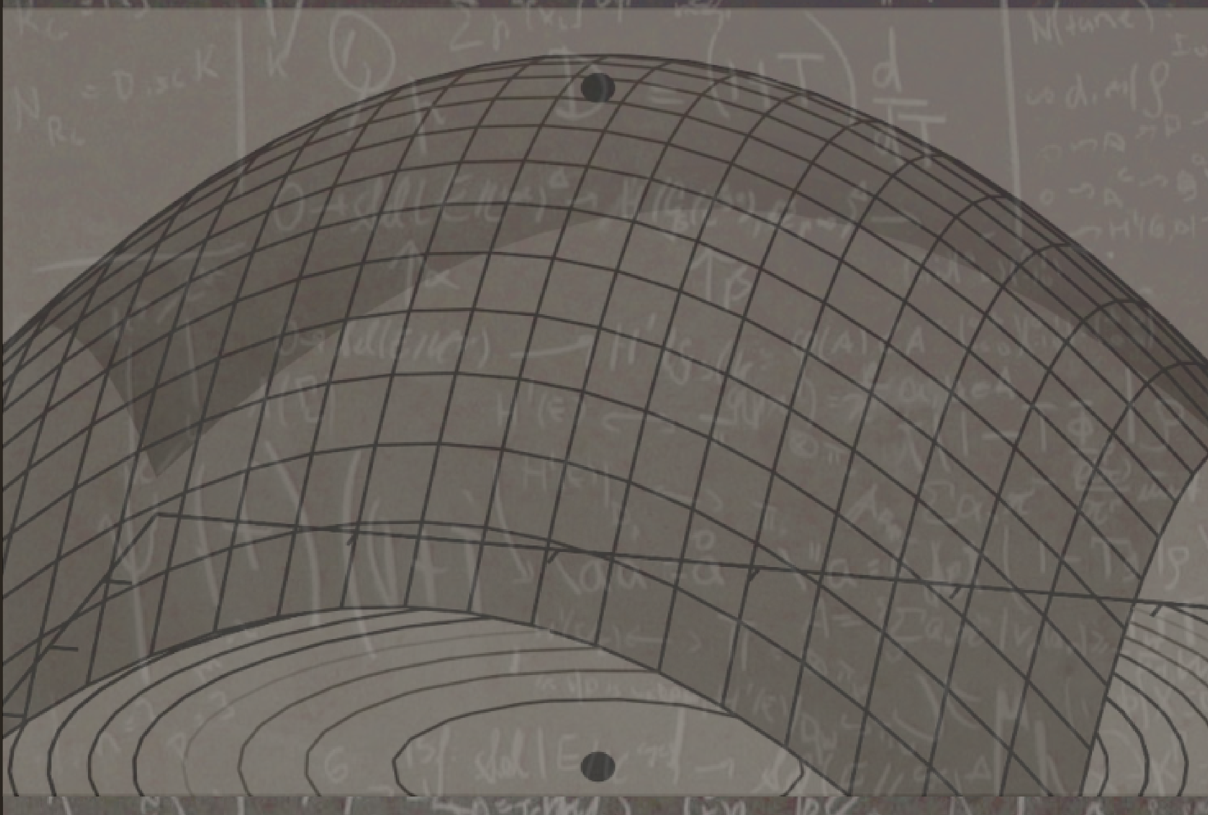
Dr. Kézi Csaba Gábor

Mathematical modelling and optimization



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UNIVERSITY OF DEBRECEN
FACULTY OF ENGINEERING DEPARTMENT
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Chapter 1

Differential calculus and its applications

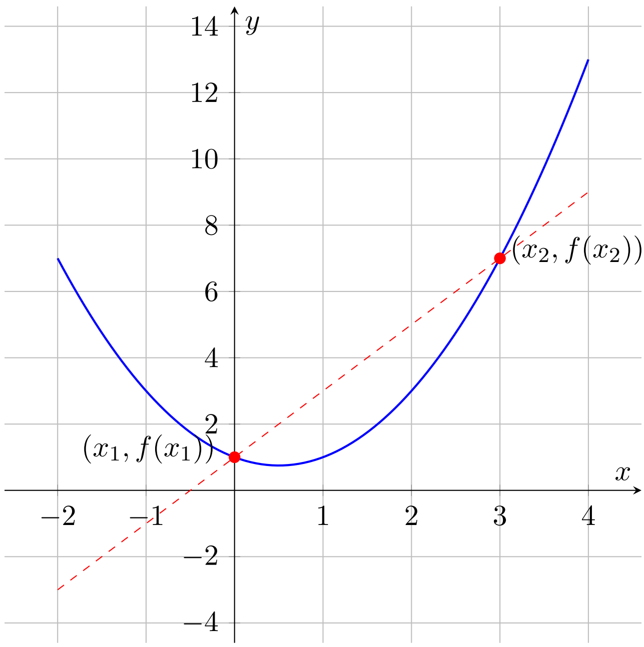
1. Introduction to differencial calculus

Theoretical summary

Definition 1.1.1. Let I be an open interval of real numbers. Consider a function $f\colon I \to \mathbb{R}$ and let $x_1 \neq x_2$ be real numbers such that $x_1, x_2 \in I$. The *difference quotient* of the function f at the points $\big(x_1; f(x_1)\big)$ and $\big(x_2; f(x_2)\big)$ is defined as

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Remark 1.1.2. The geometric meaning of the difference quotient is the slope of the secant line passing through the points $\big(x_1; f(x_1)\big)$ and $\big(x_2; f(x_2)\big)$.



Definition 1.1.3. Let $x_0 \in I$. The *difference quotient function* of a function $f\colon I \to \mathbb{R}$ at the point x_0 is defined as

$$d(x) = \frac{f(x) - f(x_0)}{x - x_0}, \qquad x \in I \setminus \{x_0\}$$

Definition 1.1.4. We say that a function $f\colon I \to \mathbb{R}$ is *differentiable* at a point $x_0 \in I$ if the limit of the difference quotient function exists and is finite at x_0, i.e., if

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and is finite. This limit is called the *derivative* of the function f at x_0. We say that the function f is *differentiable* if it is differentiable at every point of its domain.

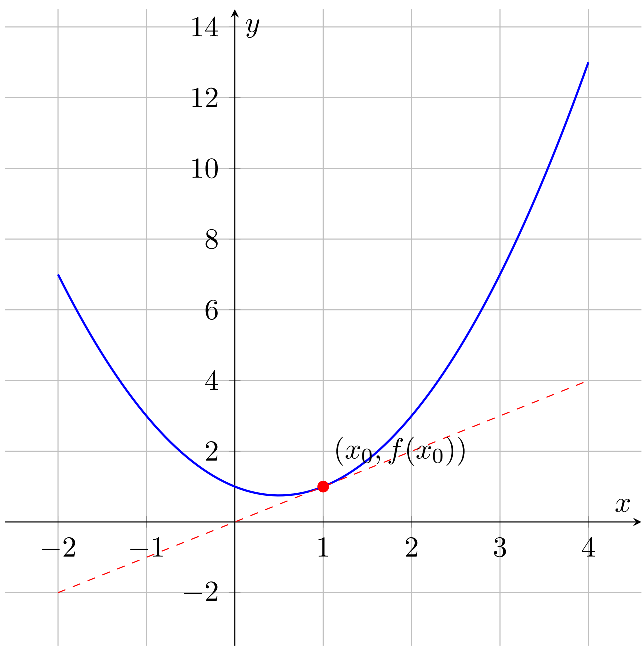
Notations: $f'(x_0)$, $\dfrac{df}{dx}(x_0)$, $\dot{f}(t)$, $\dfrac{d}{dx}f(x_0)$.

Definition 1.1.5. If a function $f\colon I \to \mathbb{R}$ is differentiable at a point $x_0 \in I$, then the line with equation

$$y = f(x_0) + f'(x_0) \cdot (x - x_0)$$

is called the *tangent line* to the graph of f at the point $\big(x_0; f(x_0)\big)$.

Remark 1.1.6. The geometric meaning of the derivative is the slope of the tangent line to the graph of the function at the point $\big(x_0; f(x_0)\big)$.



Theorem 1.1.7. The derivatives of elementary functions:

$f(x)$	D_f	$f'(x)$	$D_{f'}$
c	$\mathbb{R}$	0	$\mathbb{R}$
x	$\mathbb{R}$	1	$\mathbb{R}$
x^r	$]0;\infty[$	$r \cdot x^{r-1}$	$]0;\infty[$
$\sin x$	$\mathbb{R}$	$\cos x$	$\mathbb{R}$
$\cos x$	$\mathbb{R}$	$-\sin x$	$\mathbb{R}$
$\tan x$	$\mathbb{R} \setminus \{\frac{\pi}{2} + k \cdot \pi \,\vert\, k \in \mathbb{Z}\}$	$\frac{1}{\cos^2 x}$	$\mathbb{R} \setminus \{\frac{\pi}{2} + k \cdot \pi \,\vert\, k \in \mathbb{Z}\}$
$\cot x$	$\mathbb{R} \setminus \{\pi + k \cdot \pi \,\vert\, k \in \mathbb{Z}\}$	$-\frac{1}{\sin^2 x}$	$\mathbb{R} \setminus \{\pi + k \cdot \pi \,\vert\, k \in \mathbb{Z}\}$
e^x	$\mathbb{R}$	e^x	$\mathbb{R}$
a^x	$\mathbb{R}$	$a^x \cdot \ln a$	$\mathbb{R}$
$\ln x$	$]0;\infty[$	$\frac{1}{x}$	$]0;\infty[$
$\log_a x$	$]0;\infty[$	$\frac{1}{x \cdot \ln a}$	$]0;\infty[$
$\arcsin x$	$[-1;1]$	$\frac{1}{\sqrt{1-x^2}}$	$]-1;1[$
$\arccos x$	$[-1;1]$	$-\frac{1}{\sqrt{1-x^2}}$	$]-1;1[$
$\arctan x$	$\mathbb{R}$	$\frac{1}{1+x^2}$	$\mathbb{R}$
$\operatorname{arccot} x$	$\mathbb{R}$	$-\frac{1}{1+x^2}$	$\mathbb{R}$

$\operatorname{sh} x$	$\mathbb{R}$	$\operatorname{ch} x$	$\mathbb{R}$
$\operatorname{ch} x$	$\mathbb{R}$	$\operatorname{sh} x$	$\mathbb{R}$
$\operatorname{th} x$	$\mathbb{R}$	$\frac{1}{\operatorname{ch}^2 x}$	$\mathbb{R}$
$\operatorname{cth} x$	$\mathbb{R} \setminus \{0\}$	$-\frac{1}{\operatorname{sh}^2 x}$	$\mathbb{R} \setminus \{0\}$
$\operatorname{arsh} x$	$\mathbb{R}$	$\frac{1}{\sqrt{1+x^2}}$	$\mathbb{R}$
$\operatorname{arch} x$	$[1;\infty[$	$\frac{1}{\sqrt{x^2-1}}$	$]1;\infty[$
$\operatorname{arth} x$	$]-1;1[$	$\frac{1}{1-x^2}$	$]-1;1[$
$\operatorname{arcth} x$	$]-\infty;-1[\cup]1;\infty[$	$-\frac{1}{1-x^2}$	$]-\infty;-1[\cup]1;\infty[$

Theorem 1.1.8. If functions $f, g \colon I \to \mathbb{R}$ are differentiable at a point $x_0 \in I$, then their sum (difference) is also differentiable at x_0 and

$$(f+g)'(x_0) = f'(x_0) + g'(x_0).$$

Theorem 1.1.9. If a function $f \colon I \to \mathbb{R}$ is differentiable at a point $x_0 \in I$ and $c \in \mathbb{R}$, then $c \cdot f$ is also differentiable at x_0 and

$$(c \cdot f)'(x_0) = c \cdot f'(x_0).$$

Theorem 1.1.10. If the functions $f, g \colon I \to \mathbb{R}$ are differentiable at a point $x_0 \in I$, then their product is also differentiable at x_0 and

$$(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0).$$

Theorem 1.1.11. If the functions $f, g \colon I \to \mathbb{R}$ are differentiable at a point $x_0 \in I$, and $g(x) \neq 0$ $(x \in I)$, then $\frac{f}{g}$ is also differentiable at x_0 and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0) \cdot g(x_0) - f(x_0) \cdot g'(x_0)}{\big(g(x_0)\big)^2}.$$

Theorem 1.1.12. Let I and J be open intervals. If the function $g \colon I \to J$ is differentiable at the point x_0, and the function $f \colon J \to \mathbb{R}$ is differentiable at the point $g(x_0)$, then the composite function $f \circ g\colon I \to \mathbb{R}$ is also differentiable at x_0 and

$$(f \circ g)'(x_0) = f'\big(g(x_0)\big) \cdot g'(x_0).$$

Remark 1.1.13. According to the previous theorem, to differentiate a composite function, we first differentiate the outer function, substitute the original inner function in place of the variable, and then multiply the result by the derivative of the inner function.

Definition 1.1.14. If a function $f\colon I \to \mathbb{R}$ is differentiable on the interval I, and its derivative is differentiable at a point $x_0 \in I$, then we say that f is *twice differentiable* at x_0. Notation:

$$\big(f'(x_0)\big)' = f''(x_0).$$

Definition 1.1.15. If a function $f\colon I \to \mathbb{R}$ is $(k-1)$-times differentiable on the interval I, and its $(k-1)$-th order derivative is differentiable at a point $x_0 \in I$, then we say that f is *k-times differentiable* at x_0. Notation:

$$\big(f^{(k-1)}(x_0)\big)' = f^{(k)}(x_0).$$

Solved exercises

Exercise 1. Consider the function $f(x) = x^2$ and let $x_0 = 1$.

a) Determine the simplest form of the difference quotient function of f at the point x_0.

b) Using the result, find the value of the derivative of f at the point x_0.

c) Determine the slope of the tangent line e to the graph of f at the point $\big(x_0; f(x_0)\big)$.

d) Write the equation of the tangent line e.

e) Make a figure, which include the graph of function f and the tangent line of f at x_0.

<u>**Solution:**</u>

a) Since $f(x_0) = f(1) = 1^2 = 1$, the difference quotient function for $x \neq 1$ is:

$$d(x) = \frac{f(x) - f(x_0)}{x - x_0} = \frac{x^2 - 1^2}{x - 1} = \frac{(x-1)\cdot(x+1)}{x-1} = x + 1.$$

b) The derivative at the point x_0 is:

$$f'(1) = \lim_{x\to 1}(x+1) = 1 + 1 = 2.$$

c) The slope of the tangent line is: $m = f'(1) = 2$.

d) We have to find the equation of the line in the form

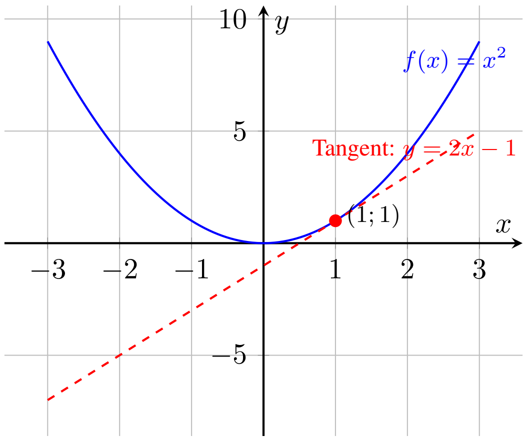
$$y = m \cdot x + b = 2x + b.$$

The line passes through the point $\big(1; f(1)\big) = (1; 1)$, thus

$$1 = 2 \cdot 1 + b \qquad \Rightarrow \qquad b = -1,$$

therefore the equation of the line is: $y = 2x - 1$.

e) The figure is as follows:



Exercise 2. Find the derivative function of $u(x) = x^3 + 2x + 1 + \sin x$.

Solution:

The derivative function of u is

$$u'(x) = (x^3)' + (2x)' + 1' + (\sin x)' = 3x^2 + 2 + \cos x.$$

Exercise 3. Find the derivative function of $u(x) = \cos x + \tan x$.

Solution:

The derivative function of u is: $u'(x) = -\sin x + \dfrac{1}{\cos^2 x}$.

Exercise 4. Find the derivative function of $u(x) = e^x + 3^x + \log_2 x$.

Solution:

The derivative function of u is $u'(x) = e^x + 3^x \cdot \ln 3 + \dfrac{1}{x \cdot \ln 2}$.

Exercise 5. Find the derivative function of $u(x) = \sqrt{x^3} + \dfrac{2}{x^2} + 5$.

Solution:

Since $\sqrt{x^3} = x^{\frac{3}{2}}$ and $\dfrac{2}{x^2} = 2x^{-2}$, we have $u(x) = x^{\frac{3}{2}} + 2x^{-2} + 5$, thus the derivative function of u is

$$u'(x) = \frac{3}{2} \cdot x^{\frac{1}{2}} - 4x^{-3} = \frac{3}{2} \cdot \sqrt{x} - \frac{4}{x^3}.$$

Exercise 6. Find the derivative function of $u(x) = x^2 \cdot \cot x$.

Solution:

The derivative function of u is

$$u'(x) = (x^2)' \cdot \cot x + x^2 \cdot (\cot x)' = 2x \cdot \cot x + x^2 \cdot \frac{-1}{\sin^2 x}.$$

Exercise 7. Find the derivative function of $u(x) = \sqrt{x} \cdot 3^x$.

Solution:

Since $\sqrt{x} = x^{\frac{1}{2}}$, we have $u(x) = x^{\frac{1}{2}} \cdot 3^x$, so the derivative function of u is

$$u'(x) = (x^{\frac{1}{2}})' \cdot 3^x + x^{\frac{1}{2}} \cdot (3^x)' = \frac{1}{2}x^{-\frac{1}{2}} \cdot 3^x + x^{\frac{1}{2}} \cdot 3^x \cdot \ln 3 =$$
$$= \frac{1}{2 \cdot \sqrt{x}} \cdot 3^x + \sqrt{x} \cdot 3^x \cdot \ln 3.$$

Exercise 8. Find the derivative function of $u(x) = \dfrac{x^3 + 3x^2 - 1}{\mathrm{e}^x}$.

Solution:

The derivative function of u is:

$$u'(x) = \frac{(x^3 + 3x^2 - 1)' \cdot \mathrm{e}^x - (x^3 + 3x^2 - 1) \cdot (\mathrm{e}^x)'}{(\mathrm{e}^x)^2} =$$
$$= \frac{(3x^2 + 6x) \cdot \mathrm{e}^x - (x^3 + 3x^2 - 1) \cdot \mathrm{e}^x}{\mathrm{e}^{2x}}.$$

Exercise 9. Find the derivative function of $u(x) = \dfrac{x}{\cos x}$.

Solution:

The derivative function of u is:

$$u'(x) = \frac{(x)' \cdot \cos x - x \cdot (\cos x)'}{\cos^2 x} = \frac{\cos x + x \cdot \sin x}{\cos^2 x}.$$

Exercise 10. Find the derivative function of

$$u(x) = \ln(\cos x)$$

Solution:

The outer function, the inner function, and their derivatives are:

$$\begin{aligned} k(x) &= \ln x & k'(x) &= \frac{1}{x} \\ b(x) &= \cos x & b'(x) &= -\sin x. \end{aligned}$$

Using these:

$$u'(x) = \frac{1}{\cos x} \cdot (-\sin x) = -\frac{\sin x}{\cos x} = -\tan x.$$

Exercise 11. Find the derivative function of

$$u(x) = \mathrm{e}^{\sin x}$$

Solution:

The outer function, the inner function, and their derivatives are:

$$\begin{aligned} k(x) &= \mathrm{e}^x & \qquad k'(x) &= \mathrm{e}^x \\ b(x) &= \sin x & \qquad b'(x) &= \cos x. \end{aligned}$$

Using these:

$$u'(x) = \mathrm{e}^{\sin x} \cdot \cos x.$$

Exercise 12. Find the derivative function of

$$u(x) = 5^{\tan x}$$

Solution:

The outer function, the inner function, and their derivatives are:

$$\begin{aligned} k(x) &= 5^x & \qquad k'(x) &= 5^x \cdot \ln 5 \\ b(x) &= \tan x & \qquad b'(x) &= \tfrac{1}{\cos^2 x}. \end{aligned}$$

Using these:

$$u'(x) = 5^{\tan x} \cdot \ln 5 \cdot \frac{1}{\cos^2 x} = \frac{5^{\tan x} \cdot \ln 5}{\cos^2 x}.$$

Exercise 13. Find the derivative function of

$$u(x) = (x^2 + 2x)^{100}$$

Solution:

The outer function, the inner function, and their derivatives are:

$$\begin{aligned} k(x) &= x^{100} & \qquad k'(x) &= 100x^{99} \\ b(x) &= x^2 + 2x & \qquad b'(x) &= 2x + 2. \end{aligned}$$

Using these:

$$u'(x) = 100 \cdot (x^2 + 2x)^{99} \cdot (2x + 2).$$

Exercise 14. Find the derivative function of

$$u(x) = \tan(x^3 + x^2 + x)$$

Solution:

The outer function, the inner function, and their derivatives are:

$$\begin{aligned} k(x) &= \tan x & k'(x) &= \frac{1}{\cos^2 x} \\ b(x) &= x^3 + x^2 + x & b'(x) &= 3x^2 + 2x + 1. \end{aligned}$$

Using these:

$$u'(x) = \frac{1}{\cos^2(x^3 + x^2 + x)} \cdot (3x^2 + 2x + 1) = \frac{3x^2 + 2x + 1}{\cos^2(x^3 + x^2 + x)}.$$

Exercise 15. Find the derivative function of

$$u(x) = \sqrt{x^2 + 5x + 6}$$

Solution:

The outer function, the inner function, and their derivatives are:

$$\begin{aligned} k(x) &= \sqrt{x} = x^{\frac{1}{2}} & k'(x) &= \frac{1}{2}x^{-\frac{1}{2}} \\ b(x) &= x^2 + 5x + 6 & b'(x) &= 2x + 5. \end{aligned}$$

Using these:

$$u'(x) = \frac{1}{2} \cdot (x^2 + 5x + 6)^{-\frac{1}{2}} \cdot (2x + 5) = \frac{2x + 5}{2 \cdot \sqrt{x^2 + 5x + 6}}.$$

Exercise 16. Find the derivative function of

$$u(x) = \ln(\sin^5 x)$$

Solution:

If $f(x) = \ln x$, $g(x) = x^5$ and $h(x) = \sin x$, then

$$u(x) = f \circ g \circ h(x) = f\Big(g\big(h(x)\big)\Big).$$

The derivative of this multiply composite function is:

$$u'(x) = f'\Big(g\big(h(x)\big)\Big) \cdot g'\big(h(x)\big) \cdot h'(x).$$

Since $f'(x) = \frac{1}{x}$, $g'(x) = 5x^4$ and $h'(x) = \cos x$, we get

$$u'(x) = \frac{1}{\sin^5 x} \cdot 5 \cdot \sin^4 x \cdot \cos x.$$

Exercise 17. Find the derivative function of

$$f(x) = \ln^3(x-1)$$

Solution:

The derivative function is:

$$f'(x) = 3 \cdot \ln^2(x-1) \cdot \frac{1}{x-1} = \frac{3 \cdot \ln^2(x-1)}{x-1}.$$

Exercise 18. Calculate the second derivative of the function

$$f(x) = x^4 + 8x^3 - 8x^2 + 3x + 10.$$

Solution:

The derivative of the function is:

$$f'(x) = 4x^3 + 24x^2 - 16x + 3.$$

The second derivative is:

$$f''(x) = 12x^2 + 48x - 16.$$

2. Properties of one-variabled function

Theoretical summary

Theorem 1.2.1. Let $f\colon I \to \mathbb{R}$ be a differentiable function. The function f is monotonically increasing on I if and only if for all $x \in I$, $f'(x) \geq 0$.

The function f is monotonically decreasing on I if and only if for all $x \in I$, $f'(x) \leq 0$.

Theorem 1.2.2. A twice differentiable function $f\colon I \to \mathbb{R}$ is convex on I if and only if $f''(x) \geq 0$ for all $x \in I$.

A twice differentiable function $f\colon I \to \mathbb{R}$ is concave on I if and only if $f''(x) \leq 0$ for all $x \in I$.

Theorem 1.2.3. Let $f\colon I \to \mathbb{R}$ be a differentiable function. If f has a local extremum at point $x_0 \in I$, then $f'(x_0) = 0$.

Remark 1.2.4. The converse of the previous theorem is not true. For example, for the function $f(x) = x^3$, the derivative at point $x_0 = 0$ is 0, however, the function f does not have a local extremum at point x_0.

Theorem 1.2.5. Let $f\colon I \to \mathbb{R}$ be a twice differentiable function. If f has an inflection point at $x_0 \in I$, then $f''(x_0) = 0$.

Remark 1.2.6. The converse of the previous theorem is not true. For example, for the function $f(x) = x^4$, the second-order derivative at point $x_0 = 0$ is 0, however, the function f does not have an inflection point at x_0.

Solved exercises

Exercise 19. Consider the function $f(x) = x^3 - 9x$.

a) Find the largest subset of real numbers on which the function is defined!

b) Calculate the zeros of the function!

c) Determine where the function is monotonically increasing and where it is monotonically decreasing and give its local extrema!

d) Characterize the function in terms of convexity and give its inflection point!

e) Calculate the limits at the boundary points of the domain!

f) Sketch the graph of the function!

g) Determine the range of the function!

Solution:

a) Domain: $x \in \mathbb{R}$.

b) The zeros of the function are found by solving the equation $f(x) = 0$, that is,
$$x^3 - 9x = 0 \qquad \Rightarrow \qquad x \cdot (x^2 - 9) = 0.$$
A product is zero if and only if one of its factors is zero, which gives us $x = 0$, or $x^2 - 9 = 0$.
The solution to the equation $x^2 - 9 = 0$ is: $x = \pm 3$.
Thus, the function has three zeros: 0; 3; -3.

c) The derivative of the function is: $f'(x) = 3x^2 - 9$. The zeros of the derivative function, that is, the solutions of the equation $3x^2 - 9 = 0$ are:
$$3x^2 - 9 = 0 \quad \Rightarrow \quad 3x^2 = 9 \quad \Rightarrow \quad x^2 = 3 \quad \Rightarrow \quad x = \pm\sqrt{3}.$$
The sign of the first derivative is summarized in a table:

x	$]-\infty; -\sqrt{3}[$	$-\sqrt{3}$	$]-\sqrt{3}; \sqrt{3}[$	$\sqrt{3}$	$]\sqrt{3}; \infty[$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	loc. max.	$\searrow$	loc. min.	$\nearrow$
$f(x)$		$6 \cdot \sqrt{3}$		$-6 \cdot \sqrt{3}$	

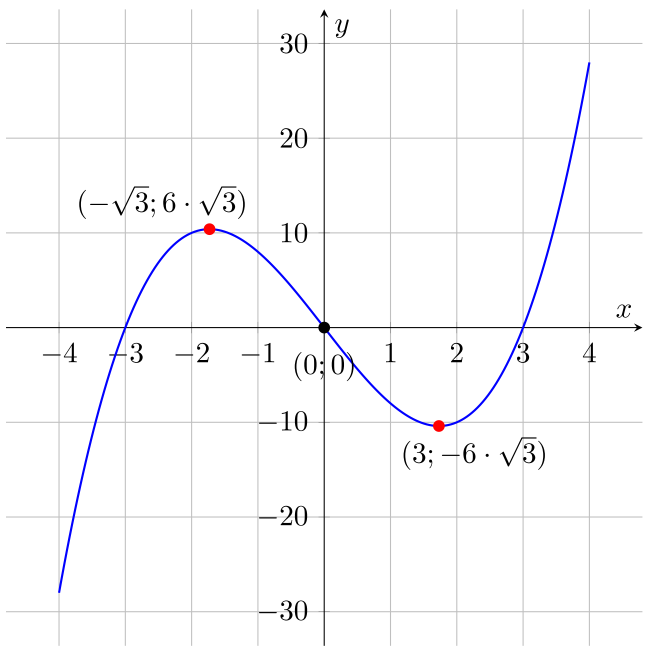
d) The second-order derivative of the function is: $f''(x) = 6x$. Its zero, that is, the solution to the equation $6x = 0$ is: $x = 0$. Accordingly, we examine the sign of the second-order derivative in a table, from which we can infer the convexity of the function:

x	$]-\infty;0[$	0	$]0;\infty[$
$f''(x)$	$-$	0	$+$
$f(x)$	concave	i.p.	convex
$f(x)$		0	

e) The boundary points of the domain are $-\infty$ and ∞.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (x^3 - 3x) = -\infty;$$
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (x^3 - 3x) = \infty.$$

f) The graph of the function:



g) Range: $y \in \mathbb{R}$.

Exercise 20. Consider the function $f(x) = x^4 - 4x^3$.

a) Find the largest subset of real numbers on which the function is defined!

b) Calculate the zeros of the function!

c) Determine where the function is monotonically increasing and where it is monotonically decreasing and give its local extrema!

d) Characterize the function in terms of convexity and give its inflection point!

e) Calculate the limits at the boundary points of the domain!

f) Sketch the graph of the function!

g) Determine the range of the function!

Solution:

a) Domain: $x \in \mathbb{R}$.

b) The zeros of the function are found by solving $f(x) = 0$:

$$x^4 - 4x^3 = 0 \qquad \Rightarrow \qquad x^3 \cdot (x - 4) = 0.$$

A product is zero if and only if one of its factors is zero, thus $x = 0$, or $x = 4$.

c) The derivative of the function is: $f'(x) = 4x^3 - 12x^2$. To determine its zeros, we solve the equation $4x^3 - 12x^2 = 0$, that is

$$x^2 \cdot (4x - 12) = 0.$$

This is only possible if $x^2 = 0$, that is, $x = 0$, or $4x - 12 = 0$, from which $4x = 12$, so $x = 3$. The sign of the first derivative is summarized in the following table:

x	$]-\infty; 0[$	0	$]0; 3[$	3	$]3; \infty[$
$f'(x)$	$-$	0	$-$	0	$+$
$f(x)$	$\searrow$	not extremum	$\searrow$	loc. min.	$\nearrow$
$f(x)$		0		27	

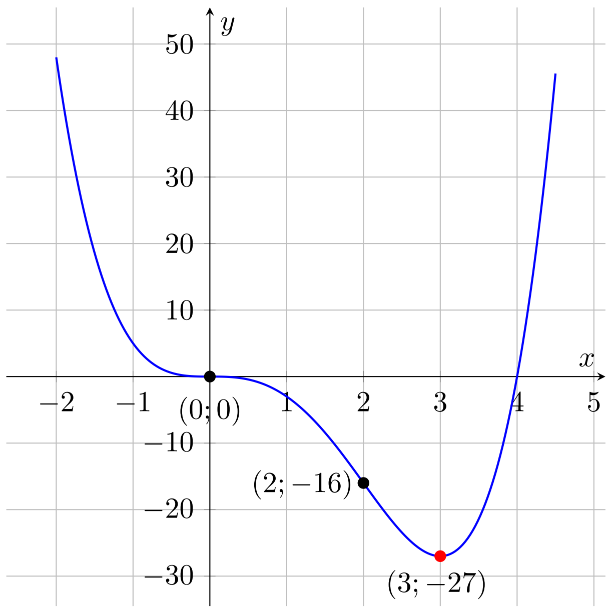
d) The second-order derivative of the function is: $f''(x) = 12x^2 - 24x$. To find its zeros, we need to solve the equation $12x^2 - 24x = 0$. Factoring out x, we get $x \cdot (12x - 24) = 0$. A product can be 0 if one of its factors is 0, thus $x = 0$ or $12x - 24 = 0$, from which $x = 2$. Accordingly, the sign of the second-order derivative is examined in a table:

x	$]-\infty;0[$	0	$]0;2[$	2	$]2;\infty[$
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	convex	i.p.	concave	i.p.	convex
$f(x)$		0		-16	

e) Limits at the boundary points of the domain:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (x^4 - 4x^3) = \infty;$$
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (x^4 - 4x^3) = \infty.$$

f) The graph of the function:



g) Range: $y \in [-27;\infty[$.

3. Optimization of one-variabled function

Theoretical summary

How we can solve optimization problems?

1. Understand the Problem: Carefully read the problem statement. Identify the given information (constraints) and the quantity to be optimized (the objective function).
2. Visualize (If Applicable): If the problem involves geometry, draw a clear and labeled diagram. This helps to visualize the relationships between variables.If we have a geometrically problem, we have to draw a figure.
3. Introduce Variables and Constraints: Define variables to represent the relevant quantities in the problem. Express all relationships (from the problem statement and/or the diagram) as equations or inequalities. These are the constraints.
4. Formulate the Objective Function: Write an equation for the quantity to be optimized (the objective function) in terms of the variables defined in step 3. If possible, use the constraints to express the objective function as a function of a single variable, say f. Specify the domain of f based on the constraints of the problem.
5. Compute the derivative function of f.
6. Find the critical points by solving the equation $f'(x) = 0$ or identifying where the derivative function is undefined.
7. Identify all potential locations of the extrema: The critical points found in step 5 that lie within the domain of f and the endpoints of the domain of f, if the domain is a closed interval.
8. Evaluate the Objective Function at each of the critical points.
9. Select the largest and smallest values of the function at the candidate points.
10. Write the answer to the qeuestions.

There is another way to give extremum point.

1. Understand the Problem: Carefully read the problem statement. Identify the given information and, most importantly, determine the quantity that needs to be maximized or minimized (optimized).

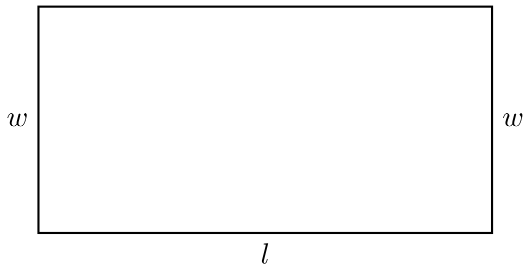
2. Draw a Diagram (if applicable): If the problem involves geometry, create a clear and labeled diagram. This will help visualize the relationships between different elements.
3. Introduce Variables and Establish Relationships: Define variables to represent the relevant quantities in the problem. Express all relationships, both from the diagram and the problem statement, as equations or algebraic expressions. Clearly identify the variable representing the quantity to be optimized.
4. Formulate the Objective Function: Write an equation for the quantity to be optimized (the "objective function"). If possible, express this quantity as a function, f, of a single variable. This is crucial for applying calculus techniques.
5. Compute the First Derivative: Calculate the derivative of the objective function, $f'(x)$, with respect to the single variable x.
6. Find Critical Points: Determine the critical points by solving the equation $f'(x) = 0$. These points are potential locations of local maxima or minima. Suppose x_0 is a critical point within the relevant domain, i.e., $a \leq x_0 \leq b$.
7. Compute the Second Derivative: Calculate the second derivative of the objective function, $f''(x)$.
8. Apply the Second Derivative Test: Evaluate the second derivative at the critical point x_0.
   - If $f''(x_0) > 0$, then x_0 corresponds to a local minimum.
   - If $f''(x_0) < 0$, then x_0 corresponds to a local maximum.

Solved exercises

Exercise 21. A rectangular garden next to a river has an area of $200\,\mathrm{m}^2$. Determine the side lengths such that the perimeter is minimized, given that fencing is required on three sides!

<u>Solution:</u>

Let and w and l represent the length and width of the rectangle, respectively.



The area of the rectangle is 200, thus

$$200 = l \cdot w \qquad \Rightarrow \qquad w = \frac{200}{l}.$$

The perimeter of the recatngular is

$$P = l + 2w \quad \Rightarrow \quad P(l) = l + 2 \cdot \frac{200}{l} = l + \frac{400}{l} = l + 400l^{-1}.$$

The derivative function of the function P is

$$P'(l) = 1 - 400l^{-2} = 1 - \frac{400}{l^2}.$$

Solving the equation $P'(l) = 0$, we have

$$1 - \frac{400}{l^2} = 0 \quad \Rightarrow \quad l^2 - 400 = 0 \quad \Rightarrow \quad l^2 = 400.$$

Since $l > 0$ for the length of a rectangle, must be 20 and

$$w = \frac{200}{l} = \frac{200}{20} = 10.$$

Since

$$P''(l) = 800l^{-3} = \frac{800}{l^3} \qquad \Rightarrow \qquad P''(10) = \frac{800}{10^3} > 0,$$

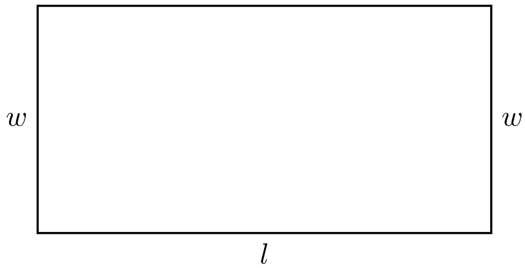
we have a minimum at $l = 5$. The minimum perimeter is

$$P = l + 2w = 20 + 2 \cdot 10 = 40\,\mathrm{m}.$$

Exercise 22. A farmer has $400\,\mathrm{m}$ of fencing and wants to fence off a rectangular field that borders a straight river. If needs no fence along the river, what are the dimensions of the field that has the largest area?

Solution:

Let's denote the sides of the rectangle by l and w.



The total length of the fence is

$$P = l + 2w.$$

Using the equation $P = 400$, we get that

$$l + 2w = 400 \qquad \Rightarrow \qquad l = 400 - 2w.$$

The area of the rectangle is

$$A = l \cdot w = (400 - 2w) \cdot w = 400w - 2w^2.$$

Thus the function we intend to maximize is

$$A(w) = 400w - 2w^2.$$

The derivative of function A is

$$A'(w) = 400 - 4w,$$

thus to find the extreme values we have to solve the equation $A'(w) = 0$, that is $400 - 4w = 0$. The solution of the equation above is $w = 100$.

The maximum value of A can be found at 100. Since $A''(w) = -4 < 0$, thus A is always concave downward and local maximum at $w = 50$ must be an absolute maximum.

The dimensions of the rectangular field are 100 and 200 meters. The maximum area is

$$A = l \cdot w = 200 \cdot 100 = 20\,000\,\mathrm{m}^2.$$

Exercise 23. The quantity of water in a water storage is given as a function of time t is

$$V(t) = 4t^3 - 42t^2 + 120t \,\text{liters},$$

where t is in the interval $[0; 6]$ and t is the number of days since the beginning of the observation.

a) How much fluid was in the storage at the beginning and end of the observation?

b) Calculate the rate of change of the quantity of water.

c) What is the rate of change at $t = 1$ and $t = 2$?

d) Give the time interval when the function is increasing and the one when it is decreasing.

e) Find the global minimum and global maximum of function V.

f) Determine the convexity of function V.

g) Sketch the graph of function $V(t)$.

Solution:

a) The beginning of the observation is

$$V(0) = 4 \cdot 0^3 - 42 \cdot 0^2 + 120 \cdot 0 = 0 \,\text{liters}.$$

At the end of the observation is

$$V(6) = 4 \cdot 6^3 - 42 \cdot 6^2 + 120 \cdot 6 = 72 \,\text{liters}.$$

b) The rate of the change of function V is

$$V'(t) = 12t^2 - 84t + 120.$$

c) The rate of change at $t = 1$ is

$$V'(1) = 12 \cdot 1^2 - 84 \cdot 1 + 120 = 48 \left[\frac{\text{liters}}{\text{day}}\right].$$

The rate of change at $t = 2$ is

$$V'(2) = 12 \cdot 2^2 - 84 \cdot 2 + 120 = 0 \left[\frac{\text{liters}}{\text{day}}\right].$$

d) We have to solve the equation $12t^2 - 84t + 120 = 0$. Simplifying this equation, we get that

$$t^2 - 7t + 10 = 0.$$

Applying the quadratic formula, we get that

$$t_{1,2} = \frac{7 \pm \sqrt{49 - 40}}{2} = \frac{7 \pm 3}{2},$$

that is $t = 2\,\mathrm{day}$ or $t = 5\,\mathrm{day}$. The sign of the first derivative of function V is shown in the table below

	$0 < t < 2$	$t = 2$	$2 < t < 5$	$t = 5$	$5 < t < 6$
$V'(t)$	$+$	0	$-$	0	$+$
$V(t)$	$\nearrow$	local max.	$\searrow$	local min.	$\nearrow$

e) The values of function V at $t = 2$ and $t = 5$ are

$$V(2) = 4 \cdot 2^3 - 42 \cdot 2^2 + 120 \cdot 2 = 104\,\mathrm{liters}$$
$$V(5) = 4 \cdot 5^3 - 42 \cdot 5^2 + 120 \cdot 5 = 50\,\mathrm{liters}$$

and

$$V(0) = 0 \quad \text{and} \quad V(6) = 72,$$

thus the minimum value is $0\,\mathrm{liters}$, the maximum value is $104\,\mathrm{liters}$.

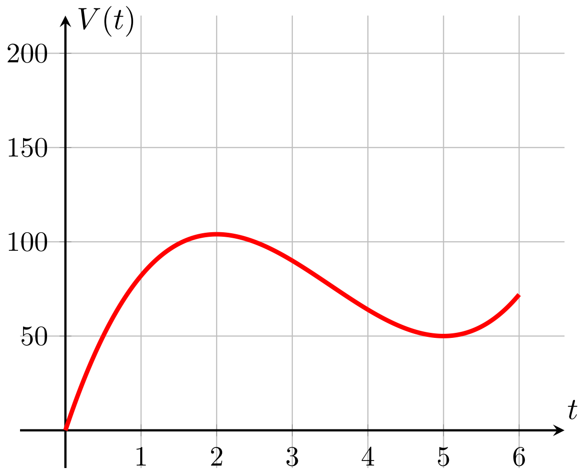
f) To find the convexity we have to calculate the function V''. Since

$$V''(t) = 24t - 84$$

then the zero of this function is the solution of equation $24t - 84 = 0$, that is $t = 3{,}5$. The sign of the second derivative of function V is shown in the table below

	$0 < t < 3{,}5$	$t = 3{,}5$	$t > 3{,}5$
$V''(t)$	$-$	0	$+$
$V(t)$	concave	infl. point	

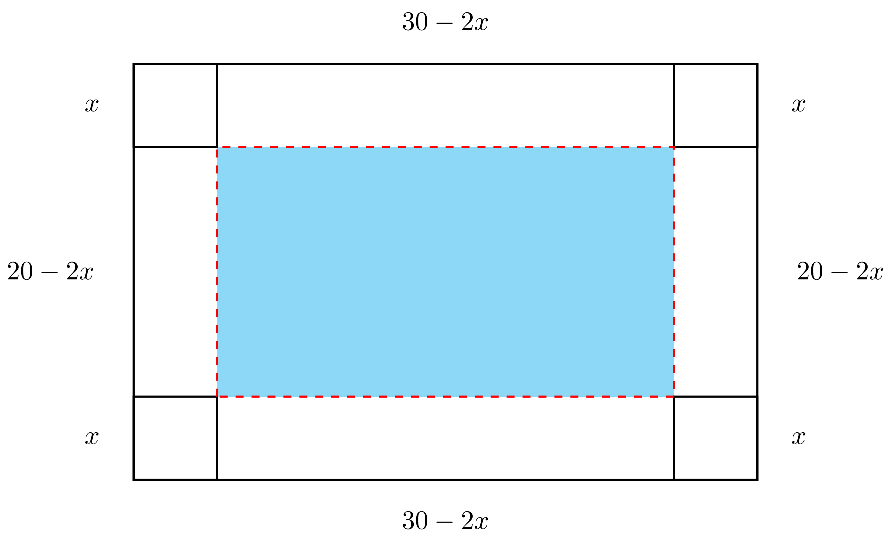
g) The graph of function V is



Exercise 24. An open box is made by cutting congruent squares from the corners of a $20\,\mathrm{cm}$ by $30\,\mathrm{cm}$ rectangular cardboard sheet. How large should the squares be so that the box has a maximum volume? What is the maximum capacity of the box?

Solution:

Let x be a side of the square in cm. In first step we make a figure according to the text.



The volume of the box is

$$\begin{aligned} V(x) &= (30-2x)\cdot(20-2x)\cdot x = (600-100x+4x^2)\cdot x = \\ &= 4x^3-100x^2+600x. \end{aligned}$$

The domain of function V is interval $[0;10]$ (since x cannot exceed half of the smaller side, which is 20 cm).
The derivative of function V is

$$V'(x) = 12x^2-200x+600,$$

thus we have to solve the equation $12x^2-200x+600=0$. If we simplify this equation, we get that

$$3x^2-50x+150=0.$$

Using the quadratic formula:

$$x_{1,2} = \frac{50\pm\sqrt{50^2-4\cdot3\cdot150}}{2\cdot3} = \frac{50\pm\sqrt{2500-1800}}{6} = \frac{50\pm\sqrt{700}}{6}$$

Computing the values:

$$x_1 \approx \frac{50+26.46}{6} \approx 12.74$$

and

$$x_2 \approx \frac{50-26.46}{6} \approx 3.92.$$

Since x must be less than 10, we only consider $x_2 \approx 3.92$ as a valid critical point. The second derivative of function V is

$$V''(x) = 24x-200 \quad \Rightarrow \quad V''(3.92) \approx 24\cdot3.92-200 \approx -106 < 0.$$

Since $V''(3.92) < 0$, at $x \approx 3.92$ function V has a maximum. The maximum volume is

$$\begin{aligned} V(3.92) &= 4\cdot(3.92)^3-100\cdot(3.92)^2+600\cdot(3.92) = \\ &= 4\cdot60.2-100\cdot15.37+600\cdot3.92 \\ &= 240.8-1537+2352 = 1055.8\,\mathrm{cm}^3. \end{aligned}$$

Therefore, the squares cut from each corner should be approximately $3.92\,\mathrm{cm}\times 3.92\,\mathrm{cm}$, and the maximum capacity of the box is approximately $1055.8\,\mathrm{cm}^3$.

Exercise 25. Engineers are designing a cylindrical can to hold 1 liter of car oil, with the minimum surface area in order to minimize costs. Find the optimal dimensions of the can.

Solution:

Let's denote the radius of the base and the height of the can by r and h respectively. Then the volume of the cylindrical can is

$$V = r^2 \cdot \pi \cdot h.$$

Since the volume is $V = 1\,000\,\mathrm{cm}^3$, we get that

$$1000 = r^2 \cdot \pi \cdot h \qquad \Rightarrow \qquad h = \frac{1000}{\pi \cdot r^2}.$$

The surface area of the can is

$$S = 2 \cdot r^2 \cdot \pi + 2 \cdot r \cdot \pi \cdot h.$$

The surface are expressed by radius r:

$$S(r) = 2 \cdot r^2 \cdot \pi + 2 \cdot r \cdot \pi \cdot \frac{1\,000}{\pi \cdot r^2} = 2 \cdot r^2 \cdot \pi + \frac{2\,000}{r}.$$

The domain of function S is interval $]0; \infty[$.
The derivative of function S is

$$S'(r) = 4 \cdot r \cdot \pi - \frac{2\,000}{r^2}.$$

We have to solve the following equation

$$4 \cdot r \cdot \pi - \frac{2\,000}{r^2} = 0.$$

Expressing r, we get that

$$4r^3 \cdot \pi - 2\,000 = 0 \qquad \Rightarrow \qquad r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42$$

The second derivative of function S is

$$S''(r) = 4\pi + \frac{4\,000}{r^3} \qquad (r > 0).$$

Since function $S''(r) > 0$, hence function S is convex for all $r > 0$, thus the global minimum is at $r \approx 5.42$. The value of h is

$$h = \frac{1\,000}{\pi \cdot r^2} \approx \frac{1\,000}{\pi \cdot 5.42^2} = 10.84\,\mathrm{cm}.$$

Exercise 26. The fish population (P) in hundreds in a lake at a time t in months is modelled by

$$P(t) = 15 + \mathrm{e}^{0.5t} - 2t \qquad (t \geq 0).$$

a) Find the initial fish population in the lake.
b) Find the function $P'(t)$.
c) Find the time at which the fish population is growing at a rate of 4 hundred per month.
d) Find the function $P''(t)$ and explain the physical significance of this quantity.
e) Find the minimum fish population, justifying that it is a minimum.

Solution:

a) Since
$$P(0) = 15 + \mathrm{e}^{0.5 \cdot 0} - 2 \cdot 0 = 15 + 1 = 16,$$
then the initial fish population in the lake is $1,600$ fish.

b) The derivative of function P is
$$P'(t) = 0.5\mathrm{e}^{0.5t} - 2.$$

c) We have to solve the equation $P'(t) = 4$, that is,
$$4 = 0.5\mathrm{e}^{0.5t} - 2.$$
Adding 2 to both sides, we have
$$6 = 0.5\mathrm{e}^{0.5t}.$$
Multiplying both sides by 2, we get that
$$12 = \mathrm{e}^{0.5t}.$$
Taking the natural logarithm of both sides, we have $\ln 12 = 0.5t$, thus
$$t = \frac{\ln 12}{0.5} = 2 \ln 12 \approx 4.97,$$
therefore the fish population is growing at a rate of 400 per month after approximately 5 months.

d) The second derivative of function P is

$$P''(t) = 0.5 \cdot 0.5\mathrm{e}^{0.5t} = 0.25\mathrm{e}^{0.5t},$$

which represents the rate of change of the growth rate of the fish population. In other words, it shows how the population growth is accelerating or decelerating over time.

e) We have to solve the equation $P'(t) = 0$, that is,

$$0.5\mathrm{e}^{0.5t} - 2 = 0 \qquad \Rightarrow \qquad 0.5\mathrm{e}^{0.5t} = 2,$$

Multiplying both sides by 2, we have $\mathrm{e}^{0.5t} = 4$.
Taking the natural logarithm of both sides:

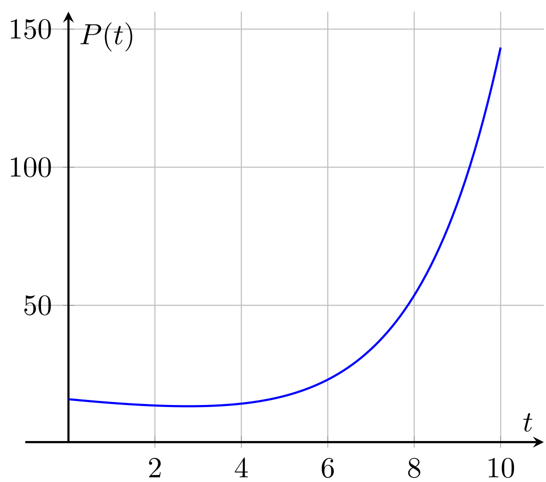
$$0.5t = \ln 4 = \ln 2^2 = 2\ln 2,$$

thus $t = \frac{2\ln 2}{0.5} = 4\ln 2 \approx 2.77$.
Second derivative of function $P(t)$ is positive for all t (since $0.25\mathrm{e}^{0.5t} > 0$ for all $t \geq 0$), thus P has a minimum at $t = 4\ln 2$.
Since

$$\begin{aligned} P(4\ln 2) &= 15 + \mathrm{e}^{0.5\cdot 4\ln 2} - 2\cdot 4\ln 2 = 15 + \mathrm{e}^{2\ln 2} - 8\ln 2 = \\ &= 15 + 2^2 - 8\ln 2 = 15 + 4 - 8\ln 2 = 19 - 8\ln 2 \approx \\ &\approx 19 - 8\cdot 0.693 \approx 19 - 5.54 \approx 13.46. \end{aligned}$$

The minimum fish population is approximately 1 346 fish.

f) The graph of function is



Exercise 27. A particle is moving in a straight line. Its position-time function is

$$s(t) = t^3 - 6t^2 + 12t + 2\,\mathrm{m} \qquad (t \in [0;4]),$$

where t is the time in seconds.

a) Find expression for the particle's velocity-time function.

b) Find expression for the particle's acceleration-time function.

c) Calculate the position, velocity and acceleration of the particle at $t = 0$.

d) Calculate the position, velocity and acceleration of the particle at $t = 2$.

e) Calculate the position, velocity and acceleration of the particle at $t = 3$.

f) Given the time interval(s) in which the speed of the particle is increasing.

g) Find the average velocity in time interval $[0;2]$.

h) Find the average acceleration in time interval $[0;2]$.

i) What is the minimum value of the velocity?

Solution:

a) The velocity-time function is the derivative of the position-time function

$$v(t) = s'(t) = 3t^2 - 12t + 12.$$

b) The acceleration-time function is the derivative of the velocity-time function

$$a(t) = v'(t) = 6t - 12.$$

c) If $t = 0$ then

$$\begin{aligned}
s(0) &= 0^3 - 6 \cdot 0^2 + 12 \cdot 0 + 2 = 2\,\mathrm{m} \\
v(0) &= 3 \cdot 0^2 - 12 \cdot 0 + 12 = 12\,\frac{\mathrm{m}}{\mathrm{s}} \\
a(0) &= 6 \cdot 0 - 12 = -12\,\frac{\mathrm{m}}{\mathrm{s}^2}.
\end{aligned}$$

d) If $t = 2$ then

$$\begin{aligned}
s(2) &= 2^3 - 6 \cdot 2^2 + 12 \cdot 2 + 2 = 8 - 24 + 24 + 2 = 10\,\mathrm{m} \\
v(2) &= 3 \cdot 2^2 - 12 \cdot 2 + 12 = 12 - 24 + 12 = 0\,\frac{\mathrm{m}}{\mathrm{s}} \\
a(2) &= 6 \cdot 2 - 12 = 12 - 12 = 0\,\frac{\mathrm{m}}{\mathrm{s}^2}.
\end{aligned}$$

e) If $t = 3$ then

$$s(3) = 3^3 - 6 \cdot 3^2 + 12 \cdot 3 + 2 = 27 - 54 + 36 + 2 = 11\,\mathrm{m}$$
$$v(3) = 3 \cdot 3^2 - 12 \cdot 3 + 12 = 27 - 36 + 12 = 3\,\frac{\mathrm{m}}{\mathrm{s}}$$
$$a(3) = 6 \cdot 3 - 12 = 18 - 12 = 6\,\frac{\mathrm{m}}{\mathrm{s}^2}.$$

f) Speed is increasing when $v(t)$ and $a(t)$ have the same sign. Since $a(t) < 0$ when $t < 2$ and $a(t) > 0$ when $t > 2$, and $v(t) > 0$ when $t < 2$ or $t > 2$ (with $v(2) = 0$), we get that speed is increasing when $t > 2$. (In the interval $t < 2$, the speed is decreasing because $v > 0$ but $a < 0$).

g) Since $s(2) = 10$ and $s(0) = 2$ the average velocity on time interval $[0; 2]\,\mathrm{s}$ is

$$\frac{s(2) - s(0)}{2 - 0} = \frac{10 - 2}{2} = 4\,\frac{\mathrm{m}}{\mathrm{s}}.$$

h) Since $v(2) = 0$ and $v(0) = 12$ the average acceleration on time interval $[0; 2]\,\mathrm{s}$ is

$$\frac{v(2) - v(0)}{2 - 0} = \frac{0 - 12}{2} = -6\,\frac{\mathrm{m}}{\mathrm{s}^2}.$$

i) To find the minimum value of velocity, we set $v'(t) = 0$ and solve:

$$a(t) = 6t - 12 = 0 \quad \Rightarrow \quad t = 2.$$

We've already calculated $v(2) = 0$. Because $a(t) < 0$ for $t < 2$ and $a(t) > 0$ for $t > 2$, this is indeed a minimum.

The minimum velocity is $v_{min} = 0\,\frac{\mathrm{m}}{\mathrm{s}}$ occurring at $t = 2$ seconds.

Exercise 28. The cost of manufacturing bicycles, in thousand units, is modeled by

$$C(x) = 3x^3 - 24x^2 + 50x \qquad (0 \le x \le 12).$$

The revenue function is modeled by

$$R(x) = 16x + 8.$$

a) Find the value of fixed cost.
b) Find the variable cost function.
c) Find the profit function.
d) Find the production level that maximizes profits.
e) Calculate the marginal cost function.

f) Find the production level, if it exists, that minimizes cost.
g) Find the average cost function.
h) Find the production level, if it exists, that minimizes average cost.

<u>Solution:</u>

a) Fixed cost is $C(0) = 0$.

b) Variable cost function is

$$VC(x) = 3x^3 - 24x^2 + 50x.$$

c) Profit function is

$$\begin{aligned}\Pi(x) = R(x) - C(x) &= 16x + 8 - (3x^3 - 24x^2 + 50x) = \\ &= -3x^3 + 24x^2 - 34x + 8.\end{aligned}$$

d) We have to solve the equation $\Pi'(x) = 0$, that is

$$-9x^2 + 48x - 34 = 0 \qquad \Rightarrow \qquad -9x^2 + 48x - 34 = 0.$$

Applying the quadratic formula, we get that

$$x_{1,2} = \frac{48 \pm \sqrt{2304 - 4 \cdot 9 \cdot 34}}{-18}.$$

The solutions of the $\Pi'(x) = 0$ equation are $x_1 \approx 0.83$ and $x_2 \approx 4.51$. The second derivative of function $\Pi(x)$ is $\Pi''(x) = -18x + 48$.

Since $\Pi''(4.51) < 0$, maximum profit occurs at a production level of $4,510$ bicycles.

e) Marginal cost function is

$$MC(x) = C'(x) = 9x^2 - 48x + 50.$$

f) We have to solve the equation $C'(x) = 0$, that is

$$9x^2 - 48x + 50 = 0.$$

Applying the quadratic formula, we get that

$$x_{1,2} = \frac{48 \pm \sqrt{2304 - 4 \cdot 9 \cdot 50}}{18} \approx \frac{48 \pm \sqrt{504}}{18}.$$

The solutions of the quadratic equation are $x_1 \approx 2.22$ and $x_2 \approx 2.45$. Since $C''(x) = 18x - 48$ and

$$C''(2.22) = 18 \cdot 2.22 - 48 < 0,$$
$$C''(2.45) = 18 \cdot 2.45 - 48 < 0,$$

we need to check the endpoints of our domain. The cost at $x = 0$ is $C(0) = 0$, at $x = 12$ is $C(12) = 3(12)^3 - 24(12)^2 + 50(12)$, which is significantly larger. Therefore the minimum cost occurs at a production level of 0 bicycles.

g) The average cost function is

$$AC(x) = \frac{C(x)}{x} = \frac{3x^3 - 24x^2 + 50x}{x} = 3x^2 - 24x + 50$$

h) The derivative of the average cost is

$$AC'(x) = 6x - 24.$$

The solution of the equation $6x - 24 = 0$ is $x = 4$, that is the production level, when the average cost is minimum is $x = 4$.

Exercise 29. A pencil cup with a capacity of $20\,\mathrm{dm}^3$ is to be constructed in the shape of a rectangular box with a square base and an open top. If the material for the sides costs 12 cents per inch squared and the material for the base costs 60 cents per inch squared, what should the dimensions of the cup be to minimize the construction cost?

Solution:

An open box with a square base can be defined as a square base prism without a top. In these premiums only two variables are defined and based on these two variables the surface area and volume can be expressed:

$$S = x^2 + 4xy \quad \text{and} \quad V = x^2 \cdot y,$$

where x is the length of the base y is the height of the box. Since

$$V = 20 \quad \Rightarrow \quad x^2 \cdot y = 20 \quad \Rightarrow \quad y = \frac{20}{x^2},$$

therefore

$$S(x) = x^2 + 4x \cdot \frac{20}{x^2} = x^2 + \frac{80}{x}.$$

We have to write the cost of the surface area as a function of one variable:

$$C(x) = 60x^2 + 12 \cdot \frac{80}{x} = 60x^2 + \frac{960}{x}.$$

Differentiate the objective function with respect to x, we get that

$$C'(x) = 120x - \frac{960}{x^2}.$$

We have to solve the equation

$$120x - \frac{960}{x^2} = 0 \quad \Rightarrow \quad 120x^3 - 960 = 0 \quad \Rightarrow \quad x^3 = 8,$$

thus $x = 2$. Since

$$C''(x) = 120 + \frac{1\,920}{x^3} > 0,$$

thus we have a minimum at $x = 2$. The height of the box is

$$y = \frac{20}{2^2} = 5\,\mathrm{dm}.$$

Therefore, the dimensions of the pencil cupthat minimize construction cost surface are:

$$x = 2\,\mathrm{dm} \quad \text{and} \quad y = 5\,\mathrm{dm}.$$

Exercise 30. An apartment complex has 250 apartments to rent. If they rent x apartments then their monthly profit, in dollars, is given by

$$P(x) = -x^2 + 400x - 1\,000.$$

How many apartments should they rent in order to maximize their profit?

Solution:

All that we are really being asked to do here is to maximize the profit subject to the constraint that x must be in the range $0 \leq x \leq 250$. The first derivative of P is

$$P'(x) = -2x + 400.$$

The zero of function P' is

$$-2x + 400 = 0 \qquad \Rightarrow \qquad x = 200.$$

Since the profit function is continuous and we have an interval with finite bounds we can find the maximum value by simply plugging in the only critical point that we have and the end points of the range:

$$\begin{aligned} P(0) &= -0^3 + 400 \cdot 0 - 1\,000 = 0 \\ P(200) &= -200^2 + 400 \cdot 200 - 1\,000 = 30\,000 \\ P(250) &= -250^2 + 400 \cdot 250 - 1\,000 = 27\,500. \end{aligned}$$

The profit function has has maximum, if they only rent out 200 of the apartments instead of all 250 of them.

Exercise 31. A concert promoter knows that if they charge \$20 per ticket, they will sell 500 tickets. For each dollar increase in price, they sell 10 fewer tickets. What ticket price maximizes revenue?

Solution:

Let x be the unit price increase. The unit price is $20 + x$, the number of tickets sold is $500 - 10x$. The revenue is

$$R(x) = (20 + x) \cdot (500 - 10x).$$

Applying the brackets, we have

$$R(x) = 10\,000 + 300x - 10x^2.$$

The derivative function is

$$R'(x) = 300 - 20x.$$

The critical point is solution of equation $R'(x) = 0$, that is

$$300 - 20x = 0 \Rightarrow x = 15.$$

The second derivative function is

$$R''(x) = -20,$$

so $x = 15$ is a local maximum. The optimal ticket price is $20 + 15 = \$35$.

Exercise 32. An open-top box with a square base is to have a volume of 10 cubic feet. Find the dimensions of the box that minimize the amount of material used.

Solution:

Let x be the side of the base, h be the height.
The volume is $V = x^2h = 10$, so $h = \frac{10}{x^2}$.
The surface area is

$$A = x^2 + 4xh.$$

Using $h = \frac{10}{x^2}$, we have

$$A(x) = x^2 + 4x\frac{10}{x^2} = x^2 + \frac{40}{x}.$$

Differentiate function A, we get that

$$A'(x) = 2x - \frac{40}{x^2}.$$

The critical point is

$$A'(x) = 0 \Rightarrow 2x = \frac{40}{x^2} \Rightarrow x^3 = 20 \Rightarrow x = \sqrt[3]{20}.$$

The second derivative function is

$$A''(x) = 2 + \frac{80}{x^3}.$$

Since $x > 0$, $A''(x) > 0$, so $x = \sqrt[3]{20}$ is a local minimum. The value of h is

$$h = \frac{10}{(\sqrt[3]{20})^2} = \frac{10}{20^{2/3}} = \frac{\sqrt[3]{20}}{2}.$$

The dimensions of the box are: $x = \sqrt[3]{20}$ feet and $h = \frac{\sqrt[3]{20}}{2}$ feet.

Exercise 33. Find the point on the graph of function $f(x) = x^2$ that is closest to the point $(3; 0)$.

Solution:

The distance between the point $(x; x^2)$ and the point $(3; 0)$ is

$$d(x) = \sqrt{(x-3)^2 + (x^2 - 0)^2} = \sqrt{x^2 - 6x + 9 + x^4}.$$

We can minimize the square of the distance

$$g(x) = f^2(x) = x^4 + x^2 - 6x + 9$$

since if function f has minimum at x_0 then the function g has minimum at x_0. The derivative of function g is

$$g'(x) = 4x^3 + 2x - 6.$$

To determine the critical point, we have to solve the equation

$$4x^3 + 2x - 6 = 0.$$

It can be seen easily that $x = 1$ is a solution of this equation, there we can factorize the left hand side:

$$4x^3 - 4 + 2x - 2 = 0 \qquad \Rightarrow \qquad 4 \cdot (x^3 - 1) + 2 \cdot (x - 1) = 0.$$

Using algebraic identity, we get that

$$4 \cdot (x-1) \cdot (x^2 + x + 1) + 2 \cdot (x-1) = 0 \quad \Rightarrow \quad 2 \cdot (x-1) \cdot (2x^2 + 2x + 3) = 0.$$

The quadratic part has no real roots, thus the only one real solution is $x = 1$. The second derivative function is

$$f''(x) = 12x^2 + 2.$$

Since $f''(1) = 14 > 0$, this $x = 1$ is a local minimum. The point that is closest to the point $(3; 0)$ is $(1; 1^2) = (1; 1)$.

Exercise 34. Engineers are designing wagons for carrying coal in a straight railway tunnel with parabolic shaped cross section. The height of the tunnel is nine meters, while its width is six meters at its bottom. If the aim of the engineers is to design wagons with the maximum possible cross section, this way transporting the maximum amount of coal at a given wagon length, what would be the height and width of the wagons?

Solution:

Regarding the text of the problem the parabola goes through points $(-3; 0)$, $(0; 9)$ and $(3; 0)$ and the function which is describing it can be written as

$$f(x) = ax^2 + bx + c.$$

Substituting the coordinates of the points above into function $f(x)$, we get the system of equations

$$\begin{aligned} 0 &= a \cdot (-3)^2 + b \cdot (-3) + c \\ 9 &= a \cdot 0^2 + b \cdot 0 + c \\ 0 &= a \cdot 3^2 + b \cdot 3 + c. \end{aligned}$$

Solving the system of equations, we get that $c = 9$, $b = 0$ and $a = -1$. Consequently, the function which describes the parabola is

$$f(x) = -x^2 + 9.$$

The cross section area of the wagon with width of $2x$ is

$$A(x) = 2x \cdot (-x^2 + 9) = -2x^3 + 18x.$$

The derivative of the function $A(x)$ is

$$A'(x) = -6x^2 + 18.$$

The zeros of the function are $\pm\sqrt{3}$. Since

$$A''(x) = -12x \qquad \Rightarrow \qquad A''(\sqrt{3}) < 0,$$

we have that function A has a local and global maximum. Consequently, the width of the wagon is

$$2x \approx 3.46\,[\mathrm{m}]$$

and its height is

$$f(\sqrt{3}) = -3 + 9 = 6\,[\mathrm{m}].$$

Exercise 35. A paper aeroplane of weight $w > 1$ will travel at a constant speed of $1 - \frac{1}{w}\left[\frac{\mathrm{m}}{\mathrm{s}}\right]$ for a time of $\frac{5}{w}$ [s]. What weight will achieve the maximum distance travelled?

Solution:

The covered distance is

$$s(w) = \left(1 - \frac{1}{\sqrt{w}}\right) \cdot \frac{5}{w} = \frac{5w^{\frac{1}{2}} - 5}{w^{\frac{3}{2}}}.$$

Derivative of function $s(w)$ is

$$s'(w) = \frac{\frac{5}{2} \cdot w^{-\frac{1}{2}} \cdot w^{\frac{3}{2}} - \left(5w^{\frac{1}{2}} - 5\right) \cdot \frac{3}{2} \cdot w^{\frac{1}{2}}}{w^3}.$$

Applying algebraic identities, we get that

$$s'(w) = \frac{\frac{5}{2} \cdot w - \frac{15}{2} \cdot w + \frac{15}{2} \cdot w^{\frac{1}{2}}}{w^3} = \frac{-5w + \frac{15}{2} \cdot \sqrt{w}}{w^3}.$$

We have to solve the equation $s'(w) = 0$, that is,

$$\frac{-5w + \frac{15}{2} \cdot \sqrt{w}}{w^3} = 0 \qquad \Rightarrow \qquad -5w + \frac{15}{2} \cdot \sqrt{w} = 0.$$

Factorizing the equation, we have

$$\sqrt{w} \cdot \left(-5\sqrt{w} + \frac{15}{2}\right).$$

Since $w \neq 0$, thus

$$-10\sqrt{w} + 15 = 0,$$

hence $w = 2.25$.

Second derivative of function $s(w)$ is

$$s''(w) = \frac{\left(-5 + \frac{15}{4\sqrt{w}}\right) \cdot w^3 - \left(5w + \frac{15}{2}\sqrt{w}\right) \cdot 3w^2}{w^6} =$$

$$= \frac{-5w + \frac{15}{4}\sqrt{w} - 5w - \frac{15}{2}\sqrt{w}}{w^4} = \frac{-10w - \frac{15}{4}\sqrt{w}}{w^4}.$$

Since $w > 1$, then $s''(w) < 0$, consequently, function $s(w)$ has a global maximum at $w = 2.25$.

Exercise 36. Architects are designing an archway for a historical building restoration. The archway must fit within a triangular space with a base of eight meters and a height of ten meters. The archway will have a semicircular top and rectangular bottom. If the goal is to create an archway with the maximum possible area for aesthetic purposes, what would be the optimal width and height of the archway?

Solution:

Let's place the triangle in a coordinate system where the base spans from $(-4, 0)$ to $(4, 0)$ and the apex is at $(0, 10)$. The lines forming the sides of the triangle can be described as: For the left side: $y = -\frac{10}{4}x + 10 = -2.5x + 10$ for $-4 \leq x \leq 0$. For the right side: $y = \frac{10}{4}x + 10 = 2.5x + 10$ for $0 \leq x \leq 4$. Denote the half-width of our archway as r. The semicircular top will have radius r. The total height of the archway will consist of two parts. The height of the rectangular part, let's call it h The height of the semicircular part, which is r. The archway must fit inside the triangle, so its top-most point $(0, h + r)$ must be below the apex of the triangle, and its sides must be within the triangular boundaries. Given the slope of the triangular sides, the maximum width of the archway at height y is

$$2x = 2 \cdot \frac{10 - y}{2.5} = \frac{8(10 - y)}{10}.$$

At the height where the semicircle begins $(y = h)$, we must have

$$2r = \frac{8(10 - h)}{10},$$

thus

$$r = \frac{4(10 - h)}{10} = 4 - \frac{2h}{5}.$$

The total area of the archway is:

$$A(h) = 2r \cdot h + \frac{\pi r^2}{2}$$

Substituting our expression for r is

$$A(h) = 2(4 - \frac{2h}{5}) \cdot h + \frac{\pi(4 - \frac{2h}{5})^2}{2}$$

Simplifying, we get that

$$A(h) = 8h - \frac{4h^2}{5} + \frac{\pi(4 - \frac{2h}{5})^2}{2}.$$

Taking the derivative with respect to h, we have

$$A'(h) = 8 - \frac{8h}{5} - \frac{\pi(4 - \frac{2h}{5}) \cdot \frac{4}{5}}{2}.$$

Setting equal to zero and solving:

$$8 - \frac{8h}{5} - \frac{4\pi(4 - \frac{2h}{5})}{10} = 0.$$

After solving this equation, we find that the optimal height of the rectangular part is $h = 3.75$ meters. With this value of h, we can calculate r:

$$r = 4 - \frac{2 \cdot 3.75}{5} = 4 - 1.5 = 2.5 \text{ meters}$$

Therefore, the optimmal width is $2r = 5$ meters and total height is

$$h + r = 3.75 + 2.5 = 6.25.$$

The maximum area of the archway is approximately 19.63 square meters.

4. Optimization of two-variabled function

Theoretical summary

Definition 1.4.1. The function $f\colon D \subset \mathbb{R}^2 \to \mathbb{R}$ has *local maximum* (*relative maximum*) at point $(x_0; y_0) \in D$ if there is disc centered at point $(x_0; y_0)$ such that $f(x; y) \leq f(x_0; y_0)$ for all point $(x; y)$ that lie inside the disc.

The function $f\colon D \subset \mathbb{R}^2 \to \mathbb{R}$ has *local minimum* (*relative minimum*) at point $(x_0; y_0) \in D$ if there is disc centered at point $(x_0; y_0)$ such that $f(x; y) \geq f(x_0; y_0)$ for all point $(x; y)$ that lie inside the disc.

Theorem 1.4.2. If $f\colon D \subset \mathbb{R}^2 \to \mathbb{R}$ is a totally differentiable function and has a local extremum at point $(x_0; y_0) \in D$ then the values of partial derivative functions at $(x_0; y_0)$ are zero, i.e. $f'_x(x_0; y_0) = 0$ and $f'_y(x_0; y_0) = 0$.

Definition 1.4.3. If $f\colon D \subset \mathbb{R}^2 \to \mathbb{R}$ is a twice differentiable function then its *Hessian-matrix* at point $(x_0; y_0) \in D$ is as follows:

$$M(x_0; y_0) = \begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{pmatrix} f''_{xx}(x_0; y_0) & f''_{xy}(x_0; y_0) \\ f''_{yx}(x_0; y_0) & f''_{yy}(x_0; y_0) \end{pmatrix} \end{array}$$

Theorem 1.4.4. If $f\colon D \subset \mathbb{R}^2 \to \mathbb{R}$ twice differentiable function and has a critical point at $(x_0; y_0) \in D$ and let the Hessian-matrix at $(x_0; y_0)$ is

$$M(x_0; y_0) = \begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{pmatrix} f''_{xx}(x_0; y_0) & f''_{xy}(x_0; y_0) \\ f''_{yx}(x_0; y_0) & f''_{yy}(x_0; y_0) \end{pmatrix} \end{array}.$$

Let denote $D_1 = f''_{xx}(x_0; y_0)$ and $D_2 = \det\big(M(x_0; y_0)\big)$.

If $D_1 > 0$ and $D_2 > 0$, then the function f has a local minimum at point $(x_0; y_0)$.

If $D_1 < 0$ and $D_2 > 0$, then the function f has a local maximum at point $(x_0; y_0)$.

If $D_2 < 0$, then the function has no local extremum at point $(x_0; y_0)$.

Remark 1.4.5. How we can find a local extremum?

1. We have to calculate the partial derivatives of function f.

2. We have to solve the system of equation
$$\left.\begin{array}{l} f'_x(x;y) = 0 \\ f'_y(x;y) = 0 \end{array}\right\} .$$
The solutions of the system are the critical points.
3. We have to calculate the second ordered partial derivatives and we have to construct the Hessian-matrix:
$$M(x;y) = \begin{array}{c c} & \begin{array}{cc} \quad x \quad & \quad\quad y \quad \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{pmatrix} f''_{xx}(x;y) & f''_{xy}(x;y) \\ f''_{yx}(x;y) & f''_{yy}(x;y) \end{pmatrix} \end{array} .$$
4. We substitute the critical points to Hessian-matrix and we apply the theorem 1.4.4.

Solved exercises

Exercise 37. Find all critical points of the function

$$f(x;y) = 3x^2 - 6x + 2y^2 - 4y.$$

Solution:

The first ordered partial derivatives of function f are

$$f'_x(x;y) = 6x - 6$$
$$f'_y(x;y) = 4y - 4.$$

We have to solve the system of equations

$$\left.\begin{array}{r} 6x - 6 = 0 \\ 4y - 4 = 0 \end{array}\right\} .$$

The solution of the system is $(1;1)$. The critical point of the function f is $P = (1;1)$.

Exercise 38. The critical point for $f(x;y) = 3x^2 - 12x + y^2 - 8y + 2$ is $P = (2;4)$. Determine if the critical point is a local maximum or minimum.

Solution:

The first ordered partial derivatives of function f are

$$f'_x(x;y) = 6x - 12$$
$$f'_y(x;y) = 2y - 8.$$

The second ordered partial derivatives are

$$f''_{xx}(x;y) = 6 \qquad f''_{xy}(x;y) = 0$$
$$f''_{yx}(x;y) = 0 \qquad f''_{yy}(x;y) = 2,$$

thus the Hessian-matrix is as follows:

$$M(x;y) = \begin{array}{c} \\ x \\ y \end{array}\!\!\begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} \end{array}.$$

If we substitute the coorinates of point P to the Hessian-matrix, we get the same matrix:

$$M(P) = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}.$$

Since $D_1 = 6$ and

$$D_2 = \det \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} = 12,$$

therefore D_1 and D_2 are positive, thus at P has a local minimum.

Exercise 39. Calculate the local extremum of the function

$$f(x;y) = 6x^3 + 6y^3 - 18xy.$$

<u>Solution:</u>

The first ordered partial derivatives of function f are

$$f'_x(x;y) = 18x^2 - 18y$$
$$f'_y(x;y) = 18y^2 - 18x.$$

We have to solve the system of equations

$$\left.\begin{aligned} 18x^2 - 18y = 0 \\ 18y^2 - 18x = 0 \end{aligned}\right\}.$$

If we simplify the equations, we get that

$$\left.\begin{aligned} x^2 - y = 0 \\ y^2 - x = 0 \end{aligned}\right\}.$$

From the second equation, we get that $x = 2y^2$. If we substitute this to the first equation, we get that $y^4 - y = 0$, thus

$$y \cdot (y^3 - 1) = 0.$$

The solutions of this equation are $y_1 = 0$ and $y_2 = 1$, hence $x_1 = 0$ and $x_2 = 1$. The critical points of the function f are $P_1 = (0;0)$ and $P_2 = (1;1)$.

The general form of Hessian-matrix for two variables function is as follows:

$$M(x;y) = \begin{array}{c} \\ x \\ y \end{array} \!\!\begin{array}{c} \begin{array}{cc} \quad x \quad\quad & \quad\quad y \end{array} \\ \begin{pmatrix} f''_{xx}(x;y) & f''_{xy}(x;y) \\ f''_{yx}(x;y) & f''_{yy}(x;y) \end{pmatrix} \end{array}.$$

In this exercise the second ordered partial derivatives are

$$f''_{xx}(x;y) = 36x \qquad\qquad f''_{xy}(x;y) = -18$$
$$f''_{yx}(x;y) = -18 \qquad\qquad f''_{yy}(x;y) = 36y,$$

thus the Hessian-matrix is as follows:

$$M(x;y) = \begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{pmatrix} 36x & -18 \\ -18 & 36y \end{pmatrix} \end{array}.$$

If we substitute the point P_1 to the Hessian-matrix, we get that

$$M(P_1) = \begin{pmatrix} 0 & -18 \\ -18 & 0 \end{pmatrix}.$$

Since $D_1 = 0$ and

$$D_2 = \det \begin{pmatrix} 0 & -18 \\ -18 & 0 \end{pmatrix} = 0 - 324 = -324.$$

therefore at point P_1 has no local extremum.

The Hessian-matrix at point P_2 is as follows:

$$M(P_2) = \begin{pmatrix} 36 & -18 \\ -18 & 36 \end{pmatrix}.$$

Since $D_1 = 36$ and

$$D_2 = \det \begin{pmatrix} 36 & -18 \\ -18 & 36 \end{pmatrix} = 36 \cdot 36 - (-18)^2 = 1\,096 - 324 = 772,$$

therefore D_1 and D_2 are positive, thus at P_2 has a local minimum. Its value is:

$$f(1;1) = 6 \cdot 1^3 + 6 \cdot 1^3 - 18 \cdot 1 \cdot 1 = -6.$$

Exercise 40. Find the local extremum of function

$$f(x;y) = x^3 - 75x + 2y^2 - 4y.$$

Solution:

The partial derivatives f are

$$f'_x(x;y) = 3x^2 - 75$$
$$f'_y(x;y) = 4y - 4.$$

The solutions of system of equation are

$$\left.\begin{aligned} 3x^2 - 75 &= 0 \\ 4y - 4 &= 0 \end{aligned}\right\}$$

$P_1 = (5;1)$ and $P_2 = (-5;1)$.

The second ordered partial derivatives of function f are

$$f''_{xx}(x;y) = 6x \qquad f''_{xy}(x;y) = 0$$
$$f''_{yx}(x;y) = 0 \qquad f''_{yy}(x;y) = 4.$$

The Hessian-matrix is

$$M(x;y) = \begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{pmatrix} 6x & 0 \\ 0 & 4 \end{pmatrix} \end{array}.$$

The Hessian-matrix at point P_1 is

$$M(P_1) = \begin{pmatrix} 30 & 0 \\ 0 & 4 \end{pmatrix}.$$

Since $D_1 = 30$ and

$$D_2 = \det \begin{pmatrix} 30 & 0 \\ 0 & 4 \end{pmatrix} = 120 - 0 = 120,$$

therefore at point P_1 has a local minimum of function f. The value of function f is

$$f(5;1) = 5^3 - 75 \cdot 5 + 2 \cdot 1^2 - 4 \cdot 1 = -252.$$

The value of matrix at point P_2 is

$$M(P_2) = \begin{pmatrix} -30 & 0 \\ 0 & 4 \end{pmatrix}.$$

Since $D_1 = -30$ and

$$D_2 = \det \begin{pmatrix} -30 & 0 \\ 0 & 4 \end{pmatrix} = -120 - 0 = -120,$$

therefore $D_2 < 0$ thus at point P_2 there is no local extremum.

Exercise 41. Let $f(x;y) = \mathrm{e}^{4xy}$ be a two variabled function. Calculate the local extremum of the function.

Solution:

The partial derivatives of function f are

$$f'_x(x;y) = \mathrm{e}^{4xy} \cdot 4y$$

$$f'_y(x;y) = \mathrm{e}^{4xy} \cdot 4x.$$

The solution of system of equations

$$\left.\begin{aligned} e^{4xy}\cdot 4y = 0 \\ e^{4xy}\cdot 4x = 0 \end{aligned}\right\}$$

is $P = (0; 0)$.

The second ordered partial derivatives f are

$$\begin{aligned} &f''_{xx}(x;y) = e^{4xy}\cdot 16y^2 && f''_{xy}(x;y) = e^{4xy}\cdot 16xy + e^{2xy}\cdot 4 \\ &f''_{yx}(x;y) = e^{2xy}\cdot 16xy + e^{2xy}\cdot 4 && f''_{yy}(x;y) = e^{2xy}\cdot 16x^2, \end{aligned}$$

thus the Hessian-matrix is

$$M(x;y) = \begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{pmatrix} e^{2xy}\cdot 16y^2 & e^{2xy}\cdot 16xy + e^{2xy}\cdot 4 \\ e^{2xy}\cdot 16xy + e^{2xy}\cdot 4 & e^{2xy}\cdot 16x^2 \end{pmatrix} \end{array}.$$

The Hessian-matrix at point P is

$$M(P) = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}.$$

Since $D_1 = 0$ and

$$D_2 = \det\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} = 0 - 16 = -16,$$

thus at point P there is no local extremum.

Exercise 42. Find three real numbers whose sum is 210 and the sum of whose squares is as small as possible.

Solution:

Let the numbers are x, y and z. The sum of three numbers is

$$x + y + z = 210.$$

If we express variable z from the previous equation, we get that

$$z = 210 - x - y,$$

thus we have to find the local minimum of the function

$$f(x;y) = x^2 + y^2 + (210 - x - y)^2.$$

The partial derivatives of function f are

$$f'_x(x;y) = 2x + 2\cdot(210 - x - y)\cdot(-1) = 4x + 2y - 420$$

$$f'_y(x;y) = 2y + 2\cdot(150 - x - y)\cdot(-1) = 2x + 4y - 420.$$

The solution of the system

$$\left.\begin{aligned} 4x + 2y - 420 &= 0 \\ 2x + 4y - 420 &= 0 \end{aligned}\right\}$$

is $P = (70; 70)$.

The second ordered partial derivatives are

$$\begin{aligned} f''_{xx}(x;y) &= 4 & \qquad f''_{xy}(x;y) &= 2 \\ f''_{yx}(x;y) &= 2 & \qquad f''_{yy}(x;y) &= 4, \end{aligned}$$

thus the Hessian-matrix is

$$M(x;y) = \begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \end{array}.$$

The Hessian-matrix at point is the same, that is

$$M(P) = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}.$$

Since $D_1 = 4$ and

$$D_2 = \det\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} = 16 - 4 = 12,$$

therefore D_1 and D_2 are positive real numbers, thus there is a minimum at point P. The value of z is

$$z = 210 - 70 - 70 = 70.$$

The sum of squares is

$$70^2 + 70^2 + 70^2 = 4\,900\cdot 3 = 14\,700.$$

Exercise 43. A small company produces speakers and subwoofers for computers that they sell through a website. After extensive research, the company has developed a revenue function,

$$R(x;y) = 120x - 4.5x^2 + 160y - 2y^2$$

thousand dollars, where x is the number of subwoofers produced and sold in thousands and y is the number of speakers produced and sold in thousands. The corresponding cost function is

$$C(x;y) = 3x^2 + 3y^2 + 5xy - 5y + 50$$

thousand dollars. Find the production levels that maximize the profit.

Solution:

By subtracting the cost from the revenue, we get the profit function:

$$\begin{aligned}\Pi(x;y) &= R(x;y) - C(x;y) = \\ &= 120x - 4.5x^2 + 160y - 2y^2 - (3x^2 + 3y^2 + 5xy - 5y + 50) = \\ &= 120x - 7.5x^2 + 165y - 5y^2 - 5xy - 50.\end{aligned}$$

The first ordered partial derivatives are

$$f'_x(x;y) = 120 - 15x - 5y$$

$$f'_y(x;y) = 165 - 10y - 5x.$$

We have to solve the system

$$\left.\begin{aligned}120 - 15x - 5y = 0 \\ 165 - 10y - 5x = 0\end{aligned}\right\}.$$

If we multiply the second equation by -3, we get that

$$\left.\begin{aligned}120 - 15x - 5y = 0 \\ -495 + 30y + 15x = 0\end{aligned}\right\}.$$

If we add the second equation to the first equation, we get that

$$375 = 25y \qquad \Rightarrow \qquad y = 15,$$

thus $x = 3$. The critical point is $P = (3;15)$.

The second ordered partial derivatives are

$$f''_{xx}(x;y) = -15 \qquad\qquad f''_{xy}(x;y) = -5$$

$$f''_{yx}(x;y) = -5 \qquad\qquad f''_{yy}(x;y) = -10,$$

thus the Hessian-matrix is

$$M(x;y) = \begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{pmatrix} -15 & -5 \\ -5 & -10 \end{pmatrix} \end{array}.$$

The Hessian-matrix at point is the same, that is

$$M(P) = \begin{pmatrix} -15 & -5 \\ -5 & -10 \end{pmatrix}.$$

Since $D_1 = -15$ and

$$D_2 = \det \begin{pmatrix} -15 & -5 \\ -5 & -10 \end{pmatrix} = 150 - 25 = 125,$$

therefore D_1 is negative and D_2 is positive real numbers, thus there is a maximum at point P. The maximum value of profit function Π is

$$\Pi(3;15) = 120 \cdot 3 - 7.5 \cdot 3^2 + 165 \cdot 15 - 5 \cdot 15^2 - 5 \cdot 3 \cdot 15 - 50 = 1\,367.5.$$

At a production level of 3 thousand subwoofers and 15 thousand speakers, the company will win 1 367.5 thousand dollars.

Exercise 44. A medium-sized electronics company manufactures headphones and earbuds that they sell through their online store. Based on market research, the company has determined that their revenue function is:

$$R(x, y) = 150x - 5x^2 + 180y - 3y^2$$

thousand dollars, where x is the number of headphones produced and sold in thousands and y is the number of earbuds produced and sold in thousands. The corresponding cost function is:

$$C(x, y) = 4x^2 + 4y^2 + 6xy - 8y + 60$$

thousand dollars. Find the production levels that maximize the profit.

Solution:

By subtracting the cost from the revenue, we get the profit function

$$\begin{aligned} \Pi(x, y) &= R(x, y) - C(x, y) = \\ &= 150x - 5x^2 + 180y - 3y^2 - (4x^2 + 4y^2 + 6xy - 8y + 60) = \\ &= 150x - 9x^2 + 188y - 7y^2 - 6xy - 60. \end{aligned}$$

The first ordered partial derivatives are:

$$\Pi'_x(x;y) = 150 - 18x - 6y$$

$$\Pi'_y(x;y) = 188 - 14y - 6x.$$

We have to solve the system

$$\left.\begin{aligned} 150 - 18x - 6y &= 0 \\ 188 - 14y - 6x &= 0 \end{aligned}\right\}$$

If we divide the first equation by -3, we get

$$\left.\begin{aligned} -50 + 6x + 2y &= 0 \\ 188 - 14y - 6x &= 0 \end{aligned}\right\}$$

If we add these two equations, we get that

$$138 - 12y = 0 \quad \Rightarrow \quad y = \frac{138}{12} = 11.5,$$

and substituting this back into the first equation, we have

$$150 - 18x - 6 \cdot 11.5 = 0 \quad \Rightarrow \quad 150 - 18x - 69 = 0 \quad \Rightarrow \quad x = 4.5$$

The critical point is $P = (4.5, 11.5)$.

The second ordered partial derivatives are:

$$\begin{aligned} \Pi''_{xx}(x;y) &= -18 & \qquad \Pi''_{xy}(x;y) &= -6 \\ \Pi''_{yx}(x;y) &= -6 & \qquad \Pi''_{yy}(x;y) &= -14, \end{aligned}$$

thus the Hessian matrix is:

$$H(x,y) = \begin{pmatrix} -18 & -6 \\ -6 & -14 \end{pmatrix}.$$

The Hessian matrix at point P is the same, that is:

$$H(P) = \begin{pmatrix} -18 & -6 \\ -6 & -14 \end{pmatrix}$$

Since $D_1 = -18$ (negative) and

$$D_2 = \det \begin{pmatrix} -18 & -6 \\ -6 & -14 \end{pmatrix} = (-18) \cdot (-14) - (-6) \cdot (-6) = 216,$$

which is positive, we have $D_1 < 0$ and $D_2 > 0$. Therefore, there is a maximum at point P. The maximum value of the profit function Π is:

$$\Pi(4.5, 11.5) = 1358.5$$

At a production level of 4.5 thousand headphones and 11.5 thousand earbuds, the company will earn a maximum profit of $1,358.5$ thousand dollars.

Exercise 45. Find the point on the plane given by $x + y - z = 1$ that is closest to the point $P = (0; -3; 2)$ and calculate their distance.

Solution:

Let the arbitrary point of the plane $x+y-z = 1$ is $Q = (x; y; z)$. The distance of point P and Q is

$$d_{PQ} = \sqrt{x^2 + (y+3)^2 + (z-2)^2}.$$

Since the point Q lies on the plane, therefore

$$x + y - z = 1 \qquad \Rightarrow \qquad z = x + y - 1,$$

thus

$$d_{PQ} = \sqrt{x^2 + (y+3)^2 + (z-2)^2} = \sqrt{x^2 + (y+3)^2 + (x+y-3)^2}.$$

We will minimize the distance squared:

$$f(x; y) = x^2 + (y+3)^2 + (x+y-3)^2$$

The partial derivatives of function f are

$$\begin{aligned} f'_x(x; y) &= 2x + 2 \cdot (x+y-3) = 4x + 2y - 6 \\ f'_y(x; y) &= 2 \cdot (y+3) + 2 \cdot (x+y-3) = 2x + 4y. \end{aligned}$$

We have to solve the system

$$\left. \begin{aligned} 4x + 2y - 6 &= 0 \\ 2x + 4y \quad &= 0 \end{aligned} \right\} .$$

The solution of the system is $S = (2; -1)$.

The second ordered partial derivatives are

$$\begin{aligned} f''_{xx}(x; y) &= 4 & \qquad f''_{xy}(x; y) &= 2 \\ f''_{yx}(x; y) &= 2 & \qquad f''_{yy}(x; y) &= 4, \end{aligned}$$

hence the Hessian-matrix is

$$M(x; y) = \begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \end{array}.$$

The Hessian-matrix at point S is

$$M(S) = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}.$$

Since $D_1 = 4$ and

$$D_2 = \det \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} = 16 - 4 = 12,$$

we get that there is a local minimum at point S. We then calculate the distance from $Q = (2; -1; 0)$ to P is $\sqrt{12}$.

Exercise 46. Find the point on the plane given by $2x + 3y + z = 5$ that is closest to the point $P = (1; 0; -2)$ and calculate their distance.

Solution:

Let the arbitrary point of the plane $2x + 3y + z = 5$ be $Q = (x; y; z)$. The distance of point P and Q is

$$d_{PQ} = \sqrt{(x-1)^2 + y^2 + (z+2)^2}.$$

Since the point Q lies on the plane, therefore

$$2x + 3y + z = 5 \qquad \Rightarrow \qquad z = 5 - 2x - 3y,$$

thus

$$d_{PQ} = \sqrt{(x-1)^2 + y^2 + (5 - 2x - 3y + 2)^2}.$$

It is enough if we will minimize the distance squared. Let

$$f(x; y) = (x-1)^2 + y^2 + (7 - 2x - 3y)^2.$$

The partial derivatives of function f are

$$f'_x(x; y) = 2 \cdot (x-1) + 2 \cdot (7 - 2x - 3y) \cdot (-2) = 10x + 12y - 30$$
$$f'_y(x; y) = 2y + 2 \cdot (7 - 2x - 3y) \cdot (-3) = 12x + 20y - 42.$$

We have to solve the system

$$\left.\begin{aligned} 10x + 12y - 30 &= 0 \\ 12x + 20y - 42 &= 0 \end{aligned}\right\}$$

If we multiply the first equation by 6 and the second equation by 5, we get

$$\left.\begin{aligned} 60x + 72y - 180 &= 0 \\ 60x + 100y - 210 &= 0 \end{aligned}\right\}.$$

If we subtract the first equation from the second equation, we get

$$28y - 30 = 0 \quad \Rightarrow \quad y = \frac{30}{28} = \frac{15}{14},$$

and substituting back into the first equation:

$$10x + 12 \cdot \frac{15}{14} - 30 = 0 \quad \Rightarrow \quad 10x = 30 - \frac{180}{14} = \frac{240}{14},$$

thus, $x = \frac{240}{140} = \frac{12}{7}$.
The second ordered partial derivatives are

$$f''_{xx}(x;y) = 10 \qquad f''_{xy}(x;y) = 12$$
$$f''_{yx}(x;y) = 12 \qquad f''_{yy}(x;y) = 20,$$

hence the Hessian-matrix is

$$M(x;y) = \begin{pmatrix} 10 & 12 \\ 12 & 20 \end{pmatrix}.$$

The Hessian-matrix at point Q is the same:

$$M(Q) = \begin{pmatrix} 10 & 12 \\ 12 & 20 \end{pmatrix}.$$

Since $D_1 = 10 > 0$ and

$$D_2 = \det \begin{pmatrix} 10 & 12 \\ 12 & 20 \end{pmatrix} = 10 \cdot 20 - 12 \cdot 12 = 200 - 144 = 56 > 0,$$

we get that there is a local minimum at point Q.
The value of z is

$$z = 5 - 2x - 3y = 5 - 2 \cdot \frac{12}{7} - 3 \cdot \frac{15}{14} = -\frac{23}{14}.$$

The point, which minimize the distance is

$$Q = \left(\frac{12}{7}; \frac{15}{14}; -\frac{23}{14} \right).$$

The distance from $Q = \left(\frac{12}{7}; \frac{15}{14}; -\frac{23}{14}\right)$ to $P = (1;0;-2)$ is

$$d_{PQ} = \sqrt{\left(\frac{12}{7} - 1\right)^2 + \left(\frac{15}{14}\right)^2 + \left(-\frac{23}{14} + 2\right)^2} = \frac{\sqrt{175}}{7 \cdot \sqrt{2}}.$$

Therefore, the distance between the points is $\frac{\sqrt{175}}{7 \cdot \sqrt{2}}$.

5. Optimization of three or more variabled function

Theoretical summary

Definition 1.5.1. The function $f\colon D \subset \mathbb{R}^n \to \mathbb{R}$ has *local maximum* (*relative maximum*) at point $P \in D$ if there is disc centered at point P such that

$$f(x_1; \dots; x_n) \leq f(P)$$

for all point $(x_1; \dots; x_n)$ that lie inside the disc.

The function $f\colon D \subset \mathbb{R}^n \to \mathbb{R}$ has *local minimum* (*relative minimum*) at point $P \in D$ if there is disc centered at point P such that

$$f(x_1; \dots; x_n) \geq f(P)$$

for all point $(x_1; \dots; x_n)$ that lie inside the disc.

Theorem 1.5.2. If $f\colon D \subset \mathbb{R}^n \to \mathbb{R}$ is a totally differentiable function and has a local extremum at point $P \in D$ then the values of partial derivative functions at P are zero.

Definition 1.5.3. If the function $f\colon D \subset \mathbb{R}^n \to \mathbb{R}$ is twice differentiable function, then its *Hessian-matrix* at point $P \in D$ is

$$M(P) = \begin{array}{c} \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & \dots & x_n \end{array} \\ \begin{pmatrix} f''_{x_1x_1}(P) & f''_{x_1x_2}(P) & \dots & f''_{x_1x_n}(P) \\ f''_{x_2x_1}(P) & f''_{x_2x_2}(P) & \dots & f''_{x_2x_n}(P) \\ \vdots & \vdots & \dots & \ddots \\ f''_{x_nx_1}(P) & f''_{x_nx_2}(P) & \dots & f''_{x_nx_n}(P) \end{pmatrix} \end{array}.$$

Theorem 1.5.4. Let $f\colon D \subset \mathbb{R}^n \to \mathbb{R}$ be twice differentiable function. Let the partial derivatives at point $P \in D$ are zero and the Hessian-matrix at point is

$$M(P) = \begin{array}{c} \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & \dots & x_n \end{array} \\ \begin{pmatrix} f''_{x_1x_1}(P) & f''_{x_1x_2}(P) & \dots & f''_{x_1x_n}(P) \\ f''_{x_2x_1}(P) & f''_{x_2x_2}(P) & \dots & f''_{x_2x_n}(P) \\ \vdots & \vdots & \dots & \ddots \\ f''_{x_nx_1}(P) & f''_{x_nx_2}(P) & \dots & f''_{x_nx_n}(P) \end{pmatrix} \end{array}.$$

Let

$$D_1 = f''_{x_1x_1}(P),$$

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$$D_2 = \det \begin{pmatrix} f''_{x_1x_1}(P) & f''_{x_1x_2}(P) \\ f''_{x_2x_1}(P) & f''_{x_2x_2}(P) \end{pmatrix},$$

and so on

$$D_i = \det \begin{pmatrix} f''_{x_1x_1}(P) & f''_{x_1x_2}(P) & \dots & f''_{x_1x_i}(P) \\ f''_{x_2x_1}(P) & f''_{x_2x_2}(P) & \dots & f''_{x_2x_i}(P) \\ \vdots & \vdots & \ddots & \vdots \\ f''_{x_ix_1}(P) & f''_{x_ix_2}(P) & \dots & f''_{x_ix_i}(P) \end{pmatrix}.$$

If $D_i > 0$ for all $i = 1, 2, \dots, n$ then the function f has a local minimum at point P.

If $(-1)^i \cdot D_i > 0$ for all $i = 1, 2, \dots, n$ then the function f has a local maximum at point P.

Remark 1.5.5. How we can find a local extremum?

1. We have to calculate the partial derivatives of function f.
2. We have to solve the system of equations

$$\left.\begin{array}{r} f'_{x_1}(x_1; x_2; \dots; x_n) = 0 \\ f'_{x_2}(x_1; x_2; \dots; x_n) = 0 \\ \dots \\ f'_{x_n}(x_1; x_2; \dots; x_n) = 0 \end{array}\right\}.$$

   The solutions of the system are the critical points.

3. We have to calculate the second ordered partial derivatives and we have to construct the Hessian-matrix:

$$M(x_1; x_2; \dots; x_n) = \begin{array}{cc} & \begin{array}{cccc} x_1 & x_2 & \dots & x_n \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} & \begin{pmatrix} f''_{x_1x_1} & f''_{x_1x_2} & \dots & f''_{x_1x_n} \\ f''_{x_2x_1} & f''_{x_2x_2} & \dots & f''_{x_2x_n} \\ \vdots & \vdots & \dots & \ddots \\ f''_{x_nx_1} & f''_{x_nx_2} & \dots & f''_{x_nx_n} \end{pmatrix} \end{array}.$$

4. We substitute the critical points to Hessian-matrix and we apply the theorem 1.5.4.

Solved exercises

Exercise 47. Find the local extremum of function
$$f(x;y;z) = x^2 - xy + y^2 + z^2 - 2z + 10.$$

Solution:

The partial derivatives of function f are
$$\begin{aligned} f'_x(x;y;z) &= 2x - y \\ f'_y(x;y;z) &= -x + 2y \\ f'_z(x;y;z) &= 2z - 2. \end{aligned}$$
We have to solve the system
$$\left.\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 0 \\ 2z - 2 &= 0 \end{aligned}\right\}.$$
The solution of the system is $P = (0;0;1)$.

The second ordered partial derivatives of f are
$$\begin{array}{lll} f''_{xx} = 2 & f''_{xy} = -1 & f''_{xz} = 0 \\ f''_{yx} = -1 & f''_{yy} = 2 & f''_{yz} = 0 \\ f''_{zx} = 0 & f''_{zy} = 0 & f''_{zz} = 2, \end{array}$$
thus the Hessian-matrix is
$$M(x;y;z) = \begin{array}{c} \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{array}.$$
If we substitute the point P to the Hessian-matrix, we get that
$$M(P) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$
Since $D_1 = 2$ and
$$D_2 = \det \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 3.$$

and

$$D_3 = \det \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 6,$$

we get that $D_i > 0$ for all $i = 1, 2, 3$ are positive real numbers, thus there is a local minimum. The value of function at point P is

$$f(0; 0; 1) = 0^2 - 0 \cdot 0 + 0^2 + 1^2 - 2 \cdot 1 + 10 = 9.$$

Exercise 48. Find the local extremum of function

$$f(x; y; z) = x^3 - 9xy + y^3 + z^2 - 2z + 10.$$

Solution:

The partial derivatives of function f are 0

$$\begin{aligned} f'_x(x; y; z) &= 3x^2 - 9y \\ f'_y(x; y; z) &= 3y^2 - 9x \\ f'_z(x; y; z) &= 2z - 2. \end{aligned}$$

We have to solve the system

$$\left.\begin{aligned} 3x^2 - 9y = 0 \\ 3y^2 - 9x = 0 \\ 2z - 2 = 0 \end{aligned}\right\} .$$

The solutions of the system are $P_1 = (0; 0; 1)$ and $P_2 = (3; 3; 1)$.

The second ordered partial derivatives of f are

$$\begin{array}{lll} f''_{xx} = 6x & f''_{xy} = -9 & f''_{xz} = 0 \\ f''_{yx} = -9 & f''_{yy} = 6y & f''_{yz} = 0 \\ f''_{zx} = 0 & f''_{zy} = 0 & f''_{zz} = 2, \end{array}$$

thus the Hessian-matrix is

$$M(x; y; z) = \begin{array}{c} \\ x \\ y \\ z \end{array} \!\! \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 6x & -9 & 0 \\ -9 & 6y & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{array}.$$

If we substitute the coordinates point P_1 to the Hessian-matrix, we get that

$$M(P_1) = \begin{pmatrix} 0 & -9 & 0 \\ -9 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Since $D_1 = 0$ and

$$D_2 = \det \begin{pmatrix} 0 & -9 \\ -9 & 0 \end{pmatrix} = -81,$$

thus at P_1 there is no extremum point.

If we substitute the coordinates point P_2 to the Hessian-matrix, we get that

$$M(P_2) = \begin{pmatrix} 18 & -9 & 0 \\ -9 & 18 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Since $D_1 = 18$ and

$$D_2 = \det \begin{pmatrix} 18 & -9 \\ -9 & 18 \end{pmatrix} = 243,$$

and

$$D_3 = \det \begin{pmatrix} 18 & -9 & 0 \\ -9 & 18 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 486,$$

thus at P_2 there is a local minimum.

The value of function at point P_2 is

$$f(3;3;1) = 3^3 - 9 \cdot 3 \cdot 3 + 3^3 + 1^2 - 2 \cdot 1 + 10 = -18.$$

Exercise 49. Find the local extremum of function

$$f(x;y;z) = x^3 + y^3 - 3xy + z^2 - 4z + 2.$$

Solution:

The partial derivates of function f are

$$f'_x(x;y;z) = 3x^2 - 3y$$
$$f'_y(x;y;z) = 3y^2 - 3x$$
$$f'_z(x;y;z) = 2z - 4.$$

We have to solve the equation

$$\left.\begin{aligned} 3x^2 - 3y &= 0 \\ 3y^2 - 3x &= 0 \\ 2z - 4 &= 0 \end{aligned}\right\}.$$

The solutions of the system are $P_1 = (0;0;2)$ and $P_2 = (1;1;2)$.
Since the second ordered partial derivatives of function f are

$$\begin{array}{lll} f''_{xx} = 6x & f''_{xy} = -3 & f''_{xz} = 0 \\ f''_{yx} = -3 & f''_{yy} = 6y & f''_{yz} = 0 \\ f''_{zx} = 0 & f''_{zy} = 0 & f''_{zz} = 2, \end{array}$$

therefore the Hessian-matrix is

$$M(x;y;z) = \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{pmatrix} 6x & -3 & 0 \\ -3 & 6y & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{array}.$$

The Hessian matrix at point P_1 is

$$M(P_1) = \begin{pmatrix} 0 & -3 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Since $D_1 = 0$ and

$$D_2 = \det\begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix} = 0 - 9 = -9,$$

then we get that at point P_1 there is no local extremum.
The Hessian-matrix at point P_2 is

$$M(P_2) = \begin{pmatrix} 6 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Since $D_1 = 6$ and

$$D_2 = \det\begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix} = 36 - 9 = 27$$

and

$$D_3 = \det \begin{pmatrix} 6 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2 \cdot (-1)^6 \cdot (36 - 9) = 54,$$

therefore the point P_2 is a local minimum. The value of function at point P_2 is

$$f(1; 1; 2) = 1^3 + 1^3 - 3 \cdot 1 \cdot 1 + 2^2 - 4 \cdot 2 + 2 = -3.$$

Exercise 50. In a manufacturing facility, Machines X, Y and Z are operated for x, y and z hours, respectively. If the daily production g is a function of x, y and z, namely

$$g(x; y; z) = 19x + 22y - 2x^2 - 3y^2 - xy - 2z^2 + 4z,$$

find the values of x, y and z that maximize g.

Solution:

The partial derivatives of function g are

$$\begin{aligned} g'_x(x; y; z) &= 19 - 4x - y \\ g'_y(x; y; z) &= 22 - 6y - x \\ g'_z(x; y; z) &= -4z + 4. \end{aligned}$$

We have to solve the system

$$\left.\begin{aligned} 19 - 4x - y = 0 \\ 22 - 6y - x = 0 \\ -4z + 4 = 0 \end{aligned}\right\}.$$

From the third equation, we immediately get $z = 1$. In this case, from the first equation, we get $y = 19 - 4x$. Substituting this into the second equation, we get that

$$\begin{aligned} 22 - 6 \cdot (19 - 4x) - x &= 0 \\ 22 - 114 + 24x - x &= 0 \\ 22 - 114 + 23x &= 0 \\ 23x &= 92 \\ x &= 4 \end{aligned}$$

Substituting back to find y, we have

$$y = 19 - 4x = 19 - 4 \cdot 4 = 3.$$

The critical point is $P = (4; 3; 1)$.

The second ordered partial derivatives are

$$\begin{array}{lll} g''_{xx}(x;y;z) = -4 & g''_{xy}(x;y;z) = -1 & g''_{xz}(x;y;z) = 0 \\ g''_{yx}(x;y;z) = -1 & g''_{yy}(x;y;z) = -6 & g''_{yz}(x;y;z) = 0 \\ g''_{zx}(x;y;z) = 0 & g''_{zy}(x;y;z) = 0 & g''_{zz}(x;y;z) = -4, \end{array}$$

hence the Hessian-matrix is

$$M(x;y;z) = \begin{pmatrix} -4 & -1 & 0 \\ -1 & -6 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

The Hessian-matrix at point P is the same:

$$M(P) = \begin{pmatrix} -4 & -1 & 0 \\ -1 & -6 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

Since $D_1 = -4 < 0$ and

$$D_2 = \det \begin{pmatrix} -4 & -1 \\ -1 & -6 \end{pmatrix} = (-4) \cdot (-6) - (-1) \cdot (-1) = 24 - 1 = 23 > 0,$$

and

$$D_3 = \det \begin{pmatrix} -4 & -1 & 0 \\ -1 & -6 & 0 \\ 0 & 0 & -4 \end{pmatrix} = (-4) \cdot 23 = -92 < 0,$$

thus the function g has a maximum at point P.

The value of function at point P is

$$\begin{aligned} g(4;3;1) &= 19 \cdot 4 + 22 \cdot 3 - 2 \cdot 4^2 - 3 \cdot 3^2 - 4 \cdot 3 - 2 \cdot 1^2 + 4 \cdot 1 \\ &= 76 + 66 - 32 - 27 - 12 - 2 + 4 = 142 - 73 = 69. \end{aligned}$$

Therefore, to maximize the daily production, Machine X should be operated for 4 hours, Machine Y for 3 hours, and Machine Z for 1 hour, resulting in a maximum production value of 69 units.

6. Review exercises - with solution

Exercise 51. The revenue per month earned by the Couture clothing chain at time t is $R(t) = N(t) \cdot S(t)$, where $N(t)$ is the number of stores and $S(t)$ is average revenue per store per month. Couture embarks on a two-part campaign: (A) to build new stores at a rate of 5 stores per month, and (B) to use advertising to increase average revenue per store at a rate of $10 000 per month. Assume that $N(0) = 50$ and $S(0) = \$150\,000$.

a) Show that total revenue will increase at the rate
$$R'(t) = 5 \cdot S(t) + 10\,000 \cdot N(t).$$
Note that the two terms in the Product Rule correspond to the separate effects of increasing the number of stores on the one hand, and the average revenue per store on the other.

b) Calculate the value $R'(0)$.

c) If Couture can implement only one leg (A or B) of its expansion at $t = 0$, which choice will grow revenue most rapidly?

Solution:

a) Given $R(t) = N(t) \cdot S(t)$, it follows that
$$R'(t) = N'(t) \cdot S(t) + N(t) \cdot S'(t).$$
We are told that $N'(t) = 5$ stores per month and $S'(t) = 10\,000$ dollars per month. Therefore,
$$R'(t) = 5 \cdot S(t) + 10\,000 \cdot N(t).$$

b) Using the previous part and the given values of $N(0)$ and $S(0)$, we can find
$$R'(0) = 5 \cdot S(0) + 10\,000 \cdot N(0) = 5 \cdot 150\,000 + 10\,000 \cdot 50 = 1\,250\,000.$$

c) From the previous part, we see that of the two terms contributing to total revenue growth, the term $5S(0)$ is larger than the term $10\,000N(0)$. Thus, if only one leg of the campaign can be implemented, it should be part A: increase the number of stores by 5 per month.

Exercise 52. A company can produce LCD digital alarm clocks at a cost of $6 each while fixed costs are $16. Therefore, the company's cost function is $C(x) = 6x + 16$.

a) Find the average cost function.

b) Find the marginal average cost function.

c) Evaluate the marginal cost at $x = 20$. Round the answer to the nearest cent.

Solution:

a) The average cost function is

$$AC(x) = \frac{C(x)}{x} = 6 + \frac{16}{x}.$$

b) The marginal average cost function is a differentiate function of the average cost function, that is

$$MAC(x) = -\frac{16}{x^2}.$$

c) The value of marcginal cost function at $x = 20$ is

$$MAC(20) = -\frac{16}{400} = -0.04,$$

that is -4 cents.

Exercise 53. A production facility is capable of producing 60 000 widgets in a day and the total daily cost of producing x widgets in a day is given by

$$C(x) = 250\,000 + 0.08x + \frac{200\,000\,000}{x}.$$

How many widgets per day should they produce in order to minimize production costs?

Solution:

Here we need to minimize the cost subject to the constraint that x must be in the range $0 \leq x \leq 60\,000$. The first derivative of the function $C(x)$ is

$$C'(x) = 0.08 - \frac{200\,000\,000}{x^2}.$$

Zeros of the function $C'(x)$ are the solutions of $C'(x) = 0$ equation

$$0.08 - \frac{200\,000\,000}{x^2} = 0$$
$$0.08x^2 = 200\,000\,000.$$

Since $x > 0$, we get that

$$x = \pm 50\,000.$$

Clearly the negative value does not make any sense in this setting and so we have a single critical point in the range of possible solutions $50\,000$. The second derivatives of the function C is

$$C''(x) = \frac{400\,000\,000}{x^3}.$$

As long as $x > 0$, the second derivative is

$$C''(50\,000) = \frac{400\,000\,000}{50\,000^3}$$

positive and thus, in the range of possible solutions the function is always concave up and so producing $50\,000$ widgets will yield the absolute minimum production cost.

Exercise 54. Let $f(x;y) = \frac{1}{x} + \frac{1}{y} + xy$ $(x > 0, y > 0)$ be a two variabled function. Calculate the local extremum of the function.

Solution:

The partial derivatives of function are

$$f'_x(x;y) = -\frac{1}{x^2} + y$$

$$f'_y(x;y) = -\frac{1}{y^2} + x.$$

We have to solve the system of equation

$$\left.\begin{aligned} -\frac{1}{x^2} + y = 0 \\ -\frac{1}{y^2} + x = 0 \end{aligned}\right\}.$$

From the first equation we get that

$$-\frac{1}{x^2} + y = 0 \qquad \Rightarrow \qquad y = \frac{1}{x^2}.$$

We substitute to the second equation, we get that

$$-\frac{1}{\left(\frac{1}{x^2}\right)^2} + x = 0,$$

thus

$$-x^4 + x = 0 \quad \Rightarrow \quad x \cdot (-x^3 + 1) = 0.$$

Since $x \neq 0$, the solution of the system is $P = (1;1)$.

The second ordered partial derivatives of function f are

$$f''_{xx}(x;y) = \frac{2}{x^3} \qquad f''_{xy}(x;y) = 1$$

$$f''_{yx}(x;y) = 1 \qquad f''_{yy}(x;y) = \frac{2}{y^3},$$

thus the Hessian-matrix is

$$M(x;y) = \begin{array}{cc} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \end{array}.$$

The Hessian-matrix at point P is

$$M(P) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Since $D_1 = 2$ and

$$D_2 = \det \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 4 - 1 = 3,$$

thus at point P has a local minimum. The value of function at point P is

$$f(1;1) = 3.$$

Exercise 55. Find the local extremum of function

$$f(x;y;z) = 2x^2 + 2y^2 + 2z^2 - 4x - 4y - 4z + 2.$$

Solution:

The partial derivatives of function f are

$$\begin{aligned} f'_x(x;y;z) &= 4x - 4 \\ f'_y(x;y;z) &= 4y - 4 \\ f'_z(x;y;z) &= 4z - 4. \end{aligned}$$

We have to solve the system

$$\left.\begin{aligned} 4x - 4 &= 0 \\ 4y - 4 &= 0 \\ 4z - 4 &= 0 \end{aligned}\right\}.$$

The solution of the system is $P = (1;1;1)$.

The second ordered partial derivatives of f are

$$\begin{array}{lll} f''_{xx} = 4 & f''_{xy} = 0 & f''_{xz} = 0 \\ f''_{yx} = 0 & f''_{yy} = 4 & f''_{yz} = 0 \\ f''_{zx} = 0 & f''_{zy} = 0 & f''_{zz} = 4, \end{array}$$

thus the Hessian-matrix is

$$M(x;y;z) = \begin{array}{c} \\ x \\ y \\ z \end{array} \!\!\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \end{array}.$$

If we substitute the point P to the Hessian-matrix, we get that

$$M(P) = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Since $D_1 = 4$ and

$$D_2 = \det \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = 16.$$

and

$$D_3 = \det \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 64,$$

we get that $D_i > 0$ for all $i = 1, 2, 3$ are positive real numbers, thus there is a local minimum. The value of function at point P is

$$f(1;1;1) = 2 \cdot 1^2 - 4 \cdot 1 + 2 \cdot 1^2 - 4 \cdot 1 + 2 \cdot 1^2 - 4 \cdot 1 + 2 = -4.$$

7. Review exercises - without solution

Exercise 56. Find the minimum value of the function

$$f(x) = x^2 - 4x + 7.$$

Exercise 57. Determine the maximum value of the function

$$g(x) = -x^2 + 6x - 5$$

on the interval $[0; 5]$.

Exercise 58. A farmer wants to fence off a rectangular field bordering a straight river. He has 1 000 feet of fencing and needs no fence along the river. What are the dimensions of the field that maximize the area?

Exercise 59. A rectangular box with a square base and no top is to be constructed from 1 200 cm^2 of material. What dimensions will maximize the volume of the box?

Exercise 60. Find the point on the line $y = 2x + 3$ that is closest to the point $(4; 2)$.

Exercise 61. A window is being built in the shape of a rectangle capped by a semicircle. If the perimeter of the window is 20 feet, find the dimensions that maximize the area of the window.

Exercise 62. A company manufactures and sells x items per week. The weekly cost function is

$$C(x) = 200 + 16x$$

and the revenue function is

$$R(x) = 50x - 0.1x^2.$$

Find the production level that maximizes profit.

Exercise 63. A cylindrical can without a top is made to contain a volume of 1000 cm^3. Find the dimensions that minimize the surface area of the can.

Exercise 64. A poster is to have an area of 180 in^2 with 1-inch margins at the bottom and sides and a 2-inch margin at the top. What dimensions will give the largest printed area?

Exercise 65. Find the absolute maximum and minimum values of the function

$$h(x) = x^3 - 6x^2 + 5$$

on the interval $[-1; 5]$.

Exercise 66. Determine the dimensions of a rectangle with perimeter 36 meters that maximize its area.

Exercise 67. A wire of length 10 meters is cut into two pieces. One piece is bent into a square and the other is bent into a circle. How should the wire be cut to minimize the sum of the areas of the square and the circle?

Exercise 68. A particle moves along the x-axis so that its position at time t is given by

$$s(t) = t^3 - 6t^2 + 9t + 1.$$

Find the maximum and minimum velocities of the particle on the interval $[0; 4]$.

Exercise 69. A lifeguard needs to reach a swimmer who is 30 meters down the beach and 10 meters out in the water. The lifeguard can run 3 m/s on the sand and swim 1 $\frac{\mathrm{m}}{\mathrm{s}}$ in the water. To what point on the beach should the lifeguard run to minimize the time it takes to reach the swimmer?

Exercise 70. Find two numbers whose sum is 10 and whose product is as large as possible.

Chapter 2

Integral calculus and its applications

8. Primitive functions I.

Theoretical summary

Remark 2.1.1. In this section I is an interval.

Definition 2.1.2. A *primitive function* (also called an antiderivative) of a function $f(x)$ is a function $F(x)$ whose derivative equals $f(x)$.
That is, if $F'(x) = f(x)$, then $F(x)$ is a primitive function of $f(x)$.

Example 2.1.3. If $f(x) = 3x^2$ then one of the primitive function is $F(x) = x^3$.

Theorem 2.1.4. If F is a primitive function of f, then $F + c$, where c is any constant, is also a primitive function of f.

Definition 2.1.5. The set of all primitive functions of $f(x)$ is denoted by:

$$\int f(x)\,dx = F(x) + C \tag{1}$$

where C is an arbitrary constant.

Theorem 2.1.6. The integral is additive, that is the integral of a sum of functions equals the sum of the integrals of each function, that is

$$\int \big(f(x) + g(x)\big)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x.$$

Example 2.1.7.

$$\int 4x^3 + 3x^2\,\mathrm{d}x = \int 4x^3\,\mathrm{d}x + \int 3x^2\,\mathrm{d}x = x^4 + x^3 + c.$$

Theorem 2.1.8. The integral is homogenous, that is

$$\int \lambda \cdot f(x)\,\mathrm{d}x = \lambda \cdot \int f(x)\,\mathrm{d}x,$$

for all $\lambda \in \mathbb{R}$.

Example 2.1.9.

$$\int 3x\,\mathrm{d}x = 3 \cdot \int x\,\mathrm{d}x = 3 \cdot \frac{x^2}{2} + c,$$

for all $c \in \mathbb{R}$.

$f(x)$	D_f	$\int f(x)\,\mathrm{d}x$
α	$\mathbb{R}$	$\alpha \cdot x + c$
1	$\mathbb{R}$	$x + c$
x^r	$]0;\infty[$	$\frac{x^{r+1}}{r+1} + c$
$\sin x$	$\mathbb{R}$	$-\cos x + c$
$\cos x$	$\mathbb{R}$	$\sin x + c$
$\frac{1}{\cos^2 x}$	$\mathbb{R} \setminus \{\frac{\pi}{2} + k \cdot \pi \mid k \in \mathbb{Z}\}$	$\tan x + c$
$-\frac{1}{\sin^2 x}$	$\mathbb{R} \setminus \{\pi + k \cdot \pi \mid k \in \mathbb{Z}\}$	$\cot x + c$
e^x	$\mathbb{R}$	$\mathrm{e}^x + c$
a^x	$\mathbb{R}$	$\frac{a^x}{\ln a} + c$
$\frac{1}{x}$	$]0;\infty[$	$\ln \lvert x \rvert + c$
$\frac{1}{x \cdot \ln a}$	$]0;\infty[$	$\log_a x + c$
$\frac{1}{\sqrt{1-x^2}}$	$]-1;1[$	$\arcsin x + c$
$-\frac{1}{\sqrt{1-x^2}}$	$]-1;1[$	$\arccos x + c$
$\frac{1}{1+x^2}$	$\mathbb{R}$	$\arctan x + c$
$\sinh x$	$\mathbb{R}$	$\cosh x + c$

$\cosh x$	$\mathbb{R}$	$\sinh x + c$
$\frac{1}{\cosh^2 x}$	$\mathbb{R}$	$\tanh x + c$
$-\frac{1}{\sinh^2 x}$	$\mathbb{R} \setminus \{0\}$	$\coth x + c$
$\frac{1}{\sqrt{1+x^2}}$	$\mathbb{R}$	$\operatorname{arsinh} x + c$
$\frac{1}{\sqrt{x^2-1}}$	$[1; \infty[$	$\operatorname{arcosh} x + c$
$\frac{1}{1-x^2}$	$]-1; 1[$	$\operatorname{artanh} x + c$
$-\frac{1}{1-x^2}$	$]-\infty; -1[\cup]1; \infty[$	$\operatorname{arcoth} x$

Theorem 2.1.10. If function F is a primitive function of function F, then

$$\int f(ax+b)\,\mathrm{d}x = \frac{1}{a} \cdot F(ax+b) + c.$$

Example 2.1.11.

$$\int \cos(3x+4)\,\mathrm{d}x = \frac{\sin(3x+4)}{3} + c.$$

Theorem 2.1.12. If $f\colon I \to \mathbb{R}$ is a differentiable function and exists the primitive function of $\frac{f'(x)}{f(x)}$ then

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln\left|f(x)\right| + c.$$

Example 2.1.13.

$$\int \frac{2x}{x^2+6}\,\mathrm{d}x = \ln(x^2+6) + c$$

Theorem 2.1.14. Let $n \in \mathbb{R} \setminus \{-1\}$ and $f\colon I \to \mathbb{R}$ there exists the primitve function of $f^n(x) \cdot f'(x)$. Then

$$\int f^n(x) \cdot f'(x)\,\mathrm{d}x = \frac{f^{n+1}}{n+1} + c.$$

Example 2.1.15.

$$\int \sin^2 x \cdot \cos x \, \mathrm{d}x = \frac{\sin^3 x}{3} + c$$

Solved exercises

Exercise 71. Solve the integrals below:

a) $\displaystyle\int 2\,\mathrm{d}x$

b) $\displaystyle\int 4x+2\,\mathrm{d}x$

c) $\displaystyle\int 3x^2+6x-8\,\mathrm{d}x$

d) $\displaystyle\int \frac{1}{x^2}\,\mathrm{d}x$

e) $\displaystyle\int \frac{2}{x^3}\,\mathrm{d}x$

f) $\displaystyle\int \sqrt{x}\,\mathrm{d}x$

g) $\displaystyle\int \sqrt[3]{x}\,\mathrm{d}x$

h) $\displaystyle\int \frac{x+1}{x}\,\mathrm{d}x$

i) $\displaystyle\int \frac{x^2+x}{x}\,\mathrm{d}x$

j) $\displaystyle\int \frac{x^2+x}{\sqrt{x}}\,\mathrm{d}x$

k) $\displaystyle\int \frac{\sqrt[3]{x}+\sqrt{x}}{x^2}\,\mathrm{d}x$

l) $\displaystyle\int 3^x+4^x\,\mathrm{d}x$

m) $\displaystyle\int \frac{x^2}{x^2+1}\,\mathrm{d}x$

n) $\displaystyle\int \frac{x^2-4}{x-2}\,\mathrm{d}x$

o) $\displaystyle\int \frac{x^2+4x+4}{x+2}\,\mathrm{d}x$

p) $\displaystyle\int \frac{x^2-6x+9}{x-3}\,\mathrm{d}x$

<u>**Solution:**</u>

a)

$$\int 2\,\mathrm{d}x = 2x + c$$

b)

$$\int 4x+2\,\mathrm{d}x = 4\cdot\int x\,\mathrm{d}x + \int 2\,\mathrm{d}x = \frac{4x^2}{2} + 2x + c = 2x^2 + 2x + c$$

c)

$$\int 3x^2+6x-8\,\mathrm{d}x = 3\cdot\int x^2\,\mathrm{d}x + 6\cdot\int x\,\mathrm{d}x - \int 8\,\mathrm{d}x =$$

$$= 3\cdot\frac{x^3}{3} + 6\cdot\frac{x^2}{2} - 8x + c = x^3 + 3x^2 - 8x + c$$

d) Since
$$\frac{1}{x^2} = x^{-2},$$
we get that
$$\int \frac{1}{x^2}\,\mathrm{d}x = \int x^{-2}\,\mathrm{d}x = \frac{x^{-1}}{-1} + c = -\frac{1}{x} + c.$$

e) Since
$$\frac{1}{x^3} = x^{-3},$$
we get that
$$\int \frac{2}{x^3}\,\mathrm{d}x = \int 2 \cdot x^{-3}\,\mathrm{d}x = 2 \cdot \frac{x^{-2}}{-2} + c = -\frac{1}{x^2} + c.$$

f) Since $\sqrt{x} = x^{\frac{1}{2}}$, we get that
$$\int \sqrt{x}\,\mathrm{d}x = \int x^{\frac{1}{2}}\,\mathrm{d}x = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3} \cdot \sqrt{x^3} + c.$$

g) Since $\sqrt[3]{x} = x^{\frac{1}{3}}$, we get that
$$\int \sqrt[3]{x}\,\mathrm{d}x = \int x^{\frac{1}{3}}\,\mathrm{d}x = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} = \frac{3}{4} \cdot \sqrt[3]{x^4} + c.$$

h)
$$\int \frac{x+1}{x}\,\mathrm{d}x = \int \frac{x}{x} + \frac{1}{x}\,\mathrm{d}x = \int 1 + \frac{1}{x}\,\mathrm{d}x = x + \ln|x| + c.$$

i)
$$\int \frac{x^2+x}{x}\,\mathrm{d}x = \int \frac{x^2}{x} + \frac{x}{x}\,\mathrm{d}x = \int x + 1\,\mathrm{d}x = \frac{x^2}{2} + x + c.$$

j)
$$\int \frac{x^2+x}{\sqrt{x}}\,\mathrm{d}x = \int \frac{x^2+x}{x^{\frac{1}{2}}}\,\mathrm{d}x = \int \frac{x^2}{x^{\frac{1}{2}}} + \frac{x}{x^{\frac{1}{2}}}\,\mathrm{d}x = \int x^{\frac{3}{2}} + x^{\frac{1}{2}}\,\mathrm{d}x =$$
$$= \frac{2}{5} \cdot x^{\frac{5}{2}} + \frac{2}{3} \cdot x^{\frac{3}{2}} + c = \frac{2}{5} \cdot \sqrt{x^5} + \frac{2}{3} \cdot \sqrt{x^3} + c.$$

k) Since $\sqrt[3]{x} = x^{\frac{1}{3}}$ and $\sqrt{x} = x^{\frac{1}{2}}$, we get that

$$\int \frac{\sqrt[3]{x} + \sqrt{x}}{x^2} \, \mathrm{d}x = \int \frac{x^{\frac{1}{3}}}{x^2} + \frac{x^{\frac{1}{2}}}{x^2} \, \mathrm{d}x = \int x^{-\frac{5}{3}} + x^{-\frac{3}{2}} \, \mathrm{d}x =$$
$$= -\frac{3}{2} \cdot x^{-\frac{2}{3}} - 2 \cdot x^{-\frac{1}{2}} + c = -\frac{3}{2} \cdot \frac{1}{\sqrt{x^3}} - \frac{2}{\sqrt{x}} + c.$$

l)

$$\int 3^x + 4^x \, \mathrm{d}x = \frac{3^x}{\ln 3} + \frac{4^x}{\ln 4} + c.$$

m)

$$\int \frac{x^2}{x^2+1} \, \mathrm{d}x = \int \frac{x^2+1-1}{x^2+1} \, \mathrm{d}x = \int 1 - \frac{1}{x^2+1} \, \mathrm{d}x =$$
$$= x - \arctan x + c.$$

n) Since $x^2 - 4 = (x-2) \cdot (x+2)$, we get that

$$\int \frac{x^2-4}{x-2} \, \mathrm{d}x = \int \frac{(x-2) \cdot (x+2)}{x-2} \, \mathrm{d}x = \int x + 2 \, \mathrm{d}x = \frac{x^2}{2} + 2x + c.$$

o) Since $x^2 + 4x + 4 = (x+2)^2$, we get that

$$\int \frac{x^2+4x+4}{x+2} \, \mathrm{d}x = \int \frac{(x+2)^2}{x+2} \, \mathrm{d}x = \int x + 2 \, \mathrm{d}x = \frac{x^2}{2} + 2x + c.$$

p) Since $x^2 - 6x + 9 = (x-3)^2$, we get that

$$\int \frac{x^2-6x+9}{x-3} \, \mathrm{d}x = \int \frac{(x-3)^2}{x-3} \, \mathrm{d}x = \int x - 3 \, \mathrm{d}x = \frac{x^2}{2} - 3x + c.$$

Exercise 72. Calculate the integrals below:

a) $\displaystyle\int \frac{3}{\cos^2 x} + \frac{2}{\sin^2 x} \, \mathrm{d}x$

b) $\displaystyle\int \tan^2 x \, \mathrm{d}x$

c) $\displaystyle\int 2 \cdot \cos x + 3 \cdot \sin x \, \mathrm{d}x$

d) $\displaystyle\int \frac{\cos 2x}{\sin x + \cos x} \, \mathrm{d}x$

e) $\displaystyle\int \cot^2 x \, \mathrm{d}x$

f) $\displaystyle\int \cot^2 x + 1 \, \mathrm{d}x$

g) $\displaystyle\int \frac{\sin 2x}{\cos x} \, \mathrm{d}x$

h) $\displaystyle\int \frac{\sin 2x}{\sin x} \, \mathrm{d}x$

i) $\displaystyle\int \tan^2 x + 1 \, \mathrm{d}x$

Solution:

a)

$$\int \frac{3}{\cos^2 x} + \frac{2}{\sin^2 x}\,\mathrm{d}x = 3\tan x - 2\cdot\cot x + c.$$

b) Since $\tan x = \frac{\sin x}{\cos x}$, we get that

$$\begin{aligned}\int \tan^2 x\,\mathrm{d}x &= \int \frac{\sin^2 x}{\cos^2 x}\,\mathrm{d}x = \int \frac{1-\cos^2 x}{\cos^2 x}\,\mathrm{d}x = \\ &= \int \frac{1}{\cos^2 x} - 1\,\mathrm{d}x = \tan x - x + c.\end{aligned}$$

c)

$$\int 2\cdot\cos x + 3\cdot\sin x\,\mathrm{d}x = 2\cdot\sin x - 3\cdot\cos x + c$$

d) Since $\cos 2x = \cos^2 x - \sin^2 x$, we get that

$$\begin{aligned}\int \frac{\cos 2x}{\sin x + \cos x}\,\mathrm{d}x &= \int \frac{\cos^2 x - \sin^2 x}{\sin x + \cos x}\,\mathrm{d}x = \\ &= \int \frac{(\cos x - \sin x)\cdot(\cos x + \sin x)}{\sin x + \cos x}\,\mathrm{d}x = \\ &= \int \cos x - \sin x\,\mathrm{d}x = \sin x + \cos x + c.\end{aligned}$$

e) Since $\cot x = \dfrac{\cos x}{\sin x}$, we get that

$$\begin{aligned}\int \cot^2 x\,\mathrm{d}x &= \int \frac{\cos^2 x}{\sin^2 x}\,\mathrm{d}x = \int \frac{1-\sin^2 x}{\sin^2 x}\,\mathrm{d}x = \\ &= \int \frac{1}{\sin^2 x} - 1\,\mathrm{d}x = -\cot x - x + c\end{aligned}$$

f) Since $\cot x = \dfrac{\cos x}{\sin x}$ and $\cos^2 x = 1 - \sin^2 x$, we get that

$$\begin{aligned}\int \cot^2 x + 1\,\mathrm{d}x &= \int \frac{\cos^2 x}{\sin^2 x} + 1\,\mathrm{d}x = \int \frac{1-\sin^2 x}{\sin^2 x} + 1\,\mathrm{d}x = \\ &= \int \frac{1}{\sin^2 x} - 1 + 1\,\mathrm{d}x = \int \frac{1}{\sin^2 x}\,\mathrm{d}x = -\cot x + c.\end{aligned}$$

g) Since $\sin 2x = 2\sin x \cdot \cos x$, we get that

$$\int \frac{\sin 2x}{\cos x}\,\mathrm{d}x = \int \frac{2\sin x \cdot \cos x}{\cos x}\,\mathrm{d}x = \int 2\sin x\,\mathrm{d}x = \\ = -2\cos x + c.$$

h) Since $\sin 2x = 2\sin x \cdot \cos x$, we get that

$$\int \frac{\sin 2x}{\sin x}\,\mathrm{d}x = \int \frac{2\sin x \cdot \cos x}{\sin x}\,\mathrm{d}x = \int 2\cos x\,\mathrm{d}x = \\ = 2\sin x + c.$$

i) Since $\tan x = \frac{\sin x}{\cos x}$ and $\sin^2 x + \cos^2 x = 1$, we get that

$$\int \tan^2 x + 1\,\mathrm{d}x = \int \frac{\sin^2 x}{\cos^2 x} + 1\,\mathrm{d}x = \int \frac{\sin^2 x + \cos^2 x}{\cos^2 x}\,\mathrm{d}x = \\ = \int \frac{1}{\cos^2 x}\,\mathrm{d}x = \tan x + c.$$

Exercise 73. Calculate the integrals below:

a) $\displaystyle\int 3^x + 6^x\,\mathrm{d}x$

b) $\displaystyle\int 4x^3 + \cos x + \mathrm{e}^x\,\mathrm{d}x$

c) $\displaystyle\int \frac{2^{2x} - 4}{2^x - 2}\,\mathrm{d}x$

d) $\displaystyle\int \left(2^x + 3\right)^2\,\mathrm{d}x$

Solution:

a)

$$\int 3^x + 6^x\,\mathrm{d}x = \frac{3^x}{\ln 3} + \frac{6^x}{\ln 6} + c.$$

b)

$$\int 4x^3 + \cos x + \mathrm{e}^x\,\mathrm{d}x = x^4 + \sin x + \mathrm{e}^x + c.$$

c) Since

$$2^{2x} - 4 = (2^x)^2 - 2^2 = (2^x - 2)\cdot(2^x + 2),$$

then

$$\int \frac{2^{2x} - 4}{2^x - 2}\,\mathrm{d}x = \int \frac{(2^x - 2)\cdot(2^x + 2)}{2^x - 2}\,\mathrm{d}x = \int 2^x + 2\,\mathrm{d}x = \\ = \frac{2^x}{\ln 2} + 2x + c.$$

d) Since

$$\left(2^x+3\right)^2 = 2^{2x} + 6 \cdot 2^x + 9 = 4^x + 6 \cdot 2^x + 9,$$

then

$$\int \left(2^x+3\right)^2 \mathrm{d}x = \int 4^x + 6 \cdot 2^x + 9 \,\mathrm{d}x = \frac{4^x}{\ln 4} + 6 \cdot \frac{2^x}{\ln 2} + 9x + c.$$

Exercise 74. Let

$$f(x) = \frac{x^2-4}{x-2}$$

and F is a primitve function of f, such that $F(2) = 16$.

Solution:

Since $x^2 - 4 = (x-2) \cdot (x+2)$, that

$$\frac{x^2-4}{x-2} = \frac{(x-2) \cdot (x+2)}{x-2} = x+2,$$

therefore

$$\int \frac{x^2-4}{x-2} \,\mathrm{d}x = \int x + 2 \,\mathrm{d}x = \frac{x^2}{2} + 2x + c.$$

We get that

$$F(x) = \frac{x^2}{2} + 2x + c.$$

Since $F(2) = 16$, thus

$$16 = \frac{2^2}{2} + 2 \cdot 2 + c \qquad \Rightarrow \qquad c = 10.$$

The primitve function is

$$F(x) = \frac{x^2}{2} + 2x + 10.$$

Exercise 75. Let $f(x) = (3x+1)^2$. Find the primitive function F of f, such that $F(1) = 6$.

Solution:

Since

$$(a+b)^2 = a^2 + 2ab + b^2$$

thus

$$(3x+1)^2 = 9x^2 + 6x + 1.$$

We get that

$$F(x) = \int 9x^2 + 6x + 1 \,\mathrm{d}x = \frac{9x^3}{3} + \frac{6x^2}{2} + x + c = 3x^3 + 3x^2 + x + c.$$

Since $F(1) = 6$, therefore

$$3 + 3 + 1 + c = 6 \qquad \Rightarrow \qquad c = -1,$$

thus the primitve function is

$$F(x) = 3x^3 + 3x^2 + x - 1.$$

Exercise 76. Calculate the integrals below:

a) $\displaystyle\int \frac{x}{x^2+3}\,\mathrm{d}x$

b) $\displaystyle\int \frac{x+2}{x^2+4x+5}\,\mathrm{d}x$

c) $\displaystyle\int \frac{1}{x\cdot\ln x}\,\mathrm{d}x$

d) $\displaystyle\int \frac{1}{\tan x\cdot\cos^2 x}\,\mathrm{d}x$

e) $\displaystyle\int \frac{1}{\cot x\cdot\sin^2 x}\,\mathrm{d}x$

f) $\displaystyle\int \frac{\mathrm{e}^x}{\mathrm{e}^x+1}\,\mathrm{d}x$

g) $\displaystyle\int \frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x}+1}\,\mathrm{d}x$

h) $\displaystyle\int 5\cdot\frac{\cos x}{\sin x}\,\mathrm{d}x$

Solution:

a) If $f(x) = x^2 + 3$, then $f'(x) = 2x$, thus

$$\int \frac{x}{x^2+3}\,\mathrm{d}x = \frac{1}{2}\cdot\int \frac{2x}{x^2+3}\,\mathrm{d}x = \frac{1}{2}\cdot\ln(x^2+3) + c.$$

b)

$$\int \frac{x+2}{x^2+4x+5}\,\mathrm{d}x = \frac{1}{2}\cdot\int \frac{2x+4}{x^2+4x+5}\,\mathrm{d}x = \frac{1}{2}\cdot\ln|x^2+4x+5| + c.$$

c) Since

$$\frac{1}{x\cdot\ln x} = \frac{1}{x}\cdot\frac{1}{\ln x},$$

therefore

$$\int \frac{1}{x\cdot\ln x}\,\mathrm{d}x = \int \frac{\frac{1}{x}}{\ln x}\,\mathrm{d}x = \ln|\ln x| + c.$$

d) Since
$$\frac{1}{\cos^2 x \cdot \tan x} = \frac{1}{\cos^2 x} \cdot \frac{1}{\tan x},$$
therefore
$$\int \frac{1}{\tan x \cdot \cos^2 x}\, \mathrm{d}x = \int \frac{\frac{1}{\cos^2 x}}{\tan x}\, \mathrm{d}x = \ln|\tan x| + c.$$

e) Since
$$\frac{1}{\sin^2 x \cdot \cot x} = \frac{1}{\sin^2 x} \cdot \frac{1}{\cot x},$$
therefore
$$\int \frac{1}{\cot x \cdot \sin^2 x}\, \mathrm{d}x = \int \frac{\frac{1}{\sin^2 x}}{\cot x}\, \mathrm{d}x = -\ln|\cot x| + c.$$

f) If $f(x) = \mathrm{e}^x + 1$, then $f'(x) = \mathrm{e}^x$, thus
$$\int \frac{\mathrm{e}^x}{\mathrm{e}^x + 1}\, \mathrm{d}x = \ln(\mathrm{e}^x + 1) + c.$$

g)
$$\int \frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x} + 1}\, \mathrm{d}x = \int \frac{1}{2} \cdot \frac{2\mathrm{e}^{2x}}{\mathrm{e}^{2x} + 1}\, \mathrm{d}x = \frac{1}{2} \cdot \int \frac{2 \cdot \mathrm{e}^{2x}}{\mathrm{e}^{2x} + 1}\, \mathrm{d}x =$$
$$= \frac{1}{2} \cdot \ln(e^{2x} + 1) + c.$$

h)
$$\int \frac{5 \cdot \cos x}{\sin x}\, \mathrm{d}x = 5 \cdot \int \frac{\cos x}{\sin x}\, \mathrm{d}x = 5 \cdot \ln|\sin x| + c.$$

Exercise 77. If $f(x) = \frac{x}{x^2+1}$ then calculate the primitive function F of function f such that $F(0) = 5$.

Solution:

Since
$$\int \frac{x}{x^2 + 1}\, \mathrm{d}x = \frac{1}{2} \cdot \int \frac{2x}{x^2 + 1}\, \mathrm{d}x = \frac{1}{2} \cdot \ln(x^2 + 1) + c,$$
thus $F(0) = 5$ and $5 = \frac{1}{2} \cdot \ln 1 + c$, therefore $c = 5$. The primitve function is
$$F(x) = \frac{1}{2} \cdot \ln(x^2 + 1) + 5.$$

Exercise 78. Calculate the integrals below:

a) $\displaystyle\int \sin^5 x \cdot \cos x \,\mathrm{d}x$

b) $\displaystyle\int \cos^5 x \cdot \sin x \,\mathrm{d}x$

c) $\displaystyle\int \frac{\sin x}{\cos^5 x} \,\mathrm{d}x$

d) $\displaystyle\int \frac{\cos x}{\sqrt[4]{\sin^3 x}} \,\mathrm{d}x$

e) $\displaystyle\int \frac{\ln^2 x}{x} \,\mathrm{d}x$

f) $\displaystyle\int \frac{x^3}{(x^4+1)^2} \,\mathrm{d}x$

g) $\displaystyle\int \frac{x}{\sqrt{1+x^2}} \,\mathrm{d}x$

h) $\displaystyle\int \frac{2x^2}{\sqrt{1+x^3}} \,\mathrm{d}x$

i) $\displaystyle\int \frac{\tan^2 x}{\cos^2 x} \,\mathrm{d}x$

j) $\displaystyle\int \frac{\cot^5 x}{\sin^2 x} \,\mathrm{d}x$

Solution:

a)

$$\int \sin^5 x \cdot \cos x \,\mathrm{d}x = \frac{\sin^6 x}{6} + c.$$

b)

$$\int \cos^5 x \cdot \sin x \,\mathrm{d}x = -\int \cos^5 x \cdot (-\sin x) \,\mathrm{d}x = -\frac{\cos^6 x}{6} + c.$$

c)

$$\int \frac{\sin x}{\cos^5 x} \,\mathrm{d}x = \int \sin x \cdot \cos^{-5} x \,\mathrm{d}x = -\int -\sin x \cdot \cos^{-5} x \,\mathrm{d}x =$$
$$= -\frac{\cos^{-4} x}{-4} + c = \frac{1}{4 \cdot \cos^4 x} + c.$$

d)

$$\int \frac{\cos x}{\sqrt[4]{\sin^3 x}} \,\mathrm{d}x = \int \cos x \cdot (\sin x)^{-\frac{3}{4}} \,\mathrm{d}x = \frac{\sin^{\frac{1}{4}} x}{\frac{1}{4}} + c = 4 \cdot \sqrt[4]{\sin x} + c.$$

e)

$$\int \frac{\ln^2 x}{x} = \int \ln^2 x \cdot \frac{1}{x} \,\mathrm{d}x = \frac{\ln^3 x}{3} + c.$$

f)

$$\int \frac{x^3}{(x^4+1)^2}\,\mathrm{d}x = \int x^3 \cdot (x^4+1)^{-2}\,\mathrm{d}x = \frac{1}{4} \cdot \int 4x^3 \cdot (x^4+1)^{-2}\,\mathrm{d}x =$$
$$= \frac{1}{4} \cdot \frac{(x^4+1)^{-1}}{-1} + c = -\frac{1}{4} \cdot \frac{1}{x^4+1} + c.$$

g)

$$\int \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x = \int x \cdot (1+x^2)^{-\frac{1}{2}}\,\mathrm{d}x = \frac{1}{2} \cdot \int 2x \cdot (1+x^2)^{-\frac{1}{2}}\,\mathrm{d}x =$$
$$= \frac{1}{2} \cdot \frac{(1+x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c = \sqrt{1+x^2} + c.$$

h)

$$\int \frac{2x^2}{\sqrt{1+x^3}}\,\mathrm{d}x = 2 \cdot \int \frac{x^2}{\sqrt{1+x^3}}\,\mathrm{d}x = \frac{2}{3} \cdot \int 3x^2 \cdot (1+x^3)^{-\frac{1}{2}} =$$
$$= \frac{2}{3} \cdot \frac{(1+x^3)^{\frac{1}{2}}}{\frac{1}{2}} + c = \frac{4}{3} \cdot \sqrt{1+x^3} + c.$$

i)

$$\int \frac{\tan^2 x}{\cos^2 x}\,\mathrm{d}x = \int \tan^2 x \cdot \frac{1}{\cos^2 x}\,\mathrm{d}x = \frac{\tan^3 x}{3} + c.$$

j)

$$\int \frac{\cot^2 x}{\sin^2 x}\,\mathrm{d}x = -\int \cot^2 x \cdot \left(-\frac{1}{\sin^2 x}\right)\,\mathrm{d}x = -\frac{\cot^3 x}{3} + c.$$

Exercise 79. Let $f(x) = \sin x \cdot \sqrt{\cos x}$ be a function. Find the primitive function F, such that $F(0) = 3$.

Solution:

Since $\sqrt{\cos x} = (\cos x)^{\frac{1}{2}}$, we have

$$\int \sin x \cdot (\cos x)^{\frac{1}{2}}\,\mathrm{d}x = -\int -\sin x \cdot (\cos x)^{\frac{1}{2}}\,\mathrm{d}x =$$
$$= -\frac{2}{3} \cdot (\cos x)^{\frac{3}{2}} + c = -\frac{2}{3} \cdot \sqrt{(\cos x)^3} + c.$$

Since $F(0) = 3$, therefore

$$3 = -\frac{2}{3} \cdot \sqrt{1} + c \qquad \Rightarrow \qquad c = \frac{11}{3}.$$

The primitive function is

$$F(x) = -\frac{2}{3} \cdot \sqrt{(\cos x)^3} + \frac{11}{3}.$$

Exercise 80. Calculate the integrals below:

a) $\displaystyle\int \cos(4x+3)\,\mathrm{d}x$

b) $\displaystyle\int \sin(2x+3)\,\mathrm{d}x$

c) $\displaystyle\int \frac{1}{\cos^2(5x-2)}\,\mathrm{d}x$

d) $\displaystyle\int \frac{1}{\sin^2(3x+6)}\,\mathrm{d}x$

e) $\displaystyle\int \mathrm{e}^{2x+5}\,\mathrm{d}x$

f) $\displaystyle\int (3x+1)^{20}\,\mathrm{d}x$

g) $\displaystyle\int (1-4x)^3\,\mathrm{d}x$

h) $\displaystyle\int \frac{1}{\sqrt{5-2x}}\,\mathrm{d}x$

<u>**Solution:**</u>

a) Since $(\sin x)' = \cos x$, we get that

$$\int \cos(4x+3)\,\mathrm{d}x = \frac{\sin(4x+3)}{4} + c.$$

b)

$$\int \sin(2x+3)\,\mathrm{d}x = -\frac{\cos(2x+3)}{2} + c$$

c)

$$\int \frac{1}{\cos^2(5x-2)}\,\mathrm{d}x = \frac{\tan(5x-2)}{5} + c.$$

d) Since $(\cot x)' = -\frac{1}{\sin^2 x}$, we get that

$$\int \frac{1}{\sin^2(3x+6)}\,\mathrm{d}x = -\frac{\cot(3x+6)}{3} + c.$$

e) Since $(\mathrm{e}^x)' = \mathrm{e}^x$, we get that

$$\int \mathrm{e}^{2x+5}\,\mathrm{d}x = \frac{\mathrm{e}^{2x+5}}{2} + c.$$

f) Since

$$\int x^{20}\,\mathrm{d}x = \frac{x^{21}}{21} + c,$$

therefore

$$\int (3x+1)^{20}\,\mathrm{d}x = \frac{(3x+1)^{21}}{21\cdot 3} + c = \frac{(3x+1)^{21}}{63} + c.$$

g) Since

$$\int x^{3}\,\mathrm{d}x = \frac{x^{4}}{4} + c,$$

therefore

$$\int (1-4x)^{3}\,\mathrm{d}x = \frac{(1-4x)^{4}}{4\cdot(-4)} = -\frac{(1-4x)^{4}}{16} + c.$$

h) Since

$$\int x^{-\frac{1}{2}}\,\mathrm{d}x = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c,$$

therefore

$$\int \frac{1}{\sqrt{5-2x}}\,\mathrm{d}x = \int (5-2x)^{-\frac{1}{2}}\,\mathrm{d}x = \frac{(5-2x)^{\frac{1}{2}}}{-2\cdot\frac{1}{2}} + c = -\sqrt{5-2x} + c.$$

Exercise 81. If the marginal revenue function for a manufacturer's product is

$$MR(q) = 3\,000 - 20q - 3q^2,$$

Find the revenue function.

Solution:

By integrating the marginal revenue function and using the initial condition $R(0) = 0$, we can find the revenue function. Since

$$\begin{aligned} R(q) &= \int 3\,000 - 20q - 3q^2\,\mathrm{d}q = \\ &= \int 3\,000\,\mathrm{d}q - \int 20q\,\mathrm{d}q - \int 3q^2\,\mathrm{d}q = \\ &= 3\,000q - 10q^2 - q^3 + c \end{aligned}$$

and $R(0) = 0$, thus

$$3\,000\cdot 0 - 10\cdot 0^2 - 0^3 + c = 0,$$

thus we get that $c = 0$. Consequently, the revenue function is

$$R(q) = -q^3 - 10q^2 + 3\,000q.$$

9. Primitive functions II. (Integral by parts)

Theoretical summary

Theorem 2.2.1.

$$\int f' \cdot g = f \cdot g - \int f \cdot g'$$

Remark 2.2.2. Type 1. Let $P(x)$ is a polynomial function. Then:

$f'(x)$	$g(x)$
$\sin x$	$P(x)$
$\cos x$	$P(x)$
e^x	$P(x)$
a^x	$P(x)$
$\sinh x$	$P(x)$
$\cosh x$	$P(x)$

Remark 2.2.3. Type 2. Let $P(x)$ is a polynomial function. Then:

$f'(x)$	$g(x)$
$P(x)$	$\ln x$
$P(x)$	$\log_a x$
$P(x)$	$\arctan x$
$P(x)$	$\arcsin x$

Solved exercises

Exercise 82. Calculate the integral

$$\int x \cdot \mathrm{e}^x \, \mathrm{d}x$$

Solution:

Let

$$\begin{array}{rclcrcl} f(x) & = & \mathrm{e}^x & \qquad & f'(x) & = & \mathrm{e}^x \\ g(x) & = & x & & g'(x) & = & 1. \end{array}$$

Apply the partial integration theorem, we get that

$$\begin{aligned} \int x \cdot \mathrm{e}^x \, \mathrm{d}x &= x \cdot \mathrm{e}^x - \int \mathrm{e}^x \, \mathrm{d}x = \\ &= x \cdot \mathrm{e}^x - \mathrm{e}^x + c = \mathrm{e}^x \cdot (x-1) + c, \end{aligned}$$

where $c \in \mathbb{R}$.

Exercise 83. Calculate the integral

$$\int x \cdot \cos x \, \mathrm{d}x.$$

Solution:

Let

$$\begin{array}{rclcrcl} f(x) & = & \sin x & \qquad & f'(x) & = & \cos x \\ g(x) & = & x & & g'(x) & = & 1. \end{array}$$

Apply the partial integration theorem, we get that

$$\begin{aligned} \int x \cdot \cos x \, \mathrm{d}x &= x \cdot \sin x - \int \sin x \, \mathrm{d}x = \\ &= x \cdot \sin x + \cos x + c, \end{aligned}$$

where $c \in \mathbb{R}$.

Exercise 84. Calculate the integral

$$\int x \cdot \sin x \, \mathrm{d}x.$$

Solution:

Let

$$\begin{aligned} f(x) &= -\cos x & f'(x) &= \sin x \\ g(x) &= x & g'(x) &= 1. \end{aligned}$$

Apply the partial integration theorem, we get that

$$\int x \cdot \sin x \, \mathrm{d}x = -x \cdot \cos x + \int \cos x \, \mathrm{d}x = \\ = -x \cdot \cos x + \sin x + c,$$

where $c \in \mathbb{R}$.

Exercise 85. Calculate the integral

$$\int (8x - 2) \cdot \mathrm{e}^x \, \mathrm{d}x.$$

Solution:

Let

$$\begin{aligned} f(x) &= \mathrm{e}^x & f'(x) &= \mathrm{e}^x \\ g(x) &= 8x - 2 & g'(x) &= 8. \end{aligned}$$

Apply the partial integration theorem, we get that

$$\int (8x - 2) \cdot \mathrm{e}^x \, \mathrm{d}x = (8x - 2) \cdot \mathrm{e}^x - \int 8 \cdot \mathrm{e}^x \, \mathrm{d}x = \\ = (8x - 2) \cdot \mathrm{e}^x - 8 \cdot \mathrm{e}^x + c = \mathrm{e}^x \cdot (8x - 10) + c,$$

where $c \in \mathbb{R}$.

Exercise 86. Calculate the integral

$$\int (2x + 3) \cdot \cos x \, \mathrm{d}x.$$

Solution:

Let

$$\begin{aligned} f(x) &= \sin x & f'(x) &= \cos x \\ g(x) &= 2x + 3 & g'(x) &= 2. \end{aligned}$$

Apply the partial integration theorem, we get that

$$\int (2x + 3) \cdot \cos x \, \mathrm{d}x = (2x + 3) \cdot \sin x - \int 2 \cdot \sin x \, \mathrm{d}x = \\ = (2x + 3) \cdot \sin x + 2 \cdot \cos x + c,$$

where $c \in \mathbb{R}$.

Exercise 87. Calculate the integral

$$\int (x^2 + 7x - 1) \cdot \cos x \,\mathrm{d}x$$

Solution:

Let

$$\begin{aligned} f_1(x) &= \sin x & f_1'(x) &= \cos x \\ g_1(x) &= x^2 + 7x - 1 & g_1'(x) &= 2x + 7. \end{aligned}$$

Apply the partial integration theorem, we get that

$$\int f_1'(x) \cdot g_1(x) \,\mathrm{d}x = f_1(x) \cdot g_1(x) - \int f_1(x) \cdot g_1'(x) \,\mathrm{d}x$$

thus

$$\int (x^2 + 7x - 1) \cdot \cos x \,\mathrm{d}x = (x^2 + 7x - 1) \cdot \sin x - \int (2x + 7) \cdot \sin x \,\mathrm{d}x.$$

The integral

$$\int (2x + 7) \cdot \sin x \,\mathrm{d}x$$

can be calculated with partial partial integration again. Let $f_2'(x) = \sin x$ and $g_2(x) = 2x + 7$. We get that

$$\begin{aligned} f_2(x) &= -\cos x & f_2'(x) &= \sin x \\ g_2(x) &= 2x + 7 & g_2'(x) &= 2. \end{aligned}$$

By the formula of partial integration we get that

$$\int f_2'(x) \cdot g_2(x) \,\mathrm{d}x = f_2(x) \cdot g_2(x) - \int f_2(x) \cdot g_2'(x) \,\mathrm{d}x$$

thus

$$\begin{aligned} \int (2x + 7) \cdot \sin x \,\mathrm{d}x &= -(2x + 7) \cdot \cos x + \int 2 \cdot \cos x \,\mathrm{d}x = \\ &= -(2x + 7) \cdot \cos x + 2 \cdot \sin x + c_1, \end{aligned}$$

where $c_1 \in \mathbb{R}$.

The final result is

$$\int (x^2+7x-1)\cdot \cos x \,\mathrm{d}x =$$

$$(x^2+7x-1)\cdot \sin x - \int (2x+7)\cdot \sin x \,\mathrm{d}x =$$

$$= (x^2+7x-1)\cdot \sin x - \big(-(2x+7)\cdot \cos x + 2\cdot \sin x + c_1\big) =$$

$$= (x^2+7x-1)\cdot \sin x + (2x+7)\cdot \cos x - 2\cdot \sin x - c_1 =$$

$$= (x^2+7x-1)\cdot \sin x + (2x+7)\cdot \cos x - 2\cdot \sin x + c.$$

Exercise 88. Calculate the integral

$$\int (x^2+2x-1)\cdot \mathrm{e}^{2x}\,\mathrm{d}x.$$

Solution:

Let $f_1'(x) = \mathrm{e}^{2x}$ and $g_1(x) = x^2+2x-1$. Thus

$$\begin{array}{rclcrcl} f_1(x) & = & \frac{\mathrm{e}^{2x}}{2} & & f_1'(x) & = & \mathrm{e}^{2x} \\ g_1(x) & = & x^2+2x-1 & & g_1'(x) & = & 2x+2. \end{array}$$

Apply the formula of partial integration, we get that

$$\int f_1'(x)\cdot g_1(x)\,\mathrm{d}x = f_1(x)\cdot g_1(x) - \int f_1(x)\cdot g_1'(x)\,\mathrm{d}x,$$

thus

$$\int (x^2+2x-1)\cdot \mathrm{e}^{2x}\,\mathrm{d}x = (x^2+2x-1)\cdot \frac{\mathrm{e}^{2x}}{2} - \int (2x+2)\cdot \frac{\mathrm{e}^{2x}}{2}\,\mathrm{d}x =$$

$$= (x^2+2x-1)\cdot \frac{\mathrm{e}^{2x}}{2} - \int (x+1)\cdot \mathrm{e}^{2x}\,\mathrm{d}x.$$

The integral

$$\int (x+1)\cdot \mathrm{e}^{2x}\,\mathrm{d}x$$

can be calculated with partial integration again.

Let $f_2'(x) = \mathrm{e}^{2x}$ and $g_2(x) = x+1$! We get that

$$\begin{array}{rclcrcl} f_2(x) & = & \frac{\mathrm{e}^{2x}}{2} & & f_2'(x) & = & \mathrm{e}^{2x} \\ g_2(x) & = & x+1 & & g_2'(x) & = & 1. \end{array}$$

By the formula of partial integration, we get that

$$\int f_2'(x) \cdot g_2(x)\,\mathrm{d}x = f_2(x) \cdot g_2(x) - \int f_2(x) \cdot g_2'(x)\,\mathrm{d}x,$$

thus

$$\begin{aligned}\int (x+1) \cdot \mathrm{e}^{2x}\,\mathrm{d}x &= (x+1) \cdot \frac{\mathrm{e}^{2x}}{2} - \int \frac{\mathrm{e}^{2x}}{2}\,\mathrm{d}x = \\ &= (x+1) \cdot \frac{\mathrm{e}^{2x}}{2} - \frac{\mathrm{e}^{2x}}{4} + c_1,\end{aligned}$$

where $c_1 \in \mathbb{R}$.

The final result is

$$\begin{aligned}&\int (x^2+2x-1) \cdot \mathrm{e}^{2x}\,\mathrm{d}x = \\ &= (x^2+2x-1) \cdot \frac{\mathrm{e}^{2x}}{2} - \left((x+1) \cdot \frac{\mathrm{e}^{2x}}{2} - \frac{\mathrm{e}^{2x}}{4} + c_1\right) = \\ &= \frac{1}{4} \cdot \mathrm{e}^{2x} \cdot \left(-3 + 2\,x + 2\,x^2\right) - c_1 = \\ &= \frac{1}{4} \cdot \mathrm{e}^{2x} \cdot \left(-3 + 2\,x + 2\,x^2\right) + c,\end{aligned}$$

where $c \in \mathbb{R}$.

Exercise 89. Calculate the integral

$$\int 2x \cdot \ln x\,\mathrm{d}x.$$

Solution:

Let $f'(x) = 2x$ and $g(x) = \ln x$. Then

$$\begin{array}{rclcrcl} g(x) &=& \ln x & \qquad & g'(x) &=& \frac{1}{x} \\ f(x) &=& x^2 & & f'(x) &=& 2x. \end{array}$$

By the formula of partial integration, we get that

$$\begin{aligned}\int 2x \cdot \ln x\,\mathrm{d}x &= x^2 \cdot \ln x - \int x^2 \cdot \frac{1}{x}\,\mathrm{d}x = \\ &= x^2 \cdot \ln x - \int x\,\mathrm{d}x = x^2 \cdot \ln x - \frac{x^2}{2} + c,\end{aligned}$$

where $c \in \mathbb{R}$.

Exercise 90. Calculate the integral

$$\int \ln x \,\mathrm{d}x.$$

Solution:

Let $f'(x) = 1$ and $g(x) = \ln x$. Thus

$$\begin{aligned} g(x) &= \ln x & \qquad g'(x) &= \tfrac{1}{x} \\ f(x) &= x & \qquad f'(x) &= 1. \end{aligned}$$

By the formula of partial integration, we get that

$$\begin{aligned} \int \ln x \,\mathrm{d}x = x \cdot \ln x - \int x \cdot \frac{1}{x} \,\mathrm{d}x = \\ = x \cdot \ln x - \int 1 \,\mathrm{d}x = x \cdot \ln x - x + c, \end{aligned}$$

where $c \in \mathbb{R}$.

Exercise 91. Calculate the integral

$$\int (3x^2 + 2x + 1) \cdot \ln x \,\mathrm{d}x.$$

Solution:

Let $f'(x) = 3x^2 + 2x + 1$ and $g(x) = \ln x$. Then

$$\begin{aligned} g(x) &= \ln x & \qquad g'(x) &= \tfrac{1}{x} \\ f(x) &= x^3 + x^2 + x & \qquad f'(x) &= 3x^2 + 2x + 1. \end{aligned}$$

Using the formula of partial integration, we get that

$$\begin{aligned} \int (3x^2 + 2x + 1) \cdot \ln x \,\mathrm{d}x &= (x^3 + x^2 + x) \cdot \ln x - \int (x^3 + x^2 + x) \cdot \frac{1}{x} \,\mathrm{d}x = \\ &= (x^3 + x^2 + x) \cdot \ln x - \int x^2 + x + 1 \,\mathrm{d}x = \\ &= (x^3 + x^2 + x) \cdot \ln x - \frac{x^3}{3} - \frac{x^2}{2} - x + c, \end{aligned}$$

where $c \in \mathbb{R}$.

10. Primitive functions III. (Integral using partial fraction decomposition)

Theoretical summary

Remark 2.3.1. Partial fraction decomposition is a technique used to integrate rational functions of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials with $\deg(P) < \deg(Q)$. This method breaks down a complex rational function into a sum of simpler rational functions that can be integrated using standard formulas.

This technique is applied when

- The integrand is a rational function $\frac{P(x)}{Q(x)}$.
- $\deg(P) < \deg(Q)$.
- The denominator $Q(x)$ can be factored completely

Theorem 2.3.2. The General Procedure is as follows:

1. Ensure $\deg(P) < \deg(Q)$. If not, perform polynomial long division.
2. Factor the denominator $Q(x)$ completely into linear and irreducible quadratic factors.
3. Decompose $\frac{P(x)}{Q(x)}$ into partial fractions based on the factors of $Q(x)$.
4. Solve for the coefficients in the partial fraction expansion.
5. Integrate each term separately using standard integration formulas.

Theorem 2.3.3. Cases in Partial Fraction Decomposition:

1. Distinct Linear Factors:

   For a factor $(ax + b)$ in the denominator, the partial fraction term has the form:
   $$\frac{A}{ax+b}$$
   where A is a constant to be determined.
2. Repeated Linear Factors:

   For a factor $(ax + b)^n$ in the denominator, the partial fraction terms have the form:
   $$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$
   where $A_1, A_2, \ldots, A_n$ are constants to be determined.

3. Irreducible Quadratic Factors:

   For a factor $(ax^2 + bx + c)$ that cannot be factored further over the real numbers, the partial fraction term has the form:
   $$\frac{Ax + B}{ax^2 + bx + c}$$
   where A and B are constants to be determined.

4. Repeated Irreducible Quadratic Factor:

   For a factor $(ax^2 + bx + c)^m$ that cannot be factored further over the real numbers, the partial fraction terms have the form:
   $$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m}$$
   where $A_1, B_1, A_2, B_2, \ldots, A_m, B_m$ are constants to be determined.

Theorem 2.3.4. After decomposition, the following standard integration formulas are typically used:

$$\int \frac{1}{ax + b}\,dx = \frac{1}{a}\ln|ax + b| + C$$
$$\int \frac{1}{(ax + b)^n}\,dx = \frac{1}{a \cdot (1 - n) \cdot (ax + b)^{n-1}} + C \quad (n \neq 1)$$
$$\int \frac{x}{ax^2 + bx + c}\,dx = \frac{1}{2a}\ln|ax^2 + bx + c| - \frac{b}{2a}\int \frac{1}{ax^2 + bx + c}\,dx$$

Solved exercises

Exercise 92. Calculate the integral

$$\int \frac{2x-3}{x^2+5x+6}\,\mathrm{d}x.$$

<u>**Solution:**</u>

Since

$$x^2+5x+6=(x+2)\cdot(x+3)$$

thus

$$\int \frac{2x-3}{x^2+5x+6}\,\mathrm{d}x=\int \frac{2x-3}{(x+2)\cdot(x+3)}\,\mathrm{d}x.$$

Applying the theorem, partial fraction decomposition, we get that

$$\frac{2x-3}{(x+2)\cdot(x+3)}=\frac{A}{x+2}+\frac{B}{x+3}.$$

Since

$$2x-3=A\cdot(x+3)+B\cdot(x+2),$$

thus

$$2x-3=x\cdot(A+B)+3A+2B.$$

We have the system of equations below

$$\left.\begin{array}{rcrcr} 2 & = & A & + & B \\ -3 & = & 3A & + & 2B \end{array}\right\}.$$

The solution of the system is $(A;B)=(-7;9)$. Thus the integal is

$$\begin{aligned}\int \frac{2x+3}{(x+2)\cdot(x+3)}\,\mathrm{d}x&=\int \frac{-7}{x+2}+\frac{9}{x+3}\,\mathrm{d}x=\\&=-7\cdot\ln|x+2|+9\cdot\ln|x+3|+c,\end{aligned}$$

where $c\in\mathbb{R}$. If we apply algebraic identity, we get that

$$\int \frac{2x-3}{(x+2)\cdot(x+3)}\,\mathrm{d}x=\ln\left|\frac{(x+3)^9}{(x+2)^7}\right|+c.$$

Exercise 93. Calculate the integral

$$\int \frac{x-2}{x^3-x}\,\mathrm{d}x.$$

Solution:

Since

$$\int \frac{x-2}{x\cdot(x^2-1)}\,\mathrm{d}x = \int \frac{x-2}{x\cdot(x+1)\cdot(x-1)}\,\mathrm{d}x,$$

therefore

$$\frac{x-2}{x\cdot(x+1)\cdot(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}.$$

If we apply algebraic identity, we get that

$$x-2 = A\cdot(x+1)\cdot(x-1) + B\cdot x\cdot(x-1) + C\cdot x\cdot(x+1),$$

thus

$$\begin{aligned} x-2 &= A\cdot(x^2-1) + B\cdot(x^2-x) + C\cdot(x^2+x) \\ x-2 &= A\cdot x^2 - A + B\cdot x^2 - B\cdot x + C\cdot x^2 + C\cdot x, \end{aligned}$$

that is

$$x-2 = (A+B+C)\cdot x^2 + (-B+C)\cdot x - A.$$

We get the system of equations below

$$\left.\begin{array}{rcrcrcr} 0 &=& A &+& B &+& C \\ 1 &=& &-& B &+& C \\ -2 &=& -A & & & & \end{array}\right\}.$$

The solution of system is $(A;B;C) = (2;-1.5;-0.5)$, thus

$$\frac{x-2}{x\cdot(x+1)\cdot(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}.$$

The integral is

$$\begin{aligned}
&\int \frac{x-2}{x\cdot(x^2-1)}\,\mathrm{d}x = \int \frac{x-2}{x\cdot(x+1)\cdot(x-1)}\,\mathrm{d}x = \\
&= \int \frac{2}{x} - \frac{1.5}{x+1} - \frac{0.5}{x-1}\,\mathrm{d}x = \\
&= \int \frac{2}{x}\,\mathrm{d}x - \int \frac{1.5}{x+1}\,\mathrm{d}x - \int \frac{0.5}{x-1}\,\mathrm{d}x = \\
&= 2\cdot\int \frac{1}{x}\,\mathrm{d}x - 1.5\cdot\int \frac{1}{x+1}\,\mathrm{d}x - 0.5\cdot\int \frac{1}{x-1}\,\mathrm{d}x = \\
&= 2\cdot\ln|x| - 1.5\cdot\ln|x+1| - 0.5\cdot\ln|x-1| + c,
\end{aligned}$$

where $c \in \mathbb{R}$.

Exercise 94. Calculate the integral

$$\int \frac{x+1}{(x-4)^2}\,\mathrm{d}x.$$

Solution:

Since

$$\frac{x+1}{(x-4)^2} = \frac{A}{x-4} + \frac{B}{(x-4)^2}$$

we get that

$$x+1 = A\cdot(x-4) + B,$$

that is

$$x+1 = Ax - 4A + B.$$

The solution of the system of equations

$$\left.\begin{array}{rcrcr} 1 & = & A & & \\ 1 & = & -4A & + & B \end{array}\right\}$$

is $(A;B) = (1;5)$, thus

$$\frac{x+1}{(x-4)^2} = \frac{1}{x-4} + \frac{5}{(x-4)^2}.$$

We get that

$$\begin{aligned}\int \frac{x+1}{(x-4)^2}\,\mathrm{d}x &= \int \frac{1}{x-4} + \frac{5}{(x-4)^2}\,\mathrm{d}x = \\ &= \int \frac{1}{x-4}\,\mathrm{d}x + 5\cdot\int \frac{1}{(x-4)^2}\,\mathrm{d}x = \\ &= \ln|x-4| + 5\cdot\int (x-4)^{-2}\,\mathrm{d}x = \\ &= \ln|x-4| + 5\cdot\frac{(x-4)^{-1}}{-1} + c = \\ &= \ln|x-4| - 5\cdot\frac{1}{x-4} + c = \ln|x-4| - \frac{5}{x-4} + c\end{aligned}$$

where $c \in \mathbb{R}$.

Exercise 95. Calculate the integral

$$\int \frac{x+2}{x^2-6x+9}\,\mathrm{d}x.$$

Solution:

Since $x^2-6x+9=(x-3)^2$, we get that

$$\frac{x+2}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2},$$

thus

$$x+2 = A\cdot(x-3)+B,$$

that is

$$x+2 = Ax - 3A + B.$$

The solution of the system of equations

$$\left.\begin{array}{rcrcr} 1 &=& A & & \\ 2 &=& -3A &+& B \end{array}\right\}$$

is $(A;B)=(1;5)$, thus

$$\frac{x+2}{(x-3)^2} = \frac{1}{x-3} + \frac{5}{(x-3)^2}.$$

The result is

$$\begin{aligned}
&\int \frac{x+2}{(x-3)^2}\,\mathrm{d}x = \int \frac{1}{x-3} + \frac{5}{(x-3)^2}\,\mathrm{d}x = \\
&= \int \frac{1}{x-3}\,\mathrm{d}x + \int \frac{5}{(x-3)^2}\,\mathrm{d}x = \\
&= \int \frac{1}{x-3}\,\mathrm{d}x + 5\cdot\int (x-3)^{-2}\,\mathrm{d}x = \\
&= \ln|x-3| + 5\cdot\frac{(x-3)^{-1}}{-1} + c = \ln|x-3| - \frac{5}{x-3} + c,
\end{aligned}$$

where $c \in \mathbb{R}$.

Exercise 96. Calculate the integral

$$\int \frac{x+2}{x^3-9x^2+27x-27}\,\mathrm{d}x.$$

<u>**Solution:**</u>

Since $x^3-9x^2+27x-27=(x-3)^3$, therefore

$$\frac{x+2}{(x-3)^3} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3},$$

thus

$$x+2 = A\cdot(x-3)^2 + B\cdot(x-3) + C,$$

that is

$$x+2 = A\cdot(x^2-6x+9) + B\cdot(x-3) + C.$$

The previous equation equvivalent to equation

$$x+2 = A\cdot x^2 + (-6A+B)\cdot x + 9A - 3B + C.$$

The solution of the system of equations

$$\left.\begin{array}{rclcccc}
0 & = & A & & & & \\
1 & = & -6A & + & B & & \\
2 & = & 9A & - & 3B & + & C
\end{array}\right\}$$

is $(A;B;C)=(0;1;5)$, thus

$$\frac{x+2}{(x-3)^3} = \frac{1}{(x-3)^2} + \frac{5}{(x-3)^3}.$$

The final result is

$$\begin{aligned}
&\int \frac{x+2}{(x-3)^3}\,\mathrm{d}x = \int \frac{1}{(x-3)^2} + \frac{5}{(x-3)^3}\,\mathrm{d}x = \\
&= \int \frac{1}{(x-3)^2}\,\mathrm{d}x + \int \frac{5}{(x-3)^2}\,\mathrm{d}x = \\
&= \int (x-3)^{-2}\,\mathrm{d}x + 5\cdot\int (x-3)^{-3}\,\mathrm{d}x = \\
&= \frac{(x-3)^{-1}}{-1} + 5\cdot\frac{(x-3)^{-2}}{-2} + c = -\frac{1}{x-3} - \frac{5}{2\cdot(x-3)^2} + c = \\
&= \frac{-2\cdot(x-3)-5}{2\cdot(x-3)^2} = \frac{-2x+1}{2\cdot(x-3)^2},
\end{aligned}$$

where $c \in \mathbb{R}$.

11. Application of Riemann-integral

Theoretical summary

Theorem 2.4.1 (Newton-Leibniz)**.** If a, b, f are as above and f is continuous, then it is Riemann integrable. Moreover, if F is a primitive function of f, then

$$\int_a^b f(x)\,\mathrm{d}x = [F(x)]_a^b \doteq F(b) - F(a).$$

Definition 2.4.2. Let $f\colon [a;b] \to \mathbb{R}$ be a continuous function. The *average value* of function f is

$$\overline{f} = \frac{1}{b-a} \cdot \int_a^b f(x)\,\mathrm{d}x.$$

Theorem 2.4.3. If $f(x)$ and $g(x)$ are continuous functions on an interval $[a;b]$ such that

$$g(x) \geq f(x)$$

for all $x \in [a;b]$ then the area of the region bounded by their graphs and by the lines $x = a,\ x = b$ can be calculated as follows:

$$A = \int_a^b \big(g(x) - f(x)\big)\,\mathrm{d}x.$$

Remark 2.4.4. Before applying the formula above it is useful to sketch the graphs of the functions. This way we can see which one is the upper $\big(g(x)\big)$ and which is the lower $\big(f(x)\big)$ function. In case if we have to determine the area of the region bounded by the graphs of two real functions f and g, sketching them also helps to find the limits of integration. To be able to do that it is necessary to find the intersection points of the graphs of the functions f and g, thus we have to solve the equation $f(x) = g(x)$. After then we have to integrate the function $g - f$ between the minimum and the maximum of the x coordinates of the intersections, and finally we get the area in question by taking the absolute value of the obtained integral.

Definition 2.4.5. The solid generated by rotating a region on a plane about an axis in that plane is called a *solid of revolution*.

Theorem 2.4.6. If we rotate the graph of a continuous function $f(x)$ $(x \in [a;b];\ a,b \in \mathbb{R},\ a \leq b)$, then the volume of the obtained solid of revolution is

$$V = \pi \cdot \int\limits_a^b \big(f(x)\big)^2 \,\mathrm{d}x.$$

Definition 2.4.7. A *surface of revolution* is formed when a curve is rotated about a line.

Theorem 2.4.8. If a function f is continuously differentiable, then the surface area of the surface obtained by rotating the graph of the function $f(x)$ about the x-axis equals

$$S = 2\pi \cdot \int\limits_a^b f(x) \cdot \sqrt{1 + \big(f'(x)\big)^2} \,\mathrm{d}x.$$

Solved exercieses

Exercise 97. Let v be a velocity-time function, such that

$$v(t) = -t^2 + 4t \qquad \big(t \in [0;4]\big).$$

The unit for time is seconds and unit for velocity is $\left[\frac{\mathrm{m}}{\mathrm{s}}\right]$. Calculate the average velocity on interval $[0;4]$.

Solution:

The average velocity on interval $[a;b]$ is

$$\overline{v} = \frac{1}{b-a} \cdot \int_a^b v(t)\,\mathrm{d}t.$$

Since

$$\int_0^4 -t^2 + 4t\,\mathrm{d}t = \left[-\frac{t^3}{3} + 2t^2\right]_0^4 = -\frac{4^3}{3} + 32 = \frac{32}{3},$$

thus the average velocity is

$$\overline{v} = \frac{1}{4} \cdot \int_0^4 -t^2 + 4t\,\mathrm{d}t = \frac{1}{4} \cdot \frac{32}{3} = \frac{8}{3}\left[\frac{\mathrm{m}}{\mathrm{s}}\right].$$

Exercise 98. Find the area of the region enclosed by the functions

$$f(x) = 2 - x^2 \quad \text{and} \quad g(x) = -x.$$

Solution:

The limits of integration are found by solving the equation

$$2 - x^2 = -x.$$

Applying an algebraic transformation, we get that

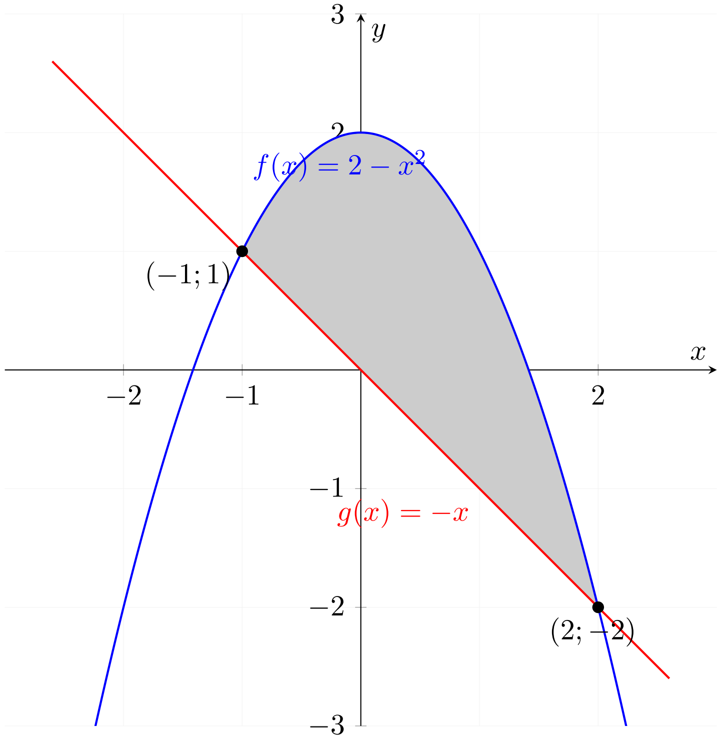
$$x^2 - x - 2 = 0.$$

By the quadratic formula, we get that

$$x_{1,2} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}.$$

Thus the solutions of the equation above: $x_1 = -1$ and $x_2 = 2$.

Consequently, the x coordinates of the leftmost and rightmost points of the region $x = -1$ and $x = 2$, thus the limits of integration are -1 and 2.



The area between the curves is

$$A = \int_{-1}^{2} 2 - x^2 - (-x)\,\mathrm{d}x = \int_{-1}^{2} 2 - x^2 + x\,\mathrm{d}x.$$

We will find a primitve function and using the Newton-Leibniz theorem, we get that

$$A = \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{2} = \left(4 - \frac{8}{3} + 2\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right) =$$

$$= 6 - \frac{8}{3} + 2 - \frac{5}{6} = \frac{9}{2},$$

thus the shaded area is $A = \dfrac{9}{2}$.

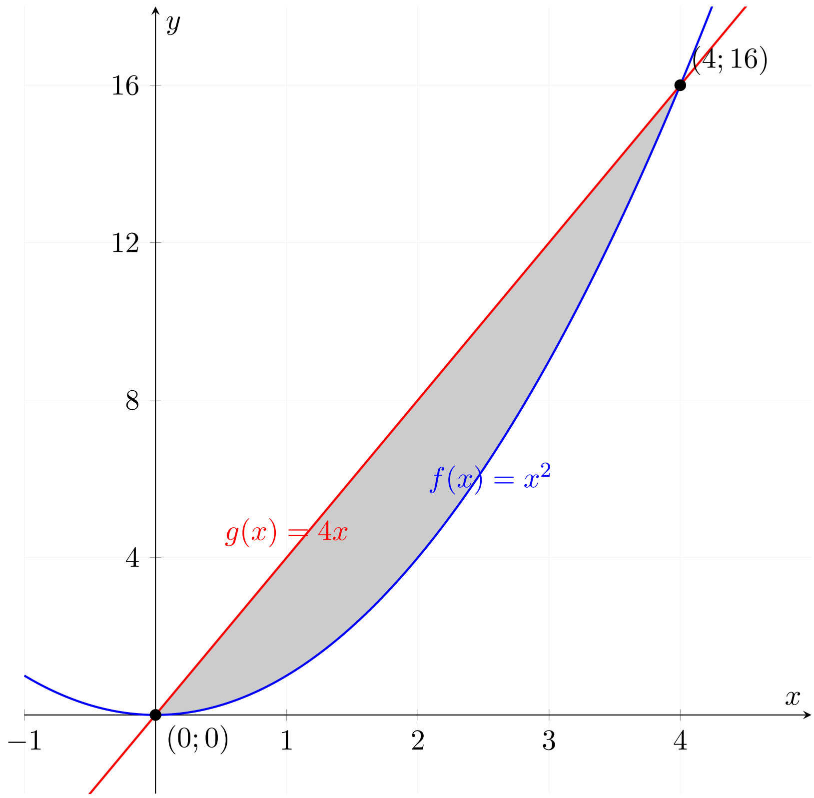
Exercise 99. Find the area of the region enclosed by the functions $f(x) = x^2$ and $g(x) = 4x$.

Solution:

The limits of integration are found by solving the equation $f(x) = g(x)$:

$$x^2 = 4x \qquad \Rightarrow \qquad x^2 - 4x = 0 \qquad \Rightarrow \qquad x \cdot (x-4) = 0.$$

The solutions of the equation above are $x = 0$ and $x = 4$.



Since

$$A = \int_0^4 4x - x^2 \,\mathrm{d}x = \left[2x^2 - \frac{x^3}{3}\right]_0^4 = 32 - \frac{64}{3} = \frac{32}{3},$$

then the area between the curves is $A = \dfrac{32}{3}$.

Exercise 100. Find the area of the region enclosed by the functions $f(x) = 2x^2$ and $g(x) = 2x + 4$.

Solution:

The limits of integration are found by solving the equation $f(x) = g(x)$:

$$2x^2 = 2x + 4.$$

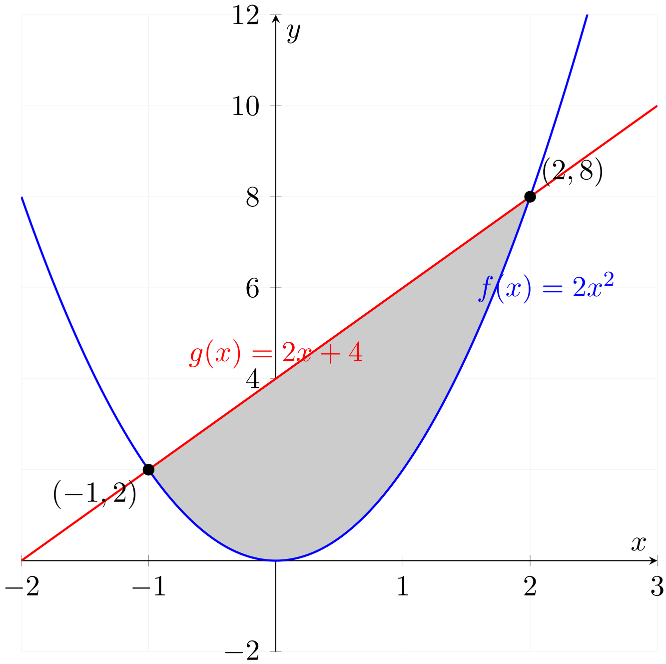
If we write the equation above in another form, we get that:

$$2x^2 - 2x - 4 = 0 \qquad \Rightarrow \qquad x^2 - x - 2 = 0.$$

Applying the quadratic formula, we get that:

$$x_{1,2} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2},$$

thus $x_1 = -1$ and $x_2 = 2$.



The area is

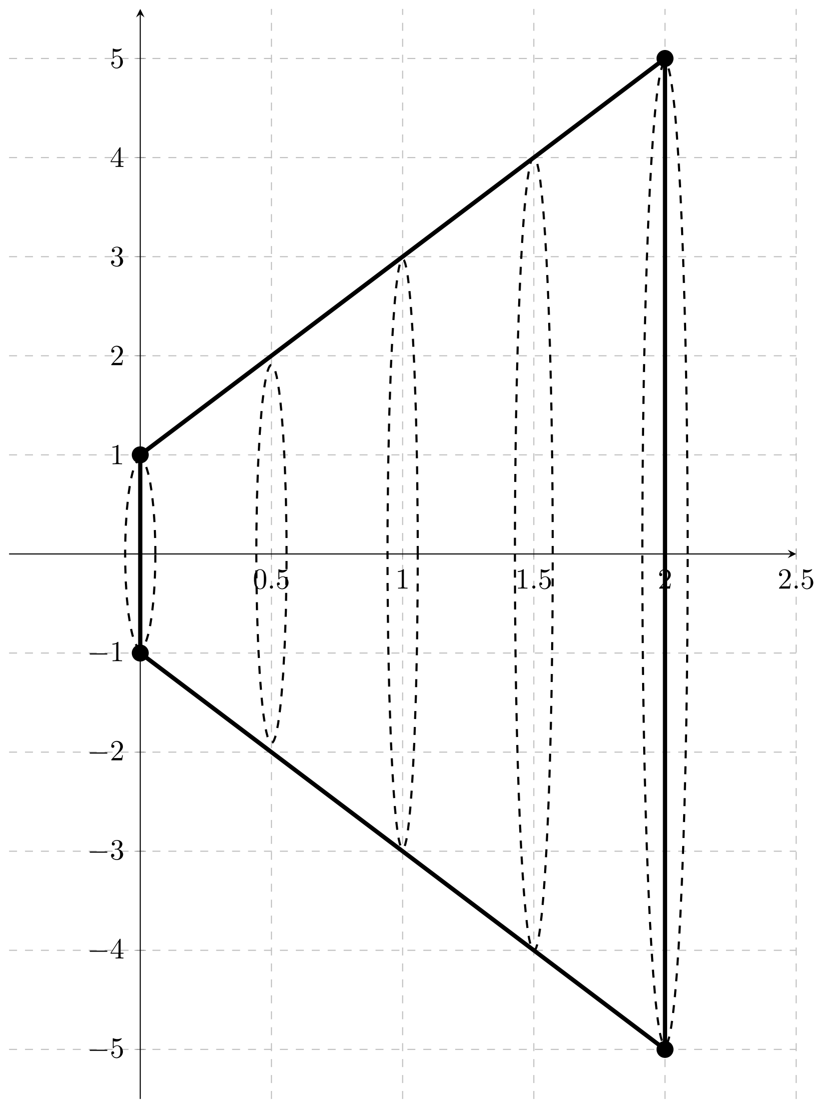
$$A = \int_{-1}^{2} 2x + 4 - 2x^2 \, \mathrm{d}x = \left[x^2 + 4x - \frac{2x^3}{3} \right]_{-1}^{2} =$$

$$= \left(2^2 + 4 \cdot 2 - \frac{2 \cdot 2^3}{3} \right) - \left((-1)^2 + 4 \cdot (-1) - \frac{2 \cdot (-1)^3}{3} \right) = 9.$$

Exercise 101. The region bounded by the graph of the function $f(x) = 2x+1$ $(x \in [0;2])$, the lines $x = 0,\ x = 2$ and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

Solution:

The figure shows the region bounded by function f the x axis on interval $[0;2]$:



The volume is:

$$V = \pi \cdot \int_0^2 (2x+1)^2 \,\mathrm{d}x = \pi \cdot \left[\frac{(2x+1)^3}{6}\right]_0^2 =$$
$$= \pi \cdot \left(\frac{(2\cdot 2+1)^3}{6}\right) - \pi \cdot \left(\frac{(2\cdot 0+1)^3}{6}\right) =$$
$$= \pi \cdot \frac{124}{6} = \frac{62}{3} \cdot \pi.$$

Exercise 102. The region bounded by the graph of the function $f(x) = 2x+6$ ($x \in [0;2]$), the lines $x = 0,\ x = 2$ and the x-axis is rotated about the x-axis to generate a geometric body. Find its surface area.

Solution:

The surface area of the surface of revolution is

$$S = 2\pi \cdot \int_0^2 (2x+6) \cdot \sqrt{5} \,\mathrm{d}x = 2\pi \cdot \sqrt{5} \cdot \left[\frac{(2x+6)^2}{2\cdot 2}\right]_0^2 =$$
$$= 2\pi \cdot \sqrt{5} \cdot \left(\frac{(2\cdot 2+6)^2}{4} - \frac{(2\cdot 0+6)^2}{4}\right) =$$
$$= 2\pi \cdot \sqrt{5} \cdot (25-18) = 14\pi \cdot \sqrt{5}.$$

Exercise 103. The velocity of the particle, which moving along the x-axis is $v(t) = \dfrac{1}{\sqrt{t}}$. At $t = 4$ its position is 2. What is the position of the particle at $t = 9$?

Solution:

The position-time function is

$$s(t) = \int \frac{1}{\sqrt{t}} \,\mathrm{d}t = \int t^{-\frac{1}{2}} \,\mathrm{d}t = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c = 2 \cdot \sqrt{t} + c.$$

Since $s(4) = 2$, we have $2 = 2 \cdot \sqrt{4} + c$, thus $c = -2$. We get that the position-time function is

$$s(t) = 2 \cdot \sqrt{t} - 2.$$

The position at $t = 9$ is $s(9) = 2 \cdot \sqrt{9} - 2 = 4$.

Exercise 104. As a particle moves along the number line, its position at time t is $s(t)$. Its velocity is $v(t)$, and its acceleration is $a(t) = 1$. If $v(3) = -3$ and

$s(2) = -10$, what is $s(4)$?

Solution:

Since $a = 1$ then
$$v(t) = \int 1 \,\mathrm{d}t = t + c.$$
Since $v(3) = -3$, thus $-3 = 3 + c$, that is $c = -6$, therefore
$$v(t) = t - 6.$$
The position-time function is
$$s(t) = \frac{t^2}{2} - 6t + C.$$
Since $s(2) = -10$, therefore $-10 = 2 - 12 + C$, that is $C = 0$, thus the position-time function is
$$s(t) = \frac{t^2}{2} - 6t.$$
The position at 4 is $s(4) = 8 - 24 = -16$.

Exercise 105. A particle moves along the x-axis from $x = 0$ to $x = 10$ m under the influence of a force $F(x) = 5x^2 - 3x + 2$ newtons. Calculate the work done by this force.

Solution:

The work done by a variable force is given by
$$W = \int_{x_1}^{x_2} F(x)\,dx$$
Substituting the given force function:
$$W = \int_0^{10} (5x^2 - 3x + 2)\,dx = \left[\frac{5x^3}{3} - \frac{3x^2}{2} + 2x\right]_0^{10} =$$
$$= \left(\frac{5 \cdot 10^3}{3} - \frac{3 \cdot 10^2}{2} + 2 \cdot 10\right) - \left(\frac{5 \cdot 0^3}{3} - \frac{3 \cdot 0^2}{2} + 2 \cdot 0\right) =$$
$$= \left(\frac{5000}{3} - 150 + 20\right) - 0 = \frac{4610}{3} \approx 1536.7 \text{ J}$$

12. Review exercises - with solution

Exercise 106. Determine the following integrals:

a) $\displaystyle\int \frac{x^3+x^2}{x}\,\mathrm{d}x$

b) $\displaystyle\int \frac{x^2+x}{\sqrt{x}}\,\mathrm{d}x$

c) $\displaystyle\int \frac{\sqrt[3]{x}+\sqrt{x}}{x^2}\,\mathrm{d}x$

d) $\displaystyle\int 4^x+5^x\,\mathrm{d}x$

e) $\displaystyle\int \frac{x^2-9}{x-3}\,\mathrm{d}x$

f) $\displaystyle\int \frac{x^2+4x+4}{x+2}\,\mathrm{d}x$

Solution:

a) By splitting the fraction into a sum of fractions and using the additive property of the integral, we get:

$$\int \frac{x^3+x^2}{x}\,\mathrm{d}x = \int \frac{x^3}{x}+\frac{x^2}{x}\,\mathrm{d}x = \int x^2+x\,\mathrm{d}x = \frac{x^3}{3}+\frac{x^2}{2}+c.$$

b) Using the properties of exponents, the identity $\sqrt{x}=x^{\frac{1}{2}}$, and splitting the fraction into a sum of fractions, then applying the additive property of integrals:

$$\int \frac{x^2+x}{\sqrt{x}}\,\mathrm{d}x = \int \frac{x^2+x}{x^{\frac{1}{2}}}\,\mathrm{d}x = \int \frac{x^2}{x^{\frac{1}{2}}}+\frac{x}{x^{\frac{1}{2}}}\,\mathrm{d}x = \int x^{\frac{3}{2}}+x^{\frac{1}{2}}\,\mathrm{d}x =$$
$$= \frac{2}{5}\cdot x^{\frac{5}{2}}+\frac{2}{3}\cdot x^{\frac{3}{2}}+c = \frac{2}{5}\cdot\sqrt{x^5}+\frac{2}{3}\cdot\sqrt{x^3}+c.$$

c) Using the identities $\sqrt[3]{x}=x^{\frac{1}{3}}$ and $\sqrt{x}=x^{\frac{1}{2}}$, and applying the properties of exponents and the additive property of integrals:

$$\int \frac{\sqrt[3]{x}+\sqrt{x}}{x^2}\,\mathrm{d}x = \int \frac{x^{\frac{1}{3}}}{x^2}+\frac{x^{\frac{1}{2}}}{x^2}\,\mathrm{d}x = \int x^{-\frac{5}{3}}+x^{-\frac{3}{2}}\,\mathrm{d}x =$$
$$= -\frac{3}{2}\cdot x^{-\frac{2}{3}}-2\cdot x^{-\frac{1}{2}}+c = -\frac{3}{2}\cdot\frac{1}{\sqrt{x^3}}-\frac{2}{\sqrt{x}}+c$$

d) Using the additive property of integrals:

$$\int 4^x+5^x\,\mathrm{d}x = \frac{4^x}{\ln 4}+\frac{5^x}{\ln 5}+c.$$

e) Using the identity $x^2 - 9 = (x-3)\cdot(x+3)$:

$$\int \frac{x^2-9}{x-3}\,\mathrm{d}x = \int \frac{(x-3)\cdot(x+3)}{x-3}\,\mathrm{d}x = \int x+3\,\mathrm{d}x = \frac{x^2}{2} + 3x + c$$

f) Using the identity $x^2 + 4x + 4 = (x+2)^2$:

$$\int \frac{x^2+4x+4}{x+2}\,\mathrm{d}x = \int \frac{(x+2)^2}{x+2}\,\mathrm{d}x = \int x+2\,\mathrm{d}x = \frac{x^2}{2} + 2x + c$$

In all of the above, $c \in \mathbb{R}$ is an arbitrary constant.

Exercise 107. Determine the following integrals:

a) $\displaystyle\int 3\cdot\cos x + 4\cdot\sin x\,\mathrm{d}x$

b) $\displaystyle\int \frac{5}{\cos^2 x} + \frac{6}{\sin^2 x}\,\mathrm{d}x$

c) $\displaystyle\int \tan^2\, x\,\mathrm{d}x$

d) $\displaystyle\int \cot^2 x\,\mathrm{d}x$

e) $\displaystyle\int \frac{\cos 2x}{\sin x + \cos x}\,\mathrm{d}x$

f) $\displaystyle\int \frac{\sin 2x}{\cos x}\,\mathrm{d}x$

<u>**Solution:**</u>

a) Using the additive and homogeneous properties of integrals:

$$\int 3\cdot\cos x + 4\cdot\sin x\,\mathrm{d}x = 3\cdot\sin x - 4\cdot\cos x + c$$

b) Using the additive and homogeneous properties of integrals:

$$\int \frac{5}{\cos^2 x} + \frac{6}{\sin^2 x}\,\mathrm{d}x = 5\tan\, x - 6\cdot\cot\, x + c.$$

c) Using the identity $\tan x = \frac{\sin x}{\cos x}$, and the identity $\sin^2 x = 1 - \cos^2 x$, and the additive property of integrals:

$$\begin{aligned}\int \tan^2\, x\,\mathrm{d}x &= \int \frac{\sin^2 x}{\cos^2 x}\,\mathrm{d}x = \int \frac{1-\cos^2 x}{\cos^2 x}\,\mathrm{d}x = \\ &= \int \frac{1}{\cos^2 x} - 1\,\mathrm{d}x = \tan\, x - x + c.\end{aligned}$$

d) Using the identity $\cot x = \dfrac{\cos x}{\sin x}$ and the identity $\cos^2 x = 1 - \sin^2 x$:

$$\begin{aligned}\int \cot^2 x \,\mathrm{d}x &= \int \frac{\cos^2 x}{\sin^2 x}\,\mathrm{d}x = \int \frac{1-\sin^2 x}{\sin^2 x}\,\mathrm{d}x = \\ &= \int \frac{1}{\sin^2 x} - 1 \,\mathrm{d}x = -\cot x - x + c\end{aligned}$$

e) Using the identity $\cos 2x = \cos^2 x - \sin^2 x$:

$$\begin{aligned}\int \frac{\cos 2x}{\sin x + \cos x}\,\mathrm{d}x &= \int \frac{\cos^2 x - \sin^2 x}{\sin x + \cos x}\,\mathrm{d}x = \\ &= \int \frac{(\cos x - \sin x)\cdot(\cos x + \sin x)}{\sin x + \cos x}\,\mathrm{d}x = \\ &= \int \cos x - \sin x \,\mathrm{d}x = \sin x + \cos x + c.\end{aligned}$$

f) Using the identity $\sin 2x = 2\sin x \cdot \cos x$:

$$\begin{aligned}\int \frac{\sin 2x}{\cos x}\,\mathrm{d}x &= \int \frac{2\sin x \cdot \cos x}{\cos x}\,\mathrm{d}x = \int 2\sin x \,\mathrm{d}x = \\ &= -2\cos x + c.\end{aligned}$$

In all of the above, $c \in \mathbb{R}$ is an arbitrary constant.

Exercise 108. If $f(x) = 5x^4 + 3x^2 - 2$ and F is the primitive function of f satisfying $F(1) = 5$, then find the value of $F(2)$.

<u>Solution:</u>

Since

$$F(x) = \int 5x^4 + 3x^2 - 2\,\mathrm{d}x = x^5 + x^3 - 2x + c,$$

then

$$F(2) = 1 + 1 - 2 + c \qquad \Rightarrow \qquad 5 = c.$$

The primitive function is

$$F(x) = x^5 + x^3 - 2x + 5.$$

Applying this result, we get that

$$F(2) = 2^5 + 2^3 - 2\cdot 2 + 5 = 32 + 8 - 4 + 5 = 41.$$

Exercise 109. If $f(x) = x \cdot \mathrm{e}^x$ and F is the primitive function of f satisfying $F(0) = 8$, then find the value of $F(2)$.

<u>Solution:</u>

Applying the partial integral theorem, we get that

$$F(x) = \int x \cdot \mathrm{e}^x \,\mathrm{d}x = x \cdot \mathrm{e}^x - \int 1 \cdot \mathrm{e}^x \,\mathrm{d}x = x \cdot \mathrm{e}^x - \mathrm{e}^x + c.$$

Since

$$F(0) = 0 - 1 + c \quad \Rightarrow \quad 8 = -1 + c \quad \Rightarrow \quad c = 9,$$

thus the primitive function is

$$F(x) = x \cdot \mathrm{e}^x - \mathrm{e}^x + 9.$$

Applying this result, we get that

$$F(2) = 2 \cdot \mathrm{e}^2 - \mathrm{e}^2 + 9 = \mathrm{e}^2 + 9.$$

Exercise 110. If $f(x) = x \cdot \cos x$ and F is the primitive function of f satisfying $F(0) = 2$, then find the value of $F(\pi)$.

Solution:

Applying the partial integral theorem, we get that

$$F(x) = \int x \cdot \cos x \,\mathrm{d}x = x \cdot \sin x - \int \sin x \,\mathrm{d}x = x \cdot \sin x + \cos x + c.$$

Since

$$F(0) = 0 + \cos 0 + c \quad \Rightarrow \quad 2 = 1 + c \quad \Rightarrow \quad c = 1,$$

therefore the primitive function is

$$F(x) = x \cdot \sin x + \cos x + 1.$$

Applying this result, we get that

$$F(\pi) = \pi \cdot \sin \pi + \cos \pi + 1 = -1 + 1 = 0.$$

Exercise 111. What is the area of the region between the graphs of functions

$$f(x) = x^2 + 2x \quad \text{and} \quad g(x) = 2x^2 + 3x - 2?$$

Solution:

In first step we have to solve the equation $f(x) = g(x)$, that is

$$x^2 + 2x = 2x^2 + 3x - 2 \quad \Rightarrow \quad x^2 + x - 2 = 0.$$

Using the quadratic formula, we get that

$$x_{1,2} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2},$$

thus the solutions of the quadratic equation are $x_1 = -2$ and $x_2 = 1$. Applying these results, we get that, the area is

$$A = \int_{-2}^{1} x^2 + 2x - (2x^2 + 3x - 2)\,\mathrm{d}x = \int_{-2}^{1} -x^2 - x + 2\,\mathrm{d}x =$$

$$= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1} = -\frac{1^3}{3} - \frac{1^2}{2} + 2 -$$

$$- \left(-\frac{(-2)^3}{3} - \frac{(-2)^2}{2} - 4\right) = \frac{9}{2}.$$

Exercise 112. What is the area of the region between the graphs of functions

$$f(x) = 3x^2 \quad \text{and} \quad g(x) = 3x + 6?$$

Solution:

In the first step we have to solve the equation $f(x) = g(x)$, that is

$$3x^2 = 3x + 6 \quad \Rightarrow \quad x^2 - x - 6 = 0.$$

Using the quadratic formula, we get that

$$x_{1,2} = \frac{1 \pm \sqrt{1 + 24}}{2} = \frac{1 \pm 5}{2},$$

thus the solutions of the quadratic equation are $x_1 = -2$ and $x_2 = 3$. Applying these results, we get that, the area is

$$A = \int_{-2}^{3} 3x + 6 - 3x^2\,\mathrm{d}x = \left[-\frac{3x^2}{2} + 6x - x^3\right]_{-2}^{3} =$$

$$= -\frac{3 \cdot 3^2}{2} + 6 \cdot 3 - 3^3 - \left(-\frac{3 \cdot (-2)^2}{2} + 6 \cdot (-2) - (-2)^3\right) = \frac{27}{2}.$$

Exercise 113. What is the area of the region between the graphs of functions

$$f(x) = \sqrt{x - 2} \quad \text{and} \quad g(x) = 14 - x \quad \text{and} \quad x = 2?$$

Solution:

In first step we have to solve the equation $f(x) = g(x)$, that is

$$\sqrt{x - 2} = 14 - x \quad \Rightarrow \quad x - 2 = x^2 - 28x + 196.$$

If we reduce to zero, we get that

$$x^2 - 29x + 198 = 0$$

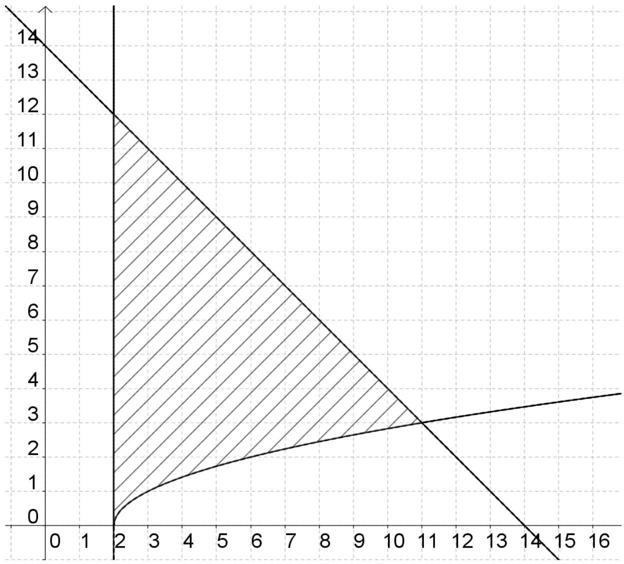
Using the quadratic formula, we get that

$$x_{1,2} = \frac{29 \pm \sqrt{49}}{2} = \frac{29 \pm 7}{2},$$

thus the solutions of the quadratic equation are $x_1 = 11$ and $x_2 = 18$.

If we check, we get that $x_2 = 18$ is not a solution of the equation $f(x) = g(x)$, thus there is only one solution $x_1 = 11$.

If we make a figure, we can see the area:



Applying these results, we get that, the area is

$$A = \int_2^{11} 14 - x - \sqrt{x-2}\,\mathrm{d}x = \left[14x - \frac{x^2}{2} - \frac{2}{3} \cdot (x-2)^{\frac{3}{2}}\right]_2^{11} =$$

$$= 14 \cdot 11 - \frac{11^2}{2} - \frac{2}{3} \cdot 27 - \left(14 \cdot 2 - \frac{2^2}{2} - \frac{2}{3} \cdot 0\right) = \frac{99}{2}.$$

Exercise 114. The velocity of the particle moving along the x-axis is $v(t) = 3\sqrt{t}$. At $t = 4$ its position is 6. What is the position of the particle at $t = 9$?

<u>**Solution:**</u>

The position-time function is

$$s(t) = \int v(t)\,\mathrm{d}t = \int 3\sqrt{t}\,\mathrm{d}t = \int 3t^{\frac{1}{2}}\,\mathrm{d}t = 3\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c = 2t^{\frac{3}{2}} + c = 2\sqrt{t^3} + c.$$

Since $s(4) = 6$, therefore

$$s(4) = 2\cdot\sqrt{4^3} + c \quad \Rightarrow \quad 6 = 16 + c,$$

thus $c = -10$. The position-time function is

$$s(t) = 2\sqrt{t^3} - 10.$$

The position at $t = 9$ is

$$s(9) = 2\cdot\sqrt{9^3} - 10 = 54 - 10 = 44.$$

Exercise 115. Calculate the primitve functions of $f(x) = \tan x$.

Solution:

Since

$$\tan x = \frac{\sin x}{\cos x},$$

thus

$$\int \tan x\,\mathrm{d}x = \int \frac{\sin x}{\cos x}\,\mathrm{d}x = -\int \frac{-\sin x}{\cos x}\,\mathrm{d}x = -\ln|\cos x| + c,$$

where c is an arbitrary real number.

Exercise 116. Calculate the primitve functions of $f(x) = \frac{6x^5}{x^6+2}$.

Solution:

Since the derivative function of $x^6 + 2$ is $6x^5$, therefore

$$\int \frac{6x^5}{x^6+2}\,\mathrm{d}x = \ln|x^6 + 2| + c = \ln(x^6 + 2) + c,$$

where c is an arbitrary real number.

Exercise 117. Find the antiderivative F of the function

$$f(x) = \frac{x^2 - 4}{x - 2},$$

such that $F(2) = 16$ is satisfied! Graph the function.

Solution:

Since $x^2 - 4 = (x-2)\cdot(x+2)$, we have:

$$\frac{x^2-4}{x-2} = \frac{(x-2)\cdot(x+2)}{x-2} = x+2,$$

Thus,

$$\int \frac{x^2-4}{x-2}\,\mathrm{d}x = \int x+2\,\mathrm{d}x = \frac{x^2}{2} + 2x + c,$$

where $c \in \mathbb{R}$. Therefore, the antiderivatives of f are:

$$F(x) = \frac{x^2}{2} + 2x + c.$$

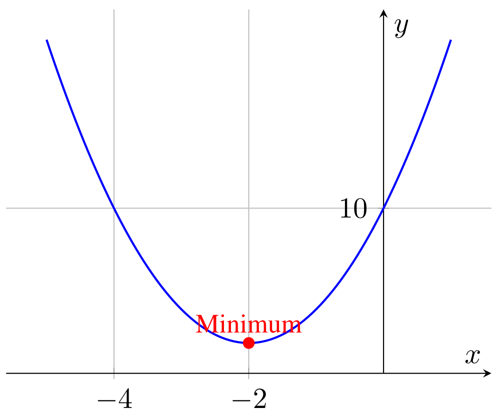
Since $F(2) = 16$, we have:

$$16 = \frac{2^2}{2} + 2\cdot 2 + c \qquad \Rightarrow \qquad c = 10.$$

Therefore, the required antiderivative is:

$$F(x) = \frac{x^2}{2} + 2x + 10 = \frac{1}{2}\cdot(x^2+4x) + 10 = \frac{1}{2}\cdot(x+2)^2 + 8$$

The graph of the function:



Exercise 118. As a particle moves along the number line, its position at time t is $s(t)$. Its velocity-time function is $v(t)$, and its acceleration is $a(t) = 1$. If $v(3) = -3$ and $s(2) = -10$, what is the value of $s(4)$?

<u>Solution:</u>

The velocity-time function is

$$v(t) = \int a(t)\,\mathrm{d}t = \int 1\,\mathrm{d}t = t + c.$$

Since $v(3) = -3$, therefore

$$v(3) = 3 + c \quad \Rightarrow \quad -3 = 3 + c,$$

thus $c = -6$. The velocity-time function is

$$v(t) = t - 6.$$

The position-time function is

$$s(t) = \int v(t)\,\mathrm{d}t = \int t - 6\,\mathrm{d}t = \frac{t^2}{2} - 6t + c.$$

Since $s(2) = -10$, therefore

$$s(2) = 2 - 12 + c \quad \Rightarrow \quad -10 = -10 + c,$$

thus $c = 0$. The position-time function is

$$s(t) = \frac{t^2}{2} - 6t.$$

The position at $t = 4$ is

$$s(4) = 8 - 24 = -16.$$

Exercise 119. Calculate the value of the integral

$$\int_1^{\mathrm{e}} \ln x\,\mathrm{d}x.$$

Solution:

Applying the partial integration formula, we get that

$$\int \ln x\,\mathrm{d}x = \int 1 \cdot \ln x\,\mathrm{d}x = x \ln x - \int x \cdot \frac{1}{x}\,\mathrm{d}x = x \cdot \ln x - x + c.$$

Using the Newton-Leibniz theorem, the final result is

$$\int_1^{\mathrm{e}} \ln x\,\mathrm{d}x = \Big[x \cdot \ln x - x\Big]_1^{\mathrm{e}} =$$
$$= \mathrm{e} \cdot \ln \mathrm{e} - \mathrm{e} - (\ln 1 - 1) = \mathrm{e} - \mathrm{e} - (0 - 1) = 1.$$

Exercise 120. Calculate the integral
$$\int \frac{x+8}{x^2-9}\,\mathrm{d}x.$$

Solution:

Since $x^2-9=(x-3)\cdot(x+3)$, therefore
$$\frac{x+8}{x^2-9}=\frac{x+8}{(x-3)\cdot(x+3)}$$
thus
$$\int \frac{x+8}{x^2-9}\,\mathrm{d}x=\int \frac{x+8}{(x-3)\cdot(x+3)}\,\mathrm{d}x.$$
Applying the partial fraction decomposition, we get that
$$\frac{x+8}{(x-3)\cdot(x+3)}=\frac{A}{x-3}+\frac{B}{x+3}.$$
Since
$$x+8=A\cdot(x+3)+B\cdot(x-3),$$
thus
$$x+8=x\cdot(A+B)+3A-3B.$$
We have the system of equations below
$$\left.\begin{array}{rcrcr} 1 & = & A & + & B \\ 8 & = & 3A & - & 3B \end{array}\right\}.$$
If we multiply by 3 in the both sides of first equation, we get that
$$\left.\begin{array}{rcrcr} 3 & = & 3A & + & 3B \\ 8 & = & 3A & - & 3B \end{array}\right\}.$$
If we add the equations, we get that $6A=11$, thus $A=\frac{11}{6}$. Since $A+B=1$, we have $B=-\frac{5}{6}$. Thus the integal is
$$\int \frac{x+8}{(x-3)\cdot(x+3)}\,\mathrm{d}x=\int \frac{\frac{11}{6}}{x-3}-\frac{\frac{5}{6}}{x+3}\,\mathrm{d}x=$$
$$=\frac{11}{6}\cdot\ln|x-3|+\frac{5}{6}\cdot\ln|x+3|+c,$$
where $c\in\mathbb{R}$.

Exercise 121. Evaluate the indefinite integral

$$\int (x^2 + 7x - 1) \cdot \cos x \, \mathrm{d}x$$

Solution:

For integration by parts, we use $f_1'(x) = \cos x$ and $g_1(x) = x^2 + 7x - 1$. Then:

$$\begin{aligned} f_1(x) &= \sin x & \qquad f_1'(x) &= \cos x \\ g_1(x) &= x^2 + 7x - 1 & \qquad g_1'(x) &= 2x + 7. \end{aligned}$$

Applying the integration by parts formula:

$$\int f_1'(x) \cdot g_1(x) \, \mathrm{d}x = f_1(x) \cdot g_1(x) - \int f_1(x) \cdot g_1'(x) \, \mathrm{d}x$$

After substitution, we get:

$$\int (x^2 + 7x - 1) \cdot \cos x \, \mathrm{d}x = (x^2 + 7x - 1) \cdot \sin x - \int (2x + 7) \cdot \sin x \, \mathrm{d}x.$$

The resulting integral

$$\int (2x + 7) \cdot \sin x \, \mathrm{d}x$$

is also calculated using integration by parts.
Let $f_2'(x) = \sin x$ and $g_2(x) = 2x + 7$. Then:

$$\begin{aligned} f_2(x) &= -\cos x & \qquad f_2'(x) &= \sin x \\ g_2(x) &= 2x + 7 & \qquad g_2'(x) &= 2. \end{aligned}$$

Applying the integration by parts formula again:

$$\int f_2'(x) \cdot g_2(x) \, \mathrm{d}x = f_2(x) \cdot g_2(x) - \int f_2(x) \cdot g_2'(x) \, \mathrm{d}x$$

After substitution, we get:

$$\begin{aligned} \int (2x + 7) \cdot \sin x \, \mathrm{d}x &= -(2x + 7) \cdot \cos x + \int 2 \cdot \cos x \, \mathrm{d}x = \\ &= -(2x + 7) \cdot \cos x + 2 \cdot \sin x + c_1, \end{aligned}$$

where $c_1 \in \mathbb{R}$ is arbitrary.

Using the results obtained above:

$$\int (x^2 + 7x - 1) \cdot \cos x \,\mathrm{d}x =$$
$$(x^2 + 7x - 1) \cdot \sin x - \int (2x + 7) \cdot \sin x \,\mathrm{d}x =$$
$$= (x^2 + 7x - 1) \cdot \sin x - \big(- (2x + 7) \cdot \cos x + 2 \cdot \sin x + c_1 \big) =$$
$$= (x^2 + 7x - 1) \cdot \sin x + (2x + 7) \cdot \cos x - 2 \cdot \sin x - c_1 =$$
$$= (x^2 + 7x - 1) \cdot \sin x + (2x + 7) \cdot \cos x - 2 \cdot \sin x + c$$

where $c \in \mathbb{R}$ is arbitrary.

Exercise 122. Determine the integral

$$\int \frac{2x - 4}{x^3 - x} \,\mathrm{d}x.$$

Solution:

The denominator can be written as a product of first-degree polynomials:

$$\int \frac{2x - 4}{x \cdot (x^2 - 1)} \,\mathrm{d}x = \int \frac{2x - 4}{x \cdot (x + 1) \cdot (x - 1)} \,\mathrm{d}x$$

We decompose the fraction into partial fractions:

$$\frac{2x - 4}{x \cdot (x + 1) \cdot (x - 1)} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{x - 1}.$$

We determine the coefficients A, B, and C. First, we multiply the equation by the common denominator:

$$2x - 4 = A \cdot (x + 1) \cdot (x - 1) + B \cdot x \cdot (x - 1) + C \cdot x \cdot (x + 1).$$

Expanding the parentheses:

$$2x - 4 = A \cdot (x^2 - 1) + B \cdot (x^2 - x) + C \cdot (x^2 + x)$$
$$2x - 4 = A \cdot x^2 - A + B \cdot x^2 - B \cdot x + C \cdot x^2 + C \cdot x.$$

Grouping terms by degree:

$$2x - 4 = (A + B + C) \cdot x^2 + (-B + C) \cdot x - A$$

Two polynomials are equal if and only if the coefficients of corresponding degrees are equal, so we get the following system of equations:

$$\left.\begin{array}{rcrcrcr} 0 & = & A & + & B & + & C \\ 2 & = & & - & B & + & C \\ -4 & = & -A & & & & \end{array}\right\}.$$

From the last equation, we get $A = 4$. Substituting this into the first and second equations:

$$\left.\begin{array}{rcrcr} -4 & = & B & + & C \\ 2 & = & -B & + & C \end{array}\right\}.$$

Adding the equations, we get $2C = -2$, so $C = -1$. Using this, from the first equation we get:

$$B - 1 = -4 \qquad \Rightarrow \qquad B = -3.$$

Substituting the values of A, B, and C into the expression

$$\frac{2x-4}{x \cdot (x+1) \cdot (x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

and using the additive and homogeneous properties of the integral, we get:

$$\int \frac{2x-4}{x \cdot (x^2-1)}\,\mathrm{d}x = \int \frac{2x-4}{x \cdot (x+1) \cdot (x-1)}\,\mathrm{d}x =$$
$$= \int \frac{4}{x} - \frac{3}{x+1} - \frac{1}{x-1}\,\mathrm{d}x = 4\ln|x| - 3\ln|x+1| - \ln|x-1| + c,$$

where $c \in \mathbb{R}$ is arbitrary.

Exercise 123. Find the antiderivative F of the function

$$f(x) = \frac{x+1}{x^3 + 3x^2 + 3x + 1},$$

such that $F(2) = 10$.

<u>Solution:</u>

Since

$$(x+1)^3 = x^3 + 3x^2 + 3x + 1,$$

we have:

$$F(x) = \int \frac{x+1}{x^3+3x^2+3x+1}\,\mathrm{d}x = \int \frac{x+1}{(x+1)^3}\,\mathrm{d}x = \int \frac{1}{(x+1)^2}\,\mathrm{d}x =$$

$$= \int (x+1)^{-2}\,\mathrm{d}x = \frac{(x+1)^{-1}}{-1} = -\frac{1}{x+1} + c,$$

where $c \in \mathbb{R}$ is arbitrary.

Since $F(2) = 10$:

$$10 = -\frac{1}{3} + c \quad \Rightarrow \quad c = \frac{31}{3}.$$

Therefore, the required antiderivative is:

$$F(x) = -\frac{1}{x+1} + \frac{31}{3}.$$

13. Review exercises - without solution

Exercise 124. Find the area of the region bounded by the curve

$$f(x) = x^2 - 4x + 5,$$

the x-axis, and the lines $x = 1$ and $x = 3$.

Exercise 125. Find the volume of the solid obtained by rotating the region bounded b $f(x) = \sqrt{x}$, the x-axis, and the line $x = 4$ around the x-axis.

Exercise 126. Calculate the work done in stretching a spring from its natural length of 10 cm to 15 cm if the force required to stretch the spring x cm beyond its natural length is $F(x) = 2x^2$ newtons.

Exercise 127. A vertical plate in the shape of a semicircle with radius 3 meters is submerged in water with its diameter along the water's surface. Calculate the hydrostatic force on one side of the plate if water has density $\rho = 1\,000$ kg/m^3 and gravitational acceleration is $g = 9.8$ m/s^2.

Exercise 128. Find the average value of the function $f(x) = e^{-x} \cdot \sin(x)$ over the interval $[0; \pi]$.

Exercise 129. A population grows at a rate of

$$\frac{dP}{dt} = 500 \cdot e^{0.03t}$$

individuals per year, where t is measured in years. If the initial population is $10\,000$, find the population after 5 years.

Chapter 3

Differential equations and their applications

14. Directly integrable differential equations

Theoretical summary

Definition 3.1.1. Let $f\colon I \to \mathbb{R}$ be a continuous function. The differential equation of the form $y'(x) = f(x)$ is called a *directly integrable first-order differential equation*.

Theorem 3.1.2. Let $f\colon I \to \mathbb{R}$ be a continuous function.

A differentiable function $y\colon I \to \mathbb{R}$ is a solution of the equation

$$y'(x) = f(x)$$

if and only if $y(x) = F(x) + c$, where $c \in \mathbb{R}$ is arbitrary and F is a primitive function of f.

Remark 3.1.3. The solution of a directly integrable differential equation is essentially obtained by integrating the function f that appears on the right side of the equation given in explicit form.

Example 3.1.4. The equation $y'(x) = \sin x$ is directly integrable. Its solution is

$$y(x) = \int \sin x \, \mathrm{d}x = -\cos x + c,$$

where $c \in \mathbb{R}$ is arbitrary.

Theorem 3.1.5. Let $f\colon I \to \mathbb{R}$ be a continuous function, $x_0 \in I$. The solution of the initial value problem $y'(x) = f(x)$, $y(x_0) = y_0$ is

$$y(x) = y_0 + \int\limits_{x_0}^{x} f(t) \, \mathrm{d}t$$

Example 3.1.6. A point mass moves along a straight line with constant acceleration a. We also know that at time $t = 0$ the velocity is $v(0) = v_0$, and at time $t = 0$ it is at the observation point, i.e., $s(0) = s_0$. We determine the velocity-time and position-time functions.

Since $\dot{v}(t) = a(t) = a$, this is a directly integrable differential equation. Therefore

$$v(t) = \int a \, \mathrm{d}t = a \cdot t + c_1,$$

where $c_1 \in \mathbb{R}$. Since $v(0) = v_0$, we have

$$v_0 = a \cdot 0 + c_1 \qquad \Rightarrow \qquad c_1 = v_0,$$

so we found that the velocity-time function is

$$v(t) = a \cdot t + v_0.$$

Since $\dot{s}(t) = v(t) = a \cdot t + v_0$, this is a directly integrable differential equation. Therefore

$$s(t) = \int a \cdot t + v_0 \,\mathrm{d}t = a \cdot \frac{t^2}{2} + v_0 \cdot t + c_2,$$

where $c_2 \in \mathbb{R}$. Since $s(0) = s_0$, we have

$$s_0 = a \cdot \frac{0^2}{2} + v_0 \cdot 0 + c_2 \qquad \Rightarrow \qquad c_2 = s_0,$$

so we found that the position-time function is

$$s(t) = \frac{a}{2} \cdot t^2 + v_0 \cdot t + s_0.$$

Remark 3.1.7. If we disregard air resistance, then a freely falling body's time to reach the ground does not depend on the body's mass. Related experiments were conducted on the Moon, where they confirmed that this is indeed the case.

The Moon has no significant atmosphere and the gravitational acceleration is also significantly less than on Earth, making it an ideal location to demonstrate that simultaneously dropped, freely falling bodies move identically regardless of their mass and reach the ground at the same time. The experiment was actually performed on the Moon by David Scott, the astronaut of Apollo 15, on August 2, 1971.

Remark 3.1.8. Since in practice a thrown (shot) body is not a point mass, other factors also influence the body's motion. One of the most significant of these is air resistance. The magnitude of the air resistance force depends on the cross-section perpendicular to the body's velocity, the magnitude of the body's velocity, the density of the air, and the shape of the body. However, such types of differential equations are no longer directly integrable; we will deal with these later.

Solved exercises

Exercise 130. Solve the

$$y'(x) = \cos 2x, \quad y(0) = 1$$

initial value problem.

Solution:

If we integrate $\cos 2x$, we get that

$$y(x) = \frac{\sin 2x}{2} + c,$$

where $c \in \mathbb{R}$. Since $y(0) = 1$, therefore

$$1 = y(0) = \frac{\sin 0}{2} + c = c,$$

thus $c = 1$. The solution of the initial value problem is

$$y(x) = \frac{\sin 2x}{2} + 1.$$

Exercise 131. Solve the

$$y'(x) = x \cdot \mathrm{e}^x, \quad y(0) = 3$$

initial value problem.

Solution:

If we apply the partial integral theorem, we get that

$$y(x) = \int x \cdot \mathrm{e}^x \, \mathrm{d}x = x \cdot \mathrm{e}^x - \int \mathrm{e}^x \, \mathrm{d}x = x \cdot \mathrm{e}^x - \mathrm{e}^x + c.$$

Since $y(0) = 3$, we get that

$$3 = y(0) = -\mathrm{e}^0 + c = -1 + c,$$

thus $c = 4$. The solution of the initial value problem is

$$y(x) = (x - 1) \cdot \mathrm{e}^x + 4.$$

Exercise 132. Solve the

$$y'(x) = \frac{2x - 16}{x^2 - x - 2}$$

initial value problem.

Solution:

The roots of the polynom $x^2 - x - 2$ are

$$x_{1,2} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2},$$

that is $x_1 = 2$, $x_2 = -1$. Applying the previous result, we get that

$$\int \frac{2x - 16}{x^2 - x - 2} \, \mathrm{d}x = \int \frac{2x - 16}{(x-2) \cdot (x+1)} \, \mathrm{d}x.$$

Using the partial fraction theorem, we get that

$$\frac{2x - 16}{(x-2) \cdot (x+1)} = \frac{A}{x-2} + \frac{B}{x+1},$$

that is

$$2x - 16 = A \cdot (x+1) + B \cdot (x-2).$$

Applying the algebraic identities, we get that

$$2x - 16 = x \cdot (A + B) + A - 2B,$$

thus we get the system of equations below

$$\begin{aligned} 2 &= A + B \\ -16 &= A - 2B. \end{aligned}$$

The soluiton of the system is $(A; B) = (-4; 6)$, therefore

$$\begin{aligned} &\int \frac{2x - 16}{(x-2) \cdot (x+1)} \, \mathrm{d}x = \int \frac{-4}{x-2} + \frac{6}{x+1} \, \mathrm{d}x = \\ &= -4 \cdot \ln|x-2| + 6 \cdot \ln|x+1| + c. \end{aligned}$$

Using the elementary properties of the logarithmic function, we get that

$$y(x) = \int \frac{2x - 16}{(x-2) \cdot (x+1)} \, \mathrm{d}x = \ln \left| \frac{(x+1)^6}{(x-2)^4} \right| + c = \ln \frac{(x+1)^6}{(x-2)^4} + c,$$

where $c \in \mathbb{R}$.

15. Separable differential equations

Theoretical summary

Definition 3.2.1. Let I and J be open intervals, and let $g\colon I \to \mathbb{R}$ and $h\colon J \to \mathbb{R}$ be continuous functions. The equation

$$y'(x) = g(x) \cdot h\big(y(x)\big)$$

is called a *differential equation with separable variables* or *separable differential equation*.

Theorem 3.2.2. Let I and J be open intervals, $g\colon I \to \mathbb{R}$ and $h\colon J \to \mathbb{R}$ be continuous functions. A function $y\colon I \to \mathbb{R}$ is a solution to the equation

$$y'(x) = g(x) \cdot h\big(y(x)\big)$$

if and only if

$$\int \frac{1}{h(y)}\,\mathrm{d}y = \int g(x)\,\mathrm{d}x,$$

that is, $H\big(y(x)\big) = G(x)$, where H is a primitive function of $1/h(x)$ and G is a primitive function of $g(x)$.

Solved exercises

Exercise 133. Are the following equations separable?

a) $y'(x) = 2x \cdot y(x) + y(x)$;

b) $y'(x) = x \cdot y(x) + x^2$;

c) $y'(x) = \mathrm{e}^{y(x)+x}$;

d) $y(x) \cdot y'(x) = \mathrm{e}^{x+y(x)} \cdot \sqrt{1+x^2}$.

Solution:

a) Since
$$y'(x) = (2x+1) \cdot y(x)$$
thus the equation is separable. Denote $g(x) = 2x+1$ and $h(y) = y$, we get the general form
$$y'(x) = g(x) \cdot h\big(y(x)\big),$$

b) No separable.

c) Since
$$y'(x) = \mathrm{e}^x \cdot \mathrm{e}^{y(x)},$$
therefore the equation is separable. Denote $g(x) = \mathrm{e}^x$ and $h(y) = \mathrm{e}^y$, we get that
$$y'(x) = g(x) \cdot h\big(y(x)\big).$$

d) Since
$$y'(x) = \frac{\mathrm{e}^x \cdot \mathrm{e}^{y(x)} \cdot \sqrt{1+x^2}}{y(x)}$$
we get that
$$g(x) = \mathrm{e}^x \cdot \sqrt{1+x^2}$$
and
$$h(y) = \frac{\mathrm{e}^y}{y}$$
thus
$$y'(x) = g(x) \cdot h\big(y(x)\big),$$
therefore the equation is separable.

Exercise 134. Solve the initial value problem below

$$y'(x) = x \cdot y(x), \quad y(0) = 1$$

Solution:

If $g(x) = x$ and $h(y) = y$ then

$$y'(x) = g(x) \cdot h(y).$$

We have to solve the equation

$$\int \frac{1}{h(y)} \,\mathrm{d}y = \int g(x) \,\mathrm{d}x$$

for y, that is

$$\int \frac{1}{y} \,\mathrm{d}y = \int x \,\mathrm{d}x.$$

If we integrate, we get that

$$\ln|y| = \frac{x^2}{2} + c,$$

thus

$$y = \mathrm{e}^{\frac{x^2}{2}+c} = \mathrm{e}^{\frac{x^2}{2}} \cdot \mathrm{e}^c = C \cdot \mathrm{e}^{\frac{x^2}{2}}.$$

By the initial value condition $y(0) = 1$, we get that

$$1 = y(0) = C \cdot \mathrm{e}^0 = C,$$

therefore the solution is

$$y(x) = \mathrm{e}^{\frac{x^2}{2}}.$$

Exercise 135. Solve the initial value problem below

$$y'(x) = \frac{y(x)+1}{2x+6}.$$

Solution:

If $g(x) = \frac{1}{2x+6}$ and $h(y) = y + 1$, then

$$y'(x) = g(x) \cdot h(y),$$

By the general theorem, we have to solve the equation

$$\int \frac{1}{h(y)} \,\mathrm{d}y = \int g(x) \,\mathrm{d}x$$

for y, that is

$$\int \frac{1}{y+1}\,\mathrm{d}y = \int \frac{1}{2x+6}\,\mathrm{d}x.$$

If we integrate, we get that

$$\ln|y+1| = \frac{1}{2}\cdot\ln|2x+6| + c$$

that is

$$y(x) = \mathrm{e}^{\ln\sqrt{2x+6}+c} - 1 = \sqrt{2x+6}\cdot\mathrm{e}^{c} - 1 = C\cdot\sqrt{2x+6} - 1,$$

where $x \geq -3$.

Exercise 136. Solve the equation

$$y'(x) = \mathrm{e}^{x+y(x)}.$$

<u>Solution:</u>

Applying the algebraic identity, we get that

$$y'(x) = \mathrm{e}^{x}\cdot\mathrm{e}^{y(x)}.$$

If $g(x) = \mathrm{e}^{x}$ and $h(y) = \mathrm{e}^{y}$, we get that

$$y'(x) = g(x)\cdot h(y).$$

We have to solve the equation

$$\int \frac{1}{h(y)}\,\mathrm{d}y = \int g(x)\,\mathrm{d}x$$

for y, that is

$$\int \frac{1}{\mathrm{e}^{y}}\,\mathrm{d}y = \int \mathrm{e}^{x}\,\mathrm{d}x.$$

If we integrate, we get that

$$-\mathrm{e}^{-y} = \mathrm{e}^{x} + c,$$

thus the solution is

$$y(x) = -\ln\big(-\mathrm{e}^{x} - c\big).$$

Exercise 137. Solve the initial value problem below

$$y'(x) = -2y^{2}(x)\cdot x, \quad y(0) = -\frac{1}{2}.$$

<u>Solution:</u>

If $g(x) = -2x$ and $h(y) = y^2$, we get that

$$y'(x) = g(x) \cdot h(y).$$

We have to solve the equation

$$\int \frac{1}{h(y)} \, \mathrm{d}y = \int g(x) \, \mathrm{d}x$$

for y, that is

$$\int \frac{1}{y^2} \, \mathrm{d}y = \int -2x \, \mathrm{d}x.$$

If we integrate, we get that

$$-\frac{1}{y} = -x^2 + c,$$

thus

$$y = \frac{1}{x^2 - c}.$$

By the initial value condition $y(0) = -\frac{1}{2}$, we get that

$$-\frac{1}{2} = \frac{1}{-c} \qquad \Rightarrow \qquad c = 2,$$

therefore the solution is

$$y(x) = \frac{1}{x^2 - 2}.$$

Exercise 138. Solve the differential equation

$$y'(x) = \sin^2\big(y(x)\big) - 1 + x \cdot \cos^2\big(y(x)\big).$$

<u>**Solution:**</u>

Since $\sin^2 y(x) = 1 - \cos^2 y(x)$, therefore

$$y'(x) = \cos^2\big(y(x)\big) \cdot (x - 1).$$

If $g(x) = x - 1$ and $h(y) = \cos^2 y$ we have to solve the equation

$$\int \frac{1}{h(y)} \, \mathrm{d}y = \int g(x) \, \mathrm{d}x$$

for y, that is

$$\int \frac{1}{\cos^2 y} \, \mathrm{d}y = \int x - 1 \, \mathrm{d}x.$$

If we integrate, we get that

$$\operatorname{tg} y = \frac{x^2}{2} - x + c,$$

that is

$$y(x) = \operatorname{arctg}\left(\frac{x^2}{2} - x + c\right),$$

where $c \in \mathbb{R}$.

Exercise 139. A fluid droplet evaporates at a rate proportional to its surface area. At the beginning of observation, a fluid droplet is spherical with a radius of $5\,[\mathrm{mm}]$.

a) Express the radius of the fluid droplet as a function of time!

b) We observed that after 5 seconds, the fluid droplet had a radius of $3\,\mathrm{mm}$. Determine the value of the coefficient that characterizes evaporation!

c) How many seconds after the observation will the fluid droplet's radius be $1\,[\mathrm{mm}]$?

Solution:

a) The rate of change of the fluid droplet's radius with respect to time is proportional to the surface area of the sphere, so we can write the

$$\dot{r}(t) = -k \cdot 4r^2(t) \cdot \pi$$

differential equation. This is a separable equation.

Let $h(r) = r^2$ and $g(t) = -4k \cdot \pi$. We need to solve the

$$\int \frac{1}{h(r)}\,\mathrm{d}r = \int g(t)\,\mathrm{d}t$$

equation for the unknown function r. After substituting the data, we get

$$\int \frac{1}{r^2}\,\mathrm{d}r = \int -k \cdot \pi\,\mathrm{d}t.$$

After performing the integrations

$$-\frac{1}{r} = -k \cdot \pi \cdot t + c$$

is obtained, from which

$$r(t) = \frac{1}{4k \cdot \pi \cdot t - c}.$$

Since $r(0) = 5$, we have

$$5 = -\frac{1}{c} \qquad \Rightarrow \qquad c = -\frac{1}{5},$$

so the radius of the fluid droplet as a function of time is

$$r(t) = \frac{1}{4k \cdot \pi \cdot t + \frac{1}{5}} = \frac{5}{20k \cdot \pi \cdot t + 1}.$$

b) Since $r(5) = 3$, we have

$$3 = \frac{5}{100k \cdot \pi + 1} \qquad \Rightarrow \qquad 300k \cdot \pi + 3 = 5,$$

from which we find that the value of the desired constant is

$$k = \frac{1}{150\pi}.$$

c) Using that $k = \frac{1}{150\pi}$, we get that the radius of the fluid droplet as a function of time is

$$r(t) = \frac{5}{20 \cdot \frac{1}{150\pi} \cdot t + 1} = \frac{5}{\frac{2}{15}t + 1} = \frac{75}{2t + 15}.$$

We are looking for the solution to the equation $r(t) = 1$, so we need to solve the

$$1 = \frac{75}{2t + 15}$$

equation. Multiplying by the common denominator, then rearranging the equation

$$2t + 15 = 75 \qquad \Rightarrow \qquad t = 30$$

is obtained, so the radius of the fluid droplet will be $1\,[\mathrm{mm}]$ after 30 seconds from the beginning of the observation.

Exercise 140. A rock sample from a mine contains $100\,[\mathrm{mg}]$ of uranium and $14\,[\mathrm{mg}]$ of lead. The half-life of uranium is known to be $4.5 \cdot 10^9$ years, and that the complete decay of 238 [g] of uranium produces 206 [g] of lead. Determine the age of the rock. (Assume that at the moment of its formation, the rock did not contain any lead.)

Solution:

If $238\,[\mathrm{g}]$ of uranium produces $206\,[\mathrm{g}]$ of lead, then $14\,[\mathrm{mg}]$ of lead is the product of the complete decay of

$$14 \cdot \frac{238}{206} = 16.1748\,[\mathrm{mg}]$$

uranium. Initially, the rock contained

$$100 + 16.1748\,[\mathrm{mg}]$$

of uranium. For the decay of uranium, we can write the

$$\dot{N}(t) = -\lambda \cdot N(t)$$

differential equation.

To solve the differential equation, let's consider the functions $g(t) = -\lambda$ and $h(N) = N$. Then the equation

$$\int \frac{1}{h(N)}\,\mathrm{d}N = \int g(t)\,\mathrm{d}t$$

must be satisfied. After substituting the data, we get

$$\int \frac{1}{N}\,\mathrm{d}N = \int -\lambda\,\mathrm{d}t.$$

Performing the integrations

$$\ln N = -\lambda \cdot t + c$$

is obtained, from which we get

$$N(t) = \mathrm{e}^{-\lambda\cdot t + c} = \mathrm{e}^{-\lambda\cdot t} \cdot \mathrm{e}^{c} = C \cdot \mathrm{e}^{-\lambda\cdot t}.$$

Since the half-life of uranium is $4.5 \cdot 10^9$ years, we have

$$N(4.5 \cdot 10^9) = \frac{N(0)}{2},$$

so

$$N(0) \cdot \mathrm{e}^{-4.5\cdot 10^9 \lambda} = \frac{N(0)}{2},$$

from which $\lambda = 1.54 \cdot 10^{-10}$. Using this

$$116.1748 \cdot \mathrm{e}^{-1.54\cdot 10^{-10} t} = 100,$$

which gives us that the rock is $t = 9.73 \cdot 10^8$ years old.

Exercise 141. A tank contains 100 liters of salt solution, which contains 8 grams of salt. Water is added to the tank at a rate of 4 liters per minute, and solution is drained at a rate of 2 liters per minute. How much salt remains in the solution after one hour?

Solution:

Let s denote the amount of salt, and t the elapsed time.

The amount of solution in the tank after t minutes is

$$100 - 2t + 4t = 100 + 2t.$$

The concentration of the solution at this time is

$$\frac{s(t)}{100 + 2t}.$$

The amount of solution flowing out during time Δt is $2\Delta t$. The change in the amount of salt during time Δt is

$$\Delta s(t) = -\frac{s(t)}{100 + 2t} \cdot 2\Delta t,$$

which rearranged gives

$$\frac{\Delta s}{\Delta t} = -\frac{2s(t)}{100 + 2t}$$

If we take the limit as $\Delta t \to 0$, then we get

$$\dot{s}(t) = -\frac{2s(t)}{100 + 2t} \qquad \Rightarrow \qquad \dot{s}(t) = -\frac{s(t)}{50 + t}.$$

Let's introduce the functions $h(s) = s$ and $g(t) = -\frac{1}{50+t}$.

According to the general theory of separable equations, we need to solve the

$$\int \frac{1}{h(s)}\,\mathrm{d}s = \int g(t)\,\mathrm{d}t$$

equation for the unknown function s. After substituting the data, we get

$$\int \frac{1}{s}\,\mathrm{d}s = \int -\frac{1}{50 + t}\,\mathrm{d}t.$$

Performing the integrations

$$\ln s = -\ln(50 + t) + c$$

is obtained, from which we get

$$s(t) = \frac{c}{50+t}.$$

Since there were 8 grams of salt in the solution at time 0, we need to consider the initial condition $s(0) = 8$. Using this

$$8 = \frac{c}{50} \qquad \Rightarrow \qquad c = 400,$$

so the amount of salt in the solution (as a function of time) is

$$s(t) = \frac{400}{50+t}.$$

After one hour (60 minutes), the amount of salt is

$$s(60) = \frac{400}{50+60} = \frac{400}{110} \approx 3.6\,[\mathrm{g}].$$

16. Linear homogeneous differential equations

Theoretical summary

Definition 3.3.1. Let $a\colon I \to \mathbb{R}$ be a continuous function. The

$$y'(x) + a(x) \cdot y(x) = 0$$

differential equation is called a *first-order, linear, homogeneous differential equation*.

Example 3.3.2. The

$$y'(x) - 6x \cdot y(x) = x^2 + 2x - 1$$

differential equation is a first-order, linear, variable coefficient, homogeneous equation.

Theorem 3.3.3. The general solution of the differential equation in the previous definition is

$$y(x) = C \cdot \mathrm{e}^{-\int a(x)\,\mathrm{d}x},$$

where $C \in \mathbb{R}$ is arbitrary.

Example 3.3.4. The equation

$$y'(x) + 2x \cdot y(x) = 0$$

is a first-order, linear, homogeneous differential equation. The function a from the general theory is in this case $a(x) = 2x$. The general solution of the equation is

$$y(x) = C \cdot \mathrm{e}^{-\int 2x\,\mathrm{d}x} = C \cdot \mathrm{e}^{-x^2}.$$

Solved exercises

Exercise 142. Solve the equation

$$y'(x) + 8x \cdot y(x) = 0$$

<u>Solution:</u>

Since

$$\int 8x \, \mathrm{d}x = 4x^2 + c$$

for all $c \in \mathbb{R}$, thus

$$y(x) = C \cdot \mathrm{e}^{-\int 8x \, \mathrm{d}x} = C \cdot \mathrm{e}^{-4x^2},$$

where $C \in \mathbb{R}$.

Exercise 143. Solve the equation

$$y'(x) + x \cdot \mathrm{e}^x \cdot y(x) = 0$$

<u>Solution:</u>

If $f'(x) = \mathrm{e}^x$ and $g(x) = x$, thus

$$f(x) = \int \mathrm{e}^x \, \mathrm{d}x = \mathrm{e}^x,$$

and $g'(x) = 1$. If we apply the formula

$$\int f'(x) \cdot g(x) \, \mathrm{d}x = f(x) \cdot g(x) - \int f(x) \cdot g'(x) \, \mathrm{d}x,$$

we get that

$$\begin{aligned} \int x \cdot \mathrm{e}^x \, \mathrm{d}x &= x \cdot \mathrm{e}^x - \int \mathrm{e}^x \, \mathrm{d}x = \\ &= x \cdot \mathrm{e}^x - \mathrm{e}^x + c = \mathrm{e}^x \cdot (x-1) + c, \end{aligned}$$

where $c \in \mathbb{R}$. The solution is

$$y(x) = C \cdot \mathrm{e}^{-\int x \cdot \mathrm{e}^x \, \mathrm{d}x} = C \cdot \mathrm{e}^{\mathrm{e}^x \cdot (1-x)},$$

where $C \in \mathbb{R}$.

Exercise 144. Solve the equation

$$y'(x) + \frac{2x+3}{x^2+5x+6} \cdot y(x) = 0$$

Solution:

In first step we have to calculate the integral below

$$\int \frac{2x+3}{x^2+5x+6}\,\mathrm{d}x$$

Since

$$x_{1,2} = \frac{-5 \pm \sqrt{25-24}}{2} = \frac{-5 \pm 1}{2},$$

that is $x_1 = -3$ and $x_2 = -2$ thus

$$x^2 + 5x + 6 = (x+2) \cdot (x+3),$$

that is

$$\int \frac{2x+3}{x^2+5x+6}\,\mathrm{d}x = \int \frac{2x+3}{(x+2)\cdot(x+3)}\,\mathrm{d}x.$$

If we apply the partial fraction decomposition, we get that

$$\frac{2x+3}{(x+2)\cdot(x+3)} = \frac{A}{x+2} + \frac{B}{x+3},$$

that is

$$2x + 3 = A \cdot (x+3) + B \cdot (x+2),$$

thus

$$2x + 3 = x \cdot (A+B) + 3A + 2B.$$

From the previous result, we get the system of equation

$$\begin{aligned} 2 &= A + B \\ 3 &= 3A + 2B. \end{aligned}$$

The solution of the system is $(A; B) = (-1; 3)$, thus

$$\begin{aligned} \int \frac{2x+3}{(x+2)\cdot(x+3)}\,\mathrm{d}x &= \int \frac{-1}{x+2} + \frac{3}{x+3}\,\mathrm{d}x = \\ &= -1 \cdot \ln|x+2| + 3 \cdot \ln|x+3| + c, \end{aligned}$$

where $c \in \mathbb{R}$ tetszőleges.

Using the elementary properties of logarithmic function, we get that

$$\int \frac{2x+3}{(x+2)\cdot(x+3)}\,\mathrm{d}x = \ln\left|\frac{(x+3)^3}{x+2}\right| + c,$$

therefore

$$y(x) = C\cdot \mathrm{e}^{-\ln\left|\frac{(x+3)^3}{x+2}\right|} = C\cdot\frac{x+2}{(x+3)^3},$$

where $C\in\mathbb{R}$.

17. Second ordered linear homogeneous differential equations

Theoretical summary

Definition 3.4.1. Let $p, q \in \mathbb{R}$. The characteristic equation of the second ordered equation

$$y''(x) + p \cdot y'(x) + q \cdot y(x) = 0$$

is

$$\lambda^2 + p \cdot \lambda + q = 0.$$

Example 3.4.2. The characteristic equation of the differential equation

$$y''(x) - y'(x) - 6y(x) = 0$$

is

$$\lambda^2 - \lambda - 6 = 0.$$

Theorem 3.4.3. Let $p, q \in \mathbb{R}$. The general solution of the differential equation

$$y''(x) + p \cdot y'(x) + q \cdot y(x) = 0$$

is

$$y(x) = c_1 \cdot \mathrm{e}^{\lambda_1 \cdot x} + c_2 \cdot \mathrm{e}^{\lambda_2 \cdot x},$$

where $c_1, c_2 \in \mathbb{R}$ is λ_1 and λ_2 are real numbers and $\lambda_1 \neq \lambda_2$.
The general solution is

$$y(x) = c_1 \cdot \mathrm{e}^{\lambda \cdot x} + c_2 \cdot x \cdot \mathrm{e}^{\lambda \cdot x},$$

where $c_1, c_2 \in \mathbb{R}$ if $\lambda_1 = \lambda_2 = \lambda$.
If λ is a complex number, whose real part is α and imaginary part is β, that is $\lambda = \alpha + i \cdot \beta$, then the general solution of the differential equation is

$$y(x) = c_1 \cdot \mathrm{e}^{\alpha \cdot x} \cdot \cos(\beta \cdot x) + c_2 \cdot \mathrm{e}^{\alpha \cdot x} \cdot \sin(\beta \cdot x),$$

where $c_1, c_2 \in \mathbb{R}$.

Example 3.4.4. Let $x \in \mathbb{R}$. The characteristic equation of the differential equation

$$y''(x) - y'(x) - 6y(x) = 0$$

is

$$\lambda^2 - \lambda - 6 = 0.$$

The soluitons of the algebraic equation are

$$\lambda_{1,2} = \frac{1 \pm \sqrt{D}}{2} = \frac{1 \pm 5}{2},$$

that is $\lambda_1 = 3$ and $\lambda_2 = -2$.

The general solution of the differential equation is

$$y(x) = c_1 \cdot \mathrm{e}^{\lambda_1 \cdot x} + c_2 \cdot \mathrm{e}^{\lambda_2 \cdot x} = c_1 \cdot \mathrm{e}^{3x} + c_2 \cdot \mathrm{e}^{-2x},$$

where $c_1, c_2 \in \mathbb{R}$.

Example 3.4.5. Let $x \in \mathbb{R}$. The characteristic equation of the differential equation

$$y''(x) + 2y'(x) + y(x) = 0$$

is

$$\lambda^2 + 2\lambda + 1 = 0.$$

The solution of the algebraic equation is $\lambda = -1$. The general solution of the equation is

$$y(x) = c_1 \cdot \mathrm{e}^{\lambda x} + c_2 \cdot x \cdot \mathrm{e}^{\lambda x} = c_1 \cdot \mathrm{e}^{-x} + c_2 \cdot x \cdot \mathrm{e}^{-x},$$

where $c_1, c_2 \in \mathbb{R}$.

Example 3.4.6. Let $x \in \mathbb{R}$. The characteristic equation of the differential equation

$$y''(x) + 4y'(x) + 5y(x) = 0$$

is

$$\lambda^2 + 4\lambda + 5 = 0.$$

The solutions of the algebraic equation are

$$\lambda_{1,2} = \frac{-4 \pm \sqrt{D}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i.$$

Let $\alpha = -2$ and $\beta = 1$. The general solution of the differential equation is

$$y(x) = c_1 \cdot \mathrm{e}^{\alpha x} \cdot \cos(\beta x) + c_2 \cdot \mathrm{e}^{\alpha x} \cdot \sin(\beta x),$$

that is

$$y(x) = c_1 \cdot \mathrm{e}^{-2x} \cdot \cos x + c_2 \cdot \mathrm{e}^{-2x} \cdot \sin x,$$

where $c_1, c_2 \in \mathbb{R}$.

Example 3.4.7. Let $x \in \mathbb{R}$. Solve the

$$y''(x) + 7y'(x) + 10y(x) = 0, \quad y(0) = 2,\, y'(0) = 5$$

initial value problem.

The characteristic equation of the differential equation is

$$\lambda^2 + 7\lambda + 10 = 0.$$

The solutions of the algebraic equation are

$$\lambda_{1,2} = \frac{-7 \pm \sqrt{D}}{2} = \frac{-7 \pm 3}{2},$$

that is $\lambda_1 = -5$ and $\lambda_2 = -2$.

The general solution of the differential equation is

$$y(x) = c_1 \cdot \mathrm{e}^{\lambda_1 \cdot x} + c_2 \cdot \mathrm{e}^{\lambda_2 \cdot x} = c_1 \cdot \mathrm{e}^{-5x} + c_2 \cdot \mathrm{e}^{-2x},$$

where $c_1, c_2 \in \mathbb{R}$.

The derivative function of y is

$$y'(x) = -5c_1 \cdot \mathrm{e}^{-5x} - 2c_2 \cdot \mathrm{e}^{-2x}.$$

By the inital value conditions, we get that

$$2 = y(0) = c_1 \cdot \mathrm{e}^{-5 \cdot 0} + c_2 \cdot \mathrm{e}^{-2 \cdot 0} = c_1 + c_2,$$

and

$$5 = y'(0) = -5c_1 \cdot \mathrm{e}^{-5 \cdot 0} - 2c_2 \cdot \mathrm{e}^{-2 \cdot 0} = -5c_1 - 2c_2.$$

Therefore we have to solve the system of equations below

$$\left.\begin{array}{rcrcr} c_1 & + & c_2 & = & 2 \\ -5c_1 & - & 2c_2 & = & 5 \end{array}\right\}.$$

The solution of the system is $(c_1; c_2) = (-3; 5)$, thus the solution of the initial value problem is

$$y(x) = -3\mathrm{e}^{-5x} + 5\mathrm{e}^{-2x}.$$

Solved exercises

Exercise 145. Let $x \in \mathbb{R}$. Solve the initial value problem

$$y''(x) + 7y'(x) + 10y(x) = 0, \quad y(0) = 2,\, y'(0) = 5.$$

Solution:

The characteristic equation of the differential equation is

$$\lambda^2 + 7\lambda + 10 = 0.$$

The solutions of the algebraic equation are

$$\lambda_{1,2} = \frac{-7 \pm \sqrt{D}}{2} = \frac{-7 \pm 3}{2},$$

that is $\lambda_1 = -5$ and $\lambda_2 = -2$.
The general solution of the differential equation is

$$y(x) = c_1 \cdot \mathrm{e}^{\lambda_1 \cdot x} + c_2 \cdot \mathrm{e}^{\lambda_2 \cdot x} = c_1 \cdot \mathrm{e}^{-5x} + c_2 \cdot \mathrm{e}^{-2x},$$

where $c_1, c_2 \in \mathbb{R}$.
The derivative function of y is

$$y'(x) = -5c_1 \cdot \mathrm{e}^{-5x} - 2c_2 \cdot \mathrm{e}^{-2x}.$$

By the initial value condition, we get that

$$2 = y(0) = c_1 \cdot \mathrm{e}^{-5 \cdot 0} + c_2 \cdot \mathrm{e}^{-2 \cdot 0} = c_1 + c_2,$$

and

$$5 = y'(0) = -5c_1 \cdot \mathrm{e}^{-5 \cdot 0} - 2c_2 \cdot \mathrm{e}^{-2 \cdot 0} = -5c_1 - 2c_2.$$

We have to solve the system of equations below

$$\left.\begin{array}{rcrcr} c_1 & + & c_2 & = & 2 \\ -5c_1 & - & 2c_2 & = & 5 \end{array}\right\}.$$

The solution of the system is $(c_1; c_2) = (-3; 5)$. The solution of the initial value problem is

$$y(x) = -3\mathrm{e}^{-5x} + 5\mathrm{e}^{-2x}.$$

Exercise 146. Let $x \in \mathbb{R}$. Solve the initial value problem

$$y''(x) + 4y'(x) + 4y(x) = 0, \quad y(0) = 1, \, y'(0) = 0.$$

Solution:

The characteristic equation of the differential equation is

$$\lambda^2 + 4\lambda + 4 = 0.$$

The solution of the algebraic equation is $\lambda = -2$. The general solution of the differential equation is

$$y(x) = c_1 \cdot \mathrm{e}^{\lambda x} + c_2 \cdot x \cdot \mathrm{e}^{\lambda x} = c_1 \cdot \mathrm{e}^{-2x} + c_2 \cdot x \cdot \mathrm{e}^{-2x},$$

where $c_1, c_2 \in \mathbb{R}$.

The derivative function of y is

$$y'(x) = -2c_1 \cdot \mathrm{e}^{-2x} + c_2 \cdot \mathrm{e}^{-2x} - 2c_2 \cdot x \cdot \mathrm{e}^{-2x}.$$

By the inital value condition we get that

$$1 = y(0) = c_1,$$

thus $c_1 = 1$. On the other hand

$$0 = y'(0) = -2c_1 \cdot \mathrm{e}^0 + c_2 \cdot \mathrm{e}^0 - 2c_2 \cdot 0 \cdot \mathrm{e}^0 = -2c_1 + c_2.$$

Therefore $c_1 = 1$ and $c_2 = 2$.

The solution of the inital value problem is

$$y(x) = \mathrm{e}^{-2x} + 2x \cdot \mathrm{e}^{-2x}.$$

Exercise 147. Let $x \in \mathbb{R}$. Solve the inital value problem

$$y''(x) + 4y(x) = 0, \quad y(0) = 1, \, y'(0) = 1$$

Solution:

The characteristic equation of the differential equation

$$\lambda^2 + 4 = 0.$$

The solution of the algebraic equation is $\lambda_1 = 2i$, $\lambda_2 = -2i$. Let $\alpha = 0$ and $\beta = 2$. The general solution of the differential equation is

$$y(x) = c_1 \cdot \mathrm{e}^{\alpha x} \cdot \cos(\beta x) + c_2 \cdot \mathrm{e}^{\alpha x} \cdot \sin(\beta x),$$

that is

$$y(x) = c_1 \cdot \cos 2x + c_2 \cdot \sin 2x,$$

where $c_1, c_2 \in \mathbb{R}$.

The derivative function of y is

$$y'(x) = -2c_1 \cdot \sin 2x + 2c_2 \cdot \cos 2x.$$

By the initial value condition, we get that

$$1 = y(0) = c_1,$$

and

$$1 = 2c_2 \qquad \Rightarrow \qquad c_2 = \frac{1}{2}.$$

The solution of the initial value problem is

$$y(x) = \cos 2x + \frac{1}{2} \cdot \sin 2x.$$

Exercise 148. The characteristic equation of the differential equation is

$$\lambda^2 - 5\lambda + 6 = 0.$$

Write down a differential equation!

Solution:

A differential equation is

$$y''(x) - 5y'(x) + 6y(x) = 0.$$

18. Review exercises - with solution

Exercise 149. A car brakes uniformly from a speed of $54\left[\frac{\mathrm{km}}{\mathrm{h}}\right]$ with a deceleration of $5\left[\frac{\mathrm{m}}{\mathrm{s}^2}\right]$. Calculate the braking distance.

Solution:

Since

$$v(0) = v_0 = 54\left[\frac{\mathrm{km}}{\mathrm{h}}\right] = 15\left[\frac{\mathrm{m}}{\mathrm{s}}\right]$$

and $a = -5\left[\frac{\mathrm{m}}{\mathrm{s}^2}\right]$, according to the previous model

$$v(t) = \int a\,\mathrm{d}t = \int -5\,\mathrm{d}t = -5t + c_1.$$

Since $v(0) = 15$, we have $c_1 = 15$, thus

$$v(t) = -5t + 15.$$

We are looking for the time when the car stops, i.e., the value of t for which $v(t) = 0$, so we need to solve the equation

$$0 = 15 - 5t$$

for which we get $t = 3$.

Using the previous model again, the position-time function of the car is

$$s(t) = \int a \cdot t + v_0\,\mathrm{d}t = \int -5 \cdot t + 15\,\mathrm{d}t = -\frac{5}{2} \cdot t^2 + 15t + c_2.$$

Since $s(0) = 0$, we have $c_2 = 0$, so we get that the position-time function of the motion is

$$s(t) = -2.5t^2 + 15t.$$

Therefore

$$s(3) = -2.5 \cdot 3^2 + 15 \cdot 3 = 22.5\,[\mathrm{m}],$$

so the braking distance is $22.5\,[\mathrm{m}]$.

Exercise 150. We shoot an arrow vertically upward from a crossbow. Let the initial velocity of the arrow be $v_0 = 49\left[\frac{\mathrm{m}}{\mathrm{s}}\right]$. How long after the shot the arrow returns to the firing position, if we disregard air resistance.

Solution:

According to the previous model, the position-time function of the vertically fired arrow is

$$h(t) = -\frac{g}{2} \cdot t^2 + v_0 \cdot t = -4.9t^2 + 49t.$$

We are looking for the time when $h(t) = 0$, so we need to solve the equation

$$-4.9t^2 + 49t = 0$$

Converting the left side to a product, we get

$$4.9t \cdot (-t + 10) = 0,$$

from which the solutions of the equation are $t_1 = 0$ and $t_2 = 10$.

So we found that the arrow returns to the firing position 10 seconds after the shot.

Exercise 151. Air at normal pressure in a cylindrical vessel of volume V_0 is compressed adiabatically to volume V. What is the work done?

Solution:

For an adiabatic state change of a gas, the pressure as a function of volume can be described using the Poisson equation:

$$\frac{p}{p_0} = \left(\frac{V_0}{V}\right)^k \qquad \Rightarrow \qquad p(V) = p_0 \cdot \left(\frac{V_0}{V}\right)^k,$$

where p_0 denotes the normal pressure and k is a constant characteristic of the gas being studied. For the work done, the

$$W'(V) = -p(V)$$

differential equation can be written, from which

$$W(V) = \int -p(V)\,\mathrm{d}V.$$

Therefore

$$W(V) = \int -p_0 \cdot \left(\frac{V_0}{V}\right)^k \mathrm{d}V = -p_0 \cdot V_0^k \cdot \frac{V^{-k+1}}{-k+1} + c,$$

where $c \in \mathbb{R}$.

If $V = V_0$, then the work done is 0, which means the initial condition $W(V_0) = 0$. Using this

$$0 = -p_0 \cdot V_0^k \cdot \frac{V_0^{-k+1}}{-k+1} + c \qquad \Rightarrow \qquad c = \frac{p_0 \cdot V_0}{1-k}.$$

Therefore the work done as a function of volume is

$$W(V) = -p_0 \cdot V_0^k \cdot \frac{V^{-k+1}}{-k+1} + \frac{p_0 \cdot V_0}{1-k} = \frac{p_0 \cdot V_0}{1-k} \cdot \left(1 - \left(\frac{V_0}{V}^{k-1}\right)\right).$$

Exercise 152. A small settlement has a population of 100 people. The growth rate is 0.3. The maximum population of the settlement cannot exceed $1,000$ people. (That is, the carrying capacity is $1,000$ people.)

a) Write the differential equation corresponding to the logistic model!
b) Solve the above differential equation, i.e., express the population of the settlement as a function of time!
c) Sketch the graph of the function!

Solution:

a) The differential equation is

$$\dot{N}(t) = 0.3 \cdot N(t) - 0.0003 \cdot N^2(t),$$

or

$$\dot{N}(t) = 0.3 \cdot N(t) \cdot \big(1 - 0.001 \cdot N(t)\big),$$

To solve this differential equation, let's introduce the functions $g(t) = 0.3$ and $h(N) = N \cdot (1 - 0.001N)$! Then the equation

$$\int \frac{1}{h(N)} \, \mathrm{d}N = \int g(t) \, \mathrm{d}t$$

must be satisfied. After substituting the data, we get

$$\int \frac{1}{N \cdot (1 - 0.001N)} \, \mathrm{d}N = \int 0.3 \, \mathrm{d}t.$$

We can evaluate the integral on the left side using the method of partial fractions. Consider the decomposition

$$\frac{1}{N \cdot (1 - 0.001 \cdot N)} = \frac{A}{N} + \frac{B}{1 - 0.001 \cdot N}$$

Multiplying the equation by the common denominator, we get

$$1 = A \cdot (1 - 0.001 \cdot N) + B \cdot N.$$

Arranging the equation by degree, we get

$$1 = N \cdot (B - 0.001A) + A$$

From this we get $A = 1$, so $B = 0.001$. Therefore

$$\frac{1}{N \cdot (1 - 0.001N)} = \frac{1}{N} + \frac{0.001}{1 - 0.001N}.$$

Using this

$$\int \frac{1}{N \cdot (1 - 0.001N)} \,\mathrm{d}N = \int \frac{1}{N} + \frac{0.001}{1 - 0.001N} \,\mathrm{d}N =$$

$$= \int \left(\frac{1}{N} - \frac{-0.001}{1 - 0.001 \cdot N} \right) \mathrm{d}N =$$

$$= (\ln |N| - \ln |1 - 0.001N|) = \ln \left| \frac{N}{1 - 0.001 \cdot N} \right|.$$

On the other hand

$$\int 0.3 \,\mathrm{d}t = 0.3t + c,$$

so we arrive at the equation

$$\ln \left| \frac{N}{1 - 0.001N} \right| = 0.3t + c$$

thus

$$\ln \left| \frac{N}{1 - 0.001N} \right| = 0.3t + c_1.$$

From this

$$\frac{N}{1 - 0.001N} = \mathrm{e}^{0.3 \cdot t} \cdot C$$

is obtained. Multiplying both sides by the denominator, we get

$$N = \mathrm{e}^{0.3 \cdot t} \cdot C \cdot (1 - 0.001N).$$

Solving for N

$$N(t) = \frac{\mathrm{e}^{0.3 \cdot t} \cdot C}{1 + 0.001 \cdot \mathrm{e}^{0.3 \cdot t} \cdot C} = \frac{C}{\mathrm{e}^{-0.3 \cdot t} + 0.001C}$$

is obtained. Using that $N(0) = 100$, we get

$$100 = \frac{C}{1 + 0.001C},$$

thus

$$100 + 0.1C = C \qquad \Rightarrow \qquad C = \frac{1,000}{9}.$$

Using this, we get

$$N(t) = \frac{\frac{1{,}000}{9}}{\mathrm{e}^{-0.3 \cdot t} + \frac{1}{9}} = \frac{1,000}{9 \cdot \mathrm{e}^{-0.3t} + 1}.$$

b) The function N has no zeros.

The limit of the function at ∞ is

$$\lim_{t \to \infty} N(t) = 1,000.$$

The limit of the function at 0 is

$$\lim_{t \to 0} N(t) = 100.$$

The derivative of the function N is

$$\dot{N}(t) = 1,000 \cdot \frac{-9 \cdot \mathrm{e}^{-0.3t} \cdot (-0.3)}{(9 \cdot \mathrm{e}^{-0.3t} + 1)^2} = 1,000 \cdot \frac{2.7 \cdot \mathrm{e}^{-0.3t}}{(9 \cdot \mathrm{e}^{-0.3t} + 1)^2}$$

Since $\dot{N}(t) > 0$, the function is strictly monotonically increasing.

The second derivative of the function N is

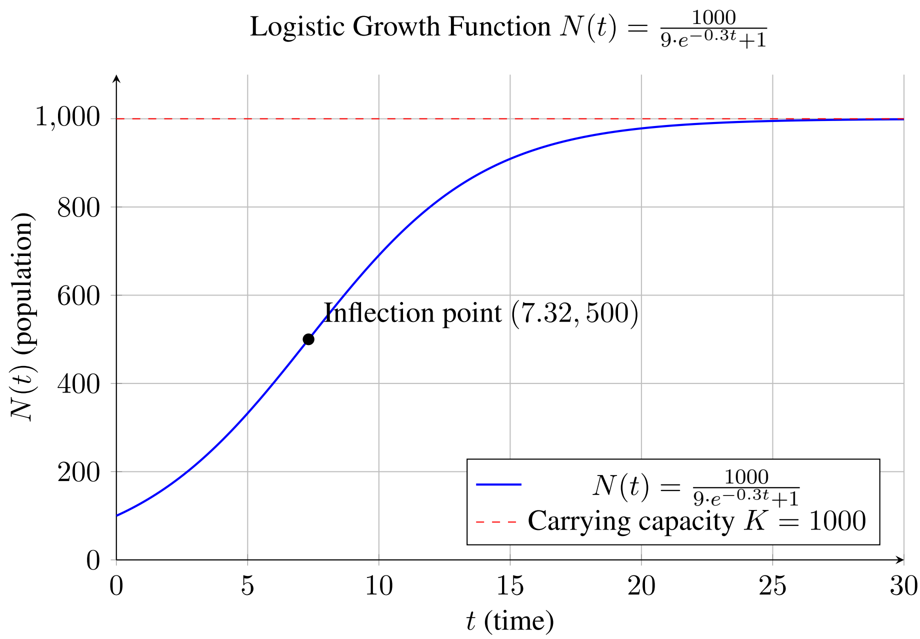
$$\begin{aligned}\ddot{N}(t) = 1,000 \cdot \frac{2.7 \cdot \mathrm{e}^{-0.3t} \cdot (-0.3) \cdot (9 \cdot \mathrm{e}^{-0.3t} + 1)^2}{(9 \cdot \mathrm{e}^{-0.3t} + 1)^4} - \\ - 1,000 \cdot \frac{2.7 \cdot \mathrm{e}^{-0.3t} \cdot 2 \cdot ((9 \cdot \mathrm{e}^{-0.3t} + 1)) \cdot 9 \cdot \mathrm{e}^{-0.3t} \cdot (-0.3)}{(9 \cdot \mathrm{e}^{-0.3t} + 1)^4} = \\ = 1,000 \cdot \frac{2.7 \cdot \mathrm{e}^{-0.3t} \cdot (-0.3) \cdot (9 \cdot \mathrm{e}^{-0.3t} + 1) \cdot (9 \cdot \mathrm{e}^{-0.3t} + 1)}{(9 \cdot \mathrm{e}^{-0.3t} + 1)^4}.\end{aligned}$$

The solution of the equation $\ddot{N}(t) = 0$ is

$$t = \frac{\ln \frac{1}{9}}{-0.3} \approx 7.32.$$

t	$]0; 7.32[$	7.32	$]7.32; \infty[$
$\ddot{N}(t)$	$+$	0	$-$
$N(t)$	convex	inflection point	concave
$N(t)$ value		500	

The graph of function is



Exercise 153. A hard-boiled egg at 98°C is placed in a large bowl of water at 18°C. The egg's temperature decreases to 38°C in 5 minutes. (The temperature of the environment does not change.)

a) Write the differential equation that describes the cooling process!

b) Express the egg's temperature as a function of time!

c) What will be the temperature of the egg 6 minutes after placing it in the bowl?

d) When will the egg's temperature be 20°C?

<u>**Solution:**</u>

a) Let's denote the temperature of the environment (in this case the water in the bowl) by T_k, the temperature of the body at time t by $T(t)$, and its known temperature at time $t = 0$ by T_0!

According to Newton's Law of Cooling, the rate of change $\dot{T}(t)$ of the body's temperature is proportional to the difference between the temperature of the body and its environment, i.e., to $T - T_k$. Using this, we get the

$$\dot{T}(t) = -\kappa \cdot \big(T(t) - T_k\big)$$

differential equation, where κ is a fixed positive constant characteristic of the material and shape of the body.

b) To determine the function T, we have the above differential equation. In addition, we know the value of the function T at a fixed point, since we know that initially the temperature of the body is T_0. Thus, we have an initial value problem for a separable differential equation.

To solve the differential equation, let's consider the functions $g(t) = -\kappa$ and $h(T) = T - T_k$. Then the equation

$$\int \frac{1}{h(T)}\,\mathrm{d}T = \int g(t)\,\mathrm{d}t$$

must be satisfied. After substituting the data, we get

$$\int \frac{1}{T - T_k}\,\mathrm{d}T = \int -\kappa\,\mathrm{d}t.$$

Performing the integrations

$$\ln(T - T_k) = -\kappa \cdot t + c$$

is obtained, from which we get

$$T(t) = T_k + \mathrm{e}^{-\kappa \cdot t + c} = T_k + \mathrm{e}^{-\kappa \cdot t} \cdot \mathrm{e}^{c} = T_k + C \cdot \mathrm{e}^{-\kappa \cdot t}.$$

Since $T(0) = T_0$, we have $T_0 - T_k = C$, thus the temperature of the body at time t is

$$T(t) = T_k + (T_0 - T_k) \cdot \mathrm{e}^{-\kappa \cdot t}.$$

Substituting the data, we get

$$T(t) = 18 + 80 \cdot \mathrm{e}^{-\kappa \cdot t}.$$

Since $T(5) = 38$, we get from the above equation

$$38 = 18 + 80 \cdot \mathrm{e}^{-5\kappa}.$$

Solving for the unknown κ, we get

$$\frac{20}{80} = \mathrm{e}^{-5\kappa} \quad \Rightarrow \quad \ln\frac{1}{4} = -5\kappa \quad \Rightarrow \quad \kappa = \frac{\ln\frac{1}{4}}{-5} = \frac{\ln 4}{5}.$$

Therefore, the temperature of the body at time t is

$$T(t) = 18 + 80 \cdot \mathrm{e}^{-\frac{\ln 4}{5} \cdot t} = 18 + 80 \cdot \left(\mathrm{e}^{\ln 4}\right)^{-\frac{1}{5} \cdot t} = 18 + 80 \cdot 4^{-\frac{t}{5}}.$$

c) The temperature of the egg 6 minutes after placing it in the bowl is

$$T(6) = 18 + 80 \cdot 4^{-\frac{6}{5}} \approx 33.16^\circ \text{ C}$$

d) We are looking for the time t when $T(t) = 20$, so we need to solve the

$$20 = 18 + 80 \cdot 4^{-\frac{t}{5}}$$

exponential equation. Subtracting 18 from both sides, then dividing both sides of the equation by 80, we get

$$\frac{1}{40} = 4^{-\frac{t}{5}}.$$

Taking the base-10 logarithm of both sides

$$\lg \frac{1}{40} = -\frac{t}{5} \cdot \lg 4$$

is obtained, thus

$$t = \frac{5 \cdot \lg 40}{\lg 4} \approx 13.3.$$

Therefore, the egg's temperature will be 20°C after 13.3 minutes.

18. Review exercises - without solution

Exercise 154. Solve the separable differential equation $\frac{dy}{dx} = \frac{x^2}{y^3}$ with the initial condition $y(1) = 2$.

Exercise 155. A population of bacteria grows at a rate proportional to the current population. If the population doubles in 3 hours, and the initial population is 500 bacteria, find an expression for the population as a function of time (in hours).

Exercise 156. A radioactive sample decays at a rate proportional to the amount present. If initially there are 100 grams, and after 5 hours there are 80 grams remaining, determine how much will remain after 24 hours.

Exercise 157. A cup of coffee at $90°\mathrm{C}$ is placed in a room with a constant temperature of $20°\mathrm{C}$. After 10 minutes, the coffee has cooled to $60°\mathrm{C}$. How long will it take for the coffee to cool to $30°\mathrm{C}$?

Exercise 158. A tank initially contains 100 liters of brine with 5 kg of dissolved salt. Fresh water flows into the tank at a rate of 3 liters per minute, and the well-mixed solution flows out at the same rate. How much salt remains in the tank after 30 minutes?

Exercise 159. A population follows the logistic model $\frac{dP}{dt} = 0.2P\left(1 - \frac{P}{1000}\right)$, where P is measured in individuals and t in years. If the initial population is 100 individuals, when will the population reach 500 individuals?

Exercise 160. In a first-order chemical reaction, the rate of decrease of the concentration of a reactant is proportional to its concentration. If the initial concentration is 0.5 mol/L and the half-life of the reaction is 20 minutes, find the concentration after 1 hour.

Exercise 161. A body falls under gravity with air resistance proportional to its velocity. The differential equation modeling this motion is $\frac{dv}{dt} = g - kv$, where $g = 9.8$ m/s^2 and k is a positive constant. If a body reaches a terminal velocity of 49 m/s, find an expression for its velocity as a function of time, assuming it starts from rest.

Exercise 162. In an RC circuit with resistance $R = 100$ ohms and capacitance $C = 0.01$ farads, the charge $q(t)$ on the capacitor satisfies the differential equation $\frac{dq}{dt} = -\frac{q}{RC}$. If the initial charge is $q(0) = 0.5$ coulombs, find the charge as a function of time.

Exercise 163. A spherical water droplet evaporates at a rate proportional to its surface area. If the initial radius is 2 mm and it takes 1 hour for the radius to decrease to 1 mm, how much longer will it take for the droplet to completely evaporate?

References

[1] Bárczy Barnabás, *Differenciálszámítás*, Műszaki Könyvkiadó, Budapest, 1994.

[2] Bíró Fatime – Vincze Szilvia, *A gazdasági matematika alapjai*, Debreceni Egyetemi Kiadó, 2010.

[3] Denkinger Géza – Gyurkó Lajos, *Analízis gyakorlatok*, Nemzeti Tankönyvkiadó, 1987.

[4] Farkas István, *Differenciálszámítás gyakorlati jegyzet*, Debreceni Egyetem, 2005.

[5] Gábos Adél – Halmos Mária, *Készüljünk az érettségire matematikából közép-, emelt szinten*, Műszaki Könyvkiadó, 2005.

[6] Kézi Csaba Gábor, *Differenciálszámítás és alkalmazásai feladatgyűjtemény*, Debreceni Egyetemi Kiadó, Debrecen, 2016.

[7] Kézi Csaba Gábor, *Differenciálszámítás és alkalmazásai feladatgyűjtemény*, Debreceni Egyetemi Kiadó, Debrecen, 2016.

[8] Kézi Csaba Gábor, *Analízis Mérnököknek*, Debreceni Egyetemi Kiadó, Debrecen, 2021.

[9] Kézi Csaba Gábor, *Közönséges elsőrendű differenciálegyenletek és alkalmazásaik*, Debreceni Egyetemi Kiadó, Debrecen, 2019.

[10] Kézi Csaba Gábor, *Közönséges elsőrendű differenciálegyenletek és alkalmazásaik feladatgyűjtemény*, Debreceni Egyetemi Kiadó, Debrecen, 2019.

[11] Kézi Csaba Gábor, *Közönséges magasabb rendű differenciálegyenletek és alkalmazásaik*, Debreceni Egyetemi Kiadó, Debrecen, 2019.

[12] Kézi Csaba Gábor, *Közönséges magasabb rendű differenciálegyenletek és alkalmazásaik feladatgyűjtemény*, Debreceni Egyetemi Kiadó, Debrecen, 2019.

[13] Kézi Csaba Gábor, *Lineáris algebra Mérnököknek*, Debreceni Egyetemi Kiadó, Debrecen, 2022.

[14] Kézi Csaba Gábor, *Vektorváltozós és vektorértékű függvények analízise*, Debreceni Egyetemi Kiadó, Debrecen, 2021.

[15] Lial M. L. – Greenwell R. N. – Ritchey N. P., *Calculus with applications*, Pearson, 2012.

[16] Mendelson E., *3000 solved problems in calculus*, McGraw-Hill Companies, 1988.

[17] Thomas G. B. – Weir M. D. – Hass J. – Giordano F. R., *Thomas féle kalkulus I. kötet*, Typotex, Budapest, 2008.

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