1949

Rigidity properties and transformations of Finsler manifolds

Doktori (PhD) értekezés

Kertész Dávid Csaba

Témavezető: Dr. Szilasi József

DEBRECENI EGYETEM
Természettudományi Doktori Tanács
Matematika- és Számítástudományok Doktori Iskola
Debrecen, 2017.

Ezen értekezést a Debreceni Egyetem Természettudományi Doktori Tanács Matematika- és Számítástudományok Doktori Iskola Matematikai analízis, függvényegyenletek és egyenlőtlenségek programja keretében készítettem a Debreceni Egyetem természettudományi doktori (PhD) fokozatának elnyerése céljából.

Debrecen, 2017. április

. . . .
Kertész Dávid Csaba
doktorjelölt

Tanúsítom, hogy Kertész Dávid Csaba doktorjelölt 2011-2017 között a fent megnevezett Doktori Iskola Matematikai analízis, függvényegyenletek és egyenlőtlenségek doktori program keretében irányításommal végezte munkáját. Az értekezésben foglalt eredményekhez a jelölt önálló alkotó tevékenységével meghatározóan hozzájárult. Az értekezés elfogadását javaslom.

Debrecen, 2017. április

. . . .
Dr. Szilasi József
témavezető

Rigidity properties and transformations of Finsler manifolds

Értekezés a doktori (PhD) fokozat megszerzése érdekében
matematika tudományágban.

Írta: Kertész Dávid Csaba okleveles matematikus

Készült a Debreceni Egyetem Matematika- és Számítástudományok Doktori Iskolája Matematikai analízis, függvényegyenletek és egyenlőtlenségek programja keretében

Témavezető: Dr. Szilasi József egyetemi docens

A doktori szigorlati bizottság:

Elnök: Dr. ....
Tagok: Dr. ....
Dr. ....

A doktori szigorlat időpontja: 2015. szeptember 18.

Az értekezés bírálói:

Dr. ....
Dr. ....

A bírálóbizottság:

Elnök: Dr. ....
Tagok: Dr. ....
Dr. ....
Dr. ....
Dr. ....

Az értekezés védésének időpontja: ....

Acknowledgements

First and foremost I would like to express my gratitude to my supervisor József Szilasi for his guidance and countless helpful suggestions.
I would also like to thank Zsolt Páles for his continuous support during my PhD studies.
Finally, I am thankful to my friends and family for tolerating my general lack of availability during the preparation of this Thesis.

Tartalomjegyzék

Introduction **1**

1 Basic concepts and tools **5**
1.1 Canonical constructions on the tangent bundle 5
1.2 Ehresmann connections and sprays 9
1.3 Linear connections and affine sprays 14
1.4 Finsler manifolds 15
1.5 The exponential map of a spray 16

2 Characterization of some spray and Finsler manifolds **19**
2.1 Projectively affine sprays 19
2.2 There are no proper Einstein–Berwald manifolds 25
2.3 Monochromatic Finsler manifolds 26

3 Affinities and isometries of spray and Finsler manifolds **33**
3.1 Affinities are smooth 33
3.2 Isometries are smooth 37
3.3 Distance coordinates 38
3.4 Relations between affinities and isometries 43
3.5 Affinely rigid Finsler manifolds 48
3.6 Finsler manifolds with many Killing vector fields 53

4 Summary **55**

5. Magyar nyelvű összefoglaló **67**

List of symbols **86**

Bibliography **90**

Introduction

Finsler manifolds are generalizations of Riemannian manifolds. In a Riemannian manifold we attach a positive definite scalar product to each of the tangent spaces. If we replace the scalar products by nonsymmetric norms, we obtain a Finsler manifold. Since the generalization happens on the tangent manifold, it is not immediately clear what new phenomena can occur on the manifold itself. This question was the main motivation behind the research whose results are presented in the Thesis. We attempt to tackle the issue from two directions: by examining Finsler manifolds that are 'close' to Riemannian manifolds, namely, Berwald and generalized Berwald manifolds, and by investigating how theorems about affinities and isometries in Riemannian geometry can be carried over into Finsler geometry.

Possibly the simplest (but most important) geometric objects attached to a Riemannian or to a Finsler manifold, who live on base manifold, are their geodesics. On a Finsler manifold, the length of a smooth curve segment $\gamma\colon [a,b] \to M$ is defined by the arclength integral

$$\int_a^b F \circ \dot{\gamma}.$$

It is invariant under orientation preserving reparametrizations of γ. A curve γ is called a geodesic if it is a stationary point of the arclength functional, and it has constant velocity, i.e., the function $F \circ \dot{\gamma}$ is constant. In some cases, the parametrization of the geodesics is not important, then we refer to them (following [8]) as oriented geodesic paths. Through the Euler–Lagrange equation, the geodesics can be obtained as solutions of a second-order differential equation. In an induced local coordinate system $((x^i)_{i=1}^n, (y^i)_{i=1}^n)$ on the tangent manifold, it takes the form

$$\ddot{\gamma}^i + 2G^i \circ \dot{\gamma} = 0, \qquad i \in \{1, \ldots, n\}, \tag{0.1}$$

where the G^is are 2^+-homogeneous functions and are smooth outside the zero vectors. In the Riemannian case, the functions G^i become smooth everywhere, and hence they are fibrewise quadratic forms. The functions G^i give a local description of the globally defined canonical spray S via the formula

$$S \underset{\text{locally}}{=} y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}.$$

Then (0.1) can be rewritten as $\ddot{\gamma} = S \circ \dot{\gamma}$. When the functions G^i are fibrewise quadratic forms, the spray S is called affine. Due to the work of Z.I. Szabó [42], Finsler manifolds with affine canonical spray, called Berwald manifolds, are well understood. Their geodesic structures carry no novelty, because for any Berwald manifold, there is a Riemannian manifold with the same geodesics. We can relax the affinity of the canonical spray by assuming only that the geodesic paths are specified by an affine spray. Since this problem requires no special parametrization of the geodesics, there is no need for a Finsler structure: it can be formulated in the framework of sprays manifolds. So we arrive at the following problem: *characterize sprays that have the same geodesic paths as an affine spray.* (Then we say that the spray is projectively equivalent to an affine spray.) This is an old problem, a local solution was given by J. Douglas [15]. He proved that if a suitable projectively invariant tensor attached to a spray S vanishes, then, locally, there exists an affine spray projectively equivalent to S. Global constructions were found by Z. Shen [40] and J. Szilasi and Sz. Vattamány [46], but they still assumed the orientability of the manifold. In section 2.1 we present a method that works without any restriction on the manifold. We construct the affine spray from the divergence of the spray with respect to a suitable volume form on the tangent manifold.

In section 2.2 we turn to the problem of finding Finslerian generalizations of Einstein manifolds. Most of the currently known Einstein–Finsler manifolds are either of Randers type or have vanishing curvature function, see e.g., [5, 9, 39, 41, 53]. We obtain a negative result by showing that among Berwald manifolds, the only non-Riemannian Einstein manifolds have vanishing curvature function.

Clearly, in a Riemannian manifold, there is a linear isomorphism between any two tangent spaces which preserves the inner product. This does not hold for Finsler manifolds in general. When it does, the Finsler manifolds is called monochromatic (an expressive term of D. Bao and Z. Shen). Y. Ichijyō proved that such Finsler manifolds are generalized Berwald manifolds. This means that the linear isomorphisms can be chosen as parallel translations with respect to a single linear connection. His proof relied on G-structures and their compatible connections, while our proof uses only simple properties of Ehresmann connections. This demonstrates that some problems in Finsler geometry become very simple if one works on the tangent manifold instead of the base manifold, and uses the techniques of tangent bundle geometry. (A detailed exposition of these tools can be found in [45, Chapter 4].) To obtain the result we assumed, and so did Ichijyō, that the Finsler manifold is monochromatic in a smooth way. Informally, this means that the linear isomorphisms 'depend smoothly' on the points. N. Bartelmeß and V. Matveev announced to prove that the condition is redundant, as it follows from the smoothness of the Finsler function. However, this result has not been published yet.

In Chapter 3 we discuss isometries and affinities of Finsler manifolds and some relations between them. A bijection on a Finsler manifold is called affine, if it preserves the geodesics as parametrized curves, and an isometry, if it preserves the Finslerian distance function. Both conditions are formulated using objects living on the base manifold. They have counterparts that are defined in terms of objects living on the tangent manifold. The analogue of an affine mapping is an automorphism of the canonical spray S, that is, a diffeomorphism $\varphi\colon M \to M$ such that $\varphi_{**} \circ S = S \circ \varphi_*$. The analogue of an isometry is a Finsler isometry, that is, a diffeomorphism $\varphi\colon M \to M$ satisfying $F \circ \varphi_* = F$, where F is the Finsler function. The 'tangent manifold counterparts' are a priori more special, as they require differentiability. F. Brickell proved in the somewhat forgotten paper [8] that if an affine transformation is also a homeomorphism, then it is necessarily smooth, and hence an automorphism of the spray. Since isometries are automatically affinities and also homeomorphisms, they are also necessarily smooth, and hence Finsler isometries. The latter result was proved also by S. Deng and Z. Hou [13] much later, independently from Brickell. In sections 3.1 and 3.2 we provide a simplified version of Brickell's argument. In section 3.3 we give a more direct proof of the smoothness of isometries. It is based on a very elegant proof from [38] of the same result for Riemannian manifolds. It utilizes coordinate systems, whose coordinate functions are distance functions. We show that such coordinate systems do exist in the Finslerian case. Using these special coordinates we show also that under some regularity condition, submetries between reversible Finsler manifolds are differentiable. This is in fact a special case of a result about submetries between metric spaces [30], however we believe it is beneficial to have a purely differential geometric proof. Our argument mostly follows [6], where the same theorem is proved for Riemannian manifolds, although in that case, the argument yields that the submetries are of class $C^{1,1}$.

We continue by examining how some theorems of Riemannian geometry about the relations between affinities and isometries can be generalized to Finsler geometry. K. Yano showed that infinitesimal affinities (i.e., affine vector fields) on a compact orientable Riemannian manifold are infinitesimal isometries (i.e., Killing vector fields) [52]; his proof is based on integral formulas. J.-I. Hano found a generalization: bounded affine vector fields on a complete Riemannian manifold are Killing vector fields. Hano's proof relies on the de Rham decomposition and special properties of irreducible Riemannian manifolds. Our Theorem 3.4.6 is a generalization of Hano's result to Finsler manifolds. Surprisingly, the proof is significantly simpler than the ones in [25, 19] for the more special Riemannian case, it utilizes only the Euler–Lagrange equation and the 'fundamental inequality' ([4, p. 7] or [45, Proposition 9.1.37]), and perhaps most notably, it does not need any concept analogous to irreducibility. This again demonstrates that some problems in Finsler geometry become very simple if one works on the tangent manifold.

Another famous theorem about affinities and isometries is due to S. Kobayashi [24]. He proved that in a complete irreducible Riemannian manifold any affinity is an isometry, except for the 1-dimensional Euclidean space. In the remainder of section 3.4 and in section 3.5, we attempt to generalize this result to Finsler manifolds. Since there is no good analogue of irreducibility in Finsler geometry, we propose to replace it by one if its consequences: if the holonomy group of a Riemannian manifold is irreducible, then it determines the metric tensor up to a scalar factor. This property can be translated to Finsler geometry: we say that a Finsler manifold is affinely rigid, if its canonical spray determines the Finsler function up to a scalar factor (this translation indeed carries the intended meaning, since the canonical spray completely determines the holonomy). Affine rigidity turns out to be a suitable replacement of irreducibility in the theorem of Kobayashi, see Corollary 3.4.10. This suggests that affine rigidity should be examined further in order to properly understand the relation between affinities and isometries. We devote section 3.5 to this topic. A simple sufficient condition for affine rigidity is the transitivity of the holonomy group on the unit sphere, see Proposition 3.5.1. However, the transitivity of the holonomy group is an overly strong assumption, even in the Riemannian case, furthermore the Finslerian holonomy groups are difficult to handle because they are potentially infinite dimensional [33, 34]. We believe that instead of the holonomy group, one should examine a smooth singular distribution constructed from horizontal vector fields. We collect our results in Theorem 3.5.7 and we formulate an open question at the end of the section.

Our concluding section 3.6 is about Finsler manifolds that admit local frame fields consisting of Killing vector fields. Such Finsler manifolds have been studied by M. Xu and S. Deng in [51]. We show that these Finsler manifolds are generalized Berwald manifolds, and hence monochromatic. We also show that if the Finsler function is compatible with the local parallelizations induced by the Killing frame fields, then the Finsler manifold is a Berwald manifold. This was already known in the special case when the whole manifold admits a Killing frame field [1]. Our arguments are quite simple, using again tools from tangent bundle geometry.

∼ ∘ ∼

Some technical remarks about the Thesis. In the first Chapter we summarize our terminology and basic facts. In the second and third Chapters we collect our results. If a theorem, lemma, corollary or proposition is not new, we write the name of the original author right at the beginning of the statement between parentheses. If a result of ours have been already published, we cite the publication also at the beginning of the theorem. The fourth and fifth chapters are the summaries, in English and in Hungarian, respectively. A list of symbols can be found at the end of the Thesis.

Chapter 1

Basic concepts and tools

1.1 Canonical constructions on the tangent bundle

1.1.1. By a manifold we mean a finite-dimensional smooth manifold whose underlying topology is second countable and Hausdorff. Mappings between manifolds are assumed to be smooth (infinitely many times differentiable), if not specified otherwise. For a manifold M we use the following notation:

(1) $C^\infty(M)$ is the ring of smooth real-valued functions on M;

(2) $\tau\colon TM \to M$ is the tangent bundle of M;

(3) $\mathring{T}M$ is the submanifold of TM consisting of the nonzero tangent vectors to M, $\mathring{\tau} := \tau \restriction \mathring{T}M$ is the slit tangent bundle of M;

(4) $\tau_{TM}\colon TTM \to TM$ is the tangent bundle of TM.

Some further notation:

(5) For a vector field X, Fl^X is its maximal flow, Fl^X_t ($t \in \mathbb{R}$) is the tth stage of the flow.

(6) For a vector bundle $\pi\colon B \to M$, $\Gamma(\pi)$ or $\Gamma(B)$ denotes the $C^\infty(M)$-module of its (smooth) sections. In particular, $\mathfrak{X}(M) := \Gamma(TM)$ is the $C^\infty(M)$-module of vector fields on M.

(7) The derivative or tangent map of a map $\varphi\colon M \to N$ is $\varphi_*\colon TM \to TN$.

Throughout the rest of the Chapter, M stands for an n-dimensional manifold, where $n \geq 1$.

1.1.2. For a function $f \in C^\infty(M)$, we define the functions f^{v} and f^{c} on TM by

$$f^{\mathsf{v}} := f \circ \tau \quad \text{and} \quad f^{\mathsf{c}}(v) := v(f), \quad v \in TM.$$

They are called the *vertical* and the *complete lift* of f, respectively.

If $(\mathcal{U}, (u^i)_{i=1}^n)$ is a chart for M, set

$$x^i := (u^i)^{\mathsf{v}}, \quad y^i := (u^i)^{\mathsf{c}}; \qquad i \in \{1, \dots, n\}. \tag{1.1}$$

Then $(\tau^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^i)_{i=1}^n))$ is a chart for TM, called the chart *induced by* $(\mathcal{U}, (u^i)_{i=1}^n)$.

For the complete lift of $f \in C^\infty(M)$ we have

$$f^{\mathsf{c}} \underset{(\mathcal{U})}{=} y^i \left(\frac{\partial f}{\partial u^i}\right)^{\mathsf{v}}. \tag{1.2}$$

In (1.2), and throughout the Thesis, we use the Einstein summation convention for expressions with repeated indicies.

1.1.3. The *vertical lift* of a vector field $X \in \mathfrak{X}(M)$ is the velocity field X^{v} of the global flow

$$(t, v) \in \mathbb{R} \times TM \mapsto v + tX(\tau(v)) \in TM$$

on TM. If $\mathrm{Fl}^X : \mathcal{D}_X \subset \mathbb{R} \times M \to M$ is the maximal flow of $X \in \mathfrak{X}(M)$ and

$$\widetilde{\mathcal{D}}_X := \{(t, v) \in \mathbb{R} \times TM \mid (t, \tau(v)) \in \mathcal{D}_X\},$$

then

$$(t, v) \in \widetilde{\mathcal{D}}_X \longmapsto (\mathrm{Fl}_t^X)_*(v) \in TM$$

is a flow on TM, whose velocity field X^{c} is called the *complete lift* of X. The vertical and the complete lifts are natural in the following sense. If M and N are manifolds, $\varphi\colon M \to N$, $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are φ-related vector fields, that is, $\varphi_* \circ X = Y \circ \varphi$, then

$$\varphi_{**} \circ X^{\mathsf{c}} = Y^{\mathsf{c}} \circ \varphi_* \qquad \text{and} \qquad \varphi_{**} \circ X^{\mathsf{v}} = Y^{\mathsf{v}} \circ \varphi_*.$$

In particular, if φ is a diffeomorphism and $\varphi_\# X := \varphi_* \circ X \circ \varphi^{-1}$ is the *push-forward* of X by φ, then

$$\varphi_{*\#}(X^{\mathsf{v}}) = (\varphi_\# X)^{\mathsf{v}} \qquad \text{and} \qquad \varphi_{*\#}(X^{\mathsf{c}}) = (\varphi_\# X)^{\mathsf{c}}. \tag{1.3}$$

The *Liouville vector field* on TM is the velocity field C of the flow of the positive dilations

$$(t, v) \in \mathbb{R} \times TM \longmapsto e^t v \in TM.$$

If $(\mathcal{U}, (u^i)_{i=1}^n)$ is a chart on M, and $X = X^i \frac{\partial}{\partial u^i}$, then locally

$$X^{\mathsf{v}} = X^i \frac{\partial}{\partial y^i}, \qquad X^{\mathsf{c}} = (X^i)^{\mathsf{v}} \frac{\partial}{\partial x^i} + (X^i)^{\mathsf{c}} \frac{\partial}{\partial y^i} \tag{1.4}$$

and

$$C \underset{(\mathcal{U})}{=} y^i \frac{\partial}{\partial y^i}. \tag{1.5}$$

In particular,

$$\left(\frac{\partial}{\partial u^i}\right)^{\mathsf{v}} = \frac{\partial}{\partial y^i}, \qquad \left(\frac{\partial}{\partial u^i}\right)^{\mathsf{c}} = \frac{\partial}{\partial x^i}, \qquad i \in \{1, \dots, n\}. \tag{1.6}$$

For any $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$ we have

$$X^{\mathsf{v}} f^{\mathsf{v}} = 0, \tag{1.7}$$
$$X^{\mathsf{v}} f^{\mathsf{c}} = (Xf)^{\mathsf{v}}, \tag{1.8}$$
$$X^{\mathsf{c}} f^{\mathsf{v}} = (Xf)^{\mathsf{v}}, \tag{1.9}$$
$$X^{\mathsf{c}} f^{\mathsf{c}} = (Xf)^{\mathsf{c}}, \tag{1.10}$$
$$C f^{\mathsf{c}} = f^{\mathsf{c}}, \tag{1.11}$$
$$(fX)^{\mathsf{c}} = f^{\mathsf{v}} X^{\mathsf{c}} + f^{\mathsf{c}} X^{\mathsf{v}}, \tag{1.12}$$
$$[X^{\mathsf{v}}, Y^{\mathsf{v}}] = 0, \tag{1.13}$$
$$[C, X^{\mathsf{v}}] = -X^{\mathsf{v}}, \tag{1.14}$$
$$[C, X^{\mathsf{c}}] = 0, \tag{1.15}$$
$$(X + Y)^{\mathsf{c}} = X^{\mathsf{c}} + Y^{\mathsf{c}}, \tag{1.16}$$
$$[X^{\mathsf{v}}, Y^{\mathsf{c}}] = [X, Y]^{\mathsf{v}}, \tag{1.17}$$
$$[X^{\mathsf{c}}, Y^{\mathsf{c}}] = [X, Y]^{\mathsf{c}}, \tag{1.18}$$
$$Y_* \circ X = X^{\mathsf{c}} \circ Y + [X, Y]^{\mathsf{v}} \circ Y. \tag{1.19}$$

For details, see, e.g., [45, 4.1].

1.1.4. Each tangent space T_pM is a submanifold of TM, thus its tangent manifold can be naturally embedded into TTM. A vector in TTM is called *vertical*, if it is tangent to a tangent space in this sense. The vertical vectors form the *vertical subbundle* of TTM, denoted by VTM. The vertical lift of a vector field in $\mathfrak{X}(M)$ is a section of VTM, and for every vertical vector w, there is a vector field $X \in \mathfrak{X}(M)$

such that $X^{\mathsf{v}}(\tau_{TM}(w)) = w$. The vertical subbundle is also the kernel of τ_*, and it restricts naturally to a subbundle of $T\mathring{T}M$. The restriction is denoted by $V\mathring{T}M$. This vector bundle is naturally isomorphic to the Finsler bundle described below.

The pull-back $\mathring{\tau}^*\tau$ of τ by $\mathring{\tau}$ is a vector bundle over $\mathring{T}M$, called the *Finsler bundle*. Its total manifold is

$$\mathring{T}M \times_M TM := \{(u,v) \in \mathring{T}M \times TM \mid \mathring{\tau}(u) = \tau(v)\},$$

and its fibre over $u \in \mathring{T}M$ is $\{u\} \times T_{\mathring{\tau}(u)}M$. Every vector field $X \in \mathfrak{X}(M)$ can be lifted to a section

$$\widehat{X} \colon \mathring{T}M \to \mathring{T}M \times_M TM, \quad \widehat{X}(u) := (u, X(\mathring{\tau}(u)))$$

of the Finsler bundle, called the *basic lift* of X. All tensors on M can be lifted similarly to the Finsler bundle.

We have a natural $C^\infty(\mathring{T}M)$-linear injection $\mathbf{i} \colon \Gamma(\mathring{\tau}^*\tau) \to \Gamma(V\mathring{T}M)$ specified by the rule $\mathbf{i}\widehat{X} = X^{\mathsf{v}}$. We agree to identify such module homomorphisms with bundle maps, so we may regard $\mathbf{i}$ as a fibrewise linear mapping from $\mathring{T}M \times_M TM$ to $V\mathring{T}M$.

On fibres, $\mathbf{i}$ is natural also in another sense. As $\mathring{T}_pM$ is an open subset of the vector space T_pM, its tangent manifold can be identified with $\mathring{T}_pM \times T_pM$. Since $\mathring{T}_pM$ is also a submanifold of $\mathring{T}M$, its tangent manifold can be naturally embedded into $T\mathring{T}M$, in fact, into $V\mathring{T}M$. This embedding is just $\mathbf{i}$.

Another natural bundle map is the *canonical surjection*

$$\mathbf{j} \colon T\mathring{T}M \to \mathring{T}M \times_M TM, \quad \mathbf{j}(w) := (\tau_{TM}w, \tau_* w).$$

Its kernel is $V\mathring{T}M$, and $\mathbf{j}X^{\mathsf{c}} = \widehat{X}$. The composition $\mathbf{J} := \mathbf{i} \circ \mathbf{j}$ is the *vertical endomorphism*. It has the characteristic properties

$$\mathbf{J}X^{\mathsf{v}} = 0, \qquad \mathbf{J}X^{\mathsf{c}} = X^{\mathsf{v}}, \qquad X \in \mathfrak{X}(M).$$

1.1.5. We denote by $\widetilde{\delta}$ the *canonical section*

$$\widetilde{\delta} \colon u \in \mathring{T}M \mapsto (u,u) \in \mathring{T}M \times_M TM$$

of $\mathring{\tau}^*\tau$. The Liouville vector field can be expressed as $\mathbf{i}\widetilde{\delta} = C$.

1.1.6. If a chart $(\mathcal{U}, (u^i)_{i=1}^n)$ is given on M, the basic lifts of the coordinate vector fields $\left(\frac{\partial}{\partial u^i}\right)_{i=1}^n$ form a local frame field $\left(\widehat{\frac{\partial}{\partial u^i}}\right)_{i=1}^n$ of the Finsler bundle. Its dual $(\widehat{du^i})_{i=1}^n$ is formed by the basic lifts of the 1-forms du^i.

1.1.7. We can construct a covariant-derivative-like operator $\nabla^{\mathsf{v}} \colon \Gamma(\mathring{\tau}^*\tau) \times \Gamma(\mathring{\tau}^*\tau) \to \Gamma(\mathring{\tau}^*\tau)$ on the Finsler bundle from the canonical differentiation on the fibres. For a section $\widetilde{X}$ of $\mathring{\tau}^*\tau$, there exists a unique map

$$\underline{X}\colon \mathring{T}M \to TM$$

such that $\widetilde{X}(u) = (u, \underline{X}(u))$ for all $u \in \mathring{T}M$. Define

$$\nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y}(u) := (u, (\underline{Y} \restriction \mathring{T}_pM)'(u)(\underline{X}(u))), \qquad p \in M,\ u \in T_pM,$$

and for a function $f \in C^\infty(\mathring{T}M)$, set $\nabla^{\mathsf{v}}_{\widetilde{X}} f := \mathbf{i}\widetilde{X} f$. Then ∇^{v} can be extended to a derivation of the tensors of $\mathring{\tau}^*\tau$ in the usual way. We call this derivation the *canonical vertical derivative*. Since basic lifts are 'constant on the fibres' we have

$$\nabla^{\mathsf{v}}\widehat{X} = 0 \qquad \text{for all } X \in \mathfrak{X}(M). \tag{1.20}$$

As a consequence,

$$\nabla^{\mathsf{v}}\nabla^{\mathsf{v}} f(\widehat{X}, \widehat{Y}) = X^{\mathsf{v}}Y^{\mathsf{v}} f. \tag{1.21}$$

In local coordinates, ∇^{v} is just 'differentiation with respect to the y^i coordinates'. For example, if $f \in C^\infty(\mathring{T}M)$, then

$$\nabla^{\mathsf{v}} f \underset{(\mathcal{U})}{=} \frac{\partial f}{\partial y^i}\widehat{du^i}, \quad \nabla^{\mathsf{v}}\nabla^{\mathsf{v}} f \underset{(\mathcal{U})}{=} \frac{\partial^2 f}{\partial y^i \partial y^j}\widehat{du^i} \otimes \widehat{du^j}, \quad \text{etc.} \tag{1.22}$$

1.1.8. A function $f \in C^\infty(\mathring{T}M)$ is k^+*-homogeneous*, if $\mathcal{L}_C f := Cf = kf$. This means that $f(\lambda v) = \lambda^k f(v)$ for all $v \in \mathring{T}M$ and positive scalar λ. A vector field $\xi \in \mathfrak{X}(\mathring{T}M)$ is k^+*-homogeneous*, if $\mathcal{L}_C\xi := [C, \xi] = (k-1)\xi$. If $\xi = \xi^i \frac{\partial}{\partial x^i} + \xi^{n+i}\frac{\partial}{\partial y^i}$, then ξ is k^+-homogeneous if and only if the functions ξ^i are $(k-1)^+$-homogeneous and the functions ξ^{n+i} are k^+-homogeneous for all $i \in \{1, \dots, n\}$.

1.2 Ehresmann connections and sprays

1.2.1. Let M be a manifold. By an *Ehresmann connection* in $\mathring{T}M$ we mean a smooth mapping

$$\mathcal{H}\colon \mathring{T}M \times_M TM \to T\mathring{T}M, \quad (u, v) \mapsto \mathcal{H}(u, v),$$

which satisfies the following conditions:

(E1) $\mathcal{H}(u, v) \in T_uTM$ for all $(u, v) \in \mathring{T}M \times_M TM$;

(E2) the mapping $v \mapsto \mathcal{H}(u, v)$ is $\mathbb{R}$-linear for each fixed $u \in \mathring{T}M$;

(E3) $\tau_*\mathcal{H}(u,v) = v$ for all $(u,v) \in \mathring{T}M \times_M TM$.

Conditions (E1) and (E3) can be briefly expressed together as $\mathbf{j} \circ \mathcal{H} = \mathrm{id}_{\mathring{T}M \times_M TM}$. The image $H\mathring{T}M := \operatorname{Im}\mathcal{H}$ of $\mathcal{H}$ is a subbundle of $T\mathring{T}M$ of rank $n = \dim M$, and it is complementary to $V\mathring{T}M$, i.e.,

$$T\mathring{T}M = V\mathring{T}M \oplus H\mathring{T}M. \tag{1.23}$$

Conversely, any subbundle of $T\mathring{T}M$ satisfying (1.23) is the image of a unique Ehresmann connection. Subbundles of $T\mathring{T}M$ satisfying (1.23) are called *horizontal.*

We have the projections $\mathbf{h}\colon T\mathring{T}M \to H\mathring{T}M$, $\mathbf{v}\colon T\mathring{T}M \to V\mathring{T}M$, characterized by

$$\operatorname{Ker}\mathbf{v} = H\mathring{T}M, \quad \operatorname{Ker}\mathbf{h} = V\mathring{T}M.$$

Then $\mathbf{h} = \mathcal{H} \circ \mathbf{j}$. We also define the *vertical mapping* $\mathcal{V} := \mathbf{i}^{-1} \circ \mathbf{v}$; it is a left inverse for $\mathbf{i}$. For a vector field $X \in \mathfrak{X}(M)$, $X^{\mathsf{h}} := \mathcal{H}\widehat{X}$ is the *horizontal lift* of X. The Ehresmann connection is *positive-homogeneous*, if the horizontal lifts are 1^+-homogeneous, that is $[C, X^{\mathsf{h}}] = 0$ for all $X \in \mathfrak{X}(M)$.

1.2.2. A spray for M is a smooth vector field on $\mathring{T}M$ satisfying the following conditions:

(S1) $\tau_* \circ S = \mathrm{id}_{\mathring{T}M}$.

(S2) $[C, S] = S$, that is, S is 2^+-homogeneous.

A *spray manifold* is a manifold M together with a spray for M. If $\mathcal{H}$ is a positive-homogeneous Ehresmann connection, then $S := \mathcal{H}\widetilde{\delta}$ is a spray. Conversely, if S is a spray, there is an Ehresmann connection $\mathcal{H}$ such that

$$\mathcal{H}(\widehat{X}) = X^{\mathsf{h}} = \frac{1}{2}(X^{\mathsf{c}} + [X^{\mathsf{v}}, S]) \qquad \text{for all } X \in \mathfrak{X}(M). \tag{1.24}$$

We call this Ehresmann connection the *Berwald connection* of the spray manifold. Then $\mathcal{H}\widetilde{\delta} = S$. For details, see [45, 7.3.3].

1.2.3. Let an Ehresmann connection $\mathcal{H}$ in $\mathring{T}M$ be given. The mappings $\mathcal{R}$ and $\mathbf{T}$ given by

$$\mathcal{R}(\widetilde{X}, \widetilde{Y}) := \mathcal{V}[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}] \tag{1.25}$$

and

$$\mathbf{T}(\widetilde{X}, \widetilde{Y}) := \mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}] - \mathcal{V}[\mathcal{H}\widetilde{Y}, \mathbf{i}\widetilde{X}] - \mathbf{j}[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}], \tag{1.26}$$

where $\widetilde{X}, \widetilde{Y} \in \Gamma(\mathring{\tau}^*\tau)$, are skew-symmetric type $(1,2)$ tensors on the Finsler bundle, called the *curvature* and the *torsion* of $\mathcal{H}$, respectively. Evaluating $\mathcal{R}$ and $\mathbf{T}$ on basic sections $\widehat{X}$ and $\widehat{Y}$, we obtain

$$\mathbf{i}\,\mathcal{R}(\widehat{X}, \widehat{Y}) = \mathbf{v}[X^{\mathsf{h}}, Y^{\mathsf{h}}] = [X^{\mathsf{h}}, Y^{\mathsf{h}}] - [X, Y]^{\mathsf{h}}, \tag{1.27}$$

$$\mathbf{i}\,\mathbf{T}(\widehat{X}, \widehat{Y}) = [X^{\mathsf{h}}, Y^{\mathsf{v}}] - [Y^{\mathsf{h}}, X^{\mathsf{v}}] - [X, Y]^{\mathsf{v}}. \tag{1.28}$$

It is easy to check that the torsion of the Berwald connection of a spray manifold vanishes. In fact, every positive-homogeneous and torsion-free Ehresmann connection is the Berwald connection of a spray (for a proof, see [45, Theorem 7.8.7]).

The *Jacobi endomorphism* of a spray manifold (M, S) is the type $(1,1)$ Finsler tensor field $\mathbf{K}$ defined by

$$\mathbf{K}(\widetilde{X}) := \mathcal{V}[S, \mathcal{H}\widetilde{X}], \quad \widetilde{X} \in \Gamma(\mathring{\tau}^*\tau). \tag{1.29}$$

If $\mathcal{H}$ is the Berwald connection of (M, S), we have

$$\mathcal{R}(\widetilde{\delta}, \widetilde{X}) = \mathcal{V}[S, \mathcal{H}\widetilde{X}] = \mathbf{K}(\widetilde{X}). \tag{1.30}$$

If $n = \dim M \geq 2$, the function

$$K := \frac{1}{n-1}\operatorname{tr}\mathbf{K} \tag{1.31}$$

is called the *curvature function* of (M, S).

1.2.4. Let $\mathcal{H}$ be a positive-homogeneous Ehresmann connection in $\mathring{T}M$, $I \subset \mathbb{R}$ an open interval and $\gamma\colon I \to M$ a smooth curve. A vector field $X\colon I \to \mathring{T}M$ along γ is said to be *parallel with respect to* $\mathcal{H}$ (briefly $\mathcal{H}$*-parallel*, or simply *parallel*) if

$$\dot{X}(t) = \mathcal{H}(X(t), \dot{\gamma}(t)) \quad \text{for all } t \in I. \tag{1.32}$$

For each $t_0 \in I$ and $v \in \mathring{T}_{\gamma(t_0)}M$, there is a unique parallel vector field X along γ such that $X(t_0) = v$. Then $X(t)$ is the *parallel transport* of v along γ to $\mathring{T}_{\gamma(t)}M$. In this way we obtain a mapping $P(\gamma)_{t_0}^t\colon \mathring{T}_{\gamma(t_0)}M \to \mathring{T}_{\gamma(t)}M$ that sends each $v \in \mathring{T}_{\gamma(t_0)}M$ to its parallel transport. Then $P(\gamma)_{t_0}^t$ is smooth and 1^+-homogeneous (in the obvious sense) for all $t \in I$. For details, see [45, 7.6]. It is natural to extend $P(\gamma)_{t_0}^t$ to $T_{\gamma(t_0)}M$ by declaring the zero vector fields to be parallel, and hence setting $P(\gamma)_{t_0}^t(0) = 0$. One should keeping in mind that this extension is only continuous, and not smooth. We will tacitly use this extension whenever necessary.

1.2.5. An Ehresmann connection $\mathcal{H}$ induces a covariant-derivative-like operator $\nabla^{\mathsf{h}}\colon \Gamma(\mathring{\tau}^*\tau) \times \Gamma(\mathring{\tau}^*\tau) \to \Gamma(\mathring{\tau}^*\tau)$, given by

$$\nabla^{\mathsf{h}}_{\widetilde{X}}\widetilde{Y} := \mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}].$$

For basic lifts we have

$$\nabla^{\mathsf{h}}_{\widehat{X}}\widehat{Y} = \mathbf{i}^{-1}[X^{\mathsf{h}}, Y^{\mathsf{v}}]. \tag{1.33}$$

From ∇^{h} and ∇^{v} we build a covariant derivative operator

$$\nabla\colon \mathfrak{X}(\mathring{T}M) \times \Gamma(\mathring{\tau}^*\tau) \to \Gamma(\mathring{\tau}^*\tau)$$

on the Finsler bundle such that

$$\nabla_{\mathcal{H}\widetilde{X}}\widetilde{Y} = \nabla^{\mathsf{h}}_{\widetilde{X}}\widetilde{Y}, \qquad \nabla_{\mathbf{i}\widetilde{X}}\widetilde{Y} = \nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y}.$$

It is called the *Berwald derivative* induced by $\mathcal{H}$. We define its curvature tensor in the usual way:

$$R^{\nabla}(\xi,\eta)\widetilde{X} := \nabla_{\xi}\nabla_{\eta}\widetilde{X} - \nabla_{\eta}\nabla_{\xi}\widetilde{X} - \nabla_{[\xi,\eta]}\widetilde{X},$$

where $\xi,\eta \in \mathfrak{X}(\mathring{T}M)$, $\widetilde{X} \in \Gamma(\mathring{\tau}^*\tau)$.

From R^{∇} some useful partial curvatures can be obtained. The first of them is the *affine curvature* (or horizontal curvature) of $\mathcal{H}$ given by

$$\begin{aligned}\mathbf{H}(\widetilde{X},\widetilde{Y})\widetilde{Z} &:= R^{\nabla}(\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y})\widetilde{Z}\\ &= \nabla_{\mathcal{H}\widetilde{X}}\nabla_{\mathcal{H}\widetilde{Y}}\widetilde{Z} - \nabla_{\mathcal{H}\widetilde{Y}}\nabla_{\mathcal{H}\widetilde{X}}\widetilde{Z} - \nabla_{[\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y}]}\widetilde{Z}.\end{aligned} \tag{1.34}$$

It is completely determined by the curvature $\mathcal{R}$ of $\mathcal{H}$:

$$\mathbf{H}(\widetilde{X},\widetilde{Y})\widetilde{Z} = -\nabla^{\mathsf{v}}_{\widetilde{Z}}\mathcal{R}(\widetilde{X},\widetilde{Y}). \tag{1.35}$$

We also have

$$\mathbf{H}(\widetilde{X},\widetilde{Y})\widetilde{\delta} = -\mathcal{R}(\widetilde{X},\widetilde{Y}). \tag{1.36}$$

For proofs, see [45, 7.14].

The second important partial curvature is the *Berwald tensor* of $\mathcal{H}$ given by

$$\begin{aligned}\mathbf{B}(\widetilde{X},\widetilde{Y})\widetilde{Z} &:= R^{\nabla}(\mathbf{i}\widetilde{X},\mathcal{H}\widetilde{Y})\widetilde{Z}\\ &= \nabla_{\mathbf{i}\widetilde{X}}\nabla_{\mathcal{H}\widetilde{Y}}\widetilde{Z} - \nabla_{\mathcal{H}\widetilde{Y}}\nabla_{\mathbf{i}\widetilde{X}}\widetilde{Z} - \nabla_{[\mathbf{i}\widetilde{X},\mathcal{H}\widetilde{Y}]}\widetilde{Z}.\end{aligned} \tag{1.37}$$

On basic lifts, it acts simply by

$$\mathbf{i}\,\mathbf{B}(\widehat{X},\widehat{Y})\widehat{Z} = -[Y^{\mathsf{h}},[Z^{\mathsf{v}},X^{\mathsf{v}}]] - [Z^{\mathsf{v}},[X^{\mathsf{v}},Y^{\mathsf{h}}]] \overset{(1.13)}{=} [[X^{\mathsf{v}},Y^{\mathsf{h}}],Z^{\mathsf{v}}], \tag{1.38}$$

see [45, 7.13].

1.2.6. Let (M,S) be a spray manifold. By condition (S1), each integral curve $\sigma\colon I \to TM$ of S is of the form $\sigma = \dot{\gamma}$, where $\gamma := \tau \circ \sigma$. Such a curve γ is called a *geodesic* of S. Equivalently, a curve $\gamma\colon I \to M$ is a geodesic of S if $S \circ \dot{\gamma} = \ddot{\gamma}$. We declare the constant curves to be geodesics as well.

Given a vector v in TM and a geodesic γ satisfying $\dot{\gamma}(t) = v$ for some $t \in I$, we say that γ is a geodesic with velocity v at t or starting at $p := \gamma(t) = \tau(v)$. From the theory of ODEs, we know that for any $v \in \mathring{T}M$, there is a unique maximal geodesic γ_v with initial velocity v at 0.

The 2^+-homogeneity of S implies that if λ and t are positive scalars such that γ_v is defined at λt, then $\gamma_{\lambda v}$ is defined at t, and

$$\gamma_{\lambda v}(t) = \gamma_v(\lambda t). \tag{1.39}$$

This implies that if γ is a geodesic, then so is $t \mapsto \gamma(at + b)$ where $a > 0$ and $b \in \mathbb{R}$. Hence an orientation preserving affine reparametrization of a geodesic is also a geodesic.

1.2.7. Let a chart $(\mathcal{U}, (u^i)_{i=1}^n)$ be given for M. An Ehresmann connection $\mathcal{H}$ in $\mathring{T}M$ can be described with the coefficients N^i_j of the horizontal lifts of the coordinate vector fields, determined by

$$\mathcal{H}\left(\widehat{\frac{\partial}{\partial u^j}}\right) = \left(\frac{\partial}{\partial u^j}\right)^{\mathsf{h}} = \frac{\partial}{\partial x^j} - N^i_j \frac{\partial}{\partial y^i}, \qquad j \in \{1, \dots, n\}. \tag{1.40}$$

Then $(N^i_j)_{i,j=1}^n$ is a family of smooth functions on $\mathring{\tau}^{-1}(\mathcal{U})$, called the *Christoffel symbols* of $\mathcal{H}$. Then the Berwald derivative induced by $\mathcal{H}$ satisfies

$$\nabla_{\frac{\partial}{\partial x^j}} \widehat{\frac{\partial}{\partial u^k}} = \frac{\partial N^i_j}{\partial y^k} \widehat{\frac{\partial}{\partial u^i}}, \qquad \nabla_{\frac{\partial}{\partial y^j}} \widehat{\frac{\partial}{\partial u^k}} = 0, \qquad j, k \in \{1, \dots, n\}. \tag{1.41}$$

The Berwald tensor has the coordinate expression

$$\mathbf{B}\left(\widehat{\frac{\partial}{\partial u^j}}, \widehat{\frac{\partial}{\partial u^k}}, \widehat{\frac{\partial}{\partial u^l}}\right) = \frac{\partial^2 N^i_k}{\partial y^l \partial y^j} \widehat{\frac{\partial}{\partial u^i}} =: B^i_{jkl} \widehat{\frac{\partial}{\partial u^i}}, \qquad j, k, l \in \{1, \dots, n\}. \tag{1.42}$$

A spray S over M can be written locally in the form

$$S \underset{(\mathcal{U})}{=} y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}. \tag{1.43}$$

Here $(G^i)_{i=1}^n$ are again smooth functions on $\mathring{\tau}^{-1}(\mathcal{U})$, called the *coefficients* of S. The Christoffel symbols of the Berwald connection of S are just

$$G^i_j := \frac{\partial G^i}{\partial y^j}, \qquad i, j \in \{1, \dots, n\}. \tag{1.44}$$

Thus, for the Berwald connection of a spray, (1.41) and (1.42) simplify to

$$\nabla_{\frac{\partial}{\partial x^j}} \widehat{\frac{\partial}{\partial u^k}} = \frac{\partial^2 G^i}{\partial y^j \partial y^k} \widehat{\frac{\partial}{\partial u^i}}, \qquad \nabla_{\frac{\partial}{\partial y^j}} \widehat{\frac{\partial}{\partial u^k}} = 0$$

and

$$\mathbf{B}\left(\widehat{\frac{\partial}{\partial u^j}}, \widehat{\frac{\partial}{\partial u^k}}, \widehat{\frac{\partial}{\partial u^l}}\right) = \frac{\partial^3 G^i}{\partial y^l \partial y^j \partial y^k} \widehat{\frac{\partial}{\partial u^i}} =: B^i_{jkl} \widehat{\frac{\partial}{\partial u^i}}. \tag{1.45}$$

This implies that the Berwald tensor is symmetric in a spray manifold.

1.3 Linear connections and affine sprays

An Ehresmann connection $\mathcal{H}$ in $\mathring{T}M$ is called *linear* if any of the following equivalent conditions holds:

(L1) $\mathcal{H}$ is positive-homogeneous and can be extended smoothly to TM.

(L2) The Christoffel symbols of $\mathcal{H}$ are fibrewise linear functions with respect to any chart of M.

(L3) The Christoffel symbols of the Berwald derivative induced by $\mathcal{H}$ are constant on the fibres with respect to any chart of M.

(L4) The Berwald tensor of $\mathcal{H}$ vanishes.

(L5) For any two vector fields fields X, Y on M, their Lie bracket $[X^{\mathsf{h}}, Y^{\mathsf{v}}]$ is a vertical lift.

(L6) There exists a unique covariant derivative D on M such that

$$(D_X Y)^{\mathsf{v}} = [X^{\mathsf{h}}, Y^{\mathsf{v}}] \overset{(1.33)}{=} \mathbf{i}\nabla_{X^{\mathsf{h}}} \widehat{Y}; \qquad X, Y \in \mathfrak{X}(M). \tag{1.46}$$

(L7) The $\mathcal{H}$-parallel translations along any smooth curve (as described in 1.2.4) are linear mappings.

We note that we defined Ehresmann connections only on $\mathring{T}M$, hence in the statements above, many objects need to be extended smoothly to the zero vectors. These extensions can be carried out in an obvious way.

From now on, we will simply refer to linear Ehresmann connections as *linear connections*.

A linear connection can be constructed directly from the covariant derivative D in (L6): for each $(v, w) \in \mathring{T}M \times_M TM$, we set

$$\mathcal{H}(v, w) := Y_*(w) - \mathbf{i}(v, D_w Y) \qquad \text{if } Y \in \mathfrak{X}(M),\ Y(\tau(v)) = v. \tag{1.47}$$

This can also be expressed as

$$X^{\mathsf{h}} \circ Y = Y_* \circ X - (D_X Y)^{\mathsf{v}} \circ Y, \qquad \text{for all } X, Y \in \mathfrak{X}(M). \tag{1.48}$$

The parallel translations with respect to $\mathcal{H}$ are the same as with respect to D. The torsion and affine curvature of $\mathcal{H}$ (see 1.2.3) can be obtained from the torsion T and curvature R of D by

$$\mathbf{i}\mathbf{T}(\widehat{X}, \widehat{Y}) = (T(X, Y))^{\mathsf{v}}, \quad X, Y \in \mathfrak{X}(M); \tag{1.49}$$

$$\mathbf{i}\mathbf{H}(\widehat{X}, \widehat{Y})\widehat{Z} = (R(X, Y)Z)^{\mathsf{v}}, \quad X, Y, Z \in \mathfrak{X}(M). \tag{1.50}$$

For a more detailed account on linear connections, we refer to sections 7.5 and 7.15 in [45].

A spray S is called *affine* if any of the following equivalent conditions holds:

(A1) S can be smoothly extended to TM.

(A2) The coefficients of S are fibrewise quadratic forms.

(A3) The Berwald connection determined by S is linear.

Condition (A3) implies that we can characterize affine sprays through their Berwald connection, by any of (L1)–(L7).

1.4 Finsler manifolds

Let M be a manifold. A function $F\colon TM \to \mathbb{R}$ is called a *Finsler function* (for M) if it satisfies the following conditions:

(F1) F is continuous on TM and smooth on $\mathring{T}M$,

(F2) F is 1^+-homogeneous,

(F3) $F(v) > 0$ if $v \neq 0$,

(F4) the *fundamental tensor*

$$g := \frac{1}{2}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}F^2 \tag{1.51}$$

is (fibrewise) positive definite.

A *Finsler manifold* is a pair (M, F) consisting of a manifold M and a Finsler function F on TM. We say that $E := \frac{1}{2}F^2$ is the *energy function* associated to F or the energy function (briefly *energy*) of (M, F). A Finsler manifold *reduces to a Riemannian manifold* if its energy is a quadratic form on each tangent space.

A smooth curve γ in M is a *geodesic* of the Finsler manifold, if it satisfies the Euler–Lagrange equation

$$\left(\frac{\partial E}{\partial y^i} \circ \dot{\gamma}\right)' - \frac{\partial E}{\partial x^i} \circ \dot{\gamma} = 0, \qquad i \in \{1, \dots, n\} \tag{1.52}$$

for any chart that intersects the image of γ. There is unique spray S for M whose geodesics are the geodesics of the Finsler manifold. We call it the *canonical spray* of (M, F). Then, using (1.4), we can rewrite (1.52) as

$$SX^{\mathsf{v}}E - X^{\mathsf{c}}E = 0 \qquad \text{for all } X \in \mathfrak{X}(M). \tag{1.53}$$

This form of the Euler–Lagrange equation is due to M. Crampin, see [11, p. 348].

The Berwald connection $\mathcal{H}$ of the canonical spray is called the *canonical connection* of the Finsler manifold. It is the only Ehresmann connection in $\mathring{T}M$ which satisfies the following conditions:

(CC1) $\mathcal{H}$ is positive-homogeneous;

(CC2) the torsion of $\mathcal{H}$ (given by (1.26)) vanishes;

(CC3) $H\mathring{T}M := \operatorname{Im}(\mathcal{H}) \subset \ker dF$.

(For details, see [45, Theorem 9.3.5].) Condition (CC3) guarantees that F is constant along parallel vector fields (see (1.32)), thus the parallel translations with respect to the canonical connection preserve the Finsler function. The proof of this is a simple calculation, as we shall see in Lemma 2.3.1. Hence the canonical connection is a direct analogue of the Levi-Civita connection of a Riemannian manifold, and in fact, if (M, F) is Riemannian, then the canonical connection is the Levi-Civita connection.

1.5 The exponential map of a spray

Throughout this section, (M, S) is a spray manifold. Let $\widetilde{TM}$ be the set of tangent vectors $v \in TM$ such that γ_v, the maximal geodesic of S with initial velocity v is defined at 1. The *exponential map* for S is the mapping

$$\exp\colon \widetilde{TM} \to M, \quad v \mapsto \exp(v) := \gamma_v(1).$$

Then $\widetilde{TM}$ is an open subset in TM [27, p. 91], and by the smooth dependence on initial conditions, exp is smooth on $\widetilde{TM} \cap \mathring{T}M$. Applying (1.39), we see that $t \in [0,1] \mapsto \exp(tv)$ is a geodesic with velocity v at 0.

We often need only the restriction of the exponential map to the tangent space at a certain point p. We denote this mapping by $\exp_p$, its domain is $\widetilde{TM}_p := \widetilde{TM} \cap T_pM$. The following properties are well-known, their proofs can be found, for example, in [40, p. 222] or [45, Lemma 5.1.45].

(i) $\exp_p$ is of class C^1 on $\widetilde{TM}_p$;

(ii) $((\exp_p)_*)_{0_p}$ is the canonical isomorphism which identifies $T_{0_p}T_pM$ with T_pM.

It follows that there is a neighbourhood of 0_p in T_pM on which $\exp_p$ is a C^1 diffeomorphism onto its image. If such a neighbourhood is also star-shaped with respect to 0_p, then it is called a *normal domain* of $\exp_p$. The image of a normal domain under $\exp_p$ is called a *normal neighbourhood* of p. Whenever we need the inverse of $\exp_p$, we use the inverse of its restriction to a normal domain. A normal neighbourhood of p has the nice property that each of its points lies on a geodesic starting from p contained in $\mathcal{U}$, and all such geodesics differ only by orientation preserving reparametrizations. This motivates the notion of an *oriented geodesic path*, a set of geodesics that differ only by orientation preserving reparametrizations. If γ is a representation of an oriented geodesic path c, and $[a,b]$ is in the domain of γ, we say that c is a geodesic path from $\gamma(a)$ to $\gamma(b)$. This terminology allows us to say: *if $\mathcal{U}$ is a normal neighbourhood of p, then for each q, there is one and only one geodesic path in $\mathcal{U}$, from p to q.*

A subset of M is called *totally normal*, if it is a normal neighbourhood of each of its points. J.H.C. Whitehead proved that there exists such a neighbourhood around any point of a spray manifold [49, 50]. Later R.E. Traber presented a simpler argument [47]. A proof based on their works, written in modern language, can be found in [45, 5.1.4].

Chapter 2

Characterization of some spray and Finsler manifolds

2.1 Projectively affine sprays

Two sprays S and $\bar{S}$ for a manifold M are *projectively related* if

$$\bar{S} = S - 2PC \tag{2.1}$$

for some function P on $\mathring{T}M$. Then we also say that $\bar{S}$ is a *projective change of* S *with factor* P. Note that P is necessarily smooth and 1^+-homogeneous. Projective relatedness means that the sprays have the same oriented geodesic paths.

In this section we prove that a spray is projectively related to an affine spray if and only if its Douglas tensor (to be defined in Fact 2.1.2) vanishes. Locally this was first proved by J. Douglas [15]. Global versions are due to Z. Shen [40, 5.2] and J. Szilasi and Sz. Vattamány [46]. However they assumed that the base manifold is orientable. In our version this assumption is eliminated.

Fact 2.1.1. *Let* (M, S) *be a spray manifold. Under a projective change* $\bar{S} := S - 2PC$ *of* S*, the Berwald tensor of* (M, S) *changes by the rule*

$$\bar{\mathbf{B}} = \mathbf{B} + (\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}P) \otimes \tilde{\delta} + (\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}P) \odot \mathrm{id}_{\Gamma(\mathring{\tau}^*\tau)} \tag{2.2}$$

where $\odot$ *is the symmetric product. The change of the trace of the Berwald tensor is given by*

$$\operatorname{tr}\bar{\mathbf{B}} = \operatorname{tr}\mathbf{B} + (n+1)\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}P. \tag{2.3}$$

Both formulae can be easily verified using (1.22) and local calculations. (An index-free proof can be found in [45].)

Fact 2.1.2. *Let (M, S) be a spray manifold, and let $\mathbf{B}$ be the Berwald tensor of (M, S). Then the tensor*

$$\mathbf{D} := \mathbf{B} - \frac{1}{n+1}\Big((\nabla^{\mathsf{v}} \operatorname{tr} \mathbf{B}) \otimes \tilde{\delta} + (\operatorname{tr} \mathbf{B}) \odot \mathrm{id}_{\Gamma(\mathring{\tau}^* \tau)}\Big) \tag{2.4}$$

is the same for all projective changes of S.

The proof is straightforward using Fact 2.1.1. The tensor $\mathbf{D}$ defined in the lemma is called the *Douglas tensor* of the spray manifold.

Fact 2.1.3. *If a spray is projectively related to an affine spray, then its Douglas tensor vanishes.*

Proof. The Berwald tensor of an affine spray vanishes, see (A3) and (L4) in section 1.3. Hence its Douglas tensor also vanishes by (2.4). Since the Douglas tensor is projectively invariant, our statement follows. □

In the remainder of the section we prove the converse: if the Douglas tensor of a spray vanishes, then it is projectively related to an affine spray. It turns out that the projective factor that produces the affine spray can be constructed directly: it is the divergence of the spray with respect to a volume form ω on the tangent manifold TM that satisfies $\mathcal{L}_{X^{\mathsf{v}}}\omega = 0$ for all $X \in \mathfrak{X}(M)$. Such volume forms are called *vertically invariant.*

We recall that given a volume form μ on a manifold M, the divergence of a vector field $X \in \mathfrak{X}(M)$ is the unique function $\operatorname{div}_\mu X$ satisfying $\mathcal{L}_X \mu = (\operatorname{div}_\mu X)\mu$. In particular, if $(\mathcal{U}, (u^i)_{i=1}^n)$ is a chart on M and

$$\mu \underset{(\mathcal{U})}{=} \sigma du^1 \wedge \cdots \wedge du^n, \quad X \underset{(\mathcal{U})}{=} \sum_{i=1}^n X^i \frac{\partial}{\partial u^i},$$

where $\sigma \in C^\infty(\mathcal{U})$, then

$$\operatorname{div}_\mu X = \sum_{k=1}^n \frac{1}{\sigma} \frac{\partial(\sigma X^k)}{\partial u^k}. \tag{2.5}$$

Lemma 2.1.4. *Let M be a manifold.*

(i) A volume form ω on TM is vertically invariant if and only if $\operatorname{div}_\omega X^{\mathsf{v}} = 0$ *for all $X \in \mathfrak{X}(M)$.*

(ii) Suppose that ω is a vertically invariant volume form on TM, and let $\tilde{\omega}$ be another volume form on TM. Then $\tilde{\omega}$ is vertically invariant if, and only if, $\tilde{\omega} = f^{\mathsf{v}}\omega$ for some smooth function f on M.

Proof. The first assertion is clear from the definition of divergence. As to the second claim, we surely have $\widetilde{\omega} = h\omega$ for some smooth function h on TM. If $\widetilde{\omega}$ is vertically invariant, then for any vector field X on M,

$$0 = \mathcal{L}_{X^{\mathsf{v}}}\widetilde{\omega} = \mathcal{L}_{X^{\mathsf{v}}}(h\omega) = (X^{\mathsf{v}}h)\omega + h\mathcal{L}_{X^{\mathsf{v}}}\omega = (X^{\mathsf{v}}h)\omega.$$

Thus $X^{\mathsf{v}}h = 0$ for all $X \in \mathfrak{X}(M)$, hence h is constant on the tangent spaces. Therefore h is a vertical lift. The converse can be shown by a similar argument. □

Remark 2.1.5. Let $(\mathcal{U}, (u^i)_{i=1}^n)$ be a chart for M and let $(\tau^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^i)_{i=1}^n))$ be its induced chart on TM. Then

$$\mu := dx^1 \wedge \dots \wedge dx^n \wedge dy^1 \wedge \dots \wedge dy^n$$

is a vertically invariant volume form on $\tau^{-1}(\mathcal{U})$.

Indeed, if $X = X^i \frac{\partial}{\partial u^i}$ is a vector field on $\mathcal{U}$, then its vertical lift is $X^{\mathsf{v}} = (X^i)^{\mathsf{v}} \frac{\partial}{\partial y^i} \in \mathfrak{X}(\tau^{-1}(\mathcal{U}))$, and

$$\operatorname{div}_\mu(X^{\mathsf{v}}) \overset{(2.5)}{=} \sum_{i=1}^n \frac{\partial (X^i)^{\mathsf{v}}}{\partial y^i} = 0,$$

so our claim follows by part (i) of the previous lemma. We say that μ is an *induced volume form* on $\tau^{-1}(\mathcal{U})$. In view of part (ii) of Lemma 2.1.4, every vertically invariant volume form on TM can locally be written as $f^{\mathsf{v}}\mu$, where f is a smooth function on M.

Lemma 2.1.6. *Let M be an n-dimensional manifold. If ω is a vertically invariant volume form on TM and $C \in \mathfrak{X}(TM)$ is the Liouville vector field, then* $\operatorname{div}_\omega C = n$.

Proof. We work locally. With the notation of the previous remark, let $\omega \underset{(\mathcal{U})}{=} f^{\mathsf{v}}\mu$. Then, since $C \underset{(\mathcal{U})}{=} y^i \frac{\partial}{\partial y^i}$,

$$\operatorname{div}_\omega C \underset{(\mathcal{U})}{\overset{(2.5)}{=}} \sum_{i=1}^n \frac{1}{f^{\mathsf{v}}} \frac{\partial (f^{\mathsf{v}} y^i)}{\partial y^i} = \sum_{i=1}^n \frac{\partial y^i}{\partial y^i} = n.$$ □

Lemma 2.1.7. *Let (M, S) be a spray manifold. If ω is a vertically invariant volume form on TM, then*

$$\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}(\operatorname{div}_\omega S) = -2 \operatorname{tr} \mathbf{B}. \tag{2.6}$$

Proof. With the notation of Remark 2.1.5, we calculate locally. Then $\omega \underset{(\mathcal{U})}{=} f^{\mathsf{v}}\mu$, where f is a smooth function on $\mathcal{U}$ and μ is the induced volume form on $\tau^{-1}(\mathcal{U})$. If $S \underset{(\mathcal{U})}{=} y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$, then

$$\operatorname{div}_\omega S \underset{(\mathcal{U})}{\overset{(2.5)}{=}} \sum_{i=1}^n \left(\frac{1}{f^{\mathsf{v}}} \frac{\partial (f^{\mathsf{v}} y^i)}{\partial x^i} - \frac{2}{f^{\mathsf{v}}} \frac{\partial (f^{\mathsf{v}} G^i)}{\partial y^i} \right)$$

$$\overset{(1.6),(1.9),(1.7)}{=} \sum_{i=1}^{n} \left(\frac{y^i}{f^{\mathsf{v}}} \left(\frac{\partial f}{\partial u^i} \right)^{\mathsf{v}} - 2\frac{\partial G^i}{\partial y^i} \right).$$

Thus, taking into account (1.2),

$$\operatorname{div}_\omega S \underset{(\mathcal{U})}{=} \frac{f^{\mathsf{c}}}{f^{\mathsf{v}}} - 2\sum_{i=1}^{n} \frac{\partial G^i}{\partial y^i} \overset{(1.44)}{=} \frac{f^{\mathsf{c}}}{f^{\mathsf{v}}} - 2G^i_i. \tag{2.7}$$

Now we turn to the proof of (2.6). For any $j, k \in \{1, \dots, n\}$,

$$\begin{aligned}
\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}(\operatorname{div}_\omega S)\left(\widehat{\frac{\partial}{\partial u^j}}, \widehat{\frac{\partial}{\partial u^k}} \right) &\overset{(2.7)}{=} \nabla^{\mathsf{v}}\nabla^{\mathsf{v}}\left(\frac{f^{\mathsf{c}}}{f^{\mathsf{v}}} - 2G^i_i \right)\left(\widehat{\frac{\partial}{\partial u^j}}, \widehat{\frac{\partial}{\partial u^k}} \right) \\
&\overset{(1.21)}{=} \frac{\partial}{\partial y^j}\left(\frac{\partial}{\partial y^k}\left(\frac{f^{\mathsf{c}}}{f^{\mathsf{v}}} - 2G^i_i \right) \right) \\
&= \left(\frac{\partial}{\partial u^j} \right)^{\mathsf{v}} \left(\left(\frac{\partial f}{\partial u^k} \right)^{\mathsf{v}} \frac{1}{f^{\mathsf{v}}} \right) - 2\frac{\partial^2 G^i_i}{\partial y^j \partial y^k} = -2\frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^i} \\
&\overset{(1.45)}{=} -2B^i_{ijk} = -2\operatorname{tr}\mathbf{B}\left(\widehat{\frac{\partial}{\partial u^j}}, \widehat{\frac{\partial}{\partial u^k}} \right).
\end{aligned}$$

□

Theorem 2.1.8 ([45]). *Let (M, S) be a spray manifold with vanishing Douglas tensor and let ω be a vertically invariant volume form on TM. Then*

$$\bar{S} := S - 2PC, \quad \textit{where} \quad P := \frac{1}{2(n+1)}\operatorname{div}_\omega S,$$

is an affine spray.

Proof. First we show that $\bar{S}$ is a spray. We only have to check that P is 1^+-homogeneous. We have

$$\begin{aligned}
CP &= \frac{1}{2(n+1)} C \operatorname{div}_\omega S \underset{(\mathcal{U})}{\overset{(2.7)}{=}} \frac{1}{2(n+1)}\left(C\frac{f^{\mathsf{c}}}{f^{\mathsf{v}}} - C2G^i_i \right) \\
&\overset{(1.11)}{=} \frac{1}{2(n+1)}\left(\frac{f^{\mathsf{c}}}{f^{\mathsf{v}}} - 2CG^i_i \right) = \frac{1}{2(n+1)}\left(\frac{f^{\mathsf{c}}}{f^{\mathsf{v}}} - 2G^i_i \right) \underset{(\mathcal{U})}{=} P,
\end{aligned}$$

taking into account that the Christoffel symbols G^i_j of the Berwald connection of (M, S) are also 1^+-homogeneous. This proves the desired homogeneity. Then notice that

$$\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}P = \frac{1}{2(n+1)}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}(\operatorname{div}_\omega S) \overset{(2.6)}{=} -\frac{1}{n+1}\operatorname{tr}\mathbf{B}. \tag{$*$}$$

For the Berwald tensor of $\bar{S}$ we obtain that

$$\begin{aligned}\bar{\mathbf{B}} &\overset{(2.2)}{=} \mathbf{B} + (\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}P)\otimes\tilde{\delta} + (\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}P)\odot \mathrm{id}_{\Gamma(\mathring{\tau}^*\tau)}\\ &\overset{(*)}{=} \mathbf{B} - \frac{1}{n+1}\Big((\nabla^{\mathsf{v}}\operatorname{tr}\mathbf{B})\otimes\tilde{\delta} + (\operatorname{tr}\mathbf{B})\odot\mathrm{id}_{\Gamma(\mathring{\tau}^*\tau)}\Big) \overset{(2.4)}{=:} \mathbf{D} = 0.\end{aligned}$$

So the Berwald tensor of $\bar{S}$ vanishes, hence it is an affine spray. □

Shen's proof is similar. However, in his construction of the projective factor a volume form on M was applied, hence his method cannot be globalized. Our construction relies on the existence of a suitable volume form on the tangent manifold, which, as the next proposition shows, always exists.

Proposition 2.1.9 ([45])**.** *The tangent manifold of a manifold admits a vertically invariant volume form.*

Proof. Let M be an n-dimensional manifold and consider its $2n$-dimensional tangent manifold TM. Suppose that $(u^i)_{i=1}^n$ and $(\bar{u}^i)_{i=1}^n$ are local coordinate systems on M with the same domain $\mathcal{U}$, and consider the induced coordinate systems $((x^i)_{i=1}^n,(y^i)_{i=1}^n)$ and $((\bar{x}^i)_{i=1}^n,(\bar{y}^i)_{i=1}^n)$ on TM.
Step 1. We show that

$$d\bar{x}^i = (A^i_j)^{\mathsf{v}}dx^j,\quad d\bar{y}^i = (A^i_j)^{\mathsf{c}}dx^j + (A^i_j)^{\mathsf{v}}dy^j,\quad i\in\{1,\dots,n\}, \tag{2.8}$$

where $A^i_j := \frac{\partial\bar{u}^i}{\partial u^j}$; $i,j\in J_n$. This can be done by an immediate calculation:

$$\begin{aligned}d\bar{x}^i &= \frac{\partial\bar{x}^i}{\partial x^j}dx^j + \frac{\partial\bar{x}^i}{\partial y^j}dy^j \overset{(1.1),(1.6)}{=} \left(\left(\frac{\partial}{\partial u^j}\right)^{\mathsf{c}}(\bar{u}^i)^{\mathsf{v}}\right)dx^j\\ &\quad+\left(\left(\frac{\partial}{\partial u^j}\right)^{\mathsf{v}}(\bar{u}^i)^{\mathsf{v}}\right)dy^j \overset{(1.9),(1.7)}{=} \left(\frac{\partial\bar{u}^i}{\partial u^j}\right)^{\mathsf{v}}dx^j = (A^i_j)^{\mathsf{v}}dx^j;\\ d\bar{y}^i &= \frac{\partial\bar{y}^i}{\partial x^j}dx^j + \frac{\partial\bar{y}^i}{\partial y^j}dy^j = \left(\left(\frac{\partial}{\partial u^j}\right)^{\mathsf{c}}(\bar{u}^i)^{\mathsf{c}}\right)dx^j + \left(\left(\frac{\partial}{\partial u^j}\right)^{\mathsf{v}}(\bar{u}^i)^{\mathsf{c}}\right)dy^j\\ &\overset{(1.10),(1.8)}{=} \left(\frac{\partial\bar{u}^i}{\partial u^j}\right)^{\mathsf{c}}dx^j + \left(\frac{\partial\bar{u}^i}{\partial u^j}\right)^{\mathsf{v}}dy^j = (A^i_j)^{\mathsf{c}}dx^j + (A^i_j)^{\mathsf{v}}dy^j.\end{aligned}$$

Step 2. Consider the induced volume forms

$$\mu = dx^1\wedge\dots\wedge dx^n\wedge dy^1\wedge\dots\wedge dy^n$$

and

$$\bar{\mu} = d\bar{x}^1\wedge\dots\wedge d\bar{x}^n\wedge d\bar{y}^1\wedge\dots\wedge d\bar{y}^n.$$

We claim that the volume forms μ and $\bar{\mu}$ represent the same orientation of the vector bundle $\tau_{TM} \upharpoonright \tau^{-1}(\mathcal{U})$. To see this, we express the volume form $\bar{\mu}$ in terms of the coordinate differentials dx^i and dy^i. Since the wedge product is $\mathcal{C}^\infty(TM)$-multilinear and $\alpha \wedge \alpha = 0$ for any one-form α, we find that

$$\begin{aligned}\bar{\mu}_1 &:= d\bar{x}^1 \wedge \dots \wedge d\bar{x}^n \overset{(2.8)}{=} (A^1_{j_1})^{\mathsf{v}} dx^{j_1} \wedge \dots \wedge (A^n_{j_n})^{\mathsf{v}} dx^{j_n} \\ &= \sum_{\sigma \in S_n} (A^1_{\sigma(1)})^{\mathsf{v}} dx^{\sigma(1)} \wedge \dots \wedge (A^n_{\sigma(n)})^{\mathsf{v}} dx^{\sigma(n)} \\ &= \left(\sum_{\sigma \in S_n} \varepsilon(\sigma) (A^1_{\sigma(1)})^{\mathsf{v}} \dots (A^n_{\sigma(n)})^{\mathsf{v}} \right) dx^1 \wedge \dots \wedge dx^n \\ &= \det((A^i_j)^{\mathsf{v}}) dx^1 \wedge \dots \wedge dx^n = (\det(A^i_j))^{\mathsf{v}} dx^1 \wedge \dots \wedge dx^n.\end{aligned}$$

Thus

$$\begin{aligned}\bar{\mu}_1 \wedge d\bar{y}^1 &= (\det(A^i_j))^{\mathsf{v}} dx^1 \wedge \dots \wedge dx^n \wedge ((A^1_j)^{\mathsf{c}} dx^j + (A^1_j)^{\mathsf{v}} dy^j) \\ &= (\det(A^i_j))^{\mathsf{v}} dx^1 \wedge \dots \wedge dx^n \wedge ((A^1_j)^{\mathsf{v}} dy^j) = \bar{\mu}_1 \wedge ((A^1_j)^{\mathsf{v}} dy^j).\end{aligned}$$

Continuing in the same way, we obtain that

$$\begin{aligned}\bar{\mu} &= \bar{\mu}_1 \wedge d\bar{y}^1 \wedge \dots \wedge d\bar{y}^n = \bar{\mu}_1 \wedge (A^1_{j_1})^{\mathsf{v}} dy^{j_1} \wedge \dots \wedge (A^n_{j_n})^{\mathsf{v}} dy^{j_n} \\ &= \bar{\mu}_1 \wedge (\det(A^i_j))^{\mathsf{v}} \wedge dy^1 \wedge \dots \wedge dy^n = ((\det(A^i_j))^2)^{\mathsf{v}} \mu.\end{aligned}$$

Here the function $((\det(A^i_j))^2)^{\mathsf{v}}$ is positive, so μ and $\bar{\mu}$ represent the same orientation.
Step 3. Let $\mathcal{A} = \{(\mathcal{U}_i, u_i) \mid i \in I\}$ be an atlas of M, and let $(f_i)_{i \in I}$ be a partition of unity on M subordinate to the covering $(\mathcal{U}_i)_{i \in I}$. Every chart $(\mathcal{U}_i, u_i)$ yields an induced volume form μ_i on $\tau^{-1}(\mathcal{U})$. Define a top form ω on TM by

$$v \in TM \mapsto \omega_v := \sum_{i \in I} (f_i)^{\mathsf{v}}(v) \mu_i(v) \in A_{2n}(T_v TM).$$

For an arbitrarily fixed vector w in TM, let

$$I(w) := \{i \in I \mid \tau(w) \in \mathcal{U}_i,\ f_i(\tau(v)) \neq 0\} \quad \text{and} \quad \mathcal{U}_w := \bigcap_{j \in I(w)} \mathcal{U}_j.$$

Then, by Step 2,

$$\mu_j \underset{(\mathcal{U}_w)}{=} (h_j)^{\mathsf{v}} \mu_{j_0}, \quad j \in I(w),$$

where $j_0 \in I(w)$ is a fixed index and h_j is a positive smooth function on $\mathcal{U}_w$. So it follows that

$$\omega \underset{(\mathcal{U}_w)}{=} \sum_{j \in I(w)} (f_j h_j)^{\mathsf{v}} \mu_{j_0},$$

and here $\sum_{j\in I(w)}(f_j h_j)^{\mathsf{v}} \in \mathcal{C}^\infty(\tau^{-1}(\mathcal{U}_w))$ is a positive function. Thus, taking into account Remark 2.1.5, ω is a vertically invariant volume form on $\tau^{-1}(\mathcal{U}_w)$. Since the vector w can be chosen arbitrarily, this completes the proof. □

This proposition and Theorem 2.1.8 yield:

Corollary 2.1.10 ([45]). *If the Douglas tensor of a spray manifold vanishes, then the spray is projectively related to an affine spray.*

2.2 There are no proper Einstein–Berwald manifolds

A *Berwald manifold* is a Finsler manifold whose canonical spray is affine. This implies that its canonical connection is linear, and hence it is generated by a (torsion-free) covariant derivative (see section 1.3).

A Finsler manifold (M, F) is called an *Einstein–Finsler manifold* if the curvature function of its canonical spray given by (1.31) is related to the Finsler function by

$$K = \lambda^{\mathsf{v}} F^2, \tag{2.9}$$

where $\lambda \in C^\infty(M)$.

Lemma 2.2.1 ([14]). *The curvature function of a Berwald manifold is a quadratic form on each tangent space.*

Proof. Notice first that the Jacobi endomorphism of the canonical spray can be expressed in terms of the affine curvature tensor:

$$\mathbf{K}_u(v) \overset{(1.30)}{=} \mathcal{R}_u(u, v) = -\mathcal{R}_u(v, u) \overset{(1.36)}{=} \mathbf{H}_u(v, u, u).$$

Let D be the covariant derivative on M that generates the canonical connection of (M, F). Then, from (1.50), for any $p \in M$, $u, v \in T_pM$ we get

$$\mathbf{K}_u(v) = (u, R_p(v, u)u), \tag{2.10}$$

where R is the curvature tensor of D. Thus

$$(n-1)K(v) = \operatorname{tr}\mathbf{K}(v) \overset{(2.10)}{=} \operatorname{tr}(u \mapsto (u, R(u, v), v)),$$

hence K is indeed a quadratic form on the tangent spaces. □

Theorem 2.2.2 ([14]). *If a connected Einstein–Finsler manifold is a Berwald manifold, then it is either a Riemannian manifold or its curvature function vanishes.*

Proof. If λ in (2.9) is everywhere zero, we are done. If there is a point $p \in M$ where λ does not vanish, then

$$F^2(u) = \frac{K(u)}{\lambda(p)} \quad \text{for all } u \in T_pM,$$

thus $F^2 \restriction T_pM$ is a quadratic form by the previous lemma. Since (M, F) is a Berwald manifold, the parallel translations are linear. These parallel translations preserve F, thus F^2 is a quadratic form on each tangent space, therefore F is Riemannian. □

The argument presented here is from [22], which is a slightly simplified version of our proof in [14].

2.3 Monochromatic Finsler manifolds

Let M be a manifold and $\mathcal{U}$ an open submanifold of M. Consider a frame field $(X_i)_{i=1}^n$ on $\mathcal{U}$. Then the $\mathbb{R}$-linear span of $(X_i)_{i=1}^n$ is called a *local parallelization (of M) over $\mathcal{U}$*. Given a local parallelization P over $\mathcal{U}$, a vector field in P is called *parallel* with respect to P. For any $v \in \tau^{-1}(\mathcal{U})$, there is a unique parallel vector field v_P (w.r.t. P), such that $v_P(\tau(v)) = v$. For each $p, q \in \mathcal{U}$, we obtain a mapping $P_{qp} \colon T_pM \to T_qM$ by setting $P_{qp}(v) := v_P(q)$. The construction guarantees that these mappings are linear. The concept of parallelization defined this way is equivalent to the one in [18, p. 174]. A (not necessarily smooth) function $f \colon \mathring{T}M \to \mathbb{R}$ is *compatible with* P, if $f \circ P_{qp} = f$ over $\mathring{T}_pM$ for all $p, q \in \mathcal{U}$. Equivalently, $f \circ X$ is constant for all $X \in P$.

We say that a (not necessarily smooth) function $f \colon \mathring{T}M \to \mathbb{R}$ is *monochromatic*, if for any $p, q \in M$, there is a linear isomorphism L_{qp} from T_pM to T_qM such that $f \circ L_{qp} = f$ over $\mathring{T}_pM$. We say that f is *smoothly monochromatic*, if for any point $p \in M$ there is a local parallelization over an open neighbourhood of p, which is compatible with f. It is easy to see that smoothly monochromatic functions are monochromatic if M is connected.

Similarly, we say that a (not necessarily smooth) function $f \colon \mathring{T}M \to \mathbb{R}$ is *compatible with a homogeneous Ehresmann connection* $\mathcal{H}$ if the parallel translations with respect to $\mathcal{H}$ preserve f. In other words, f is constant along $\mathcal{H}$-parallel vector fields (cf. section 1.2.4).

For functions of class C^1, the compatibility with an Ehresmann connection has the following simple characterization.

Lemma 2.3.1. *A function $f \colon \mathring{T}M \to \mathbb{R}$ of class C^1 is compatible with a homogeneous Ehresmann connection if and only if the corresponding horizontal subbundle is a subset of* $\ker df$.

Proof. Let $X\colon I \to TM$ be a parallel vector field along a curve $\gamma\colon I \to M$. Then

$$(f \circ X)'(t) = \dot{X}(t)f \overset{(1.32)}{=} \mathcal{H}(X(t), \dot{\gamma}(t))f = df(\mathcal{H}(X(t), \dot{\gamma}(t)))$$

for all $t \in I$. So $f \circ X$ is constant if and only if $\mathcal{H}(X(t), \dot{\gamma}(t))$ is in $\ker df$ for all $t \in I$. Since $(X(t), \dot{\gamma}(t))$ can be any point in $\mathring{T}M \times_M TM$, this proves the desired equivalence. □

Notice that the lemma above also shows that a Finsler function is compatible with its canonical connection.

A Finsler manifold (M, F) is a *generalized Berwald manifold* if there is a linear connection in $\mathring{T}M$ compatible with F. If this linear connection is torsion-free, then it is necessarily the canonical connection, hence (M, F) is a Berwald manifold.

We prove that if a smoothly monochromatic function $f\colon \mathring{T}M \to \mathbb{R}$ is of class C^1, then it is compatible with a linear connection. In particular, if F is a smoothly monochromatic Finsler function, then (M, F) is a generalized Berwald manifold. The latter result is due to Y. Ichijyō [21, Theorem 2]. We note that his proof relied on G-structures and their compatible connections. Our proof relies merely on simple properties of Ehresmann connections and utilizes only the differentiability of the Finsler function, and so is very simple. Another generalization of Y. Ichijyō's result, in which even differentiability is omitted, can be found in [2].

Lemma 2.3.2. *Any affine combination of Ehresmann connections is an Ehresmann connection. More precisely, if $\mathcal{H}_1, \dots, \mathcal{H}_k$ are Ehresmann connections in $\mathring{T}M$ and $f_1, \dots, f_k \in C^\infty(\mathring{T}M)$ such that $\sum_{i=1}^k f_i = 1$, then*

$$\mathcal{H}\colon \mathring{T}M \times_M TM \to TTM, \quad \mathcal{H}(u, v) := \sum_{i=1}^k f_i(u)\mathcal{H}_i(u, v)$$

is an Ehresmann connection in $\mathring{T}M$. If $\mathcal{H}_1, \dots, \mathcal{H}_k$ are linear and $f_1, \dots, f_k$ are vertical lifts, then $\mathcal{H}$ is also linear.

Proof. To prove the first claim, we have to show that $\mathcal{H}$ satisfies conditions (E1)–(E3) from 1.2.1. Clearly, $\mathcal{H}(u, v)$ is in T_uTM, and $\mathcal{H}(u, v)$ is $\mathbb{R}$-linear in v, so (E1) and (E2) are satisfied. Since

$$\begin{aligned}\tau_*\mathcal{H}(u, v) &= \tau_*\left(\sum_{i=1}^k f_i(u)\mathcal{H}_i(u, v)\right) \\ &= \sum_{i=1}^k f_i(u)\tau_*(\mathcal{H}_i(u, v)) \overset{(E3)}{=} \sum_{i=1}^k f_i(u)v = v,\end{aligned}$$

(E3) is also satisfied. We prove the second claim using (L5). We have to check that $[\mathcal{H}\widehat{X}, Y^{\mathsf{v}}]$ is a vertical lift. However, this is easy:

$$[\mathcal{H}\widehat{X}, Y^{\mathsf{v}}] = \left[\sum_{i=1}^{k} f_i^{\mathsf{v}}\mathcal{H}_i\widehat{X}, Y^{\mathsf{v}}\right] = \sum_{i=1}^{k}[f_i^{\mathsf{v}}\mathcal{H}_i\widehat{X}, Y^{\mathsf{v}}] \overset{(1.7)}{=} \sum_{i=1}^{k} f_i^{\mathsf{v}}[\mathcal{H}_i\widehat{X}, Y^{\mathsf{v}}].$$

The Ehresmann connections $\mathcal{H}_1, \dots, \mathcal{H}_k$ are linear, hence the right-hand side here is a vertical lift. □

Theorem 2.3.3. *If a function $f\colon \mathring{T}M \to \mathbb{R}$ of class C^1 is smoothly monochromatic, then it is compatible with a linear connection.*

Proof. Let P be a local parallelization over $\mathcal{U}$, compatible with f. Define a covariant derivative D on $\mathcal{U}$ by setting

$$DX = 0 \tag{2.11}$$

for all $X \in P$. Let $\mathcal{H}$ be the linear connection in $\mathring{T}\mathcal{U}$ generated by D. Then $\mathcal{H}$ is compatible with f by Lemma 2.3.1. Indeed, for any $v, w \in \mathring{T}M \times_M TM$,

$$\mathcal{H}(v, w)f \overset{(1.47)}{=} v_{P*}(w)f - \mathbf{i}(v, D_w v_P)f \overset{(2.11)}{=} v_{P*}(w)f = w(f \circ v_P) = 0,$$

since f is constant along the vector fields in P.

By condition, there is an open cover $(\mathcal{U}_\alpha)_{\alpha\in\mathcal{A}}$ of M such that for each $\alpha \in \mathcal{A}$, there is a local parallelization P^α over $\mathcal{U}_\alpha$, compatible with f. Then, through the construction above, we obtain a linear connection $\mathcal{H}^\alpha$ in each $\mathring{\tau}^{-1}(\mathcal{U}_\alpha)$, compatible with f. Let $(\phi_\alpha)_{\alpha\in\mathcal{A}}$ be a partition of unity subordinate to our covering. By Lemma 2.3.2, the affine combination $\sum_{\alpha\in\mathcal{A}} \phi_\alpha^{\mathsf{v}}\mathcal{H}^\alpha$ yields a linear connection in $\mathring{T}M$. Then Lemma 2.3.1 implies that this new Ehresmann connection is still compatible with f, because $\ker df$ is closed under affine combinations. □

The converse of Theorem 2.3.3 is true without any restriction on the function. To show this, we need a few technical results. The first is well-known.

Fact 2.3.4 (Smooth Dependence Theorem)**.** *Let $\mathcal{V}$ be an open subset of a finite-dimensional real vector space, and let $\psi\colon \mathcal{V} \to \mathfrak{X}(M)$ be a mapping, such that*

$$\widetilde{\psi}\colon \mathcal{V} \times M \to TM,\ (v, p) \mapsto \widetilde{\psi}(v, p) := \psi(v)(p)$$

is a smooth mapping ('the vector field $\psi(v)$ depends smoothly on v'). Then for each $(p_0, v_0) \in M \times \mathcal{V}$ there exist

an open neighbourhood $\mathcal{U}$ of p_0 in M,

an open interval I containing 0,

an open neighbourhood $\mathcal{W}$ *of* v_0 *in* $\mathcal{V}$*,*

a smooth mapping $\varphi\colon I \times \mathcal{U} \times \mathcal{W} \to M$

such that for each $(p, v) \in \mathcal{U} \times \mathcal{W}$ *the curve*

$$\varphi_p^v\colon I \to M,\ t \mapsto \varphi_p^v(t) := \varphi(t, p, v)$$

is an integral curve of the vector field $\psi(v)$ *starting at* p*, i.e.,*

$$\dot{\varphi}_p^v = \psi(v) \circ \varphi_p^v, \quad \varphi_p^v(0) = p.$$

For a proof, see [20, Proposition 7.5.15]. The next lemma, and lemmas of similar flavour are considered 'simple' consequences of the smooth dependence theorem, however, the technical details are not straightforward.

Lemma 2.3.5. *Let* $\mathcal{H}$ *be a linear connection in* $\mathring{T}M$ *and let* S *be an affine spray. Fix a point* p *in* M*, let* $\mathcal{V}$ *be a normal domain of* $\exp_p$ *(with respect to* S*), and abbreviate* $\exp := \exp_p \restriction \mathcal{V}$*. For any* $b \in T_pM$*, there exist a positive real number* t_0 *and an open neighbourhood* $\mathcal{U}_0 \subset \mathcal{U} := \exp(\mathcal{V})$ *of* p *such that the mapping*

$$\begin{cases} X\colon \mathcal{U}_0 \to TM,\ q \mapsto X(q) := P_0^{t_0}(\gamma_q)(b), \\ \gamma_q\colon [0, t_0] \to \mathcal{U}_0, \quad \gamma_q(t) = \exp_p(t \exp^{-1}(q)) \end{cases} \tag{2.12}$$

is a (smooth) vector field on $\mathcal{U}_0$*.*

Proof. Notice that exp is smooth everywhere, since S is an affine spray.
Step 1. Define a mapping

$$Z\colon \mathcal{V} \times \mathcal{U} \subset T_pM \times M \to \tau^{-1}(\mathcal{U}) \subset TM$$

by

$$Z(v, q) := \exp_*\Big(\mathbf{i}(\exp^{-1}(q), v)\Big); \quad v \in \mathcal{V}, q \in \mathcal{U},$$

We show that for any fixed $v \in \mathcal{V}$, *the mapping*

$$Z(v)\colon \mathcal{U} \to T\mathcal{U},\ q \mapsto Z(v)(q) = Z(v, q)$$

is a vector field on $\mathcal{U}$ *whose integral curve starting at* p *is just the geodesic* γ_v *of* S *with initial velocity* v*.*

Indeed, the smoothness of $Z(v)$ is guaranteed by the construction, and it is easy to check that $\tau \circ Z(v) = 1_{\mathcal{U}}$, so $Z(v) \in \mathfrak{X}(\mathcal{U})$. Further, let t be an element of the domain of γ_v, and define $\alpha\colon s \in \mathbb{R} \mapsto sv \in T_pM$. Then we have $\gamma_v = \exp_p \circ \alpha$ and

$$Z(v)(\gamma_v(t)) = \exp_*\Big(\mathbf{i}(\exp^{-1}(\gamma_v(t)), v)\Big) = \exp_*(\mathbf{i}(\alpha(t), v))$$

$$= \exp_*(\dot{\alpha}(t)) = (\exp_p \circ \alpha)^{\cdot}(t) = \dot{\gamma}_v(t),$$

as desired.

Step 2. With the help of the mapping Z, define a further mapping

$$\widetilde{Z}\colon \mathcal{V} \times \tau^{-1}(\mathcal{U}) \subset T_pM \times TM \to T(\tau^{-1}(\mathcal{U})),\ (v,w) \mapsto \widetilde{Z}(v,w)$$

by

$$\widetilde{Z}(v,w) := (Z(v))^{\mathsf{h}}(w) = \mathcal{H}\Big(w, Z(v)(\tau(w))\Big).$$

Then $\widetilde{Z}$ satisfies the conditions of Fact 2.3.4, and so there exist

an open neighbourhood $\mathcal{U}_1$ of b in TM,

an open interval I containing 0,

an open neighbourhood $\mathcal{W}$ of 0_p in $\mathcal{V}$,

a smooth mapping $\varphi\colon I \times \mathcal{U}_1 \times \mathcal{W} \to \tau^{-1}(\mathcal{U})$

such that for each $(w,v) \in \mathcal{U}_1 \times \mathcal{W}$ the curve

$$\varphi^v_w\colon I \to T\mathcal{U},\ t \mapsto \varphi^v_w(t) := \varphi(t,w,v)$$

is an integral curve of $(Z(v))^{\mathsf{h}}$ starting at w, i.e.,

$$(Z(v))^{\mathsf{h}} \circ \varphi^v_w = \dot{\varphi}^v_w,\ \varphi^v_w(0) = w. \qquad (*)$$

Step 3. Consider, in particular, the curve φ^v_b, where b is the given vector in T_pM and $v \in W$. We show that

$$\tau \circ \varphi^v_b = \gamma_v. \qquad (**)$$

To see this, we calculate the velocity vector of $\tau \circ \varphi^v_b$ at a point $t \in I$. We find that

$$\dot{\overline{\tau \circ \varphi^v_b}}(t) = \tau_*(\dot{\varphi}^v_b(t)) \overset{(*)}{=} \tau_* \circ (Z(v))^{\mathsf{h}} \circ \varphi^v_b(t) \overset{\text{(E3)}}{=} Z(v) \circ \tau \circ \varphi^v_b(t),$$

therefore $\tau \circ \varphi^v_b$ is the integral curve of $Z(v)$ starting at p. Thus, by Step 1, we obtain $(**)$. From this it follows that φ^v_b is a vector field along γ_v.

Step 4. We show that φ^v_b *is an* $\mathcal{H}$*-parallel vector field along* γ_v. Indeed, for every $t \in I$ we have

$$\begin{aligned}\dot{\varphi}^v_b(t) &\overset{(*)}{=} (Z(v))^{\mathsf{h}}(\varphi^v_b(t)) = \mathcal{H}(\varphi^v_b(t), Z(v) \circ \tau \circ \varphi^v_b(t)) \\ &\overset{(**)}{=} \mathcal{H}(\varphi^v_b(t), Z(v) \circ \gamma_v(t)) \overset{\text{Step 1}}{=} \mathcal{H}(\varphi^v_b(t), \dot{\gamma}_v(t)),\end{aligned}$$

so for any fixed $t_0 \in I$ we obtain

$$P_0^{t_0}(\gamma_v)(b) = \varphi_b^v(t_0).$$

Now let $\mathcal{U}_0 := \exp(\mathcal{W})$. Then for every $q = \exp(v) \in \mathcal{U}_0$,

$$\begin{aligned} X(q) &:= P_0^{t_0}(\gamma_q)(b) = P_0^{t_0}(\gamma_v)(b) = \varphi_b^v(t_0) \\ &:= \varphi(t_0, b, v) = \varphi(t_0, b, \exp^{-1}(q)), \end{aligned}$$

which proves the smoothness of X. □

Notice that in the above lemma no relation between $\mathcal{H}$ and S is assumed.

Proposition 2.3.6. *If a not necessarily smooth function $f\colon \mathring{T}M \to \mathbb{R}$ is compatible with a linear connection $\mathcal{H}$, then f is smoothly monochromatic.*

Proof. Fix a point $p \in M$. For each $v \in T_pM$, let v_P denote the smooth vector field constructed in Lemma 2.3.5 (with respect to $\mathcal{H}$ and any affine spray S). If $(v_i)_{i=1}^n$ is a basis of T_pM, then $(v_{iP})_{i=1}^n$ is a frame field on some neighbourhood $\mathcal{U}$ of p, because the parallel translations $P(\gamma_q)_0^{t_0}$ are linear isomorphisms. For the same reason, the $\mathbb{R}$-linear span of $(v_{iP})_{i=1}^n$ is $\{v_P \in \mathfrak{X}(\mathcal{U}) \mid v \in T_pM\} =: P$, thus P is a local parallelization over $\mathcal{U}$. For each vector field $X \in P$, $f \circ X$ is constant, because f is compatible with $\mathcal{H}$. □

This and Theorem 2.3.3 yields

Theorem 2.3.7 (Y. Ichijyō [21])**.** *A Finsler manifold is a generalized Berwald manifold if and only if it is smoothly monochromatic.*

Finally we have a modern reformulation of a classical result that goes back to H. Weyl. The idea of the proof is from [28, p. 185].

Remark 2.3.8. Recall that if (M, F) is a Berwald manifold, then there is a Riemannian metric g on M, whose Levi-Civita derivative induces the same parallel translations as the canonical connection of (M, F). This result is due to Z.I. Szabó [42], other proofs can be found in, e.g., [48, 44, 31], see also [10]. Each proof relies on the construction of a Riemannian metric g on M with the following property: for all $p, q \in M$, given a linear isomorphism $L_{qp}\colon T_pM \to T_qM$ such that $F \circ L_{qp} = F$ over T_pM, L_{qp} also satisfies $g_q(L_{qp}(v), L_{qp}(w)) = g_p(v, w)$ for any $v, w \in T_pM$. Then the Levi-Civita derivative of g indeed induces the same parallel translations as the canonical connection of (M, F) because of the uniqueness of the Levi-Civita derivative.

Theorem 2.3.9 (Weyl)**.** *Let $f\colon T_pM \to \mathbb{R}$ be a Finsler norm. (Then f is continuous, 1^+-homogeneous, smooth and positive on $T_pM \setminus \{0\}$, and $f''(v)$ is positive definite for all $v \in T_pM \setminus \{0\}$.) For each parallelization P over an open subset $\mathcal{U} \subset M$ containing p, define a Finsler function F_P for $\mathcal{U}$ by*

$$F_P(v) := f(v_P(p)).$$

Assume that for each such parallelization P, $(\mathcal{U}, F_P)$ is a Berwald manifold. Then f is an Euclidean norm.

Proof. Let P be a parallelization over an open subset $\mathcal{U} \subset M$ with $p \in \mathcal{U}$. Then, since F_P is compatible with P, $F_P \circ P_{qp} = F_P$ over T_pM. Also, $(\mathcal{U}, F_P)$ is a Berwald manifold by assumption. Then there is a Riemannian metric g^P on $\mathcal{U}$ such that the Levi-Civita derivative of g^P induces the same parallel translations as the canonical connection of F_P. Furthermore, g^P is compatible with the parallelization P, that is, $g^P_q(P_{qp}(v), P_{qp}(w)) = g^P_p(v, w)$ for all $p, q \in \mathcal{U}$, $v, w \in T_pM$ (see Remark 2.3.8).

Now let $\mathcal{U}$ be an open neighbourhood of p that admits a Riemannian metric g of constant nonzero curvature. We may obtain such a Riemannian metric using a diffeomorphism from a suitable open subset of the sphere S^n to $\mathcal{U}$. Using the freedom of the chosen diffeomorphism, we can guarantee that $g_p = g^P_p$. Let $(X_i)_{i=1}^n$ be an orthonormed frame field (w.r.t. g) on a neighbourhood of p. After a restriction, we may assume that this neighbourhood is $\mathcal{U}$. Consider the parallelization P over $\mathcal{U}$ given by the $\mathbb{R}$-linear span of the frame field $(X_i)_{i=1}^n$. Then $(\mathcal{U}, F_P)$ is a Berwald manifold by assumption, and its canonical covariant derivative D is the Levi-Civita derivative of g^P. The linear isomorphisms P_{qp} preserve g^P. But they also preserve g, because P was constructed from an orthonormed frame field. Then, since $g_p = g^P_p$, we have $g = g^P$. Therefore D is also the Levi-Civita derivative of g. But g has constant curvature, so its holonomy group is transitive on the unit sphere, hence the unit sphere of f is an ellipsoid. This concludes the proof. □

Chapter 3

Affinities and isometries of spray and Finsler manifolds

This Chapter is about affine and isometric transformations of spray and Finsler manifolds. Given a Finsler manifold (M, F) or a spray manifold (M, S), we define an *affinity* as a (not necessarily smooth) bijective mapping $\varphi\colon M \to M$ that preserves geodesics. More precisely, φ is an affinity if for any geodesic γ, $\varphi \circ \gamma$ is also a geodesic. Similarly, for a Finsler manifold (M, F), a (not necessarily smooth) bijective mapping $\varphi\colon M \to M$ is an *isometry* if $\varrho(\varphi(p), \varphi(q)) = \varrho(p, q)$ for all $p, q \in M$, where ϱ is the (generally nonsymmetric) Finslerian distance function on M. This distance function is described in more detail at the beginning of section 3.2.

For simplicity, we deal mostly with transformations only, although many of the concepts and results in the Chapter translate naturally to mappings between different manifolds. For example, if (M_1, S_1) and (M_2, S_2) are spray manifolds, a bijective mapping $\varphi\colon M_1 \to M_2$ is *affine* if for any geodesic γ of S_1, $\varphi \circ \gamma$ is a geodesic of S_2.

3.1 Affinities are smooth

F. Brickell proved [8] that an affine homeomorphism of a spray manifold is necessarily smooth. We present a proof following Brickell's ideas, however we slightly streamlined his argument with our Lemma 3.1.1, which may be interesting in its own right.

Throughout the section, (M, S) is a spray manifold. The argument is of local nature, so we may assume that M is an open subset of $\mathbb{R}^n$. Thus we identify TM with $M \times \mathbb{R}^n$. We endow $\mathbb{R}^n$ with a norm $\|\cdot\|$, and we set $W_p := \exp_p^{-1}(M)$. We also assume that M is totally normal, thus for any pair of points (p, q) of M, there is a unique oriented geodesic path from p to q.

Lemma 3.1.1. *Let (M,S) be a spray manifold, p a point in M. Suppose that (a_n) and (b_n) are sequences converging to p, and let*

$$L_n := \exp_{a_n}^{-1}(b_n), \quad A_n := \exp_p^{-1}(a_n), \quad B_n := \exp_p^{-1}(b_n); \qquad n \in \mathbb{N}^*.$$

Then

$$\lim_{n\to\infty} \frac{B_n - A_n - L_n}{\|L_n\|} = 0,$$

and hence $\frac{\|B_n - A_n\|}{\|L_n\|} \to 1$. Furthermore, if one of the sequences

$$\left(\frac{B_n - A_n}{\|B_n - A_n\|}\right), \qquad \left(\frac{L_n}{\|L_n\|}\right)$$

converges, then so does the other, and their limits coincide.

Proof. For each $q \in M$, define the mapping

$$\mathcal{A}_q := \exp_p^{-1} \circ \exp_q \colon W_q \to W_p := \exp_p^{-1}(M).$$

Then the mapping $(q,v) \mapsto \mathcal{A}_q(v)$ is of class C^1, and we have

$$\mathcal{A}_p = \mathrm{id}_{W_p}, \quad \mathcal{A}_{a_n}(0) = A_n, \quad \mathcal{A}_{a_n}(L_n) = B_n.$$

For all n, $s \mapsto \mathcal{A}_{a_n}(sL_n)$ is the inverse image of the geodesic from a_n to b_n under $\exp_p$, and it runs from A_n to B_n. Its initial velocity is $A'_{a_n}(0)L_n$. We have

$$\begin{aligned}\lim_{n\to\infty} \frac{\|L_n - \mathcal{A}'_{a_n}(0)L_n\|}{\|L_n\|} &\le \lim_{n\to\infty} \frac{\|\,\mathrm{id}_{\mathbb{R}^n} - \mathcal{A}'_{a_n}(0)\|\|L_n\|}{\|L_n\|} \\ &= \lim_{n\to\infty} \|\,\mathrm{id}_{\mathbb{R}^n} - \mathcal{A}'_{a_n}(0)\| = 0,\end{aligned}$$

since $\mathcal{A}_p = \mathrm{id}_{W_p}$. So $(L_n/\|L_n\|)$ and $(\mathcal{A}'_{a_n}(0)L_n/\|L_n\|)$ converge to the same vector, provided that at least one of them is convergent. We prove the same for the pair of sequences $(\mathcal{A}'_{a_n}(0)L_n/\|L_n\|)$ and $((B_n - A_n)/\|L_n\|)$. We define (for each $n \in \mathbb{N}^*$) the curve

$$\alpha_n \colon s \mapsto \mathcal{A}_{a_n}(sL_n) - A_n - s\mathcal{A}'_n(0)L_n,$$

which measures how much $s \mapsto \mathcal{A}_{a_n}(sL_n)$ deviates from its tangent line at A_n. Using the main value theorem, we obtain

$$\begin{aligned}\|B_n - A_n - \mathcal{A}'_{a_n}(0)L_n\| &= \|\alpha_n(1)\| = \|\alpha_n(1) - \alpha_n(0)\| \le \sup_{\theta\in[0,1]} \|\alpha'(\theta)\| \\ &= \sup_{\theta\in[0,1]} \|\mathcal{A}'_{a_n}(\theta L_n)L_n - \mathcal{A}'_{a_n}(0)L_n\|\end{aligned}$$

$$\leq \sup_{\theta\in[0,1]} \|\mathcal{A}'_{a_n}(\theta L_n) - \mathcal{A}'_{a_n}(0)\| \|L_n\|.$$

Hence

$$\lim_{n\to\infty} \frac{\|B_n - A_n - \mathcal{A}'_{a_n}(0)L_n\|}{\|L_n\|} \leq \lim_{n\to\infty} \sup_{\theta\in[0,1]} \|\mathcal{A}'_{a_n}(\theta L_n) - \mathcal{A}'_{a_n}(0)\| = 0,$$

since both $(\mathcal{A}'_{a_n}(\theta L_n))$ and $(\mathcal{A}'_{a_n}(0))$ tend to $\mathrm{id}_{\mathbb{R}^n}$. This proves our first claim. Then

$$\left|\frac{\|B_n - A_n\|}{\|L_n\|} - 1\right| = \frac{|\|B_n - A_n\| - \|L_n\||}{\|L_n\|} \leq \frac{\|B_n - A_n - L_n\|}{\|L_n\|} \to 0,$$

hence $\frac{\|B_n - A_n\|}{\|L_n\|} \to 1$, and so the last claim follows. □

Corollary 3.1.2 (F. Brickell [8])**.** *Let (M, S) be a spray manifold, $p \in M$. Choose two vectors $A, B \in W_p$ and let the curves $a, b\colon [0,1] \to M$ be defined by*

$$a(t) := \exp_p(tA), \qquad b(t) := \exp_p(tB).$$

Let $s \mapsto c_t(s)$ be the geodesic with $c_t(0) = a(t)$, $c_t(1) = b(t)$. Consider the mapping

$$C\colon (t,s) \in [0,1]^2 \mapsto C(t,s) := \frac{1}{t}\exp_p^{-1} c_t(s) \in M.$$

Then

$$\lim_{t\to 0^+} C(t,s) = (1-s)A + sB.$$

Proof. For any $t \in [0,1]$, let

$$L(t) := \exp_{a(t)}^{-1}(b(t)) \in T_{a(t)}M,$$

which is the initial velocity of c_t. Fix a sequence (t_n) converging to 0^+, and apply Lemma 3.1.1 for $a_n := a(t_n)$, and $b_n := b(t_n)$. Then

$$1 = \lim_{n\to\infty} \frac{\|\exp_p^{-1}(b(t_n)) - \exp_p^{-1}(a(t_n))\|}{\|L(t_n)\|} = \lim_{n\to\infty} \frac{\|t_nA - t_nB\|}{\|L(t_n)\|}. \tag{3.1}$$

The sequence $(\frac{t_nB - t_nA}{\|t_nB - t_nA\|})$ converges if and only if $(\frac{L(t_n)}{\|L(t_n)\|})$ does, and they have the same limit. The first one obviously converges to $v := \frac{B-A}{\|B-A\|}$, so $\frac{L(t_n)}{\|L(t_n)\|} \to v$. Now apply Lemma 3.1.1 to $(a(t_n))$ and $(c_{t_n}(s))$. We get

$$1 = \lim_{n\to\infty} \frac{\|\exp_p^{-1}(c_{t_n}(s)) - \exp_p^{-1}(a(t_n))\|}{\|sL(t_n)\|} = \lim_{n\to\infty} \frac{\|t_nC(t_n,s) - t_nA\|}{\|sL(t_n)\|},$$

and, since $(\frac{sL(t_n)}{\|sL(t_n)\|})$ converges (to v), we find that

$$\begin{aligned}\frac{B-A}{\|B-A\|} = v = \lim_{n\to\infty}\frac{L(t_n)}{\|L(t_n)\|} &= \lim_{n\to\infty}\frac{sL(t_n)}{\|sL(t_n)\|}\\ &= \lim_{n\to\infty}\frac{t_nC(t_n,s)-t_nA}{s\|L(t_n)\|} \overset{(3.1)}{=} \lim_{n\to\infty}\frac{t_nC(t_n,s)-t_nA}{s\|t_nA-t_nB\|}\\ &= \lim_{n\to\infty}\frac{C(t_n,s)-A}{s\|A-B\|}.\end{aligned}$$

Thus

$$\lim_{n\to\infty} C(t_n,s) = A + s(B-A) = (1-s)A + sB. \qquad \square$$

Theorem 3.1.3 (F. Brickell [8])**.** *An affine homeomorphism of a spray manifold is a smooth diffeomorphism.*

Proof. Let $\varphi\colon M\to M$ be an affine homeomorphism. Choose a point $p\in M$, a vector $v\in W_p$, and let γ_v be the geodesic with initial velocity v. Then, since φ is affine, $\varphi\circ\gamma_v$ is a geodesic starting from $\varphi(p)$. We denote its initial velocity by $\varphi'(v)$. Then $v\mapsto\varphi'(v)$ is actually the mapping

$$\varphi' := \exp^{-1}_{\varphi(p)}\circ\varphi\circ\exp_p \tag{3.2}$$

from W_p to $W_{\varphi(p)}$. It is continuous, and, since geodesics are invariant under orientation preserving affine reparametrizations, φ' is 1^+-homogeneous.

We show that it is in fact linear. Consider two vectors v,w in a convex subset of W_p, and construct the mapping c_t as in Corollary 3.1.2. Repeat the construction for $\varphi'(v)$ and $\varphi'(w)$ as well. Then $\varphi\circ c_t$ takes the role of c_t, since it is defined by concatenating geodesics and φ is affine. Then we perform the following calculation:

$$\begin{aligned}\frac{1}{2}\varphi'(v)+\frac{1}{2}\varphi'(w) &= \lim_{t\to0^+}\frac{1}{t}\left(\exp^{-1}_{\varphi(p)}\circ\varphi\circ c_t\left(\frac{1}{2}\right)\right)\\ &= \lim_{t\to0^+}\frac{1}{t}\left(\varphi'\circ\exp_p^{-1}\circ c_t\left(\frac{1}{2}\right)\right)\\ &= \varphi'\left(\lim_{t\to0^+}\frac{1}{t}\exp_p^{-1}\left(c_t\left(\frac{1}{2}\right)\right)\right)\\ &= \varphi'\left(\frac{1}{2}v+\frac{1}{2}w\right).\end{aligned}$$

This, together with the 1^+ homogeneity, implies that φ' is linear not only on W_p but on the whole T_pM. So, in particular, it is smooth. Now, rearranging (3.2),

$$\varphi = \exp_{\varphi(p)}\circ\varphi'\circ\exp_p^{-1}.$$

Here the right-hand side is a smooth diffeomorphism on $M\setminus\{p\}$, hence so is φ. But the choice of p was arbitrary, therefore φ is smooth everywhere. $\square$

3.2 Isometries are smooth

There is a natural (generally nonsymmetric) distance function on Finsler manifolds, defined as follows.

Let $\gamma\colon [a,b] \to M$ be a smooth curve segment. The *Finslerian length* of γ is

$$\ell_F(\gamma) = \int_a^b F \circ \dot\gamma. \tag{3.3}$$

If γ is a piecewise smooth curve segment, then its Finslerian length is

$$\ell_F(\gamma) := \sum_{i=1}^{k} \ell_F(\gamma \restriction [t_{i-1}, t_i]).$$

Given a point $(p,q) \in M \times M$, let $\Omega(p,q)$ be the set of all piecewise smooth curve segments in M from p to q. The *Finslerian distance* $\varrho(p,q)$ from p to q is the greatest lower bound of $\{\ell_F(\gamma) \in \mathbb{R} \mid \gamma \in \Omega(p,q)\}$, i.e.,

$$\varrho(p,q) := \inf\{\ell_F(\gamma) \in \mathbb{R} \mid \gamma \in \Omega(p,q)\}. \tag{3.4}$$

Then the function

$$\varrho\colon M \times M \to \mathbb{R}, \quad (p,q) \mapsto \varrho(p,q)$$

is a quasi-distance on M: it is non-negative, definite, satisfies the triangle inequality, but it is not necessarily symmetric. The *forward metric balls*

$$\mathcal{B}_r^+(a) := \{p \in S \mid \varrho(a,p) < r\}$$

generate the topology of M [4, 6.2 C].

A curve $\gamma \in \Omega(p,q)$ is *distance minimizing* if $\varrho(p,q) = \ell_F(\gamma)$, i.e., if it makes the right-hand side of (3.4) minimal. Distance minimizing curves are geodesics. Conversely, 'short' geodesics are distance minimizing. More precisely, we have

Fact 3.2.1. *Let (M,F) be a Finsler manifold, $p \in M$ a fixed point and let r and ε be positive numbers such that $\exp_p$ restricted to*

$$B_{r+\varepsilon}(p) := \{v \in T_pM \mid F(v) < r + \varepsilon\}$$

is a diffeomorphism. Then for each $v \in T_pM$ with $F(v) = 1$, the geodesic

$$\gamma\colon [0,r] \to M, \quad \gamma(t) := \exp_p(tv)$$

is distance minimizing, i.e.,

$$\varrho(p, \exp_p(tv)) = t. \tag{3.5}$$

Furthermore, any minimizing curve in $\Omega(p, \exp(rv))$ is a reparametrization of γ. From (3.5) we also deduce that

$$tF(w) = \varrho(p, \exp_p(tw)) \qquad \text{for all } w \in B_r(p) \text{ and } t \in [0,1]. \tag{3.6}$$

This is essentially [4, Theorem 6.3.1].

Using Fact 3.2.1 it is easy to show that isometries (i.e., distance preserving transformations) of a Finsler manifold are affinities. Furthermore, isometries are continuous, since forward balls generate the topology of M. Thus, an isometry is a homeomorphism. Hence from Theorem 3.1.3 we obtain immediately

Theorem 3.2.2 (F. Brickell [8])**.** *Isometries of Finsler manifolds are smooth.*

3.3 Distance coordinates

Throughout the section we use the shorthand

$$\varrho_p\colon M \to \mathbb{R}, \quad q \mapsto \varrho_p(q) := \varrho(p,q)$$

for any $p \in M$. We say that ϱ_p is the *distance function* at p. Note that if $\mathcal{U}$ is a normal neighbourhood of p, then $\varrho_p \overset{(3.6)}{=} F \circ \exp_p^{-1}$ over $\mathcal{U}$, and hence ϱ_p is smooth on $\mathcal{U} \setminus \{p\}$.

If $(p_i)_{i=1}^n$ is a family of points in M such that the mapping $u_\varrho := (\varrho_{p_1}, \dots, \varrho_{p_n})$ is a diffeomorphism from an open subset $\mathcal{U}$ of M onto an open subset of $\mathbb{R}^n$, then $(\mathcal{U}, u_\varrho)$ is called a *distance chart* and $(\varrho_{p_1}, \dots, \varrho_{p_n})$ is a *distance coordinate system* for (M, F). Then the points $p_1, \dots, p_n$ are called the *base points* of the chart (or of the coordinate system).

Lemma 3.3.1 ([3])**.** *Any point of a Finsler manifold has a neighbourhood that admits a distance chart.*

Proof. Let (M, F) be a Finsler manifold, and fix a point $p \in M$. Throughout the proof, for any $v \in \mathring{T}M$, γ_v denotes the maximal geodesic with $\dot{\gamma}_v(0) = v$. We construct a family $(p_i)_{i=1}^n$ of base points in M and a basis $(v_i)_{i=1}^n$ of T_pM such that

$$\begin{cases} v_j(\varrho_{p_j}) > 0 \text{ for all } j \in \{1, \dots, n\}, \\ v_j(\varrho_{p_i}) = 0 \text{ if } i < j;\ i, j \in \{1, \dots, n\}. \end{cases} \tag{$*$}$$

We begin with a general observation. Let $v \in T_pM$ be fixed. From basic ODE theory we know that the domain of γ_v contains an open interval containing 0. Thus there is a positive real number δ such that $a := \gamma(-\delta)$ is defined. We may also assume that a is in a totally normal neighbourhood $\mathcal{U}$ of p. We call such a point an *emanating point of* v. Because of (1.39), affine reparametrizations of geodesics are also geodesics, thus if we set $w := \dot{\gamma}(-\delta)$, then $\gamma_w(t) = \gamma_v(t - \delta)$, and hence $\gamma_w(0) = a$, $\gamma_w(\delta) = p$, and $\dot{\gamma}_w(\delta) = v$. This implies that $v(\varrho_a) > 0$. Indeed, we have

$$v(\varrho_a) = \dot{\gamma}_w(\delta)(\varrho_a) = (\varrho_a \circ \gamma_w)'(\delta).$$

For any real number s such that $\gamma_w(s)$ is in $\mathcal{U}$, we find that

$$\varrho_a \circ \gamma_w(s) = \varrho(a, \gamma_w(s)) = \varrho(a, \exp_a(sw)) \overset{(3.6)}{=} F(sw) = sF(w),$$

hence

$$v(\varrho_a) = (\varrho_a \circ \gamma_w)'(\delta) = F(w) > 0.$$

Now we turn to the proof. Choose any nonzero vector $v_1 \in T_pM$ and let p_1 be an emanating point for v_1. Since $(d\varrho_{p_1})_p(v_1) = v_1(\varrho_{p_1}) > 0$, we have $\dim(\operatorname{Im}(d\varrho_{p_1})_p) = 1$, and hence $\dim(\operatorname{Ker}(d\varrho_{p_1})_p) = n-1$.

In the next step we select a vector $v_2 \in \operatorname{Ker}(d\varrho_{p_1})_p \setminus \{0\}$ together with an emanating point p_2 for v_2. Then

$$v_2(\varrho_{p_2}) > 0 \text{ and } v_2(\varrho_{p_1}) = (d\varrho_{p_1})_p(v_2) = 0.$$

Furthermore, we have

$$\dim(\operatorname{Ker}(d\varrho_{p_i})_p) = n-1, \quad i \in \{1,2\},$$

and Sylvester's rank inequality gives

$$\dim\Big(\operatorname{Ker}(d\varrho_{p_1})_p \cap \operatorname{Ker}(d\varrho_{p_2})_p\Big) \geq (n-1)+(n-1)-n = n-2.$$

We proceed by induction. Let $k \in \{1,\dots,n\}$, and suppose that we have a sequence $(v_i)_{i=1}^k$ of nonzero tangent vectors in T_pM together with a sequence $(p_i)_{i=1}^k$ of corresponding emanating points such that $v_j(\varrho_{p_i}) = 0$ if $i < j$, $i,j \in \{1,\dots,k\}$. As above, by Sylvester's rank inequality,

$$\dim\left(\bigcap_{i=1}^k \operatorname{Ker}(d\varrho_{p_k})_p\right) \geq n-k,$$

so, as long as $k < n$, we can choose a vector

$$v_{k+1} \in \bigcap_{i=1}^k \operatorname{Ker}(d\varrho_{p_k})_p \setminus \{0\},$$

together with an emanating point p_{k+1} for v_{k+1}. Then we have $v_{k+1}(\varrho_{p_{k+1}}) > 0$ and $v_{k+1}(\varrho_{p_i}) = 0$ for all $i < k+1$. This proves that there exist families $(v_i)_{i=1}^n$ of vectors in T_pM and families $(p_i)_{i=1}^n$ of points in M satisfying $(*)$.

Now consider the mapping $u_\varrho := (\varrho_{p_1},\dots,\varrho_{p_n})$. Then u_ϱ is smooth in an open neighbourhood of p. We claim that $\Big(((u_\varrho)_*)_p(v_i)\Big)_{i=1}^n$ is a basis of $T_{u_\varrho(p)}\mathbb{R}^n$, and hence

$(v_i)_{i=1}^n$ is a basis of T_pM. Let $(e^i)_{i=1}^n$ be the canonical coordinate system on $\mathbb{R}^n$, i.e., the dual of the canonical basis of $\mathbb{R}^n$. Since

$$((u_\varrho)_*)_p(v_j)(e^i) = v_j(e^i \circ u_\varrho) = v_j(\varrho_{p_i})$$

for all $i, j, \in \{1, \dots, n\}$, the well-known basis theorem [37, Theorem 1.12] gives

$$((u_\varrho)_*)_p(v_j) = ((u_\varrho)_*)_p(v_j)(e^i)\left(\frac{\partial}{\partial e^i}\right)_{u_\varrho(p)} = v_j(\varrho_{p_i})\left(\frac{\partial}{\partial e^i}\right)_{u_\varrho(p)}.$$

By $(*)$, the matrix $(v_j(\varrho_{p_i}))$ is invertible, whence our claim. Now the inverse mapping theorem guarantees that there exists an open neighbourhood $\mathcal{U}$ of p such that $u_\varrho \restriction \mathcal{U}$ is a diffeomorphism onto its image. This concludes the proof. □

Using distance coordinates, we give, as we promised, a simple new proof for the smoothness of isometries which does not rely on the smoothness of affinities. The idea is borrowed from the proof in P. Petersen's book [38, Ch. 5.10].

Proof of Theorem 3.2.2 Choose and fix a point $p \in M$, and let $q := \varphi(p)$. Forward balls constitute a basis for the topology of M, so we can find a positive number r such that $\mathcal{B}_r^+(p)$ is contained in a totally normal neighbourhood of p. Since φ is a distance preserving bijection, we have $\mathcal{B}_r^+(q) = \varphi(\mathcal{B}_r^+(p))$. Let

$$(\mathcal{U}, u_\varrho) = (\mathcal{U}, (\varrho_{q_1}, \dots, \varrho_{q_n}))$$

be a distance chart for N at q such that the base points $q_1, \dots, q_n$ are elements of $\mathcal{B}_r^+(q)$. This can be achieved because for any nonzero tangent vector in T_qN we can find an emanating point in N which is arbitrarily close to q. Let

$$p_i := \varphi^{-1}(q_i), \quad i \in \{1, \dots, n\}.$$

Then $\{p_1, \dots, p_n\} \subset \mathcal{B}_r^+(p)$, and for every point $a \in \varphi^{-1}(\mathcal{U}) \subset M$ we have

$$\varrho_{p_i}(a) = \varrho(p_i, a) = \varrho(\varphi(p_i), \varphi(a)) = \varrho(q_i, \varphi(a)) = \varrho_{q_i} \circ \varphi(a).$$

Therefore,

$$u_\varrho \circ \varphi = (\varrho_{q_1} \circ \varphi, \dots, \varrho_{q_n} \circ \varphi) = (\varrho_{p_1}, \dots, \varrho_{p_n}).$$

Since $\mathcal{B}_r^+(p)$ is contained in a totally normal neighbourhood of p, the function ϱ_{p_i} is smooth on $\mathcal{B}_r^+(p) \setminus \{p_i\}$ for each $i \in \{1, \dots, n\}$. Furthermore, u_ϱ is a diffeomorphism on a neighbourhood of q, hence φ is smooth at p. □

As a further application of distance coordinates, we show that regular submetries between reversible Finsler manifolds are differentiable (Theorem 3.3.3 below).

This result is known in the more general setting of metric spaces (see [30]). We believe, however, that our proof is more accessible and more interesting for experts in differential geometry. First we clarify the terminology.

Let M_1 and M_2 be metric spaces. We denote by $\mathcal{B}$ the metric balls in both of them. We say that a mapping $\varphi\colon M_1 \to M_2$ is a *submetry* if for any p in M_1, there is a positive number δ such that for every $\varepsilon \in\,]0, \delta[$ we have

$$\varphi(\mathcal{B}_\varepsilon(p)) = \mathcal{B}_\varepsilon(\varphi(p)).$$

For each $p \in M_1$, the supremum of these positive numbers δ will be denoted by δ_p (note that δ_p can be infinite). We say that a submetry is *regular* if for any compact set $\mathcal{K} \subset M_1$ we have

$$\delta_{\mathcal{K}} := \inf_{p\in\mathcal{K}} \delta_p > 0. \tag{3.7}$$

The following properties of submetries can be immediately deduced from the definition:

(i) submetries are continuous;

(ii) composition of submetries is a submetry;

(iii) composition of regular submetries is a regular submetry.

Recall that a Finsler manifold is *reversible*, if $F(-v) = F(v)$ for every $v \in TM$. In case of such a Finsler manifold, the distance function ϱ is symmetric, and (M, ϱ) is a metric space. Hence, for reversible Finsler manifolds, we omit the '+' from the notation $\mathcal{B}^+_r(p)$ of forward balls.

Lemma 3.3.2 ([3])**.** *Let (M, F) be a reversible Finsler manifold, p a point in M, and $\mathcal{U}$ a normal neighbourhood of p. Then the distance function ϱ_p at p restricted to $\mathcal{U} \setminus \{p\}$ is a regular submetry into $\mathbb{R}$ with respect to its canonical distance.*

Proof. Choose a point q in $\mathcal{U} \setminus \{p\}$. Let δ be the minimum of the two numbers $\varrho(p, q)$ and $\varrho(q, M \setminus \mathcal{U}) := \inf\{\varrho(q, \tilde{q}) \mid \tilde{q} \in M \setminus \mathcal{U}\}$. Fix $\varepsilon \in\,]0, \delta[$ and $a \in \mathcal{B}_\varepsilon(q)$. The Finslerian distance satisfies the triangle inequality, therefore

$$\varrho(p, a) \le \varrho(p, q) + \varrho(q, a),$$
$$\varrho(p, q) \le \varrho(p, a) + \varrho(a, q).$$

Rearranging these inequalities and using the symmetry of ϱ, we obtain

$$|\varrho(p, a) - \varrho(p, q)| \le \varrho(q, a) < \varepsilon.$$

Consequently,

$$\varrho_p(\mathcal{B}_\varepsilon(q)) \subset \,]\varrho_p(q)-\varepsilon, \varrho_p(q)+\varepsilon[\, =: B_\varepsilon(\varrho_p(q)).$$

Now we show that ϱ_p maps $\mathcal{B}_\varepsilon(q)$ onto $B_\varepsilon(\varrho_p(q))$. Let γ be the maximal unit speed geodesic starting at p passing through q. For any positive t such that $\gamma([0,t])$ is contained in $\mathcal{U}$, we have

$$\varrho_p(\gamma(t)) = \varrho(p, \gamma(t)) \overset{\text{Fct.3.2.1}}{=} t. \tag{3.8}$$

Since $\varepsilon < \varrho_p(q)$, the interval $B_\varepsilon(\varrho_p(q))$ contains only positive numbers. Furthermore, $\mathcal{B}_\varepsilon(q) \subset \mathcal{U}$ because $\varepsilon < \varrho(q, M \setminus \mathcal{U})$. Thus, according to (3.8), we only have to show that $\gamma(t) \in \mathcal{B}_\varepsilon(q)$ if $t \in B_\varepsilon(\varrho_p(q))$.

First assume that $t \in [\varrho_p(q), \varrho_p(q)+\varepsilon[$. The curve γ is of unit speed, so for any $t \in [\varrho_p(q), \varrho_p(q)+\varepsilon[$, the Finslerian length of the curve segment $\gamma \restriction [\varrho_p(q), t]$ is equal to $t - \varrho_p(q)$. Notice that (3.8) implies $\gamma(\varrho_p(q)) = q$. Then, since γ has unit speed, we have

$$\varrho(q, \gamma(t)) = \varrho(\gamma(\varrho_p(q)), \gamma(t)) \le t - \varrho_p(q) < \varepsilon.$$

Now suppose that $t \in \,]\varrho_p(q)-\varepsilon, \varrho_p(q)]$. Using the reversibility of F, we obtain similarly that $\varrho(q, \gamma(t)) \le \varrho_p(q) - t < \varepsilon$, as was to be shown.

The regularity of ϱ_p follows from the fact that disjoint closed and compact sets have positive distance. □

Now we are in a position to present a second application of distance coordinates. We shall use some ideas of [6] where the analogous result is proved in Riemannian setting.

Theorem 3.3.3 (A. Lytchak [30],[3]). *A surjective regular submetry between reversible Finsler manifolds is differentiable.*

Proof. Let (M, F) and $(N, \bar{F})$ be reversible Finsler manifolds, and let φ be a surjective regular submetry from M to N. Choose a point $p \in M$ and a distance coordinate system $(\mathcal{D}, (\varrho_{p_i})_{i=1}^n)$ at $\varphi(p)$ such that the base points $(p_i)_{i=1}^n$ and $\mathcal{D}$ are contained in a totally normal neighbourhood of $\varphi(p)$. The functions ϱ_{p_i} are regular submetries on $\mathcal{D}$ by the previous lemma, hence the functions $\varrho_{p_i} \circ \varphi$ are also regular submetries on $\varphi^{-1}(\mathcal{D})$. If these functions are differentiable, then φ is also differentiable, because $(\mathcal{D}, (\varrho_{p_i})_{i=1}^n)$ is a chart. Consequently, we only have to show that regular submetries from (M, F) into $\mathbb{R}$ are differentiable.

To do this, let f be such a submetry, and let q be a point in M. Choose an open neighbourhood D of q with compact closure and a number $\delta \in \left]0, \delta_{\mathrm{cl}(D)}\right[$ (where $\delta_{\mathrm{cl}(D)}$ is defined as (3.7)) such that the closure of $\mathcal{B}_\delta(q)$ is contained in D. Then for any point $p \in \mathrm{cl}(\mathcal{B}_\delta(q))$ we have $\delta_p > \delta$.

Consider the fibres $H^+ := f^{-1}(\{f(q)+\delta\})$ and $H^- := f^{-1}(\{f(q)-\delta\})$. Since f is a submetry, there is a point $b \in H^+$ and a point $a \in H^-$ such that $\varrho(q,b) = \varrho(q,a) = \delta$. Then $\delta_a > \delta$ and $\delta_b > \delta$, so $\mathcal{B}_{\delta_a}(a) \cap \mathcal{B}_{\delta_b}(b)$ is an open neighbourhood of q. Define the functions f_a and f_b on this set by $f_a(u) := f(a)+\varrho_a(u)$ and $f_b(u) := f(b)-\varrho_b(u)$. These functions have the following properties:

(i) $f_b \leq f \leq f_a$;
(ii) $f_b(q) = f(q) = f_a(q)$.

Indeed, since f is submetry,

$$\varrho_a(u) = \varrho(a,u) \geq |f(u) - f(a)| \geq f(u) - f(a).$$

Similarly, $\varrho_b(u) \geq f(b) - f(u)$, and we obtain (i). Furthermore,

$$f_a(q) = f(a) + \varrho_a(q) \overset{a \in H^-}{=} f(q) - \delta + \varrho(q,a) = f(q),$$
$$f_b(q) = f(b) - \varrho_b(q) \overset{b \in H^+}{=} f(q) + \delta - \varrho(q,b) = f(q),$$

so (ii) is also true. The function $f_a - f_b$ is differentiable on $\mathcal{B}_{\delta_a}(a) \cap \mathcal{B}_{\delta_b}(b)$, nonnegative and vanishes at q, so it has a local minimum at that point. Therefore its differential vanishes at q, which implies that $(df_b)_q = (df_a)_q$. Now let σ be a differentiable curve in M with $\sigma(0) = q$. Then, taking into account (i) and (ii), we find that

$$\lim_{t\to 0} \frac{f \circ \sigma(t) - f \circ \sigma(0)}{t} \geq \lim_{t\to 0} \frac{f_b \circ \sigma(t) - f_b \circ \sigma(0)}{t} = (df_b)_q(\dot{\sigma}(0)),$$

$$\lim_{t\to 0} \frac{f \circ \sigma(t) - f \circ \sigma(0)}{t} \leq \lim_{t\to 0} \frac{f_a \circ \sigma(t) - f_a \circ \sigma(0)}{t} = (df_a)_q(\dot{\sigma}(0)),$$

therefore f is differentiable at q. □

3.4 Relations between affinities and isometries

In the previous sections we have seen that an affine homeomorphism φ of a spray manifold (M,S) is smooth. Then it is clear that φ is a *automorphism* of S, that is, a diffeomorphism satisfying $\varphi_{**} \circ S = S \circ \varphi_*$. The converse is obvious, automorphisms of sprays are affine. Similarly, we proved that an isometry φ of a Finsler manifold (M,F) is smooth. It can be shown that then φ is a *Finsler isometry* of F, that is, a diffeomorphism satisfying $F \circ \varphi_* = F$. In the Riemannian case, S.B. Myers and N. Steenrod [35] proved that the two types of isometries are the same. The Finslerian version of the theorem is due to F. Brickell [8]. S. Deng and Z. Hou rediscovered it four decades later [13]. In both proofs, the crucial point is the smoothness of isometries, which we have already established. Hence we obtain the Finslerian version of the Myers–Steenrod theorem easily.

Theorem 3.4.1 (F. Brickell [8]). *An isometry of a Finsler manifold is a Finsler isometry.*

Proof. Let (M, F) be a Finsler manifold with Finslerian distance ϱ. Suppose that $\varphi\colon M \to M$ is an isometry. Since φ is necessarily smooth, we only have to show $F \circ \varphi_* = F$. Let v be any vector in TM and let γ be the maximal geodesic with $\dot\gamma(0) = v$. Then $\varphi \circ \gamma$ is a geodesic and $(\varphi \circ \gamma)^{\cdot}(0) = \varphi_*(v)$. So

$$F(v) \overset{(3.6)}{=} \lim_{t\to 0^+} \frac{1}{t}\varrho(\gamma(0), \gamma(t)) = \lim_{t\to 0^+} \frac{1}{t}\varrho\Big(\varphi(\gamma(0)), \varphi(\gamma(t))\Big) \overset{(3.6)}{=} F(\varphi_* v). \qquad \square$$

The converse is clearly true. To summarize: *Finsler isometries of a Finsler manifold are the same as its isometries, and automorphisms of a spray manifold are the same as its affine homeomorphisms.* From now on, we will simply refer to them as isometries and affinities.

Now we turn to the relations between isometries and affinities of a Finsler manifold. It is more or less obvious, that isometries are affinities, since if a transformation preserves the Finsler function, it should preserve anything derived from it. However, this heuristic argument could not replace the precise proof.

Proposition 3.4.2 ([3]). *A Finsler isometry is an affinity.*

Proof. Let (M, F) be a Finsler manifold, $\varphi\colon M \to M$ a diffeomorphism satisfying $F \circ \varphi_* = F$. Then we also have $E \circ \varphi_* = E$. We show that the push-forward $\varphi_{*\#}S := \varphi_{**} \circ S \circ \varphi_*^{-1}$ of the canonical spray S also satisfies the Euler–Lagrange equation (1.53).

One can easily check that for any $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$ we have

$$((\varphi_\# X)f) \circ \varphi = X(f \circ \varphi), \tag{3.9}$$

and hence

$$(\varphi_\# X)f = X(f \circ \varphi) \circ \varphi^{-1}. \tag{3.10}$$

Now let $X \in \mathfrak{X}(M)$ be arbitrary. Then $X = \varphi_\# Y$ for some $Y \in \mathfrak{X}(M)$, and

$$\begin{aligned}
\varphi_{*\#}S(X^{\mathsf{v}}E) &\overset{(3.10)}{=} S(X^{\mathsf{v}}E \circ \varphi_*) \circ \varphi_*^{-1} \overset{(1.3)}{=} S((\varphi_{*\#}Y^{\mathsf{v}}E) \circ \varphi_*) \circ \varphi_*^{-1} \\
&\overset{(3.9)}{=} S(Y^{\mathsf{v}}(E \circ \varphi_*)) \circ \varphi_*^{-1} = S(Y^{\mathsf{v}}E) \circ \varphi_*^{-1} \\
&\overset{(1.53)}{=} Y^{\mathsf{c}}E \circ \varphi_*^{-1} = Y^{\mathsf{c}}(E \circ \varphi_*) \circ \varphi_*^{-1} \\
&\overset{(3.10)}{=} (\varphi_{*\#}Y^{\mathsf{c}})E \overset{(1.3)}{=} X^{\mathsf{c}}E.
\end{aligned} \qquad \square$$

The converse of Proposition 3.4.2 is clearly not true in general. There are affinities that are not isometries, for example, in Euclidean spaces. In Riemannian geometry, some celebrated results give sufficient conditions for the equivalence of affinities and isometries (see, e.g., [25, pp. 242–244]). In the remainder of the section, we present some Finslerian counterparts of these results.

Our first result is about infinitesimal affinities and isometries. These are vector fields on the base manifold, whose stages have the corresponding property. Infinitesimal affinities are called *affine vector fields*, and infinitesimal isometries are called *Killing vector fields*. Both properties can be expressed in terms of the complete lifts of vector fields: if (M, F) is a Finsler manifold with canonical spray S, then $X \in \mathfrak{X}(M)$ is affine if and only if $\mathcal{L}_{X^{\mathsf{c}}} S := [X^{\mathsf{c}}, S] = 0$, and it is a Killing vector field if and only if $\mathcal{L}_{X^{\mathsf{c}}} F := X^{\mathsf{c}} F = 0$.

First we generalize a theorem of J.-I. Hano to Finsler manifolds. Hano proved that *bounded affine vector fields on a complete Riemannian manifold are Killing fields.* The proof of this in the book [25] relies on the de Rham decomposition theorem of simply connected and complete Riemannian manifolds [25, Theorem 6.2], and on the following important result:

Fact 3.4.3 (S. Kobayashi [24])**.** *The affinities and isometries of a complete irreducible Riemannian manifold coincide, except for the* 1*-dimensional Euclidean space.*

Hano did not use explicitly the second result in his proof [19], but he followed similar ideas as Kobayashi.

Rather surprisingly, Hano's theorem can be proved for Finsler manifolds without introducing any analogue of irreducibility (or decomposition). In fact, our proof is elementary, it relies merely on the Euler–Lagrange equation (1.53) and the following well-known

Fact 3.4.4 (Fundamental inequality)**.** *If V is a finite-dimensional real vector space, $f \colon V \to \mathbb{R}$ is Finsler norm, then $f'(p)(v) \leq f(v)$ for all $v \in V$, and equality holds if and only if v is a non-negative scalar multiple of p.*

For a proof, see, e.g., [4, p. 7] or [45, Proposition 9.1.37]. It is clear that a Finsler function restricts to a Finsler norm on each tangent space.

The key to our proof is the following simple observation.

Lemma 3.4.5 ([23])**.** *If X is an affine vector field on a Finsler manifold (M, F) and γ is a geodesic, then for all t and t_0 in the domain of γ we have*

$$X^{\mathsf{v}} E(\dot{\gamma}(t)) = X^{\mathsf{v}} E(\dot{\gamma}(t_0)) + (t - t_0) X^{\mathsf{c}} E(\dot{\gamma}(t_0)).$$

Proof. Since X is affine, we have $[X^{\mathsf{c}}, S] = 0$. Geodesics have constant speed, hence $SE = 0$. From these we get

$$0 = [X^{\mathsf{c}}, S]E = X^{\mathsf{c}}(SE) - S(X^{\mathsf{c}}E) = -S(X^{\mathsf{c}}E).$$

Since γ is a geodesic, $S \circ \dot{\gamma} = \ddot{\gamma}$, and we have

$$(X^{\mathsf{v}}E \circ \dot{\gamma})' = S(X^{\mathsf{v}}E) \circ \dot{\gamma} \overset{(1.53)}{=} X^{\mathsf{c}}E \circ \dot{\gamma},$$
$$(X^{\mathsf{v}}E \circ \dot{\gamma})'' = (X^{\mathsf{c}}E \circ \dot{\gamma})' = S(X^{\mathsf{c}}E) \circ \dot{\gamma} = 0.$$

Therefore $X^{\mathsf{v}}E \circ \dot{\gamma}$ is an affine function, and our claim follows. □

This lemma is in fact a disguised special case of Exercise 5.4.3(c) from [4], which states the following: if J is a Jacobi field along a geodesic γ, then

$$g_{\dot{\gamma}}(J, \dot{\gamma})(t) = g_{\dot{\gamma}}(J(t_0), \dot{\gamma}(t_0)) + (t - t_0)g_{\dot{\gamma}}(J'(t_0), \dot{\gamma}(t_0)),$$

where J' is the covariant derivative of J along γ with respect to the Berwald or the Chern derivative. However, it is easier to prove the lemma directly, than from the quoted exercise.

Recall that a Finsler manifold is *forward complete* if the domains of its maximal geodesics are not bounded from above, and *complete*, if the domain of its maximal geodesics is $\mathbb{R}$.

Theorem 3.4.6 ([23])**.** *Let (M, F) be a Finsler manifold, X an affine vector field on M, and suppose that one of the following conditions holds:*

(1) $F \circ X$ is bounded, and (M, F) is complete;

(2) $F \circ X$ and $F \circ (-X)$ are bounded, and (M, F) is forward complete.

Then X is a Killing vector field.

Proof. First we show that $X^{\mathsf{v}}E$ is bounded from above on $U(TM) := F^{-1}(\{1\})$ if (1) holds, and it is bounded from above and from below if (2) holds. For any $v \in U(TM)$, setting $p := \tau(v)$, we have

$$X^{\mathsf{v}}E(v) = F(v)X^{\mathsf{v}}F(v) = X^{\mathsf{v}}F(v) = (F \restriction T_pM)'(v)(X(p)) \overset{\text{Fact 3.4.4}}{\leq} F(X(p)).$$

In a similar way, we obtain

$$X^{\mathsf{v}}E(v) = (F \restriction T_pM)'(v)(X(p)) = -(F \restriction T_pM)'(v)(-X(p)) \geq -F(-X(p)).$$

Over $U(TM)$ these two inequalities give $-F \circ (-X) \circ \tau \leq X^{\mathsf{v}} E \leq F \circ X \circ \tau$, from which our first assertion follows.

Now we show that $X^{\mathsf{c}} E = 0$, and hence X is a Killing field. It suffices to prove it on $U(TM)$, because $X^{\mathsf{c}} E$ is 2^{+}-homogeneous. Indeed:

$$C(X^{\mathsf{c}} E) = [C, X^{\mathsf{c}}]E + X^{\mathsf{c}}(CE) \overset{(1.15)}{=} 2X^{\mathsf{c}} E.$$

We fix $v \in U(TM)$ and consider the maximal geodesic γ starting at $\tau(v)$ with initial velocity v. Then Lemma 3.4.5 gives

$$X^{\mathsf{v}} E(\dot{\gamma}(t)) = X^{\mathsf{v}} E(\dot{\gamma}(0)) + tX^{\mathsf{c}} E(\dot{\gamma}(0)) = X^{\mathsf{v}} E(v) + tX^{\mathsf{c}} E(v)$$

for any real number t in case (1) and for any positive real number t in case (2). Geodesics have constant speed, hence $\dot{\gamma}$ remains inside $U(TM)$, and the left-hand side of the above formula has to be bounded from above in case (1), and it has to be bounded from above and below in case (2), which is possible only if $X^{\mathsf{c}} E(v) = 0$. □

As a simple corollary we have

Theorem 3.4.7 (J.-I. Hano)**.** *Let (M, g) be a complete Riemannian manifold, and X an affine vector field on M such that the function $g(X, X)$ is bounded. Then X is a Killing vector field.*

The proof is immediate if we apply Theorem 3.4.6 to the Finsler function given by $F(v) := \sqrt{g(v, v)}$, $v \in TM$. Since compact Finsler manifolds are complete, we also have

Theorem 3.4.8 ([23])**.** *An affine vector field on a compact Finsler manifold is a Killing vector field.*

The Riemannian version of the last theorem is due to K. Yano [52].

To conclude the section, we propose a possible generalization of Fact 3.4.3 to Finsler manifolds, by replacing irreducibility with affine rigidity, defined as follows. We say that a spray S for M is *uniquely metrizable*, if it is the canonical spray of a Finsler function F for M, and if $\tilde{F}$ is another Finsler function whose canonical spray is also S, then on each component of M, $\tilde{F}$ is a scalar multiple of F (in other words, $d(F/\bar{F}) = 0$). We say that a Finsler manifold (M, F) is *affinely rigid*, if its canonical spray S is uniquely metrizable. The following result is a direct Finslerian analogue of [25, p. 242, Lemma 1].

Lemma 3.4.9. *An affinity of a connected affinely rigid Finsler manifold is a homothety.*

Proof. Let (M, F) be such a Finsler manifold and S its canonical spray. Let $\varphi\colon M \to M$ be an affinity, and consider the Finsler function $\tilde{F} := F \circ \varphi_*$. Then φ is an isometry from (M, F) to $(M, \tilde{F})$, hence the canonical spray of $(M, \tilde{F})$ is the push-forward $\varphi_{*\#}S := \varphi_{**} \circ S \circ \varphi_*^{-1}$. However, φ is also an affine transformation of (M, F), so we have $\varphi_{*\#}S = S$. Thus F and $\tilde{F}$ have the same canonical spray. Since (M, F) is affinely rigid, $\tilde{F} = F \circ \varphi_*$ is a constant multiple of F, therefore φ is a homothety. □

Corollary 3.4.10. *The affinities and isometries of a connected forward complete affinely rigid Finsler manifold coincide.*

Proof. From the previous lemma we know that the affinities of such a Finsler manifold are homotheties. However, the only forward complete connected Finsler manifolds admitting proper homotheties are the Finsler vector spaces [29]. But Finsler vector spaces are clearly not affinely rigid, so our claim follows. □

This generalization of Fact 3.4.3 is not very efficient, as little is known about affinely rigid Finsler manifolds. In the next section we explore some sufficient conditions.

3.5 Affinely rigid Finsler manifolds

In this section we attempt to give some sufficient conditions for a Finsler manifold to be affinely rigid (defined at the end of the previous section). Throughout, (M, F) is connected Finsler manifold.

We may construct the holonomy group of a Finsler manifold, analogously to that of a Riemannian manifold, using the parallel translation with respect to the canonical connection (see 1.2.4). In this way, for a fixed $p \in M$, we obtain a subgroup Hol_p of the group of diffeomorphisms of $\mathring{T}_pM$. Each element of Hol_p is a 1^+-homogeneous diffeomorphism of $\mathring{T}_pM$, which preserves the Finsler function. If the Finsler manifold is connected, Hol_p and Hol_q are isomorphic for any p and q in M, so we may speak of *the* holonomy group of a (connected) Finsler manifold. Finslerian holonomy groups can be vastly different from the Riemannian holonomy groups, as they can be infinite-dimensional (see, e.g., [33, 34]). Non-Berwald Landsberg manifolds have non-Riemannian, but finite dimensional holonomy groups [26], however, it is still not known whether such Finsler manifolds exist.

A simple sufficient condition for affine rigidity of a Finsler manifold is given by the following

Proposition 3.5.1. *If* Hol_p *acts transitively on the unit sphere*

$$U(T_pM) := F^{-1}(\{1\}) \cap T_pM,$$

then (M, F) is affinely rigid.

Proof. Let $\bar{F}$ be a Finsler function for M that shares its canonical spray with F. Then F and $\bar{F}$ have the same canonical connection, parallel translations and holonomy groups. Fix a point $p \in M$ and a vector $v \in U(T_pM)$, and let $c := \bar{F}(v)$. Then $\bar{F}(w) = c$ for any $w \in U(T_pM)$, since there is an element in Hol_p that maps v into w, and the elements of the holonomy group preserve both Finsler functions. Now choose another point $q \in M$ and a smooth curve γ with $\gamma(0) = p$ and $\gamma(1) = q$. Then $P(\gamma)_0^1$ maps $U(T_pM)$ onto $U(T_qM)$, and it satisfies $\bar{F} \circ P(\gamma)_0^1 = \bar{F}$ over T_pM. Thus $\bar{F}$ has the constant value c over $U(T_qM)$. Since q was arbitrary, the homogeneity of the Finsler functions imply $\bar{F} = cF$. □

It is well-known that irreducible Riemannian manifolds are affinely rigid. By Berger's holonomy theorem [7, 36] there are irreducible Riemannian manifolds whose holonomy groups are not transitive on the unit sphere. This implies that the converse of Proposition 3.5.1 is not true, and the transitivity of Hol_p should be replaced by a weaker condition.

It is worth noting that Riemannian manifolds are 'much more rigid' than Finsler manifolds. In the Riemannian case, the norms on the tangent spaces are required to be quadratic functions, thus they are uniquely determined by their Hessian at any point. Finsler functions allow much more freedom, therefore characterizing affinely rigid Finsler manifolds is expected to be more difficult, then the Riemannian ones.

To prepare the formulation of our results, we recall a few concepts and results about singular distributions from [32]. Given a manifold M, we denote by $\mathfrak{X}_{loc}(M)$ the set of vector fields that are defined only on an open subset of M. Suppose that for each $p \in M$ we have selected a subspace $\mathcal{E}_p$ of T_pM. Then the disjoint union $\mathcal{E} = \bigsqcup_{p \in M} \mathcal{E}_p$ is a *singular distribution* on M. We denote by $\mathfrak{X}_{\mathcal{E}}$ the set of (smooth) local vector fields in $\mathfrak{X}_{loc}(M)$, that take values only in $\mathcal{E}$. We say that a subset $\mathcal{V}$ of $\mathfrak{X}_{\mathcal{E}}$ *spans* $\mathcal{E}$, if at each $p \in M$, $\mathcal{E}_p$ is the linear span of $\{X(p) \in T_pM \mid X \in \mathcal{V}\}$. (We agree that the linear span of the empty set is $\{0\}$.) We say that $\mathcal{E}$ is *smooth* if it is spanned by $\mathfrak{X}_{\mathcal{E}}$.

An *integral manifold* of a smooth singular distribution $\mathcal{E}$ on M is an immersed submanifold $i \colon N \to M$ such that $i_*(T_pN) = \mathcal{E}_{i(p)}$ for all $p \in N$. It turns out that these integral manifolds are actually initial submanifolds [32, 2.13], so we need not to specify the immersion i.

A subset $\mathcal{V}$ of $\mathfrak{X}_{loc}(M)$ is *stable* if for any $X, Y \in \mathcal{V}$, the local vector field $(\mathrm{FL}_t^X)_\#(Y)$ is also in $\mathcal{V}$. For a set $\mathcal{W} \subset \mathfrak{X}_{loc}(M)$, $\mathcal{S}(\mathcal{W})$ denotes the set of local vector fields of the form

$$(\mathrm{Fl}_{t_1}^{X_1} \circ \cdots \circ \mathrm{Fl}_{t_k}^{X_k})_\# Y,$$

where $X_1, \dots, X_k, Y \in \mathcal{W}$, $k \in \mathbb{N}$. Then $\mathcal{S}(\mathcal{W})$ is the smallest stable subset of $\mathfrak{X}_{loc}(M)$ that contains $\mathcal{W}$. According to [32, 3.24 Lemma], the smooth singular distribution $\mathcal{E}$ spanned by $\mathcal{S}(\mathcal{W})$ is *integrable* in the sense that any point of M is contained in an integral manifold of $\mathcal{E}$.

We are going to study $\mathcal{S}(\mathfrak{X}_{H\mathring{T}M})$, and the smooth singular distribution $\mathcal{D}^h$ spanned by $\mathcal{S}(\mathfrak{X}_{H\mathring{T}M})$, where $H\mathring{T}M$ is the horizontal subbundle corresponding to the canonical connection of (M, F).

Lemma 3.5.2. *With the introduced notation, $\mathcal{D}^h \subset \ker(dF)$.*

Proof. We know that $H\mathring{T}M \subset \ker dF$, because $H\mathring{T}M$ is the image of the canonical connection of (M, F). Then it suffices to show that if ξ and η are vector fields on $\mathring{T}M$ satisfying $\eta F = \xi F = 0$, then we also have $(\mathrm{Fl}^\xi_{t\,\#}\eta)F = 0$.

Let $c\colon I \to TM$ be an integral curve of ξ. Then F is constant along c, because

$$(F \circ c)'(t) = \dot{c}(t)F = \xi(c(t))F = 0.$$

This implies that $F \circ \mathrm{Fl}^\xi_t = F$, whenever both sides are defined. Then

$$(\mathrm{Fl}^\xi_{t\,\#}\eta)F \overset{(3.10)}{=} (\eta(F \circ \mathrm{Fl}^\xi_t)) \circ \mathrm{Fl}^\xi_{-t} = (\eta F) \circ \mathrm{Fl}^\xi_{-t} = 0. \qquad \square$$

Corollary 3.5.3. *The Finsler function F is constant on the connected integral manifolds of $\mathcal{D}^h$.*

Thus it suffices to examine $\mathcal{D}^h$ on a single level set of F. The following observation is from [16]. We provide a proof for the reader's convenience

Fact 3.5.4. *If $\mathcal{E}$ is a smooth singular distribution, then the function $p \mapsto \dim \mathcal{E}_p$ is lower semi-continuous.*

Proof. Fix a point $p \in M$, and set $k = \dim \mathcal{E}_p$. We show that p has a neighbourhood $\mathcal{U}$, such that $\dim \mathcal{E}_q \geq k$ for all $q \in \mathcal{U}$, from which our claim follows.

There are vector fields $(X_i)_{i=1}^k$ in $\mathfrak{X}_\mathcal{E}$ such that $(X_i(p))$ spans $\mathcal{E}_p$. For an arbitrary frame field $(F_j)_{i=1}^n$ around p, we have $X_i = X_i^j F_j$ where (X_i^j) is a family of smooth functions on a neighbourhood of p. The rank of the matrix $(X_i^j(p))$ is k. Since the rank function of matrices is lower semi-continuous, there is a neighbourhood $\mathcal{U}$ of p such that the rank of $(X_i^j(q))$ is at least k for all $q \in \mathcal{U}$. So $(X_i(q))$ span a subspace of dimension at least k. Since $X_i(q) \in \mathcal{E}_q$, this proves that $\dim \mathcal{E}_q \geq k$. $\square$

Proposition 3.5.5. *If $\mathcal{D}^h$ has dimension $2n - 1$ over a dense subset of $\mathring{T}M$, then (M, F) is affinely rigid.*

Proof. Let $\bar{F}$ be a Finsler function for M which has the same canonical spray as F. Fix a point $v \in \mathring{T}M$ such that $\mathcal{D}^h_v$ has dimension $2n - 1$. Without loss of generality, we may assume that $F(v) = 1$, and hence $v \in U(TM) := F^{-1}(\{1\})$. By Fact 3.5.4,

$\mathcal{D}^h$ has dimension $2n-1$ on an open neighbourhood of v. Also, v is contained in an integral manifold N of $\mathcal{D}^h$, which has dimension $2n-1$. Since F is constant on N, by Corollary 3.5.3 we can assume that N is an open submanifold of $U(TM)$. However, $\bar{F}$ is also constant on N, thus $d(\bar{F} \restriction U(TM))_v = d(\bar{F} \restriction N)_v = 0$. Such points v in $U(TM)$ form a dense set, therefore $d(\bar{F} \restriction U(TM)) = 0$. This and the homogeneity of F and $\bar{F}$ imply that $F/\bar{F}$ is constant on TM. □

Proposition 3.5.6. *If $U(TM) := F^{-1}(\{1\})$ contains countably many maximal integral manifolds of $\mathcal{D}^h$, then (M,F) is affinely rigid.*

Proof. Let $\bar{F}$ be a Finsler function for M which has the same canonical spray as F. Let $\mathcal{O}$ be the set of integral manifolds of $\mathcal{D}^h$ contained in $U(TM)$. Then $\bar{F}$ is constant on each of these, thus $\bar{F}$ can have at most countably many different values on $U(TM)$. However, $\bar{F}$ is continuous, so this is possible only if $\bar{F}$ is constant on $U(TM)$. □

To summarize, from the results of this section, Lemma 3.4.9 and Corollary 3.4.10 we obtain:

Theorem 3.5.7. *Let (M,F) be a connected Finsler manifold satisfying any of the following conditions:*

(1) Hol_p *acts transitively on the unit sphere $U(T_pM) := F^{-1}(\{1\}) \cap T_pM$;*

(2) $\mathcal{D}^h$ *has dimension $2n-1$ over a dense subset of $\mathring{T}M$;*

(3) $U(TM) := F^{-1}(\{1\})$ *contains countably many maximal integral manifolds of $\mathcal{D}^h$.*

Then any affine transformation of (M,F) is a homothety. If, in addition, (M,F) is forward complete, then any affine transformation of it is an isometry.

Remark 3.5.8. Some special cases of these results have been appeared in the literature. J. Szenthe in [43] considered (in our terminology) the singular distribution spanned by the vector fields $\mathbf{v}[\xi,\eta]$ where $\xi,\eta \in \mathfrak{X}_{H\mathring{T}M}$. He proved that if it is a distribution and its rank is $n-1$, then any affine transformation is a homothety. This is a special case of our result, because $\mathcal{S}(\mathfrak{X}_{H\mathring{T}M})$ is closed under Lie brackets by [32, 3.27 Lemma], and the dimension of $\mathcal{D}^h$ is $2n-1$ if and only if the dimension of $\mathbf{v}\mathcal{D}^h$ is $n-1$. In [17], the authors considered the singular distribution spanned by all the successive Lie brackets of the vector fields in $\mathfrak{X}_{H\mathring{T}M}$, and assumed that it is a distribution. They have found a connection between the codimension of this distribution and the maximal number of functionally independent Finsler functions that have the same canonical spray as F. As a special case, they obtained that if the codimension is 1, then the canonical spray is uniquely metrizable. Our Proposition 3.5.5 is a generalization of this result, because we allow the distribution to be singular, and it is also larger, thus it has a better chance to have the maximal dimension $2n-1$ (almost) everywhere.

The following converse of Proposition 3.5.5 is quite tempting:

> *If the dimension of $\mathcal{D}^h$ is less than $2n-1$ on an open subset of $\mathring{T}M$, then (M,F) is* not *affinely rigid.*

If $\mathcal{D}^h$ has a nonmaximal dimension on an open subset, it can have (uncountably) many integral manifolds, which forces less rigidity on the Finsler functions that metrize the canonical spray. However, even if a smooth singular distribution has nonmaximal dimension on an open subset, it can still uniquely determine the continuous functions that are constant on the integral manifolds. For example, there is a smooth singular distribution on $\mathbb{R}^2$ (endowed with the canonical coordinate system (x,y)) whose maximal integral manifolds are

(a) the half-planes given by the inequalities $y<0$ and $y>1$;

(b) the 'vertical' line segment $\{(x_0,y)\in\mathbb{R}^2 \mid 0<y<1\}$, for each $x_0\in\mathbb{R}$;

(c) each point of the straight lines given by $y=0$ and $y=1$.

It is easy to see that the only continuous functions that are constant on each of these integral manifolds are the constant functions. However, we do not know whether a similar configuration can occur or not in the case of $\mathcal{D}^h$.

For the sake of completeness, we show that there is indeed a smooth singular distribution on $\mathbb{R}^2$ with the integral manifolds described above. Let $\varphi,\psi\colon\mathbb{R}\to\mathbb{R}$ be smooth, nonnegative functions such that

$$\varphi(0)=\varphi(1)=0,\ \text{and it is positive everywhere else;}$$
$$\psi \text{ vanishes on } [0,1], \text{ and it is positive everywhere else.}$$

Consider the smooth singular distribution $\mathcal{E}$ spanned by the vector fields

$$X=(\psi\circ x)\frac{\partial}{\partial x},\qquad Y=(\varphi\circ y)\frac{\partial}{\partial y}.$$

Then

$$\dim(\mathcal{E}_p)=\begin{cases}2 \text{ if } y(p)>1 \text{ or } y(p)<0\\ 1 \text{ if } 0<y(p)<1\\ 0 \text{ if } y(p)=1 \text{ or } y(p)=0\end{cases},$$

and $\mathcal{E}$ has the prescribed integral manifolds.

3.6 Finsler manifolds with many Killing vector fields

Let (M, F) be a Finsler manifold. A local parallelization P over an open subset of M (see section 2.3) is called a *Killing parallelization* if it is the $\mathbb{R}$-linear span of Killing vector fields. Then, actually, all vector fields in P are Killing vector fields.

We say that a Finsler manifold (M, F) has the *Killing property* (cf. [51]), if each point in M has an open neighbourhood that admits a Killing parallelization. We say that (M, F) has the *constant Killing property* if it has the Killing property, and the Killing parallelizations are also compatible with F.

These concepts were motivated by the paper [12] where the authors examined Riemannian manifolds which admit local orthonormal frame fields consisting of Killing vector fields. Finsler manifolds with constant Killing property are direct generalizations of these Riemannian manifolds. We show that a Finsler manifold with Killing property is a generalized Berwald manifold, and Finsler manifolds with the constant Killing property are Berwald manifolds. We note that the second result is a generalization of a corollary of Theorems 3.21 and 3.22 in [1].

Theorem 3.6.1. *A Finsler manifold with Killing property is a generalized Berwald manifold.*

Proof. Let (M, F) be a Finsler manifold with Killing property, and let P be a local Killing parallelization over $\mathcal{U}$. Define a covariant derivative D (over $\mathcal{U}$) by setting $\nabla_X Y := [X, Y]$ for any $X, Y \in P$. Then the Ehresmann connection in $\mathring{\tau}^{-1}(\mathcal{U})$ induced by D is given by

$$\begin{aligned}\mathcal{H}(v, w) &\overset{(1.47)}{:=} v_{P*}(w) - \mathbf{i}(v, (D_w v_P)) \\ &= v_{P*} \circ w_P(p) - (D_{w_P} v_P)^{\mathsf{v}} \circ v_P(p) \\ &= v_{P*} \circ w_P(p) - [w_P, v_P]^{\mathsf{v}} \circ v_P(p) \\ &\overset{(1.19)}{=} w_P^{\mathsf{c}}(v_P(p))\end{aligned}$$

for all $p \in \mathcal{U}$, $v, w \in \mathring{\tau}_p^{-1}(\mathcal{U})$. Since w_P is a Killing vector field, we have $w_P^{\mathsf{c}} F = 0$, so the above calculation gives $\mathcal{H}(v, w)F = 0$. In other words, $\operatorname{Im}\mathcal{H} \subset \ker dF$, that is, $\mathcal{H}$ is compatible with F.

We may choose an open covering $(\mathcal{U}_\alpha)_{\alpha \in \mathcal{A}}$ of M such that there is a local Killing parallelization P_α over each $\mathcal{U}_\alpha$. Then the above construction gives a linear connection $\mathcal{H}^\alpha$ on $\mathring{\tau}^{-1}(\mathcal{U}_\alpha)$, compatible with F. The rest of the proof is the same as that of Theorem 2.3.3. □

Theorem 3.6.2. *A Finsler manifold with constant Killing property is a Berwald manifold.*

Proof. Let (M, F) be a Finsler manifold with constant Killing property and P a local Killing parallelization over $\mathcal{U} \subset M$ such that for all $X \in P$, $F \circ X$ is constant. As in the proof of Theorem 2.3.3, we may define a covariant derivative D^- on $\mathcal{U}$ such that $D^-_X Y = 0$ for all $X, Y \in P$ and the corresponding linear connection $\mathcal{H}^-$ in $\mathring{\tau}^{-1}(\mathcal{U})$ is compatible with F. However, as in the proof of Theorem 3.6.1, there is a covariant derivative D^+ on $\mathcal{U}$ such that $D^+_X Y = [X, Y]$ for all $X, Y \in P$, and then the corresponding linear connection $\mathcal{H}^+$ in $\mathring{\tau}^{-1}(\mathcal{U})$ is also compatible with F. Then $\mathcal{H} := \frac{1}{2}\mathcal{H}^+ + \frac{1}{2}\mathcal{H}^-$ is also a linear connection (Lemma 2.3.2), and it is still compatible with F, because $\ker dF$ is closed under affine combinations. It is torsion-free, because for each $X, Y \in P$ we have

$$\begin{aligned}
\mathbf{i}\,\mathbf{T}(\widehat{X}, \widehat{Y}) &\overset{(1.28)}{=} [\mathcal{H}\widehat{X}, Y^{\mathsf{v}}] - [\mathcal{H}\widehat{Y}, X^{\mathsf{v}}] - [X, Y]^{\mathsf{v}} \\
&= \frac{1}{2}\left([\mathcal{H}^+\widehat{X}, Y^{\mathsf{v}}] + [\mathcal{H}^-\widehat{X}, Y^{\mathsf{v}}] - [\mathcal{H}^+\widehat{Y}, X^{\mathsf{v}}] - [\mathcal{H}^-\widehat{Y}, X^{\mathsf{v}}]\right) - [X, Y]^{\mathsf{v}} \\
&\overset{(1.46)}{=} \frac{1}{2}\left(D^+_X Y + D^-_X Y - D^+_Y X - D^-_Y X\right)^{\mathsf{v}} - [X, Y]^{\mathsf{v}} \\
&= \frac{1}{2}\left([X, Y] - [Y, X]\right)^{\mathsf{v}} - [X, Y]^{\mathsf{v}} = 0.
\end{aligned}$$

Now we may proceed in the same way as in the proof of Theorem 2.3.3, using partition of unity to glue together the linear connections compatible with F. It only remains to check that if $(\mathcal{H}^\alpha)_{\alpha \in \mathcal{A}}$ is a family if torsion-free linear connections and $(\phi_\alpha)_{\alpha \in \mathcal{A}}$ is a family of smooth functions on M such that $\sum_{\alpha \in \mathcal{A}} \phi_\alpha = 1$, then $\mathcal{H} := \sum_{\alpha \in \mathcal{A}} \phi^{\mathsf{v}}_\alpha \mathcal{H}^\alpha$ is also torsion-free. This is a straightforward calculation:

$$\begin{aligned}
\mathbf{i}\mathbf{T}(\widehat{X}, \widehat{Y}) &\overset{(1.28)}{=} [X^{\mathsf{h}}, Y^{\mathsf{v}}] - [Y^{\mathsf{h}}, X^{\mathsf{v}}] - [X, Y]^{\mathsf{v}} \\
&= \left[\sum_{\alpha \in \mathcal{A}} \phi^{\mathsf{v}}_\alpha \mathcal{H}^\alpha \widehat{X}, Y^{\mathsf{v}}\right] - \left[\sum_{\alpha \in \mathcal{A}} \phi^{\mathsf{v}}_\alpha \mathcal{H}^\alpha \widehat{Y}, X^{\mathsf{v}}\right] - [X, Y]^{\mathsf{v}} \\
&\overset{(1.7)}{=} \sum_{\alpha \in \mathcal{A}} \phi_\alpha \left([\mathcal{H}^\alpha \widehat{X}, Y^{\mathsf{v}}] - [\mathcal{H}^\alpha \widehat{Y}, X^{\mathsf{v}}] - [X, Y]^{\mathsf{v}}\right) = 0.
\end{aligned}$$

□

Chapter 4

Summary

In this summary we collect our main results. Since in Chapter 1 we have already summarized our basic tools and concepts, we do not repeat them here. All results are quoted from the Thesis with the same numbering. If a theorem, lemma, corollary or proposition is not new, we write the name of the original author right at the beginning of the statement between parentheses. If a result of ours have been already published, we cite the publication also at the beginning of the theorem.

Characterization of some Finsler and spray manifolds

Projectively affine sprays

Two sprays S and $\bar{S}$ for a manifold M are *projectively related* if $\bar{S} = S - 2PC$ for some function P on $\mathring{T}M$. Then we also say that $\bar{S}$ is a *projective change of S with factor P*.

It can be shown by simple calculations that if a spray is projectively related to an affine spray, then its *Douglas tensor*

$$\mathbf{D} := \mathbf{B} - \frac{1}{n+1}\Big((\nabla^{\mathsf{v}} \operatorname{tr} \mathbf{B}) \otimes \widetilde{\delta} + (\operatorname{tr} \mathbf{B}) \odot \operatorname{id}_{\Gamma(\mathring{\tau}^*\tau)}\Big)$$

vanishes. Our aim is to prove the converse. To do this, we need the notion of *vertically invariant volume forms*. These are volume forms on the tangent manifold of M, whose Lie-derivatives with respect to vertical lifts vanish.

Theorem 2.1.8 ([45]). *Let (M, S) be a spray manifold with vanishing Douglas tensor and let ω be a vertically invariant volume form on TM. Then*

$$\bar{S} := S - 2PC, \quad where \quad P := \frac{1}{2(n+1)} \operatorname{div}_\omega S,$$

is an affine spray.

Proposition 2.1.9 ([45]). *The tangent manifold of a manifold admits a vertically invariant volume form.*

Together these yield:

Corollary 2.1.10 ([45]). *If the Douglas tensor of a spray manifold vanishes, then the spray is projectively related to an affine spray.*

Locally this was first proved by J. Douglas [15]. Global versions are due to Z. Shen [40, 5.2] and J. Szilasi and Sz. Vattamány [46]. However they assumed that the base manifold orientable. In our version this assumption is eliminated.

There are no proper Einstein–Berwald manifolds

A *Berwald manifold* is a Finsler manifold whose canonical spray is affine, or, equivalently, whose canonical connection is linear.

A Finsler manifold (M, F) is called an *Einstein–Finsler manifold* if the curvature function of its canonical spray given by (1.31) is related to the Finsler function by $K = (\lambda \circ \tau)F^2$, where $\lambda \in C^\infty(M)$. We have obtained a negative result on the existence of Einstein–Finsler manifolds among Berwald manifolds.

Lemma 2.2.1 ([14]). *The curvature function of a Berwald manifold is a quadratic form on each tangent space.*

Theorem 2.2.2 ([14]). *If a connected Einstein–Finsler manifold is a Berwald manifold, then it is either a Riemannian manifold or its curvature function vanishes.*

Monochromatic Finsler manifolds

Let M be a manifold and $\mathcal{U}$ an open submanifold of M. Consider a frame field $(X_i)_{i=1}^n$ on $\mathcal{U}$. Then the $\mathbb{R}$-linear span of $(X_i)_{i=1}^n$ is called a *local parallelization (of M) over $\mathcal{U}$*. Given a local parallelization P over $\mathcal{U}$, a vector field in P is called *parallel* with respect to P. For any $v \in \tau^{-1}(\mathcal{U})$, there is a unique parallel vector field v_P (w.r.t. P), such that $v_P(\tau(v)) = v$. For each $p, q \in \mathcal{U}$, we obtain a mapping $P_{qp} \colon T_pM \to T_qM$ by setting $P_{qp}(v) := v_P(q)$. The construction guarantees that these mappings are linear. The concept of parallelization defined this way is equivalent to the one in [18, p. 174]. A (not necessarily smooth) function $f \colon \mathring{T}M \to \mathbb{R}$ is *compatible with* P, if $f \circ P_{qp} = f$ over $\mathring{T}_pM$ for all $p, q \in \mathcal{U}$. Equivalently, $f \circ X$ is constant for all $X \in P$.

We say that a (not necessarily smooth) function $f \colon \mathring{T}M \to \mathbb{R}$ is *monochromatic*, if for any $p, q \in M$, there is a linear mapping L_{qp} from T_pM to T_qM such that

$f \circ L_{qp} = f$ over $\mathring{T}_pM$. We say that f is *smoothly monochromatic*, if for any point $p \in M$ there is a local parallelization over an open neighbourhood of p, which is compatible with f. It is easy to see that smoothly monochromatic functions are monochromatic if M is connected.

Similarly, we say that a (not necessarily smooth) function $f \colon \mathring{T}M \to \mathbb{R}$ is *compatible with a homogeneous Ehresmann connection* $\mathcal{H}$ if the parallel translations with respect to $\mathcal{H}$ preserve f. In other words, f is constant along $\mathcal{H}$-parallel vector fields (cf. section 1.2.4).

Lemma 2.3.1. *A function $f \colon \mathring{T}M \to \mathbb{R}$ of class C^1 is compatible with a homogeneous Ehresmann connection if and only if the corresponding horizontal subbundle is a subset of* $\ker df$.

We establish some basic relations between functions compatible with linear connections and monochromatic functions.

Theorem 2.3.3. *If a function $f \colon \mathring{T}M \to \mathbb{R}$ of class C^1 is smoothly monochromatic, then it is compatible with a linear connection.*

The converse is true without any restriction on the function.

Proposition 2.3.6. *If a not necessarily smooth function $f \colon \mathring{T}M \to \mathbb{R}$ is compatible with a linear connection, then f is smoothly monochromatic.*

A Finsler manifold (M, F) is a *generalized Berwald manifold* if there is a linear connection in $\mathring{T}M$ compatible with F. From our two results above, the following characterization of generalized Berwald manifolds follows immediately.

Theorem 2.3.7 (Y. Ichijyō [21])**.** *A Finsler manifold is a generalized Berwald manifold if and only if it is smoothly monochromatic.*

We note that Y. Ichijyō's proof relied on G-structures and their compatible connections. Our proof relies merely on simple properties of Ehresmann connections and utilizes only the differentiability of the Finsler function, and so is very simple.

Finally we have a modern reformulation of a classical result that goes back to H. Weyl.

Theorem 2.3.9 (Weyl)**.** *Let $f \colon T_pM \to \mathbb{R}$ be a Finsler norm. (Then f is continuous, 1^+-homogeneous, smooth and positive on $T_pM \setminus \{0\}$, and $f''(v)$ is positive definite for all $v \in T_pM \setminus \{0\}$.). For each parallelization P over an open subset $\mathcal{U} \subset M$ containing p, define a Finsler function F_P for $\mathcal{U}$ by $F_P(v) := f(v_P(p))$. Assume that for each such parallelization P, $(\mathcal{U}, F_P)$ is a Berwald manifold. Then f is an Euclidean norm.*

Affinities and isometries of spray and Finsler manifolds

This chapter is devoted to affine and isometric transformations of spray and Finsler manifolds. Given a Finsler manifold (M, F) or a spray manifold (M, S), we define an *affinity* as a (not necessarily smooth) bijective mapping $\varphi\colon M \to M$ that preserves geodesics. More precisely, φ is an affinity if for any geodesic γ, $\varphi \circ \gamma$ is also a geodesic. Similarly, for a Finsler manifold (M, F), a (not necessarily smooth) bijective mapping $\varphi\colon M \to M$ is an *isometry* if $\varrho(\varphi(p), \varphi(q)) = \varrho(p, q)$, for all $p, q \in M$. Here $\varrho(p, q)$ is the (generally nonsymmetric) Finslerian distance, defined as the infimum of the set of lengths of the piecewise smooth curves connecting p with q (in this order).

Affinities and isometries are smooth

F. Brickell proved [8] that an affine homeomorphism of a spray manifold is necessarily smooth. His proof can be simplified using the lemma below, which may be interesting in its own right.

Lemma 3.1.1. *Let (M, S) be a spray manifold, p a point in M. Suppose that (a_n) and (b_n) are sequences converging to p, and let*

$$L_n := \exp_{a_n}^{-1}(b_n), \quad A_n := \exp_p^{-1}(a_n), \quad B_n := \exp_p^{-1}(b_n); \qquad n \in \mathbb{N}^*.$$

Then

$$\lim_{n\to\infty} \frac{B_n - A_n - L_n}{\|L_n\|} = 0,$$

and hence $\frac{\|B_n - A_n\|}{\|L_n\|} \to 1$. Furthermore, if one of the sequences

$$\left(\frac{B_n - A_n}{\|B_n - A_n\|}\right), \qquad \left(\frac{L_n}{\|L_n\|}\right)$$

converges, then so does the other, and their limits coincide.

Here $\|\cdot\|$ is the norm with respect to an arbitrarily chosen Riemannian metric on M.

Theorem 3.1.3 (F. Brickell [8])**.** *An affine homeomorphism of a spray manifold is a smooth diffeomorphism.*

Using the minimizing property of 'short geodesics' [4, Theorem 6.3.1] it is easy to show that isometries of a Finsler manifold are affinities. Furthermore, isometries are continuous, since forward balls generate the topology of M. Thus, an isometry is a homeomorphism. Hence from Theorem 3.1.3 we obtain immediately

Theorem 3.2.2 (F. Brickell [8]). *Isometries of Finsler manifolds are smooth.*

Distance coordinates

Let (M, F) be a Finsler manifold and $(p_i)_{i=1}^n$ a family of points in M such that the mapping

$$u_\varrho \colon q \mapsto (\varrho(p_1, q), \dots, \varrho(p_n, q))$$

is a diffeomorphism from an open subset $\mathcal{U}$ of M onto an open subset of $\mathbb{R}^n$. Then $(\mathcal{U}, u_\varrho)$ is called a *distance chart* for (M, F).

Lemma 3.3.1 ([3]). *Any point of a Finsler manifold has a neighbourhood that admits a distance chart.*

Using distance charts, we give a simple new proof for the smoothness of isometries which does not rely on the smoothness of affinities. We borrowed the idea of proof from P. Petersen's book [38, Ch. 5.10].

As a further application of distance charts, we show that regular submetries between reversible Finsler manifolds are differentiable (Theorem 3.3.3 below). This result is known in the more general setting of metric spaces (see [30]). We believe, however, that our proof is more accessible and more interesting for experts in differential geometry. First we clarify the terminology.

Let M_1 and M_2 be metric spaces. We denote by $\mathcal{B}$ the metric balls in both of them. We say that a mapping $\varphi \colon M_1 \to M_2$ is a *submetry* if for any p in M_1, there is a positive number δ such that for every $\varepsilon \in \,]0, \delta[$ we have $\varphi(\mathcal{B}_\varepsilon(p)) = \mathcal{B}_\varepsilon(\varphi(p))$. For each $p \in M_1$, the supremum of these positive numbers δ will be denoted by δ_p (note that δ_p can be infinite). We say that a submetry is *regular* if for any compact set $\mathcal{K} \subset M_1$ we have

$$\delta_{\mathcal{K}} := \inf_{p \in \mathcal{K}} \delta_p > 0.$$

Recall that a Finsler manifold is *reversible*, if $F(-v) = F(v)$ for every $v \in TM$. In case of such a Finsler manifold, the distance function ϱ is symmetric, and hence (M, ϱ) is a metric space.

Lemma 3.3.2 ([3]). *Let (M, F) be a reversible Finsler manifold, p a point in M, and $\mathcal{U}$ a normal neighbourhood of p. Then the distance function ϱ_p at p restricted to $\mathcal{U} \setminus \{p\}$ is a regular submetry into $\mathbb{R}$ with respect to its canonical distance.*

Theorem 3.3.3 (A. Lytchak [30],[3]). *A surjective regular submetry between reversible Finsler manifolds is differentiable.*

In the proof we borrowed some ideas from [6] where the analogous result is proved in Riemannian setting.

Relations between affinities and isometries

In the previous sections we have seen that an affine homeomorphism φ of a spray manifold (M, S) is smooth. Then it is clear that φ is a *automorphism* of S, that is, a diffeomorphism satisfying $\varphi_{**} \circ S = S \circ \varphi_*$. The converse is obvious, automorphisms of sprays are affine. Similarly, we proved that an isometry φ of a Finsler manifold (M, F) is smooth. It can be shown that then φ is a *Finsler isometry* of F, that is, a diffeomorphism satisfying $F \circ \varphi_* = F$. In the Riemannian case, S.B. Myers and N. Steenrod [35] proved that the two types of isometries are the same. The Finslerian version of the theorem is due to F. Brickell [8]. S. Deng and Z. Hou rediscovered it four decades later [13]. In both proofs, the crucial point is the smoothness of isometries, which we have already established. Hence we obtain the Finslerian version of the Myers–Steenrod theorem easily.

Theorem 3.4.1 (F. Brickell [8])**.** *An isometry of a Finsler manifold is a Finsler isometry.*

The converse is clearly true. To summarize: *Finsler isometries of a Finsler manifold are the same as its isometries, and automorphisms of a spray manifold are the same as its affine homeomorphisms.* From now on, we will simply refer to them as isometries and affinities.

Now we turn to the relations between isometries and affinities of a Finsler manifold. It is more or less obvious, that isometries are affinities, since if a transformation preserves the Finsler function, it should preserve anything derived from it. However, this heuristic argument could not replace the precise proof.

Proposition 3.4.2 ([3])**.** *A Finsler isometry is an affinity.*

The reverse is clearly not true in general. There are affinities that are not isometries, for example, in Euclidean spaces. In Riemannian geometry, some celebrated results give sufficient conditions for the equivalence of affinities and isometries (see, e.g., [25, pp. 242–244]). We present some Finslerian counterparts.

Our first result is about infinitesimal affinities and isometries. These are vector fields on the base manifold, whose stages have the corresponding property. Infinitesimal affinities are called *affine vector field*s, and infinitesimal isometries are called *Killing vector field*s. Both property can be characterized with the help of complete lift: if (M, F) is a Finsler manifold with canonical spray S, then $X \in \mathfrak{X}(M)$ is affine if and only if $\mathcal{L}_{X^{\mathsf{c}}} S := [X^{\mathsf{c}}, S] = 0$, and it is a Killing vector field if and only if $\mathcal{L}_{X^{\mathsf{c}}} F := X^{\mathsf{c}} F = 0$.

First we generalize a theorem of J.-I. Hano to Finsler manifolds. Hano proved that *bounded affine vector fields on a complete Riemannian manifold are Killing fields.* The proof of this in the book [25] relies on the de Rham decomposition theorem of simply connected and complete Riemannian manifolds [25, Theorem 6.2], and on the following important result:

Fact 3.4.3 (S. Kobayashi, [24])**.** *The affinities and isometries of a complete irreducible Riemannian manifold coincide, except for the* 1*-dimensional Euclidean space.*

Hano did not use explicitly the second result in his proof [19], but he followed similar ideas as Kobayashi.

Rather surprisingly, Hano's theorem can be proved for Finsler manifolds without introducing any analogue of irreducibility (or decomposition). In fact, our proof is elementary, it relies merely on the Euler–Lagrange equation (1.53). The key is the following simple observation.

Lemma 3.4.5 ([23])**.** *If X is an affine vector field on a Finsler manifold (M, F) and γ is a geodesic, then for all t and t_0 in the domain of γ we have*

$$X^{\mathsf{v}}E(\dot\gamma(t)) = X^{\mathsf{v}}E(\dot\gamma(t_0)) + (t - t_0)X^{\mathsf{c}}E(\dot\gamma(t_0)).$$

This lemma is in fact a disguised special case of Exercise 5.4.3(c) from [4], which states the following: if J is a Jacobi field along a geodesic γ, then

$$g_{\dot\gamma}(J, \dot\gamma)(t) = g_{\dot\gamma}(J(t_0), \dot\gamma(t_0)) + (t - t_0)g_{\dot\gamma}(J'(t_0), \dot\gamma(t_0)),$$

where J' is the covariant derivative of J along γ with respect to the Berwald or the Chern derivative. However, it is easier to prove the lemma directly, than from the quoted exercise.

Theorem 3.4.6 ([23])**.** *Let (M, F) be a Finsler manifold, X an affine vector field on M, and suppose that one of the following conditions holds:*

(1) $F \circ X$ is bounded, and (M, F) is complete;

(2) $F \circ X$ and $F \circ (-X)$ are bounded, and (M, F) is forward complete.

Then X is a Killing vector field.

As a simple corollary we have

Theorem 3.4.7 (J.-I. Hano)**.** *Let (M, g) be a complete Riemannian manifold, and X an affine vector field on M such that the function $g(X, X)$ is bounded. Then X is a Killing vector field.*

The proof is immediate if we apply Theorem 3.4.6 to the Finsler function given by $F(v) := \sqrt{g(v, v)}$, $v \in TM$. Since compact Finsler manifolds are complete, we also have

Theorem 3.4.8 ([23])**.** *An affine vector field on a compact Finsler manifold is a Killing vector field.*

The Riemannian version of the last theorem is due to K. Yano [52].

We propose a possible generalization of Theorem 3.4.3 to Finsler manifolds, by replacing irreducibility with affine rigidity, defined as follows. We say that a spray S for M is *uniquely metrizable*, if it is the canonical spray of a Finsler function F for M, and if $\tilde{F}$ is another Finsler function whose canonical spray is also S, then on each component of M, $\tilde{F}$ is a scalar multiple of F (in other words, $d(F/\bar{F}) = 0$). We say that a Finsler manifold (M, F) is *affinely rigid*, if its canonical spray S is uniquely metrizable. The following result is a direct Finslerian analogue of [25, p. 242, Lemma 1].

Lemma 3.4.9. *An affinity of a connected affinely rigid Finsler manifold is a homothety.*

Here by a homothety we mean a diffeomorphism φ satisfying $F \circ \varphi_* = cF$ for some constant c. Since the only forward complete Finsler manifold admitting homotheties that are not isometries are the Finsler vector spaces [29], we have:

Corollary 3.4.10. *The affinities and isometries of a connected forward complete affinely rigid Finsler manifold coincide.*

This generalization of Theorem 3.4.3 is not very efficient, as little is known about affinely rigid Finsler manifolds. We continue with exploring some sufficient conditions.

Affinely rigid Finsler manifolds

Assume that (M, F) is connected Finsler manifold.

We may construct the holonomy group of a Finsler manifold, analogously to that of a Riemannian manifold, using the parallel translation with respect to the canonical connection (see 1.2.4). In this way, for a fixed $p \in M$, we obtain a subgroup Hol_p of the group of diffeomorphisms of $\mathring{T}_pM$.

A simple sufficient condition for affine rigidity of a Finsler manifold is given by the following

Proposition 3.5.1. *If* Hol_p *acts transitively on the unit sphere*

$$U(T_pM) := F^{-1}(\{1\}) \cap T_pM,$$

then (M, F) *is affinely rigid.*

It is well-known that irreducible Riemannian manifolds are affinely rigid. By Berger's holonomy theorem [7, 36] there are irreducible Riemannian manifolds whose holonomy groups are not transitive on the unit sphere. This implies that the converse of Proposition 3.5.1 is not true, and the transitivity of Hol_p should be replaced by a weaker condition.

To prepare the formulation of our results, we have to recall a few concepts and results about singular distributions from [32]. Given a manifold M, we denote by $\mathfrak{X}_{loc}(M)$ the set of vector fields that are defined only on an open subset of M. Suppose that for each $p \in M$ we have selected a subspace $\mathcal{E}_p$ of T_pM. Then the disjoint union $\mathcal{E} = \bigsqcup_{p\in M} \mathcal{E}_p$ is a *singular distribution* on M. We denote by $\mathfrak{X}_{\mathcal{E}}$ the set of (smooth) local vector fields in $\mathfrak{X}_{loc}(M)$, that take values only in $\mathcal{E}$. We say that a subset $\mathcal{V}$ of $\mathfrak{X}_{\mathcal{E}}$ *spans* $\mathcal{E}$, if at each $p \in M$, $\mathcal{E}_p$ is the linear span of $\{X(p) \in T_pM \mid X \in \mathcal{V}\}$. (We agree that the linear span of the empty set is $\{0\}$.) We say that $\mathcal{E}$ is *smooth* if it is spanned by $\mathfrak{X}_{\mathcal{E}}$.

An *integral manifold* of a smooth singular distribution $\mathcal{E}$ on M is an immersed submanifold $i\colon N \to M$ such that $i_*(T_pN) = \mathcal{E}_{i(p)}$ for all $p \in N$. It turns out that these integral manifolds are actually initial submanifolds [32, 2.13], so we need not to specify the immersion i.

For a set $\mathcal{W} \subset \mathfrak{X}_{loc}(M)$, $\mathcal{S}(\mathcal{W})$ denotes the set of local vector fields of the form

$$(\mathrm{Fl}^{X_1}_{t_1} \circ \cdots \circ \mathrm{Fl}^{X_k}_{t_k})_\# Y,$$

where $X_1, \ldots, X_k, Y \in \mathcal{W}$, $k \in \mathbb{N}$.

Let $\mathcal{D}^h$ be the smooth singular distribution spanned by $\mathcal{S}(\mathfrak{X}_{H\mathring{T}M})$, where $H\mathring{T}M$ is the horizontal subbundle corresponding to the canonical connection of (M, F).

Lemma 3.5.2. *With the introduced notation,* $\mathcal{D}^h \subset \ker(dF)$.

Corollary 3.5.3. *The Finsler function F is constant on the connected integral manifolds of $\mathcal{D}^h$.*

Proposition 3.5.5. *If $\mathcal{D}^h$ has dimension $2n-1$ over a dense subset of $\mathring{T}M$, then (M, F) is affinely rigid.*

Proposition 3.5.6. *If $U(TM) := F^{-1}(\{1\})$ contains countably many maximal integral manifolds of $\mathcal{D}^h$, then (M, F) is affinely rigid.*

To summarize, from the results of this section, Lemma 3.4.9 and Corollary 3.4.10 we obtain:

Theorem 3.5.7. *Let (M, F) be a connected Finsler manifold satisfying any of the following conditions:*

(1) Hol_p *acts transitively on the unit sphere* $U(T_pM) := F^{-1}(\{1\}) \cap T_pM$*;*

(2) $\mathcal{D}^h$ *has dimension* $2n-1$ *over a dense subset of* $\mathring{T}M$*;*

(3) $U(TM) := F^{-1}(\{1\})$ *contains countably many maximal integral manifolds of* $\mathcal{D}^h$*.*

Then any affine transformation of (M,F) *is a homothety. If, in addition,* (M,F) *is forward complete, then any affine transformation of it is an isometry.*

Remark 3.5.8. Some special cases of these results have been appeared in the literature. J. Szenthe in [43] considered (in our terminology) the singular distribution spanned by the vector fields $\mathbf{v}[\xi,\eta]$ where $\xi,\eta \in \mathfrak{X}_{H\mathring{T}M}$. He proved that if it is a distribution and its rank is $n-1$, then any affine transformation is a homothety. This is a special case of our result, because $\mathcal{S}(\mathfrak{X}_{H\mathring{T}M})$ is closed under Lie brackets by [32, 3.27 Lemma], and the dimension of $\mathcal{D}^h$ is $2n-1$ if and only if the dimension of $\mathbf{v}\mathcal{D}^h$ is $n-1$. In [17], the authors considered the singular distribution spanned by all the successive Lie brackets of the vector fields in $\mathfrak{X}_{H\mathring{T}M}$, and assumed that it is a distribution. They have found a connection between the codimension of this distribution and the maximal number of functionally independent Finsler functions that have the same canonical spray as F. As a special case, they obtained that if the codimension is 1, then the canonical spray is uniquely metrizable. Our Proposition 3.5.5 is a generalization of this result, because we allow the distribution to be singular, and it is also larger, thus it has a better chance to have the maximal dimension $2n-1$ (almost) everywhere.

The following converse of Proposition 3.5.5 is quite tempting:

> *If the dimension of* $\mathcal{D}^h$ *is less than* $2n-1$ *on an open subset of* $\mathring{T}M$*, then* (M,F) *is* not *affinely rigid.*

If $\mathcal{D}^h$ has a nonmaximal dimension on an open subset, it can have (uncountably) many integral manifolds, which forces less rigidity on the Finsler functions that metrize the canonical spray. However, even if a smooth singular distribution has nonmaximal dimension on an open subset, it can still uniquely determine the continuous functions that are constant on the integral manifolds. For example, there is a smooth singular distribution on $\mathbb{R}^2$ (endowed with the canonical coordinate system (x,y)) whose maximal integral manifolds are

(a) the half-planes given by the inequalities $y<0$ and $y>1$;

(b) the 'vertical' line segment $\{(x_0,y) \in \mathbb{R}^2 \mid 0<y<1\}$, for each $x_0 \in \mathbb{R}$;

(c) each point of the straight lines given by $y=0$ and $y=1$.

It is easy to see that the only continuous functions that are constant on each of these integral manifolds are the constant functions. However, we do not know whether a similar configuration can occur or not in the case of $\mathcal{D}^h$.

Finsler manifolds with many Killing vector fields

Let (M, F) be a Finsler manifold. A local parallelization P over an open subset of M (see section 4) is called a *Killing parallelization* if it is the $\mathbb{R}$-linear span of Killing vector fields.

We say that a Finsler manifold (M, F) has the *Killing property* (cf. [51]), if each point in M has an open neighbourhood that admits a Killing parallelization. We say that (M, F) has the *constant Killing property* if it has the Killing property, and the Killing parallelizations are also compatible with F.

These concepts were motivated by the paper [12] where the authors examined Riemannian manifolds which admit local orthonormal frame fields consisting of Killing vector fields. Finsler manifolds with constant Killing property are direct generalizations of these Riemannian manifolds. We give simple proofs for the following two theorems.

Theorem 3.6.1. *A Finsler manifold with Killing property is a generalized Berwald manifold.*

Theorem 3.6.2. *A Finsler manifold with constant Killing property is a Berwald manifold.*

We note that the last result is a generalization of a corollary of Theorems 3.21 and 3.22 in [1].

5. fejezet

Magyar nyelvű összefoglaló (Summary in Hungarian)

Ebben az összefoglalóban ismertetjük az értekezés fontosabb tételeit és a kimondásukhoz szükséges fogalmakat. Valamennyi eredmény azzal a sorszámmal szerepel, mint a disszertációban. Ha egy tétel, lemma, következmény vagy állítás nem új, az eredeti szerző nevét zárójelben jelezzük a tétel legelején. A publikált új eredményeink kimondásakor szintén zárójelben jelezzük a föllelhetőségüket.

1. fejezet: Fogalmi háttér és eszközök

Kanonikus objektumok az érintőnyalábon

A disszertációban megszámlálható bázisú, sima Hausdorff-sokaságokon dolgozunk, ezeket az egyszerűség kedvéért sokaságoknak hívjuk. Ha mást nem mondunk, egy sokaságok közötti leképezést simának (végtelen sokszor differenciálhatónak) tételezünk fel. Tetszőleges M sokaság esetén használjuk az alábbi jelöléseket:

(1) $C^\infty(M)$ az M-en értelmezett valós értékű sima függvények gyűrűje;

(2) $\tau\colon TM \to M$ az M sokaság érintőnyalábja;

(3) $\mathring{T}M \subset TM$ a nemnulla érintővektorok részsokasága, $\mathring{\tau} := \tau \upharpoonright \mathring{T}M$ a hasított érintőnyaláb;

(4) $\tau_{TM}\colon TTM \to TM$ a TM érintősokaság érintőnyalábja.

További jelölések:

(5) Fl^X az X vektormező folyama, Fl^X_t $(t \in \mathbb{R})$ a t-edik stádiuma.

(6) $\Gamma(\pi)$ vagy $\Gamma(B)$ jelöli egy $\pi\colon E \to M$ vektornyaláb (sima) szeléseinek $C^\infty(M)$-modulusát. Speciálisan $\mathfrak{X}(M) := \Gamma(TM)$ az M sokaság vektormezőinek $C^\infty(M)$-modulusa.

(7) $\varphi_*\colon TM \to TN$ egy sokaságok közötti $\varphi\colon M \to N$ leképezés deriváltja .

Tetszőleges $X \in \mathfrak{X}(M)$ vektormezőnek értelmezzük a *vertikális lift*jét, a

$$(t, v) \in \mathbb{R} \times TM \longmapsto v + tX(\tau(v)) \in TM.$$

folyam sebesség-vektormezőjeként. Ez egy vektormező a TM érintősokaságon, amelyet X^{v}-vel jelölünk. Szintén alkalmazzuk X *teljes lift*jét, a

$$(t, v) \in \widetilde{\mathcal{D}}_X \longmapsto (\mathrm{Fl}_t^X)_*(v) \in TM$$

folyam sebesség-vektormezőjét. Itt $\mathrm{Fl}^X\colon \mathcal{D}_X \subset \mathbb{R} \times M \to M$ az X vektormező folyama és

$$\widetilde{\mathcal{D}}_X := \{(t, v) \in \mathbb{R} \times TM \mid (t, \tau(v)) \in \mathcal{D}_X\}.$$

A teljes lift szintén egy vektormező az érintősokaságon, jelölése X^{c}.

A TM érintősokaság

$$(t, v) \in \mathbb{R} \times TM \longmapsto e^t v \in TM$$

pozitív nyújtásai által generált vektormezőt *Liouville-vektormező*nek hívjuk és C-vel jelöljük.

A TTM érintőnyaláb *vertikális résznyaláb*ja $VTM := \ker(\tau_*)$, ennek $\mathring{T}M$-re való természetes leszűkítését $V\mathring{T}M$-mel jelöljük.

A $\mathring{\tau}^*\tau$ pull-back nyalábot *Finsler-nyaláb*nak nevezzük. Ennek alapsokasága $\mathring{T}M$, teljes sokasága

$$\mathring{T}M \times_M TM := \{(u, v) \in \mathring{T}M \times TM \mid \mathring{\tau}(u) = \tau(v)\},$$

fibruma egy $u \in \mathring{T}M$ pont fölött az $\{u\} \times T_{\mathring{\tau}(u)}M$ vektortér. Az M sokaság egy X vektormezőjéből természetes módon származtatható a Finsler-nyaláb

$$\widehat{X}\colon \mathring{T}M \to \mathring{T}M \times_M TM, \quad \widehat{X}(u) := (u, X(\mathring{\tau}(u)))$$

szelése. Ezt az X vektormező *bázikus lift*jének nevezzük. Hasonló módon az M sokaság minden tenzora felemelhető a Finsler-nyalábra.

A Finsler-nyaláb és $V\mathring{T}M$ nyaláb szeléseinek modulusai természetes módon izomorfak. Ezt az $\mathbf{i}$-vel jelölt izomorfizmust egyértelműen meghatározza az $\mathbf{i}\widehat{X} = X^{\mathsf{v}}$ összefüggés. Hallgatólagosan minden ilyen típusú modulus-homomorfizmust nyalábleképezésnek tekintünk, ha szükséges. Ily módon $\mathbf{i}$ felfogható egy fibrumonként lineáris leképezésnek is $\mathring{T}M \times_M TM$-ből $V\mathring{T}M$-be.

A Finsler-nyaláb *kanonikus szelése*

$$\widetilde{\delta}\colon \mathring{T}M \to \mathring{T}M \times_M TM, \qquad \widetilde{\delta}(u) := (u, u),$$

erre teljesül, hogy $\mathbf{i}\widetilde{\delta} = C$.

A Finsler-nyaláb ellátható egy természetes derivációval, amely lényegében a fibrumokon adott természetes deriválásból származik. A precíz megadásához a Finsler-nyaláb minden $\widetilde{X}$ szeléséhez tekintjük ennek $\underline{X}\colon \mathring{T}M \to TM$ fő részét, amelyet az $\widetilde{X}(u) = (u, \underline{X}(u))$ előírás ad meg. Ezután definiálunk egy

$$\nabla^{\mathsf{v}}\colon \Gamma(\mathring{\tau}^*\tau) \times \Gamma(\mathring{\tau}^*\tau) \to \Gamma(\mathring{\tau}^*\tau)$$
$$\nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y}(u) := (u, (\underline{Y} \restriction \mathring{T}_pM)'(u)(\underline{X}(u)))$$

leképezést. Ez a klasszikus tenzorderivációkhoz hasonló módon kiterjeszthető a Finsler-tenzorok nyalábjára, ha megállapodunk abban, hogy tetszőleges $f \in C^\infty(\mathring{T}M)$ függvény esetén $\nabla^{\mathsf{v}}_{\widetilde{X}} f := \mathbf{i}\widetilde{X}f$. Ezt a derivációt *kanonikus vertikális deriváltnak* nevezzük. A lokális számolásokban ∇^{v} egyszerűen 'y^i-szerinti deriválásként' jelentkezik. Például: adva egy $f \in C^\infty(\mathring{T}M)$ függvényt,

$$\nabla^{\mathsf{v}} f \underset{(\mathcal{U})}{=} \frac{\partial f}{\partial y^i}\widehat{du^i}, \quad \nabla^{\mathsf{v}}\nabla^{\mathsf{v}} f \underset{(\mathcal{U})}{=} \frac{\partial^2 f}{\partial y^i \partial y^j}\widehat{du^i} \otimes \widehat{du^j}, \quad \text{stb.} \tag{5.1}$$

Ehresmann-konnexiók és sprayk

Legyen M egy sokaság. Egy

$$\mathcal{H}\colon \mathring{T}M \times_M TM \to T\mathring{T}M, \quad (u, v) \mapsto \mathcal{H}(u, v),$$

leképezést $\mathring{T}M$-en adott *Ehresmann-konnexiónak* nevezünk, ha

(E1) $\mathcal{H}(u, v) \in T_uTM$ minden $(u, v) \in \mathring{T}M \times_M TM$ esetén;

(E2) a $v \mapsto \mathcal{H}(u, v)$ leképezés $\mathbb{R}$-lineáris minden $u \in \mathring{T}M$ esetén;

(E3) $\tau_*\mathcal{H}(u, v) = v$ minden $(u, v) \in \mathring{T}M \times_M TM$ esetén.

A $H\mathring{T}M := \operatorname{Im}\mathcal{H}$ képhalmaz a $T\mathring{T}M$ egy $n = \dim M$-rangú résznyalábja, melyre

$$T\mathring{T}M = V\mathring{T}M \oplus H\mathring{T}M$$

teljesül. Ezt a $\mathcal{H}$-hoz tartozó *horizontális résznyalábnak* nevezzük.

Értelmezzük a $\mathbf{h}\colon T\mathring{T}M \to H\mathring{T}M$ és $\mathbf{v}\colon T\mathring{T}M \to V\mathring{T}M$ projekciókat oly módon, hogy $\operatorname{Ker}\mathbf{v} = H\mathring{T}M$, $\operatorname{Ker}\mathbf{h} = V\mathring{T}M$. Szintén definiáljuk a $\mathcal{V} := \mathbf{i}^{-1} \circ \mathbf{v}$ *vertikális leképezést*, amely az $\mathbf{i}$ egy balinverze. Egy $X \in \mathfrak{X}(M)$ vektormező, *horizontális liftje* az $X^{\mathsf{h}} := \mathcal{H}\widehat{X} \in \mathfrak{X}(\mathring{T}M)$ vektormező. Az Ehresmann-konnexió *pozitív-homogén*, ha a horizontális liftek 1^+-homogének, azaz $[C, X^{\mathsf{h}}] = 0$ minden $X \in \mathfrak{X}(M)$ esetén.

Azt mondjuk, hogy $S \in \mathfrak{X}(\mathring{T}M)$ *sprayje* M-nek, ha

(S1) $\tau_* \circ S = \mathrm{id}_{\mathring{T}M}$;

(S2) $[C, S] = S$, azaz S 2^+-homogén.

Ekkor az (M, S) párt *spray-sokaság*nak nevezzük. Ha $\mathcal{H}$ pozitív-homogén Ehresmann-konnexió, akkor $S := \mathcal{H}\widetilde{\delta}$ spray. Megfordítva, ha S egy spray, akkor létezik olyan $\mathcal{H}$ Ehresmann-konnexió, hogy

$$\mathcal{H}(\widehat{X}) = X^{\mathsf{h}} = \frac{1}{2}(X^{\mathsf{c}} + [X^{\mathsf{v}}, S])$$

minden $X \in \mathfrak{X}(M)$ esetén. Ezt az Ehresmann-konnexiót a spray-sokaság *Berwald-konnexió*jának nevezzük. Ekkor $\mathcal{H}\widetilde{\delta} = S$ teljesül. A részleteket illetően ld. [45, 7.3.3].

Egy Ehresmann-konnexió *torziójának* nevezzük azt az $(1,2)$-típusú $\mathbf{T}$ Finsler-tenzormezőt, amelyre

$$\mathbf{i}\,\mathbf{T}(\widehat{X}, \widehat{Y}) = [X^{\mathsf{h}}, Y^{\mathsf{v}}] - [Y^{\mathsf{h}}, X^{\mathsf{v}}] - [X, Y]^{\mathsf{v}}; \qquad X, Y \mathfrak{X}(M)$$

teljesül. Könnyen látható, hogy egy spray-sokaság Berwald-konnexiójának a torziója eltűnik. Megmutatható, hogy minden torziómentes Ehresmann-konnexió egy spraynek a Berwald-konnexiója.

Egy (M, S) spray-sokaság *Jacobi-endomorfizmus*a a

$$\mathbf{K}(\widetilde{X}) := \mathcal{V}[S, \mathcal{H}\widetilde{X}], \quad \widetilde{X} \in \Gamma(\mathring{\tau}^*\tau)$$

előírással definiált $(1,1)$-típusú Finsler-tenzormező. Ha $n = \dim M \geq 2$, akkor a

$$K := \frac{1}{n-1}\,\mathrm{tr}\,\mathbf{K} \tag{5.2}$$

függvény a spray-sokaság *görbületi függvény*e.

Legyen $\mathcal{H}$ egy $\mathring{T}M$-en adott pozitív-homogén Ehresmann-konnexió, $I \subset \mathbb{R}$ egy nyílt intervallum, és $\gamma\colon I \to M$ egy sima görbe. Egy γ-menti $X\colon I \to \mathring{T}M$ vektormező *párhuzamos* $\mathcal{H}$ *szerint* (röviden $\mathcal{H}$*-párhuzamos* vagy *párhuzamos*), ha

$$\dot{X}(t) = \mathcal{H}(X(t), \dot{\gamma}(t)) \qquad (t \in I).$$

Minden $t_0 \in I$ és $v \in \mathring{T}_{\gamma(t_0)}M$ esetén létezik egy és csak egy X párhuzamos vektormező γ mentén úgy, hogy $X(t_0) = v$. Ekkor $X(t)$ a v *párhuzamos eltolt*ja γ mentén $\mathring{T}_{\gamma(t)}M$-be. Így értelmezhetünk egy

$$P(\gamma)^t_{t_0}\colon \mathring{T}_{\gamma(t_0)}M \to \mathring{T}_{\gamma(t)}M$$

leképezést, amely minden $v \in \mathring{T}_{\gamma(t_0)}M$ vektort a párhuzamos eltoltjába visz át. Ekkor $P(\gamma)^t_{t_0}$ sima és 1^+-homogén minden $t \in I$ esetén [45, 7.6]. A $P(\gamma)^t_{t_0}$ leképezés természetes módon kiterjeszthető az egész $T_{\gamma(t_0)}M$ érintőtérre, ha a zérus vektormezőt párhuzamosnak tekintjük. Így $P(\gamma)^t_{t_0}(0) = 0$ adódik.

Egy $\mathcal{H}$ Ehresmann-konnexió birtokában bevezetünk egy

$$\nabla^{\mathsf{h}}\colon \Gamma(\mathring{\tau}^*\tau) \times \Gamma(\mathring{\tau}^*\tau) \to \Gamma(\mathring{\tau}^*\tau)$$

operátort, a $\nabla^{\mathsf{h}}_{\widetilde{X}}\widetilde{Y} := \mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}]$ előírással. Bázikus liftekre ekkor azt kapjuk, hogy

$$\nabla^{\mathsf{h}}_{\widehat{X}}\widehat{Y} = \mathbf{i}^{-1}[X^{\mathsf{h}}, Y^{\mathsf{v}}]. \tag{5.3}$$

A ∇^{h} és ∇^{v} operátorokból előállítható egy $\nabla\colon \mathfrak{X}(\mathring{T}M) \times \Gamma(\mathring{\tau}^*\tau) \to \Gamma(\mathring{\tau}^*\tau)$ kovariáns deriválás a Finsler-nyalábon. Ezt jellemzik a

$$\nabla_{\mathcal{H}\widetilde{X}}\widetilde{Y} = \nabla^{\mathsf{h}}_{\widetilde{X}}\widetilde{Y}, \qquad \nabla_{\mathbf{i}\widetilde{X}}\widetilde{Y} = \nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y}$$

összefüggések. A ∇ kovariáns deriválást a $\mathcal{H}$ által indukált *Berwald-deriválás*-nak nevezzük. Az Ehresmann-konnexió *Berwald-tenzor*át a

$$\mathbf{B}(\widetilde{X}, \widetilde{Y})\widetilde{Z} := \nabla^{\mathsf{v}}_{\widetilde{X}}\nabla^{\mathsf{h}}_{\widetilde{Y}}\widetilde{Z} - \nabla^{\mathsf{h}}_{\widetilde{Y}}\nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Z} - \nabla_{[\mathbf{i}\widetilde{X}, \mathcal{H}\widetilde{Y}]}\widetilde{Z}.$$

képlet definiálja. A bázikus lifteken ez az $\mathbf{i}\,\mathbf{B}(\widehat{X}, \widehat{Y})\widehat{Z} = [[X^{\mathsf{v}}, Y^{\mathsf{h}}], Z^{\mathsf{v}}]$ alakra redukálódik [45, 7.13].

Legyen (M, S) egy spray-sokaság. Ha $\sigma\colon I \to TM$ S-nek integrálgörbéje, akkor az (S1) feltétel alapján a $\gamma := \tau \circ \sigma$ görbe sebességvektormezője éppen σ. Az ilyen γ görbéket nevezzük S *geodetikus*ainak. Egy $\gamma\colon I \to M$ görbe pontosan akkor geodetikusa S-nek, ha $S \circ \dot{\gamma} = \ddot{\gamma}$. A konstans görbéket szintén geodetikusoknak tekintjük.

Lineáris konnexiók és affin sprayk

Egy $\mathring{T}M$-en adott Ehresmann-konnexiót *lineáris*nak mondjuk, ha az alábbi ekvivalens feltételek valamelyike (és ezért mindegyike) teljesül.

(L1) $\mathcal{H}$ pozitív-homogén és létezik sima kiterjesztése TM-re.

(L2) $\mathcal{H}$ Christoffel-szimbólumai lineáris függvények a fibrumokon.

(L3) A $\mathcal{H}$ által indukált Berwald-deriválás Christoffel-szimbólumai konstans függvények a fibrumokon.

(L4) $\mathcal{H}$ Berwald-tenzora eltűnik.

(L5) Bármely $X, Y \in \mathfrak{X}(M)$ esetén $[X^{\mathsf{h}}, Y^{\mathsf{v}}] \in \mathfrak{X}(\mathring{T}M)$ megkapható vertikális liftként.

(L6) Egyértelműen létezik olyan D kovariáns deriválás M-en, amelyre

$$(D_X Y)^{\mathsf{v}} = [X^{\mathsf{h}}, Y^{\mathsf{v}}] \overset{(5.3)}{=} \mathbf{i}\nabla_{X^{\mathsf{h}}}\widehat{Y}; \qquad X, Y \in \mathfrak{X}(M).$$

(L7) A $\mathcal{H}$-párhuzamos eltolások lineárisak.

Mivel az Ehresmann-konnexiókat csak $\mathring{T}M$-en értelmeztük, a fenti feltételekben több objektumot is ki kell terjeszteni a zérus vektorokra, ez azonban mindig természetes módon lehetséges.

A továbbiakban lineáris Ehresmann-konnexió helyett egyszerűen *lineáris konnexió*ról szólunk.

Egy S sprayt *affin*nak hívunk ha az alábbi ekvivalens feltételek valamelyike teljesül.

(A1) S-nek van sima kiterjesztése TM-re.

(A2) S együtthatói kvadratikus formák a fibrumokon.

(A3) Az S-ből származó Berwald-konnexió lineáris.

(A3) alapján az affin sprayk a Berwald-konnexiójuk segítségével is jellemezhetőek, az (L1)–(L7) feltételek bármelyikével.

Finsler-sokaságok

Legyen M sokaság. Egy $F\colon TM \to \mathbb{R}$ függvényt *Finsler-függvény*nek nevezzük (M fölött) ha:

(F1) F folytonos TM-en, és sima $\mathring{T}M$-en,

(F2) F 1^+-homogén,

(F3) $F(v) > 0$ ha $v \neq 0$,

(F4) a $g := \frac{1}{2}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}F^2$ *fundamentális tenzor* (fibrumonként) pozitív definit.

Ekkor az (M, F) párt *Finsler-sokaság*nak hívjuk. $E := \frac{1}{2}F^2$ az F-hez csatolt *energia-függvény*. Azt mondjuk, hogy egy Finsler-sokaság *Riemann-sokaságra redukálódik*, ha energia-függvénye minden érintőtéren kvadratikus forma.

Egy $\gamma\colon I \to M$ sima görbe a Finsler-sokaság *geodetikus*a, ha teljesíti a

$$\left(\frac{\partial E}{\partial y^i} \circ \dot{\gamma}\right)' - \frac{\partial E}{\partial x^i} \circ \dot{\gamma} = 0, \qquad i \in \{1, \dots, n\} \tag{5.4}$$

Euler–Lagrange egyenletet minden, a γ képét metsző térkép esetén. Az (M, F) geodetikusai szintén a geodetikusai egy egyértelműen meghatározott spraynek, ezt nevezzük az Finsler-sokaság *kanonikus spray*jének. S-sel jelölve a kanonikus sprayt, (5.4) az

$$SX^{\mathsf{v}}E - X^{\mathsf{c}}E = 0, \qquad X \in \mathfrak{X}(M) \tag{5.5}$$

alakba írható. Az Euler–Lagrange egyenletnek ezt a formáját M. Crampin találta [11, 348. oldal].

A kanonikus spray Berwald-konnexióját a Finsler-sokaság *kanonikus konnexiójá*nak nevezzük. Egy $\mathring{T}M$-en adott $\mathcal{H}$ Ehresmann-konnexió pontosan akkor a kanonikus konnexiója (M, F)-nek, ha

(CC1) $\mathcal{H}$ pozitív-homogén,

(CC2) $\mathcal{H}$ torziója eltűnik,

(CC3) $H\mathring{T}M := \operatorname{Im}(\mathcal{H}) \subset \ker dF$.

(részletekért lásd [45, Theorem 9.3.5]). A (CC3) feltétel garantálja, hogy F konstans a párhuzamos vektormezők mentén, és így a $\mathcal{H}$-párhuzamos eltolások megőrzik a Finsler-függvényt (lásd Lemma 2.3.1). Tehát a kanonikus konnexió analóg a Riemann-sokaságok Levi-Civita konnexiójával.

Az exponenciális leképezés

Legyen (M, S) spray-sokaság. Tetszőleges $v \in TM$ esetén jelölje γ_v azt a maximális geodetikust, amely kezdősebessége v. Legyen $\widetilde{TM}$ azon $v \in TM$ érintővektorok halmaza, amelyekre γ_v értelmezve van az 1-ben. Ekkor

$$\exp\colon \widetilde{TM} \to M, \quad v \mapsto \exp(v) := \gamma_v(1)$$

az S által generált *exponenciális leképezés*.

Projektíven affin sprayk

Legyenek S és $\bar{S}$ sprayk egy M sokaságon. S és $\bar{S}$ *projektíven csatolt*, ha $\bar{S} = S - 2PC$, ahol P egy függvény $\mathring{T}M$-en. Ekkor azt is mondjuk, hogy $\bar{S}$ az S *spray egy projektív változtatása, P faktorral.*

Egyszerű számolással igazolható, hogy ha egy spray projektíven csatolt egy affin sprayhez, akkor a

$$\mathbf{D} := \mathbf{B} - \frac{1}{n+1}\Big((\nabla^{\mathsf{v}} \operatorname{tr} \mathbf{B}) \otimes \widetilde{\delta} + (\operatorname{tr} \mathbf{B}) \odot \mathrm{id}_{\Gamma(\mathring{\tau}^* \tau)}\Big)$$

képlettel definiált *Douglas-tenzor* eltűnik. *Megmutattuk, hogy a megfordítás is igaz.* Ehhez szükségünk volt a *vertikálisan invariáns* térfogati formák bevezetésére. Ezek olyan térfogati formák az M érintősokaságán, amelyeknek a vertikális liftek szerinti Lie-deriváltjai eltűnnek.

2.1.8. Tétel ([45]). *Ha (M, S) eltűnő Douglas-tenzorú spray-sokaság, és ω egy vertikálisan invariáns térfogati forma TM-en, akkor*

$$\bar{S} := S - 2PC \quad \textit{ahol} \quad P := \frac{1}{2(n+1)} \operatorname{div}_\omega S$$

affin spray.

2.1.9. Állítás ([45]). *Bármely sokaság érintősokaságán létezik vertikálisan invariáns térfogati forma.*

Ezekből adódik:

2.1.10. Következmény ([45]). *Ha egy spray-sokaság Douglas-tenzora eltűnik, akkor a spray projektíven csatolt egy affin sprayhez.*

Ezt az állítást lokálisan J. Douglas igazolta [15]. Globálisan Z. Shen [40, 5.2] továbbá Szilasi József és Vattamány Szabolcs adtak bizonyítást [46], de feltételezték a sokaság irányíthatóságát. A mi változatunk erősebb annyiban, hogy ez a feltétel nem szerepel.

Nem léteznek valódi Einstein–Berwald-sokaságok

Egy *Berwald-sokaság* olyan Finsler-sokaság, amelynek a kanonikus sprayje affin, vagy – ekvivalens módon – a kanonikus konnexiója lineáris.

Egy (M, F) Finsler-sokaság *Einstein–Finsler-sokaság*, ha az (5.2) görbületi függvény és a Finsler-függvény között fennáll a $K = (\lambda \circ \tau)F^2$ összefüggés, ahol λ sima függvény M-en. Megmutattuk, Berwald-sokaságok között csak nagyon speciális Einstein–Finsler sokaságok léteznek.

2.2.1. Lemma ([14]). *Egy Berwald-sokaság görbületi függvénye kvadratikus forma minden érintőtéren.*

2.2.2. Tétel ([14]). *Ha egy összefüggő Einstein–Finsler-sokaság egyben Berwald-sokaság is, akkor vagy Riemann-sokaságra redukálódik, vagy pedig a görbületi függvénye eltűnik.*

Monokromatikus Finsler-sokaságok

Legyen M sokaság, $\mathcal{U}$ pedig az M-nek nyílt részsokasága. Tekintsünk egy $(X_i)_{i=1}^n$ n-élmezőt $\mathcal{U}$-n. Ekkor az $(X_i)_{i=1}^n$ $\mathbb{R}$-lineáris burkát *M $\mathcal{U}$ fölötti lokális párhuzamosításának* nevezzük. Egy $\mathcal{U}$ fölötti P lokális párhuzamosítás elemei a *P-szerinti párhuzamos vektormezők*. Minden $v \in \tau^{-1}(\mathcal{U})$ esetén egyértelműen létezik olyan $v_P \in P$ vektormező, melyre $v_P(\tau(v)) = v$. Ha p és q $\mathcal{U}$-nak pontjai, akkor a $P_{qp}(v) := v_P(q)$ előírással adott $P_{qp}\colon T_pM \to T_qM$ leképezés lineáris. Az így definiált párhuzamosítás-fogalom ekvivalens a [18] könyvben találhatóval. Egy nem feltétlenül sima $f\colon \mathring{T}M \to \mathbb{R}$ függvény *kompatibilis P-vel*, ha $f \circ P_{qp} = f \restriction \mathring{T}_pM$ minden $p, q \in \mathcal{U}$ esetén. Ezzel ekvivalens, hogy $f \circ X$ konstans, ha $X \in P$.

Egy nem feltétlenül sima $f\colon \mathring{T}M \to \mathbb{R}$ függvény *monokromatikus*, ha minden $p, q \in M$ esetén létezik egy $L_{qp}\colon T_pM \to T_qM$ lineáris izomorfizmus úgy, hogy $f \circ L_{qp} = f \restriction \mathring{T}_pM$. Az f függvény *simán monokromatikus*, ha minden $p \in M$ pontnak van olyan környezte, amely fölött létezik f-fel kompatibilis lokális párhuzamosítás. Könnyen látható, hogy egy simán monokromatikus függvény monokromatikus is, ha M is összefüggő.

Hasonlóan, egy nem feltétlenül sima $f\colon \mathring{T}M \to \mathbb{R}$ függvény is *kompatibilis egy $\mathcal{H}$ homogén Ehresmann-konnexióval*, ha a $\mathcal{H}$ szerinti párhuzamos eltolások megőrzik f-et. Ezzel ekvivalens, hogy f konstans a $\mathcal{H}$-párhuzamos vektormezők mentén.

2.3.1. Lemma. *Egy C^1-osztályú $f\colon \mathring{T}M \to \mathbb{R}$ függvény pontosan akkor kompatibilis egy homogén Ehresmann-konnexióval, ha $\ker df$ tartalmazza a konnexióhoz tartozó horizontális résznyalábbot.*

A fenti lemma szerint minden Finsler-függvény kompatibilis a kanonikus konnexiójával.

2.3.3. Tétel. *Ha egy C^1-osztályú $f\colon \mathring{T}M \to \mathbb{R}$ függvény simán monokromatikus, akkor kompatibilis egy lineáris konnexióval.*

A megfordítás teljesül anélkül, hogy bármit feltennénk a függvényről.

2.3.6. Állítás. *Ha egy nem feltétlenül sima $f\colon \mathring{T}M \to \mathbb{R}$ függvény kompatibilis egy lineáris konnexióval, akkor f simán monokromatikus.*

Egy (M, F) Finsler-sokaság *általánosított Berwald-sokaság*, ha F kompatibilis egy $\mathring{T}M$-en adott lineáris konnexióval. A fenti két eredményből azonnal adódik az általánosított Berwald-sokaságok alábbi jellemzése.

2.3.7. Tétel (Y. Ichijyō [21])**.** *Egy Finsler-sokaság pontosan akkor általánosított Berwald-sokaság, ha simán monokromatikus.*

Y. Ichijyō bizonyítása a G-struktúrákkal kompatibilis konnexiókra támaszkodott. A mi bizonyításunkban csak az Ehresmann-konnexiók egyszerű tulajdonságaira, és a Finsler-függvény differenciálhatóságára van szükség.

E szakasz záró eredménye H. Weyl egy fontos, de kissé elfeledett tételének modern átfogalmazása.

2.3.9. Tétel (Weyl)**.** *Legyen M sokaság, f pedig egy Finsler-norma egy T_pM érintőtéren (azaz f folytonos, 1^+-homogén, a $T_pM \setminus \{0\}$ halmazon sima és pozitív, továbbá $f''(v)$ pozitív definit bármely $v \in T_pM \setminus \{0\}$ esetén). Minden $\mathcal{U} \subset M$ feletti P párhuzamosítás esetén (ahol $p \in \mathcal{U}$), definiáljunk egy $F_P\colon T\mathcal{U} \to \mathbb{R}$ Finsler-függvényt az*

$$F_P(v) := f(v_P(p))$$

előírással. Ha minden ilyen P párhuzamosítás esetén $(\mathcal{U}, F_P)$ Berwald-sokaság, akkor f euklideszi norma.

3. fejezet: Spray- és Finsler-sokaságok affinitásai és izometriái

Legyen adva egy (M, F) Finsler-sokaság vagy egy (M, S) spray sokaság. Egy (nem feltétlenül sima) bijektív $\varphi\colon M \to M$ leképezést *affinitásnak* nevezünk, ha megőrzi a geodetikusokat. Pontosabban: φ affinitás, ha minden γ geodetikus esetén $\varphi \circ \gamma$ is geodetikus. Hasonlóan, ha (M, F) Finsler-sokaság, egy (nem feltétlenül sima) $\varphi\colon M \to M$ leképezés *izometria*, ha $\varrho(\varphi(p), \varphi(q)) = \varrho(p, q)$ minden $p, q \in M$ esetén. Itt ϱ az (általában nem szimmetrikus) Finsler-távolság M-en, amelyet a p-ből q-ba futó szakaszonként sima görbék ívhosszai halmazának infimumaként definiálunk.

Az affinitások és az izometriák simák

F. Brickell megmutatta [8], hogy egy spray-sokaság affin homeomorfizmusa szükségképpen sima. Megmutattuk, hogy bizonyítása egyszerűsíthető az alábbi (önmagában is érdekes) lemma segítségével.

3.1.1. Lemma. *Legyen (M, S) spray-sokaság, $p \in M$. Legyenek (a_n) és (b_n) a p ponthoz konvergáló sorozatok, és definiáljuk a*

$$L_n := \exp_{a_n}^{-1}(b_n), \quad A_n := \exp_p^{-1}(a_n), \quad B_n := \exp_p^{-1}(b_n), \qquad n \in \mathbb{N}^*$$

sorozatokat. Ekkor

$$\lim_{n\to\infty} \frac{B_n - A_n - L_n}{\|L_n\|} = 0,$$

és így $\frac{\|B_n - A_n\|}{\|L_n\|} \to 1$*. Továbbá, ha a*

$$\left(\frac{B_n - A_n}{\|B_n - A_n\|}\right), \qquad \left(\frac{L_n}{\|L_n\|}\right)$$

sorozatok egyike konvergens, akkor a másik is az, és a határértékük megegyezik.

Itt $\|\cdot\|$ a norma függvény egy M-en adott tetszőleges Riemann-metrikára nézve.

3.1.3. Tétel (F. Brickell [8]). *Egy spray-sokaság affin homeomorfizmusa sima diffeomorfizmus.*

A 'rövid geodetikusok' minimalizáló tulajdonsága segítségével [4, Theorem 6.3.1] könnyen látható, hogy egy Finsler-sokaság izometriái affinitások. Az izometriák folytonosak is, mivel a 'forward' gömbök generálják M topológiáját. Tehát minden izometria homeomorfizmus. Emiatt a 3.1.3 Tételből azonnal adódik:

3.2.2. Tétel (F. Brickell [8]). *A Finsler-sokaságok izometriái simák.*

Távolság-koordinátarendszerek

Legyen (M, F) Finsler-sokaság, és $(p_i)_{i=1}^n$ az M pontjainak egy olyan sorozata amelyre teljesül, hogy az $u_\varrho \colon q \mapsto (\varrho(p_1, q), \dots, \varrho(p_n, q))$ leképezés diffeomorfizmus M egy $\mathcal{U}$ nyílt részhalmazából $\mathbb{R}^n$ egy nyílt részhalmazába. Ekkor $(\mathcal{U}, u_\varrho)$-t *távolság-koordinátarendszer*nek nevezzük. Megmutattuk:

3.3.1. Lemma ([3]). *Egy Finsler-sokaság minden pontja körül létezik távolság-koordinátarendszer.*

Távolságkoordináták segítségével új, egyszerű bizonyítást adtunk az izometriák simaságára, nem használva fel az affinitások simaságát. A bizonyítás ötlete P. Petersen Riemann-esetre való bizonyításából származik [38, Ch. 5.10].

A távolság-koordinátarendszereknek egy további alkalmazásaként igazoltuk, hogy *reverzibilis Finsler-sokaságok közötti reguláris szubmetriák differenciálhatóak* (3.3.3 Tétel). Ez az állítás következik egy metrikus terekben ismert tételből is (lásd [30]), de az általunk talált bizonyítás közvetlenebb és differenciálgeometriai szemléletű. Először tisztáznunk kell egyes elnevezéseket.

Legyenek M_1 és M_2 metrikus terek. Mindkettőben $\mathcal{B}$-vel jelöljük a metrikus gömböket. Egy $\varphi \colon M_1 \to M_2$ leképezést *szubmetriá*nak hívunk, ha M_1 minden p pontja esetén létezik olyan δ pozitív szám, melyre $\varphi(\mathcal{B}_\varepsilon(p)) = \mathcal{B}_\varepsilon(\varphi(p))$, ha $\varepsilon \in\,]0, \delta[$. Az ilyen δ számok szupremumát jelölje δ_p (ez lehet végtelen is). A szubmetria *reguláris*, ha minden $\mathcal{K} \subset M_1$ kompakt halmaz esetén $\delta_\mathcal{K} := \inf_{p \in \mathcal{K}} \delta_p > 0$.

Egy Finsler-sokaság *reverzibilis*, ha $F(-v) = F(v)$ bármely $v \in TM$ vektor esetén. Ebben az esetben ϱ szimmetrikus, és így (M, ϱ) metrikus tér.

3.3.2. Lemma ([3]). *Legyen (M, F) reverzibilis Finsler-sokaság, p M-nek egy pontja, és $\mathcal{U}$ a p-nek egy normálkörnyezete. Ekkor a ϱ_p távolságfüggvény leszűkítése az $\mathcal{U} \setminus \{p\}$ halmazra reguláris szubmetria az $\mathbb{R}$ kanonikus távolságára nézve.*

3.3.3. Tétel (A. Lytchak [30],[3]). *Egy reverzibilis Finsler-sokaságok közötti reguláris szubmetria differenciálható.*

A bizonyítás fő lépéseiben a [6] dolgozatot követi, amelyben az analóg állítás a Riemann-esetben lett igazolva.

Kapcsolatok affinitások és izometriák között

A korábbiakban láttuk, hogy egy (M, S) spray-sokaság egy φ affin homeomorfizmusa sima. Könnyen adódik, hogy ekkor φ *automorfizmus*a S-nek, azaz diffeomorfizmus, amelyre $\varphi_{**} \circ S = S \circ \varphi_*$ teljesül. A megfordítás nyilvánvaló: egy spray automorfizmusai affin transzformációk. Hasonlóan, tudjuk, hogy egy (M, F) Finsler-sokaság φ izometriája sima. Ebből könnyen adódik, hogy φ *Finsler-izometria*, azaz φ diffeomorfizmus és $F \circ \varphi_* = F$. A Riemann-esetben ezt S.B. Myers és N. Steenrod igazolta [35], a Finsler-esetben pedig először F. Brickell [8]. S. Deng és Z. Hou négy évtizeddel később újra felfedezte ezt az eredményt [13]. Mindkét bizonyításban a kritikus lépés a φ simaságának igazolása. Ezt már tisztáztuk, így a Myers–Steenrod tétel Finsler-változata könnyen adódik:

3.4.1. Tétel (F. Brickell [8]). *Egy Finsler-sokaság izometriái Finsler-izometriák.*

A megfordítás nyilván igaz. Összefoglalva: *egy Finsler-sokaság Finsler-izometriái éppen az izometriák, és egy spray-sokaság automorfizmusai éppen az affin homeomorfizmusok.* Mostantól egyszerűen úgy hivatkozunk rájuk, mint „izometriák” és „affinitások”.

A továbbiakban Finsler-sokaságok izometriái és affinitásai közötti kapcsolattal foglalkozunk. Nyilvánvalónak tűnik, hogy az izometriák affinitások, mivel ha egy transzformáció megőrzi a Finsler-függvényt, akkor minden belőle származó objektumot is meg kell őriznie. Ez az érvelés azonban nem helyettesítheti a precíz bizonyítást.

3.4.2. Állítás ([3]). *Egy Finsler-sokaság minden izometriája affinitás.*

A megfordítás általában nyilván nem igaz. Létezek affinitások amelyek nem izometriák, például az euklideszi terekben. A Riemann-geometriában több nevezetes tétel ad elégséges feltételt arra, hogy az affinitások és az izometriák egybeessenek (lásd például [25, 242–244. oldal]). Két ilyen tétel Finsler-geometriai általánosítását adjuk meg az alábbiakban.

Az első tételünk infinitezimális affinitásokról és izometriákról szól. Ezek olyan vektormezők az alapsokaságon, melyek stádiumai a megfelelő típusú leképezések. Az infinitezimális affinitásokat *affin vektormező*knek, az infinitezimális izometriákat *Killing-vektormező*knek hívjuk. Mindkét tulajdonság kifejezhető a vektormező teljes liftjével: ha (M, F) egy Finsler-sokaság és S a kanonikus spray, akkor $X \in \mathfrak{X}(M)$ pontosan akkor affin, ha $\mathcal{L}_{X^c} S := [X^c, S] = 0$, és pontosan akkor Killing-vektormező, ha $\mathcal{L}_{X^c} F := X^c F = 0$.

Általánosítását adtuk J.-I. Hano tételének, amely szerint egy teljes Riemann-sokaság minden korlátos affin-vektormezője Killing-vektormező. Ennek olyan bizonyítása szerepel a [25] könyvben, amely a de Rham-felbontáson [25, Theorem 6.2], és az alábbi fontos tételen alapszik:

3.4.3. Tény (S. Kobayashi, [24])**.** *Egy teljes és irreducibilis Riemann-sokaság affinitásai és izometriák egybeesnek, kivéve az* 1*-dimenziós euklideszi teret.*

Hano bizonyítása [19] nem használja ez utóbbi tételt, de a bizonyításában szereplőkhöz hasonló elgondolásokat követ.

Hano tétele Finsler-sokaságokra is átvihető, s meglepő módon egészen elemi eszközökkel, a de Rham-felbontás és az irreducibilitásnak megfelelő fogalom bevezetése nélkül. Kulcsfontosságú a következő észrevétel:

3.4.5. Lemma ([23])**.** *Legyen* (M, F) *Finsler-sokaság,* X *egy affin vektormező és* γ *egy geodetikus. Ekkor*

$$X^v E(\dot{\gamma}(t)) = X^v E(\dot{\gamma}(t_0)) + (t - t_0) X^c E(\dot{\gamma}(t_0))$$

teljesül a γ *görbe értelmezési tartományának minden* t *és* t_0 *pontjára.*

Bár nem nyilvánvaló, ez a lemma valójában egy speciális esete az 5.4.3(c) feladatnak a [4] könyvből. Ez a következőt állítja: ha J Jacobi-mező egy γ geodetikus mentén, akkor

$$g_{\dot{\gamma}}(J, \dot{\gamma})(t) = g_{\dot{\gamma}}(J(t_0), \dot{\gamma}(t_0)) + (t - t_0) g_{\dot{\gamma}}(J'(t_0), \dot{\gamma}(t_0)),$$

ahol J' a J a γ-menti kovariáns deriváltja a Berwald- vagy a Chern-deriválásra nézve. Azonban a lemmánkat egyszerűbb közvetlenül igazolni, mint levezetni idézett feladatból.

3.4.6. Tétel ([23])**.** *Legyen* (M, F) *Finsler-sokaság,* X *egy affin vektormező, és tegyük fel, hogy az alábbi két feltétel valamelyike teljesül.*

(1) $F \circ X$ *korlátos és* (M, F) *teljes;*

(2) $F \circ X$ *és* $F \circ (-X)$ *korlátos, továbbá* (M, F) *'forward' teljes.*

Ekkor X Killing-vektormező.

Ebből következik:

3.4.7. Tétel (J.-I. Hano)**.** *Legyen (M, g) egy teljes Riemann-sokaság és X egy olyan affin vektormező M-en, hogy a $g(X, X)$ függvény korlátos. Ekkor X Killing-vektormező.*

Valóban, elegendő a 3.4.6 Tételt az $F(v) := \sqrt{g(v, v)}$, $v \in TM$ Finsler-függvényre alkalmazni. Mivel a kompakt Finsler-sokaságok teljesek, szintén azonnal adódik a következő:

3.4.8. Tétel ([23])**.** *Egy kompakt Finsler-sokaság affin vektormezői Killing-vektormezők.*

Az utóbbi tételt a Riemann-esetben először K. Yano igazolta [52].

A következőkben a 3.4.3 Tétel Finsler-általánosításait tárgyaljuk. Ennek az állításnak már a megfogalmazása is problémát okoz a Finsler-esetben, mert az irreducibilitásnak nincs egyértelmű megfelelője. Egy lehetséges út az irreducibilitási feltétel kicserélése az *affin merevség* feltételére. Ennek definíciója a következő. Egy M sokaság S sprayje *egyértelműen metrizálható*, ha S kanonikus sprayje egy F Finsler-függvénynek, és ha $\tilde{F}$ olyan további Finsler-függvény, amelynek szintén S a kanonikus sprayje, akkor M minden komponensén $\tilde{F}$ az F-nek skalárszorosa (más szóval, $d(F/\bar{F}) = 0$). Egy Finsler-sokaság *affin merev*, ha a kanonikus sprayje egyértelműen metrizálható. A következő eredmény a [25, p. 242, Lemma 1] közvetlen megfelelője.

3.4.9. Lemma. *Egy összefüggő affin merev Finsler-sokaság minden affinitása homotécia.*

Itt a homotécia olyan φ diffeomorfizmust jelent, melyre $F \circ \varphi_* = cF$ teljesül, ahol c konstans. Mivel a 'forward' teljes Finsler-sokaság között egyedül a Finsler-vektortereken léteznek olyan homotéciák amelyek nem izometriák [29], adódik a következő:

3.4.10. Következmény. *Egy összefüggő 'forward' teljes affin merev Finsler-sokaság affinitásai és izometriák egybeesnek.*

Sajnos a 3.4.3 Tételnek ez az általánosítása nem túl hasznos, mert kevés elégséges feltétel ismert a Finsler-sokaságok affin merevségére. A folytatásban ezzel a problémával foglalkozunk.

Affin merev Finsler-sokaságok

Ebben a szakaszban (M, F) végig egy összefüggő Finsler-sokaság.

A kanonikus konnexió párhuzamos eltolásait felhasználva egy Finsler-sokaságnak is értelmezhetjük a holonómia-csoportját, ugyanúgy, mint a Riemann-esetben. Rögzítve egy $p \in M$ pontot, Hol_p a $\mathring{T}_pM$ sima 1^+-homogén diffeomorfizmusainak egy részcsoportja.

3.5.1. Állítás. *Ha* Hol_p *tranzitíven hat az* $U(T_pM) := F^{-1}(\{1\}) \cap T_pM$ *egységgömbön, akkor* (M, F) *affin merev.*

Jól ismert, hogy az irreducibilis Riemann-sokaságok affin merevek. Berger holonómia-tétele [7, 36] szerint léteznek olyan irreducibilis Riemann-sokaságok, melyek holonómia-csoportja nem hat tranzitíven az egységgömbön. Ebből következik, hogy a 3.5.1 Állítás megfordítása nem igaz, tehát Hol_p tranzitivitása gyengíthető.

A további eredményeink ismertetéséhez fel kell idéznünk néhány szinguláris disztribúciókkal kapcsolatos fogalmat [32]. Rögzítve egy M sokaságot, rendeljünk hozzá minden $p \in M$ ponthoz egy $\mathcal{E}_p \subset T_pM$ alteret. Ekkor az $\mathcal{E} = \bigsqcup_{p\in M} \mathcal{E}_p$ diszjunkt uniót *szinguláris disztribúció*nak nevezzük M-en. Jelölje $\mathfrak{X}_{loc}(M)$ azon vektormezők halmazát, amelyek csak M egy nyílt részhalmazán vannak értelmezve, $\mathfrak{X}_{\mathcal{E}}$ pedig azon vektormezőket $\mathfrak{X}_{loc}(M)$-ben melyek csak $\mathcal{E}$-ben vesznek fel értékeket. A $\mathfrak{X}_{\mathcal{E}}$ egy $\mathcal{V}$ részhalmaza *kifeszíti* $\mathcal{E}$-t, ha minden $p \in M$ esetén $\mathcal{E}_p$ az $\{X(p) \in T_pM \mid X \in \mathcal{V}\}$ lineáris burka. (Megállapodunk abban, hogy az üres halmaz lineáris burka a vektortér zérus eleme.) Azt mondjuk hogy $\mathcal{E}$ *sima*, ha $\mathfrak{X}_{\mathcal{E}}$ kifeszíti.

Egy $\mathcal{E}$ sima szinguláris disztribúció *integrálsokaságá*a olyan $i\colon N \to M$ immergált részsokaság, amelyre $i_*(T_pN) = \mathcal{E}_{i(p)}$ minden $p \in N$ esetén. Valójában az ilyen részsokaságok szükségképpen gyengén beágyazottak, és így nem kell megadni az i immerziót.

Rögzítve $\mathfrak{X}_{loc}(M)$ egy $\mathcal{W}$ részhalmazát, jelölje $\mathcal{S}(\mathcal{W})$ a

$$(\mathrm{Fl}^{X_1}_{t_1} \circ \cdots \circ \mathrm{Fl}^{X_k}_{t_k})_\# Y$$

alakú lokális vektormezők halmazát, ahol $X_1, \dots, X_k, Y \in \mathcal{W}$, $k \in \mathbb{N}$.

Legyen $\mathcal{D}^h$ az a sima szinguláris disztribúció, amelyet $\mathcal{S}(\mathfrak{X}_{H\mathring{T}M})$ feszít fel, ahol $H\mathring{T}M$ az (M, F) kanonikus konnexiójához tartozó horizontális disztribúció.

3.5.2. Lemma. *A bevezetett jelölésekkel* $\mathcal{D}^h \subset \ker(dF)$ *teljesül.*

3.5.3. Következmény. *Az* F *Finsler-függvény konstans a* $\mathcal{D}^h$ *szinguláris disztribúció összefüggő integrálsokaságain.*

3.5.5. Állítás. *Ha* $\mathcal{D}^h$ *dimenziója* $2n - 1$ $\mathring{T}M$ *egy sűrű részhalmaza felett, akkor* (M, F) *affin merev.*

3.5.6. Állítás. *Ha $U(TM) := F^{-1}(\{1\})$ $\mathcal{D}^h$-nak megszámlálható részsokaságát tartalmazza, akkor (M, F) affin merev.*

A szakasz eredményeit, a korábbi 3.4.9 Lemmát és 3.4.10 Következményt összefoglalva kapjuk az alábbi tételt.

3.5.7. Tétel. *Legyen (M, F) összefüggő Finsler-sokaság amelyre a következők valamelyike teljesül.*

(1) Hol_p *tranzitíven hat a $U(T_pM) := F^{-1}(\{1\}) \cap T_pM$ egységgömbön;*

(2) $\mathcal{D}^h$ *dimenziója $2n-1$ $\mathring{T}M$ egy sűrű részhalmaza felett;*

(3) *$U(TM) := F^{-1}(\{1\})$ $\mathcal{D}^h$-nak megszámlálható részsokaságát tartalmazza.*

Ekkor (M, F) minden affinitása homotécia. Ha ráadásul (M, F) 'forward' teljes is, akkor minden affinitás izometria.

3.5.8. Megjegyzés. Egyes fenti eredmények speciális esetei már előfordultak az irodalomban. Szenthe János a [43] dolgozatban a $\mathbf{v}[\xi, \eta]$ vektormezők által felfeszített disztribúciót vizsgálta, ahol $\xi, \eta \in \mathfrak{X}_{H\mathring{T}M}$. Megmutatta, hogy ha ennek a disztribúciónak a rangja mindenütt $n-1$, akkor minden affin transzformáció homotécia. Ez következik a 3.5.7 Tétel (2) esetéből, mert $\mathcal{S}(\mathfrak{X}_{H\mathring{T}M})$ zárt a Lie-zárójel műveletére nézve (lásd [32, 3.27 Lemma]), és a $\mathcal{D}^h$ dimenziója pontosan akkor $2n-1$, ha $\mathbf{v}\mathcal{D}^h$ dimenziója $n-1$. A [17] dolgozatban a szerzők a $\mathfrak{X}_{H\mathring{T}M}$ által generált Lie-algebrát vizsgálták, pontosabban az általa kifeszített disztribúciót. Kapcsolatot találtak ennek a disztribúciónak a dimenziója és a kanonikus sprayt metrizáló funkcionálisan független Finsler-függvények maximális száma között. Speciálisan igazolták, hogy ha a disztribúció kodimenziója 1, akkor a kanonikus spray egyértelműen metrizálható. A 3.5.5 Állítás ennek közvetlen általánosítása, egyrészt mert bővebb disztribúciót tekintünk, másrészt mert nem tesszük fel, hogy a dimenzió mindenütt $2n-1$.

A 3.5.5 Állítás alábbi megfordítása igaznak tűnik:

> *Ha $\mathcal{D}^h$ dimenziója $\mathring{T}M$ egy nyílt részhalmazán kisebb mint $2n-1$, akkor (M, F) nem affin merev.*

Indoklás: ha $\mathcal{D}^h$ dimenziója a maximálisnál kisebb egy nyílt részhalmazon, akkor megszámlálhatónál több integrálsokasága létezhet, és ez kisebb merevséget kényszerít a kanonikus sprayt metrizáló a Finsler-függvényekre. Azonban nem ez a helyzet: egy sima szinguláris disztribúció, amelynek dimenziója maximálisnál kisebb egy nyílt halmazon, még egyértelműen meghatározhatja azokat a folytonos függvényeket, amelyek konstansok az integrálsokaságain. Például létezik olyan sima szinguláris disztribúció a kanonikus (x, y) koordináta-rendszerrel ellátott $\mathbb{R}^2$ sokaságon, melynek maximális integrálsokaságai az alábbiak:

(a) az $y < 0$ és $y > 1$ egyenlőtlenségekkel megadott félsíkok;

(b) az $\{(x_0, y) \in \mathbb{R}^2 \mid 0 < y < 1\}$ ‘vertikális’ szakaszok, ahol $x_0 \in \mathbb{R}$ tetszőleges;

(c) az $y = 0$ és $y = 1$ egyenletű egyenesek pontjai, mint nulla dimenziós sokaságok.

Könnyen látható, hogy a folytonos függvények között csak a mindenütt konstans függvények konstansok az összes integrálsokaságon. Az nyitott kérdés, hogy ilyen konfiguráció előfordulhat-e $\mathcal{D}^h$ esetében.

Finsler-sokaságok sok Killing-vektormezővel

Legyen (M, F) Finsler-sokaság. Egy M egy nyílt halmaza fölötti P párhuzamosítást *Killing-párhuzamosítás*nak nevezünk, ha Killing-vektormezők $\mathbb{R}$-lineáris burkaként áll elő.

Egy (M, F) Finsler-sokaság *Killing-tulajdonságú* (v.ö. [51]), ha M minden pontja körül van Killing-párhuzamosítható nyílt környezet. Továbbá egy (M, F) Finsler-sokaság *konstans Killing-tulajdonságú*, ha M minden pontja körül van olyan Killing-párhuzamosítható nyílt környezet, amely kompatibilis F-fel.

A fenti fogalmakat a [12] dolgozat motiválta, ahol a szerzők olyan Riemann-sokaságokat vizsgáltak, melyekben minden pont körül van Killing-vektormezőkből álló orthonormált n-élmező. A konstans Killing-tulajdonságú Finsler-sokaságok ezek közvetlen általánosításai.

3.6.1. Tétel. *Egy Killing-tulajdonságú Finsler-sokaság általánosított Berwald-sokaság.*

3.6.2. Tétel. *Egy konstans Killing-tulajdonságú Finsler-sokaság Berwald-sokaság.*

Megjegyezzük, hogy az utóbbi tétel [1, Theorem 3.21 és 3.22] egy közös következményének egy általánosítása.

List of symbols

P_{qp}, 26
X^{c}, complete lift of vector field X, 6
X^{v}, vertical lift of vector field X, 6
$\widehat{X}$, basic lift of vector field X on M, 8
φ_*, tangent map of φ, 5
$\varphi_\#$, push-forward w.r.t. φ, 6
f^{c}, complete lift of function f, 6
f^{v}, vertical lift of function f, 6
v_P, vector field in parallelization P generated by v, 26

$\mathcal{B}_r^+(a)$, forward metric ball, 37
$\mathbf{B}$, Berwald tensor, 12

$C^\infty(M)$, ring of smooth real-valued functions on M, 5

$\mathbf{D}$, Douglas tensor, 20
$\widetilde{\delta}$, canonical section, 8
$\mathcal{D}^h$, 50

$E := \frac{1}{2}F^2$, energy of a Finsler manifold (M, F), 16
exp, exponential map, 16

Fl^X, the maximal flow of a vector field X, 5

g, fundamental tensor of a Finsler manifold, 15
$\Gamma(\pi)$, sections of vector bundle π, 5

$\mathbf{h}$, horizontal projection, 10
$\mathbf{H}$, affine curvature, 12
$\mathcal{H}$, Ehresmann connection, 9, 69
$H\mathring{T}M$, horizontal subbundle, 10

$\mathbf{i}$, canonical injection, 8

j, canonical surjection, 8

K, Jacobi endomorphism, 11
$K := \frac{1}{n-1}\mathrm{tr}\mathbf{K}$, curvature function of a spray, 11

$\ell_F(\gamma)$, Finslerian length of a curve γ, 37

(M, F), Finsler manifold, 16

∇, Berwald derivative, 12
∇^{h}, horizontal part of Berwald derivative, 11
∇^{v}, canonical vertical derivative, 9

$\Omega(p, q)$, set of piecewise smooth curves from p to q, 37

$P(\gamma)_{t_0}^{t}$, parallel translation along γ from $T_{\gamma(t_0)}M$ to $T_{\gamma(t)}M$, 11

$\mathcal{R}$, curvature of an Ehresmann connection, 10
$\varrho(p, q)$, Finslerian distance from p to q, 37
$\varrho_p := q \mapsto \varrho(p, q)$ distance function at p, 38

$\mathcal{S}(\mathcal{W})$, 49

T, torsion of an Ehresmann connection, 10
$\tau\colon TM \to M$, tangent bundle of M, 5
$\mathring{\tau}\colon \mathring{T}M \to M$, the slit tangent bundle of M, 5
$\tau_{TM}\colon TTM \to TM$ the tangent bundle of TM, 5
TM, tangent manifold of M, 5
$\widetilde{TM}$, domain of exp, 16
$\mathring{T}M$, the slit tangent manifold, 5
$\mathring{T}M \times_M TM$, total manifold of a Finsler bundle, 8

v, vertical projection, 10
VTM, vertical subbundle, 7

$\mathfrak{X}(M)$, module of vector fields on M, 5
$\mathfrak{X}_{\mathcal{E}}$, vector fields taking value in singular distribution $\mathcal{E}$, 49

Irodalomjegyzék

[1] B. Aradi, *Parallelisms and Finsler structures*, PhD thesis, University of Debrecen, 2015.

[2] B. Aradi and D. Cs. Kertész, *A characterization of holonomy invariant functions on tangent bundles*, Balkan J. Geom. Appl., **19** (2014), 1–10.

[3] ——, *Isometries, submetries and distance coordinates on Finsler manifolds*, Acta Math. Hungar., **143** (2014), 337–350.

[4] D. Bao, S.-S. Chern, and Z. Shen, *An Introduction to Riemann–Finsler geometry*, Springer, 2000.

[5] D. Bao, C. Robles, and Z. Shen, *Zermelo navigation on Riemannian manifolds*, J. Differential Geom., **66** (2004), 377–435.

[6] V. Berestovskii and L. Guijarro, *A metric characterization of Riemannian submersions*, Annals of Global Analysis and Geometry, **18** (2000), 577–588.

[7] M. Berger, *Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes*, Bull. Soc. Math. France, **83** (1955), 279–330.

[8] F. Brickell, *On the differentiability of affine and projective transformations*, Proc. Amer. Math. Soc., **16** (1965), 567–574.

[9] X. Cheng, Z. Shen, and Y. Tian, *A class of Einstein (α, β)-metrics*, Israel J. Math., **192** (2012), 221–249.

[10] M. Crampin, *On the construction of Riemannian metrics for Berwald spaces by averaging*, Houston J. Math., **40** (2014), 737–750.

[11] M. Crampin and F. A. E. Pirani, *Applicable differential geometry*, vol. 59, Cambridge University Press, Cambridge, 1986.

[12] J. E. D'Atri and H. K. Nickerson, *The existence of special orthonormal frames*, J. Differential Geometry, **2** (1968), 393–409.

[13] S. Deng and Z. Hou, *The group of isometries of a Finsler space*, Pac. J. Math., **207** (2002), 149–155.

[14] S. Deng, D. Cs. Kertész, and Z. Yan, *There are no proper Berwald–Einstein manifolds*, Publ. Math. Debrecen, **86** (2015), 245–249.

[15] J. Douglas, *The general geometry of paths*, Ann. of Math., **29** (1927/28), 143–168.

[16] L. D. Drager, J. M. Lee, E. Park, and K. Richardson, *Smooth distributions are finitely generated*, Ann. Global Anal. Geom., **41** (2012), 357–369.

[17] S. G. Elgendi and Z. Muzsnay, *Freedom of h(2)-variationality and metrizability of sprays*, Diff. Geom. Appl., (2017).

[18] W. Greub, S. Halperin, and R. Vanstone, *Connections, Curvature and Cohomology*, vol. 1, Academic Press, New York and London, 1972.

[19] J.-i. Hano, *On affine transformations of a Riemannian manifold*, Nagoya Math. J., **9** (1955), 99–109.

[20] J. Hilgert and K.-H. Neeb, *Structure and Geometry of Lie Groups*, Springer, 2012.

[21] Y. Ichijyō, *Finsler manifolds modeled on Minkowski spaces*, J. Math. Kyoto Univ., **16** (1976), 639–652.

[22] M. Kell, *A note on non-negatively curved Berwald spaces*, arXiv:1502.03764, (2015).

[23] D. Cs. Kertész and R. L. Lovas, *A generalization and short proof of a theorem of Hano on affine vector fields*, submitted, (2016).

[24] S. Kobayashi, *A theorem on the affine transformation group of a Riemannian manifold*, Nagoya Math. J., **9** (1955), 39–41.

[25] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol. I*, John Wiley & Sons, Inc., New York, 1996.

[26] L. Kozma, *Holonomy structures in Finsler geometry*, in Handbook of Finsler geometry. Vol. 1, 2, Kluwer Acad. Publ., Dordrecht, 2003, 445–488.

[27] S. Lang, *Differential and Riemannian Manifolds*, vol. 160 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1995.

[28] D. Laugwitz, *Differential and Riemannian geometry*, Translated by Fritz Steinhardt, Academic Press, New York-London, 1965.

[29] R. L. Lovas and J. Szilasi, *Homotheties of Finsler manifolds*, SUT J. Math., **46** (2010), 23–34.

[30] A. Lytchak, *Differentiation in metric spaces*, St. Petersburg Math. J., **16** (2004), 1017–1041.

[31] V. S. Matveev, *Riemannian metrics having common geodesics with Berwald metrics*, Publ. Math. Debrecen, **74** (2009), 405–416.

[32] P. W. Michor, *Topics in Differential Geometry*, vol. 93 of Graduate Studies in Mathematics, AMS, 2008.

[33] Z. Muzsnay and P. T. Nagy, *Finsler manifolds with non-Riemannian holonomy*, Houston J. Math., **38** (2012), 77–92.

[34] ——, *Finsler 2-manifolds with maximal holonomy group of infinite dimension*, Differential Geom. Appl., **39** (2015), 1–9.

[35] S. B. Myers and N. Steenrod, *The group of isometries of a Riemannian manifold*, Ann. of Math., **40** (1939), 400–416.

[36] C. Olmos, *A geometric proof of the Berger holonomy theorem*, Ann. of Math. (2), **161** (2005), 579–588.

[37] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.

[38] P. Petersen, *Riemannian Geometry, 2nd ed.*, Springer, 2006.

[39] M. Rafie-Rad and B. Rezaei, *On Einstein Matsumoto metrics*, Nonlinear Anal. Real World Appl., **13** (2012), 882–886.

[40] Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, Dordrecht, 2001.

[41] ——, *On projectively related Einstein metrics in Riemann-Finsler geometry*, Math. Ann., **320** (2001), 625–647.

[42] Z. I. Szabó, *Positive definite Berwald spaces (Structure theorems on Berwald spaces)*, Tensor N.S., **35** (1981), 25–39.

[43] J. Szenthe, *Symmetries of Lagrangian fields are homotheties in the homogeneous case*, in Differential geometry and applications (Brno, 1995), Masaryk Univ., Brno, 1996, 321–328.

[44] J. Szilasi, R. L. Lovas, and D. Cs. Kertész, *Several ways to a Berwald manifold – and some steps beyond*, Extracta Mathematicae, **26** (2011), 89–130.

[45] ——, *Connections, Sprays and Finsler Structures*, World Scientific, 2014.

[46] J. Szilasi and S. Vattamány, *Erratum to: „On the projective geometry of sprays"[Differential Geom. Appl. 12 (2000) 185–206]*, Differential Geometry and its Applications, **13** (2000), 97–118.

[47] R. E. Traber, *A fundamental lemma on normal coordinates and its applications*, Quart. J. Math. Oxford, **8** (1937), 142–147.

[48] Cs. Vincze, *A new proof of Szabó's theorem on the Riemann-metrizability of Berwald manifolds*, Acta Math. Acad. Paedagog. Nyházi.(NS), **21** (2005), 199–204.

[49] J. H. C. Whitehead, *Convex regions in the geometry of paths*, Quart. J. Math. Oxford, **3** (1932), 33–42.

[50] ——, *Convex regions in the geometry of paths – addendum*, Quart. J. Math. Oxford, **4** (1933), 226–227.

[51] M. Xu and S. Deng, *Killing frames and S-curvature of homogeneous Finsler spaces*, Glasg. Math. J., **57** (2015), 457–464.

[52] K. Yano, *On harmonic and Killing vector fields*, Ann. of Math. (2), **55** (1952), 38–45.

[53] Y. Yu and Y. You, *On Einstein m-th root metrics*, Differential Geom. Appl., **28** (2010), 290–294.