$, $i = 1, 2$ and $B = \Sigma\Sigma^\top$ we obtain

$$\begin{aligned}
\mathsf{d}\xi_1^2(t) &= 2\xi_1(t)\,\mathsf{d}\xi_1(t) + b_{1,1}\xi_1^2(t)\,\mathsf{d}t,\\
\mathsf{d}\xi_2^2(t) &= 2\xi_2(t)\,\mathsf{d}\xi_2(t) + b_{2,2}\xi_1^2(t)\,\mathsf{d}t,\\
\mathsf{d}(\xi_1(t)\xi_2(t)) &= \xi_2(t)\,\mathsf{d}\xi_1(t) + \xi_1(t)\,\mathsf{d}\xi_2(t) + b_{1,2}\xi_1^2(t)\,\mathsf{d}t.
\end{aligned}$$

Taking the expectation of the integrated forms of these equations we get

$$\begin{aligned}
\mathsf{E}\xi_1^2(t) &= 1 + 2\,(a_1 + b_{1,1}) \int_0^t \mathsf{E}\xi_1^2(s)\,\mathsf{d}s,\\
\mathsf{E}\xi_2^2(t) &= b_{2,2} \int_0^t \mathsf{E}\xi_1^2(s)\,\mathsf{d}s + (2a_2 + b_{1,2}) \int_0^t \mathsf{E}(\xi_1(s)\xi_2(s))\,\mathsf{d}s,\\
\mathsf{E}(\xi_1(t)\xi_2(t)) &= \left(a_2 + \frac{3}{2}b_{1,2}\right) \int_0^t \mathsf{E}\xi_1^2(s)\,\mathsf{d}s\\
&\quad + \left(a_1 + \frac{b_{1,1}}{2}\right) \int_0^t \mathsf{E}(\xi_1(s)\xi_2(s))\,\mathsf{d}s.
\end{aligned} \tag{3.3.4}$$

Indeed, we check that for all $t \geqslant 0$

$$\begin{aligned}
\mathsf{E}\left(\int_0^t \xi_1^2(s)\ \mathsf{d}B_1(s)\right) &= \mathsf{E}\left(\int_0^t \xi_1^2(s)\ \mathsf{d}B_2(s)\right) = 0,\\
\mathsf{E}\left(\int_0^t \xi_1(s)\xi_2(s)\ \mathsf{d}B_1(s)\right) &= \mathsf{E}\left(\int_0^t \xi_1(s)\xi_2(s)\ \mathsf{d}B_2(s)\right) = 0.
\end{aligned}$$

For this it is enough to show that for all $t \geqslant 0$,

$$\mathsf{E}\left(\int_0^t \xi_1^4(s)\ \mathsf{d}s\right) < \infty, \tag{3.3.5}$$

$$\mathsf{E}\left(\int_0^t \xi_1^2(s)\xi_2^2(s)\ \mathsf{d}s\right) < \infty. \tag{3.3.6}$$

The proof of (3.3.5) is similar to the proof of (3.3.1). If

$$\begin{aligned}
\xi_1(t) &= \mathsf{e}^{W_1(t)}, \quad t \geqslant 0,\\
\xi_2(t) &= \int_0^t \mathsf{e}^{W_1(s)}\ \mathsf{d}W_2(s), \quad t \geqslant 0,
\end{aligned}$$

where W_1 and W_2 are independent standard Wiener processes in $\mathbb{R}$, then, by Karatzas–Shreve [34, Proposition 3.2.10],

$$\begin{aligned}\mathsf{E}(\xi_1^2(s)\xi_2^2(s)) &= \mathsf{E}\left(\mathsf{e}^{2W_1(s)}\left(\int_0^s \mathsf{e}^{W_1(u)}\,\mathsf{d}W_2(u)\right)^2\right)\\ &= \mathsf{E}\left(\int_0^s \mathsf{e}^{W_1(s)+W_1(u)}\,\mathsf{d}W_2(u)\right)^2 = \int_0^s \mathsf{E}(\mathsf{e}^{2(W_1(s)+W_1(u))})\,\mathsf{d}u.\end{aligned}$$

Hence

$$\mathsf{E}\left(\int_0^t \xi_1^2(s)\xi_2^2(s)\,\mathsf{d}s\right) = \int_0^t\int_0^s \mathsf{E}(\mathsf{e}^{2(W_1(s)+W_1(u))})\,\mathsf{d}u\,\mathsf{d}s < \infty, \qquad (3.3.7)$$

since the function $(s,u)\in[0,t]\times[0,t]\mapsto \mathsf{E}(\mathsf{e}^{2(W_1(s)+W_1(u))})$ is continuous. For the general case it is enough to check that

$$\mathsf{E}\left(\int_0^t \mathsf{e}^{2W_1(s)}\left(\int_0^s \mathsf{e}^{W_1(u)}\,\mathsf{d}(W_2(u)+u)\right)^2\mathsf{d}s\right) < \infty.$$

Indeed, for all $s\in[0,t]$

$$\left(\int_0^s \mathsf{e}^{W_1(u)}\,\mathsf{d}(W_2(u)+u)\right)^2 \leqslant 2\left(\int_0^s \mathsf{e}^{W_1(u)}\,\mathsf{d}W_2(u)\right)^2 + 2\left(\int_0^s \mathsf{e}^{W_1(u)}\,\mathsf{d}u\right)^2,$$

and hence using (3.3.7) it is enough to check that

$$\mathsf{E}\left(\int_0^t \mathsf{e}^{2W_1(s)}\left(\int_0^s \mathsf{e}^{W_1(u)}\,\mathsf{d}u\right)^2\mathsf{d}s\right) = \int_0^t \mathsf{E}\left(\int_0^s \mathsf{e}^{W_1(s)+W_1(u)}\,\mathsf{d}u\right)^2\mathsf{d}s < \infty.$$

For this we show that the function

$$s\in[0,t]\mapsto \mathsf{E}\left(\int_0^s \mathsf{e}^{W_1(s)+W_1(u)}\,\mathsf{d}u\right)^2 \qquad (3.3.8)$$

is bounded. Indeed, for all $0\leqslant s\leqslant t$,

$$\mathsf{E}\left(\int_0^s \mathsf{e}^{W_1(s)+W_1(u)}\,\mathsf{d}u\right)^2 = \int_0^s\int_0^s \mathsf{E}\left(\mathsf{e}^{2W_1(s)+W_1(u)+W_1(v)}\right)\,\mathsf{d}u\,\mathsf{d}v.$$

Since the function $(u,v)\in[0,s]\times[0,s]\mapsto \mathsf{E}\left(\mathsf{e}^{2W_1(s)+W_1(u)+W_1(v)}\right)$ is continuous and hence bounded for all $s\in[0,t]$, the function in (3.3.8) is bounded. Hence (3.3.6) is valid.

It can be easily checked that the unique solutions of the equations (3.3.4) are the following

$$\mathsf{E}\xi_1^2(t) = \mathsf{e}^{2(a_1+b_{1,1})t}\,, \tag{3.3.9}$$

$$\mathsf{E}(\xi_1(t)\xi_2(t)) = \left(a_2 + \frac{3}{2}b_{1,2}\right)\mathsf{e}^{(a_1+b_{1,1}/2)t}\int_0^t \mathsf{e}^{(a_1+3b_{1,1}/2)s}\,\mathsf{d}s, \tag{3.3.10}$$

$$\begin{aligned}\mathsf{E}\xi_2^2(t) = (2a_2+b_{1,2})\left(a_2+\frac{3}{2}b_{1,2}\right)\int_0^t \mathsf{e}^{(a_1+b_{1,1}/2)s}\left(\int_0^s \mathsf{e}^{(a_1+3b_{1,1}/2)u}\,\mathsf{d}u\right)\mathsf{d}s \\ + b_{2,2}\int_0^t \mathsf{e}^{2(a_1+b_{1,1})s}\,\mathsf{d}s.\end{aligned} \tag{3.3.11}$$

Using (3.3.2) and (3.3.9) with $t=1$ we have

$$\begin{cases} a_1 + \frac{b_{1,1}}{2} & = \log \mathsf{E}\xi_1(1), \\ 2(a_1+b_{1,1}) & = \log \mathsf{E}\xi_1^2(1). \end{cases}$$

This system of linear equations has a unique solution for a_1 and $b_{1,1}$. Substituting a_1 and $b_{1,1}$ into (3.3.3) and (3.3.10) with $t=1$ we obtain a system of linear equations for a_2 and $b_{1,2}$ which has again a unique solution. Equation (3.3.11) yields that $b_{2,2}$ is unique, too. So the infinitesimal generator of the Gauss semigroup $(\mu_t)_{t\geqslant 0}$ is uniquely determined by μ_1. □

3.4 Support of a Gauss measure

Let $(\mu_t)_{t\geqslant 0}$ be a Gauss semigroup on the affine group F with infinitesimal generator $\widetilde{N}$. Siebert [54] showed that according to the structure of $\widetilde{N}$ we can distinguish five different types of Gauss semigroups:

(i) $\widetilde{N} = \widetilde{Y} + \frac{1}{2}(\widetilde{X}_1^2 + \widetilde{X}_2^2)$ with X_1 and X_2 linearly independent. Then the semigroup is absolutely continuous, it has a strictly positive analytic density and $\mathsf{supp}\,(\mu_t) = G$ for all $t>0$. Moreover, $\mathsf{rank}\,(B) = 2$.

(ii) $\widetilde{N} = \widetilde{Y} + \frac{1}{2}\widetilde{X}_1^2$ with Y and X_1 linearly independent and $[X_1, e_2] \neq 0$. Then the semigroup is absolutely continuous. Moreover, $\mathsf{rank}\,(B) = 1$.

(iii) $\widetilde{N} = \widetilde{Y} + \frac{1}{2}\widetilde{X}_1^2$ with Y and X_1 linearly independent and $[X_1, e_2] = 0$. Then the semigroup is singular. Moreover, $\mathsf{rank}\,(B) = 1$.

(iv) $\widetilde{N} = \widetilde{Y} + \frac{1}{2}\widetilde{X}_1^2$ with Y and X_1 linearly dependent. Then the semigroup is singular and is supported by the proper closed subgroup $\exp(\mathbb{R} \cdot X_1)$. Moreover, $\mathsf{rank}\,(B) = 1$.

(v) $\widetilde{N} = \widetilde{Y}$. Then the semigroup is singular and consists of Dirac measures, namely, $\mu_t = \delta_{\exp(tY)}$ for all $t \geqslant 0$.

Our aim is to determine the supports of Gauss semigroups of type (ii) and (iii). In special cases (when $\widetilde{N} = \widetilde{e}_2 + \widetilde{e}_1^2$ and $\widetilde{N} = \widetilde{e}_1 + \widetilde{e}_2^2$) Siebert [54] has already described them.

Let $\mathcal{M}$ denote the Lie subalgebra generated by $\{X_i : 1 \leqslant i \leqslant r\}$. We will use Siebert's support formula

$$\mathsf{supp}\,(\mu_t) = \overline{\bigcup_{n=1}^{\infty} \Big(M \exp \frac{tY}{n}\Big)^n} \qquad \text{for all} \qquad t > 0,$$

where M is the analytic subgroup of G corresponding to $\mathcal{M}$ (see Siebert [54]).

3.4.1 Theorem. *Let $(\mu_t)_{t\geqslant 0}$ be a Gauss semigroup on the affine group F with infinitesimal generator $\widetilde{N}$.*

(a) *If $\widetilde{N}$ is of type* (ii) *then for all $t > 0$, the measure μ_t is supported by*

$$\begin{cases} \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0,\, b \geqslant \frac{b_{2,1}}{b_{1,1}}(a-1) \right\} & \text{if} \quad a_2 b_{1,1} - a_1 b_{2,1} > 0, \\ \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0,\, b \leqslant \frac{b_{2,1}}{b_{1,1}}(a-1) \right\} & \text{if} \quad a_2 b_{1,1} - a_1 b_{2,1} < 0. \end{cases}$$

(b) *If $\widetilde{N}$ is of type* (iii) *then the measure μ_t is supported by $\exp(ta_1 e_1)\exp(\mathbb{R} \cdot e_2)$ for all $t > 0$.*

Proof. In both cases we have $r = 1$ and $\widetilde{N} = \widetilde{Y} + \frac{1}{2}\widetilde{X}_1^2$, where $Y = a_1 e_1 + a_2 e_2$ and $X_1 = \sigma_{1,1} e_1 + \sigma_{2,1} e_2$.

(a). Now $\sigma_{1,1} e_2 = [X_1, e_2] \neq 0$, and Y and X_1 are linearly independent, hence $a_1 \sigma_{2,1} - a_2 \sigma_{1,1} \neq 0$, which implies $a_1 b_{2,1} - a_2 b_{1,1} \neq 0$.

First consider the case $a_1 = 0$. By induction,

$$\begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}^k = \begin{pmatrix} \alpha^k & \alpha^{k-1}\beta \\ 0 & 0 \end{pmatrix}, \qquad k = 1, 2, \ldots,$$

hence

$$\exp\left\{\begin{pmatrix}\alpha & \beta\\ 0 & 0\end{pmatrix}\right\} = \begin{cases}\begin{pmatrix}\mathrm{e}^{\alpha} & \beta\cdot\dfrac{\mathrm{e}^{\alpha}-1}{\alpha}\\ 0 & 1\end{pmatrix}, & \text{for } \alpha\neq 0,\\[2ex] \begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}, & \text{for } \alpha = 0.\end{cases}$$

Using this formula it can be easily checked by induction that the elements of the set $\left(M\exp\frac{tY}{n}\right)^n$ have the form $S=(s_{i,j})_{1\leqslant i,j\leqslant 2}$, where

$$\begin{cases} s_{1,1} &= \mathrm{e}^{(\alpha_1+\cdots+\alpha_n)\sigma_{1,1}},\\ s_{1,2} &= \frac{t}{n}a_2\mathrm{e}^{(\alpha_1+\cdots+\alpha_n)\sigma_{1,1}} + \frac{\sigma_{2,1}}{\sigma_{1,1}}\left(\mathrm{e}^{(\alpha_1+\cdots+\alpha_n)\sigma_{1,1}}-1\right)\\ &\quad + \frac{t}{n}a_2\left(\mathrm{e}^{\alpha_1\sigma_{1,1}}+\cdots+\mathrm{e}^{(\alpha_1+\cdots+\alpha_{n-1})\sigma_{1,1}}\right),\\ s_{2,1} &= 0,\\ s_{2,2} &= 1,\end{cases}$$

and $\alpha_1,\ldots,\alpha_n\in\mathbb{R}$, $n\in\mathbb{N}$ can be arbitrary. The term $\mathrm{e}^{\alpha_1\sigma_{1,1}}+\cdots+\mathrm{e}^{(\alpha_1+\cdots+\alpha_{n-1})\sigma_{1,1}}$ attends every positive number. Hence $s_{1,2}\geqslant\frac{t}{n}a_2s_{1,1}+\frac{\sigma_{2,1}}{\sigma_{1,1}}(s_{1,1}-1)$ if $a_2>0$, and $s_{1,2}\leqslant\frac{t}{n}a_2s_{1,1}+\frac{\sigma_{2,1}}{\sigma_{1,1}}(s_{1,1}-1)$ if $a_2<0$. Using Siebert's supports formula and the facts that $\frac{\sigma_{2,1}}{\sigma_{1,1}}=\frac{b_{2,1}}{b_{1,1}}$ and $b_{1,1}>0$ we obtain the assertion.

If $a_1\neq 0$ then again by induction we obtain that the elements of the set $\left(M\exp\frac{tY}{n}\right)^n$ have the form $S=(s_{i,j})_{1\leqslant i,j\leqslant 2}$, where

$$\begin{cases} s_{1,1} &= \mathrm{e}^{(\alpha_1+\cdots+\alpha_n)\sigma_{1,1}+ta_1},\\ s_{1,2} &= \left(a_2\frac{1-\mathrm{e}^{-ta_1/n}}{a_1}+\frac{\sigma_{2,1}}{\sigma_{1,1}}\mathrm{e}^{-ta_1/n}\right)\mathrm{e}^{(\alpha_1+\cdots+\alpha_n)\sigma_{1,1}+ta_1}-\frac{\sigma_{2,1}}{\sigma_{1,1}}\\ &\quad+\frac{\mathrm{e}^{ta_1/n}-1}{a_1}\left(a_2-\frac{\sigma_{2,1}}{\sigma_{1,1}}a_1\right)\left(\mathrm{e}^{\alpha_1\sigma_{1,1}}+\cdots+\mathrm{e}^{(\alpha_1+\cdots+\alpha_{n-1})\sigma_{1,1}+(n-2)ta_1/n}\right),\\ s_{2,1} &= 0,\\ s_{2,2} &= 1,\end{cases}$$

and $\alpha_1,\ldots,\alpha_n\in\mathbb{R}$, $n\in\mathbb{N}$ can be arbitrary. The term $\mathrm{e}^{\alpha_1\sigma_{1,1}}+\cdots+e^{(\alpha_1+\cdots+\alpha_{n-1})\sigma_{1,1}+(n-2)ta_1/n}$ attends every positive number. Using the fact that $\frac{\mathrm{e}^{ta_1/n}-1}{a_1}>0$ we have

$$s_{1,2}\geqslant\left(a_2\frac{1-\mathrm{e}^{-ta_1/n}}{a_1}+\frac{\sigma_{2,1}}{\sigma_{1,1}}\mathrm{e}^{-ta_1/n}\right)s_{1,1}-\frac{\sigma_{2,1}}{\sigma_{1,1}}\qquad\text{if } a_2b_{1,1}-a_1b_{2,1}>0,$$

$$s_{1,2} \leqslant \left(a_2 \frac{1 - \mathsf{e}^{-ta_1/n}}{a_1} + \frac{\sigma_{2,1}}{\sigma_{1,1}} \mathsf{e}^{-ta_1/n} \right) s_{1,1} - \frac{\sigma_{2,1}}{\sigma_{1,1}} \qquad \text{if } \ a_2 b_{1,1} - a_1 b_{2,1} < 0.$$

Since

$$a_2 \frac{1 - \mathsf{e}^{-ta_1/n}}{a_1} + \frac{\sigma_{2,1}}{\sigma_{1,1}} \mathsf{e}^{-ta_1/n} > \frac{\sigma_{2,1}}{\sigma_{1,1}} \qquad \text{if } \ a_2 b_{1,1} - a_1 b_{2,1} > 0,$$

$$a_2 \frac{1 - \mathsf{e}^{-ta_1/n}}{a_1} + \frac{\sigma_{2,1}}{\sigma_{1,1}} \mathsf{e}^{-ta_1/n} < \frac{\sigma_{2,1}}{\sigma_{1,1}} \qquad \text{if } \ a_2 b_{1,1} - a_1 b_{2,1} < 0,$$

and

$$\lim_{n \to \infty} \frac{\mathsf{e}^{ta_1/n} - 1}{a_1} = 0,$$

we get the assertion.

(b). Now $\sigma_{1,1} e_2 = [X_1, e_2] = 0$. Moreover, Y and X_1 are linearly independent, hence $a_1 \sigma_{2,1} - a_2 \sigma_{1,1} \neq 0$, which implies $a_1 \neq 0$. The elements of the set $\big(M \exp \frac{tY}{n}\big)^n$ have the form

$$\begin{pmatrix} \mathsf{e}^{ta_1} & \frac{a_2}{a_1} (\mathsf{e}^{ta_1} - 1) + \sigma_{2,1} \Big(\alpha_1 + \alpha_2 \mathsf{e}^{ta_1/n} + \cdots + (\alpha_1 + \cdots + \alpha_n) \mathsf{e}^{(n-1)a_1 t/n} \Big) \\ 0 & 1 \end{pmatrix},$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. Using Siebert's support formula we get

$$\mathsf{supp}\,(\mu_t) = \left\{ \begin{pmatrix} \mathsf{e}^{ta_1} & \beta \\ 0 & 1 \end{pmatrix} : \beta \in \mathbb{R} \right\} \qquad \text{for all } \ t > 0,$$

that is $\mathsf{supp}\,(\mu_t) = \exp(ta_1 e_1 + \mathbb{R} \cdot e_2) = \exp(ta_1 e_1) \exp(\mathbb{R} \cdot e_2)$ for all $t > 0$. □

3.4.2 Remark. In case (ii) the semigroup $(\mu_t)_{t \geqslant 0}$ is absolutely continuous and $\mathsf{supp}\,(\mu_t)$ is the same closed subsemigroup of G for all $t > 0$. In case (iii) the semigroup $(\mu_t)_{t \geqslant 0}$ is singular and $\mathsf{supp}\,(\mu_t)$ is a proper coset of the same closed normal subgroup $\exp(\mathbb{R} \cdot e_2)$ for all $t > 0$.

We recall that a measure ν on the affine group F is called symmetric if $\nu = \nu^\star$, where $\nu^\star(B) := \nu(B^{-1})$ for all Borel subsets B of F. A process $(\xi(t))_{t \geqslant 0}$ with values in F is called symmetric if the distribution of $\xi(t)$ is symmetric for all $t \geqslant 0$. Similarly as in Remark 2.6.10 one can check that a Gauss Lévy process in F with infinitesimal generator (3.2.1) is symmetric if and only if $a_1 = a_2 = 0$.

3.4.3 Remark. Let $(\xi(t))_{t\geqslant 0}$ be the Gauss Lévy process in the affine group F with infinitesimal generator $\widetilde{N}$ of type (iii), i.e., $\widetilde{N} = a_1\widetilde{e}_1 + a_2\widetilde{e}_2 + \frac{1}{2}\sigma_{2,1}^2\widetilde{e}_2^2$, where $a_1 \neq 0$ and $\sigma_{2,1} \neq 0$. By Theorem 3.4.1, the distribution of $\xi(t)$ is singular for all $t > 0$. Since $a_1 \neq 0$, the distribution of $\xi(t)$ is not symmetric for any $t > 0$. But

$$\xi(t) = \eta\left(\frac{\mathsf{e}^{2a_1t} - 1}{2a_1}\right) x(t), \qquad t \geqslant 0,$$

where

$$x(t) = \exp(ta_1e_1 + ta_2e_2) = \begin{pmatrix} \mathsf{e}^{a_1t} & a_2\frac{\mathsf{e}^{a_1t}-1}{a_1} \\ 0 & 1 \end{pmatrix},$$

and $(\eta(t))_{t\geqslant 0}$ is a symmetric Gauss Lévy process with infinitesimal generator $\frac{1}{2}\sigma_{2,1}^2\widetilde{e}_2^2$. Indeed, by Theorem 3.2.1

$$\xi(t) = \begin{pmatrix} \mathsf{e}^{a_1t} & \int_0^t \mathsf{e}^{a_1s}\,\mathsf{d}(a_2s + \sigma_{2,1}W(s)) \\ 0 & 1 \end{pmatrix}, \quad \eta(t) = \begin{pmatrix} 1 & \sigma_{2,1}\widetilde{W}(t) \\ 0 & 1 \end{pmatrix}, \quad t \geqslant 0,$$

where $(W(t))_{t\geqslant 0}$ and $(\widetilde{W}(t))_{t\geqslant 0}$ are standard Wiener processes in $\mathbb{R}$. Clearly

$$\eta\left(\frac{\mathsf{e}^{2a_1t} - 1}{2a_1}\right) x(t) = \begin{pmatrix} \mathsf{e}^{a_1t} & a_2\frac{\mathsf{e}^{a_1t}-1}{a_1} + \sigma_{2,1}\widetilde{W}\left(\frac{\mathsf{e}^{2a_1t}-1}{2a_1}\right) \\ 0 & 1 \end{pmatrix}, \qquad t \geqslant 0.$$

Both processes

$$\left(\int_0^t \mathsf{e}^{a_1s}\,\mathsf{d}(a_2s + \sigma_{2,1}W(s))\right)_{t\geqslant 0}, \qquad \left(a_2\frac{\mathsf{e}^{a_1t} - 1}{a_1} + \sigma_{2,1}\widetilde{W}\left(\frac{\mathsf{e}^{2a_1t} - 1}{2a_1}\right)\right)_{t\geqslant 0}$$

are processes with independent increments in $\mathbb{R}$ starting from 0 and their increments on the interval $[s,t] \subset [0,\infty)$ have a normal distribution with mean $a_2\frac{\mathsf{e}^{a_1t}-\mathsf{e}^{a_1s}}{a_1}$ and variance $\sigma_{2,1}^2\frac{\mathsf{e}^{2a_1t}-\mathsf{e}^{2a_1s}}{2a_1}$, hence the assertion. The process $(\eta(t))_{t\geqslant 0}$ can be considered as the symmetric counterpart of process $(\xi(t))_{t\geqslant 0}$. In fact, $(x(t))_{t\geqslant 0}$ is a deterministic Lévy process on the affine group F, which can be considered as the shift part of the process $(\xi(t))_{t\geqslant 0}$. We note that using Trotter's formula of Hazod [27], Siebert [54] showed that the distribution of $\xi(t)$ and $\eta\left(\frac{\mathsf{e}^{2a_1t}-1}{2a_1}\right)x(t)$ coincide for all $t \geqslant 0$ in the special case $a_1 = 1$, $a_2 = 0$ and $\sigma_{2,1} = 2$.

Moreover, it can be checked that if the infinitesimal generator of a Gauss Lévy process $(\xi(t))_{t\geqslant 0}$ is of type different from (iii) then the decomposition $\xi(t) = \eta\big(c(t)\big)x(t)$, $t \geqslant 0$, does not hold with any function $c : [0,\infty) \to [0,\infty)$.

Chapter 4

Limit theorems on LCA2 groups

First we recall the most important notions and known results in the theory of probability measures on locally compact Abelian groups. Then we prove (central) limit theorems for row sums of a rowwise independent infinitesimal array of random elements with values in a locally compact Abelian group. We give a proof of Gaiser's theorem on convergence of triangular arrays [23, Satz 1.3.6], since it does not have an easy access and it is not complete (see Theorem 4.3.1). This theorem gives sufficient conditions for convergence of the row sums of a rowwise independent infinitesimal array of random elements with values in an LCA2 group, but the limit measure can not have a nondegenerate idempotent factor, i.e., a nondegenerate Haar measure on some compact subgroup as its factor.

As new results we prove necessary and sufficient conditions for convergence of the row sums of symmetric arrays and Bernoulli arrays, where the limit measure can also be a nondegenerate normalized Haar measure on a compact subgroup (see Theorem 4.4.2 and Theorem 4.5.1). Then we investigate special LCA2 groups: the torus group (see Section 4.6), the group of p-adic integers (see Section 4.7) and the p-adic solenoid (see Section 4.8).

Besides proving limit theorems, we give a construction of an arbitrary weakly infinitely divisible probability measure on the torus group and the group of p-adic integers (see Theorems 4.6.4 and 4.7.4). On the p-adic solenoid we give a construction of weakly infinitely divisible probability measures without nondegenerate idempotent factors (see Theorem 4.8.4). In our constructions we only use real valued random variables. We note that, as a special case of our results, we have a new construction of the normalized Haar measure on the group of p-adic integers and the p-adic solenoid.

The results of this chapter are contained in our submitted papers [7] and [8].

4.1 Motivation

Let G be a second countable locally compact Abelian T_0-topological group (LCA2 group). The group operation in G will be denoted by $+$. Let $\mathcal{B}(G)$ denote the σ-algebra of Borel sets in G. Let $\mathcal{M}^1(G)$ denote the set of probability measures on $\mathcal{B}(G)$. For $\mu, \nu \in \mathcal{M}^1(G)$, the *convolution* $\mu * \nu$ is the unique measure in $\mathcal{M}^1(G)$ defined by

$$(\mu * \nu)(A) := \int_G \mu(Ax^{-1})\, \nu(\mathsf{d}x), \qquad A \in \mathcal{B}(G).$$

Then $\mathcal{M}^1(G)$ is an Abelian topological semigroup with the product $(\mu, \nu) \in \mathcal{M}^1(G) \times \mathcal{M}^1(G) \mapsto \mu * \nu$ and the topology induced by weak convergence.

The main question of limit problems on G can be formulated as follows. Let $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \ldots, K_n\}$ be a triangular array of rowwise independent random elements with values in G satisfying the infinitesimality condition

$$\lim_{n\to\infty} \max_{1\leqslant k\leqslant K_n} \mathrm{P}(X_{n,k} \in G \setminus U) = 0$$

for all Borel neighbourhoods U of the identity e of G. One searches for conditions on the array so that the convergence in distribution

$$\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \mu \qquad \text{as} \quad n \to \infty$$

to a probability measure μ on G holds. For a sequence $\{X_n : n \in \mathbb{N}\}$ of random elements in G and for a probability measure μ on G, the notation $X_n \xrightarrow{\mathcal{D}} \mu$ means weak convergence $\mathrm{P}_{X_n} \xrightarrow{\mathsf{w}} \mu$ of the distributions P_{X_n} of X_n, $n \in \mathbb{N}$ towards μ. Moreover, for a random element X in G, the notation $X \stackrel{\mathcal{D}}{=} \mu$ means that the distribution P_X of X is μ.

Let $\mathcal{L}(G)$ denote the set of all possible limits of row sums of rowwise independent infinitesimal triangular arrays in G. The following problems arise:

(P1) How to parametrize the set $\mathcal{L}(G)$, i.e., to give a bijection between $\mathcal{L}(G)$ and an appropriate parameter set $\mathcal{P}(G)$;

(P2) How to associate suitable quantities q_n to the rows $\{X_{n,k} : 1 \leqslant k \leqslant K_n\}$, $n \in \mathbb{N}$ so that

$$\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \mu \quad \Longleftrightarrow \quad q_n \to q,$$

where $q \in \mathcal{P}(G)$ corresponds to the limiting distribution μ, and the convergence $q_n \to q$ is meant in an appropriate sense.

The problem (P1) has been solved by Parthasarathy (see Chapter IV, Corollary 7.1 in [46] and Remark 4.2.7 in Section 4.2). Gaiser [23] gave a partial solution to the problem (P2). His theorem (see Section 4.3) gives only some sufficient conditions for the convergence $\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \mu$, which does not include the case where μ has a nondegenerate idempotent factor, i.e., a nondegenerate Haar measure on a compact subgroup of G as its factor. For a survey of results on limit theorems on a general locally compact Abelian group, see Bingham [10].

We prove necessary and sufficient conditions for some limit theorems to hold on general locally compact Abelian groups. Our results complete the results of Gaiser [23]. In our theorems the limit measure can also be a nondegenerate normalized Haar measure on a compact subgroup of G.

We also specify our results considering some classical topological groups such as the torus group, the group of p-adic integers and the p-adic solenoid. Here we apply Gaiser's theorem as well. For completeness, we present a proof of this theorem, since Gaiser's dissertation does not have an easy access and Gaiser's proof is not complete. Concerning limit problems on totally disconnected Abelian groups, like the group of p-adic integers, we mention Telöken [57].

Besides proving limit theorems, we give a construction of an arbitrary weakly infinitely divisible probability measure on the torus group and the group of p-adic integers. On the p-adic solenoid we give a construction of weakly infinitely divisible probability measures without nondegenerate idempotent factors. In our constructions we only use real valued random variables. Let us consider a probability measure μ on G and an infinitesimal rowwise independent array $\{X_{n,k} : n \in \mathbb{N}, k = 1, \ldots, K_n\}$ of random elements with values in G. If the row sums $\sum_{k=1}^{K_n} X_{n,k}$ of this array converge in distribution to μ then μ is necessarily weakly infinitely divisible (see, e.g., Parthasarathy [46, Chapter IV, Theorem 5.2]). Moreover, Parthasarathy [46, Chapter IV, Corollary 7.1] gives a representation of an arbitrary weakly infinitely divisible probability measure on G in terms of a Haar measure, a Dirac measure, a symmetric Gauss measure and a generalized Poisson measure on G (for the definitions, see Section 4.2).

In this chapter we consider special cases: the torus group, the group of p-adic integers and the p-adic solenoid. For each of the three groups, first we find a measurable homomorphism φ from an appropriate Abelian topological group (which is a certain product of some subgroups of $\mathbb{R}$) onto the group in question. Then we consider an arbitrary weakly infinitely divisible probability measure μ on the group in question (without a nondegenerate idempotent factor in case of the p-adic solenoid) and we find real valued random variables $Z_0, Z_1, \ldots$ such that the distribution of $\varphi(Z_0, Z_1, \ldots)$ is μ. Since φ is a homomorphism, the building blocks of μ (Haar measure, Dirac measure, symmetric Gauss measure and generalized Poisson measure) can be handled separately.

We note that, as a special case of our results, we have a new construction of the normalized Haar measure on the group of p-adic integers and the p-adic solenoid. Another kind of description of the normalized Haar measure on the group of p-adic integers can also be found in Hewitt and Ross [29, p. 220]. One can find a construction of the normalized Haar measure on the p-adic solenoid in Chistyakov [14, Section 3]. It is based on Hausdorff measures and rather sophisticated, while our simpler construction (see Theorem 4.8.4) is based on a probabilistic method and reflects the structure of the p-adic solenoid.

4.2 Parametrization of weakly infinitely divisible measures

Let $\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the set of nonnegative integers ant the set of nonnegative real numbers, respectively. The expression "a measure μ on G" means a measure μ on the σ-algebra of Borel subsets of G, i.e., on $\mathcal{B}(G)$. The Dirac measure at a point $x \in G$ will be denoted by δ_x.

4.2.1 Definition. A probability measure μ on G is called **infinitely divisible** if for all $n \in \mathbb{N}$ there exists a probability measure μ_n on G such that $\mu = \mu_n^{*n}$, where μ_n^{*n} denotes the n-times convolution.

4.2.2 Definition. A probability measure μ on G is called **weakly infinitely divisible** if for all $n \in \mathbb{N}$ there exist a probability measure μ_n on G and an element $x_n \in G$ such that $\mu = \mu_n^{*n} * \delta_{x_n}$. The collection of all weakly infinitely divisible measures on G will be denoted by $\mathcal{I}_{\mathsf{w}}(G)$.

According to Parthasarathy [46, Chapter IV, Theorem 5.2], $\mathcal{L}(G) \subset \mathcal{I}_{\mathsf{w}}(G)$. Now we recall the building blocks of weakly infinitely divisible measures. The main tool for their description is the Fourier transform. A function $\chi : G \to \mathbb{C}$ is said to be a *character* of G if it is bounded, continuous, not identically zero and $\chi(x+y) = \chi(x)\chi(y)$ for all $x, y \in G$. Note that $|\chi(g)| = 1$ for all characters χ of G and for all $g \in G$. The group of all characters of G is called the character group of G and is denoted by $\widehat{G}$. The character group $\widehat{G}$ of G is also a second countable locally compact Abelian T_0-topological group (see Theorems 23.15 and 24.14 in Hewitt–Ross [29]). For every bounded measure μ on G, let $\widehat{\mu} : \widehat{G} \to \mathbb{C}$ be defined by

$$\widehat{\mu}(\chi) := \int_G \chi \, \mathsf{d}\mu, \qquad \chi \in \widehat{G}.$$

This function $\widehat{\mu}$ is called the *Fourier transform* of μ. Note that for each character $\chi \in \widehat{G}$, the mapping $x \in G \mapsto T_{\chi(x)}$, where $T_{\chi(x)}(z) := \chi(x)z$, $z \in \mathbb{C}, x \in G$, is a one-dimensional unitary representation of G in the group of unitary operators of $\mathbb{C}$. Hence the definition of the Fourier transform of a measure on a locally compact Abelian group is in accordance with the definition of the Fourier transform of a measure on a general locally compact group. The basic properties of the Fourier transformation can be found, e.g., in Heyer [30, Theorem 1.3.8, Theorem 1.4.2], in Hewitt and Ross [29, Theorem 23.10] and in Parthasarathy [46, Chapter IV, Theorem 3.3]. We only mention that the Fourier transformation is injective.

If H is a compact subgroup of G then ω_H will denote the Haar measure on H (considered as a measure on G) normalized by the requirement $\omega_H(H) = 1$. The normalized Haar measures of compact subgroups of G are the only idempotents in the semigroup of probability measures on G (see, e.g., Wendel [60, Theorem 1]). It can be checked that for all $\chi \in \widehat{G}$,

$$\widehat{\omega}_H(\chi) = \begin{cases} 1 & \text{if } \chi(x) = 1 \text{ for all } x \in H, \\ 0 & \text{otherwise,} \end{cases} \tag{4.2.1}$$

i.e., $\widehat{\omega}_H = \mathbb{1}_{H^\perp}$, where

$$H^\perp := \big\{ \chi \in \widehat{G} : \chi(x) = 1 \text{ for all } x \in H \big\}$$

is the annihilator of H. Clearly $\omega_H \in \mathcal{I}_{\mathsf{w}}(G)$, since $\omega_H * \omega_H = \omega_H$. Sazonov and Tutubalin [51] proved that $\omega_H \in \mathcal{L}(G)$.

Obviously $\delta_x \in \mathcal{I}_{\mathsf{w}}(G)$ for all $x \in G$, and one can easily check that $\delta_x \in \mathcal{L}(G)$ for all $x \in G^{\mathrm{arc}}$, where G^{arc} denotes the arc-component of the identity e. By the arc-component G^{arc} of e we mean

$$G^{\mathrm{arc}} := \bigcup \Big\{ \varphi(\mathbb{R}) : \varphi \in \mathrm{Hom}(\mathbb{R}, G) \Big\},$$

where $\mathrm{Hom}(\mathbb{R}, G)$ denotes the set of all continuous homomorphisms from the additive group $\mathbb{R}$ into G.

A *quadratic form* on $\widehat{G}$ is a nonnegative continuous function $\psi : \widehat{G} \to \mathbb{R}_+$ such that

$$\psi(\chi_1\chi_2) + \psi(\chi_1\chi_2^{-1}) = 2(\psi(\chi_1) + \psi(\chi_2)) \qquad \text{for all } \chi_1, \chi_2 \in \widehat{G}.$$

The set of all quadratic forms on $\widehat{G}$ will be denoted by $\mathsf{q}_+(\widehat{G})$.

For any quadratic form $\psi \in \mathsf{q}_+(\widehat{G})$, there exists a unique probability measure γ_ψ on G determined by

$$\widehat{\gamma_\psi}(\chi) = \mathsf{e}^{-\psi(\chi)/2} \qquad \text{for all } \chi \in \widehat{G},$$

see, e.g., Theorem 5.2.8 in Heyer [30]. We check that γ_ψ is a symmetric Gauss measure on G (in the sense of the definition of a Gauss measure on a (not necessarily Abelian) locally compact group given in Section 3.1 in Chapter 3). Theorem 3.7 in Heyer–Pap [31] implies that if ν is a probability measure on G such that there exists a quadratic form $\psi_\nu \in \mathsf{q}_+(\widehat{G})$ and a continuously embeddable element $m_\nu \in G$ with

$$\widehat{\nu}(\chi) = \chi(m_\nu)\mathsf{e}^{-\psi_\nu(\chi)/2} \qquad \text{for all } \chi \in \widehat{G},$$

then ν is a Gauss measure on G. Using that the identity e of G is continuously embeddable into the continuous one-parameter subsemigroup $(x_t)_{t\geqslant 0}$ in G, where $x_t = e$ for all $t \geqslant 0$, and $\chi(e) = 1$ for all $\chi \in \widehat{G}$, we obtain that γ_ψ is a Gauss measure on G. To prove the symmetry of γ_ψ, by definition, we have to check that $\gamma_\psi^\star = \gamma_\psi$, where $\gamma_\psi^\star(B) := \gamma_\psi(B^{-1})$ for all $B \in \mathcal{B}(G)$. This follows from

$$\widehat{\gamma_\psi^\star}(\chi) = \overline{\widehat{\gamma_\psi}(\chi)} = \widehat{\gamma_\psi}(\chi) \qquad \text{for all } \chi \in \widehat{G},$$

where $\overline{z}$ denotes the conjugate of an element $z \in \mathbb{C}$. Obviously $\gamma_\psi \in \mathcal{L}(G)$, since $\gamma_\psi = \gamma_{\psi/n}^{*n}$ for all $n \in \mathbb{N}$ and $\gamma_{\psi/n} \xrightarrow{\mathsf{w}} \delta_e$ as $n \to \infty$. (Recall that $\xrightarrow{\mathsf{w}}$ denotes weak convergence of bounded measures on G.)

For a bounded measure η on G, the *compound Poisson measure* $\mathsf{e}(\eta)$ is the probability measure on G defined by

$$\mathsf{e}(\eta) := \mathsf{e}^{-\eta(G)} \left(\delta_e + \eta + \frac{\eta * \eta}{2!} + \frac{\eta * \eta * \eta}{3!} + \cdots \right).$$

The Fourier transform of a compound Poisson measure $\mathsf{e}(\eta)$ is

$$(\mathsf{e}(\eta))\widehat{\ }(\chi) = \exp\left\{ \int_G (\chi(x) - 1)\, \eta(\mathsf{d}x) \right\}, \qquad \chi \in \widehat{G}. \tag{4.2.2}$$

Clearly $\mathsf{e}(\eta) \in \mathcal{L}(G)$, since $\mathsf{e}(\eta) = \big(\mathsf{e}(\eta/n)\big)^{*n}$ for all $n \in \mathbb{N}$ and $\mathsf{e}(\eta/n) \xrightarrow{\mathsf{w}} \delta_e$ as $n \to \infty$. In order to introduce generalized Poisson measures, we recall the notions of a local inner product and a Lévy measure. Let $\mathcal{N}_e$ denote the collection of all Borel neighbourhoods of e.

4.2.3 Definition. A continuous function $g : G \times \widehat{G} \to \mathbb{R}$ is called a **local inner product** for G if

(i) for every compact subset C of $\widehat{G}$, there exists $U \in \mathcal{N}_e$ such that

$$\chi(x) = \mathsf{e}^{ig(x,\chi)} \qquad \text{for all } x \in U, \quad \chi \in C,$$

(ii) for all $x \in G$ and $\chi, \chi_1, \chi_2 \in \widehat{G}$,

$$g(x, \chi_1\chi_2) = g(x, \chi_1) + g(x, \chi_2), \qquad g(-x, \chi) = -g(x, \chi),$$

(iii) for every compact subset C of $\widehat{G}$,

$$\sup_{x\in G} \sup_{\chi\in C} |g(x, \chi)| < \infty, \qquad \lim_{x\to e} \sup_{\chi\in C} |g(x, \chi)| = 0.$$

Parthasarathy [46, Chapter IV, Lemma 5.3] proved the existence of a local inner product for an arbitrary second countable locally compact Abelian T_0-topological group.

4.2.4 Definition. An extended real valued measure η on G is said to be a **Lévy measure** if $\eta(\{e\}) = 0$, $\eta(G \setminus U) < \infty$ for all $U \in \mathcal{N}_e$, and $\int_G (1 - \mathsf{Re}\,\chi(x))\, \eta(\mathsf{d}x) < \infty$ for all $\chi \in \widehat{G}$. The set of all Lévy measures on G will be denoted by $\mathbb{L}(G)$.

It can be checked that every Lévy measure on G is σ-finite. We note that for all $\chi \in \widehat{G}$ there exists $U \in \mathcal{N}_e$ such that

$$\frac{1}{4}g(x,\chi)^2 \leqslant 1 - \mathsf{Re}\,\chi(x) \leqslant \frac{1}{2}g(x,\chi)^2, \qquad x \in U, \tag{4.2.3}$$

(see, e.g., Heyer [30, p. 344]), thus the requirement $\int_G (1 - \mathsf{Re}\,\chi(x))\,\eta(\mathsf{d}x) < \infty$ can be replaced by $\int_G g(x,\chi)^2\,\eta(\mathsf{d}x) < \infty$ for some (and then necessarily for any) local inner product g.

For a Lévy measure $\eta \in \mathbb{L}(G)$ and for a local inner product g for G, the *generalized Poisson measure* $\pi_{\eta,\,g}$ is the probability measure on G defined by

$$\widehat{\pi}_{\eta,\,g}(\chi) = \exp\left\{\int_G \big(\chi(x) - 1 - ig(x,\chi)\big)\,\eta(\mathsf{d}x)\right\} \qquad \text{for all } \chi \in \widehat{G}$$

(see, e.g., Parthasarathy [46, Chapter IV, Theorem 7.1]). Obviously $\pi_{\eta,\,g} \in \mathcal{L}(G)$, since $\pi_{\eta,\,g} = \pi_{\eta/n,\,g}^{*n}$ for all $n \in \mathbb{N}$ and $\pi_{\eta/n,\,g} \xrightarrow{\mathsf{w}} \delta_e$ as $n \to \infty$.

4.2.5 Definition. For a bounded measure η on G and for a local inner product g for G, the **local mean** of η with respect to g is the uniquely defined element $m_g(\eta) \in G$ given by

$$\chi(m_g(\eta)) = \exp\left\{i\int_G g(x,\chi)\,\eta(\mathsf{d}x)\right\} \qquad \text{for all } \chi \in \widehat{G}.$$

The existence and uniqueness of a local mean is guaranteed by Pontryagin's duality theorem. If η coincides with the distribution P_X of a random element X in G, we will use the notation $m_g(X)$ instead of $m_g(\mathrm{P}_X)$. Remark that $\chi(m_g(X)) = \mathsf{e}^{i\,\mathsf{E}\,g(X,\chi)}$ for all $\chi \in \widehat{G}$.

Note that for a bounded measure η on G with $\eta(\{e\}) = 0$ we have $\eta \in \mathbb{L}(G)$ and $\mathsf{e}(\eta) = \pi_{\eta,\,g} * \delta_{m_g(\eta)}$.

Let $\mathcal{P}(G)$ be the set of all quadruplets (H, a, ψ, η), where H is a compact subgroup of G, $a \in G$, $\psi \in \mathsf{q}_+(\widehat{G})$ and $\eta \in \mathbb{L}(G)$. Parthasarathy [46, Chapter IV, Corollary 7.1] proved the following parametrization for weakly infinitely divisible measures on G.

4.2.6 Theorem. (Parthasarathy) *Let g be a fixed local inner product for G. If $\mu \in \mathcal{I}_{\mathsf{w}}(G)$ then there exists a quadruplet $(H, a, \psi, \eta) \in \mathcal{P}(G)$ such that*

$$\mu = \omega_H * \delta_a * \gamma_\psi * \pi_{\eta,\,g}. \tag{4.2.4}$$

Conversely, if $(H, a, \psi, \eta) \in \mathcal{P}(G)$ then $\omega_H * \delta_a * \gamma_\psi * \pi_{\eta,\,g} \in \mathcal{I}_{\mathsf{w}}(G)$.

4.2.7 Remark. In general, this parametrization is not one-to-one (see Parthasarathy [46, p.112, Remark 3]), but the compact subgroup H is uniquely determined in (4.2.4) by μ (more precisely, H is the annihilator of the open subgroup $\{\chi \in \widehat{G} : \widehat{\mu}(\chi) \neq 0\}$). If $H = \{e\}$ then the quadratic form ψ in (4.2.4) is also uniquely determined by μ. In order to obtain one-to-one parametrization one can take equivalence classes of quadruplets related to the equivalence relation $\approx$ defined by

$$(H, a_1, \psi_1, \eta_1) \approx (H, a_2, \psi_2, \eta_2) \iff \omega_H * \delta_{a_1} * \gamma_{\psi_1} * \pi_{\eta_1,\, g} = \omega_H * \delta_{a_2} * \gamma_{\psi_2} * \pi_{\eta_2,\, g}.$$

4.3 Gaiser's limit theorem

Let $\mathcal{C}(G)$, $\mathcal{C}_0(G)$ and $\mathcal{C}_0^{\mathrm{u}}(G)$ denote the spaces of real valued bounded continuous functions on G, the set of all functions in $\mathcal{C}(G)$ vanishing in some $U \in \mathcal{N}_e$, and the set of all uniformly continuous functions in $\mathcal{C}_0(G)$, respectively. Gaiser [23, Satz 1.3.6] proved the following limit theorem.

4.3.1 Theorem. (Gaiser) *Let g be a fixed local inner product for G. Let $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \ldots, K_n\}$ be a rowwise independent infinitesimal array of random elements in G. Suppose that there exists a quadruplet $(\{e\}, a, \psi, \eta) \in \mathcal{P}(G)$ such that*

(i) $\displaystyle\sum_{k=1}^{K_n} m_g(X_{n,k}) \to a$ *as* $n \to \infty$,

(ii) $\displaystyle\sum_{k=1}^{K_n} \operatorname{Var} g(X_{n,k}, \chi) \to \psi(\chi) + \int_G g(x,\chi)^2\, \eta(\mathrm{d}x)$ *as* $n \to \infty$ *for all* $\chi \in \widehat{G}$,

(iii) $\displaystyle\sum_{k=1}^{K_n} \mathsf{E}\, f(X_{n,k}) \to \int_G f \,\mathrm{d}\eta$ *as* $n \to \infty$ *for all* $f \in \mathcal{C}_0(G)$.

Then

$$\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \delta_a * \gamma_\psi * \pi_{\eta,\, g} \qquad \text{as } n \to \infty. \tag{4.3.1}$$

4.3.2 Remark. If either $a \neq e$ or $\psi \neq 0$ or $\eta \neq 0$ then the infinitesimality of $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \ldots, K_n\}$ and (4.3.1) imply $K_n \to \infty$.

4.3.3 Remark. Condition (i) is equivalent to

$$\text{(i}'\text{)}\quad \exp\left\{ i \sum_{k=1}^{K_n} \mathsf{E}\, g(X_{n,k}, \chi) \right\} \to \chi(a) \quad \text{as} \quad n \to \infty \quad \text{for all} \quad \chi \in \widehat{G}.$$

Concerning condition (iii) we mention the following version of the well-known portmanteau theorem.

4.3.4 Theorem. *Let* $\{\eta_n : n \in \mathbb{Z}_+\}$ *be a sequence of extended real valued measures on* G *such that* $\eta_n(G \setminus U) < \infty$ *for all* $U \in \mathcal{N}_e$ *and for all* $n \in \mathbb{Z}_+$*. Then the following assertions are equivalent:*

(a) $\displaystyle\int_G f \,\mathsf{d}\eta_n \to \int_G f \,\mathsf{d}\eta_0$ *as* $n \to \infty$ *for all* $f \in \mathcal{C}_0(G)$,

(b) $\displaystyle\int_G f \,\mathsf{d}\eta_n \to \int_G f \,\mathsf{d}\eta_0$ *as* $n \to \infty$ *for all* $f \in \mathcal{C}_0^{\mathrm{u}}(G)$,

(c) $\eta_n(G \setminus U) \to \eta_0(G \setminus U)$ *as* $n \to \infty$ *for all* $U \in \mathcal{N}_e$ *with* $\eta_0(\partial U) = 0$,

(d) $\displaystyle\int_{G\setminus U} f \,\mathsf{d}\eta_n \to \int_{G\setminus U} f \,\mathsf{d}\eta_0$ *as* $n \to \infty$ *for all* $f \in \mathcal{C}(G)$, $U \in \mathcal{N}_e$ *with* $\eta_0(\partial U) = 0$,

(e) $\eta_n|_{G\setminus U} \xrightarrow{\mathsf{w}} \eta_0|_{G\setminus U}$ *as* $n \to \infty$ *for all* $U \in \mathcal{N}_e$ *with* $\eta_0(\partial U) = 0$.

(Here and in the sequel $\eta|_B$ denotes the restriction of a measure η onto a Borel subset B of G, considered as a measure on G.)

For the proof of Theorem 4.3.4, see Theorem 5.2.1 and Remark 5.2.2 in Chapter 5. Theorem 4.3.4 is a consequence of Theorem 5.2.1 in Chapter 5.

4.3.5 Remark. Due to Theorem 4.3.4, condition (iii) of Theorem 4.3.1 is equivalent to

(iii$'$) $\displaystyle\sum_{k=1}^{K_n} \mathrm{P}(X_{n,k} \in G \setminus U) \to \eta(G \setminus U)$ as $n \to \infty$ for all $U \in \mathcal{N}_e$ with $\eta(\partial U) = 0$.

In order to prove Theorem 4.3.1, first we recall a theorem about convergence of weakly infinitely divisible measures without idempotent factors (see Gaiser [23, Satz 1.2.1]).

4.3.6 Theorem. *For each* $n \in \mathbb{Z}_+$, *let* $\mu_n \in \mathcal{I}_{\mathsf{w}}(G)$ *be such that* (4.2.4) *holds for* μ_n *with a quadruplet* $(\{e\}, a_n, \psi_n, \eta_n)$. *If there exists a local inner product* g *for* G *such that*

(i) $a_n \to a_0$ *as* $n \to \infty$,

(ii) $\psi_n(\chi) + \int_G g(x,\chi)^2 \, \eta_n(\mathsf{d}x) \to \psi_0(\chi) + \int_G g(x,\chi)^2 \, \eta_0(\mathsf{d}x)$ *as* $n \to \infty$ *for all* $\chi \in \widehat{G}$,

(iii) $\int_G f \, \mathsf{d}\eta_n \to \int_G f \, \mathsf{d}\eta_0$ *as* $n \to \infty$ *for all* $f \in \mathcal{C}_0(G)$,

then $\mu_n \xrightarrow{\mathsf{w}} \mu_0$ *as* $n \to \infty$.

Proof. It suffices to show $\widehat{\mu}_n(\chi) \to \widehat{\mu}_0(\chi)$ as $n \to \infty$ for all $\chi \in \widehat{G}$. Let

$$h(x,\chi) := \chi(x) - 1 - ig(x,\chi) + \frac{1}{2} g(x,\chi)^2$$

for all $x \in G$ and all $\chi \in \widehat{G}$. Then

$$\widehat{\mu}_n(\chi) = \chi(a_n) \exp\left\{ -\frac{1}{2}\left(\psi_n(\chi) + \int_G g(x,\chi)^2 \, \eta_n(\mathsf{d}x) \right) + \int_G h(x,\chi) \, \eta_n(\mathsf{d}x) \right\}$$

for all $n \in \mathbb{Z}_+$ and all $\chi \in \widehat{G}$. Taking into account assumptions (i) and (ii), it is enough to show that

$$\int_G h(x,\chi) \, \eta_n(\mathsf{d}x) \to \int_G h(x,\chi) \, \eta_0(\mathsf{d}x) \quad \text{as } n \to \infty \text{ for all } \chi \in \widehat{G}. \tag{4.3.2}$$

For each $\chi \in \widehat{G}$, there exists $U \in \mathcal{N}_e$ such that $\chi(x) = \mathsf{e}^{ig(x,\chi)}$ for all $x \in U$. Using the inequality

$$\left| \mathsf{e}^{iy} - 1 - iy + \frac{y^2}{2} \right| \leqslant \frac{|y|^3}{6} \qquad \text{for all } y \in \mathbb{R}, \tag{4.3.3}$$

we obtain $|h(x,\chi)| \leqslant |g(x,\chi)|^3/6$ for all $x \in U$. Consequently, for all $V \in \mathcal{N}_e$ with $V \subset U$,

$$\left| \int_G h(x,\chi) \, \eta_n(\mathsf{d}x) - \int_G h(x,\chi) \, \eta_0(\mathsf{d}x) \right| \leqslant I_n^{(1)}(V) + I_n^{(2)}(V),$$

where

$$I_n^{(1)}(V) := \frac{1}{6} \int_V |g(x,\chi)|^3 \, (\eta_n + \eta_0)(\mathsf{d}x),$$

$$I_n^{(2)}(V) := \left| \int_{G \setminus V} h(x,\chi) \, \eta_n(\mathsf{d}x) - \int_{G \setminus V} h(x,\chi) \, \eta_0(\mathsf{d}x) \right|.$$

We have

$$I_n^{(1)}(V) \leqslant \frac{1}{6} \sup_{x \in V} |g(x,\chi)| \int_V g(x,\chi)^2 \, (\eta_n + \eta_0)(\mathsf{d}x).$$

By assumption (ii),

$$\sup_{n \in \mathbb{Z}_+} \int_V g(x,\chi)^2 \, \eta_n(\mathsf{d}x) \leqslant \sup_{n \in \mathbb{Z}_+} \left(\psi_n(\chi) + \int_G g(x,\chi)^2 \, \eta_n(\mathsf{d}x) \right) < \infty.$$

Theorem 8.3 in Hewitt and Ross [29] yields existence of a metric d on G compatible with the topology of G. The function $t \mapsto \eta_0(\{x \in G : d(x,e) \geqslant t\})$ from $(0,\infty)$ into $\mathbb{R}$ is non-increasing, hence the set $\big\{t \in (0,\infty) : \eta_0(\{x \in G : d(x,e) = t\}) > 0\big\}$ of its discontinuities is countable. Consequently, for arbitrary $\varepsilon > 0$, there exists $t > 0$ such that $V_1 := \{x \in G : d(x,e) < t\} \in \mathcal{N}_e$, $V_1 \subset U$, $\eta_0(\partial V_1) = 0$ and

$$\sup_{y \in V_1} |g(x,\chi)| < \frac{3\varepsilon}{2 \sup\limits_{n \in \mathbb{Z}_+} \int_V g(x,\chi)^2 \, \eta_n(\mathsf{d}x)},$$

thus $I_n^{(1)}(V_1) < \varepsilon/2$. By assumption (iii) and Theorem 4.3.4, $I_n^{(2)}(V_1) < \varepsilon/2$ for all sufficiently large n, hence we obtain

$$\left| \int_G h(x,\chi) \, \eta_n(\mathsf{d}x) - \int_G h(x,\chi) \, \eta_0(\mathsf{d}x) \right| < \varepsilon$$

for all sufficiently large n, which implies (4.3.2). □

The notion of a special local inner product is also needed.

4.3.7 Definition. A local inner product g for G is called **special** if it is uniformly continuous in its first variable, i.e., if for all $\chi \in \widehat{G}$ and for all $\varepsilon > 0$ there exists $U \in \mathcal{N}_e$ such that $|g(x,\chi) - g(y,\chi)| < \varepsilon$ for all $x, y \in G$ with $x - y \in U$.

Gaiser [23, Satz 1.1.4] proved the existence of a special local inner product for an arbitrary second countable locally compact Abelian T_0-topological group. The proof goes along the lines of the proof of the existence of a local inner product in Heyer [30, Theorem 5.1.10].

Proof of Theorem 4.3.1. First we show that it is enough to prove the statement for a special local inner product, namely, to prove that if the statement is true for some local inner product g, then it is true for any local inner product $\widetilde{g}$. Suppose that assumptions (i)–(iii) hold for $\widetilde{g}$ with a quadruplet $(\{e\}, a, \psi, \eta)$. We show that they hold for g with the quadruplet $(\{e\}, a + m_{g,\, g}(\eta), \psi, \eta)$, where the element $m_{g,\, g}(\eta) \in G$ is uniquely determined by

$$\chi(m_{g,\, g}(\eta)) = \exp\left\{ i \int_G (g(x,\chi) - \widetilde{g}(x,\chi))\, \eta(\mathsf{d}x) \right\} \qquad \text{for all } \chi \in \widehat{G}.$$

(Note that $g(\cdot,\chi) - \widetilde{g}(\cdot,\chi) \in \mathcal{C}_0(G)$ can be checked easily.) Hence we want to prove

$$\text{(i}'\text{)} \quad \sum_{k=1}^{K_n} m_g(X_{n,k}) \to a + m_{g,\, g}(\eta) \quad \text{as } n \to \infty,$$

$$\text{(ii}'\text{)} \quad \sum_{k=1}^{K_n} \operatorname{Var} g(X_{n,k},\chi) \to \psi(\chi) + \int_G g(x,\chi)^2\, \eta(\mathsf{d}x) \quad \text{as } n \to \infty \text{ for all } \chi \in \widehat{G},$$

$$\text{(iii}'\text{)} \quad \sum_{k=1}^{K_n} \mathsf{E}\, f(X_{n,k}) \to \int_G f\, \mathsf{d}\eta \quad \text{as } n \to \infty \text{ for all } f \in \mathcal{C}_0(G).$$

Clearly (iii$'$) holds, since it is identical with assumption (iii).

By assumption (i), in order to prove (i$'$) we have to show

$$\chi\left(\sum_{k=1}^{K_n} m_g(X_{n,k}) - \sum_{k=1}^{K_n} m_g(X_{n,k}) \right) \to \chi(m_{g,\, g}(\eta)) \qquad \text{for all } \chi \in \widehat{G}.$$

We have

$$\chi\left(\sum_{k=1}^{K_n} m_g(X_{n,k}) - \sum_{k=1}^{K_n} m_g(X_{n,k})\right) = \prod_{k=1}^{K_n} \frac{\chi(m_g(X_{n,k}))}{\chi(m_g(X_{n,k}))}$$

$$= \prod_{k=1}^{K_n} \frac{\mathsf{e}^{i\mathsf{E}\, g(X_{n,k},\chi)}}{\mathsf{e}^{i\mathsf{E}\, g(X_{n,k},\chi)}} = \exp\left\{ i \sum_{k=1}^{K_n} \mathsf{E}\big(g(X_{n,k},\chi) - \widetilde{g}(X_{n,k},\chi)\big)\right\}$$

$$\to \exp\left\{ i \int_G (g(x,\chi) - \widetilde{g}(x,\chi))\, \eta(\mathsf{d}x)\right\},$$

where we applied assumption (iii) with the function $g(\cdot,\chi) - \widetilde{g}(\cdot,\chi) \in \mathcal{C}_0(G)$.

By assumption (ii), in order to prove (ii$'$) we have to show

$$\sum_{k=1}^{K_n} \operatorname{Var} g(X_{n,k},\chi) - \sum_{k=1}^{K_n} \operatorname{Var} \widetilde{g}(X_{n,k},\chi) \to \int_G \big(g(x,\chi)^2 - \widetilde{g}(x,\chi)^2\big)\, \eta(\mathsf{d}x) \tag{4.3.4}$$

for all $\chi \in \widehat{G}$, where $g(\cdot,\chi)^2 - \widetilde{g}(\cdot,\chi)^2 \in \mathcal{C}_0(G)$ can be checked easily. We have

$$\sum_{k=1}^{K_n} \operatorname{Var} g(X_{n,k},\chi) - \sum_{k=1}^{K_n} \operatorname{Var} \widetilde{g}(X_{n,k},\chi) = A_n - B_n,$$

where

$$A_n := \sum_{k=1}^{K_n} \mathsf{E}\,\big(g(X_{n,k},\chi)^2 - \widetilde{g}(X_{n,k},\chi)^2\big),$$

$$B_n := \sum_{k=1}^{K_n} \big[(\mathsf{E}\, g(X_{n,k},\chi))^2 - (\mathsf{E}\, \widetilde{g}(X_{n,k},\chi))^2\big].$$

Applying assumption (iii) with the function $g(\cdot,\chi)^2 - \widetilde{g}(\cdot,\chi)^2 \in \mathcal{C}_0(G)$, we obtain

$$A_n \to \int_G \big(g(x,\chi)^2 - \widetilde{g}(x,\chi)^2\big)\, \eta(\mathsf{d}x). \tag{4.3.5}$$

Moreover,

$$B_n = \sum_{k=1}^{K_n} \mathsf{E}\big(g(X_{n,k},\chi) - \widetilde{g}(X_{n,k},\chi)\big)\, \mathsf{E}\big(g(X_{n,k},\chi) + \widetilde{g}(X_{n,k},\chi)\big)$$

implies

$$|B_n| \leqslant \max_{1\leqslant k\leqslant K_n} \mathsf{E}\big(|g(X_{n,k},\chi)| + |\widetilde{g}(X_{n,k},\chi)|\big) \sum_{k=1}^{K_n} \mathsf{E}\,|g(X_{n,k},\chi) - \widetilde{g}(X_{n,k},\chi)|.$$

Using assumption (iii) with the function $|g(\cdot,\chi) - \widetilde{g}(\cdot,\chi)| \in \mathcal{C}_0(G)$, we get

$$\sum_{k=1}^{K_n} \mathsf{E}\,|g(X_{n,k},\chi) - \widetilde{g}(X_{n,k},\chi)| \to \int_G |g(x,\chi) - \widetilde{g}(x,\chi)|\,\eta(\mathsf{d}x). \tag{4.3.6}$$

Infinitesimality of $\{X_{n,k} : n \in \mathbb{N},\, k = 1,\dots,K_n\}$ implies

$$\max_{1\leqslant k\leqslant K_n} \mathsf{E}\,|g(X_{n,k},\chi)| \to 0 \qquad \text{for all } \ \chi \in \widehat{G}. \tag{4.3.7}$$

Indeed,

$$\max_{1\leqslant k\leqslant K_n} \mathsf{E}\,|g(X_{n,k},\chi)| \leqslant \sup_{x\in U} |g(x,\chi)| + \sup_{x\in G} |g(x,\chi)| \cdot \max_{1\leqslant k\leqslant K_n} \mathrm{P}(X_{n,k} \in G\setminus U)$$

for all $U \in \mathcal{N}_e$ and for all $\chi \in \widehat{G}$, and (iii) of Definition 4.2.3 implies $\sup_{x\in U}|g(x,\chi)| \to 0$ as $U \downarrow \{e\}$. Clearly (4.3.6) and (4.3.7) imply $B_n \to 0$, hence, by (4.3.5), we obtain (4.3.4).

We conclude that assumptions (i)–(iii) hold for the local inner product g with the quadruplet $(\{e\}, a + m_{g,\,g}(\eta), \psi, \eta)$. Since we supposed that the statement is true for g, we get

$$\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \delta_{a+m_{g,\,g}(\eta)} * \gamma_\psi * \pi_{\eta,\,g}.$$

Hence

$$\begin{aligned}\mathsf{E}\chi\Bigg(\sum_{k=1}^{K_n} X_{n,k}\Bigg) &\to \chi(a + m_{g,\,g}(\eta)) \exp\Bigg\{-\frac{1}{2}\psi(\chi) + \int_G \big(\chi(x) - 1 - ig(x,\chi)\big)\,\eta(\mathsf{d}x)\Bigg\}\\ &= \chi(a) \exp\Bigg\{-\frac{1}{2}\psi(\chi) + \int_G \big(\chi(x) - 1 - i\widetilde{g}(x,\chi)\big)\,\eta(\mathsf{d}x)\Bigg\}\end{aligned}$$

for all $\chi \in \widehat{G}$, which implies

$$\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \delta_a * \gamma_\psi * \pi_{\eta,\,g}.$$

Thus we may suppose that g is a special local inner product. Let $Y_{n,k} := X_{n,k} - m_g(X_{n,k})$ for all $n \in \mathbb{N}$, $k = 1, \dots, K_n$. We show that $\{Y_{n,k} : n \in \mathbb{N},\, k = 1, \dots, K_n\}$ is an infinitesimal array of rowwise independent random elements in G, and

(i″) $\displaystyle\sum_{k=1}^{K_n} m_g(Y_{n,k}) \to e$ as $n \to \infty$,

(ii″) $\displaystyle\sum_{k=1}^{K_n} \mathsf{E}\big(g(Y_{n,k},\chi)^2\big) \to \psi(\chi) + \int_G g(x,\chi)^2\,\eta(\mathsf{d}x)$ as $n \to \infty$ for all $\chi \in \widehat{G}$,

(iii″) $\displaystyle\sum_{k=1}^{K_n} \mathsf{E}\, f(Y_{n,k}) \to \int_G f\,\mathsf{d}\eta$ as $n \to \infty$ for all $f \in \mathcal{C}_0(G)$.

Infinitesimality of $\{Y_{n,k} : n \in \mathbb{N},\, k = 1, \dots, K_n\}$ is equivalent to

$$\max_{1 \leqslant k \leqslant K_n} |\mathsf{E}\,\chi(Y_{n,k}) - 1| \to 0 \qquad \text{for all } \chi \in \widehat{G}. \tag{4.3.8}$$

We have

$$\begin{aligned} |\mathsf{E}\,\chi(Y_{n,k}) - 1| &= \left| \frac{\mathsf{E}\,\chi(X_{n,k})}{\chi(m_g(X_{n,k}))} - 1 \right| = \left| \frac{\mathsf{E}\,\chi(X_{n,k})}{\mathsf{e}^{i\mathsf{E}\,g(X_{n,k},\chi)}} - 1 \right| \\ &= \left| \mathsf{E}\,\chi(X_{n,k}) - \mathsf{e}^{i\mathsf{E}\,g(X_{n,k},\chi)} \right| \leqslant |\mathsf{E}\,\chi(X_{n,k}) - 1| + \left| \mathsf{e}^{i\mathsf{E}\,g(X_{n,k},\chi)} - 1 \right|. \end{aligned}$$

Infinitesimality of $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \dots, K_n\}$ implies

$$\max_{1 \leqslant k \leqslant K_n} |\mathsf{E}\,\chi(X_{n,k}) - 1| \to 0 \qquad \text{for all } \chi \in \widehat{G}. \tag{4.3.9}$$

Infinitesimality of $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \dots, K_n\}$ implies (4.3.7) as well, hence using the inequality $|\mathsf{e}^{iy} - 1| \leqslant |y|$ for all $y \in \mathbb{R}$, we get

$$\max_{1 \leqslant k \leqslant K_n} \left| \mathsf{e}^{i\mathsf{E}\,g(X_{n,k},\chi)} - 1 \right| \to 0 \qquad \text{for all } \chi \in \widehat{G},$$

and we obtain (4.3.8).

For (i″), it is enough to show

$$\sum_{k=1}^{K_n} \mathsf{E}\, g(Y_{n,k},\chi) \to 0 \qquad \text{for all } \chi \in \widehat{G}.$$

Let $\chi \in \widehat{G}$ be fixed. Infinitesimality of $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \ldots, K_n\}$ implies that for all $V \in \mathcal{N}_e$ and for all sufficiently large n we have $m_g(X_{n,k}) \in V$ for $k = 1, \ldots, K_n$. Consequently, using (4.3.7) as well, we conclude that for all sufficiently large n we have

$$g(m_g(X_{n,k}), \chi) = \mathsf{E}\, g(X_{n,k}, \chi) \qquad \text{for } \ k = 1, \ldots, K_n. \tag{4.3.10}$$

Infinitesimality of $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \ldots, K_n\}$ and properties of the local inner product g imply also the existence of $U \in \mathcal{N}_e$ such that $\eta(\partial U) = 0$ and for all $x \in U$, $k = 1, \ldots, K_n$,

$$g(x - m_g(X_{n,k}), \chi) - g(x, \chi) = -g(m_g(X_{n,k}), \chi) \tag{4.3.11}$$

for all sufficiently large n (see Parthasarathy [46, page 91]). Consequently, for all sufficiently large n, we obtain

$$\begin{aligned}
\left|\sum_{k=1}^{K_n} \mathsf{E}\, g(Y_{n,k}, \chi)\right| &= \left|\sum_{k=1}^{K_n} \mathsf{E}\Big(g(Y_{n,k}, \chi) - g(X_{n,k}, \chi) + g(m_g(X_{n,k}), \chi)\Big) \mathbb{1}_{G\setminus U}(X_{n,k})\right| \\
&\leqslant \Big(\max_{1\leqslant k\leqslant K_n} \sup_{x\in G} |g(x - m_g(X_{n,k}), \chi) - g(x, \chi)|\Big) \sum_{k=1}^{K_n} \mathrm{P}(X_{n,k} \in G \setminus U) \\
&\quad + \max_{1\leqslant k\leqslant K_n} |g(m_g(X_{n,k}), \chi)| \sum_{k=1}^{K_n} \mathrm{P}(X_{n,k} \in G \setminus U) \to 0.
\end{aligned}$$

Indeed,

$$\max_{1\leqslant k\leqslant K_n} \sup_{x\in G} |g(x - m_g(X_{n,k}), \chi) - g(x, \chi)| \to 0 \qquad \text{as } \ n \to \infty, \tag{4.3.12}$$

since g is uniformly continuous in its first variable and for all $V \in \mathcal{N}_e$ and for all sufficiently large n we have $m_g(X_{n,k}) \in V$ for $k = 1, \ldots, K_n$. Moreover, (4.3.7) and (4.3.10) imply

$$\max_{1\leqslant k\leqslant K_n} |g(m_g(X_{n,k}), \chi)| \to 0 \qquad \text{as } \ n \to \infty, \tag{4.3.13}$$

and assumption (iii) implies

$$\sup_{n\in\mathbb{N}} \sum_{k=1}^{K_n} \mathrm{P}(X_{n,k} \in G \setminus U) < \infty. \tag{4.3.14}$$

To prove (ii$''$), we have to show

$$\sum_{k=1}^{K_n} \Big(\mathsf{E}\left(g(Y_{n,k},\chi)^2\right) - \operatorname{Var} g(X_{n,k},\chi) \Big) \to 0 \qquad \text{for all } \ \chi \in \widehat{G}.$$

Consider again a neighbourhood $U \in \mathcal{N}_e$ such that $\eta(\partial U) = 0$ and (4.3.11) holds for all sufficiently large n. We have

$$\mathsf{E}\left(g(Y_{n,k},\chi)^2\right) - \operatorname{Var} g(X_{n,k},\chi) = C_{n,k} + D_{n,k},$$

where

$$\begin{aligned} C_{n,k} &:= \mathsf{E}\Big(g(Y_{n,k},\chi)^2 - g(X_{n,k},\chi)^2\Big)\mathbb{1}_U(X_{n,k}) + \big(\mathsf{E}\, g(X_{n,k},\chi)\big)^2, \\ D_{n,k} &:= \mathsf{E}\Big(g(Y_{n,k},\chi)^2 - g(X_{n,k},\chi)^2\Big)\mathbb{1}_{G\setminus U}(X_{n,k}). \end{aligned}$$

For all sufficiently large n we have (4.3.10), hence

$$\begin{aligned} C_{n,k} &= \mathsf{E}\Big(\big(g(X_{n,k},\chi) - g(m_g(X_{n,k}),\chi)\big)^2 - g(X_{n,k},\chi)^2\Big)\mathbb{1}_U(X_{n,k}) \\ &\quad + \big(\mathsf{E}\, g(X_{n,k},\chi)\big)^2 \\ &= g(m_g(X_{n,k}),\chi)^2\, \mathrm{P}(X_{n,k} \in U) - 2g(m_g(X_{n,k}),\chi)\, \mathsf{E}\left(g(X_{n,k},\chi)\mathbb{1}_U(X_{n,k})\right) \\ &\quad + \big(\mathsf{E}\, g(X_{n,k},\chi)\big)^2 \\ &= 2\mathsf{E}\, g(X_{n,k},\chi)\, \mathsf{E}\left(g(X_{n,k},\chi)\mathbb{1}_{G\setminus U}(X_{n,k})\right) - \big(\mathsf{E}\, g(X_{n,k},\chi)\big)^2\, \mathrm{P}(X_{n,k} \in G\setminus U). \end{aligned}$$

Consequently, again by (4.3.10),

$$|C_{n,k}| \leqslant \mathrm{P}(X_{n,k} \in G\setminus U)\left(2|\mathsf{E}\, g(X_{n,k},\chi)| \sup_{x\in G}|g(x,\chi)| + |\mathsf{E}\, g(X_{n,k},\chi)|^2\right). \tag{4.3.15}$$

Moreover,

$$D_{n,k} = \mathsf{E}\left(g(Y_{n,k},\chi) - g(X_{n,k},\chi)\right)\left(g(Y_{n,k},\chi) + g(X_{n,k},\chi)\right)\mathbb{1}_{G\setminus U}(X_{n,k}),$$

thus

$$\begin{aligned} |D_{n,k}| &\leqslant 2\mathrm{P}(X_{n,k} \in G\setminus U) \sup_{x\in G}|g(x,\chi)| \\ &\qquad \times \max_{1\leqslant k\leqslant K_n} \sup_{x\in G}\big|g(x - m_g(X_{n,k}),\chi) - g(x,\chi)\big|. \end{aligned} \tag{4.3.16}$$

Now (4.3.15) and (4.3.16), using (4.3.12), (4.3.13) and (4.3.14), imply (ii$''$).

To prove (iii$''$), it is enough to show

$$\sum_{k=1}^{K_n} \mathsf{E}\, f(Y_{n,k}) - \sum_{k=1}^{K_n} \mathsf{E}\, f(X_{n,k}) \to 0 \tag{4.3.17}$$

for all $f \in \mathcal{C}_0^{\mathrm{u}}(G)$ (see Theorem 4.3.4). Choose $V \in \mathcal{N}_e$ such that $f(x) = 0$ for all $x \in V$. Then choose $U \in \mathcal{N}_e$ such that $U - U \subset V$, where $U - U := \{x - y : x, y \in U\}$. Infinitesimality of $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \dots, K_n\}$ implies that for all sufficiently large n we have $m_g(X_{n,k}) \in U$ for $k = 1, \dots, K_n$, hence

$$f(Y_{n,k}) - f(X_{n,k}) = \big(f(Y_{n,k}) - f(X_{n,k})\big)\mathbb{1}_{G\setminus U}(X_{n,k}).$$

Consequently,

$$\left|\sum_{k=1}^{K_n} \mathsf{E}\, f(Y_{n,k}) - \sum_{k=1}^{K_n} \mathsf{E}\, f(X_{n,k})\right| \leqslant \sup_{x\in G} \big|f(x - m_g(X_{n,k})) - f(x)\big| \sum_{k=1}^{K_n} \mathrm{P}(X_{n,k} \in G\setminus U),$$

and uniform continuity of f and (4.3.14) imply (4.3.17).

Now consider the shifted compound Poisson measures

$$\nu_n := \mathsf{e}\left(\sum_{k=1}^{K_n} \mathrm{P}_{Y_{n,k}}\right) * \delta_{\sum_{k=1}^{K_n} m_g(X_{n,k})}, \qquad n \in \mathbb{N}.$$

Clearly $\nu_n \in \mathcal{I}_{\mathsf{w}}(G)$ such that (4.2.4) holds for ν_n with the quadruplet

$$\left(\{e\}, \sum_{k=1}^{K_n} m_g(X_{n,k}) + \sum_{k=1}^{K_n} m_g(Y_{n,k}),\, 0,\, \sum_{k=1}^{K_n} \mathrm{P}_{Y_{n,k}}\right).$$

By Theorem 4.3.6, using (i) and (i$''$)–(iii$''$), we obtain

$$\nu_n \xrightarrow{\mathsf{w}} \delta_a * \gamma_\psi * \pi_{\eta,\, g}.$$

Applying a theorem on the accompanying Poisson array due to Parthasarathy [46, Chapter IV, Theorem 5.1], we conclude the statement. □

4.4 Limit theorems for symmetric arrays

A random element X in G is called symmetric if $X \stackrel{\mathcal{D}}{=} -X$. By a symmetric array we mean an array of symmetric random elements in G.

The following theorem is an easy consequence of Theorem 4.3.1.

4.4.1 Theorem. (CLT for symmetric array) *Let* g *be a fixed local inner product for* G. *Let* $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \dots, K_n\}$ *be a rowwise independent array of random elements in* G *such that* $X_{n,k} \stackrel{\mathcal{D}}{=} -X_{n,k}$ *for all* $n \in \mathbb{N}$, $k = 1, \dots, K_n$. *Suppose that there exists a quadratic form* ψ *on* $\widehat{G}$ *such that*

(i) $\displaystyle\sum_{k=1}^{K_n} \operatorname{Var} g(X_{n,k}, \chi) \to \psi(\chi)$ *as* $n \to \infty$ *for all* $\chi \in \widehat{G}$,

(ii) $\displaystyle\sum_{k=1}^{K_n} \mathrm{P}(X_{n,k} \in G \setminus U) \to 0$ *as* $n \to \infty$ *for all* $U \in \mathcal{N}_e$.

Then the array $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \dots, K_n\}$ *is infinitesimal and*

$$\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \gamma_\psi \qquad \text{as } n \to \infty.$$

The next theorem gives necessary and sufficient conditions in case of a rowwise independent and identically distributed (i.i.d.) symmetric array. It turns out that in this special case conditions of Theorem 4.4.1 are not only sufficient but necessary as well. If G is compact then the limit measure can be the normalized Haar measure on G.

4.4.2 Theorem. (Limit theorem for rowwise i.i.d. symmetric array) *Let* $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \dots, K_n\}$ *be an infinitesimal, rowwise i.i.d. array of random elements in* G *such that* $K_n \to \infty$ *and* $X_{n,k} \stackrel{\mathcal{D}}{=} -X_{n,k}$ *for all* $n \in \mathbb{N}$, $k = 1, \dots, K_n$.

If g *is a local inner product for* G *and* ψ *is a quadratic form on* $\widehat{G}$, *then the following statements are equivalent:*

(i) $\displaystyle\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \gamma_\psi$ *as* $n \to \infty$,

(ii) $K_n\big(1 - \mathsf{Re}\,\mathsf{E}\,\chi(X_{n,1})\big) \to \frac{\psi(\chi)}{2}$ *as* $n \to \infty$ *for all* $\chi \in \widehat{G}$,

(iii) $K_n \operatorname{Var} g(X_{n,1}, \chi) \to \psi(\chi)$ *as* $n \to \infty$ *for all* $\chi \in \widehat{G}$ *and* $K_n\, \mathrm{P}(X_{n,1} \in G \setminus U) \to 0$ *as* $n \to \infty$ *for all* $U \in \mathcal{N}_e$.

If G is compact then

$$\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \omega_G \quad \Longleftrightarrow \quad K_n\big(1 - \operatorname{Re} \mathsf{E}\, \chi(X_{n,1})\big) \to \infty \ \ \textit{for all} \ \ \chi \in \widehat{G} \setminus \{\mathbb{1}_G\}.$$

For the proof of Theorem 4.4.2, we need the following simple observation.

4.4.3 Lemma. *Let $\{\alpha_n : n \in \mathbb{N}\}$ be a sequence of real numbers such that $\alpha_n \geqslant -n$ for all sufficiently large n, and let $\alpha \in \mathbb{R} \cup \{-\infty, \infty\}$. Then*

$$\Big(1 + \frac{\alpha_n}{n}\Big)^n \to \mathsf{e}^\alpha \qquad \Longleftrightarrow \qquad \alpha_n \to \alpha,$$

where $\mathsf{e}^{-\infty} := 0$ and $\mathsf{e}^{\infty} := \infty$.

Proof. If $\alpha_n \to \alpha \in \mathbb{R}$ then $\alpha_n/n \to 0$, hence L'Hospital's rule gives

$$\log\Big[\Big(1 + \frac{\alpha_n}{n}\Big)^n\Big] = \alpha_n \cdot \frac{\log(1 + \alpha_n/n)}{\alpha_n/n} \to \alpha.$$

Now suppose that $\alpha_n \to -\infty$. By the assumptions, we can choose $n_0 \in \mathbb{N}$ such that $\alpha_n \geqslant -n$ for all $n \geqslant n_0$, hence $1 + \alpha_n/n \geqslant 0$ for all $n \geqslant n_0$, implying $\liminf\limits_{n\to\infty}(1 + \alpha_n/n)^n \geqslant 0$. For each $M \in \mathbb{R}$ there exists $n_M \in \mathbb{N}$ such that $\alpha_n \leqslant M$ for all $n \geqslant n_M$. Then $(1+\alpha_n/n)^n \leqslant (1+M/n)^n$ for all $n \geqslant \max(n_0, n_M)$, which implies

$$\limsup_{n\to\infty} \Big(1 + \frac{\alpha_n}{n}\Big)^n \leqslant \limsup_{n\to\infty} \bigg(1 + \frac{M}{n}\bigg)^n = \mathsf{e}^M.$$

Since M is arbitrary, we obtain $\limsup\limits_{n\to\infty}(1+\alpha_n/n)^n \leqslant 0$, and finally $\lim\limits_{n\to\infty}(1+\alpha_n/n)^n = 0$. The case of $\alpha_n \to \infty$ can be handled similarly.

If $(1+\alpha_n/n)^n \to \mathsf{e}^\alpha$ and $\alpha_n \not\to \alpha$ then there exist subsequences (n') and (n'') and $\alpha', \alpha'' \in \mathbb{R} \cup \{-\infty, \infty\}$ with $\alpha' \neq \alpha''$ such that $\alpha_{n'} \to \alpha'$ and $\alpha_{n''} \to \alpha''$. Then $(1 + \alpha_{n'}/n')^{n'} \to \mathsf{e}^{\alpha'}$ and $(1 + \alpha_{n''}/n'')^{n''} \to \mathsf{e}^{\alpha''}$ lead to a contradiction. $\square$

Proof of Theorem 4.4.2. (i) $\Longleftrightarrow$ (ii). Statement (i) is equivalent to

$$\mathsf{E}\,\chi\bigg(\sum_{k=1}^{K_n} X_{n,k}\bigg) \to \widehat{\gamma}_\psi(\chi) \qquad \text{for all } \ \chi \in \widehat{G}. \tag{4.4.1}$$

We have $\widehat{\gamma}_\psi(\chi) = \mathsf{e}^{-\psi(\chi)/2}$. Clearly $X_{n,k} \stackrel{\mathcal{D}}{=} -X_{n,k}$ implies $\mathsf{E}\,\chi(X_{n,k}) = \operatorname{Re}\mathsf{E}\,\chi(X_{n,k})$, hence

$$\mathsf{E}\,\chi\biggl(\sum_{k=1}^{K_n} X_{n,k}\biggr) = \bigl(\operatorname{Re}\mathsf{E}\,\chi(X_{n,1})\bigr)^{K_n} = \left(1 + \frac{K_n\bigl(\operatorname{Re}\mathsf{E}\,\chi(X_{n,1}) - 1\bigr)}{K_n}\right)^{K_n}. \tag{4.4.2}$$

Infinitesimality of $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \dots, K_n\}$ implies $\mathsf{E}\,\chi(X_{n,1}) \to 1$ (see (4.3.8)), thus $\operatorname{Re}\mathsf{E}\,\chi(X_{n,1}) - 1 \geqslant -1$ for all sufficiently large $n \in \mathbb{N}$. Hence by $K_n \to \infty$ and by Lemma 4.4.3 we conclude that (4.4.1) and (ii) are equivalent.

(ii) $\Longrightarrow$ (iii). We have already proved that (ii) implies (i), hence, by Theorem 5.4.2 in Heyer [30], (ii) implies $K_n\,\mathrm{P}(X_{n,1} \in G\backslash U) \to 0$ for all $U \in \mathcal{N}_e$. Clearly $X_{n,k} \stackrel{\mathcal{D}}{=} -X_{n,k}$ implies $\mathsf{E}\,g(X_{n,k}, \chi) = 0$, thus $\operatorname{Var} g(X_{n,1}, \chi) = \mathsf{E}\left(g(X_{n,1}, \chi)^2\right)$. Consequently, it is enough to show

$$K_n\left(\operatorname{Re}\mathsf{E}\,\chi(X_{n,1}) - 1 + \frac{1}{2}\mathsf{E}\left(g(X_{n,1}, \chi)^2\right)\right) \to 0 \qquad \text{for all } \chi \in \widehat{G}. \tag{4.4.3}$$

For $\chi \in \widehat{G}$, choose $U \in \mathcal{N}_e$ such that $\chi(x) = \mathsf{e}^{ig(x,\chi)}$ and (4.2.3) hold for all $x \in U$. Then

$$K_n\left(\operatorname{Re}\mathsf{E}\,\chi(X_{n,1}) - 1 + \frac{1}{2}\mathsf{E}\left(g(X_{n,1}, \chi)^2\right)\right) = A_n + B_n,$$

where

$$A_n := K_n \operatorname{Re}\mathsf{E}\left(\mathsf{e}^{ig(X_{n,1},\chi)} - 1 - ig(X_{n,1}, \chi) + \frac{1}{2}g(X_{n,1}, \chi)^2\right)\mathbb{1}_U(X_{n,1}),$$
$$B_n := K_n \operatorname{Re}\mathsf{E}\left(\chi(X_{n,1}) - 1 + \frac{1}{2}g(X_{n,1}, \chi)^2\right)\mathbb{1}_{G\backslash U}(X_{n,1}).$$

By (4.3.3) and (4.2.3) we get

$$|A_n| \leqslant \frac{1}{6}K_n\,\mathsf{E}\left(|g(X_{n,1}, \chi)|^3\,\mathbb{1}_U(X_{n,1})\right) \leqslant \frac{4\bigl(K_n(1 - \operatorname{Re}\mathsf{E}\,\chi(X_{n,1}))\bigr)^{3/2}}{3K_n^{1/2}},$$

hence $K_n \to \infty$ and assumption (ii) yield $A_n \to 0$. Moreover,

$$|B_n| \leqslant \left(2 + \frac{1}{2}\sup_{x \in G} g(x, \chi)^2\right) K_n\,\mathrm{P}(X_{n,1} \in G \setminus U) \to 0,$$

thus we obtain (4.4.3).

(iii) $\Longrightarrow$ (i) follows from Theorem 4.4.1.

If G is compact then every Haar measure on G is finite (see, e.g., Hewitt–Ross [29, Theorem 15.9]). Hence the normalized Haar measure ω_G on G is a probability measure and the Fourier transform $\widehat{\omega}_G$ is defined. Convergence $\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \omega_G$ is equivalent to

$$\mathsf{E}\,\chi\Bigg(\sum_{k=1}^{K_n} X_{n,k}\Bigg) \to \widehat{\omega}_G(\chi) \qquad \text{for all } \ \chi \in \widehat{G}. \tag{4.4.4}$$

Using (4.4.2), (4.2.1) and Lemma 4.4.3, one can easily show that (4.4.4) holds if and only if $K_n\big(1 - \operatorname{Re}\mathsf{E}\,\chi(X_{n,1})\big) \to \infty$ for all $\chi \in \widehat{G} \setminus \{\mathbb{1}_G\}$. □

A random element X in G is called Rademacher if $P(X = e) = 1$ or there exists an element $x \in G$, $x \neq e$ such that $P(X = x) = P(X = -x) = 1/2$. By a Rademacher array we mean an array of Rademacher random elements in G. The next statement is a special case of Theorem 4.4.2.

4.4.4 Theorem. (Limit theorem for rowwise i.i.d. Rademacher array) *Let $x_n \in G$, $n \in \mathbb{N}$ such that $x_n \to e$. Let $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \dots, K_n\}$ be a rowwise i.i.d. array of random elements in G such that $K_n \to \infty$ and*

$$\mathrm{P}(X_{n,k} = x_n) = \mathrm{P}(X_{n,k} = -x_n) = \frac{1}{2}.$$

Then the array $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \dots, K_n\}$ is infinitesimal.

If ψ is a quadratic form on $\widehat{G}$ then

$$\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \gamma_\psi \quad \Longleftrightarrow \quad K_n\big(1 - \operatorname{Re}\chi(x_n)\big) \to \frac{\psi(\chi)}{2} \ \text{ for all } \ \chi \in \widehat{G}.$$

If G is compact then

$$\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \omega_G \quad \Longleftrightarrow \quad K_n\big(1 - \operatorname{Re}\chi(x_n)\big) \to \infty \ \text{ for all } \ \chi \in \widehat{G} \setminus \{\mathbb{1}_G\}.$$

Note that in Theorem 4.4.4 the expression $1 - \operatorname{Re}\chi(x_n)$ can be replaced in both places by $\frac{1}{2}g(x_n, \chi)^2$, where g is an arbitrary local inner product for G (see the proof of (4.4.3) and the inequalities in (4.2.3)).

4.5 Limit theorem for Bernoulli arrays

A random element X in G is called Bernoulli if there exists an element $x \in G$, $x \neq e$ such that $P(X = x) = p$, $P(X = e) = 1 - p$ with some $p \in [0, 1]$. By a Bernoulli array we mean an array of Bernoulli random elements in G. In the following limit theorem the limit measure can be the normalized Haar measure on the smallest closed subgroup of G containing a single element provided that this subgroup is compact.

4.5.1 Theorem. (Limit theorem for rowwise i.i.d. Bernoulli array) *Let $x \in G$ such that $x \neq e$. Let $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \dots, K_n\}$ be a rowwise i.i.d. array of random elements in G such that $K_n \to \infty$,*

$$\mathrm{P}(X_{n,k} = x) = p_n, \qquad \mathrm{P}(X_{n,k} = e) = 1 - p_n,$$

and $p_n \to 0$. Then the array $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \dots, K_n\}$ is infinitesimal.

If λ is a nonnegative real number then

$$\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \mathsf{e}(\lambda\delta_x) \quad \iff \quad K_n\, p_n \to \lambda.$$

If the smallest closed subgroup H of G containing x is compact then

$$\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \omega_H \quad \iff \quad K_n\, p_n \to \infty.$$

Proof. First we suppose $K_n\, p_n \to \lambda$ and show convergence $\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \mathsf{e}(\lambda\delta_x)$. We need to prove

$$\mathsf{E}\,\chi\Bigg(\sum_{k=1}^{K_n} X_{n,k}\Bigg) \to (\mathsf{e}(\lambda\delta_x))\widehat{\ }(\chi) \qquad \text{for all } \chi \in \widehat{G}. \tag{4.5.1}$$

We have $(\mathsf{e}(\lambda\delta_x))\widehat{\ }(\chi) = \mathsf{e}^{\lambda(\chi(x)-1)}$ and

$$\mathsf{E}\,\chi\Bigg(\sum_{k=1}^{K_n} X_{n,k}\Bigg) = (p_n\chi(x) + 1 - p_n)^{K_n} = \left(1 + \frac{K_n\, p_n(\chi(x) - 1)}{K_n}\right)^{K_n}. \tag{4.5.2}$$

If $\{z_n : n \in \mathbb{N}\}$ is a sequence of complex numbers such that $z_n \to z \in \mathbb{C}$ then $\left(1 + \frac{z_n}{n}\right)^n \to \mathsf{e}^z$. Consequently, $K_n\, p_n \to \lambda$ and $K_n \to \infty$ imply (4.5.1).

Next we suppose $K_n p_n \to \infty$ and show that $\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \omega_H$. (Since H is compact we can consider the normalized Haar measure ω_H on G.) We need to prove

$$\mathsf{E}\,\chi\biggl(\sum_{k=1}^{K_n} X_{n,k}\biggr) \to \widehat{\omega}_H(\chi) \qquad \text{for all } \chi \in \widehat{G}.$$

Since H is the smallest closed subgroup of G containing x, Remarks 23.24 (a) in Hewitt–Ross [29] implies $\{x\}^\perp = H^\perp$, and thus by (4.2.1) we are left to check

$$\mathsf{E}\,\chi\biggl(\sum_{k=1}^{K_n} X_{n,k}\biggr) \to \begin{cases} 1 & \text{if } \chi \in \{x\}^\perp, \\ 0 & \text{otherwise.} \end{cases} \tag{4.5.3}$$

If $\chi \in \{x\}^\perp$ then $\chi(x) = 1$, hence

$$\mathsf{E}\,\chi\biggl(\sum_{k=1}^{K_n} X_{n,k}\biggr) = (p_n\chi(x) + 1 - p_n)^{K_n} = 1,$$

and we obtain (4.5.3). To handle the case $\chi \notin \{x\}^\perp$, consider the equality

$$\begin{aligned} \left|\mathsf{E}\,\chi\biggl(\sum_{k=1}^{K_n} X_{n,k}\biggr)\right| &= |p_n\chi(x) + 1 - p_n|^{K_n} \\ &= \Bigl(\bigl(1 + p_n(\operatorname{Re}\chi(x) - 1)\bigr)^2 + p_n^2\bigl(\operatorname{Im}\chi(x)\bigr)^2\Bigr)^{K_n/2} \\ &= \left(1 + \frac{K_n\,p_n\Bigl(2(\operatorname{Re}\chi(x) - 1) + p_n|1-\chi(x)|^2\Bigr)}{K_n}\right)^{K_n/2}. \end{aligned}$$

Clearly $\chi \notin \{x\}^\perp$ implies $\chi(x) \neq 1$, and by $|\chi(x)| = 1$ we get $\operatorname{Re}\chi(x) - 1 < 0$. Hence, by Lemma 4.4.3, we conclude that $K_n\,p_n \to \infty$, $K_n \to \infty$ and $p_n \to 0$ imply (4.5.3).

Now we suppose $\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \mathsf{e}(\lambda\delta_x)$ and derive $K_n\,p_n \to \lambda$. If $K_n\,p_n \not\to \lambda$ then either there exists a subsequence (n') such that $K_{n'}\,p_{n'} \to \infty$, or there exist subsequences (n'') and (n''') and two distinct nonnegative real numbers λ'' and λ''' such that $K_{n''}\,p_{n''} \to \lambda''$ and $K_{n'''}\,p_{n'''} \to \lambda'''$. In the first case we would obtain $\sum_{k=1}^{K_{n'}} X_{n',k} \xrightarrow{\mathcal{D}} \omega_H$, which contradicts to $\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \mathsf{e}(\lambda\delta_x)$. In the second case we would obtain

$\sum_{k=1}^{K_{n''}} X_{n'',k} \xrightarrow{\mathcal{D}} \mathsf{e}(\lambda''\delta_x)$ and $\sum_{k=1}^{K_{n'''}} X_{n''',k} \xrightarrow{\mathcal{D}} \mathsf{e}(\lambda'''\delta_x)$ which again contradicts to $\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \mathsf{e}(\lambda\delta_x)$.

Finally we suppose $\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \omega_H$ and prove $K_n\, p_n \to \infty$. If $K_n\, p_n \not\to \infty$ then there exists a subsequence (n') and a nonnegative real number λ' such that $K_{n'}\, p_{n'} \to \lambda'$. Then we would obtain $\sum_{k=1}^{K_{n'}} X_{n',k} \xrightarrow{\mathcal{D}} \mathsf{e}(\lambda'\delta_x)$, which contradicts to $\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \omega_H$. □

4.6 Limit theorems on the torus

The set $\mathbb{T} := \{\mathsf{e}^{ix} : -\pi \leqslant x < \pi\}$ equipped with the usual multiplication of complex numbers and with the relative topology as a subset of complex numbers is a second countable compact Abelian T_0-topological group. In fact, $\mathbb{T}$ is a Lie group and it is called the one-dimensional torus group. Its character group is $\widehat{\mathbb{T}} = \{\chi_\ell : \ell \in \mathbb{Z}\}$, where

$$\chi_\ell(y) := y^\ell, \qquad y \in \mathbb{T}, \quad \ell \in \mathbb{Z}.$$

Hence $\widehat{\mathbb{T}}$ is topologically isomorphic with the additive group of integers $\mathbb{Z}$.

The set of all quadratic forms on $\widehat{\mathbb{T}}$ is $\mathsf{q}_+\big(\widehat{\mathbb{T}}\big) = \{\psi_b : b \in \mathbb{R}_+\}$, where

$$\psi_b(\chi_\ell) := b\ell^2, \qquad \ell \in \mathbb{Z}, \quad b \in \mathbb{R}_+.$$

Let us define the functions $\arg : \mathbb{T} \to [-\pi, \pi[$ and $h : \mathbb{R} \to \mathbb{R}$ by

$$\arg(\mathsf{e}^{ix}) := x, \qquad -\pi \leqslant x < \pi,$$

$$h(x) := \begin{cases} 0 & \text{if } x < -\pi \text{ or } x \geqslant \pi, \\ -x-\pi & \text{if } -\pi \leqslant x < -\pi/2, \\ x & \text{if } -\pi/2 \leqslant x < \pi/2, \\ -x+\pi & \text{if } \pi/2 \leqslant x < \pi. \end{cases}$$

The function $g_{\mathbb{T}} : \mathbb{T} \times \widehat{\mathbb{T}} \to \mathbb{R}$, defined by

$$g_{\mathbb{T}}(y, \chi_\ell) := \ell h(\arg y), \qquad y \in \mathbb{T}, \quad \ell \in \mathbb{Z},$$

is a local inner product for $\mathbb{T}$. An extended real valued measure η on $\mathbb{T}$ is a Lévy measure if and only if $\eta(\{e\}) = 0$ and $\int_{\mathbb{T}} (\arg y)^2\, \eta(\mathsf{d}y) < \infty$.

Theorem 4.3.1 has the following consequence on the torus.

4.6.1 Theorem. (Gauss–Poisson limit theorem) *Let* $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \dots, K_n\}$ *be a rowwise independent array of random elements in* $\mathbb{T}$*. Suppose that there exists a quadruplet* $(\{e\}, a, \psi_b, \eta) \in \mathcal{P}(\mathbb{T})$ *such that*

(i) $\displaystyle\max_{1 \leqslant k \leqslant K_n} \mathrm{P}(|\arg(X_{n,k})| > \varepsilon) \to 0$ *as* $n \to \infty$ *for all* $\varepsilon > 0$,

(ii) $\displaystyle\exp\left\{ i \sum_{k=1}^{K_n} \mathsf{E}\, h(\arg(X_{n,k})) \right\} \to a$ *as* $n \to \infty$,

(iii) $\displaystyle\sum_{k=1}^{K_n} \operatorname{Var} h(\arg(X_{n,k})) \to b + \int_{\mathbb{T}} \big(h(\arg y)\big)^2 \,\eta(\mathsf{d}y)$ *as* $n \to \infty$,

(iv) $\displaystyle\sum_{k=1}^{K_n} \mathsf{E}\, f(X_{n,k}) \to \int_{\mathbb{T}} f \,\mathsf{d}\eta$ *as* $n \to \infty$ *for all* $f \in \mathcal{C}_0(\mathbb{T})$.

Then the array $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \dots, K_n\}$ *is infinitesimal and*

$$\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \delta_a * \gamma_{\psi_b} * \pi_{\eta,\, g_{\mathbb{T}}} \qquad \text{as } n \to \infty.$$

The next theorem shows that if the limit measure in Theorem 4.6.1 has no generalized Poisson factor $\pi_{\eta,\, g_{\mathbb{T}}}$ then the truncation function h can be omitted.

4.6.2 Theorem. (CLT) *Let* $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \dots, K_n\}$ *be a rowwise independent array of random elements in* $\mathbb{T}$*. Suppose that there exist an element* $a \in \mathbb{T}$ *and a nonnegative real number* b *such that*

(i) $\displaystyle\exp\left\{ i \sum_{k=1}^{K_n} \mathsf{E} \arg(X_{n,k}) \right\} \to a$ *as* $n \to \infty$,

(ii) $\displaystyle\sum_{k=1}^{K_n} \operatorname{Var} \arg(X_{n,k}) \to b$ *as* $n \to \infty$,

(iii) $\displaystyle\sum_{k=1}^{K_n} \mathrm{P}(|\arg(X_{n,k})| > \varepsilon) \to 0$ *as* $n \to \infty$ *for all* $\varepsilon > 0$.

Then the array $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \dots, K_n\}$ *is infinitesimal and*

$$\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \delta_a * \gamma_{\psi_b} \qquad \textit{as} \quad n \to \infty.$$

Proof. In view of Theorem 4.6.1 and Remark 4.3.5, it is enough to check

(i′) $\exp\left\{ i \sum_{k=1}^{K_n} \mathsf{E}\, h(\arg(X_{n,k})) \right\} \to a$ as $n \to \infty$,

(ii′) $\sum_{k=1}^{K_n} \operatorname{Var} h(\arg(X_{n,k})) \to b$ as $n \to \infty$,

(iii′) $\sum_{k=1}^{K_n} \mathrm{P}(|\arg(X_{n,k})| > \varepsilon) \to 0$ as $n \to \infty$ for all $\varepsilon > 0$.

Clearly (iii′) and assumption (iii) are identical. In order to prove (i′) it is sufficient to show

$$\sum_{k=1}^{K_n} \mathsf{E}\, h(\arg(X_{n,k})) - \sum_{k=1}^{K_n} \mathsf{E} \arg(X_{n,k}) \to 0,$$

since $|\mathrm{e}^{iy_1} - \mathrm{e}^{iy_2}| = |\mathrm{e}^{i(y_1 - y_2)} - 1| \leqslant |y_1 - y_2|$ for all $y_1, y_2 \in \mathbb{R}$. We have $|h(y) - y| \leqslant \pi \mathbb{1}_{[-\pi,-\pi/2] \cup [\pi/2,\pi]}(y)$ for all $y \in [-\pi, \pi]$, hence

$$\left| \sum_{k=1}^{K_n} \mathsf{E}\, h(\arg(X_{n,k})) - \sum_{k=1}^{K_n} \mathsf{E} \arg(X_{n,k}) \right| \leqslant \pi \sum_{k=1}^{K_n} \mathrm{P}(|\arg(X_{n,k})| \geqslant \pi/2) \to 0$$

by condition (iii). In order to check (ii′) it is enough to prove

$$\sum_{k=1}^{K_n} \operatorname{Var} h(\arg(X_{n,k})) - \sum_{k=1}^{K_n} \operatorname{Var} \arg(X_{n,k}) \to 0.$$

We have

$$\begin{aligned}
\left|\sum_{k=1}^{K_n} \operatorname{Var} h(\arg(X_{n,k})) - \sum_{k=1}^{K_n} \operatorname{Var} \arg(X_{n,k})\right| \\
\leqslant \sum_{k=1}^{K_n} \mathsf{E}\left|\big(h(\arg(X_{n,k}))\big)^2 - (\arg(X_{n,k}))^2\right| \\
+ \sum_{k=1}^{K_n} \left|\big(\mathsf{E}\, h(\arg(X_{n,k}))\big)^2 - (\mathsf{E} \arg(X_{n,k}))^2\right| \\
\leqslant 2\pi^2 \sum_{k=1}^{K_n} \mathrm{P}(|\arg(X_{n,k})| \geqslant \pi/2) \to 0,
\end{aligned}$$

as desired. □

Theorem 4.4.4 has the following consequence on the torus.

4.6.3 Theorem. (Limit theorem for rowwise i.i.d. Rademacher array) *Let* $x_n \in \mathbb{T}$, $n \in \mathbb{N}$ *such that* $x_n \to e$. *Let* $\{X_{n,k} : n \in \mathbb{N}, k = 1, \dots, K_n\}$ *be a rowwise i.i.d. array of random elements in* $\mathbb{T}$ *such that* $K_n \to \infty$ *and*

$$\mathrm{P}(X_{n,k} = x_n) = \mathrm{P}(X_{n,k} = -x_n) = \frac{1}{2}.$$

Then the array $\{X_{n,k} : n \in \mathbb{N}, k = 1, \dots, K_n\}$ *is infinitesimal.*

If b *is a nonnegative real number then*

$$\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \gamma_{\psi_b} \qquad \Longleftrightarrow \qquad K_n(\arg x_n)^2 \to b.$$

Moreover,

$$\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \omega_{\mathbb{T}} \qquad \Longleftrightarrow \qquad K_n(\arg x_n)^2 \to \infty.$$

In the rest of this section we consider the question of giving a construction of an arbitrary weakly infinitely divisible measure on $\mathbb{T}$ using only real valued random variables. We show that for a weakly infinitely divisible measure μ on $\mathbb{T}$ there exist independent real valued random variables U and Z such that U is uniformly distributed on a suitable subset of $\mathbb{R}$, Z has an infinitely divisible distribution on $\mathbb{R}$, and $\mathsf{e}^{i(U+Z)} \stackrel{\mathcal{D}}{=} \mu$. We note that $\mathbb{R}$ is a locally compact Abelian T_0-topological group, its character group is $\widehat{\mathbb{R}} = \{\chi_y : y \in \mathbb{R}\}$, where $\chi_y(x) := \mathsf{e}^{iyx}$. The function $g_{\mathbb{R}} : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{R}$, defined by $g_{\mathbb{R}}(x, \chi_y) := yh(x)$, is a local inner product for $\mathbb{R}$.

For the parametrization of an arbitrary weakly infinitely divisible measure on $\mathbb{T}$ we need to know all the compact subgroups of $\mathbb{T}$. The compact subgroups of $\mathbb{T}$ are

$$H_r := \{\mathsf{e}^{2\pi i j/r} : j = 0, 1, \dots, r-1\}, \qquad r \in \mathbb{N},$$

and $\mathbb{T}$ itself.

4.6.4 Theorem. *If* $(H, a, \psi_b, \eta) \in \mathcal{P}(\mathbb{T})$ *then*

$$\mathsf{e}^{i(U+\arg a+X+Y)} \stackrel{\mathcal{D}}{=} \omega_H * \delta_a * \gamma_{\psi_b} * \pi_{\eta,\, g_{\mathbb{T}}},$$

where U, X *and* Y *are independent real valued random variables such that* U *is uniformly distributed on* $[0, 2\pi]$ *if* $H = \mathbb{T}$, U *is uniformly distributed on* $\{2\pi j/r : j = 0, 1, \dots, r-1\}$ *if* $H = H_r$ *for some* $r \in \mathbb{N}$, X *has a normal distribution on* $\mathbb{R}$ *with zero mean and variance* b, *and the distribution of* Y *is the generalized Poisson measure* $\pi_{\arg\circ\eta,\, g_{\mathbb{R}}}$ *on* $\mathbb{R}$, *where the measure* $\arg\circ\eta$ *on* $\mathbb{R}$ *is defined by* $(\arg\circ\eta)(B) := \eta\big(\{x \in \mathbb{T} : \arg(x) \in B\}\big)$ *for all Borel subsets* B *of* $\mathbb{R}$.

Proof. Let U be a real valued random variable which is uniformly distributed on $[0, 2\pi]$. Then for all $\chi_\ell \in \widehat{\mathbb{T}}$, $\ell \in \mathbb{Z}$, $\ell \neq 0$,

$$\mathsf{E}\,\chi_\ell(\mathsf{e}^{iU}) = \mathsf{E}\,\mathsf{e}^{i\ell U} = \frac{1}{2\pi}\int_0^{2\pi} \mathsf{e}^{i\ell x}\,\mathsf{d}x = 0.$$

Hence $\mathsf{E}\,\chi_\ell(\mathsf{e}^{iU}) = \widehat{\omega}_{\mathbb{T}}(\chi_\ell)$ for all $\chi_\ell \in \widehat{\mathbb{T}}$, $\ell \in \mathbb{Z}$, and we obtain $\mathsf{e}^{iU} \stackrel{\mathcal{D}}{=} \omega_{\mathbb{T}}$.

Now let U be a real valued random variable which is uniformly distributed on $\{2\pi j/r : j = 0, 1, \dots, r-1\}$ with some $r \in \mathbb{N}$. Then for all $\chi_\ell \in \widehat{\mathbb{T}}$, $\ell \in \mathbb{Z}$,

$$\mathsf{E}\,\chi_\ell(\mathsf{e}^{iU}) = \mathsf{E}\,\mathsf{e}^{i\ell U} = \frac{1}{r}\sum_{j=0}^{r-1} \mathsf{e}^{2\pi i\ell j/r} = \begin{cases} 1 & \text{if } r|\ell, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\mathsf{E}\,\chi_\ell(\mathsf{e}^{iU}) = \widehat{\omega}_{H_r}(\chi_\ell)$ for all $\chi_\ell \in \widehat{\mathbb{T}}$, $\ell \in \mathbb{Z}$, and we obtain $\mathsf{e}^{iU} \stackrel{\mathcal{D}}{=} \omega_{H_r}$.

For $a \in \mathbb{T}$, we have $a = \mathsf{e}^{i\arg a}$, hence $\mathsf{e}^{i\arg a} \stackrel{\mathcal{D}}{=} \delta_a$.

For $b \in \mathbb{R}_+$, the Fourier transform of the symmetric Gauss measure γ_{ψ_b} has the form

$$\widehat{\gamma}_{\psi_b}(\chi_\ell) = \mathsf{e}^{-b\ell^2/2}, \qquad \chi_\ell \in \widehat{\mathbb{T}}, \quad \ell \in \mathbb{Z}.$$

For all $\chi_\ell \in \widehat{\mathbb{T}}$, $\ell \in \mathbb{Z}$,

$$\mathsf{E}\,\chi_\ell(\mathsf{e}^{iX}) = \mathsf{E}\,\mathsf{e}^{i\ell X} = \mathsf{e}^{-b\ell^2/2}.$$

Hence $\mathsf{E}\,\chi_\ell(\mathsf{e}^{iX}) = \gamma_{\psi_b}(\chi_\ell)$ for all $\chi_\ell \in \widehat{\mathbb{T}}$, $\ell \in \mathbb{Z}$, and we obtain $\mathsf{e}^{iX} \stackrel{\mathcal{D}}{=} \gamma_{\psi_b}$.

For a Lévy measure $\eta \in \mathbb{L}(\mathbb{T})$, the Fourier transform of the generalized Poisson measure $\pi_{\eta,\, g_\mathbb{T}}$ has the form

$$\widehat{\pi}_{\eta,\, g_\mathbb{T}}(\chi_\ell) = \exp\left\{\int_\mathbb{T} \big(y^\ell - 1 - i\ell h(\arg y)\big)\,\eta(\mathsf{d}y)\right\}, \qquad \chi_\ell \in \widehat{\mathbb{T}}, \quad \ell \in \mathbb{Z}.$$

An extended real valued measure $\widetilde{\eta}$ on $\mathbb{R}$ is a Lévy measure if and only if $\widetilde{\eta}(\{0\}) = 0$ and $\int_\mathbb{R} \min\{1, x^2\}\,\widetilde{\eta}(\mathsf{d}x) < \infty$. Consequently, $\arg\circ\eta$ is a Lévy measure on $\mathbb{R}$, and for all $\chi_\ell \in \widehat{\mathbb{T}}$, $\ell \in \mathbb{Z}$,

$$\begin{aligned}\mathsf{E}\,\chi_\ell(\mathsf{e}^{iY}) = \mathsf{E}\,\mathsf{e}^{i\ell Y} &= \exp\left\{\int_\mathbb{R} \big(\mathsf{e}^{i\ell x} - 1 - i\ell h(x)\big)\,(\arg\circ\eta)(\mathsf{d}x)\right\} \\ &= \exp\left\{\int_\mathbb{T} \big(y^\ell - 1 - i\ell h(\arg y)\big)\,\eta(\mathsf{d}y)\right\}.\end{aligned}$$

Hence $\mathsf{E}\,\chi_\ell(\mathsf{e}^{iY}) = \widehat{\pi}_{\eta,\, g_\mathbb{T}}(\chi_\ell)$ for all $\chi_\ell \in \widehat{\mathbb{T}}$, $\ell \in \mathbb{Z}$, and we obtain $\mathsf{e}^{iY} \stackrel{\mathcal{D}}{=} \pi_{\eta,\, g_\mathbb{T}}$.

Finally, independence of U, X and Y implies

$$\begin{aligned}&\mathsf{E}\,\chi(\mathsf{e}^{i(U+\arg a+X+Y)}) = \mathsf{E}\,\chi(\mathsf{e}^{iU}) \cdot \chi(\mathsf{e}^{i\arg a}) \cdot \mathsf{E}\,\chi(\mathsf{e}^{iX}) \cdot \mathsf{E}\,\chi(\mathsf{e}^{iY}) \\ &= \widehat{\omega}_H(\chi)\,\widehat{\delta}_a(\chi)\,\widehat{\gamma}_{\psi_b}(\chi)\,\widehat{\pi}_{\eta,\, g_\mathbb{T}}(\chi) = (\omega_H * \delta_a * \gamma_{\psi_b} * \pi_{\eta,\, g_\mathbb{T}})\widehat{\ }(\chi)\end{aligned}$$

for all $\chi \in \widehat{\mathbb{T}}$, hence we obtain the statement. $\square$

4.7 Limit theorems on the group of p-adic integers

Let p be a prime. The group of p-adic integers is

$$\Delta_p := \big\{(x_0, x_1, \dots) : x_j \in \{0, 1, \dots, p-1\} \text{ for all } j \in \mathbb{Z}_+\big\},$$

where the sum $z := x + y \in \Delta_p$ for $x, y \in \Delta_p$ is uniquely determined by the relationships

$$\sum_{j=0}^{d} z_j p^j \equiv \sum_{j=0}^{d} (x_j + y_j) p^j \qquad \bmod p^{d+1} \qquad \text{for all } d \in \mathbb{Z}_+.$$

Equivalently, the operation $+$ in Δ_p can be given in the following way. For $x, y \in \Delta_p$, let their sum z be defined as follows. Write $x_0 + y_0 = t_0 p + z_0$, where $z_0 \in \{0, \dots, p-1\}$ and t_0 is an integer. Suppose that $z_0, z_1, \dots, z_k$ and $t_0, t_1, \dots, t_k$ have been defined. Then write $x_{k+1} + y_{k+1} + t_k = t_{k+1} p + z_{k+1}$, where $z_{k+1} \in \{0, \dots, p-1\}$ and t_{k+1} is an integer. This defines by induction a sequence $z = (z_n)_{n \geqslant 0}$ in Δ_p. We define the sum $x + y$ to be z. To complete the definition of addition in Δ_p, we define $0 + x = x + 0 = x$ for all $x \in \Delta_p$, where 0 is the identically zero sequence in Δ_p. (Definition 10.2 in Hewitt–Ross [29] contains this introduction of the group operation in Δ_p.)

For each $r \in \mathbb{Z}_+$, let

$$\Lambda_r := \{x \in \Delta_p : x_j = 0 \text{ for all } j \leqslant r - 1\}.$$

The family of sets $\{x + \Lambda_r : x \in \Delta_p,\ r \in \mathbb{Z}_+\}$ is an open subbasis for a topology on Δ_p under which Δ_p is a second countable compact Abelian T_0-topological group (see Theorems 4.5 and 10.5 in Hewitt–Ross [29]). Note that Δ_p is not a Lie group.

We show that Δ_p is totally disconnected. By definition, we have to check that every component of Δ_p consists of one point. Let C_0 be the component of the identity 0 in Δ_p. By Theorem 7.2 in Hewitt–Ross [29], for all $x \in \Delta_p$, $x + C_0$ is the component of x. So it is enough to prove that $C_0 = \{0\}$. By Theorem 7.8 in Hewitt–Ross [29], C_0 is the intersection of all open subgroups of Δ_p. Since Λ_r is an open subgroup of Δ_p for all $r \in \mathbb{Z}_+$, we have

$$C_0 \subset \bigcap_{r=0}^{\infty} \Lambda_r = \{0\}.$$

Since $0 \in C_0$, we have $C_0 = \{0\}$.

The character group of Δ_p is $\widehat{\Delta}_p = \{\chi_{d,\ell} : d \in \mathbb{Z}_+,\ \ell = 0, 1, \dots, p^{d+1} - 1\}$, where

$$\chi_{d,\ell}(x) := \mathrm{e}^{2\pi i \ell (x_0 + p x_1 + \dots + p^d x_d)/p^{d+1}}, \quad x \in \Delta_p, \quad d \in \mathbb{Z}_+, \quad \ell = 0, 1, \dots, p^{d+1} - 1,$$

see, e.g., Hewitt–Ross [29, p. 403].

Since the group Δ_p is totally disconnected, the only quadratic form on $\widehat{\Delta}_p$ is $\psi = 0$, and the function $g_{\Delta_p} : \Delta_p \times \widehat{\Delta}_p \to \mathbb{R}$, $g_{\Delta_p} = 0$ is a local inner product for Δ_p (see Parthasarathy [46, p. 109, Remark 1]).

An extended real valued measure η on Δ_p is a Lévy measure if and only if $\eta(\{e\}) = 0$ and $\eta(\Delta_p \setminus \Lambda_r) < \infty$ for all $r \in \mathbb{Z}_+$.

Theorem 4.3.1 has the following consequence on the group Δ_p of p-adic integers.

4.7.1 Theorem. (Poisson limit theorem) *Let $\{X_{n,k} : n \in \mathbb{N}, k = 1, \ldots, K_n\}$ be a rowwise independent array of random elements in Δ_p. Suppose that there exists a Lévy measure $\eta \in \mathbb{L}(\Delta_p)$ such that*

(i) $\max\limits_{1 \leqslant k \leqslant K_n} \mathrm{P}\Big(\big((X_{n,k})_0, \ldots, (X_{n,k})_d\big) \neq 0\Big) \to 0$ *as* $n \to \infty$ *for all* $d \in \mathbb{Z}_+$,

(ii) $\sum\limits_{k=1}^{K_n} \mathrm{P}\big((X_{n,k})_0 = \ell_0, \ldots, (X_{n,k})_d = \ell_d\big)$
$\to \eta(\{x \in \Delta_p : x_0 = \ell_0, \ldots, x_d = \ell_d\})$ *as* $n \to \infty$ *for all* $d \in \mathbb{Z}_+$, $\ell_0, \ldots, \ell_d \in \{0, \ldots, p-1\}$ *with* $(\ell_0, \ldots, \ell_d) \neq 0$.

Then the array $\{X_{n,k} : n \in \mathbb{N}, k = 1, \ldots, K_n\}$ is infinitesimal and

$$\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \pi_{\eta,\, g_{\Delta_p}} \qquad \textit{as} \ \ n \to \infty.$$

For the proof of Theorem 4.7.1, we use the following lemma.

4.7.2 Lemma. *Let $\{\eta_n : n \in \mathbb{Z}_+\}$ be extended real valued measures on Δ_p such that $\eta_n(\Delta_p \setminus \Lambda_r) < \infty$ for all $n, r \in \mathbb{Z}_+$. Then the following statements are equivalent:*

(a) $\eta_n(x + \Lambda_r) \to \eta_0(x + \Lambda_r)$ *as* $n \to \infty$ *for all* $r \in \mathbb{N}$, $x \in \Delta_p \setminus \Lambda_r$,

(b) $\displaystyle\int_{\Delta_p} f \, \mathsf{d}\eta_n \to \int_{\Delta_p} f \, \mathsf{d}\eta_0$ *as* $n \to \infty$ *for all* $f \in \mathcal{C}_0(\Delta_p)$.

Proof. By Theorem 4.3.4, (b) is equivalent to

(b′) $\eta_n|_{\Delta_p \setminus U} \xrightarrow{\mathsf{w}} \eta_0|_{\Delta_p \setminus U}$ as $n \to \infty$ for all $U \in \mathcal{N}_e$ with $\eta_0(\partial U) = 0$.

It can be checked that if $\eta_n|_{\Delta_p \setminus U} \xrightarrow{\mathsf{w}} \eta_0|_{\Delta_p \setminus U}$ holds for some $U \in \mathcal{N}_e$ with $\eta_0(\partial U) = 0$ then $\eta_n|_{\Delta_p \setminus V} \xrightarrow{\mathsf{w}} \eta_0|_{\Delta_p \setminus V}$ holds for all $V \in \mathcal{N}_e$ with $V \supset U$ and $\eta_0(\partial V) = 0$. Hence, using that $\{\Lambda_r : r \in \mathbb{N}\}$ is an open neighbourhood basis of e and $\partial \Lambda_r = \emptyset$ for all $r \in \mathbb{Z}_+$, (b$'$) is equivalent to

(b$''$) $\eta_n|_{\Delta_p \setminus \Lambda_r} \xrightarrow{\mathsf{w}} \eta_0|_{\Delta_p \setminus \Lambda_r}$ as $n \to \infty$ for all $r \in \mathbb{N}$.

For distinct elements $x, y \in \Delta_p$, let $\varrho(x, y)$ be the number 2^{-m}, where m is the least nonnegative integer for which $x_m \neq y_m$. For all $x \in \Delta_p$, let $\varrho(x, x) := 0$. Then ϱ is an invariant metric on Δ_p compatible with the topology of Δ_p (see Theorem 10.5 in Hewitt and Ross [29]). Let $d(x, y) := \sum_{k=0}^{\infty} 2^{-k} \mathbb{1}_{\{x_k \neq y_k\}}$ for all $x, y \in \Delta_p$. Then d is a metric on Δ_p equivalent to ϱ, since $\varrho(x, y) \leqslant d(x, y) \leqslant 2\varrho(x, y)$ for all $x, y \in \Delta_p$. Hence the original topology of Δ_p and the topology on Δ_p induced by the metric d coincide. Then weak convergence of bounded measures on the locally compact group Δ_p can be considered as weak convergence of bounded measures on the metric space Δ_p equipped with the metric d.

We show that the set

$$M := \{\mathbb{1}_{x+\Lambda_c} : c \in \mathbb{N},\, x \in \Delta_p\}$$

is convergence determining for the weak convergence of probability measures on Δ_p. For this one can check that Proposition 4.6 in Ethier and Kurtz [20] is applicable with the following choices: $S := \Delta_p$ equipped with the metric d, S_k is the set $\{0, 1, \ldots, p-1\}$ for all $k \in \mathbb{N}$, d_k is the discrete metric on S_k, $k \in \mathbb{N}$, and

$$M_k := \{f_{c_k} : c_k \in S_k\}, \quad k \in \mathbb{N},$$

where

$$f_{c_k}(x) := \begin{cases} 1 & \text{if } x = c_k, \\ 0 & \text{if } x \neq c_k, \end{cases} \quad x \in S_k, \quad k \in \mathbb{N}.$$

For checking we note that for each $c \in \mathbb{N}$ and $x \in \Delta_p$, the function $\mathbb{1}_{x+\Lambda_c}$ is bounded and continuous, since the set $x + \Lambda_c$ is open and closed. Moreover, for each $k \in \mathbb{N}$, S_k with the metric d_k is a complete separable metric space.

It is easy to check that M is a convergence determining set for the weak convergence of bounded measures on Δ_p as well. Consequently, (b$''$) is equivalent to

(b$'''$) $\displaystyle\int_{\Delta_p} \mathbb{1}_{x+\Lambda_c}\, \eta_n|_{\Delta_p \setminus \Lambda_r}(\mathrm{d}x) \to \int_{\Delta_p} \mathbb{1}_{x+\Lambda_c}\, \eta_0|_{\Delta_p \setminus \Lambda_r}(\mathrm{d}x)$ as $n \to \infty$ for all $x \in \Delta_p$ and for all $c, r \in \mathbb{N}$.

Clearly, this is equivalent to

(b'''') $\eta_n\big((x+\Lambda_c)\cap(\Delta_p\setminus\Lambda_r)\big)\to\eta_0\big((x+\Lambda_c)\cap(\Delta_p\setminus\Lambda_r)\big)$ as $n\to\infty$ for all $x\in\Delta_p$ and for all $c,r\in\mathbb{N}$.

We have

$$(x+\Lambda_c)\cap(\Delta_p\setminus\Lambda_r)=\begin{cases}\Lambda_c\setminus\Lambda_r & \text{if } r\geqslant c \text{ and } x\in\Lambda_c,\\ \emptyset & \text{if } r<c \text{ and } x\in\Lambda_r,\\ x+\Lambda_c & \text{otherwise.}\end{cases}$$

If $r\geqslant c$ then $\Lambda_c\setminus\Lambda_r$ can be written as a union of $p^{r-c}-1$ disjoint sets of the form $y+\Lambda_r$ with $y\in\Lambda_c\setminus\Lambda_r$. Consequently, (b'''') and (a) are equivalent. □

Proof of Theorem 4.7.1. The local mean of any random element with values in Δ_p is e (with respect to the local inner product $g_{\Delta_p}=0$). Moreover, for each $U\in\mathcal{N}_e$, there exists $r\in\mathbb{Z}_+$ such that $\Lambda_r\subset U$. Hence, in view of Theorem 4.3.1, it is enough to check that

(i') $\max\limits_{1\leqslant k\leqslant K_n}\mathrm{P}(X_{n,k}\in\Delta_p\setminus\Lambda_r)\to 0$ as $n\to\infty$ for all $r\in\mathbb{Z}_+$,

(ii') $\displaystyle\sum_{k=1}^{K_n}\mathsf{E}\,f(X_{n,k})\to\int_{\Delta_p}f\,\mathsf{d}\eta$ as $n\to\infty$ for all $f\in\mathcal{C}_0(\Delta_p)$.

Clearly $\{x\in\Delta_p:(x_0,x_1,\dots,x_d)\neq 0\}=\Delta_p\setminus\Lambda_{d+1}$, hence (i') and (i) are identical. Applying Lemma 4.7.2 for $\eta_n:=\sum_{k=1}^{K_n}\mathrm{P}_{X_{n,k}}$ and $\eta_0:=\eta$, we conclude that (ii'') and (ii) are equivalent. □

4.7.3 Remark. Theorem 4.4.4 has the following consequence on Δ_p. If $x_n\in\Delta_p$, $n\in\mathbb{N}$ such that $x_n\to e$, and $\{X_{n,k}:n\in\mathbb{N},\,k=1,\dots,K_n\}$ is a rowwise i.i.d. array of random elements in Δ_p such that $K_n\to\infty$ and $\mathrm{P}(X_{n,k}=x_n)=\mathrm{P}(X_{n,k}=-x_n)=\frac{1}{2}$, then the array $\{X_{n,k}:n\in\mathbb{N},\,k=1,\dots,K_n\}$ is infinitesimal and $\sum_{k=1}^{K_n}X_{n,k}\xrightarrow{\mathcal{D}}\delta_e$.

In the rest of this section we consider the question of giving a construction of an arbitrary weakly infinitely divisible measure on Δ_p using only real valued random variables. We show that for a weakly infinitely divisible measure μ on Δ_p there exist integer valued random variables $U_0,U_1,\dots$ and $Z_0,Z_1,\dots$ such that $U_0, U_1, \dots$ are independent of each other and of the sequence $Z_0, Z_1, \dots$, moreover, $U_0, U_1, \dots$ are uniformly distributed on a suitable subset of $\mathbb{Z}$, $(Z_0, \dots, Z_n)$ has a weakly infinitely divisible distribution on $\mathbb{Z}^{n+1}$ for all $n \in \mathbb{Z}_+$, and $\varphi(U_0 + Z_0, U_1 + Z_1, \dots) \stackrel{\mathcal{D}}{=} \mu$, where the mapping $\varphi : \mathbb{Z}^\infty \to \Delta_p$, uniquely defined by the relationships

$$\sum_{j=0}^{d} y_j p^j \equiv \sum_{j=0}^{d} \varphi(y)_j p^j \qquad \mathrm{mod}\ p^{d+1} \qquad \text{for all } d \in \mathbb{Z}_+, \tag{4.7.1}$$

is a measurable homomorphism from the Abelian topological group $\mathbb{Z}^\infty$ (furnished with the product topology) onto Δ_p. (Note that $\mathbb{Z}^\infty$ is not locally compact.) Measurability of φ follows from

$$\varphi^{-1}(x + \Lambda_r) = \{y \in \mathbb{Z}^\infty : (y_0, y_1, \dots, y_{r-1}) \in F_{x,r}\}$$

for all $x \in \Delta_p$, $r \in \mathbb{Z}_+$, where $F_{x,r}$ is a suitable finite subset of $\mathbb{Z}^r$.

For the parametrization of an arbitrary weakly infinitely divisible measure on Δ_p we need to know all the compact subgroups of Δ_p. For all $r \in \mathbb{Z}_+$, Λ_r is a compact subgroup of Δ_p and Example 10.16 (a) in Hewitt–Ross [29] shows that there is no compact subgroup of Δ_p which differs from Λ_r, $r \geqslant 0$.

4.7.4 Theorem. *If* $(\Lambda_r, a, 0, \eta) \in \mathcal{P}(\Delta_p)$ *then*

$$\varphi(U_0 + a_0 + Y_0,\, U_1 + a_1 + Y_1, \dots) \stackrel{\mathcal{D}}{=} \omega_{\Lambda_r} * \delta_a * \pi_{\eta,\, g_{\Delta_p}},$$

where $U_0, U_1, \dots$ *and* $Y_0, Y_1, \dots$ *are integer valued random variables such that* $U_0, U_1, \dots$ *are independent of each other and of the sequence* $Y_0, Y_1, \dots$, *moreover,* $U_0 = \dots = U_{r-1} = 0$ *and* $U_r, U_{r+1}, \dots$ *are uniformly distributed on* $\{0, 1, \dots, p-1\}$, *and the distribution of* $(Y_0, \dots, Y_n)$ *is the compound Poisson measure* $\mathsf{e}(\eta_{n+1})$ *on* $\mathbb{Z}^{n+1}$ *for all* $n \in \mathbb{Z}_+$, *where the measure* η_{n+1} *on* $\mathbb{Z}^{n+1}$ *is defined by* $\eta_{n+1}(\{0\}) := 0$ *and* $\eta_{n+1}(\ell) := \eta(\{x \in \Delta_p : (x_0, x_1, \dots, x_n) = \ell\})$ *for all* $\ell \in \mathbb{Z}^{n+1} \setminus \{0\}$.

Proof. Since $U_0, U_1, \dots$ and $Y_0, Y_1, \dots$ are integer valued random variables and the mapping $\varphi : \mathbb{Z}^\infty \to \Delta_p$ is measurable, we obtain that $\varphi(U_0 + a_0 + Y_0,\, U_1 + a_1 + Y_1, \dots)$ is a random element with values in Δ_p.

First we show $\varphi(U) \stackrel{\mathcal{D}}{=} \omega_{\Lambda_r}$, where $U := (U_0, U_1, \dots)$. By (4.7.1) we obtain

$$\begin{aligned}
&\mathsf{E}\,\chi_{d,\ell}(\varphi(U)) = \mathsf{E}\,\mathsf{e}^{2\pi i\ell(\varphi(U)_0 + p\varphi(U)_1 + \dots + p^d\varphi(U)_d)/p^{d+1}} \\
&= \mathsf{E}\,\mathsf{e}^{2\pi i\ell(U_0 + pU_1 + \dots + p^d U_d)/p^{d+1}} \qquad (4.7.2)\\
&= \begin{cases} \dfrac{1}{p^{d-r+1}} \displaystyle\sum_{j_r=0}^{p-1} \cdots \sum_{j_d=0}^{p-1} \mathsf{e}^{2\pi i\ell(p^r j_r + \dots + p^d j_d)/p^{d+1}} = 0 & \text{if } d \geqslant r \text{ and } p^{d+1-r} \not| \ell, \\ 1 & \text{otherwise} \end{cases}
\end{aligned}$$

for all $d \in \mathbb{Z}_+$ and $\ell = 0, 1, \dots, p^{d+1} - 1$. Hence $\mathsf{E}\,\chi_{d,\ell}(\varphi(U)) = \widehat{\omega}_{\Lambda_r}(\chi_{d,\ell})$ for all $d \in \mathbb{Z}_+$ and $\ell = 0, 1, \dots, p^{d+1} - 1$, and we obtain $\varphi(U) \stackrel{\mathcal{D}}{=} \omega_{\Lambda_r}$.

For $a \in \Delta_p$, we have $a = \varphi(a_0, a_1, \dots)$, hence $\varphi(a_0, a_1, \dots) \stackrel{\mathcal{D}}{=} \delta_a$.

For a Lévy measure $\eta \in \mathbb{L}(\Delta_p)$, the Fourier transform of the generalized Poisson measure $\pi_{\eta,\, g_{\Delta_p}}$ has the form

$$\widehat{\pi}_{\eta,\, g_{\Delta_p}}(\chi_{d,\ell}) = \exp\left\{ \int_{\Delta_p} \big(\mathsf{e}^{2\pi i\ell(x_0 + px_1 + \dots + p^d x_d)/p^{d+1}} - 1\big)\eta(\mathsf{d}x) \right\}$$

for all $d \in \mathbb{Z}_+$ and $\ell = 0, 1, \dots, p^{d+1} - 1$. Then $\eta_{n+1}(\mathbb{Z}^{n+1}) = \eta(\Delta_p \setminus \Lambda_{n+1}) < \infty$, hence η_{n+1} is a bounded measure on $\mathbb{Z}^{n+1}$, and the compound Poisson measure $\mathsf{e}(\eta_{n+1})$ on $\mathbb{Z}^{n+1}$ is defined. The character group of $\mathbb{Z}^{n+1}$ is $(\mathbb{Z}^{n+1})\widehat{\ } = \{\chi_{z_0,z_1,\dots,z_n} : z_0, z_1, \dots, z_n \in \mathbb{T}\}$, where $\chi_{z_0,z_1,\dots,z_n}(\ell_0, \ell_1, \dots, \ell_n) := z_0^{\ell_0} z_1^{\ell_1} \cdots z_n^{\ell_n}$ for all $(\ell_0, \ell_1, \dots, \ell_n) \in \mathbb{Z}^{n+1}$.

We show that the family of measures $\{\mathsf{e}(\eta_{n+1}) : n \in \mathbb{Z}_+\}$ satisfies the consistency property: $\mathsf{e}(\eta_{n+2})(B \times \mathbb{Z}) = \mathsf{e}(\eta_{n+1})(B)$ for all subsets B of $\mathbb{Z}^{n+1}$ and for all $n \in \mathbb{Z}_+$. For this it is enough to check that

$$(\mathsf{e}(\eta_{n+1}))\widehat{\ }(\chi_{z_0,z_1,\dots,z_n}) = \widehat{\mu}(\chi_{z_0,z_1,\dots,z_n}) \tag{4.7.3}$$

for all $z_0, z_1, \dots, z_n \in \mathbb{T}$, where μ is the probability measure on $\mathbb{Z}^{n+1}$ defined by $\mu(B) := \mathsf{e}(\eta_{n+2})(B \times \mathbb{Z})$, $B \subset \mathbb{Z}^{n+1}$. Then

$$(\mathsf{e}(\eta_{n+1}))\widehat{\ }(\chi_{z_0,z_1,\dots,z_n}) = \exp\left\{ \int_{\mathbb{Z}^{n+1}} (z_0^{\ell_0} z_1^{\ell_1} \cdots z_n^{\ell_n} - 1)\, \eta_{n+1}(\mathsf{d}\ell_0, \mathsf{d}\ell_1, \dots, \mathsf{d}\ell_n) \right\},$$

and

$$\begin{aligned}
\widehat{\mu}(\chi_{z_0,z_1,\dots,z_n}) &= \int_{\mathbb{Z}^{n+1}} \chi_{z_0,z_1,\dots,z_n}(\ell_0,\ell_1,\dots,\ell_n)\,\mu(\mathsf{d}\ell_0,\mathsf{d}\ell_1,\dots,\mathsf{d}\ell_n) \\
&= \sum_{\ell\in\mathbb{Z}^{n+1}} z_0^{\ell_0} z_1^{\ell_1}\cdots z_n^{\ell_n}\mu(\{\ell\}) = \sum_{\ell\in\mathbb{Z}^{n+1}} z_0^{\ell_0} z_1^{\ell_1}\cdots z_n^{\ell_n}\mathsf{e}(\eta_{n+2})(\{\ell\}\times\mathbb{Z}) \\
&= \sum_{k\in\mathbb{Z}^{n+2}} z_0^{k_0} z_1^{k_1}\cdots z_n^{k_n}\mathsf{e}(\eta_{n+2})(\{k\}) = (\mathsf{e}(\eta_{n+2}))\widehat{\ }(\chi_{z_0,z_1,\dots,z_n,1}).
\end{aligned}$$

Since

$$\begin{aligned}
&(\mathsf{e}(\eta_{n+2}))\widehat{\ }(\chi_{z_0,z_1,\dots,z_n,1}) \\
&\qquad = \exp\left\{\int_{\mathbb{Z}^{n+2}} (z_0^{\ell_0} z_1^{\ell_1}\cdots z_n^{\ell_n}-1)\,\eta_{n+2}(\mathsf{d}\ell_0,\mathsf{d}\ell_1,\dots,\mathsf{d}\ell_n,\mathsf{d}\ell_{n+1})\right\},
\end{aligned}$$

to prove (4.7.3) it is enough to check that

$$\begin{aligned}
&\int_{\mathbb{Z}^{n+2}} (z_0^{\ell_0} z_1^{\ell_1}\cdots z_n^{\ell_n}-1)\,\eta_{n+2}(\mathsf{d}\ell_0,\mathsf{d}\ell_1,\dots,\mathsf{d}\ell_n,\mathsf{d}\ell_{n+1}) \\
&\qquad\qquad = \int_{\mathbb{Z}^{n+1}} (z_0^{\ell_0} z_1^{\ell_1}\cdots z_n^{\ell_n}-1)\,\eta_{n+1}(\mathsf{d}\ell_0,\mathsf{d}\ell_1,\dots,\mathsf{d}\ell_n).
\end{aligned}$$

We show that both sides of the above equation are equal to

$$\int_{\Delta_p} (z_0^{x_0} z_1^{x_1}\cdots z_n^{x_n}-1)\,\eta(\mathsf{d}x).$$

This integral is finite, since

$$\begin{aligned}
\int_{\Delta_p} |z_0^{x_0} z_1^{x_1}\cdots z_n^{x_n}-1|\,\eta(\mathsf{d}x) &= \int_{\Delta_p\setminus\Lambda_{n+1}} |z_0^{x_0} z_1^{x_1}\cdots z_n^{x_n}-1|\,\eta(\mathsf{d}x) \\
&\leqslant 2\eta(\Delta_p\setminus\Lambda_{n+1}) < \infty.
\end{aligned}$$

Using the notation $\Lambda_{n+1}(\ell) := \{x\in\Delta_p : (x_0,x_1,\dots,x_n)=\ell\}$ for all $\ell\in\mathbb{Z}^{n+1}$, we get

$$\begin{aligned}
\int_{\Delta_p} (z_0^{x_0} z_1^{x_1}\cdots z_n^{x_n}-1)\,\eta(\mathsf{d}x) &= \sum_{\ell\in\mathbb{Z}^{n+1}} \int_{\Lambda_{n+1}(\ell)} (z_0^{x_0} z_1^{x_1}\cdots z_n^{x_n}-1)\,\eta(\mathsf{d}x) \\
&= \sum_{\ell\in\mathbb{Z}^{n+1}} (z_0^{\ell_0} z_1^{\ell_1}\cdots z_n^{\ell_n}-1)\eta_{n+1}(\{\ell\}) \\
&= \int_{\mathbb{Z}^{n+1}} (z_0^{\ell_0} z_1^{\ell_1}\cdots z_n^{\ell_n}-1)\,\eta_{n+1}(\mathsf{d}\ell_0,\mathsf{d}\ell_1,\dots,\mathsf{d}\ell_n).
\end{aligned}$$

A similar computation shows that

$$\int_{\Delta_p} (z_0^{x_0} z_1^{x_1} \cdots z_n^{x_n} - 1)\, \eta(\mathsf{d}x) = \int_{\mathbb{Z}^{n+2}} (z_0^{\ell_0} z_1^{\ell_1} \cdots z_n^{\ell_n} - 1)\, \eta_{n+2}(\mathsf{d}\ell_0, \mathsf{d}\ell_1, \ldots, \mathsf{d}\ell_n, \mathsf{d}\ell_{n+1}).$$

Hence (4.7.3) is satisfied.

By Kolmogorov's Consistency Theorem (see, e.g., Shiryaev [52, p.163, Theorem 3]), there exists a sequence $Y_0, Y_1, \ldots$ of integer valued random variables such that the distribution of $(Y_0, \ldots, Y_n)$ is the compound Poisson measure $\mathsf{e}(\eta_{n+1})$ on $\mathbb{Z}^{n+1}$ for all $n \in \mathbb{Z}_+$. For all $d \in \mathbb{Z}_+$ and $\ell = 0, 1, \ldots, p^{d+1} - 1$ we have

$$\begin{aligned}
\mathsf{E}\, \chi_{d,\ell}(\varphi(Y_0, Y_1, \ldots)) &= \mathsf{E}\, \mathsf{e}^{2\pi i \ell (Y_0 + pY_1 + \cdots + p^d Y_d)/p^{d+1}} \\
&= \exp\left\{ \int_{\mathbb{Z}^{d+1}} \left(\mathsf{e}^{2\pi i \ell (\ell_0 + p\ell_1 + \cdots + p^d \ell_d)/p^{d+1}} - 1 \right) \eta_{d+1}(\mathsf{d}\ell_0, \mathsf{d}\ell_1, \ldots, \mathsf{d}\ell_d) \right\} \\
&= \exp\left\{ \int_{\Delta_p} \left(\mathsf{e}^{2\pi i \ell (x_0 + px_1 + \cdots + p^d x_d)/p^{d+1}} - 1 \right) \eta(\mathsf{d}x) \right\}.
\end{aligned}$$

Hence $\mathsf{E}\, \chi_{d,\ell}(\varphi(Y_0, Y_1, \ldots)) = \widehat{\pi}_{\eta,\, g_{\Delta_p}}(\chi_{d,\ell})$ for all $d \in \mathbb{Z}_+$ and $\ell = 0, 1, \ldots, p^{d+1} - 1$, and we obtain $\varphi(Y_0, Y_1, \ldots) \stackrel{\mathcal{D}}{=} \pi_{\eta,\, g_{\Delta_p}}$.

Since the sequences $U_0, U_1, \ldots$ and $Y_0, Y_1, \ldots$ are independent and the mapping $\varphi : \mathbb{Z}^\infty \to \Delta_p$ is a homomorphism, we have

$$\begin{aligned}
&\mathsf{E}\, \chi(\varphi(U_0 + a_0 + Y_0,\, U_1 + a_1 + Y_1, \ldots)) \\
&= \mathsf{E}\, \chi(\varphi(U_0, U_1, \ldots)) \cdot \chi(\varphi(a_0, a_1, \ldots)) \cdot \mathsf{E}\, \chi(\varphi(Y_0, Y_1, \ldots)) \\
&= \widehat{\omega}_{\Lambda_r}(\chi)\, \widehat{\delta}_a(\chi)\, \widehat{\pi}_{\eta,\, g_{\Delta_p}}(\chi) = (\omega_{\Lambda_r} * \delta_a * \pi_{\eta,\, g_{\Delta_p}})\widehat{\ }(\chi)
\end{aligned}$$

for all $\chi \in \widehat{\Delta}_p$, and we obtain the statement. □

4.8 Limit theorems on the p-adic solenoid

Let p be a prime. The p-adic solenoid is a subgroup of $\mathbb{T}^\infty$, namely,

$$S_p := \left\{ (y_0, y_1, \ldots) \in \mathbb{T}^\infty : y_j = y_{j+1}^p \text{ for all } j \in \mathbb{Z}_+ \right\},$$

furnished with the relative topology as a subset of the locally compact T_0-topological group $\mathbb{T}^\infty$. Then S_p is a second countable compact connected Abelian T_0-topological group. For an equivalent introduction of the p-adic solenoid, see Hewitt–Ross [29, Definition 10.12]. Note that S_p is not a Lie group. By Theorems 23.21 and 24.11 in Hewitt–Ross [29], the character group of S_p is $\widehat{S}_p = \{\chi_{d,\ell} : d \in \mathbb{Z}_+,\, \ell \in \mathbb{Z}\}$, where

$$\chi_{d,\ell}(y) := y_d^\ell, \qquad y \in S_p, \quad d \in \mathbb{Z}_+, \quad \ell \in \mathbb{Z}.$$

The set of all quadratic forms on $\widehat{S}_p$ is $\mathsf{q}_+\big(\widehat{S}_p\big) = \{\psi_b : b \in \mathbb{R}_+\}$, where

$$\psi_b(\chi_{d,\ell}) := \frac{b\ell^2}{p^{2d}}, \qquad d \in \mathbb{Z}_+, \quad \ell \in \mathbb{Z}, \quad b \in \mathbb{R}_+,$$

see, e.g., Heyer–Pap [31, Section 5.4]. The function $g_{S_p} : S_p \times \widehat{S}_p \to \mathbb{R}$,

$$g_{S_p}(y, \chi_{d,\ell}) := \frac{\ell h(\arg y_0)}{p^d}, \qquad y \in S_p, \quad d \in \mathbb{Z}_+, \quad \ell \in \mathbb{Z},$$

is a local inner product for S_p. An extended real valued measure η on S_p is a Lévy measure if and only if $\eta(\{e\}) = 0$ and $\int_{S_p} (\arg y_0)^2\, \eta(\mathsf{d}y) < \infty$.

Theorem 4.3.1 has the following consequence on the p-adic solenoid S_p.

4.8.1 Theorem. (Gauss–Poisson limit theorem) *Let $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \dots, K_n\}$ be a rowwise independent array of random elements in S_p. Suppose that there exists a quadruplet $(\{e\}, a, \psi_b, \eta) \in \mathcal{P}(S_p)$ such that*

(i) $\displaystyle\max_{1 \leqslant k \leqslant K_n} \mathrm{P}(\exists\, j \leqslant d : |\arg((X_{n,k})_j)| > \varepsilon) \to 0$ *as* $n \to \infty$ *for all* $d \in \mathbb{Z}_+$ *and for all* $\varepsilon > 0$,

(ii) $\displaystyle\exp\left\{ \frac{i}{p^d} \sum_{k=1}^{K_n} \mathsf{E}\, h(\arg((X_{n,k})_0)) \right\} \to a_d$ *as* $n \to \infty$ *for all* $d \in \mathbb{Z}_+$,

(iii) $\displaystyle\sum_{k=1}^{K_n} \operatorname{Var} h(\arg((X_{n,k})_0)) \to b + \int_{S_p} h(\arg(y_0))^2\, \eta(\mathsf{d}y)$ *as* $n \to \infty$,

(iv) $\displaystyle\sum_{k=1}^{K_n} \mathsf{E}\, f(X_{n,k}) \to \int_{S_p} f\, \mathsf{d}\eta$ *as* $n \to \infty$ *for all* $f \in \mathcal{C}_0(S_p)$.

Then the array $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \ldots, K_n\}$ *is infinitesimal and*

$$\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \delta_a * \gamma_{\psi_b} * \pi_{\eta,\, g_{S_p}} \qquad \textit{as} \;\; n \to \infty.$$

The next theorem shows that if the limit measure in Theorem 4.8.1 has no generalized Poisson factor $\pi_{\eta,\, g_{S_p}}$ then the truncation function h can be omitted. The proof of this fact can be carried out as in case of Theorem 4.6.2.

4.8.2 Theorem. (CLT) *Let* $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \ldots, K_n\}$ *be a rowwise independent array of random elements in* S_p. *Suppose that there exist an element* $a \in S_p$ *and a nonnegative real number* b *such that*

(i) $\exp\left\{ \dfrac{i}{p^d} \displaystyle\sum_{k=1}^{K_n} \mathsf{E} \arg((X_{n,k})_0) \right\} \to a_d$ *as* $n \to \infty$ *for all* $d \in \mathbb{Z}_+$,

(ii) $\displaystyle\sum_{k=1}^{K_n} \operatorname{Var} \arg((X_{n,k})_0) \to b$ *as* $n \to \infty$,

(iii) $\displaystyle\sum_{k=1}^{K_n} \mathrm{P}(\exists\, j \leqslant d : |\arg((X_{n,k})_j)| > \varepsilon) \to 0$ *as* $n \to \infty$ *for all* $d \in \mathbb{Z}_+$ *and for all* $\varepsilon > 0$.

Then the array $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \ldots, K_n\}$ *is infinitesimal and*

$$\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \delta_a * \gamma_{\psi_b}.$$

Theorem 4.4.4 has the following consequence on S_p.

4.8.3 Theorem. (Limit theorem for rowwise i.i.d. Rademacher array) *Let* $x^{(n)} \in S_p$, $n \in \mathbb{N}$ *such that* $x^{(n)} \to e$. *Let* $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \ldots, K_n\}$ *be a rowwise i.i.d. array of random elements in* S_p *such that* $K_n \to \infty$ *and*

$$\mathrm{P}(X_{n,k} = x^{(n)}) = \mathrm{P}(X_{n,k} = -x^{(n)}) = \frac{1}{2}.$$

Then the array $\{X_{n,k} : n \in \mathbb{N},\, k = 1, \ldots, K_n\}$ *is infinitesimal.*

If b *is a nonnegative real number then*

$$\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \gamma_{\psi_b} \qquad \Longleftrightarrow \qquad K_n\big(\arg(x_0^{(n)})\big)^2 \to b.$$

Moreover,

$$\sum_{k=1}^{K_n} X_{n,k} \xrightarrow{\mathcal{D}} \omega_{S_p} \qquad \Longleftrightarrow \qquad K_n\big(\arg(x_0^{(n)})\big)^2 \to \infty.$$

In the rest of this section we consider the question of giving a construction of a weakly infinitely divisible measure on S_p without a nondegenerate idempotent factor using only real valued random variables. We show that for a weakly infinitely divisible measure μ on S_p without an idempotent factor there exist real valued random variables $Z_0, Z_1, \dots$ such that $(Z_0, \dots, Z_n)$ has a weakly infinitely divisible distribution on $\mathbb{R} \times \mathbb{Z}^n$ for all $n \in \mathbb{Z}_+$, and $\varphi(Z_0, Z_1, \dots) \stackrel{\mathcal{D}}{=} \mu$, where the mapping $\varphi : \mathbb{R} \times \mathbb{Z}^\infty \to S_p$, defined by

$$\begin{aligned}
&\varphi(y_0, y_1, y_2, \dots) \\
&\quad := \big(\mathrm{e}^{iy_0},\, \mathrm{e}^{i(y_0+2\pi y_1)/p},\, \mathrm{e}^{i(y_0+2\pi y_1+2\pi y_2 p)/p^2},\, \mathrm{e}^{i(y_0+2\pi y_1+2\pi y_2 p+2\pi y_3 p^2)/p^3}, \dots\big)
\end{aligned}$$

for $(y_0, y_1, y_2, \dots) \in \mathbb{R} \times \mathbb{Z}^\infty$, is a measurable homomorphism from the Abelian topological group $\mathbb{R} \times \mathbb{Z}^\infty$ (furnished with the product topology) onto S_p. Note that $\mathbb{R} \times \mathbb{Z}^\infty$ is not locally compact, but $\mathbb{R} \times \mathbb{Z}^n$ is a second countable locally compact Abelian T_0-topological group for all $n \in \mathbb{Z}_+$. The character group of $\mathbb{R} \times \mathbb{Z}^n$ is $(\mathbb{R} \times \mathbb{Z}^n)\widehat{\ } = \{\chi_{y,z} : y \in \mathbb{R},\, z \in \mathbb{T}^n\}$, where $\chi_{y,z}(x, \ell) := \mathrm{e}^{iyx} z_1^{\ell_1} \cdots z_n^{\ell_n}$ for all $x, y \in \mathbb{R}$, $z = (z_1, \dots, z_n) \in \mathbb{T}^n$ and $\ell = (\ell_1, \dots, \ell_n) \in \mathbb{Z}^n$. The function $g_{\mathbb{R}\times\mathbb{Z}^n}\big((x,\ell), \chi_{y,z}\big) := yh(x)$ is a local inner product for $\mathbb{R} \times \mathbb{Z}^n$.

We also find independent real valued random variables $U_0, U_1, \dots$ such that $U_0, U_1, \dots$ are uniformly distributed on suitable subsets of $\mathbb{R}$ and $\varphi(U_0, U_1, \dots) \stackrel{\mathcal{D}}{=} \omega_{S_p}$.

4.8.4 Theorem. *If* $(\{e\}, a, \psi_b, \eta) \in \mathcal{P}(S_p)$ *then*

$$\begin{aligned}
&\varphi(\tau(a)_0 + X_0 + Y_0,\, \tau(a)_1 + Y_1,\, \tau(a)_2 + Y_2, \dots) \\
&\quad = \Big(a_0 \mathrm{e}^{i(X_0+Y_0)},\, a_1 \mathrm{e}^{i(X_0+Y_0+2\pi Y_1)/p},\, a_2 \mathrm{e}^{i(X_0+Y_0+2\pi Y_1+2\pi Y_2 p)/p^2}, \dots\Big) \\
&\quad \stackrel{\mathcal{D}}{=} \delta_a * \gamma_{\psi_b} * \pi_{\eta,\, g_{S_p}},
\end{aligned}$$

where the mapping $\tau : S_p \to \mathbb{R} \times \mathbb{Z}^\infty$ *is defined by*

$$\tau(x) := \left(\arg x_0, \, \frac{p \arg x_1 - \arg x_0}{2\pi}, \, \frac{p \arg x_2 - \arg x_1}{2\pi}, \dots \right)$$

for $x = (x_0, x_1, \dots) \in S_p$, X_0, Y_0 *are real valued random variables and* $Y_1, Y_2, \dots$ *are integer valued random variables such that* X_0 *is independent of the sequence* $Y_0, Y_1, \dots,$ *the variable* X_0 *has a normal distribution with zero mean and variance* b, *and the distribution of* $(Y_0, \dots, Y_n)$ *is the generalized Poisson measure* $\pi_{\eta_{n+1}, \, g_{\mathbb{R}\times\mathbb{Z}^n}}$ *on* $\mathbb{R} \times \mathbb{Z}^n$ *for all* $n \in \mathbb{Z}_+$, *where the measure* η_{n+1} *on* $\mathbb{R} \times \mathbb{Z}^n$ *is defined by* $\eta_{n+1}(\{0\}) := 0$ *and*

$$\eta_{n+1}(B \times \{\ell\}) := \eta\big(\big\{x \in S_p : \tau(x)_0 \in B, \, (\tau(x)_1, \dots, \tau(x)_n) = \ell\big\}\big)$$

for all Borel subsets B *of* $\mathbb{R}$ *and for all* $\ell \in \mathbb{Z}^n$ *with* $0 \notin B \times \{\ell\}$.

Moreover,

$$\varphi(U_0, \, U_1, \dots) \stackrel{\mathcal{D}}{=} \omega_{S_p},$$

where $U_0, U_1, \dots$ *are independent real valued random variables such that* U_0 *is uniformly distributed on* $[0, 2\pi]$ *and* $U_1, U_2, \dots$ *are uniformly distributed on* $\{0, 1, \dots, p-1\}$.

Proof. Since X_0, Y_0 and $U_0, U_1, \dots$ are real valued random variables and $Y_1, Y_2, \dots$ are integer valued random variables and the mapping $\varphi : \mathbb{R} \times \mathbb{Z}^\infty \to S_p$ is measurable, we obtain that $\varphi(\tau(a)_0 + X_0 + Y_0, \, \tau(a)_1 + Y_1, \, \tau(a)_2 + Y_2, \dots)$ and $\varphi(U_0, \, U_1, \dots)$ are random elements with values in S_p.

For $a \in S_p$, we have $a = \varphi(\tau(a))$, hence $\varphi(\tau(a)) \stackrel{\mathcal{D}}{=} \delta_a$.

For $b \in \mathbb{R}_+$, the Fourier transform of the Gauss measure γ_{ψ_b} has the form

$$\widehat{\gamma}_{\psi_b}(\chi_{d,\ell}) = \exp\left\{ -\frac{b\ell^2}{2p^{2d}} \right\}, \qquad d \in \mathbb{Z}_+, \quad \ell \in \mathbb{Z}.$$

For all $d \in \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$,

$$\mathsf{E}\, \chi_{d,\ell}(\varphi(X_0, 0, 0, \dots)) = \mathsf{E}\, \mathrm{e}^{i\ell X_0/p^d} = \exp\left\{ -\frac{b\ell^2}{2p^{2d}} \right\}.$$

Hence $\mathsf{E}\, \chi_{d,\ell}(\varphi(X_0, 0, 0, \dots)) = \widehat{\gamma}_{\psi_b}(\chi_{d,\ell})$ for all $d \in \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$, and we obtain $\varphi(X_0, 0, 0, \dots) \stackrel{\mathcal{D}}{=} \gamma_{\psi_b}$.

For a Lévy measure $\eta \in \mathbb{L}(S_p)$, the Fourier transform of the generalized Poisson measure $\pi_{\eta,\, g_{S_p}}$ has the form

$$\widehat{\pi}_{\eta,\, g_{S_p}}(\chi_{d,\ell}) = \exp\left\{ \int_{S_p} \left(y_d^\ell - 1 - i\ell h(\arg y_0)/p^d \right) \eta(\mathsf{d}y) \right\}$$

for all $d \in \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$. An extended real valued measure $\widetilde{\eta}$ on $\mathbb{R} \times \mathbb{Z}^n$ is a Lévy measure if and only if $\widetilde{\eta}(\{0\}) = 0$, $\widetilde{\eta}(\{(x,\ell) \in \mathbb{R} \times \mathbb{Z}^n : |x| \geqslant \varepsilon \text{ or } \ell \neq 0\}) < \infty$ for all $\varepsilon > 0$, and $\int_{\mathbb{R}\times\mathbb{Z}^n} h(x)^2 \, \widetilde{\eta}(\mathsf{d}x, \mathsf{d}\ell) < \infty$. We have

$$\begin{aligned} &\eta_{n+1}(\{(x,\ell) \in \mathbb{R} \times \mathbb{Z}^n : |x| \geqslant \varepsilon \text{ or } \ell \neq 0\}) \\ &\quad = \eta(\{y \in S_p : |\arg y_0| \geqslant \varepsilon \text{ or } (\tau(y)_1, \ldots, \tau(y)_n) \neq 0\}) = \eta(S_p \setminus N_{\varepsilon,n}) < \infty \end{aligned}$$

for all $\varepsilon \in (0, \pi)$, where

$$N_{\varepsilon,n} := \{y \in S_p : |\arg y_0| < \varepsilon,\ |\arg y_1| < \varepsilon/p, \ldots, |\arg y_n| < \varepsilon/p^n\}.$$

Moreover, $\int_{\mathbb{R}\times\mathbb{Z}^n} h(x)^2 \, \eta_{n+1}(\mathsf{d}x, \mathsf{d}\ell) = \int_{S_p} h(\arg y_0)^2 \, \eta(\mathsf{d}y) < \infty$, since η is a Lévy measure on S_p. Hence, η_{n+1} is a Lévy measure on $\mathbb{R} \times \mathbb{Z}^n$. The family of measures $\{\pi_{\eta_{n+1},\, g_{\mathbb{R}\times\mathbb{Z}^n}} : n \in \mathbb{Z}_+\}$ is consistent, since $\pi_{\eta_{n+2},\, g_{\mathbb{R}\times\mathbb{Z}^{n+1}}}(\{x\} \times \mathbb{Z}) = \pi_{\eta_{n+1},\, g_{\mathbb{R}\times\mathbb{Z}^n}}(\{x\})$ for all $x \in \mathbb{R} \times \mathbb{Z}^{n+1}$ and $n \in \mathbb{Z}_+$. Indeed, this is a consequence of

$$(\pi_{\eta_{n+2},\, g_{\mathbb{R}\times\mathbb{Z}^{n+1}}})\widehat{\ }(\chi_{y,z_1,\ldots,z_n,1}) = (\pi_{\eta_{n+1},\, g_{\mathbb{R}\times\mathbb{Z}^n}})\widehat{\ }(\chi_{y,z_1,\ldots,z_n})$$

for all $y \in \mathbb{R}$, $z_1, \ldots, z_n \in \mathbb{T}$, which follows from

$$\begin{aligned} &\int_{\mathbb{R}\times\mathbb{Z}^{n+1}} \left(\mathsf{e}^{iyx} z_1^{\ell_1} \cdots z_n^{\ell_n} - 1 - iyh(x) \right) \eta_{n+2}(\mathsf{d}x, \mathsf{d}\ell_1, \ldots, \mathsf{d}\ell_n, \mathsf{d}\ell_{n+1}) \\ &= \int_{\mathbb{R}\times\mathbb{Z}^n} \left(\mathsf{e}^{iyx} z_1^{\ell_1} \cdots z_n^{\ell_n} - 1 - iyh(x) \right) \eta_{n+1}(\mathsf{d}x, \mathsf{d}\ell_1, \ldots, \mathsf{d}\ell_n) \end{aligned}$$

for all $y \in \mathbb{R}$, $z_1, \ldots, z_n \in \mathbb{T}$, where both sides are equal to

$$\begin{aligned} I := \int_{S_p} \Big(&\mathsf{e}^{iy \arg x_0} z_1^{(p \arg x_1 - \arg x_0)/(2\pi)} \cdots z_n^{(p \arg x_n - \arg x_{n-1})/(2\pi)} \\ &- 1 - iyh(\arg x_0) \Big) \, \eta(\mathsf{d}x). \end{aligned}$$

This integral is finite. Indeed, for all $x \in N_{\varepsilon,n}$ and $0 < \varepsilon < \pi/2$ we have $p \arg x_k = \arg x_{k-1}$ for each $k = 1, \ldots, n$, hence

$$|I| \leqslant (2 + \pi|y|)\, \eta(S_p \setminus N_{\varepsilon,n}) + \int_{N_{\varepsilon,n}} |\mathsf{e}^{iy \arg x_0} - 1 - iy \arg x_0|\, \eta(\mathsf{d}x)$$

$$\leqslant (2 + \pi|y|)\, \eta(S_p \setminus N_{\varepsilon,n}) + \frac{1}{2} \int_{N_{\varepsilon,n}} (\arg x_0)^2\, \eta(\mathsf{d}x) < \infty,$$

since η is a Lévy measure on S_p. By Kolmogorov's Consistency Theorem (see, e.g., Shiryaev [52, p.163, Theorem 3]), there exist a real valued random variable Y_0 and a sequence $Y_1,\, Y_2,\, \ldots$ of integer valued random variables such that the distribution of $(Y_0, \ldots, Y_n)$ is the generalized Poisson measure $\pi_{\eta_{n+1},\, g_{\mathbb{R}\times\mathbb{Z}^n}}$ for all $n \in \mathbb{Z}_+$. For all $d \in \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$,

$$\mathsf{E}\, \chi_{d,\ell}(\varphi(Y_0, Y_1, \ldots)) = \mathsf{E}\, \mathsf{e}^{i\ell(Y_0 + 2\pi Y_1 + \cdots + 2\pi Y_d p^{d-1})/p^d}$$

$$= \exp\left\{ \int_{\mathbb{R}\times\mathbb{Z}^d} \left(\mathsf{e}^{i\ell(x + 2\pi \ell_1 + \cdots + 2\pi \ell_d p^{d-1})/p^d} - 1 - i\ell h(x)/p^d \right) \eta_{d+1}(\mathsf{d}x, \mathsf{d}\ell_1, \ldots, \mathsf{d}\ell_d) \right\}$$

$$= \exp\left\{ \int_{S_p} \left(y_d^\ell - 1 - i\ell h(\arg y_0)/p^d \right) \eta(\mathsf{d}y) \right\}.$$

Hence $\mathsf{E}\, \chi_{d,\ell}(\varphi(Y_0, Y_1, \ldots)) = \widehat{\pi}_{\eta,\, g_{S_p}}(\chi_{d,\ell})$ for all $d \in \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$, and we obtain $\varphi(Y_0, Y_1, \ldots) \stackrel{\mathcal{D}}{=} \pi_{\eta,\, g_{S_p}}$.

Since the sequence $Y_0, Y_1, \ldots$ and the random variable X_0 are independent and the mapping $\varphi : \mathbb{R} \times \mathbb{Z}^\infty \to S_p$ is a homomorphism, we get

$$\begin{aligned}
&\mathsf{E}\, \chi(\varphi(\tau(a)_0 + X_0 + Y_0,\, \tau(a)_1 + Y_1,\, \tau(a)_2 + Y_2, \ldots)) \\
&= \chi(\varphi(\tau(a)_0, \tau(a)_1, \ldots)) \cdot \mathsf{E}\, \chi(\varphi(X_0, 0, 0, \ldots)) \cdot \mathsf{E}\, \chi(\varphi(Y_0, Y_1, \ldots)) \\
&= \widehat{\delta}_a(\chi)\, \widehat{\gamma}_{\psi_b}(\chi)\, \widehat{\pi}_{\eta,\, g_{S_p}}(\chi) = (\delta_a * \gamma_{\psi_b} * \pi_{\eta,\, g_{S_p}})\widehat{\ }(\chi)
\end{aligned}$$

for all $\chi \in \widehat{S}_p$, and we obtain the first statement.

For all $d \in \mathbb{Z}_+$ and $\ell \in \mathbb{Z} \setminus \{0\}$,

$$\mathsf{E}\, \chi_{d,\ell}(\varphi(U_0, U_1, \ldots)) = \mathsf{E}\, \mathsf{e}^{i\ell(U_0 + 2\pi U_1 + \cdots + 2\pi U_d p^{d-1})/p^d}$$

$$= \frac{1}{2\pi p^d} \int_0^{2\pi} \mathsf{e}^{i\ell x/p^d}\, \mathsf{d}x \sum_{j_0=0}^{p-1} \cdots \sum_{j_{d-1}=0}^{p-1} \mathsf{e}^{2\pi i\ell(j_0 + j_1 p + \cdots + j_{d-1} p^{d-1})/p^d}.$$

Using (4.7.2), we get $\mathsf{E}\,\chi_{d,\ell}(\varphi(U_0, U_1, \dots)) = 0$ for all $d \in \mathbb{Z}_+$ and $\ell \in \mathbb{Z} \setminus \{0\}$. Hence $\mathsf{E}\,\chi_{d,\ell}(\varphi(U_0, U_1, \dots)) = \widehat{\omega}_{S_p}(\chi_{d,\ell})$ for all $d \in \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$, and we obtain $\varphi(U_0, U_1, \dots) \stackrel{\mathcal{D}}{=} \omega_{S_p}$. $\square$

Chapter 5

Portmanteau theorem for unbounded measures

In this chapter we prove an analogue of the portmanteau theorem on weak convergence of probability measures allowing measures which are finite on the complement of any Borel neighbourhood of a fixed element of an underlying metric space. We use this result in proving Gaiser's limit theorem (Theorem 4.3.1). We present this separately, because it can be formulated in a more general setting than it is needed in proving Gaiser's theorem.

The results of this chapter are contained in our submitted paper [9].

5.1 Motivation

Weak convergence of probability measures on a metric space has a very important role in probability theory. The well-known *portmanteau theorem* due to A. D. Alexandroff (see, e.g., Dudley [19, Theorem 11.1.1]) provides useful conditions equivalent to weak convergence of probability measures; any of them could serve as the definition of weak convergence. Proposition 1.2.13 in the book of Meerschaert and Scheffler [39] gives an analogue of the portmanteau theorem for bounded measures on $\mathbb{R}^d$. Moreover, Proposition 1.2.19 in Meerschaert and Scheffler [39] gives an analogue for special unbounded measures on $\mathbb{R}^d$, more precisely, for extended real valued measures which are finite on the complement of any Borel neighbourhood of $0 \in \mathbb{R}^d$.

By giving counterexamples we show that some parts of Propositions 1.2.13

and 1.2.19 in Meerschaert and Scheffler [39] are not true, namely, the equivalence of (c) and (d) in their propositions is not valid (see Remark 5.2.3 and Remark 5.2.4). We reformulate Proposition 1.2.19 in Meerschaert and Scheffler [39] in a more detailed form adding new equivalent assertions to it (see Theorem 5.2.1). Moreover, we note that Theorem 5.2.1 generalizes the equivalence of (a) and (b) in Theorem 11.3.3 of Dudley [19] in two aspects. On the one hand, the equivalence is extended allowing not necessarily finite measures which are finite on the complement of any Borel neighbourhood of a fixed element of an underlying metric space. On the other hand, we do not assume the separability of the underlying metric space to prove the equivalence. But we mention that this latter fact is hiddenly contained in Problem 3, p. 312 in Dudley [19]. For completeness we give a detailed proof of Theorem 5.2.1. Our proof goes along the lines of the proof of the original portmanteau theorem (Dudley [19, Theorem 11.1.1]) and differs from the proof of Proposition 1.2.19 in Meerschaert and Scheffler [39].

To shed some light on the sense of the analogue of the portmanteau theorem, let us consider the question of weak convergence of infinitely divisible probability measures μ_n, $n \in \mathbb{N}$ towards an infinitely divisible probability measure μ_0 in case of the real line $\mathbb{R}$. Theorem 2.9, p. 355 in Jacod–Shiryayev [33] gives equivalent conditions for weak convergence $\mu_n \xrightarrow{\mathsf{w}} \mu_0$. Among these conditions we have

$$\int_{\mathbb{R}} f \, \mathsf{d}\eta_n \to \int_{\mathbb{R}} f \, \mathsf{d}\eta_0 \qquad \text{for all } \ f \in \mathcal{C}_2(\mathbb{R}), \tag{5.1.1}$$

where η_n, $n \in \mathbb{Z}_+$ are nonnegative, extended real valued measures on $\mathbb{R}$ with $\eta_n(\{0\}) = 0$ and $\int_{\mathbb{R}} (x^2 \wedge 1) \, \eta_n(\mathsf{d}x) < \infty$, (i.e., Lévy measures on $\mathbb{R}$) corresponding to μ_n, and $\mathcal{C}_2(\mathbb{R})$ is the set of all real valued bounded continuous functions f on $\mathbb{R}$ vanishing on some Borel neighbourhood of 0 and having a limit at infinity. The analogue of the portmanteau theorem is about the equivalent reformulations of (5.1.1) when it holds for all real valued bounded continuous functions on $\mathbb{R}$ vanishing on some Borel neighbourhood of 0.

5.2 An analogue of the portmanteau theorem

Let $\mathbb{Z}_+$ denote the set of nonnegative integers. Let (X, d) be a metric space and x_0 be a fixed element of X. Let $\mathcal{B}(X)$ denote the σ-algebra of Borel subsets of X. A Borel neighbourhood U of x_0 is an element of $\mathcal{B}(X)$ for which there exists an open subset $\widetilde{U}$ of X such that $x_0 \in \widetilde{U} \subset U$. Let $\mathcal{N}_{x_0}$ denote the set of all Borel neighbourhoods of x_0, and the set of bounded measures on X is denoted by $\mathcal{M}^b(X)$. The expression "a measure μ on X" means a measure μ on the σ-algebra $\mathcal{B}(X)$.

Let $\mathcal{C}(X)$, $\mathcal{C}_{x_0}(X)$ and $\mathrm{BL}_{x_0}(X)$ denote the spaces of all real valued bounded continuous functions on X, the set of all elements of $\mathcal{C}(X)$ vanishing on some Borel neighbourhood of x_0, and the set of all real valued bounded Lipschitz functions vanishing on some Borel neighbourhood of x_0, respectively.

For a measure η on X and for a Borel subset $B \in \mathcal{B}(X)$, let $\eta|_B$ denote the restriction of η onto B, i.e., $\eta|_B(A) := \eta(B \cap A)$ for all $A \in \mathcal{B}(X)$.

Let μ_n, $n \in \mathbb{Z}_+$ be bounded measures on X. We say that $\mu_n \xrightarrow{\mathsf{w}} \mu$ if $\mu_n(A) \to \mu(A)$ for all $A \in \mathcal{B}(X)$ with $\mu(\partial A) = 0$. This is called *weak convergence of bounded measures* on X.

The well-known portmanteau theorem (see, e.g., Dudley [19, Theorem 11.1.1]) gives equivalent reformulations of weak convergence of probability measures.

Now we formulate and prove an analogue of the portmanteau theorem for unbounded measures.

5.2.1 Theorem. *Let (X, d) be a metric space and x_0 be a fixed element of X. Let η_n, $n \in \mathbb{Z}_+$, be measures on X such that $\eta_n(X \setminus U) < \infty$ for all $U \in \mathcal{N}_{x_0}$ and for all $n \in \mathbb{Z}_+$. Then the following assertions are equivalent:*

(i) $\int_{X\setminus U} f \, \mathsf{d}\eta_n \to \int_{X\setminus U} f \, \mathsf{d}\eta_0$ *for all* $f \in \mathcal{C}(X)$ *and for all* $U \in \mathcal{N}_{x_0}$ *with* $\eta_0(\partial U) = 0$,

(ii) $\eta_n|_{X\setminus U} \xrightarrow{\mathsf{w}} \eta_0|_{X\setminus U}$ *for all* $U \in \mathcal{N}_{x_0}$ *with* $\eta_0(\partial U) = 0$,

(iii) $\eta_n(X \setminus U) \to \eta_0(X \setminus U)$ *for all* $U \in \mathcal{N}_{x_0}$ *with* $\eta_0(\partial U) = 0$,

(iv) $\int_X f \, \mathsf{d}\eta_n \to \int_X f \, \mathsf{d}\eta_0$ *for all* $f \in \mathcal{C}_{x_0}(X)$,

(v) $\int_X f \, \mathsf{d}\eta_n \to \int_X f \, \mathsf{d}\eta_0$ *for all* $f \in \mathrm{BL}_{x_0}(X)$,

(vi) *the following inequalities hold:*

  (a) $\limsup_{n\to\infty} \eta_n(X\setminus U) \leqslant \eta_0(X\setminus U)$ *for all open neighbourhoods* U *of* x_0,

  (b) $\liminf_{n\to\infty} \eta_n(X\setminus V) \geqslant \eta_0(X\setminus V)$ *for all closed neighbourhoods* V *of* x_0.

Proof. First we show the equivalence of (i),(ii) and (iii).
(i)⇒(ii): Suppose that (i) holds. Let U be an element of $\mathcal{N}_{x_0}$ with $\eta_0(\partial U) = 0$. Note that $\eta_n|_{X\setminus U} \in \mathcal{M}^b(X)$, $n \in \mathbb{Z}_+$. By the equivalence of

(a) and (b) in Proposition 1.2.13 in Meerschaert and Scheffler [39], to prove $\eta_n|_{X\setminus U} \xrightarrow{\mathsf{w}} \eta_0|_{X\setminus U}$ it is enough to check

$$\int_X f \, \mathsf{d}\eta_n|_{X\setminus U} \to \int_X f \, \mathsf{d}\eta_0|_{X\setminus U} \quad \text{for all} \quad f \in \mathcal{C}(X).$$

For this it is enough to show that for all real valued bounded measurable functions h on X, for all $A \in \mathcal{B}(X)$ and for all $n \in \mathbb{Z}_+$ we have

$$\int_X h \, \mathsf{d}\eta_n|_A = \int_A h \, \mathsf{d}\eta_n. \tag{5.2.1}$$

Using Beppo-Levi's theorem, a standard measure-theoretic argument shows that (5.2.1) is valid.

(ii)⇒(iii): Suppose that (ii) holds. Let U be an element of $\mathcal{N}_{x_0}$ with $\eta_0(\partial U) = 0$. By (ii), we have $\eta_n|_{X\setminus U} \xrightarrow{\mathsf{w}} \eta_0|_{X\setminus U}$. Since $\eta_0|_{X\setminus U}(\partial X) = \eta_0|_{X\setminus U}(\emptyset) = 0$, we get $\eta_n(X \setminus U) = \eta_n|_{X\setminus U}(X) \to \eta_0|_{X\setminus U}(X) = \eta_0(X \setminus U)$, as desired.

(iii)⇒(ii): Suppose that (iii) holds. Let U be an element of $\mathcal{N}_{x_0}$ with $\eta_0(\partial U) = 0$ and let $B \in \mathcal{B}(X)$ be such that $\eta_0|_{X\setminus U}(\partial B) = 0$. We have to show that $\eta_n|_{X\setminus U}(B) \to \eta_0|_{X\setminus U}(B)$.

Since $\eta_n|_{X\setminus U}(B) = \eta_n(B \cap (X \setminus U))$, $n \in \mathbb{Z}_+$ and

$$B \cap (X \setminus U) = X \setminus [X \setminus (B \cap (X \setminus U))],$$

by (iii), it is enough to check that $\eta_0\big(\partial\big(X \setminus (B \cap (X \setminus U))\big)\big) = 0$. First we show that

$$\partial\big(B \cap (X \setminus U)\big) \subset \big(\partial B \cap (X \setminus U)\big) \cup \partial U, \tag{5.2.2}$$

for all subsets B, U of X. Let x be an element of $\partial\big(B \cap (X \setminus U)\big)$ and $(y_n)_{n\geqslant 1}$, $(z_n)_{n\geqslant 1}$ be two sequences such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = x$ and $y_n \in B \cap (X \setminus U)$, $z_n \in X \setminus (B \cap (X \setminus U))$, $n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$ we have one or two of the following possibilities:

- $y_n \in B$, $y_n \in X \setminus U$ and $z_n \in X \setminus B$,
- $y_n \in B$, $y_n \in X \setminus U$ and $z_n \in U$.

Then we get $x \in \big(\partial B \cap ((X \setminus U) \cup \partial U)\big) \cup \big(\partial U \cap (B \cup \partial B)\big) \cup \big(\partial B \cap \partial U\big)$. Since $\partial B \cap ((X \setminus U) \cup \partial U) \subset (\partial B \cap (X \setminus U)) \cup \partial U$, we have $x \in \big(\partial B \cap (X \setminus U)\big) \cup \partial U$, as desired.

Using (5.2.2) we get $\eta_0\big(\partial\big(X\setminus(B\cap(X\setminus U))\big)\big) \leqslant \eta_0\big(\partial B\cap(X\setminus U)\big)+\eta_0(\partial U) = 0$. Indeed, by the assumptions $\eta_0\big(\partial B\cap(X\setminus U)\big) = 0$ and $\eta_0(\partial U) = 0$. Hence $\eta_0\big(\partial\big(X\setminus(B\cap(X\setminus U))\big)\big) = 0$.

(ii)⇒(i): Using again the equivalence of (a) and (b) in Proposition 1.2.13 in Meerschaert and Scheffler [39] and (5.2.1) we obtain (i).

(iii)⇒(iv): Suppose that (iii) holds. Let f be an element of $\mathcal{C}_{x_0}(X)$. Then there exists $A\in\mathcal{N}_{x_0}$ such that $f(x)=0$ for all $x\in A$ and $\eta_0(\partial A)=0$. Indeed, using that the function $t\mapsto\eta_0\big(\{x\in X : d(x,x_0)\geqslant t\}\big)$ from $(0,\infty)$ into $\mathbb{R}$ is monotone decreasing, we get the set $\big\{t\in(0,\infty) : \eta_0(\{x\in X : d(x,x_0)=t\})>0\big\}$ of its discontinuities is at most countable. Consequently, for all $\widetilde{U}\in\mathcal{N}_{x_0}$ there exists some $t>0$ such that $U:=\{x\in X : d(x,x_0)<t\}\in\mathcal{N}_{x_0}$, $U\subset\widetilde{U}$ and $\eta_0(\partial U)=0$. (Note that at this step we use that an element $\widetilde{U}$ of $\mathcal{N}_{x_0}$ contains an open subset of X containing x_0.) This implies the existence of A. We show that the set

$$D:=\Big\{t\in\mathbb{R} : \eta_0\big(\{x\in X : f(x)=t\}\big)>0\Big\}$$

is at most countable. The function $F:\mathbb{R}\to[0,\eta_0(X\setminus A)]$, defined by

$$F(t):=\eta_0\big(\{x\in X\setminus A : f(x)<t\}\big),\quad t\in\mathbb{R},$$

is monotone increasing and left-continuous, so it has at most countable many discontinuity points. (Note that $\eta_0(X\setminus A)<\infty$, by the assumption on η_0.) And $t_0\in\mathbb{R}$ is a discontinuity point of F if and only if $F(t_0+0)>F(t_0)$, i.e., $\eta_0\big(\{x\in X\setminus A : f(x)=t_0\}\big)>0$. If $t_0\neq 0$, then

$$\{x\in X : f(x)=t_0\}=\{x\in X\setminus A : f(x)=t_0\},$$

which implies that $t_0\neq 0$ is a discontinuity point of F if and only if $\eta_0(\{x\in X : f(x)=t_0\})>0$. Hence if $t\in D$ then $t=0$ or t is a discontinuity point of F, which yields that D is at most countable. Since f is bounded and D is at most countable, there exists a real number $M>0$ such that $-M,M\notin D$ and $|f(x)|<M$ for $x\in X$. Let $\varepsilon>0$ be arbitrary, but fixed. Choose real numbers t_i, $i=0,\dots,k$ such that $-M=t_0<t_1<\cdots<t_k=M$, $t_i\notin D$, $i=0,\dots,k$ and $\max_{0\leqslant i\leqslant k-1}(t_{i+1}-t_i)<\varepsilon$. The countability of D implies the existence of t_i, $i=0,\dots,k$. Let

$$B_i:=f^{-1}\big([t_i,t_{i+1})\big)\cap(X\setminus A)=\Big\{x\in X\setminus A : t_i\leqslant f(x)<t_{i+1}\Big\},\quad i=0,\dots,k-1.$$

Then B_i, $i = 0, \dots, k-1$, are pairwise disjoint Borel sets and $X \setminus A = \bigcup_{i=0}^{k-1} B_i$. Since f is continuous, the boundary $\partial(f^{-1}(H))$ of the set $f^{-1}(H)$ is a subset of the set $f^{-1}(\partial H)$ for all subsets H of $\mathbb{R}$. Using (5.2.2) this implies that

$$\partial(X \setminus B_i) = \partial B_i \subset f^{-1}(\{t_i\}) \cup f^{-1}(\{t_{i+1}\}) \cup \partial A, \quad i = 0, \dots, k-1.$$

Since $t_i \notin D$, $i = 0, \dots, k$, $\eta_0(\partial A) = 0$, and

$$\eta_0(\partial(X \setminus B_i)) \leqslant \eta_0\big(\{x \in X : f(x) = t_i\}\big) + \eta_0\big(\{x \in X : f(x) = t_{i+1}\}\big) + \eta_0(\partial A),$$

we get $\eta_0(\partial(X \setminus B_i)) = 0$, $i = 0, \dots, k-1$. Since $A \subset X \setminus B_i$, we have $X \setminus B_i \in \mathcal{N}_{x_0}$ for all $i = 0, \dots, k-1$. Hence condition (iii) implies that $\eta_n(B_i) \to \eta_0(B_i)$ as $n \to \infty$, $i = 0, \dots, k-1$. Then

$$\begin{aligned}
\Big| \int_X f \, \mathsf{d}\eta_n - \int_X f \, \mathsf{d}\eta_0 \Big| &= \Big| \int_{X \setminus A} f \, \mathsf{d}\eta_n - \int_{X \setminus A} f \, \mathsf{d}\eta_0 \Big| \\
&\leqslant \Big| \int_{X \setminus A} f \, \mathsf{d}\eta_n - \sum_{i=0}^{k-1} t_i \eta_n(B_i) \Big| + \Big| \sum_{i=0}^{k-1} t_i \big(\eta_n(B_i) - \eta_0(B_i)\big) \Big| \\
&\quad + \Big| \sum_{i=0}^{k-1} t_i \eta_0(B_i) - \int_{X \setminus A} f \, \mathsf{d}\eta_0 \Big| \\
&\leqslant \sum_{i=0}^{k-1} \int_{B_i} |f(x) - t_i| \, \eta_n(\mathsf{d}x) + \Big| \sum_{i=0}^{k-1} t_i \big(\eta_n(B_i) - \eta_0(B_i)\big) \Big| \\
&\quad + \sum_{i=0}^{k-1} \int_{B_i} |f(x) - t_i| \, \eta_0(\mathsf{d}x) \\
&\leqslant 2 \max_{0 \leqslant i \leqslant k-1} (t_{i+1} - t_i) + \Big| \sum_{i=0}^{k-1} t_i \big(\eta_n(B_i) - \eta_0(B_i)\big) \Big|.
\end{aligned}$$

Hence

$$\limsup_{n \to \infty} \Big| \int_X f \, \mathsf{d}\eta_n - \int_X f \, \mathsf{d}\eta_0 \Big| \leqslant 2 \max_{0 \leqslant i \leqslant k-1} (t_{i+1} - t_i) < 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (iv) holds.

(iv)⇒(v): It is trivial, since $\mathrm{BL}_{x_0}(X) \subset \mathcal{C}_{x_0}(X)$.

(v)⇒(vi): Suppose that (v) holds. First let U be an open neighbourhood of x_0. Let $\varepsilon > 0$ be arbitrary, but fixed. We show that there exists a closed neighbourhood U_ε of x_0 such that $U_\varepsilon \subset U$ and $\eta_0(U \setminus U_\varepsilon) < \varepsilon$, and a function $f \in \mathrm{BL}_{x_0}(X)$ such that $f(x) = 0$ for $x \in U_\varepsilon$, $f(x) = 1$ for $x \in X \setminus U$ and $0 \leqslant f(x) \leqslant 1$ for $x \in X$.

For all $B \in \mathcal{B}(X)$ and for all $\lambda > 0$ we use the notation $B^\lambda := \Big\{x \in X : d(x, B) < \lambda\Big\}$, where $d(x, B) := \inf\{d(x, z) : z \in B\}$. Since U is open, we get $U = \bigcup_{n=1}^\infty F_n$, where $F_n := X \setminus (X \setminus U)^{1/n}$, $n \in \mathbb{N}$. Then $F_n \subset F_{n+1}$, $n \in \mathbb{N}$, F_n is a closed subset of X for all $n \in \mathbb{N}$ and $\bigcap_{n=1}^\infty (X \setminus F_n) = X \setminus U$. We also have $\eta_0(X \setminus F_N) < \infty$ for some sufficiently large $N \in \mathbb{N}$ and $X \setminus F_n \supset X \setminus F_{n+1}$ for all $n \in \mathbb{N}$, and hence the continuity of the measure η_0 implies that $\lim_{n\to\infty} \eta_0(X \setminus F_n) = \eta_0(X \setminus U)$. Since $\eta_0(X \setminus U) < \infty$, there exists some $n_0 \in \mathbb{N}$ such that $\eta_0(X \setminus F_{n_0}) - \eta_0(X \setminus U) < \varepsilon$. Set $U_\varepsilon := F_{n_0}$. Since

$$\eta_0(X \setminus F_{n_0}) - \eta_0(X \setminus U) = \eta_0\big((X \setminus F_{n_0}) \setminus (X \setminus U)\big) = \eta_0(U \setminus F_{n_0}),$$

we have U_ε is a closed neighborhood of x_0, $U_\varepsilon \subset U$ and $\eta_0(U \setminus U_\varepsilon) < \varepsilon$.

We show that the function $f : X \to \mathbb{R}$, defined by $f(x) := \min(1, n_0 d(x, U_\varepsilon))$, $x \in X$, is an element of $\mathrm{BL}_{x_0}(X)$, $f(x) = 0$ for $x \in U_\varepsilon$, $f(x) = 1$ for $x \in X \setminus U$ and $0 \leqslant f(x) \leqslant 1$ for $x \in X$.
Note that if $x \in U_\varepsilon$ then $d(x, U_\varepsilon) = 0$, hence $f(x) = 0$. And if $x \in X \setminus U$ then $d(x, U_\varepsilon) \geqslant d(X \setminus U, U_\varepsilon) \geqslant 1/n_0$, hence $f(x) = 1$. The fact that $0 \leqslant f(x) \leqslant 1$, $x \in X$ is obvious. To prove that f is Lipschitz, we check that

$$|f(x) - f(y)| \leqslant n_0 d(x, y) \qquad \text{for all } x, y \in X.$$

If $x, y \in X$ with $d(x, y) \geqslant 1/n_0$ then $|f(x) - f(y)| \leqslant 1 \leqslant n_0 d(x, y)$. If $x, y \in X$ with $d(x, y) < 1/n_0$ then we have to consider the following four cases apart from changing the role of x and y :

- $x \in X \setminus U$, $y \in U \setminus U_\varepsilon$,
- $x \in U_\varepsilon$, $y \in U \setminus U_\varepsilon$,
- $x, y \in U \setminus U_\varepsilon$,
- $x, y \in U_\varepsilon$ or $x, y \in X \setminus U$.

If $x \in X \setminus U$, $y \in U \setminus U_\varepsilon$ and $f(y) = n_0 d(y, U_\varepsilon)$ then $d(y, U_\varepsilon) \leqslant 1/n_0$ and we get $|f(x) - f(y)| = 1 - n_0 d(y, U_\varepsilon) \leqslant n_0 d(x, y)$. Indeed,

$$1/n_0 \leqslant d(X \setminus U, U_\varepsilon) \leqslant d(x, U_\varepsilon) \leqslant d(x, y) + d(y, U_\varepsilon).$$

If $x \in X \setminus U$, $y \in U \setminus U_\varepsilon$ and $f(y) = 1$ then $|f(x) - f(y)| = 0 \leqslant n_0 d(x, y)$.
If $x \in U_\varepsilon$, $y \in U \setminus U_\varepsilon$ and $f(y) = 1$ then $d(y, U_\varepsilon) \geqslant 1/n_0$ and we get $|f(x) - f(y)| = 1 \leqslant n_0 d(x, y)$. Indeed, $d(x, y) \geqslant d(U_\varepsilon, y) \geqslant 1/n_0$. If $x \in U_\varepsilon$, $y \in U \setminus U_\varepsilon$ and $f(y) = n_0 d(y, U_\varepsilon)$ then $d(y, U_\varepsilon) \leqslant 1/n_0$ and we get $|f(x) - f(y)| = n_0 d(y, U_\varepsilon) \leqslant n_0 d(x, y)$.
If $x, y \in U \setminus U_\varepsilon$ and $f(x) = 1$, $f(y) = n_0 d(y, U_\varepsilon)$ then $d(x, U_\varepsilon) \geqslant 1/n_0$, $d(y, U_\varepsilon) \leqslant 1/n_0$ and we get $|f(x) - f(y)| = 1 - n_0 d(y, U_\varepsilon) \leqslant n_0 d(x, y)$. Indeed, $1/n_0 \leqslant d(x, U_\varepsilon) \leqslant d(x, y) + d(y, U_\varepsilon)$. The case $x, y \in U \setminus U_\varepsilon$ and $f(y) = 1$, $f(x) = n_0 d(x, U_\varepsilon)$ can be handled similarly. If $x, y \in U \setminus U_\varepsilon$ and $f(x) = n_0 d(x, U_\varepsilon)$, $f(y) = n_0 d(y, U_\varepsilon)$ then

$$|f(x) - f(y)| = n_0 |d(x, U_\varepsilon) - d(y, U_\varepsilon)| \leqslant n_0 d(x, y).$$

Indeed, since U_ε is closed, we have $|d(x, U_\varepsilon) - d(y, U_\varepsilon)| \leqslant d(x, y)$. If $x, y \in U \setminus U_\varepsilon$ and $f(x) = f(y) = 1$ then $|f(x) - f(y)| = 0 \leqslant n_0 d(x, y)$.
If $x, y \in U_\varepsilon$ or $x, y \in X \setminus U$ then $|f(x) - f(y)| = 0 \leqslant n_0 d(x, y)$. Hence $f \in \mathrm{BL}_{x_0}(X)$.
Then we get

$$\int_X f \, \mathsf{d}\eta_0 = \int_{X \setminus U_\varepsilon} f \, \mathsf{d}\eta_0 \leqslant \eta_0(X \setminus U_\varepsilon) = \eta_0(X \setminus U) + \eta_0(U \setminus U_\varepsilon) < \eta_0(X \setminus U) + \varepsilon,$$
$$\int_X f \, \mathsf{d}\eta_n \geqslant \int_{X \setminus U} f \, \mathsf{d}\eta_n = \eta_n(X \setminus U).$$

Hence by condition (v) we have

$$\begin{aligned} \limsup_{n \to \infty} \eta_n(X \setminus U) &\leqslant \limsup_{n \to \infty} \int_X f \, \mathsf{d}\eta_n = \lim_{n \to \infty} \int_X f \, \mathsf{d}\eta_n = \int_X f \, \mathsf{d}\eta_0 \\ &< \eta_0(X \setminus U) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get (a).
Now let V be a closed neighbourhood of x_0. Let $\varepsilon > 0$ be arbitrary, but fixed. We show that there exists an open neighbourhood V_ε of x_0 such that $V \subset V_\varepsilon$ and $\eta_0(V_\varepsilon \setminus V) < \varepsilon$ and a function $f \in \mathrm{BL}_{x_0}(X)$ such that $f(x) = 0$ for $x \in V$, $f(x) = 1$ for $x \in X \setminus V_\varepsilon$ and $0 \leqslant f(x) \leqslant 1$ for $x \in X$.

Since V is closed, we get $V = \bigcap_{n=1}^\infty V_n$, where $V_n := V^{1/n}$, $n \in \mathbb{N}$. Then $V_{n+1} \subset V_n$, $n \in \mathbb{N}$, V_n is an open subset of X for all $n \in \mathbb{N}$ and $\bigcup_{n=1}^\infty X \setminus V_n = X \setminus V$. Since $X \setminus V_{n+1} \supset X \setminus V_n$, $n \in \mathbb{N}$, the continuity of the measure η_0 implies that $\lim_{n \to \infty} \eta_0(X \setminus V_n) = \eta_0(X \setminus V)$. Since $\eta_0(X \setminus V) < \infty$, there exists some $n_0 \in \mathbb{N}$ such that $\eta_0(X \setminus V) - \eta_0(X \setminus V_{n_0}) < \varepsilon$. Set $V_\varepsilon := V_{n_0}$.

Since $\eta_0(X \setminus V) - \eta_0(X \setminus V_{n_0}) = \eta_0\big((X \setminus V) \setminus (X \setminus V_{n_0})\big) = \eta_0(V_{n_0} \setminus V)$, we have V_ε is an open neighbourhood of x_0, $V \subset V_\varepsilon$ and $\eta_0(V_\varepsilon \setminus V) < \varepsilon$.

As earlier one can check that the function $f : X \to \mathbb{R}$, defined by $f(x) := \min(1, n_0 d(x, V))$, $x \in X$, is an element of $\mathrm{BL}_{x_0}(X)$, $f(x) = 0$ for $x \in V$, $f(x) = 1$ for $x \in X \setminus V_\varepsilon$ and $0 \leqslant f(x) \leqslant 1$ for $x \in X$. Then we get

$$\begin{aligned}\int_X f \,\mathsf{d}\eta_0 = \int_{X\setminus V} f \,\mathsf{d}\eta_0 &= \eta_0(X \setminus V_\varepsilon) + \int_{V_\varepsilon \setminus V} f \,\mathsf{d}\eta_0 \\ &\geqslant \eta_0(X \setminus V) - \eta_0(V_\varepsilon \setminus V) > \eta_0(X \setminus V) - \varepsilon,\end{aligned}$$

and $\int_X f \,\mathsf{d}\eta_n = \int_{X\setminus V} f \,\mathsf{d}\eta_n \leqslant \eta_n(X \setminus V)$. Hence by condition (v) we have

$$\begin{aligned}\liminf_{n\to\infty} \eta_n(X \setminus V) &\geqslant \liminf_{n\to\infty} \int_X f \,\mathsf{d}\eta_n = \lim_{n\to\infty} \int_X f \,\mathsf{d}\eta_n = \int_X f \,\mathsf{d}\eta_0 \\ &> \eta_0(X \setminus V) - \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain (b). Hence we proved that (a) and (b) are valid.

(vi)$\Rightarrow$ (iii): Suppose that (vi) holds. Let A be an element of $\mathcal{N}_{x_0}$ with $\eta_0(\partial A) = 0$. Then for the interior A° and the closure $\overline{A}$ of A we have $\eta_0((X \setminus A^\circ) \setminus (X \setminus \overline{A})) = \eta_0(\overline{A} \setminus A^\circ) = 0$. Then A° is an open and $\overline{A}$ is a closed neighbourhood of x_0. Indeed, the fact that A is in $\mathcal{N}_{x_0}$ yields that A° is nonempty and contains x_0. Hence we get

$$\begin{aligned}\eta_0(X \setminus A^\circ) &\geqslant \limsup_{n\to\infty} \eta_n(X \setminus A^\circ) \geqslant \limsup_{n\to\infty} \eta_n(X \setminus A) \geqslant \liminf_{n\to\infty} \eta_n(X \setminus A) \\ &\geqslant \liminf_{n\to\infty} \eta_n(X \setminus \overline{A}) \geqslant \eta_0(X \setminus \overline{A}).\end{aligned}$$

Since $\eta_0(X\setminus A^\circ) = \eta_0(X\setminus\overline{A}) = \eta_0(X\setminus A)$, we have the limit $\lim_{n\to\infty} \eta_n(X\setminus A)$ exists and $\lim_{n\to\infty} \eta_n(X \setminus A) = \eta_0(X \setminus A)$. □

5.2.2 Remark. The assertion (v) in Theorem 5.2.1 can be replaced by

$$\int_X f \,\mathsf{d}\eta_n \to \int_X f \,\mathsf{d}\eta_0 \quad \text{for all } f \in \mathcal{C}^u_{x_0}(X),$$

where $\mathcal{C}^u_{x_0}(X)$ denotes the set of all uniformly continuous functions in $\mathcal{C}_{x_0}(X)$. Indeed, $\mathcal{C}^u_{x_0}(X) \subset \mathcal{C}_{x_0}(X)$ and $\mathrm{BL}_{x_0}(X) \subset \mathcal{C}^u_{x_0}(X)$.

5.2.3 Remark. By giving a counterexample we show that the equivalence of (a) and (b) in condition (vi) of Theorem 5.2.1 is not valid. For all $n \in \mathbb{N}$ let η_n be the Dirac measure δ_2 on $\mathbb{R}$ concentrated on 2 and let η_0 be the Dirac measure δ_0 on $\mathbb{R}$ concentrated on 0. Then $\eta_0(\mathbb{R} \setminus V) = 0$ for all closed neighbourhoods V of 0, hence (b) in condition (vi) of Theorem 5.2.1 is satisfied. But (a) in condition (vi) of Theorem 5.2.1 is not satisfied. Indeed, $U := (-1, 1)$ is an open neighbourhood of 0, and

$$\eta_n(\mathbb{R} \setminus U) = \eta_n\big((-\infty, -1] \cup [1, \infty)\big) = 1, \quad n \in \mathbb{N},$$

hence $\limsup_{n\to\infty} \eta_n(\mathbb{R} \setminus U) = 1$. But $\eta_0(\mathbb{R} \setminus U) = 0$, which yields that (a) in condition (vi) of Theorem 5.2.1 is not satisfied. This counterexample also implies that the equivalence of (c) and (d) in Proposition 1.2.19 in Meerschaert and Scheffler [39] is not valid.

5.2.4 Remark. By giving a counterexample we show that the equivalence of (c) and (d) in Proposition 1.2.13 in Meerschaert and Scheffler [39] is not valid. For all $n \in \mathbb{N}$ let μ_n be the measure $2\delta_{1/n}$ on $\mathbb{R}$ and μ be the Dirac measure δ_0 on $\mathbb{R}$. We check that $\mu(A) \leqslant \liminf_{n\to\infty} \mu_n(A)$ for all open subsets A of $\mathbb{R}$, but there exists some closed subset F of $\mathbb{R}$ such that $\limsup_{n\to\infty} \mu_n(F) > \mu(F)$. If A is an open subset of $\mathbb{R}$ such that $0 \in A$ then $\mu(A) = 1$ and $\mu_n(A) = 2$ for all sufficiently large n, which implies that $\mu(A) \leqslant \liminf_{n\to\infty} \mu_n(A)$. If A is an open subset of $\mathbb{R}$ such that $0 \notin A$ then $\mu(A) = 0$, hence $\mu(A) \leqslant \liminf_{n\to\infty} \mu_n(A)$ is valid. Let F be the closed interval $[-1, 1]$. Then $\mu(F) = 1$ and $\mu_n(F) = 2$, $n \in \mathbb{N}$, which yields that $\limsup_{n\to\infty} \mu_n(F) = 2$. Hence $\limsup_{n\to\infty} \mu_n(F) > \mu(F)$.

Summary

This dissertation deals with some questions of probability theory on special locally compact groups. We consider two more or less independent topics in four chapters. First we investigate questions concerning Gauss measures on special noncommutative Lie groups, such as on the Heisenberg group and on the affine group (Chapter 2 and Chapter 3). In Chapter 2 one of our main interests is to describe the distribution of the convolution of two Gauss measures on the 3-dimensional Heisenberg group. In Chapter 3 we show that a Gauss measure on the affine group can be embedded only in a uniquely determined Gauss semigroup. Then we deal with proving (central) limit theorems for infinitesimal triangular arrays of random elements with values in a locally compact Abelian group, such as in the torus, in the group of p-adic integers and in the p-adic solenoid (Chapter 4). We also consider the problem of representation of weakly infinitely divisible probability measures on these groups (Chapter 4). Finally, we prove an analogue of the portmanteau theorem on weak convergence of probability measures (Chapter 5). Chapter 5 can be considered as an auxiliary result for Chapter 4. The reason for presenting it separately is that its main result can be formulated in a more general setting than it is needed in Chapter 4.

In Chapter 2 we consider the 3-dimensional Heisenberg group $\mathbb{H}$ which can be obtained by furnishing $\mathbb{R}^3$ with its natural topology and with the product

$$(g_1, g_2, g_3)(h_1, h_2, h_3) = \Big(g_1 + h_1, g_2 + h_2, g_3 + h_3 + \frac{1}{2}(g_1 h_2 - g_2 h_1)\Big).$$

Then $\mathbb{H}$ is a nilpotent Lie group. The Schrödinger representations $\{\pi_{\pm\lambda} : \lambda > 0\}$ of $\mathbb{H}$ are representations in the group of unitary operators of the complex Hilbert space $L^2(\mathbb{R})$ given by

$$[\pi_{\pm\lambda}(g)u](x) := \mathrm{e}^{\pm i(\lambda g_3 + \sqrt{\lambda} g_2 x + \lambda g_1 g_2/2)} u(x + \sqrt{\lambda} g_1)$$

for $g = (g_1, g_2, g_3) \in \mathbb{H}$, $u \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$. The value of the Fourier transform of a probability measure μ on $\mathbb{H}$ at the Schrödinger representation $\pi_{\pm\lambda}$ is the bounded linear operator $\widehat{\mu}(\pi_{\pm\lambda}) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given by

$$\widehat{\mu}(\pi_{\pm\lambda})u := \int_{\mathbb{H}} \pi_{\pm\lambda}(g)u \, \mu(\mathrm{d}g), \qquad u \in L^2(\mathbb{R}).$$

A family $(\mu_t)_{t\geqslant 0}$ of probability measures on $\mathbb{H}$ is said to be a *continuous convolution semigroup* if we have $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \geqslant 0$, and $\mu_t \xrightarrow{\mathsf{w}} \mu_0 = \delta_e$ as $t \downarrow 0$, where δ_e denotes the Dirac measure concentrated on the unit element $e = (0, 0, 0)$ of $\mathbb{H}$. (Here the notation $\xrightarrow{\mathsf{w}}$ means weak convergence.) A convolution semigroup $(\mu_t)_{t\geqslant 0}$ is called a *Gauss semigroup* if $\lim_{t\downarrow 0} t^{-1}\mu_t(\mathbb{H} \setminus U) = 0$ for all Borel neighbourhoods U of e. A probability measure μ on $\mathbb{H}$ is called *continuously embeddable* if there exists a continuous convolution semigroup $(\mu_t)_{t\geqslant 0}$ of probability measures on $\mathbb{H}$ such that $\mu_1 = \mu$. A probability measure on $\mathbb{H}$ is called a *Gauss measure* if it is continuously embeddable into a Gauss semigroup.

In Chapter 2 an explicit formula is derived for the Fourier transform of a Gauss measure on the 3-dimensional Heisenberg group at the Schrödinger representation. Using this explicit formula, we give necessary and sufficient conditions for the convolution of two Gauss measures to be a Gauss measure. It turns out that a convolution of Gauss measures on $\mathbb{H}$ is almost never a Gauss measure. We also give the Fourier transform of the convolution of two Gauss measures on the Heisenberg group including the case when the convolution is not a Gauss measure. The structure of Chapter 2 is similar to Pap [45]. Our main theorems are generalizations of the corresponding results for symmetric Gauss measures on $\mathbb{H}$ due to Pap [45].

The results of Chapter 2 are contained in our accepted paper [6].

In Chapter 3 we consider the 2-dimensional affine group F which can be realized as the matrix group

$$F := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0, \, b \in \mathbb{R} \right\}.$$

Then F is a Lie group which is not nilpotent. It is shown that a Gauss measure on the affine group can be embedded only in a uniquely determined Gauss semigroup. The starting point of the proof is the fact that a Gauss Lévy process in the affine group satisfies a certain stochastic differential equation (SDE). We also give the solution of this SDE. Moreover, we give a complete description of supports of Gauss measures on the affine group using Siebert's support formula.

The results of Chapter 3 appeared in our paper [5].

In Chapter 4 we deal with proving (central) limit theorems on second countable locally compact Abelian groups (LCA2 groups). We also consider the question of giving a construction of weakly infinitely divisible probability measures on special LCA2 groups using only real valued random variables. We prove limit theorems for row sums of a rowwise independent infinitesimal array of random elements with values in an LCA2 group. We give a proof of Gaiser's theorem on convergence of triangular arrays [23, Satz 1.3.6], since it does not have an easy access and it is not complete. This theorem gives sufficient conditions for convergence of the row sums of a rowwise independent infinitesimal array of random elements with values in an LCA2 group, but the limit measure can not have a nondegenerate idempotent factor, i.e., a nondegenerate Haar measure on some compact subgroup as its factor.

As new results we prove necessary and sufficient conditions for convergence of the row sums of symmetric arrays and Bernoulli arrays, where the limit measure can also be a nondegenerate Haar measure on a compact subgroup. Then we investigate special LCA2 groups: the torus group, the group of p-adic integers and the p-adic solenoid.

The set $\mathbb{T} := \{\mathrm{e}^{ix} : -\pi \leqslant x < \pi\}$ equipped with the usual multiplication of complex numbers and with the relative topology as a subset of complex numbers is a compact Abelian group. This is called the one-dimensional torus group.

Let p be a prime. The group of p-adic integers is

$$\Delta_p := \big\{(x_0, x_1, \dots) : x_j \in \{0, 1, \dots, p-1\} \quad \text{for all} \quad j \in \mathbb{Z}_+\big\},$$

where the sum $z := x + y \in \Delta_p$ for $x, y \in \Delta_p$ is uniquely determined by the relationships

$$\sum_{j=0}^{d} z_j p^j \equiv \sum_{j=0}^{d} (x_j + y_j) p^j \qquad \mod p^{d+1} \qquad \text{for all} \quad d \in \mathbb{Z}_+.$$

(Here $\mathbb{Z}_+$ denotes the set of nonnegative integers.) For each $r \in \mathbb{Z}_+$, let

$$\Lambda_r := \{x \in \Delta_p : x_j = 0 \quad \text{for all} \quad j \leqslant r-1\}.$$

The family of sets $\{x + \Lambda_r : x \in \Delta_p,\ r \in \mathbb{Z}_+\}$ is an open subbasis for a topology on Δ_p under which Δ_p is a compact, totally disconnected Abelian group.

The p-adic solenoid is a subgroup of $\mathbb{T}^\infty$, namely,

$$S_p := \{(y_0, y_1, \dots) \in \mathbb{T}^\infty : y_j = y_{j+1}^p \ \text{ for all } \ j \in \mathbb{Z}_+\},$$

furnished with the relative topology as a subset of the locally compact group $\mathbb{T}^\infty$. Then S_p is a compact connected Abelian group.

On the above mentioned LCA2 groups, we derive limit theorems applying Gaiser's theorem and our general results for symmetric and Bernoulli arrays.

Besides proving limit theorems, we give a construction of an arbitrary weakly infinitely divisible probability measure on the torus group and the group of p-adic integers. On the p-adic solenoid we give a construction of weakly infinitely divisible probability measures without nondegenerate idempotent factors. In our constructions we only use real valued random variables. For each of the three groups, first we find a measurable homomorphism φ from an appropriate Abelian topological group (which is a certain product of some subgroups of $\mathbb{R}$) onto the group in question. Then we consider an arbitrary weakly infinitely divisible probability measure μ on the group in question (without a nondegenerate idempotent factor in case of the p-adic solenoid) and we find real valued random variables $Z_0, Z_1, \dots$ such that the distribution of $\varphi(Z_0, Z_1, \dots)$ is μ. We note that, as a special case of our results, we have a new construction of the normalized Haar measure on the group of p-adic integers and the p-adic solenoid.

The results of Chapter 4 are contained in our submitted papers [7] and [8].

In Chapter 5 we prove an analogue of the *portmanteau theorem* on weak convergence of probability measures allowing measures which are finite on the complement of any Borel neighbourhood of a fixed element of an underlying metric space. Our theorem is a reformulation of Proposition 1.2.19 in Meerschaert–Scheffler [39] in a more detailed form adding new equivalent assertions to it. Our proof differs from the proof of Meerschaert and Scheffler, and we use our result in proving Gaiser's limit theorem [23, Satz 1.3.6]. We present our theorem separately in a new chapter, since it can be formulated in a more general setting than it is needed in proving Gaiser's limit theorem.

We remark that, by giving counterexamples, we show that some parts of Propositions 1.2.13 and 1.2.19 in Meerschaert–Scheffler [39] are not true, namely, the equivalence of (c) and (d) in their propositions is not valid.

The results of Chapter 5 are contained in our submitted paper [9].

Összefoglaló (Hungarian summary)

Disszertációm a valószínűségszámítás azon területéhez kapcsolódik, mely lokálisan kompakt csoportokon értelmezett valószínűségi mértékek tulajdonságait vizsgálja. Két, többé-kevésbé független témával foglalkozunk a disszertáció négy fejezetében. Először speciális nemkommutatív Lie-csoportokon, a Heisenberg-csoporton és az affin-csoporton értelmezett Gauss-mértékekkel kapcsolatos kérdéseket tárgyalunk (2. és 3. fejezet). A 2. fejezetben egyik fő célunk, hogy megadjuk két, a 3-dimenziós Heisenberg-csoporton értelmezett Gauss-mérték konvolúciójának eloszlását. A 3. fejezetben megmutatjuk, hogy egy affin-csoporton értelmezett Gauss-mérték egyértelműen ágyazható be egy Gauss konvolúciós félcsoportba. Ezt követően lokálisan kompakt Abel-csoportbeli értékű véletlen elemekből álló infinitezimális háromszögrendszerekre vonatkozóan bizonyítunk (centrális) határeloszlás-tételeket (4. fejezet). Speciális esetekként a tórusz, a p-adikus egészek és a p-adikus szolenoid esetét tárgyaljuk. Foglalkozunk ezeken a csoportokon értelmezett gyengén korlátlanul osztható valószínűségi mértékek reprezentációjának kérdésével is (4. fejezet). Az utolsó fejezetben a valószínűségi mértékek gyenge konvergenciájára vonatkozó portmanteau-tétel egy analógját bizonyítjuk be (5. fejezet). Az 5. fejezet a 4. fejezet kiegészítéseként, segédleteként tekinthető, s főként azért szerepeltetjük külön, mert a fejezet fő eredménye sokkal általánosabban is igaz, mint amire a 4. fejezetben szükségünk van.

A 2. fejezetben a 3-dimenziós Heisenberg-csoporttal foglalkozunk. Ellátva $\mathbb{R}^3$-at a szokásos topológiával és a

$$(g_1, g_2, g_3)(h_1, h_2, h_3) = \Big(g_1 + h_1, g_2 + h_2, g_3 + h_3 + \frac{1}{2}(g_1h_2 - g_2h_1)\Big)$$

szorzással a 3-dimenziós Heisenberg-csoportot kapjuk, melyet $\mathbb{H}$-val jelölünk. Ismert, hogy $\mathbb{H}$ egy nilpotens Lie-csoport. A $\{\pi_{\pm\lambda} : \lambda > 0\}$ Schrödinger-reprezentációk $\mathbb{H}$ reprezentációi a $L^2(\mathbb{R})$ komplex Hilbert-tér unitér operátorainak csoportjában, melyek értelmezése

$$[\pi_{\pm\lambda}(g)u](x) := \mathrm{e}^{\pm i(\lambda g_3+\sqrt{\lambda}g_2x+\lambda g_1g_2/2)}u(x+\sqrt{\lambda}g_1),$$

$g = (g_1, g_2, g_3) \in \mathbb{H}$, $u \in L^2(\mathbb{R})$ és $x \in \mathbb{R}$ esetén. Egy $\mathbb{H}$-n adott μ valószínűségi mérték Fourier-transzformáltja a $\pi_{\pm\lambda}$ Schrödinger-reprezentációban a $\widehat{\mu}(\pi_{\pm\lambda}) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$,

$$\widehat{\mu}(\pi_{\pm\lambda})u := \int_{\mathbb{H}} \pi_{\pm\lambda}(g)u\,\mu(\mathrm{d}g), \qquad u \in L^2(\mathbb{R}),$$

korlátos lineáris operátor. A $\mathbb{H}$ Heisenberg-csoporton értelmezett valószínűségi mértékek $(\mu_t)_{t\geqslant 0}$ családját *folytonos konvolúciós félcsoport*nak nevezzük, ha $\mu_s * \mu_t = \mu_{s+t}$ minden $s, t \geqslant 0$ esetén és $\mu_t \xrightarrow{\mathsf{w}} \mu_0 = \delta_e$ amint $t \downarrow 0$, ahol δ_e az $e = (0, 0, 0) \in \mathbb{H}$ pontra koncentrálódó Dirac-mértéket, $\xrightarrow{\mathsf{w}}$ pedig a gyenge konvergenciát jelöli. Valószínűségi mértékek $(\mu_t)_{t\geqslant 0}$ konvolúciós félcsoportját *Gauss-félcsoport*nak nevezzük, ha $\lim_{t\downarrow 0} t^{-1}\mu_t(\mathbb{H}\setminus U) = 0$ az e pont összes U Borel-környezetére. Azt mondjuk, hogy egy $\mathbb{H}$-n adott μ valószínűségi mérték *folytonosan beágyazható,* ha létezik olyan $\mathbb{H}$-n adott valószínűségi mértékekből álló $(\mu_t)_{t\geqslant 0}$ folytonos konvolúciós félcsoport, hogy $\mu_1 = \mu$. Egy $\mathbb{H}$-n adott valószínűségi mértéket *Gauss-mérték*nek nevezzük, ha folytonosan beágyazható egy Gauss-félcsoportba.

A 2. fejezetben explicit képletet adunk a $\mathbb{H}$ Heisenberg-csoporton értelmezett Gauss-mértékek Fourier-transzformáltjára a Schrödinger-reprezentációban. Ezen explicit képletet felhasználva szükséges és elegendő feltételeket származtatunk arra vonatkozóan, hogy mikor lesz két, a Heisenberg-csoporton értelmezett Gauss-mérték konvolúciója újra Gauss-mérték. Kiderül, hogy Heisenberg-csoporton értelmezett Gauss-mértékek konvolúciója szinte sohasem Gauss-mérték. Megadjuk Gauss-mértékek konvolúciójának Fourier-transzformáltját abban az esetben is, mikor a konvolúció nem Gauss-mérték. A 2. fejezet felépítése hasonló a Pap [45] cikkhez. Tételeink a Pap [45] cikkben szereplő szimmetrikus Gauss-mértékekre vonatkozó megfelelő eredmények általánosításai.

A 2. fejezet eredményei elfogadott [6] cikkünkben jelennek meg.

A 3. fejezetben a 2-dimenziós affin-csoportot tekintjük, melyen az alábbi mátrix-csoportot értjük

$$F := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0,\, b \in \mathbb{R} \right\}.$$

Ismert, hogy F egy Lie-csoport, mely nem nilpotens. Megmutatjuk, hogy egy affin-csoporton értelmezett Gauss-mérték egyértelműen ágyazható be egy Gauss-félcsoportba. Ezen tény bizonyításának kiindulópontja, hogy egy affin-csoportbeli értékű Gauss–Lévy-folyamat kielégít egy sztochasztikus differenciálegyenletet. Ezen differenciálegyenlet megoldása is szerepel a 3. fejezetben. Továbbá az affin-csoporton értelmezett Gauss-mértékek tartójának teljes leírását is megadjuk, Siebert tartó-formuláját felhasználva.

A 3. fejezet eredményei [5] cikkünkben jelentek meg.

A 4. fejezetben (centrális) határeloszlás-tételek bizonyításával foglalkozunk második megszámlálható lokálisan kompakt Abel-csoportok (LCA2-csoportok) esetében. Foglalkozunk speciális LCA2-csoportokon értelmezett gyengén korlátlanul osztható valószínűségi mértékek konstrukciójának megadásával is csak valós értékű valószínűségi változókat felhasználva. Lokálisan kompakt Abel-csoportbeli értékű véletlen elemekből álló soronként független, infinitezimális háromszögrendszerek esetén bizonyítunk határeloszlás-tételeket. Szerepeltetjük Gaiser háromszögrendszerek konvergenciájára vonatkozó tételének [23, Satz 1.3.6] bizonyítását, mivel a bizonyítás nehezen hozzáférhető és nem teljes. Gaiser tétele elégséges feltételeket fogalmaz meg arra vonatkozóan, hogy egy lokálisan kompakt Abel-csoportbeli értékű véletlen elemekből álló soronként független, infinitezimális háromszögrendszer sorösszegei eloszlásban konvergáljanak. Azonban a szóbanforgó elégséges feltételek teljesülése esetén a határeloszlásnak nem lehet nemdegenerált idempotens faktora, azaz valamely kompakt részcsoport nemdegenerált Haar-mértéke nem fordulhat elő faktoraként.

Új eredményként szükséges és elegendő feltételeket bizonyítunk szimmetrikus-, illetve ún. Bernoulli-háromszögrendszerek sorösszegeinek eloszlásban való konvergenciájára vonatkozóan. Esetünkben a határeloszlás lehet valamilyen kompakt részcsoport nemdegenerált normalizált Haar-mértéke is. Ezt követően speciális LCA2-csoportokat vizsgálunk: a tóruszt, a p-adikus egészek csoportját és a p-adikus szolenoidot.

A $\mathbb{T} := \{\mathrm{e}^{ix} : -\pi \leqslant x < \pi\}$ halmaz, felruházva a komplex számok szokásos szorzásával és a komplex számok halmazától örökölt topológiával, egy kompakt Abel-csoport, az ún. 1-dimenziós tórusz csoport.

Legyen p egy prímszám. A p-adikus számok csoportja a

$$\Delta_p := \big\{(x_0, x_1, \dots) : \ x_j \in \{0, 1, \dots, p-1\} \quad \forall \ j \in \mathbb{Z}_+\big\}$$

halmaz, ahol tetszőleges $x, y \in \Delta_p$ esetén a $z := x + y \in \Delta_p$ összeg az alábbi kongruenciák által egyértelműen meghatározott:

$$\sum_{j=0}^{d} z_j p^j \equiv \sum_{j=0}^{d} (x_j + y_j) p^j \qquad \text{mod } p^{d+1}, \qquad \forall \ d \in \mathbb{Z}_+.$$

(Itt $\mathbb{Z}_+$ a nemnegatív egész számok halmazát jelöli.) Minden $r \in \mathbb{Z}_+$ esetén legyen

$$\Lambda_r := \{x \in \Delta_p : \ x_j = 0 \quad \forall \ j \leqslant r - 1\}.$$

Az $\{x + \Lambda_r : x \in \Delta_p, r \in \mathbb{Z}_+\}$ alakú halmazok nyílt szubbázisát alkotják egy topológiának Δ_p-n. A fenti művelettel és topológiával Δ_p egy kompakt, teljesen szétesső Abel-csoport.

A p-adikus szolenoid a következő részcsoportja $\mathbb{T}^\infty$-nek:

$$S_p := \big\{(y_0, y_1, \dots) \in \mathbb{T}^\infty : \ y_j = y_{j+1}^p, \quad \forall \ j \in \mathbb{Z}_+\big\},$$

felruházva a $\mathbb{T}^\infty$ lokálisan kompakt csoporttól örökölt topológiával. Ekkor S_p egy kompakt Abel-csoport.

A 4. fejezetben vizsgáljuk azt a kérdést, hogy milyen következményei vannak Gaiser tételének és az általunk bizonyított szimmetrikus-, illetve Bernoulli-háromszögrendszerekre vonatkozó határeloszlás-tételeknek az előbb említett LCA2-csoportokon.

Határeloszlás-tételek bizonyításán kívül foglalkozunk még a 4. fejezetben az előbb említett LCA2-csoportokon értelmezett gyengén korlátlanul osztható valószínűségi mértékek olyan konstrukciójának megadásával is, mely csak valós értékű valószínűségi változókat használ. Mindhárom csoport esetén először egy φ mérhető homomorfizmust keresünk, mely egy alkalmas Abel-csoportot (ami $\mathbb{R}$ bizonyos részcsoportjainak szorzata) képez a szóbanforgó topologikus csoportra. Ezután tekintve egy tetszőleges μ gyengén korlátlanul osztható valószínűségi mértéket a szóbanforgó topologikus csoporton (nemdegenerált idempotens faktor nélkülit a p-adikus szolenoid esetén), olyan valós értékű $Z_0, Z_1, \dots$ valószínűségi változókat keresünk, hogy $\varphi(Z_0, Z_1, \dots)$ eloszlása μ legyen. Megjegyezzük, hogy eredményeink speciális eseteként új előállítását kapjuk a p-adikus egészek csoportján, illetve a p-adikus szolenoidon értelmezett normalizált Haar-mértéknek.

A 4. fejezet eredményeit a közlésre benyújtott [7] és [8] cikkeink tartalmazzák.

Az 5. fejezetben a valószínűségi mértékek gyenge konvergenciájára vonatkozó *portmanteau-tétel* egy analógját bizonyítjuk be, megengedve olyan mértékeket is, melyek végesek egy alapul vett metrikus tér valamely rögzített pontja tetszőleges Borel-környezetének komplementerén. Tételünk a Meerschaert és Scheffler [39] könyv 1.2.19 Állításának újrafogalmazása és kiegészítése, az eredetitől eltérő bizonyítással. Eredményünket Gaiser tételének [23, Satz 1.3.6] bizonyításánál használjuk, s főként azért szerepeltetjük külön fejezetben, mert eredményünk sokkal általánosabban is igaz, mint amire a Gaiser-tétel bizonyításánál szükségünk van.

Megjegyezzük, hogy a fejezetben ellenpéldát adva megmutatjuk, hogy a Meerschaert és Scheffler [39] könyv 1.2.19 Állításában és 1.2.13 Állításában szereplő (c) és (d) részek ekvivalenciája nem teljesül.

Az 5. fejezet eredményeit a közlésre benyújtott [9] cikkünk tartalmazza.

Bibliography

[1] D. Applebaum and H. Kunita, Lévy flows on manifolds and Lévy processes on Lie groups. *J. Math. Kyoto Univ.* **33** (1993), 1103–1123.

[2] R. B. Ash, *Real analysis and probability.* Academic Press, New York, 1972.

[3] A. Baker, *Matrix groups, An introduction to Lie group theory.* Springer, 2001.

[4] P. Baldi, Unicité du plongement d'une mesure de probabilité dans un semi-groupe de convolution gaussien. Cas non-abélien. *Math. Z.* **188** (1985), 411–417.

[5] M. Barczy and G. Pap, Gaussian measures on the affine group: uniqueness of embedding and supports. *Publ. Math. Debrecen* **63**(1-2) (2003), 221–234.

[6] M. Barczy and G. Pap, Fourier transform of a Gaussian measure on the Heisenberg group, to appear in *Annales de L'Institut Henri Poincaré Probabilités et Statistiques.*

[7] M. Barczy, A. Bendikov and G. Pap, Limit theorems on locally compact Abelian groups, submitted to *Mathematische Nachrichten.*

[8] M. Barczy and G. Pap, Weakly infinitely divisible measures on some locally compact Abelian groups, submitted to *Bulletin of the Australian Mathematical Society.*

[9] M. Barczy and G. Pap, Portmanteau theorem for unbounded measures, submitted to *Statistics & Probability Letters.*

[10] M. S. Bingham, Central limit theory on locally compact abelian groups. In: Probability measures on groups and related structures, XI. Proceedings Oberwolfach, 1994, pp. 14–37, World Sci. Publishing, NJ, 1995.

[11] S. Bochner, General analytic setting for the central limit theory of probability. In: *Bulletin Calcutta Mathematical Society Golden Jubilee Commemoration Volume, Part I* (1958), pp. 111–128.

[12] S. Bochner, *Harmonic analysis and the theory of probability.* Berkeley: University of California Press, 1960.

[13] M. Chaleyat-Maurel, Densités des diffusions invariantes sur certains groupes nilpotents. Calcul d'aprés B. Gaveau. *Astérisque* **84–85** (1981), 203–214.

[14] D. V. Chistyakov, Fractal geometry of images of continuous embeddings of p-adic numbers and solenoids into Euclidean spaces. *Theoret. and Math. Phys.* **109**(3) (1996), 1495–1507.

[15] L. Corwin and F. P. Greenleaf, *Representations of nilpotent Lie groups and their applications, Part 1: Basic theory and examples.* Cambridge University Press, 1990.

[16] E. B. Davies, *Heat kernels and spectral theory.* Cambridge University Press, 1989.

[17] P. Diaconis, *Group representations in probability and statistics.* Institute of Mathematical Statistics Lecture Notes-Monograph Series **11**, 1988.

[18] T. Drisch and L. Gallardo, Stable laws on the Heisenberg groups. In: H. Heyer ed., Probability Measures on Groups VII. Proceedings, Oberwolfach 1983, *Lecture Notes in Math.* **1064**, pp. 56–79, Springer, Berlin–Heidelberg–New York, 1984.

[19] R. M. Dudley, *Real analysis and probability.* The Wadsworth & Brooks Cole Mathematics Series, Pacific Grove, 1989.

[20] S. N. Ethier and T. G. Kurtz, *Markov processes.* John Wiley & Sons, New York, 1986.

[21] Ph. Feinsilver and R. Schott, Operators, stochastic processes, and Lie groups. In: H. Heyer ed., Probability Measures on Groups IX. Proceedings, Oberwolfach 1988, *Lecture Notes in Math.* **1379**, pp. 75–78, Springer, Berlin–Heidelberg–New York, 1989.

[22] PH. FEINSILVER and R. SCHOTT, An operator approach to processes on Lie groups. In: Probability Theory on Vector Spaces IV. Proceedings, Łancut 1987, *Lecture Notes in Math.* **1391**, pp. 59–65, Springer, Berlin–Heidelberg–New York, 1989.

[23] J. GAISER, Konvergenz stochastischer prozesse mit werten in einer lokalkompakten Abelschen gruppe. *Ph.D. Thesis*, Universität Tübingen, 1994.

[24] I. S. GRADSHTEYN and I. M. RYZHIK, *Table of integrals, series, and products.* Academic Press, New York, 1965.

[25] U. GRENANDER, *Probabilities on algebraic structures.* Almquist & Wiksell, Stockholm, 1963.

[26] E. J. HANNAN, *Group representations and applied probability.* Methuen's Review Series in Applied Probability Vol. 3, London: Methuen & Co. Ltd., 1965.

[27] W. HAZOD, Stetige Faltungshalbgruppen von Wahrscheinlichkeitsmaßen und erzeugende Distributionen. *Lecture Notes in Math.* **595**, Springer, Berlin-Heidelberg-New York, 1977.

[28] W. HAZOD and E. SIEBERT, *Stable probability measures on Euclidean spaces and on locally compact groups. Structural properties and limit theorems.* Kluwer Academic Publishers, Dordrecht, 2001.

[29] E. HEWITT and K. A. ROSS, *Abstract harmonic analysis I.* Springer, 1963.

[30] H. HEYER, *Probability measures on locally compact groups.* Springer, 1977.

[31] H. HEYER and G. PAP, On infinite divisibility and embedding of probability measures on a locally compact Abelian group. *Infinite Dimensional Harmonic Analysis III* (Proc. of the Third German–Japanese Symposium, Tübingen, 2003), pp. 99–118, World Sci. Publishing, River Edge, NJ, 2005.

[32] K. ITO and Y. KAWADA, On the probability distribution on a compact group I. *Proc. Phys. -Math. Soc. Japan* **22** (1940), 977–998.

[33] J. JACOD and A. N. SHIRYAYEV, *Limit theorems for stochastic processes.* Springer, 1987.

[34] I. Karatzas and S. E. Shreve, *Brownian motion and stochastic calculus, 2nd ed.* Springer, 1991.

[35] D. Kelly-Lyth and M. McCrudden, Supports of Gauss measures on semisimple Lie groups. *Math. Z.* **221** (1996), 633–645.

[36] M. McCrudden and R. M. Wood, On the support of absolutely continuous Gauss measures. In: H. Heyer ed., Probability Measures on Groups, VII. Proceedings, Oberwolfach 1983. *Lecture Notes in Math.* **1064**, pp. 379–397, Springer, Berlin–Heidelberg–New York, 1984.

[37] M. McCrudden, On the supports of absolutely continuous Gauss measures on connected Lie groups. *Monatsh. Math.* **98** (1984), 295-310.

[38] M. McCrudden, An example of a solvable Lie group admitting an absolutely continuous Gauss semigroup with incomparable supports. In: H. Heyer ed., Probability Measures on Groups X. Proceedings, Oberwolfach 1990, pp. 293–297, Plenum Press, New York, 1991.

[39] M. M. Meerschaert and H.-P. Scheffler, *Limit distributions for sums of independent random vectors. Heavy tails in theory and practice.* John Wiley & Sons, Inc., New York, 2001.

[40] D. Neuenschwander, Probabilities on the Heisenberg group: Limit theorems and Brownian motion. *Lecture Notes in Math.* **1630**, Springer, Berlin Heidelberg New-York, pp. 379–397, 1996.

[41] D. Neuenschwander, On the uniqueness problem for continuous semigroups of probability measures on simply connected nilpotent Lie groups. *Publ. Math. Debrecen* **53** (1998), 415–422.

[42] D. Neuenschwander, Uniqueness properties of convolution roots of p-adic and probability measures on simply connected nilpotent Lie groups. *C.R. Acad. Sci. Paris.* **330** (2000), 1025–1030.

[43] S. Nobel, Limit theorems for probability measures on simply connected nilpotent Lie groups. *J. Theoret. Probab.* **4** (1991), 261-284.

[44] G. Pap, Uniqueness of embedding into a Gaussian semigroup on a nilpotent Lie group. *Arch. Math.* **62** (1994), 282–288.

[45] G. Pap, Fourier transform of symmetric Gauss measures on the Heisenberg group. *Semigroup Forum* **64** (2002), 130–158.

[46] K. R. Parthasarathy, *Probability measures on metric spaces.* Academic Press, New York, 1967.

[47] B. Roynette, Croissance et mouvements browniens d'un groupe de Lie nilpotent et simplement connexe. *Z. Wahr. Verw. Gebiete* **32** (1975), 133–138.

[48] I. Z. Ruzsa and G. J. Székely, *Algebraic probability theory.* Wiley Series in Probability and Mathematical Statistics, 1988.

[49] J. D. Salinger, *The cathcher in the rye.* Penguin Books, London, 1994.

[50] J. D. Salinger, *Zabhegyező.* Európa Könyvkiadó, Budapest, 2004.

[51] V. Sazonov and V. N. Tutubalin, Probability distributions on topological groups. *Theory Probab. Appl.* **11** (1966), 1–45.

[52] A. N. Shiryaev, *Probability, 2nd ed.* Springer, 1996.

[53] E. Siebert, Fourier analysis and limit theorems for convolution semigroups on a locally compact group. *Adv. Math.* **39** (1981), 111–154.

[54] E. Siebert, Absolute continuity, singularity, and supports of Gauss semigroups on a Lie group. *Monatsh. Math.* **93** (1982), 239-253.

[55] E. Siebert, Jumps of stochastic processes with values in a topological group. *Probab. Math. Stat.* **5** (1985), 197–209.

[56] M. E. Taylor, *Noncommutative harmonic analysis,* Math. Surveys Monogr. **22**, American Mathematical Society, Providence, RI, 1986.

[57] K. Telöken, Grenzwertsätze für wahrscheinlichkeitsmasse auf total unzusammenhängenden gruppen. *Ph.D. Thesis*, Universität Dortmund, 1995.

[58] W. Tomé, *The representation independent propagator for general Lie groups.* World Scientific, Singapore, 1998.

[59] N. N. Vakhania, V. I. Tarieladze and S. A. Chobanyan, *Probability distributions on Banach spaces.* D. Reidel Publishing Company, Dordrecht, 1987.

[60] J. G. Wendel, Haar measure and the semigroup of measures on a compact group. *Proc. Amer. Math. Soc.* **5** (1954), 923–929.

[61] W. Woess, *Random walks on infinite graphs and groups.* Cambridge University Press, Cambridge, 2000.

Appendix A

List of papers of the author and citations to these papers

1. M. Barczy and M. Tóth, Local automorphisms of the sets of states and effects on a Hilbert space. *Rep. Math. Phys.* **48** (2001), 289-298.

    - M. Győry, Preserver problems and reflexivity problems on operator algebras and on function algebras. *Ph.D. Thesis*, University of Debrecen, 2003.
    - L. Molnár, Preserver problems on algebraic structures of linear operators and on function spaces. *Dissertation for the D.Sc. degree of the Hungarian Academy of Sciences*, 2005.
    - S. O. Kim, Automorphisms of Hilbert space effect algebras. *Linear Algebra Appl.* **402** (2005), 193–198.

2. M. Barczy and G. Pap, Gaussian measures on the affine group: uniqueness of embedding and supports. *Publ. Math. Debrecen* **63**(1-2) (2003), 221–234.

3. L. Molnár and M. Barczy, Linear maps on the space of all bounded observables preserving maximal deviation. *J. Funct. Anal.* **205** (2003), 380-400.

- M. Győry, Preserver problems and reflexivity problems on operator algebras and on function algebras. *Ph.D. Thesis*, University of Debrecen, 2003.
- L. Molnár, Preserver problems on algebraic structures of linear operators and on function spaces. *Dissertation for the D.Sc. degree of the Hungarian Academy of Sciences*, 2005.
- M. A. Chebotar, K. Wen-Fong and L. Pjek-Hwee, Maps preserving zero Jordan products on Hermitian operators. *Illinois J. Math.* **49**(2) (2005), 445–452 (electronic).

4. M. Barczy and G. Pap, Connection between deriving bridges and radial parts from multidimensional Ornstein-Uhlenbeck processes. *Periodica Mathematica Hungarica* Vol. **50**(1-2) (2005), 47-60.

5. M. Barczy and G. Pap, Fourier transform of a Gaussian measure on the Heisenberg group, to appear in *Annales de L'Institut Henri Poincaré Probabilités et Statistiques.*

6. M. Barczy, A. Bendikov and G. Pap, Limit theorems on locally compact Abelian groups, submitted to *Mathematische Nachrichten.*

   - P. Becker-Kern, Explicit representation of roots on p-adic solenoids and non-uniqueness of embeddability into rational one-parameter subgroups. *Preprint,* URL: http://www.mathematik.uni-dortmund.de/lsiv/becker-kern/solenoid.pdf

7. M. Barczy and G. Pap, Weakly infinitely divisible measures on some locally compact Abelian groups, submitted to *Bulletin of the Australian Mathematical Society.*

   - P. Becker-Kern, Explicit representation of roots on p-adic solenoids and non-uniqueness of embeddability into rational one-parameter subgroups. *Preprint,* URL: http://www.mathematik.uni-dortmund.de/lsiv/becker-kern/solenoid.pdf

8. M. Barczy and G. Pap, Portmanteau theorem for unbounded measures, submitted to *Statistics & Probability Letters.*

Appendix B

List of talks of the author

I participated and gave a talk in the following international conferences with the following titles:

1. Convolution of Gauss measures on Heisenberg group, *XXI Seminar on Stability Problems of Stochastic Models,* Eger, Hungary, January 2001.
2. Convolution of Gauss measures on Heisenberg group, *The 12th European Young Statisticians Meeting,* Jánska Dolina, Slovakia, September 2001.
3. Brownian motions on the affine group, *International Conference on Probability Theory on Algebriac Topological Structures,* Bommerholz, Germany, March 2003.
4. By *"The research in pairs program (RiP)",* I was in Oberwolfach, Germany during August 2003 with Alexander Bendikov and Gyula Pap.
5. Central limit theorems in locally compact Abelian groups, *Conference on probability measures on groups and related structures on the occassion of Herbert Heyer's retirement,* Budapest, Hungary, August 2004.
6. Some questions of Markov bridges, *25th European Meeting of Statisticians,* Oslo, Norway, July 2005.

Appendix C

Acknowledgements

"Bernice meet me at recess I have something very very important to tell you."

J.D. Salinger: The catcher in the rye [1]

I would like to thank Prof. Gyula Pap for being an excellent supervisor. He has spent endless hours to teach me and he has been much more than an advisor: I could always turn to him with questions far beyond academic life.

I am grateful to Prof. Lajos Molnár for our joint works in functional analysis which resulted in two papers about linear preserver problems and local automorphisms. The discussions with him always inspire me.

Thanks my friends, Péter Diviánszky, István Járási and Zoltán Szegedi for the enjoyable conversations.

Last, but not least, I thank my father and my mother for having been able to teach me.

[1] See [49].

Appendix D

Köszönetnyilvánítások

"Bernice, találkozzunk a szünetben, valami nagyon fontosat akarok mondani."

J.D. Salinger: Zabhegyező [1]

Köszönöm Pap Gyulának, hogy kiváló témavezetőm volt. Véget nem érő konzultációk során tanított engem, s számomra sokkal többet jelentett, mint pusztán témavezető: bátran fordulhattam hozzá kérdéseimmel és gondolataimmal, nemcsak az egyetemi életet illetően.

Hálás vagyok Molnár Lajosnak a vele folytatott közös kutató munkáért a funkcionálanalízis területén, melynek gyümölcseként két cikk is született a lineáris megőrzési problémákkal és lokális automorfizmusokkal kapcsolatban. A vele való beszélgetések mindig lelkesítenek.

Köszönet barátaimnak, Diviánszky Péternek, Járási Istvánnak és Szegedi Zoltánnak, az élvezetes beszélgetésekért.

Végül, de nem utolsó sorban, köszönöm édesapámnak és édesanyámnak, hogy lehetőséget teremtettek tanulmányaimhoz.

[1] Fordította Gyepes Judit (lásd [50]).

Some questions of probability theory on special topological groups

Értekezés a doktori (Ph.D.) fokozat megszerzése érdekében
a matematika tudományágban.

Írta: Barczy Mátyás okleveles matematikus.

Készült a Debreceni Egyetem Matematika- és számítástudományok doktori iskolája (Valószínűségelmélet, matematikai statisztika és alkalmazott matematika alprogramja) keretében.

Témavezető: Dr. Pap Gyula

A doktori szigorlati bizottság:

elnök: Dr. ....

tagok: Dr. ....

Dr. ....

A doktori szigorlat időpontja: 200... ....

Az értekezés bírálói:

Dr. ....

Dr. ....

Dr. ....

A bírálóbizottság:

elnök: Dr. ....

tagok: Dr. ....

Dr. ....

Dr. ....

Dr. ....

Az értekezés védésének időpontja: 200... ....