

Short thesis for the degree
of Doctor of Philosophy (PhD)

Approximately Monotone, Hölder, Convex and Affine Functions

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Notation and Symbols

\mathbb{R}	the set of real numbers
\mathbb{R}_+	the set of nonnegative real numbers
I	a nonempty open interval
$\ell(I)$	the length of the interval I
$\mathcal{E}(I)$	the class of error functions on I
$\mathcal{E}_0(I)$	the class of functions $\Phi \in \mathcal{E}(I)$ with $\Phi(0) = 0$
$\mathcal{E}^\Gamma(I)$	the class of error functions with the Γ -property
$\mathcal{E}_0^\Gamma(I)$	the class of functions $\Phi \in \mathcal{E}^\Gamma(I)$ with $\Phi(0) = 0$
Φ, Ψ	symbols for error functions
Φ^σ	the subadditive envelope of Φ
Φ^α	the absolutely subadditive envelope of Φ
$\mathcal{M}_\Phi(I)$	class of Φ -monotone functions
$\mathcal{M}(I)$	class of all approximately monotone functions
$\overline{\mathcal{M}}_\Phi(f)$	the upper Φ -monotone envelop for f
$\underline{\mathcal{M}}_\Phi(f)$	the lower Φ -monotone envelope for f
Φ_f^σ	the individual error function for Φ -monotone function f
$\mathcal{H}_\Phi(I)$	class of Φ -Hölder functions
$\mathcal{H}(I)$	class of all approximately Hölder functions
$\overline{\mathcal{H}}_\Phi(f)$	the upper Φ -Hölder envelope for f
$\underline{\mathcal{H}}_\Phi(f)$	the lower Φ -Hölder envelope for f
Φ_f^α	the individual error function for Φ -Hölder function f
$\mathcal{C}_\Phi(I)$	class of Φ -convex functions
$\underline{\mathcal{C}}_\Phi(f)$	the lower Φ -convex envelope for f
$\mathcal{A}_\Phi(I)$	class of Φ -affine functions
$V^\Phi(f; \tau)$	the Φ -variation of f w.r.t. the partition τ
$V_{[a,b]}^\Phi f$	the total Φ -variation of f on $[a, b]$

In this thesis we summarize the most important results of our PhD dissertation. We mention some of the interesting lemmas, propositions, theorems, and corollaries based on our research which can be found in full details in the papers [13–15].

We start with the definition of the four approximate classes of functions (monotone, Hölder, convex and affine). Next we will go through some of the basic structural and inclusion properties of these function classes. Then we will obtain precise formulas for the optimal error functions for each of the pre-mentioned function classes. We also give formulas for the minorants and majorants. Besides, a few sandwich type theorems are also presented. Towards the end, we characterize these approximate function classes and also we are able to find the building blocks for some of them. In the application part we primarily focus on associating our results with *Jordan-type decompositions*, *Hermite–Hadamard-type inequalities* and *Ostrowski-type inequalities*.

Introduction

Let I be a nonempty open real interval throughout this thesis and let $\ell(I) \in]0, \infty[$ denote its length. The symbols \mathbb{R} and \mathbb{R}_+ denote the sets of real and nonnegative real numbers, respectively.

The class of all functions $\Phi : [0, \ell(I)[\rightarrow \mathbb{R}_+$, called *error functions*, will be denoted by $\mathcal{E}(I)$. Obviously, $\mathcal{E}(I)$ is a convex cone, i.e., it is closed with respect to addition and multiplication by nonnegative scalars. The subset of $\mathcal{E}(I)$ whose elements also satisfy $\Phi(0) = 0$ will be denoted by $\mathcal{E}_0(I)$. In what follows, we are going to define four concepts related to an error function $\Phi \in \mathcal{E}(I)$.

A function $f : I \rightarrow \mathbb{R}$ will be called Φ -*monotone* if, for all $x, y \in I$ with $x \leq y$,

$$f(x) \leq f(y) + \Phi(y - x).$$

If this inequality is satisfied with the identically zero error function Φ , then we say that f is *monotone (increasing)*. The class of Φ -monotone functions on I will be denoted by $\mathcal{M}_\Phi(I)$.

A function $f : I \rightarrow \mathbb{R}$ will be called Φ -*Hölder* if, for all $x, y \in I$,

$$|f(x) - f(y)| \leq \Phi(|x - y|).$$

The class of Φ -Hölder functions on I will be denoted by $\mathcal{H}_\Phi(I)$.

We say that a function $f : I \rightarrow \mathbb{R}$ is Φ -*convex* if, for all $x, y \in I$ and $t \in [0, 1]$,

$$\begin{aligned} f(tx + (1 - t)y) &\leq tf(x) + (1 - t)f(y) \\ &\quad + t\Phi((1 - t)|x - y|) + (1 - t)\Phi(t|x - y|). \end{aligned}$$

If the above inequality is satisfied with the identically zero error function Φ , then we say that f is *convex*. The class of Φ -convex functions on I will be denoted by $\mathcal{C}_\Phi(I)$.

A function $f : I \rightarrow \mathbb{R}$ is said to be Φ -*affine* if, for all $x, y \in I$ and $t \in [0, 1]$,

$$\begin{aligned} |f(tx + (1 - t)y) - tf(x) - (1 - t)f(y)| \\ \leq t\Phi((1 - t)|x - y|) + (1 - t)\Phi(t|x - y|). \end{aligned}$$

If, in particular, the above inequality is satisfied with the identically zero error function Φ , then we say that f is *affine*. The class of Φ -affine functions on I will be denoted by $\mathcal{A}_\Phi(I)$.

Structural and Inclusion Properties

Below we describe some of the basic structural properties by dealing with various operations such as sup, limsup, inf, liminf etc.

PROPOSITION. *Let $\Phi_1, \dots, \Phi_n \in \mathcal{E}(I)$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}_+$. Then*

$$\begin{aligned}\alpha_1 \mathcal{M}_{\Phi_1}(I) + \dots + \alpha_n \mathcal{M}_{\Phi_n}(I) &\subseteq \mathcal{M}_{\alpha_1 \Phi_1 + \dots + \alpha_n \Phi_n}(I), \\ \alpha_1 \mathcal{C}_{\Phi_1}(I) + \dots + \alpha_n \mathcal{C}_{\Phi_n}(I) &\subseteq \mathcal{C}_{\alpha_1 \Phi_1 + \dots + \alpha_n \Phi_n}(I).\end{aligned}$$

In particular, for all functions $\Phi \in \mathcal{E}(I)$, the classes $\mathcal{M}_\Phi(I)$ and $\mathcal{C}_\Phi(I)$ are convex.

PROPOSITION. *Let $\Phi_1, \dots, \Phi_n \in \mathcal{E}(I)$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. Then*

$$\begin{aligned}\alpha_1 \mathcal{H}_{\Phi_1}(I) + \dots + \alpha_n \mathcal{H}_{\Phi_n}(I) &\subseteq \mathcal{H}_{|\alpha_1 \Phi_1 + \dots + \alpha_n \Phi_n|}(I), \\ \alpha_1 \mathcal{A}_{\Phi_1}(I) + \dots + \alpha_n \mathcal{A}_{\Phi_n}(I) &\subseteq \mathcal{A}_{|\alpha_1 \Phi_1 + \dots + \alpha_n \Phi_n|}(I).\end{aligned}$$

In particular, for all functions $\Phi \in \mathcal{E}(I)$, the classes $\mathcal{H}_\Phi(I)$ and $\mathcal{A}_\Phi(I)$ are convex and central symmetric, i.e., $\mathcal{H}_\Phi(I)$ and $\mathcal{A}_\Phi(I)$ are closed with respect to multiplication by (-1) .

We say that a family \mathcal{F} of real valued functions is *closed with respect to the pointwise supremum* if $\{f_\gamma : I \rightarrow \mathbb{R} \mid \gamma \in \Gamma\}$ is a subfamily of \mathcal{F} with a pointwise supremum $f : I \rightarrow \mathbb{R}$, i.e.,

$$f(x) = \sup_{\gamma \in \Gamma} f_\gamma(x) \quad (x \in I),$$

then $f \in \mathcal{F}$. Similarly, we can define that a family \mathcal{F} of real valued functions is *closed with respect to the pointwise infimum*.

A family $\{f_\gamma : I \rightarrow \mathbb{R} \mid \gamma \in \Gamma\}$ is called a *chain* if, for all $\alpha, \beta \in \Gamma$, either $f_\alpha \leq f_\beta$ or $f_\beta \leq f_\alpha$ holds on I . We say that a family \mathcal{F} of real valued functions is *closed with respect to the pointwise chain supremum (chain infimum)* if $\{f_\gamma : I \rightarrow \mathbb{R} \mid \gamma \in \Gamma\} \subseteq \mathcal{F}$ is a chain with a pointwise supremum (infimum) $f : I \rightarrow \mathbb{R}$, then $f \in \mathcal{F}$.

PROPOSITION. *Let $\Phi \in \mathcal{E}(I)$. Then the class $\mathcal{M}_\Phi(I)$ is closed under pointwise infimum and supremum and the class $\mathcal{C}_\Phi(I)$ is closed under pointwise supremum and pointwise chain infimum. Furthermore, $\mathcal{M}_\Phi(I)$ is closed with respect to the pointwise liminf and limsup operations, and $\mathcal{C}_\Phi(I)$ is closed with respect to the limsup operation.*

PROPOSITION. *Let $\Phi \in \mathcal{E}(I)$. Then the class $\mathcal{H}_\Phi(I)$ is closed under pointwise infimum and pointwise supremum, furthermore the class $\mathcal{A}_\Phi(I)$ is closed under pointwise chain infimum and pointwise chain supremum. Consequently, $\mathcal{H}_\Phi(I)$ is closed with respect to the pointwise liminf and limsup operations, and $\mathcal{A}_\Phi(I)$ is closed with respect to the pointwise limit operation.*

The upcoming proposition indicates the relations and inclusions among these classes of functions.

PROPOSITION. *Let $\Phi \in \mathcal{E}(I)$. Then*

$$\mathcal{H}_\Phi(I) = \mathcal{M}_\Phi(I) \cap (-\mathcal{M}_\Phi(I)) \quad \text{and} \quad \mathcal{A}_\Phi(I) = \mathcal{C}_\Phi(I) \cap (-\mathcal{C}_\Phi(I)).$$

Furthermore,

$$\mathcal{H}_\Phi(I) \subseteq \mathcal{A}_\Phi(I).$$

A Short Study of Error Functions

Now we are going to briefly study the class $\mathcal{E}(I)$ of error functions. We begin with subadditivity and describe the optimal subadditive minorant for a given error function. Next, we introduce a new class of error function, termed as absolute subadditive functions. A similar type of study is conducted on it as well. Finally, we also define a new type of functional property called the Γ -property. We will also look at some of the basic behavioral properties for this subclass of error function.

In what follows, a function $\Phi \in \mathcal{E}(I)$ will be called *subadditive* if, for all $u, v \in \mathbb{R}_+$ with $u + v < \ell(I)$, the inequality

$$\Phi(u + v) \leq \Phi(u) + \Phi(v)$$

holds. It is easy to see that a decreasing function $\Phi \in \mathcal{E}(I)$ is automatically subadditive.

The simplest but important error functions are of the form

$$(1) \quad \Phi_p(0) := 0, \quad \Phi_p(u) := u^p \quad (u > 0),$$

where $p \in \mathbb{R}$. Their subadditivity is characterized by the following statement.

PROPOSITION. *Let $p \in \mathbb{R}$. Then Φ_p is subadditive on \mathbb{R}_+ if and only if $p \in]-\infty, 1]$.*

PROPOSITION. *Let $\Phi \in \mathcal{E}(I)$ be an arbitrary function. Define the function $\Phi^\sigma : [0, \ell(I)[\rightarrow \mathbb{R}_+$ by*

$$\Phi^\sigma(u) := \inf \left\{ \Phi(u_1) + \cdots + \Phi(u_n) \mid \right. \\ \left. n \in \mathbb{N}, u_1, \dots, u_n \in \mathbb{R}_+ : u_1 + \cdots + u_n = u \right\}.$$

Then Φ^σ is the largest subadditive function which satisfies the inequality $\Phi^\sigma \leq \Phi$ on $[0, \ell(I)[$. Furthermore, $\Phi^\sigma(0) = \Phi(0)$ and, additionally, if Φ is increasing, then Φ^σ is also increasing.

A stronger property of a function $\Phi \in \mathcal{E}(I)$ is its *absolute subadditivity*, which is defined as follows: for all $u, v \in \mathbb{R}$ with $|u|, |v|, |u + v| < \ell(I)$, the inequality

$$\Phi(|u + v|) \leq \Phi(|u|) + \Phi(|v|)$$

is satisfied. It is clear that absolutely subadditive functions are automatically subadditive. For a reversed implication we have the following

LEMMA. *If $\Phi \in \mathcal{E}(I)$ is increasing and subadditive, then it is absolutely subadditive.*

PROPOSITION. Let $\Phi \in \mathcal{E}(I)$ be an arbitrary function. Define the function $\Phi^\alpha : [0, \ell(I)[\rightarrow \mathbb{R}_+$ by

$$\Phi^\alpha(u) := \inf \left\{ \Phi(|u_1|) + \cdots + \Phi(|u_n|) \mid n \in \mathbb{N}, \right. \\ \left. |u_1|, \dots, |u_n| < \ell(I), u_1 + \cdots + u_n = u \right\}.$$

Then Φ^α is the largest absolutely subadditive function which satisfies the inequality $\Phi^\alpha \leq \Phi$ and hence $\Phi^\alpha \leq \Phi^\sigma$ on $[0, \ell(I)[$.

We say that an error function $\Phi \in \mathcal{E}(I)$ possesses the property Γ if it satisfies the inequality

$$\Phi(x + y) \leq \Phi(x) + \frac{2x + y}{y} \Phi(y) \quad (x \geq 0, y > 0, x + y < \ell(I)).$$

The subclass of error functions in $\mathcal{E}(I)$ with the property Γ will be denoted by $\mathcal{E}^\Gamma(I)$. The subset of $\mathcal{E}^\Gamma(I)$ whose elements also satisfy $\Phi(0) = 0$ will be denoted by $\mathcal{E}_0^\Gamma(I)$.

One can easily see that any subadditive error function possesses property Γ , however, as we will see later, the reversed implication is not true.

The next result establishes some necessary and some sufficient conditions for the Γ property.

THEOREM. Let $\Phi \in \mathcal{E}^\Gamma(I)$. Then $\sqrt{\Phi}$ and the map $t \mapsto t^{-1}\Phi(t)$ is subadditive on $]0, \ell(I)[$. If, in addition, $\varphi : [0, \ell(I)[\rightarrow \mathbb{R}_+$ is decreasing on $]0, \ell(I)[$, then $\varphi \cdot \Phi \in \mathcal{E}^\Gamma(I)$. In particular, if $\Psi \in \mathcal{E}(I)$ and $t \mapsto t^{-2}\Psi(t)$ is decreasing on $]0, \ell(I)[$, then $\Psi \in \mathcal{E}^\Gamma(I)$.

COROLLARY. Let $p \in \mathbb{R}$. Then $\Phi_p \in \mathcal{E}^\Gamma(\mathbb{R}_+)$ if and only if $p \leq 2$.

PROPOSITION. Let $\Phi \in \mathcal{E}^\Gamma(I)$. Then, for all $n \in \mathbb{N}$ and for all $u_1 \geq 0, u_2, \dots, u_n > 0$ with $u_1 + \cdots + u_n < \ell(I)$, the following inequality holds

$$\Phi(u_1 + \cdots + u_n) \leq \Phi(u_1) + \frac{2u_1 + u_2}{u_2} \Phi(u_2) + \dots \\ \dots + \frac{2(u_1 + \cdots + u_{n-1}) + u_n}{u_n} \Phi(u_n).$$

PROPOSITION. The classes $\mathcal{E}^\Gamma(I)$ and $\mathcal{E}_0^\Gamma(I)$ are closed with respect to addition, multiplication by nonnegative scalars, pointwise maximum, pointwise chain infimum and the limsup operation.

In view of this proposition, for every error function $\Phi \in \mathcal{E}(I)$ there exists a largest error function below Φ which possesses the Γ property. This will be denoted by Φ^Γ . To give a formula for this function, for any error function $\Phi \in \mathcal{E}_0(I)$, we define its γ -transform $\Phi^\gamma : [0, \ell(I)[\rightarrow \mathbb{R}_+$ of Φ by

$$\Phi^\gamma(0) := 0,$$

and

$$\Phi^\gamma(u) := \inf \left\{ \Phi(x) + \frac{2x + y}{y} \Phi(y) \mid x \geq 0, y > 0: x + y = u \right\} \quad (u > 0).$$

It follows from $\Phi(0) = 0$ that $\Phi^\gamma \leq \Phi$ on $[0, \ell(I)[$.

THEOREM. Let $\Phi \in \mathcal{E}_0(I)$ and define the sequence $(\Phi_n)_{n=0}^\infty$ by

$$\Phi_1 := \Phi, \quad \Phi_{n+1} := \Phi_n^\gamma \quad (n \in \mathbb{N}).$$

Then (Φ_n) is a pointwise decreasing sequence on $[0, \ell(I)[$ whose pointwise limit function equals Φ^Γ on $[0, \ell(I)[$.

Effect of Lowering Error Functions

We are now ready to present the optimal error functions for our approximate classes of functions.

THEOREM. Let $\Phi \in \mathcal{E}(I)$. Then

$$\mathcal{M}_\Phi(I) = \mathcal{M}_{\Phi^\sigma}(I) \quad \text{and} \quad \mathcal{H}_\Phi(I) = \mathcal{H}_{\Phi^\sigma}(I)$$

Additionally if I is unbounded, then

$$\mathcal{H}_\Phi(I) = \mathcal{H}_{\Phi^\alpha}(I)$$

THEOREM. Let $\Phi \in \mathcal{E}_0(I)$. Then

$$\mathcal{C}_\Phi(I) = \mathcal{C}_{\Phi^\Gamma}(I) \quad \text{and} \quad \mathcal{A}_\Phi(I) = \mathcal{A}_{\Phi^\Gamma}(I).$$

COROLLARY. Let $p > 2$ and define Φ_p as in (1). Then

$$\mathcal{C}_{\Phi_p}(I) = \mathcal{C}_0(I) \quad \text{and} \quad \mathcal{A}_{\Phi_p}(I) = \mathcal{A}_0(I).$$

Majorants, Minorants, Sandwich-type Theorems

Now, in this part of thesis, for a given function we focus on the construction of the best minorants and majorants with respect to the one of the approximate classes of functions. Moreover, a few interesting sandwich types theorems are also presented.

PROPOSITION. Let $\Phi \in \mathcal{E}_0(I)$ and let $f : I \rightarrow \mathbb{R}$ be a function which admits a Φ -monotone minorant. Then the function $\underline{M}_\Phi(f)$ defined by

$$\underline{M}_\Phi(f)(x) := \inf_{x \leq y} (f(y) + \Phi^\sigma(y - x)) \quad (x \in I)$$

is real-valued and is the largest Φ -monotone function which is smaller than or equal to f . Analogously, if f admits a Φ -monotone majorant, then the function $\overline{M}_\Phi(f)$ defined by

$$\overline{M}_\Phi(f)(x) := \sup_{y \leq x} (f(y) - \Phi^\sigma(x - y)) \quad (x \in I)$$

is real-valued and is the smallest Φ -monotone function which is bigger than or equal to f .

COROLLARY. Let $\Phi \in \mathcal{E}_0(I)$ and let $g, h : I \rightarrow \mathbb{R}$. Then in order that there exist a Φ -monotone function $f : I \rightarrow \mathbb{R}$ between g and h it is necessary and sufficient that, for all $x, y \in I$ with $x \leq y$, the inequality

$$g(x) \leq h(y) + \Phi^\sigma(y - x)$$

be valid.

PROPOSITION. Let I be an unbounded interval, $\Phi \in \mathcal{E}_0(I)$ and let $f : I \rightarrow \mathbb{R}$ be a function which admits a Φ -Hölder minorant. Then the function $\underline{H}_\Phi(f)$ defined by

$$\underline{H}_\Phi(f)(x) := \inf_{y \in I} (f(y) + \Phi^\alpha(|y - x|)) \quad (x \in I)$$

is real-valued and is the largest Φ -Hölder function which is smaller than or equal to f . Analogously, if f admits a Φ -Hölder majorant, then the function $\overline{H}_\Phi(f)$ defined by

$$\overline{H}_\Phi(f)(x) := \sup_{y \in I} (f(y) - \Phi^\alpha(|x - y|)) \quad (x \in I)$$

is real-valued and is the smallest Φ -Hölder function which is bigger than or equal to f .

COROLLARY. Let I be an unbounded interval, let $\Phi \in \mathcal{E}_0(I)$ and let $g, h : I \rightarrow \mathbb{R}$. Then in order that there exist a Φ -Hölder function $f : I \rightarrow \mathbb{R}$ between g and h it is necessary and sufficient that, for all $x, y \in I$, the inequality

$$g(x) \leq h(y) + \Phi^\alpha(|y - x|)$$

be valid.

For $\Phi \in \mathcal{E}_0(I)$ and $f : I \rightarrow \mathbb{R}$, define $\underline{C}_\Phi(f) : I \rightarrow [-\infty, \infty)$ by

$$\underline{C}_\Phi(f)(u) := \inf_{\substack{x, y \in I, t \in [0, 1] \\ u = tx + (1-t)y}} t(f(x) + \Phi((1-t)|y - x|)) + (1-t)(f(y) + \Phi(t|y - x|)).$$

THEOREM. Let $\Phi \in \mathcal{E}_0(I)$. Then the operator \underline{C}_Φ is monotone and concave in the pointwise sense, i.e., for $f, g : I \rightarrow \mathbb{R}$ and $\lambda \in [0, 1]$, we have

$$\lambda \underline{C}_\Phi(f) + (1 - \lambda) \underline{C}_\Phi(g) \leq \underline{C}_\Phi(\lambda f + (1 - \lambda)g).$$

Furthermore, for all $f : I \rightarrow \mathbb{R}$, the inequality $\underline{C}_\Phi(f) \leq f$ holds and here equality is valid if and only if f is Φ -convex. In other words, the fixed points of \underline{C}_Φ are exactly the Φ -convex functions.

LEMMA. Let $\Phi \in \mathcal{E}_0(I)$ be increasing and subadditive. Then $\Psi(u) := -\Phi(|u|)$ is Φ -convex on $J := I - I$. In particular, for $p \in [0, 1]$, $\Psi_p(u) := -\Phi_p(|u|)$ is Φ_p -convex on \mathbb{R} .

THEOREM. Let $\Phi \in \mathcal{E}_0(I)$ such that $\Psi(u) := -\Phi(|u|)$ is Φ -convex on $J := I - I$. Let $f : I \rightarrow \mathbb{R}$ be a function which admits a Φ -convex minorant. Then the function $\underline{C}_\Phi(f)$ is the largest Φ -convex function which is smaller than or equal to f .

COROLLARY. Let $\Phi \in \mathcal{E}_0(I)$ be nondecreasing and subadditive. Let $f : I \rightarrow \mathbb{R}$ be a function which admits a Φ -convex minorant. Then the function $\underline{C}_\Phi(f)$ is the largest Φ -convex function which is smaller than or equal to f .

THEOREM. Let $\Phi \in \mathcal{E}_0(I)$ and let $f, g : I \rightarrow \mathbb{R}$ such that $g \leq h \leq f$ for some Φ -convex function $h : I \rightarrow \mathbb{R}$. Then, for all $x, y \in I$ and $t \in [0, 1]$, the functional inequality

$$(2) \quad \begin{aligned} g(tx + (1-t)y) &\leq tf(x) + (1-t)f(y) \\ &\quad + t(\Phi(1-t)|y-x|) + (1-t)\Phi(t|y-x|) \end{aligned}$$

holds. Conversely, if $\Psi(u) := -\Phi(|u|)$ is Φ -convex on $J := I - I$ and (2) is valid for all $x, y \in I$ and $t \in [0, 1]$, then there exists a Φ -convex function $h : I \rightarrow \mathbb{R}$ such that $g \leq h \leq f$ holds on I .

COROLLARY. Let $\Phi \in \mathcal{E}_0(I)$ be nondecreasing, subadditive and let $f, g : I \rightarrow \mathbb{R}$. Then the inequalities $g \leq h \leq f$ hold for some Φ -convex function $h : I \rightarrow \mathbb{R}$ if and only if, for all $x, y \in I$ and $t \in [0, 1]$, the functional inequality (2) is satisfied.

The net result establishes a formula for the Φ -convex envelope of real function.

THEOREM. Let $\Phi \in \mathcal{E}_0(I)$. Let $f : I \rightarrow \mathbb{R}$ be a function which admits a Φ -convex minorant and define the sequence $f_n : I \rightarrow \mathbb{R}$ by

$$f_1 := f, \quad f_{n+1} := \underline{C}_\Phi(f_n) \quad (n \in \mathbb{N}).$$

Then the sequence (f_n) is pointwise decreasing and its limit function is the largest Φ -convex function which is smaller than or equal to f .

Characterization of Φ -Monotone and Φ -Hölder Functions

In this part, we describe how using the notions of upper and lower interpolations, we can establish a characterization for approximately monotone and Hölder functions. We also see the construction of Φ -monotone and Φ -Hölder functions from elementary ones, which could be termed the building blocks for this class.

If $\Phi \in \mathcal{E}(I)$, $f : I \rightarrow \mathbb{R}$ and $p \in I$, then define the three functions $h, h_p, h^p : I \rightarrow [-\infty, \infty]$ as follows:

$$h(x) := \begin{cases} \inf_{[x,p]} f & \text{if } x < p, \\ f(p) & \text{if } x = p, \\ \sup_{[p,x]} f & \text{if } p < x. \end{cases}$$

$$h_p(x) := \begin{cases} h(x) & \text{if } x \leq p, \\ h(p) - \Phi(x-p) & \text{if } p < x, \end{cases}$$

and

$$h^p(x) := \begin{cases} h(p) + \Phi(p-x) & \text{if } x < p, \\ h(x) & \text{if } p \leq x. \end{cases}$$

PROPOSITION. *Let $\Phi \in \mathcal{E}(I)$ be subadditive and nondecreasing and $h : I \rightarrow [-\infty, \infty]$ be nondecreasing. Then, for all $p \in I$, the functions h_p and h^p defined above are Φ -monotone.*

If $\Phi \in \mathcal{E}(I)$, $f : I \rightarrow \mathbb{R}$ and $p \in I$, then we say that f can be interpolated at p by a Φ -monotone function from below [resp. from above] if there exists a Φ -monotone function $h : I \rightarrow \mathbb{R}$ such that $h(p) = f(p)$ and $h \leq f$ [resp. $f \leq h$].

PROPOSITION. *Let $\Phi \in \mathcal{E}(I)$ be a subadditive and nondecreasing function, let $f : I \rightarrow \mathbb{R}$ and $p \in I$ be fixed. Then f can be interpolated at p by a Φ -monotone function from below if and only if, for all $x \in I$,*

$$-\infty < \inf_{[x,p]} f \quad \text{if } x \leq p \quad \text{and} \quad f(p) \leq f(x) + \Phi(x - p) \quad \text{if } p < x.$$

Analogously, f can be interpolated at p by a Φ -monotone function from above if and only if, for all $x \in I$,

$$f(x) \leq f(p) + \Phi(p - x) \quad \text{if } x < p \quad \text{and} \quad \sup_{[p,x]} f < +\infty \quad \text{if } p \leq x.$$

THEOREM. *Let $\Phi \in \mathcal{E}(I)$ be a subadditive and nondecreasing function and $f : I \rightarrow \mathbb{R}$. Then the following assertions are equivalent.*

- (i) f is Φ -monotone.
- (ii) There exists a function $H : I \times I \rightarrow \mathbb{R}$ such that H satisfies the functional equations

$$\begin{aligned} \min(H(x, y), H(y, z)) &= H(x, z) & \text{and} \\ \max(H(z, y), H(y, x)) &= H(z, x) \end{aligned}$$

for all $x, y, z \in I$ with $x \leq y \leq z$ and, for all $p \in I$, the function $h := H(\cdot, p)$ is nondecreasing and h_p and h^p are Φ -monotone interpolations for f at p from below and from above, respectively.

- (iii) For every $p \in I$, there exists a nondecreasing function $h : I \rightarrow \mathbb{R}$ such that h_p is a Φ -monotone interpolation of f at p from below.
- (iv) For every $p \in I$, there exists a nondecreasing function $h : I \rightarrow \mathbb{R}$ such that h^p is a Φ -monotone interpolation of f at p from above.

If $\Phi \in \mathcal{E}(I)$, $f : I \rightarrow \mathbb{R}$ and $p \in I$, then we say that f can be interpolated at p by a Φ -Hölder function from below [resp. from above] if there exists a Φ -Hölder function $h : I \rightarrow \mathbb{R}$ such that $h(p) = f(p)$ and $h \leq f$ [resp. $f \leq h$].

In what follows, given an error function $\Phi \in \mathcal{E}(I)$ and $p \in I$, we define the function $\Phi_p : I \rightarrow \mathbb{R}$ by

$$\Phi_p(x) := \Phi(|x - p|) \quad (x \in I).$$

PROPOSITION. *Let $\Phi \in \mathcal{E}_0(I)$ be an absolutely subadditive function. Then, for all $p \in I$, the function Φ_p is Φ -Hölder on I .*

PROPOSITION. Let $\Phi \in \mathcal{E}_0(I)$ be an absolutely subadditive function and let $f : I \rightarrow \mathbb{R}$ and $p \in I$ be fixed. Then f can be interpolated at p by a Φ -Hölder function from below if and only if, for all $x \in I$,

$$f(p) \leq f(x) + \Phi(|x - p|).$$

Analogously, f can be interpolated at p by a Φ -Hölder function from above if and only if, for all $x \in I$,

$$f(x) \leq f(p) + \Phi(|x - p|).$$

THEOREM. Let $\Phi \in \mathcal{E}_0(I)$ be absolutely subadditive function and $f : I \rightarrow \mathbb{R}$. Then the following assertions are equivalent to each other:

- (i) f is Φ -Hölder.
- (ii) For every $p \in I$, the functions $f(p) - \Phi_p$ and $f(p) + \Phi_p$ are Φ -Hölder interpolations of f at p from below and above, respectively.
- (iii) For every $p \in I$, f possesses a Φ -Hölder interpolation from below.
- (iv) For every $p \in I$, f possesses a Φ -Hölder interpolation from above.

Characterization of Φ -Convex and Φ -Affine Functions

The theorems mentioned below characterize the Φ -convex and Φ -affine functions, respectively. The first one was partly established in the paper [34].

THEOREM. Let $\Phi \in \mathcal{E}(I)$ and $f : I \rightarrow \mathbb{R}$. Then the following conditions are equivalent to each other:

- (i) f is Φ -convex;
- (ii) For all $x, u, y \in I$ with $x < u < y$,

$$\frac{f(u) - f(x) - \Phi(u - x)}{u - x} \leq \frac{f(y) - f(u) + \Phi(y - u)}{y - u};$$

- (iii) There exists a function $\varphi : I \rightarrow \mathbb{R}$ such that, for all $x, u \in I$,

$$(3) \quad f(u) + (x - u)\varphi(u) \leq f(x) + \Phi(|u - x|);$$

- (iv) For all $n \in \mathbb{N}$, $x_1, \dots, x_n \in I$, $t_1, \dots, t_n \geq 0$ with $t_1 + \dots + t_n = 1$,

$$f(t_1x_1 + \dots + t_nx_n) \leq \sum_{i=1}^n t_i (f(x_i) + \Phi(|(t_1x_1 + \dots + t_nx_n) - x_i|)).$$

THEOREM. Let $\Phi \in \mathcal{E}(I)$ and $f : I \rightarrow \mathbb{R}$. Then the following conditions are equivalent to each other:

- (i) f is Φ -affine;
- (ii) For all $x, u, y \in I$ with $x < u < y$,

$$\left| \frac{f(u) - f(x)}{u - x} - \frac{f(y) - f(u)}{y - u} \right| \leq \frac{\Phi(u - x)}{u - x} + \frac{\Phi(y - u)}{y - u};$$

(iii) For all $n \in \mathbb{N}$, $x_1, \dots, x_n \in I$, $t_1, \dots, t_n \geq 0$ with $t_1 + \dots + t_n = 1$,

$$\begin{aligned} \left| f(t_1x_1 + \dots + t_nx_n) - \sum_{i=1}^n t_i(f(x_i)) \right| \\ \leq \sum_{i=1}^n t_i \Phi(|(t_1x_1 + \dots + t_nx_n) - x_i|). \end{aligned}$$

Provided that Φ is increasing, each of the properties (i), (ii) and (iii) is also equivalent to the following condition:

(iv) There exists a function $\varphi : I \rightarrow \mathbb{R}$ such that, for all $x, u \in I$,

$$(4) \quad |f(u) - f(x) - (u - x)\varphi(u)| \leq \Phi(|u - x|).$$

The following results of slope functions tell us about the connections of Φ -convexity with Φ -monotone function and Φ -affinity with Φ -Hölder function.

PROPOSITION. Let $\Phi \in \mathcal{E}(I)$, $f : I \rightarrow \mathbb{R}$ be a Φ -convex function and $\varphi : I \rightarrow \mathbb{R}$ be a Φ -slope function for f , i.e., it satisfies (3) Then φ is Φ^* -monotone, where

$$\Phi^*(t) := \begin{cases} 2 \frac{\Phi(t)}{t} & \text{if } 0 < t < \ell(I), \\ 0 & \text{if } t = 0. \end{cases}$$

PROPOSITION. Let $\Phi \in \mathcal{E}(I)$ be increasing, $f : I \rightarrow \mathbb{R}$ be a Φ -affine function and $\varphi : I \rightarrow \mathbb{R}$ be a Φ -slope function for f , i.e., it satisfies (4) Then φ is Φ^* -Hölder, where

$$\Phi^*(t) := \begin{cases} 2 \frac{\Phi(t)}{t} & \text{if } 0 < t < \ell(I), \\ 0 & \text{if } t = 0. \end{cases}$$

Application of Φ -Monotone and Φ -Hölder Functions

Now we can discuss about some of the application of these approximate monotone classes.

Let $\Phi \in \mathcal{E}(I)$. Then a function $f : I \rightarrow \mathbb{R}$ is called *delta- Φ -monotone* if it is the difference of two Φ -monotone functions. In what follows, we shall extend the celebrated Jordan Decomposition Theorem for delta- Φ -monotone functions. For this purpose, we extend the notion of total variation to this more general setting.

Let $[a, b] \subseteq I$ and let $\tau = (t_0, \dots, t_n)$ be a partition of the interval $[a, b]$ (i.e., $a = t_0 < t_1 < \dots < t_n = b$). Then the Φ -variation of f with respect to τ is defined by

$$V^\Phi(f; \tau) := \sum_{i=1}^n (|f(t_i) - f(t_{i-1})| - \Phi(t_i - t_{i-1})).$$

Finally, the *total Φ -variation of f on the interval $[a, b]$* is defined by

$$V_{[a,b]}^\Phi f := \sup\{V^\Phi(f; \tau) \mid \tau \text{ is a partition of } [a, b]\}.$$

LEMMA. Let $\Phi \in \mathcal{E}(I)$. Then, for all $f : I \rightarrow \mathbb{R}$ and $a < b < c$ in I , we have

$$V_{[a,b]}^{\Phi} f + V_{[b,c]}^{\Phi} f \leq V_{[a,c]}^{\Phi} f.$$

THEOREM. Let $\Phi \in \mathcal{E}(I)$. Then $V_{[a,b]}^{\Phi} f \leq 0$ holds for all $a < b$ in I if and only if f is a Φ -Hölder function.

THEOREM. Let $\Phi, \Psi \in \mathcal{E}(I)$. If $f : I \rightarrow \mathbb{R}$ is the difference of a Φ -monotone and a Ψ -monotone functions, then the total $2 \max(\Phi, \Psi)$ -variation of f is finite on every compact subinterval of I .

The particular case $\Phi = \Psi$ of the above result yields the following statement.

COROLLARY. Let $\Phi \in \mathcal{E}(I)$. If $f : I \rightarrow \mathbb{R}$ is a delta- Φ -monotone function, then the total 2Φ -variation of f is finite on every compact subinterval of I .

THEOREM. Let $\Phi \in \mathcal{E}(I)$ let $f : I \rightarrow \mathbb{R}$ such that the total 2Φ -variation of f on is finite on every compact subinterval of I . Then, for all $a \in I$, f is a delta- Φ -monotone function on $I \cap]a, \infty[$.

The classical Hermite–Hadamard inequality states that "The integral mean of a convex function in any subinterval of its domain is sandwich between the functional value of the midpoint and the average of the two functional values at the extreme points." Likewise, we can deal with locally integrable Φ -monotone functions and give a proper bound to their integral mean.

In the sequel, a function defined on an interval will be called locally integrable if it has a finite Lebesgue integral over every compact subinterval of its domain.

For the description of our subsequent results, we now introduce the following notation and terminology: If $a, b \in I$ then the convex hull of $\{a, b\}$, i.e., the smallest interval containing a and b , will denoted by $\langle a, b \rangle$. If, additionally, $f : \langle a, b \rangle \rightarrow \mathbb{R}$ is Lebesgue integrable, then the integral average of f over $\langle a, b \rangle$ is defined by

$$\mathcal{A}(f, \langle a, b \rangle) := \int_0^1 f(ta + (1-t)b) dt.$$

One can easily see that the following equality holds:

$$\langle a, b \rangle := \begin{cases} [a, b] & \text{if } a < b, \\ \{a\} & \text{if } a = b, \\ [b, a] & \text{if } a > b, \end{cases} \quad \text{and } \mathcal{A}(f, \langle a, b \rangle) = \begin{cases} \frac{1}{b-a} \int_a^b f & \text{if } a < b, \\ f(a) & \text{if } a = b, \\ \frac{1}{a-b} \int_b^a f & \text{if } a > b. \end{cases}$$

The inequalities stated in the following results summarize both Hermite–Hadamard and Ostrowski-type inequalities for the Φ -monotone as well as for the Φ -Hölder settings.

THEOREM. Let $\Phi \in \mathcal{E}(I)$ and $f : I \rightarrow \mathbb{R}$ be locally Lebesgue integrable functions. If f is Φ -monotone, then, for all $u, v, w, z \in I$ with $u \leq w$ and $v \leq z$,

$$\mathcal{A}(f, \langle u, v \rangle) \leq \mathcal{A}(f, \langle w, z \rangle) + \mathcal{A}(\Phi, \langle w - u, z - v \rangle).$$

If f is Φ -Hölder, then, for all $u, v, w, z \in I$,

$$|\mathcal{A}(f, \langle u, v \rangle) - \mathcal{A}(f, \langle w, z \rangle)| \leq \mathcal{A}(\Phi \circ |\cdot|, \langle w - u, z - v \rangle).$$

Assuming Φ -monotonicity, we deduce a monotonicity type integral inequality which we will call the *lower and upper Hermite–Hadamard inequalities for Φ -monotone functions*.

THEOREM. Let $\Phi \in \mathcal{E}(I)$ and $f : I \rightarrow \mathbb{R}$ be a Φ -monotone. Assume that both functions are locally Lebesgue integrable. Then, for every $x < y$ in I , the following two inequalities hold:

$$f(x) - \frac{1}{y-x} \int_0^{y-x} \Phi \leq \frac{1}{y-x} \int_x^y f \leq f(y) + \frac{1}{y-x} \int_0^{y-x} \Phi.$$

Furthermore, if Φ is subadditive and nondecreasing, then, for all $x < y$ in I ,

$$\begin{aligned} \sup_{f \in \mathcal{M}_\Phi(I)} \left(f(x) - \frac{1}{y-x} \int_x^y f \right) &= \sup_{f \in \mathcal{M}_\Phi(I)} \left(\frac{1}{y-x} \int_x^y f - f(y) \right) \\ &= \frac{1}{y-x} \int_0^{y-x} \Phi. \end{aligned}$$

COROLLARY. Let $p \in [0, 1]$, $c \in [0, \infty[$ and $f : I \rightarrow \mathbb{R}$ be a $c(\cdot)^p$ -monotone locally Lebesgue integrable function. Then, for every $x < y$ in I , the following two inequalities hold:

$$f(x) - \frac{c}{p+1}(y-x)^p \leq \frac{1}{y-x} \int_x^y f \leq f(y) + \frac{c}{p+1}(y-x)^p.$$

Furthermore, for all $x < y$ in I ,

$$\begin{aligned} \sup_{f \in \mathcal{M}_\Phi(I)} \left(f(x) - \frac{1}{y-x} \int_x^y f \right) &= \sup_{f \in \mathcal{M}_\Phi(I)} \left(\frac{1}{y-x} \int_x^y f - f(y) \right) \\ &= \frac{c}{p+1}(y-x)^p. \end{aligned}$$

LEMMA. Let $\Psi \in \mathcal{E}(I)$ and assume that the map $t \mapsto \Psi(t)/t$ is locally integrable on $]0, \ell(I)[$ and define $\Phi \in \mathcal{E}(I)$ by

$$(5) \quad \Phi(u) = \Psi(u) + \int_0^u \frac{\Psi(t)}{t} dt \quad (u \in]0, \ell(I)[).$$

Then Φ is locally integrable and satisfies the following equation:

$$\Psi(u) + \frac{1}{u} \int_0^u \Phi = \Phi(u) \quad (u \in]0, \ell(I)[).$$

THEOREM. Let $\Psi \in \mathcal{E}(I)$ and assume that the map $t \mapsto \Psi(t)/t$ is locally integrable on $[0, \ell(I)[$ and define $\Phi \in \mathcal{E}(I)$ by (5). If $f : I \rightarrow \mathbb{R}$ is an upper semicontinuous solution of

$$f(u) \leq \frac{1}{v-u} \int_u^v f + \Psi(v-u) \quad (u, v \in I, u < v),$$

then f is Φ -monotone on I .

The following result is a counterpart of previous theorem. It can be proved directly in an analogous way, however, in our dissertation we deduce it from this theorem using a sign transformation.

THEOREM. Let $\Psi \in \mathcal{E}(I)$ and assume that the map $t \mapsto \Psi(t)/t$ is locally integrable on $[0, \ell(I)[$ and define $\Phi \in \mathcal{E}(I)$ by (5). If $f : I \rightarrow \mathbb{R}$ is a lower semicontinuous solution of

$$\frac{1}{v-u} \int_u^v f \leq f(v) + \Psi(v-u) \quad (u, v \in I, u < v),$$

then f is Φ -monotone on I .

COROLLARY. Let $p \in]0, 1]$, $c \in [0, \infty[$. If $f : I \rightarrow \mathbb{R}$ is an upper semicontinuous solution of

$$f(u) \leq \frac{1}{v-u} \int_u^v f + c(v-u)^p \quad (u, v \in I, u < v),$$

then f is $\frac{c(p+1)}{p}(\cdot)^p$ -monotone on I . In particular, f is increasing if $p > 1$.

COROLLARY. Let $p \in]0, 1]$, $c \in [0, \infty[$. If $f : I \rightarrow \mathbb{R}$ is a lower semicontinuous solution of

$$\frac{1}{v-u} \int_u^v f \leq f(v) + c(v-u)^p \quad (u, v \in I, u < v),$$

then f is $\frac{c(p+1)}{p}(\cdot)^p$ -monotone on I . In particular, f is increasing if $p > 1$

The result in the next theorem will be called an Ostrowski-type inequality for Φ -Hölder functions. The Ostrowski's inequality provides a sharp upper bound between the difference of integral mean of a bounded differentiable function and functional value at any point in that interval.

THEOREM. Let $\Phi \in \mathcal{E}(I)$ and $f : I \rightarrow \mathbb{R}$ be a Φ -Hölder. Assume that both functions are locally Lebesgue integrable. Then, for every $x < y$ in I , the following inequality hold:

$$\left| f(p) - \frac{1}{y-x} \int_x^y f \right| \leq \frac{1}{y-x} \left(\int_0^{p-x} \Phi + \int_0^{y-p} \Phi \right) \quad (p \in [x, y]).$$

Furthermore, if Φ is subadditive and nondecreasing with $\Phi(0) = 0$, then, for all $x < y$ in I ,

$$\sup_{f \in \mathcal{H}_{\Phi}(I)} \left| f(p) - \frac{1}{y-x} \int_x^y f \right| = \frac{1}{y-x} \left(\int_0^{p-x} \Phi + \int_0^{y-p} \Phi \right) \quad (u \in [x, y]).$$

The inequalities obtained in the particular cases $p = x$ and $p = y$ will be called the *lower and upper Hermite–Hadamard-type inequalities for Φ -Hölder functions*.

COROLLARY. Let $q \in [0, 1]$, $c \in [0, \infty[$ and $f : I \rightarrow \mathbb{R}$ be a $c(\cdot)^q$ -Hölder locally Lebesgue integrable function. Then, for every $x < y$ in I ,

$$\left| f(p) - \frac{1}{y-x} \int_x^y f \right| \leq \frac{c}{q+1} ((p-x)^q + (y-p)^q) \quad (p \in [x, y]).$$

Furthermore, for all $x < y$ in I ,

$$\sup_{f \in \mathcal{H}_{\Phi}(I)} \left| f(p) - \frac{1}{y-x} \int_x^y f \right| = \frac{c}{q+1} ((p-x)^q + (y-p)^q) \quad (p \in [x, y]).$$

THEOREM. Let $\Psi \in \mathcal{E}(I)$ and assume that the map $t \mapsto \Psi(t)/t$ is locally integrable on $[0, \ell(I)[$ and define $\Phi \in \mathcal{E}(I)$ by (5). If $f : I \rightarrow \mathbb{R}$ is a continuous solution of

$$\left| f(u) - \frac{1}{v-u} \int_u^v f \right| \leq \Psi(v-u) \quad (u, v \in I, u < v),$$

then f is Φ -Hölder on I .

THEOREM. Let $\Psi \in \mathcal{E}(I)$ and assume that the map $t \mapsto \Psi(t)/t$ is locally integrable on $[0, \ell(I)[$ and define $\Phi \in \mathcal{E}(I)$ by (5). If $f : I \rightarrow \mathbb{R}$ is a continuous solution of

$$\left| f(v) - \frac{1}{v-u} \int_u^v f \right| \leq \Psi(v-u) \quad (u, v \in I, u < v),$$

then f is Φ -Hölder on I .

COROLLARY. Let $p \in]0, 1]$, $c \in [0, \infty[$. If $f : I \rightarrow \mathbb{R}$ is a continuous solution of

$$\left| f(u) - \frac{1}{v-u} \int_u^v f \right| \leq c(v-u)^p \quad (u, v \in I, u < v),$$

then f is $\frac{c(p+1)}{p}(\cdot)^p$ -Hölder on I . In particular, f is constant if $p > 1$.

COROLLARY. Let $p \in]0, 1]$, $c \in [0, \infty[$. If $f : I \rightarrow \mathbb{R}$ is a continuous solution of

$$\left| f(v) - \frac{1}{v-u} \int_u^v f \right| \leq c(v-u)^p \quad (u, v \in I, u < v),$$

then f is $\frac{c(p+1)}{p}(\cdot)^p$ -Hölder on I . In particular, f is constant if $p > 1$.

List of Scientific Events

Conferences with contributinal talks

- (1) The Nineteenth Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities, Zakopane, Poland, January 30–February 2, 2019.
- (2) The 15th International Student Conference on Analysis, Gwarek, Ustroń, Poland, February 2–February 5, 2019
- (3) The Second Spring Seminar on Analysis, University of Debrecen, Hungary, 4 April, 2019
- (4) 2019 Szeged Workshop on Convexity, Bolyai Institute, University of Szeged, Hungary, April 5–6, 2019
- (5) The Twentieth Debrecen—Katowice Winter Seminar on Functional Equations and Inequalities, Hajdúszoboszló, Hungary, January 29–February 1, 2020

Conferences without contributinal talks

- (1) The First Spring Workshop on Analysis, University of Debrecen, Hungary, 4 April, 2018
- (2) Scientific meeting on "Fourier Analysis and Applications", MTA headquarters, Budapest, Hungary, May 15, 2019
- (3) Síkfökút Seminar of the Department of Analysis, Síkfökút, Hungary, 24–26 May, 2019
- (4) Spring and Winter Seminar Series, University of Debrecen, Hungary, 2017–2021
- (5) Workshop in Geometry and Analysis, IMPAN, Poland, 07–18 October, 2019
- (6) The 15th International Student Conference on Analysis, Síkfökút, Hungary, 01-04 February, 2020

Summer Schools

- (1) Summer School on High Dimensional Probability, Cortona, Italy, July, 2018
- (2) BMS Summer School "Mathematics of Deep Learning" Zuse Institute, Berlin, Germany, August 19–30, 2019
- (3) Summer School on Functional Analysis, Perugia, Italy, July–August, 2020

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Doctoral School: Doctoral School of Mathematical and Computational Sciences

List of publications related to the dissertation

Foreign language scientific articles in Hungarian journals (1)

1. **Goswami, A. R.**, Páles, Z.: On approximately monotone and approximately Hölder functions.
Period. Math. Hung. 81 (1), 65-87, 2020. ISSN: 0031-5303.
DOI: <http://dx.doi.org/10.1007/s10998-020-00351-0>
IF: 0.693 (2019)

Foreign language scientific articles in international journals (2)

2. **Goswami, A. R.**, Páles, Z.: Characterization of approximately monotone and approximately Hölder functions.
Math. Inequal. Appl. 24 (1), 247-264, 2021. ISSN: 1331-4343.
DOI: <http://dx.doi.org/10.7153/mia-2021-24-18>
IF: 1.51 (2019)
3. **Goswami, A. R.**, Páles, Z.: On approximately convex and affine functions.
J. Math. Inequal. Accepted by publisher, 1-22, 2021. ISSN: 1846-579X.
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Total IF of journals (all publications): 3,422

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