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On a family of analytic diassociative loops

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Abstract. A loop is called diassociative if any two elements generate a subgroup, an anti-commutative algebra is binary Lie if any two elements are contained in a Lie subalgebra. In binary Lie algebras the elements generate one-parameter subgroups in the corresponding Lie groups, so the exponential and logarithm maps are locally well defined. The logarithm of the product of the exponential images defines the multiplication of a local analytic diassociative loop, represented by the classical Baker-Campbell-Hausdorff series. We study a family of binary Lie algebras for which the closed form of the Baker-Campbell-Hausdorff series defines the multiplication function of an analytic diassociative loop on the entire binary Lie algebra. These algebras are semidirect sums of the two-dimensional non-abelian and an abelian Lie algebra, the Lie subalgebras generated by the 2-frames have dimension 2 or 3. We express the group multiplications of the exponential images of elements of the matrix Lie algebras that are isomorphic to these Lie subalgebras. By transforming back to the binary Lie algebra, we express the analytic diassociative loop multiplication. Our result contributes to the Lie theory of diassociative loops, since no fully developed examples of the correspondence between binary Lie algebras and global analytic diassociative loops are known so far.

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Keywords. Global analytic loop, Baker-Campbell-Hausdorff series, Diassociative loop, Binary Lie algebra.

1. Introduction

The power series of two non-commuting variables

$$\begin{aligned} Z(x,y) = \log(e^x e^y) = \sum_{k=1}^{\infty} Z_k(x,y) = \\ = x + y + \frac{1}{2}[x,y] + \frac{1}{12}([x,[x,y]] + [y,[y,x]]) + ..., \end{aligned} \tag{1.1}$$

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is called the *Baker-Campbell-Hausdorff* series, where $Z_1(x,y) = x + y$, $Z_2 = \frac{1}{2}[x,y]$, $Z_3 = \frac{1}{12}([x,[x,y]] + [y,[y,x]]), \ldots$ are the homogeneous terms. The Baker-Campbell-Hausdorff series expansion has wide applications in both mathematics and physics: e.g. in the theory of Lie algebras and Lie groups, group theory, analysis of differential equations, quantum field theories, etc. If x, y are elements of a Lie algebra $\mathfrak{g}$, then the series (1.1) converges in a neighborhood of $0 \in \mathfrak{g}$ and gives the local Lie group multiplication $\exp^{-1}(\exp x \cdot \exp y)$ in a neighborhood of the identity element. Unfortunately, this series in general quickly becomes extremely unwieldy, hence it is not suitable for computing the corresponding group multiplication. In the last decades, motivated by problems in physics, various approaches have been pursued to find explicit formulas for the Baker-Campbell-Hausdorff series (cf. e.g. [3], [20], [28], [29]).

The homogeneous terms $Z_k(x,y)$ of the power series $Z(x,y)$ for a local analytic loop multiplication in a suitable coordinate system define multilinear operations in the tangent space at the identity element. For Lie groups these operations are expressed by compositions of the commutator, as can be seen from the series (1.1). The construction of a non-associative version of the Baker-Campbell-Hausdorff formula is a challenging problem (cf. [9], [21], [22], [30]). The reconstruction of the local loop multiplication via a Baker-Campbell-Hausdorff series seems to be possible for special classes of loops, for example for Bruck loop [23], for commutative automorphic loop [15], for geodesic loop [26].

The important correspondence between Lie groups and their tangent Lie algebras can be extended to Moufang loops and their tangent Malcev algebras or to the more general diassociative loops and their tangent binary Lie algebras. The loops in which any two elements generate a subgroup are called *diassociative*. Binary Lie algebras are defined by the property that any two elements generate a Lie subalgebra. Since the classical Baker-Campbell-Hausdorff formula only depends on two elements, it is well defined for binary Lie algebras and leads to diassociative local loop multiplications (cf. [1], Ch. 4. §5 in [10], [18]). The mentioned structures play an essential role in non-associative Lie theory (cf. e.g. [25]), the development of which was initiated by the fundamental work [19] by A. I. Mal'cev in 1955, the history of this theory is well described in the article [27] by L. V. Sabinin. Nowadays, the theory of Malcev algebras and Moufang loops has almost reached the level of the theory of Lie algebras and Lie groups, but there are many interesting open questions about binary Lie algebras and diassociative loops. For example, although every locally analytic Moufang loop can be uniquely embedded in a connected simply connected global loop (cf. [16], [17], [24]), this is not generally true for local analytic diassociative loops (cf. [14]).

The systematic study of binary Lie algebras started with the work [7] of A. T. Gainov in 1957. He found that 4 is the minimum dimension of non-Malcev binary Lie algebras and classified them in [8]. An interesting class of 4-dimensional anticommutative algebras is studied and the 5-dimensional binary Lie algebras with analogous properties are recently classified by the authors in [5], [6].

The basic theory was further developed by A. N. Grishkov (cf. [11], [12], [13]), who initiated in [14] and in his manuscript *Solvable groups and Lie algebras over rings* the study of the existence or non-existence of embedding of local diassociative analytic loops into global loops. He illustrated the Lie theory of binary Lie algebras and diassociative loops with interesting analytic and algebraic examples and counterexamples. In addition, he formulated the theorem in [14]: *If the exponential map of a binary Lie algebra is a diffeomorphism, then there exists a global analytic diassociative loop corresponding to the given binary Lie algebra* and supported his claim with an idea of a proof.

The goal of our paper is to treat the above statement for a family of binary Lie algebras with a constructive method, expressing the multiplication function as a continuation of a real analytic function with removable singularities, formed by arithmetic operations on constants, variables x, y and functions e^x, e^y. The explicit form of the multiplication function $(x, y) \mapsto \exp^{-1}(\exp x \cdot \exp y)$ defines a global analytic diassociative loop on the binary Lie algebra. Our construction is based on the joining of partial Lie group multiplications on the Lie subalgebras of the binary Lie algebra.

In Section 2 we prove that if the adjoint map of a binary Lie algebra has no nonzero purely imaginary eigenvalues, then the exponential map is globally defined and determines a diassociative loop multiplication on the entire binary Lie algebra. The studied class of binary Lie algebras is presented in Section 3. In Section 4 we describe the Lie subalgebras generated by 2-frames of the binary Lie algebra. In Section 5 we find matrix Lie algebras that are isomorphic to the 2- and 3-dimensional Lie subalgebras obtained in Section 4 and matrix Lie groups that are isomorphic to their bijective exponential images. We determine the Lie group multiplications given by the Baker-Campbell-Hausdorff series in these matrix Lie algebras in Section 6. We determine the isomorphic Lie group multiplications on the Lie subalgebras of the binary Lie algebra in Sections 7. We show in Section 8 that the obtained Lie subgroup multiplications define the explicit form of the global analytic multiplication on the entire binary Lie algebra corresponding to the Baker-Campbell-Hausdorff series.

2. Preliminaries

In the following we investigate Lie subalgebras in binary Lie algebras generated by pairs of linearly independent elements, called 2-frames. Let $\mathfrak{b}$ denote a binary Lie algebra. We denote by $St_2(\mathfrak{b})$ the Stiefel manifold of 2-frames in $\mathfrak{b}$. Let $(\mathbf{x}, \mathbf{y}) \mapsto \mathfrak{h}(\mathbf{x}, \mathbf{y})$ be the map that assigns to elements of $St_2(\mathfrak{b})$ the generated Lie subalgebra $\mathfrak{h}(\mathbf{x}, \mathbf{y})$ in $\mathfrak{b}$. We define the partial multiplication on the binary Lie algebra $\mathfrak{b}$ by

$$\mathbf{x} \circ \mathbf{y} = \exp^{-1}_{\mathfrak{h}(\mathbf{x},\mathbf{y})} \left(\exp_{\mathfrak{h}(\mathbf{x},\mathbf{y})}(\mathbf{x}) \cdot \exp_{\mathfrak{h}(\mathbf{x},\mathbf{y})}(\mathbf{y}) \right). \tag{2.1}$$

The power series expansion of this multiplication results the Baker-Campbell-Hausdorff series, defined on a neighborhood of $\mathbf{0}$ in the Lie algebra $\mathfrak{h}(\mathbf{x},\mathbf{y})$.

Lemma 2.1. *If on a binary Lie algebra $\mathfrak{b}$ the map $\mathrm{ad}_{\mathbf{u}} : \mathbf{u} \mapsto [\mathbf{u},\mathbf{x}]$ has no nonzero purely imaginary eigenvalues for all $\mathbf{u} \in \mathfrak{b}$, then the exponential map is globally defined on $\mathfrak{b}$ and the multiplication (2.1) yields a diassociative loop defined on $\mathfrak{b}$.*

Proof. According to Corollary 1.8.4 (i) and (ii) in [2] the Lie subalgebras of $\mathfrak{b}$ are exponential, particularly any subalgebra $\mathfrak{h}(\mathbf{x},\mathbf{y})$ of $\mathfrak{b}$ generated by arbitrary two elements $\mathbf{x},\mathbf{y} \in \mathfrak{b}$ is exponential, i.e. its exponential map $\exp_{\mathfrak{h}(\mathbf{x},\mathbf{y})} : \mathfrak{h}(\mathbf{x},\mathbf{y}) \to H(\mathbf{x},\mathbf{y})$ bijectively maps the Lie subalgebra $\mathfrak{h}(\mathbf{x},\mathbf{y})$ onto the corresponding connected and simply connected Lie group $H(\mathbf{x},\mathbf{y})$. Since for any $\mathbf{u},\mathbf{v} \in \mathfrak{h}(\mathbf{x},\mathbf{y})$ the subgroup $H(\mathbf{u},\mathbf{v})$ is a connected closed exponential Lie subgroup of $H(\mathbf{x},\mathbf{y})$ (cf. Corollary 1.8.5 (1) in [2]), the multiplication (2.1) satisfies

$$\begin{aligned} &\exp^{-1}_{\mathfrak{h}(\mathbf{x},\mathbf{y})} \left(\exp_{\mathfrak{h}(\mathbf{x},\mathbf{y})}(\mathbf{u}) \cdot \exp_{\mathfrak{h}(\mathbf{x},\mathbf{y})}(\mathbf{v}) \right) = \\ &= \exp^{-1}_{\mathfrak{h}(\mathbf{u},\mathbf{v})} \left(\exp_{\mathfrak{h}(\mathbf{u},\mathbf{v})}(\mathbf{u}) \cdot \exp_{\mathfrak{h}(\mathbf{u},\mathbf{v})}(\mathbf{v}) \right) \end{aligned}$$

for all $\mathbf{u},\mathbf{v} \in \mathfrak{h}(\mathbf{x},\mathbf{y})$. It follows that (2.1) determines a Lie group $(\mathfrak{h}(\mathbf{x},\mathbf{y}),\circ)$ on the Lie subalgebra $\mathfrak{h}(\mathbf{x},\mathbf{y})$ and the map $\exp_{\mathfrak{h}(\mathbf{x},\mathbf{y})} : \mathfrak{h}(\mathbf{x},\mathbf{y}) \to H(\mathbf{x},\mathbf{y})$ yields a Lie group isomorphism of $(\mathfrak{h}(\mathbf{x},\mathbf{y}),\circ) \to H(\mathbf{x},\mathbf{y})$.
Let $B_{\mathfrak{b}}$ denote the multiplicative structure defined by (2.1) on the binary Lie algebra $\mathfrak{b}$. It follows from the previous discussion that any two elements of $B_{\mathfrak{b}}$ generate a subgroup, i.e. $B_{\mathfrak{b}}$ is diassociative. We prove that (2.1) is a loop multiplication on $B_{\mathfrak{b}}$. For any fixed $\mathbf{a},\mathbf{b} \in B_{\mathfrak{b}}$ the equation $\mathbf{a} \cdot \mathbf{x} = \mathbf{b}$ or $\mathbf{y} \cdot \mathbf{a} = \mathbf{b}$ can be solved in the subgroup $H(\mathbf{a},\mathbf{b})$ with $\mathbf{x} = \mathbf{a}^{-1} \cdot \mathbf{b}$ or $\mathbf{y} = \mathbf{b} \cdot \mathbf{a}^{-1}$, respectively. Moreover, if $\mathbf{a} \cdot \mathbf{x_1} = \mathbf{a} \cdot \mathbf{x_2} = \mathbf{b}$, then $\mathbf{b}$ is contained in $\mathfrak{h}(\mathbf{a},\mathbf{x_1}) \cap \mathfrak{h}(\mathbf{a},\mathbf{x_2})$, consequently $\mathbf{a}^{-1} \cdot \mathbf{b} \in \mathfrak{h}(\mathbf{a},\mathbf{x_1}) \cap \mathfrak{h}(\mathbf{a},\mathbf{x_2})$, hence $\mathbf{x_1} = \mathbf{x_2}$. Similarly we get that the equation $\mathbf{y} \cdot \mathbf{a} = \mathbf{b}$ has unique solution. It follows that $B_{\mathfrak{b}}$ is a loop defined on $\mathfrak{b}$ and the identity map is the exponential map. □

We note that the multiplication (2.1) is glued together from partial Lie group multiplications, its analytic property does not follow for general solvable binary Lie algebras from the previous thought process. In the case of nilpotent binary Lie algebras we get:

Proposition 2.2. *Let $\mathfrak{b}$ be a nilpotent binary Lie algebra. Then there exists a connected and simply connected analytic nilpotent diassociative loop B with* *tangent algebra* $\mathfrak{b}$ *and the exponential map gives a diffeomorphism between* $\mathfrak{b}$ *and* B.

Proof. The loop multiplication (2.1) is determined by the Baker-Campbell-Hausdorff polynomials defining an analytic loop multiplication. □

In the following we construct a family of solvable binary Lie algebras, where the multiplication (2.1) is defined on the entire binary Lie algebra by an analytic function. In these cases the Lie subalgebras of the solvable binary Lie algebras and the corresponding simply connected Lie groups have faithful linear representations. With their aid we can express the Baker-Campbell-Hausdorff series giving the analytic multiplication (2.1) in a closed form.

3. The anti-commutative $\mathfrak{bl}(\mu)$-algebras

In the following the multiplication $(\mathbf{x}, \mathbf{y}) \mapsto [\mathbf{x}, \mathbf{y}]$ of anti-commutative algebras will be given by the non-vanishing multiplication relations with respect to a suitable basis.

Proposition 3.1. *Let* $\mathfrak{bl}(\mu)$ *be the anti-commutative algebra defined on* $\mathbb{R}^{n+1} = \mathbb{R}^1 \oplus \mathbb{R}^2 \oplus \mathbb{R}^{n-3} \oplus \mathbb{R}^1$ *by the multiplication*

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_n,\ [\mathbf{e}_0, \mathbf{e}_i] = \mathbf{e}_i,\ \ i = 1, \dots, n-1,\ [\mathbf{e}_0, \mathbf{e}_n] = \mu \mathbf{e}_n, \quad \mu \in \mathbb{R},\ n \geq 3.$$

The algebra $\mathfrak{bl}(\mu)$ *is a*

- *(a) semidirect sum* $\mathfrak{aff}(\mathbb{R}) \oplus_l \mathfrak{i}$ *with respect to the bilinear map* $l : \mathfrak{aff}(\mathbb{R}) \times \mathfrak{i} \to \mathfrak{i}$, *where* $\mathfrak{aff}(\mathbb{R})$ *is the 2-dimensional non-abelian Lie algebra with multiplication* $\mathbf{e}_0\mathbf{e}_1 = \mathbf{e}_1$ *and* $\mathfrak{i} = \langle \mathbf{e}_2, \cdots, \mathbf{e}_n \rangle$ *is an abelian algebra,*
- *(b) binary Lie algebra for any* $\mu \in \mathbb{R}$, *which is a Malcev algebra if* $\mu \in \{2, -1\}$ *and Lie algebra if* $\mu = 2$.

Proof. Let $\xi = \xi_0\mathbf{e}_0 + \xi_1\mathbf{e}_1,\ \eta = \eta_0\mathbf{e}_0 + \eta_1\mathbf{e}_1,\ \zeta = \zeta_0\mathbf{e}_0 + \zeta_1\mathbf{e}_1 \in \mathfrak{aff}(\mathbb{R})$ be the decomposition of the vectors $\xi, \eta, \zeta \in \mathfrak{aff}(\mathbb{R})$ and denote by $l_i = l_{\mathbf{e}_i},\ i = 0, 1$ the map induced on the abelian ideal $\mathfrak{i}$ by the left multiplication $L_{(\mathbf{e}_i, \mathbf{0})} : \mathfrak{aff}(\mathbb{R}) \oplus_l \mathfrak{i} \to \mathfrak{aff}(\mathbb{R}) \oplus_l \mathfrak{i}$ for $i = 0, 1$. The maps l_0, l_1 are given by the matrices

$$l_0 = \begin{bmatrix} 1\ 0\ 0 \cdots 0 \\ 0\ 1\ 0 \cdots 0 \\ \vdots \\ 0\ 0\ 0 \cdots \mu \end{bmatrix}, \quad l_1 = \begin{bmatrix} 0\ 0\ 0 \cdots 0 \\ 0\ 0\ 0 \cdots 0 \\ \vdots \\ 1\ 0\ 0 \cdots 0 \end{bmatrix}.$$

According to Theorem 10.5. in [6] a semidirect sum $\mathfrak{aff}(\mathbb{R}) \oplus_l \mathfrak{i}$ is a

(a) binary Lie algebra if and only if

$$l_0l_1l_0 + l_1 - l_1l_0^2 - l_0l_1 = 0, \quad l_0l_1^2 - l_1l_0l_1 - l_1^2 = 0,$$

(b) Malcev algebra if and only if it is binary Lie and

$$l_1l_0^2 - l_0^2l_1 + l_0l_1 + l_1l_0 = 0, \quad l_1^2l_0 - l_1l_0l_1 + l_1^2 = 0,$$

(c) Lie algebra if and only if $l_1 = l_0l_1 - l_1l_0$.

Computing

$$l_0l_1 = l_0l_1l_0 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \\ \mu & 0 & 0 & \cdots & 0 \end{bmatrix}, \; l_1 = l_1l_0 = l_1l_0^2 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

we get from $l_0l_1^2 = l_1l_0l_1 = l_1^2 = 0$, that $\mathfrak{bl}(\mu)$ is binary Lie algebra for any $\mu \in \mathbb{R}$. Moreover, the equations $l_1l_0^2 - l_0^2l_1 + l_0l_1 + l_1l_0 = 2l_1 - l_0^2l_1 + l_0l_1 = 0$ and $l_1l_0l_1 = l_1^2 = 0$ give the condition $\mu^2 - \mu = 2$. Hence $\mathfrak{bl}(\mu)$ is a Malcev algebra if and only if $\mu \in \{2, -1\}$. In particular, $\mathfrak{bl}(\mu)$ is a Lie algebra if and only if $\mu = 2$. □

4. Lie subalgebras of $\mathfrak{bl}(\mu)$-algebras

In the following we investigate Lie subalgebras in $\mathfrak{bl}(\mu)$-algebras generated by 2-frames $(\mathbf{x}, \mathbf{y})$. We denote by $St_2(\mathfrak{bl}(\mu))$ the Stiefel manifold of 2-frames in $\mathfrak{bl}(\mu)$. For any 2-frame $(\mathbf{x}, \mathbf{y}) \in St_2(\mathfrak{bl}(\mu))$ the generated subalgebra $\mathfrak{h}(\mathbf{x}, \mathbf{y})$ is 2- or 3-dimensional solvable Lie algebra. Let $\mathbf{x} = (\xi, X_1, X_2, x)$ and $\mathbf{y} = (\eta, Y_1, Y_2, y)$ be the vectors consisting of components $\xi, x, \eta, y \in \mathbb{R}$, $X_1, Y_1 \in \mathbb{R}^2$, $X_2, Y_2 \in \mathbb{R}^{n-3}$ with respect to the decomposition $\mathbb{R}^{n+1} = \mathbb{R}^1 \oplus \mathbb{R}^2 \oplus \mathbb{R}^{n-3} \oplus \mathbb{R}^1$.

Lemma 4.1. *The Lie subalgebra $\mathfrak{h}(\mathbf{x}, \mathbf{y})$ for $(\mathbf{x}, \mathbf{y}) \in St_2(\mathfrak{bl}(\mu))$ is isomorphic to the*

(a) abelian algebra $\mathfrak{a}_2$: $[\mathbf{e}, \mathbf{f}] = \mathbf{0}$, if $\xi = \eta = [X_1, Y_1] = 0$,
(b) nilpotent Heisenberg algebra $\mathfrak{h}_3$: $[\mathbf{e}, \mathbf{f}] = \mathbf{g}$, if $\xi = \eta = 0$, $[X_1, Y_1] \neq 0$,
(c) 2-dimensional non-abelian Lie algebra $\mathfrak{aff}(\mathbb{R})$: $[\mathbf{e}, \mathbf{f}] = \mathbf{f}$, if $(\xi, \eta) \neq (0, 0)$ and $(\mu - 1)(\xi y - \eta x) + [X_1, Y_1] = 0$,
(d) 3-dimensional non-nilpotent solvable Lie algebra $\mathfrak{l}_\mu$ given by

$$\begin{aligned} &[\mathbf{e}, \mathbf{f}] = \mathbf{f}, \; [\mathbf{e}, \mathbf{g}] = \mu\mathbf{g}, \qquad \text{for } \mu \neq 1, \\ &[\mathbf{e}, \mathbf{f}] = \mathbf{f} + \mathbf{g}, \; [\mathbf{e}, \mathbf{g}] = \mathbf{g}, \quad \text{for } \mu = 1, \end{aligned} \tag{4.1}$$

if $(\xi, \eta) \neq (0, 0)$ and $(\mu - 1)(\xi y - \eta x) + [X_1, Y_1] \neq 0$.

Proof. Let $\mathbf{x} = (\xi, X_1, X_2, x)$, $\mathbf{y} = (\eta, Y_1, Y_2, y)$ be linearly independent elements of $\mathfrak{bl}(\mu)$ and denote $\mathbf{z}(\mathbf{x}, \mathbf{y}) = (0, 0, 0, (\mu-1)(\xi y - \eta x) + [X_1, Y_1])$ satisfying $\mathbf{z}(\mathbf{y}, \mathbf{x}) = -\mathbf{z}(\mathbf{x}, \mathbf{y})$. One has

$$[\mathbf{x}, \mathbf{y}] = [(\xi, X_1, X_2, x), (\eta, Y_1, Y_2, y)] =$$
$$= (0, \xi Y_1 - \eta X_1, \xi Y_2 - \eta X_2, \mu(\xi y - \eta x) + [X_1, Y_1]) = \xi \mathbf{y} - \eta \mathbf{x} + \mathbf{z}(\mathbf{x}, \mathbf{y}).$$

If $\xi = \eta = 0$, then $\mathfrak{h}(\mathbf{x}, \mathbf{y})$ is a Lie subalgebra of the commutator algebra of $\mathfrak{bl}(\mu)$. Moreover, it is a subalgebra of the Heisenberg algebra $\mathfrak{h}_3$ defined by $[\mathbf{e}, \mathbf{f}] = \mathbf{g}$ with respect to a basis $(\mathbf{e}, \mathbf{f}, \mathbf{g})$ of $\mathfrak{h}_3$. If $\mathfrak{h}(\mathbf{x}, \mathbf{y})$ is two-dimensional, then we obtain the case (a) and if three-dimensional, then we get (b). Now, we consider the case $\xi^2 + \eta^2 \neq 0$ and $\mu \neq 1$. Assume $\xi \neq 0$. The vectors

$$\mathbf{g} = \mathbf{z}(\mathbf{x}, \mathbf{y}) = (0, 0, 0, (\mu - 1)(\xi y - \eta x) + [X_1, Y_1]), \mathbf{e} = \frac{1}{\xi}\mathbf{x} = \frac{1}{\xi}(\xi, X_1, X_2, x),$$
$$\mathbf{f} = \xi \mathbf{y} - \eta \mathbf{x} + \frac{1}{1-\mu}\mathbf{z}(\mathbf{x}, \mathbf{y}) = \big(0, \xi Y_1 - \eta X_1, \xi Y_2 - \eta X_2, \frac{1}{1-\mu}[X_1, Y_1]\big) \tag{4.2}$$

satisfy $[\mathbf{e}, \mathbf{g}] = \frac{1}{\xi}[(\xi, X_1, X_2, x), (0, 0, 0, (\mu - 1)(\xi y - \eta x) + [X_1, Y_1])] = \mu \mathbf{g}$, $[\mathbf{e}, \mathbf{f}] = \left[\frac{1}{\xi}\mathbf{x}, \xi \mathbf{y} - \eta \mathbf{x} + \frac{1}{1-\mu}\mathbf{z}(\mathbf{x}, \mathbf{y})\right] = \xi \mathbf{y} - \eta \mathbf{x} + \mathbf{z}(\mathbf{x}, \mathbf{y}) + \frac{\mu}{1-\mu}\mathbf{z}(\mathbf{x}, \mathbf{y}) = \mathbf{f}$, or shortly

$$[\mathbf{e}, \mathbf{f}] = \mathbf{f}, \quad [\mathbf{e}, \mathbf{g}] = \mu \mathbf{g}. \tag{4.3}$$

If $\eta \neq 0$, changing the vectors $\mathbf{x} \leftrightarrow \mathbf{y}$ in the equations (4.2), we get

$$\mathbf{g} = \mathbf{z}(\mathbf{y}, \mathbf{x}) = (0, 0, 0, (1 - \mu)(\xi y - \eta x) - [X_1, Y_1]), \mathbf{e} = \frac{1}{\eta}\mathbf{y} = \frac{1}{\eta}(\eta, Y_1, Y_2, y),$$
$$\mathbf{f} = -\xi \mathbf{y} + \eta \mathbf{x} + \frac{1}{1-\mu}\mathbf{z}(\mathbf{y}, \mathbf{x}) = -\big(0, \xi Y_1 - \eta X_1, \xi Y_2 - \eta X_2, \frac{1}{1-\mu}[X_1, Y_1]\big).$$

It follows $[\mathbf{e}, \mathbf{g}] = \frac{1}{\eta}[(\eta, Y_1, Y_2, y), (0, 0, 0, (1 - \mu)(\xi y - \eta x) - [X_1, Y_1])] = \mu \mathbf{g}$, $[\mathbf{e}, \mathbf{f}] = \left[\frac{1}{\eta}\mathbf{y}, -\xi \mathbf{y} + \eta \mathbf{x} + \frac{1}{1-\mu}\mathbf{z}(\mathbf{y}, \mathbf{x})\right] = -\xi \mathbf{y} + \eta \mathbf{x} + \mathbf{z}(\mathbf{y}, \mathbf{x}) + \frac{\mu}{1-\mu}\mathbf{z}(\mathbf{y}, \mathbf{x}) = \mathbf{f}$. In the case $\xi = 0$ these equations yield

$$\begin{aligned} \mathbf{g} &= \mathbf{z}(\mathbf{y}, \mathbf{x}) = (0, 0, 0, (\mu - 1)\eta x - [X_1, Y_1]), \mathbf{e} = \frac{1}{\eta}\mathbf{y} = \frac{1}{\eta}(\eta, Y_1, Y_2, y), \\ \mathbf{f} &= \eta \mathbf{x} + \frac{1}{1-\mu}\mathbf{z}(\mathbf{y}, \mathbf{x}) = \big(0, \eta X_1, \eta X_2, -\frac{1}{1-\mu}[X_1, Y_1]\big). \end{aligned} \tag{4.4}$$

It follows for $\mathbf{z}(\mathbf{x}, \mathbf{y}) \neq \mathbf{0}$, that $\mathfrak{h}(\mathbf{x}, \mathbf{y})$ is isomorphic to the 3-dimensional non-nilpotent solvable Lie algebra $\mathfrak{l}_\mu$ given by (4.3).
Now, assume $\mu = 1$. Putting

$$\begin{aligned} \mathbf{e} &= \frac{1}{\xi}\mathbf{x}, \quad \mathbf{f} = \xi \mathbf{y} - \eta \mathbf{x}, \quad \mathbf{g} = \mathbf{z}(\mathbf{x}, \mathbf{y}), \quad \text{if} \quad \xi \neq 0, \\ \mathbf{e} &= \frac{1}{\eta}\mathbf{y}, \quad \mathbf{f} = \eta \mathbf{x}, \quad \mathbf{g} = \mathbf{z}(\mathbf{y}, \mathbf{x}), \quad \text{if} \quad \xi = 0, \ \eta \neq 0, \end{aligned} \tag{4.5}$$

we get the equations

$$[\mathbf{e},\mathbf{f}] = \mathbf{f}+\mathbf{g}, \quad [\mathbf{e},\mathbf{g}] = \mathbf{g}. \tag{4.6}$$

Hence for $\mathbf{z}(\mathbf{x},\mathbf{y}) \neq \mathbf{0}$ the subalgebra $\mathfrak{h}(\mathbf{x},\mathbf{y})$ is isomorphic to the 3-dimensional non-nilpotent solvable Lie algebra $\mathfrak{l}_1$ given by (4.6) proving assertion (d).
In the case $\mathbf{z}(\mathbf{x},\mathbf{y}) = \mathbf{g} = \mathbf{0}$, the previous formulas show that $\mathfrak{h}(\mathbf{x},\mathbf{y})$ is isomorphic to the two-dimensional non-abelian Lie algebra $\mathfrak{aff}(\mathbb{R})$. Hence (c) is true. □

Corollary 4.2. *The determination of Lie subalgebras generated by 2-frames of the binary Lie algebras $\mathfrak{bl}(\mu)$ shows that $\mathfrak{bl}(\mu)$ satisfies the condition of Lemma 2.1.*

Remark 4.3. The Lie algebras $\mathfrak{l}_\mu$, $0 \neq \mu \in \mathbb{R}$, are non-decomposable, $\mathfrak{l}_0$ is the direct sum $\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$. The Lie algebra $\mathfrak{l}_\mu$ is isomorphic to $\mathfrak{l}_\nu$ if and only if either $\mu = \nu$ or $\mu = \nu^{-1}$ (cf. [4], Ex. 3.2.).

Remark 4.4. The set of 2-frames $(\mathbf{x},\mathbf{y}) \in St_2(\mathfrak{bl}(\mu))$ generating a solvable Lie algebra isomorphic to $\mathfrak{l}_\mu$ is an open submanifold in the Stiefel manifold $St_2(\mathfrak{bl}(\mu))$.

5. Exponential maps $\mathfrak{l}_\mu \to \mathcal{L}_\mu$, $\mathfrak{aff}(\mathbb{R}) \to \mathbf{Aff}(\mathbb{R})$, $\mathfrak{h}_3 \to \mathbf{H}_3$

The real analytic function $\frac{e^t-1}{t}$ has a removable singularity at $t=0$, hence we can consider its analytic continuation $E:\mathbb{R}\to\mathbb{R}$ defined by the power series

$$E(t) = 1+\frac{t}{2}+\frac{t^2}{3!}+\cdots = \sum_{k=0}^{\infty}\frac{t^k}{(k+1)!} = \begin{cases} \frac{e^t-1}{t}, & t\neq 0 \\ 1, & t=0 \end{cases}, \quad t\in\mathbb{R}. \tag{5.1}$$

One has $E(t)>0$ for all $t\in\mathbb{R}$, since $e^t-1>0$ if $t>0$ and $e^t-1<0$ if $t<0$, moreover

$$E(s+t)(s+t) = e^tE(s)s+E(t)t = e^sE(t)t+E(s)s \quad \text{for all} \quad s,t\in\mathbb{R}, \tag{5.2}$$

$$E'(t) = \begin{cases} \frac{1}{t}(e^t-E(t)), & t\neq 0, \\ \frac{1}{2}, & t=0. \end{cases} \tag{5.3}$$

We consider the matrices

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathbf{F} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$[\mathbf{E},\mathbf{F}] = \mathbf{F}, \ [\mathbf{E},\mathbf{G}] = \mu\mathbf{G}, \ [\mathbf{F},\mathbf{G}] = \mathbf{0} \quad \text{for } \mu\neq 1$$

and

$$\mathbf{E} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathbf{F} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$[\mathbf{E},\mathbf{F}]=\mathbf{F}+\mathbf{G},\ [\mathbf{E},\mathbf{G}]=\mathbf{G},\ [\mathbf{F},\mathbf{G}]=\mathbf{0}\quad \text{for } \mu=1.$$

The Lie algebras $\mathfrak{l}_\mu$, $\mu\in\mathbb{R}$, have the linear representations

$$\mathfrak{l}_\mu \cong \left\{ \begin{bmatrix} x & 0 & y\\ 0 & \mu x & z\\ 0 & 0 & 0\end{bmatrix}; x,y,z\in\mathbb{R}\right\},\ \mu\neq 1,\ \mathfrak{l}_1 \cong \left\{ \begin{bmatrix} x & x & z\\ 0 & x & y\\ 0 & 0 & 0\end{bmatrix}; x,y,z\in\mathbb{R}\right\}. \tag{5.4}$$

The corresponding simply connected matrix Lie groups $\mathcal{L}_\mu$, $\mu\in\mathbb{R}$, are

$$\begin{aligned}\mathcal{L}_\mu &\cong \left\{ \begin{bmatrix} e^x & 0 & y\\ 0 & e^{\mu x} & z\\ 0 & 0 & 1\end{bmatrix}; x,y,z\in\mathbb{R}\right\},\ \mu\neq 1,\\ \mathcal{L}_1 &\cong \left\{ \begin{bmatrix} e^x & xe^x & z\\ 0 & e^x & y\\ 0 & 0 & 1\end{bmatrix}; x,y,z\in\mathbb{R}\right\}.\end{aligned} \tag{5.5}$$

Proposition 5.1. *Identifying the Lie algebras $\mathfrak{l}_\mu$ and the Lie groups $\mathcal{L}_\mu$, $\mu\in\mathbb{R}$, with their linear representations (5.4) and (5.5), the exponential map* $\exp : \mathfrak{l}_\mu \to \mathcal{L}_\mu$ *is determined by the one-parameter subgroups*

$$\begin{aligned}\exp\left(\begin{bmatrix} x & 0 & y\\ 0 & \mu x & z\\ 0 & 0 & 0\end{bmatrix} t\right) &= \begin{bmatrix} e^{xt} & 0 & E(xt)yt\\ 0 & e^{\mu xt} & E(\mu xt)zt\\ 0 & 0 & 1\end{bmatrix},\ \textit{if}\ \mu\neq 1,\\ \exp\left(\begin{bmatrix} x & x & z\\ 0 & x & y\\ 0 & 0 & 0\end{bmatrix} t\right) &= \begin{bmatrix} e^{xt} & xte^{xt} & (e^{xt}-E(xt))yt+E(xt)zt\\ 0 & e^{xt} & E(xt)yt\\ 0 & 0 & 1\end{bmatrix},\ \textit{if}\ \mu=1,\end{aligned} \tag{5.6}$$

where $t\in\mathbb{R}$ and $x,y,z\in\mathbb{R}$.

Proof. For $\mu\neq 1$ it follows from the equations

$$\begin{aligned} e^{xs}E(xt)yt+E(xs)ys &= E(x(s+t))y(s+t),\\ e^{\mu xs}E(\mu xt)yt+E(\mu xs)ys &= E(\mu x(s+t))y(s+t),\end{aligned}$$

that for all $t,s\in\mathbb{R}$ one has $\exp(\mathbf{x}t)\exp(\mathbf{x}s)=\exp(\mathbf{x}(t+s))$, $\mathbf{x}\in\mathfrak{l}_\mu$, and hence the image set $\{\exp(\mathbf{x}t); t\in\mathbb{R}\}$ given by (5.6) is a one-parameter subgroup. An analogous computation yields the one-parameter subgroups for $\mu=1$. □

The Lie algebra $\mathfrak{aff}(\mathbb{R})$ and the corresponding simply connected Lie group $\mathrm{Aff}(\mathbb{R})$ have the linear representation

$$\mathfrak{aff}(\mathbb{R})\cong\left\{\begin{bmatrix} x & y\\ 0 & 0\end{bmatrix}; x,y\in\mathbb{R}\right\},\quad \mathrm{Aff}(\mathbb{R})\cong\left\{\begin{bmatrix} e^x & y\\ 0 & 1\end{bmatrix}; x,y\in\mathbb{R}\right\}. \tag{5.7}$$

Lemma 5.2. *Identifying* $\mathfrak{aff}(\mathbb{R})$ *and* $Aff(\mathbb{R})$ *with their linear representation* (5.7) *the exponential map* $\exp : \mathfrak{aff}(\mathbb{R}) \to Aff(\mathbb{R})$ *is determined by the one-parameter subgroups*

$$\exp\left(\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} t\right) = \begin{bmatrix} e^{xt} & E(xt)yt \\ 0 & 1 \end{bmatrix}, \; t \in \mathbb{R}, \quad \textit{for any } x, y \in \mathbb{R}.$$

The Heisenberg Lie algebra $\mathfrak{h}_3$ and the corresponding simply connected Heisenberg group H_3 have the linear representation

$$\mathfrak{h}_3 \cong \left\{ \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix}; x, y, z \in \mathbb{R} \right\}, \quad \mathrm{H}_3 \cong \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}; x, y, z \in \mathbb{R} \right\}.$$

Lemma 5.3. *The exponential map* $\exp : \mathfrak{h}_3 \to H_3$ *is determined by the one-parameter subgroups*

$$\exp\left(\begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} t\right) = \begin{bmatrix} 1 & xt & zt + \frac{1}{2}xyt^2 \\ 0 & 1 & yt \\ 0 & 0 & 1 \end{bmatrix}, \; t \in \mathbb{R}, \quad \textit{for any } x, y, z \in \mathbb{R}. \tag{5.8}$$

Corollary 5.4. *The exponential maps* $\exp : \mathfrak{l}_\mu \to \mathcal{L}_\mu$, $\exp : \mathfrak{aff}(\mathbb{R}) \to Aff(\mathbb{R})$, $\exp : \mathfrak{h}_3 \to H_3$ *are diffeomorphisms.*

6. Multiplication in the Lie algebras $\mathfrak{l}_\mu$, $\mathfrak{aff}(\mathbb{R})$, $\mathfrak{h}_3$

Since the exponential map $\exp : \mathfrak{l}_\mu \to \mathcal{L}_\mu$ is a bijective map, the multiplication

$$(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} \circ \mathbf{v} = \exp^{-1}(\exp(\mathbf{u}) \cdot \exp(\mathbf{v})), \; \mathbf{u}, \mathbf{v} \in \mathfrak{l}_\mu, \tag{6.1}$$

on $\mathfrak{l}_\mu$ defines a Lie group isomorphic to $\mathcal{L}_\mu$.
Let $\mathbf{e}, \mathbf{f}, \mathbf{g}$ be a fixed basis in $\mathfrak{l}_\mu$ satisfying (4.1).

We denote

$$\mathbf{u} = x_1\mathbf{e} + y_1\mathbf{f} + z_1\mathbf{g} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad \mathbf{v} = x_2\mathbf{e} + y_2\mathbf{f} + z_2\mathbf{g} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}. \tag{6.2}$$

Proposition 6.1. *The multiplication* (6.1) *in the Lie algebra* $\mathfrak{l}_\mu$ *is expressed by*

$$\mathbf{u} \circ \mathbf{v} = \begin{bmatrix} x_1 + x_2 \\ \frac{e^{x_1} E(x_2)y_2 + E(x_1)y_1}{E(x_1+x_2)} \\ \frac{e^{\mu x_1} E(\mu x_2)z_2 + E(\mu x_1)z_1}{E(\mu(x_1+x_2))} \end{bmatrix}, \quad \textit{if } \mu \neq 1, \tag{6.3}$$

$$\mathbf{u}\circ\mathbf{v}=\begin{bmatrix} x_1+x_2 \\ \frac{E(x_1)y_1+e^{x_1}E(x_2)y_2}{E(x_1+x_2)} \\ \frac{E(x_1)z_1+e^{x_1}E(x_2)z_2}{E(x_1+x_2)}+\frac{e^{x_1}}{E(x_1+x_2)}\left(1-\frac{e^{x_2}E(x_1)}{E(x_1+x_2)}\right)y_1+ \\ +\left[\frac{e^{x_1+x_2}}{E(x_1+x_2)}\left(1-\frac{e^{x_1}E(x_2)}{E(x_1+x_2)}\right)+\frac{x_1e^{x_1}E(x_2)}{E(x_1+x_2)}\right]y_2 \end{bmatrix},\ \textit{if}\quad \mu=1. \tag{6.4}$$

Proof. We consider the correspondence between the bases $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ of $\mathfrak{l}_\mu$ and $\{\mathbf{E},\mathbf{F},\mathbf{G}\}$ of its linear representation. In the case $\mu\neq 1$ we get the natural identification $\begin{bmatrix} x\\ y\\ z\end{bmatrix}\mapsto\begin{bmatrix} x & 0 & y\\ 0 & \mu x & z\\ 0 & 0 & 0\end{bmatrix}$. Since we have

$$\exp\left(\begin{bmatrix} x_1 & 0 & y_1\\ 0 & \mu x_1 & z_1\\ 0 & 0 & 0\end{bmatrix}\right)\exp\left(\begin{bmatrix} x_2 & 0 & y_2\\ 0 & \mu x_2 & z_2\\ 0 & 0 & 0\end{bmatrix}\right)=$$
$$=\exp\left(\begin{bmatrix} x_1+x_2 & 0 & \frac{e^{x_1}E(x_2)y_2+E(x_1)y_1}{E(x_1+x_2)}\\ 0 & \mu(x_1+x_2) & \frac{e^{\mu x_1}E(\mu x_2)z_2+E(\mu x_1)z_1}{E(\mu(x_1+x_2))}\\ 0 & 0 & 0\end{bmatrix}\right),$$

the assertion follows.

If $\mu=1$, using the identification $x\mathbf{e}+y\mathbf{f}+z\mathbf{g}\mapsto\begin{bmatrix} x\\ y\\ z\end{bmatrix}\mapsto\begin{bmatrix} x & x & z\\ 0 & x & y\\ 0 & 0 & 0\end{bmatrix}$, we obtain

$$\exp\left(\begin{bmatrix} x_1 & x_1 & z_1\\ 0 & x_1 & y_1\\ 0 & 0 & 0\end{bmatrix}\right)\exp\left(\begin{bmatrix} x_2 & x_2 & z_2\\ 0 & x_2 & y_2\\ 0 & 0 & 0\end{bmatrix}\right)=$$
$$=\exp\left(\begin{bmatrix} x_1+x_2 & x_1+x_2 & \begin{array}{c}\frac{E(x_1)z_1+e^{x_1}E(x_2)z_2}{E(x_1+x_2)}+\\ +\frac{e^{x_1}}{E(x_1+x_2)}\left(1-\frac{e^{x_2}E(x_1)}{E(x_1+x_2)}\right)y_1+\\ +\left[\frac{e^{x_1+x_2}}{E(x_1+x_2)}\left(1-\frac{e^{x_1}E(x_2)}{E(x_1+x_2)}\right)+\frac{x_1e^{x_1}E(x_2)}{E(x_1+x_2)}\right]y_2\end{array}\\ 0 & x_1+x_2 & \frac{E(x_1)y_1+e^{x_1}E(x_2)y_2}{E(x_1+x_2)}\\ 0 & 0 & 0\end{bmatrix}\right).$$

This proves the assertion. □

Since the Lie algebra $\mathfrak{aff}(\mathbb{R})$ is isomorphic to the factor algebra $\mathfrak{l}_\mu/\mathfrak{i}$, where $\mathfrak{i}$ is the ideal $\mathfrak{i}=\mathbb{R}\mathbf{g}$, we get:

Lemma 6.2. *The multiplication* (6.1) *in the Lie algebra* $\mathfrak{aff}(\mathbb{R})$ *is expressed by*

$$\mathbf{u}\circ\mathbf{v}=\begin{bmatrix} x_1\\ y_1\end{bmatrix}\circ\begin{bmatrix} x_2\\ y_2\end{bmatrix}=\begin{bmatrix} x_1+x_2\\ \frac{e^{x_1}E(x_2)y_2+E(x_1)y_1}{E(x_1+x_2)}\end{bmatrix}=(x_1+x_2)\mathbf{e}+\frac{e^{x_1}E(x_2)y_2+E(x_1)y_1}{E(x_1+x_2)}\mathbf{f},$$

where $\{\mathbf{e},\mathbf{f}\}$ *is a basis in* $\mathfrak{aff}(\mathbb{R})$ *satisfying* $[\mathbf{e},\mathbf{f}]=\mathbf{f}$.

In the three-dimensional Heisenberg matrix group the multiplication is given by

$$\begin{bmatrix}1 & x_1 & z_1\\ 0 & 1 & y_1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & x_2 & z_2\\ 0 & 1 & y_2\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & x_1+x_2 & z_1+x_1y_2+z_2\\ 0 & 1 & y_1+y_2\\ 0 & 0 & 1\end{bmatrix}.$$

So using the formula (5.8) for the exponential map we have:

Lemma 6.3. *The multiplication* (6.1) *in the Heisenberg algebra* $\mathfrak{h}_3$ *is expressed by*

$$\mathbf{u}\circ\mathbf{v} = \begin{bmatrix}x_1\\ y_1\\ z_1\end{bmatrix}\circ\begin{bmatrix}x_2\\ y_2\\ z_2\end{bmatrix} = \begin{bmatrix}x_1+x_2\\ y_1+y_2\\ z_1+z_2+\frac{1}{2}(x_1y_2-x_2y_1)\end{bmatrix} =$$
$$= (x_1+x_2)\mathbf{e} + (y_1+y_2)\mathbf{f} + \left(z_1+z_2+\frac{1}{2}(x_1y_2-y_1x_2)\right)\mathbf{g},$$

where $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ *is a basis in* $\mathfrak{h}_3$ *satisfying* $[\mathbf{e},\mathbf{f}]=\mathbf{g}$, $[\mathbf{e},\mathbf{g}]=[\mathbf{f},\mathbf{g}]=\mathbf{0}$.

7. Multiplication in the Lie subalgebra $\mathfrak{h}(\mathbf{u},\mathbf{v})$

Let be $\xi,\eta\in\mathbb{R}$. Using the function $E:\mathbb{R}\to\mathbb{R}$ introduced in (5.1) we define

$$F(\xi,\eta) = \frac{1}{\eta}\sum_{k=1}^{\infty}\frac{(\xi+\eta)^k-\xi^k}{(k+1)!} = \begin{cases}\frac{E(\xi+\eta)-E(\xi)}{\eta}, & \eta\neq 0\\ E'(\xi), & \eta=0\end{cases},\quad G(\xi,\eta)=\frac{F(\xi,\eta)}{E(\xi+\eta)}.$$

The function $F(\xi,\eta)$ is obviously analytic on $\mathbb{R}^2$, and since $E(t)\neq 0$ for all $t\in\mathbb{R}$, then so $G(\xi,\eta)$ is analytic too. Using (5.3) one has

$$F(0,\eta) = \begin{cases}\frac{E(\eta)-1}{\eta} = \frac{e^{\eta}-\eta-1}{\eta^2}, & \eta\neq 0\\ \frac{1}{2}, & \eta=0\end{cases},\qquad F(\xi,0)=E'(\xi)$$

and

$$G(0,\eta) = \frac{F(0,\eta)}{E(\eta)} = \begin{cases}\frac{e^{\eta}-1-\eta}{\eta(e^{\eta}-1)}, & \eta\neq 0\\ \frac{1}{2}, & \eta=0\end{cases},$$
$$G(\xi,0) = \frac{F(\xi,0)}{E(\xi)} = \begin{cases}\frac{\xi e^{\xi}-e^{\xi}+1}{\xi(e^{\xi}-1)}, & \xi\neq 0\\ \frac{1}{2}, & \xi=0\end{cases}.$$

Theorem 7.1. *The product* $\mathbf{x}\circ\mathbf{y}$ *of the vectors* $\mathbf{x}=(\xi,X_1,X_2,x)$, $\mathbf{y}=(\eta,Y_1,Y_2,y)$ *in* $\mathfrak{h}(\mathbf{x},\mathbf{y})$ *is expressed by*

$$\begin{aligned}\mathbf{x}\circ\mathbf{y} =& \exp^{-1}(\exp(\mathbf{x})\cdot\exp(\mathbf{y})) =\\ &=(1-\eta\,G(\xi,\eta))\mathbf{x} + (1+\xi\,G(\xi,\eta))\,\mathbf{y} +\\ &+\begin{cases}\frac{1}{1-\mu}\left(G(\xi,\eta)-\mu G(\mu\xi,\mu\eta)\right)\mathbf{z}(\mathbf{x},\mathbf{y}), & \mu\neq 1,\\ \left(1+\xi\,G(\xi,\eta)-\frac{e^{\xi+\eta}G(\xi,\eta)}{E(\xi+\eta)}\right)\mathbf{z}(\mathbf{x},\mathbf{y}), & \mu=1.\end{cases}\end{aligned}$$

Proof. First, we consider the case $\mu \neq 1$.
Assume $(\xi,\eta) \neq (0,0)$ and $\mathbf{z}(\mathbf{x},\mathbf{y}) = (0,0,0,(\mu-1)(\xi y - \eta x) + [X_1, Y_1]) \neq \mathbf{0}$.
If $\xi \neq 0$, then using the notations of (4.2) and (6.2) we get

$$\mathbf{x} = \xi\mathbf{e} = \begin{bmatrix}\xi\\0\\0\end{bmatrix}, \quad \mathbf{y} = \eta\mathbf{e} + \frac{1}{\xi}\mathbf{f} - \frac{1}{(1-\mu)\xi}\mathbf{g} = \begin{bmatrix}\eta\\ \frac{1}{\xi}\\ \frac{-1}{(1-\mu)\xi}\end{bmatrix}.$$

From (6.3) we obtain

$$\mathbf{x}\circ\mathbf{y} = \begin{bmatrix}\xi+\eta\\ \frac{e^{\xi}E(\eta)}{\xi E(\xi+\eta)}\\ \frac{-e^{\mu\xi}E(\mu\eta)}{(1-\mu)\xi E(\mu(\xi+\eta))}\end{bmatrix} = (\xi+\eta)\mathbf{e} + \frac{e^{\xi}E(\eta)}{\xi E(\xi+\eta)}\mathbf{f} - \frac{e^{\mu\xi}E(\mu\eta)}{(1-\mu)\xi E(\mu(\xi+\eta))}\mathbf{g}.$$

Since $\mathbf{e} = \frac{1}{\xi}\mathbf{x}$, $\mathbf{f} = \xi\mathbf{y} - \eta\mathbf{x} + \frac{1}{1-\mu}\mathbf{z}(\mathbf{x},\mathbf{y})$, $\mathbf{g} = \mathbf{z}(\mathbf{x},\mathbf{y})$, we receive

$$\begin{aligned}\mathbf{x}\circ\mathbf{y} &= (\xi+\eta)\frac{1}{\xi}\mathbf{x} + \frac{e^{\xi}E(\eta)}{\xi E(\xi+\eta)}(\xi\mathbf{y} - \eta\mathbf{x} + \frac{1}{1-\mu}\mathbf{z}(\mathbf{x},\mathbf{y})) - \\ &\quad - \frac{e^{\mu\xi}E(\mu\eta)}{(1-\mu)\xi E(\mu(\xi+\eta))}\mathbf{z}(\mathbf{x},\mathbf{y}) = \frac{E(\xi+\eta)(\xi+\eta) - e^{\xi}E(\eta)\eta}{\xi E(\xi+\eta)}\mathbf{x} + \\ &\quad + \frac{e^{\xi}E(\eta)}{E(\xi+\eta)}\mathbf{y} + \frac{1}{(1-\mu)\xi}\left(\frac{e^{\xi}E(\eta)}{E(\xi+\eta)} - \frac{e^{\mu\xi}E(\mu\eta)}{E(\mu(\xi+\eta))}\right)\mathbf{z}(\mathbf{x},\mathbf{y}).\end{aligned}$$

According to the identity (5.2) we get

$$\begin{aligned}\mathbf{x}\circ\mathbf{y} &= \frac{E(\xi)\xi}{\xi E(\xi+\eta)}\mathbf{x} + \frac{E(\xi+\eta)(\xi+\eta) - E(\xi)\xi}{\eta E(\xi+\eta)}\mathbf{y} + \\ &\quad + \frac{1}{(1-\mu)\xi}\left(\frac{E(\xi+\eta)(\xi+\eta) - E(\xi)\xi}{\eta E(\xi+\eta)}\right)\mathbf{z}(\mathbf{x},\mathbf{y}) - \\ &\quad - \frac{1}{(1-\mu)\xi}\left(\frac{E(\mu(\xi+\eta))\mu(\xi+\eta) - E(\mu\xi)\mu\xi}{\mu\eta E(\mu(\xi+\eta))}\right)\mathbf{z}(\mathbf{x},\mathbf{y}) = \\ &= \left(1 - \eta\frac{E(\xi+\eta) - E(\xi)}{\eta E(\xi+\eta)}\right)\mathbf{x} + \left(1 + \xi\frac{E(\xi+\eta) - E(\xi)}{\eta E(\xi+\eta)}\right)\mathbf{y} + \\ &\quad + \frac{1}{1-\mu}\left(\frac{E(\xi+\eta) - E(\xi)}{\eta E(\xi+\eta)} - \mu\frac{E(\mu(\xi+\eta)) - E(\mu\xi)}{\mu\eta E(\mu(\xi+\eta))}\right)\mathbf{z}(\mathbf{x},\mathbf{y}),\end{aligned}$$

hence we obtain $\mathbf{x}\circ\mathbf{y} =$

$$= (1 - \eta\, G(\xi,\eta))\mathbf{x} + (1 + \xi\, G(\xi,\eta))\,\mathbf{y} + \frac{1}{1-\mu}\left(G(\xi,\eta) - \mu G(\mu\xi,\mu\eta)\right)\mathbf{z}(\mathbf{x},\mathbf{y}). \tag{7.1}$$

If $\xi = 0$ and $\eta \neq 0$, then equations (4.4) yield $\mathbf{e} = \frac{1}{\eta}\mathbf{y}$, $\mathbf{f} = \eta\mathbf{x} + \frac{1}{1-\mu}\mathbf{z}(\mathbf{y},\mathbf{x})$, $\mathbf{g} = \mathbf{z}(\mathbf{y},\mathbf{x}) = -\mathbf{z}(\mathbf{x},\mathbf{y})$, $\mathbf{x} = \frac{1}{\eta}\mathbf{f} - \frac{1}{(1-\mu)\eta}\mathbf{g}$, $\mathbf{y} = \eta\mathbf{e}$. Using (6.3) we can express

$$\begin{aligned}\mathbf{x}\circ\mathbf{y} &= \begin{bmatrix} 0 \\ \frac{1}{\eta} \\ -\frac{1}{(1-\mu)\eta}\end{bmatrix} \circ \begin{bmatrix} \eta \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \eta \\ \frac{1}{\eta E(\eta)} \\ -\frac{1}{(1-\mu)\eta E(\mu\eta)}\end{bmatrix} = \\ &= \mathbf{y} + \frac{1}{E(\eta)}\mathbf{x} + \frac{1}{(1-\mu)\eta}\left(\frac{E(\eta)-1}{E(\eta)} - \frac{E(\mu\eta)-1}{E(\mu\eta)}\right)\mathbf{z}(\mathbf{x},\mathbf{y}) = \\ &= (1-\eta\, G(0,\eta))\mathbf{x} + \mathbf{y} + \frac{1}{1-\mu}\left(G(0,\eta) - \mu G(0,\mu\eta)\right)\mathbf{z}(\mathbf{x},\mathbf{y}),\end{aligned}$$

giving (7.1) with $\xi = 0$. If $\mathbf{z}(\mathbf{x},\mathbf{y}) = \mathbf{0}$, then due to the isomorphism $\mathfrak{aff}(\mathbb{R}) \cong \mathfrak{l}_\mu/\mathbb{R}\mathbf{g}$, we get (7.1) with vanishing $\mathbf{z}(\mathbf{x},\mathbf{y})$. Consequently, the equation (7.1) is satisfied for any $(\xi,\eta) \neq (0,0)$.
If $(\xi,\eta) = (0,0)$, then $[\mathbf{x},\mathbf{y}] = [(0,X_1,X_2,x),(0,Y_1,Y_2,y)] = (0,0,0,[X_1,Y_1]) = \mathbf{z}(\mathbf{x},\mathbf{y})$. Denoting $\mathbf{e} = \mathbf{x}$, $\mathbf{f} = \mathbf{y}$, $\mathbf{g} = \mathbf{z}(\mathbf{x},\mathbf{y})$ we have

$$\mathbf{x}\circ\mathbf{y} = \begin{bmatrix}1\\0\\0\end{bmatrix} \circ \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\\frac{1}{2}\end{bmatrix} = \mathbf{x} + \mathbf{y} + \frac{1}{2}\mathbf{z}(\mathbf{x},\mathbf{y}). \tag{7.2}$$

We obtain (7.1) with $\xi = \eta = 0$, since $\frac{1}{1-\mu}\left(G(0,0) - \mu G(0,0)\right) = \frac{1}{2}$. Hence the assertion is proved for $\mu \neq 1$.
We consider the case $\mu = 1$. If $\xi \neq 0$, then it follows from (4.5) that $\mathbf{e} = \frac{1}{\xi}\mathbf{x}$, $\mathbf{f} = \xi\mathbf{y} - \eta\mathbf{x}$, $\mathbf{g} = \mathbf{z}(\mathbf{x},\mathbf{y})$, $\mathbf{x} = \xi\mathbf{e}$, $\mathbf{y} = \eta\mathbf{e} + \frac{1}{\xi}\mathbf{f}$. We obtain from (6.4) the equation

$$\begin{aligned}\mathbf{x}\circ\mathbf{y} &= \begin{bmatrix}\xi\\0\\0\end{bmatrix} \circ \begin{bmatrix}\eta\\\frac{1}{\xi}\\0\end{bmatrix} = \begin{bmatrix}\xi+\eta \\ \frac{e^{\xi}E(\eta)}{\xi E(\xi+\eta)} \\ \frac{e^{\xi+\eta}}{\xi E(\xi+\eta)}\left(1 - \frac{e^{\xi}E(\eta)}{E(\xi+\eta)}\right) + \frac{e^{\xi}E(\eta)}{E(\xi+\eta)}\end{bmatrix} = \\ &= (\xi+\eta)\mathbf{e} + \frac{e^{\xi}E(\eta)}{\xi E(\xi+\eta)}\mathbf{f} + \left(\frac{e^{\xi+\eta}}{\xi E(\xi+\eta)}\left(1 - \frac{e^{\xi}E(\eta)}{E(\xi+\eta)}\right) + \frac{e^{\xi}E(\eta)}{E(\xi+\eta)}\right)\mathbf{g} = \\ &= \frac{E(\xi+\eta)(\xi+\eta) - e^{\xi}E(\eta)\eta}{\xi E(\xi+\eta)}\mathbf{x} + \frac{e^{\xi}E(\eta)}{E(\xi+\eta)}\mathbf{y} + \\ &\quad + \left(\frac{e^{\xi+\eta}(E(\xi+\eta) - e^{\xi}E(\eta))}{\xi E(\xi+\eta)^2} + \frac{e^{\xi}E(\eta)}{E(\xi+\eta)}\right)\mathbf{z}(\mathbf{x},\mathbf{y}).\end{aligned}$$

Applying the identity (5.2) we get the coefficients of $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}(\mathbf{x},\mathbf{y})$, similarly as in (7.1):

$$\begin{aligned}\mathbf{x}\circ\mathbf{y} &= (1-\eta\, G(\xi,\eta))\mathbf{x} + (1+\xi\, G(\xi,\eta))\,\mathbf{y}+\\ &\quad+\left(\frac{e^{\xi+\eta}(E(\xi)-E(\xi+\eta))}{E(\xi+\eta)^2\eta}+1+\xi\, G(\xi,\eta)\right)\mathbf{z}(\mathbf{x},\mathbf{y}) =\\ &= (1-\eta\, G(\xi,\eta))\mathbf{x} + (1+\xi\, G(\xi,\eta))\,\mathbf{y}+\\ &\quad+\left(1+\xi\, G(\xi,\eta)-\frac{e^{\xi+\eta}}{E(\xi+\eta)}G(\xi,\eta)\right)\mathbf{z}(\mathbf{x},\mathbf{y}).\end{aligned} \tag{7.3}$$

If $\xi=0$ and $\eta\neq 0$, then equations (4.5) yield $\mathbf{e}=\frac{1}{\eta}\mathbf{y}$, $\mathbf{f}=\eta\mathbf{x}$, $\mathbf{g}=\mathbf{z}(\mathbf{y},\mathbf{x})=-\mathbf{z}(\mathbf{x},\mathbf{y})$, $\mathbf{x}=\frac{1}{\eta}\mathbf{f}$, $\mathbf{y}=\eta\mathbf{e}$ and from (6.4) it follows

$$\begin{aligned}\mathbf{x}\circ\mathbf{y} &= \begin{bmatrix}0\\ \frac{1}{\eta}\\ 0\end{bmatrix}\circ\begin{bmatrix}\eta\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}\eta\\ \frac{1}{\eta E(\eta)}\\ \frac{E(\eta)-e^{\eta}}{E(\eta)^2\eta}\end{bmatrix} = \mathbf{y}+\frac{1}{E(\eta)}\mathbf{x}+\frac{e^{\eta}-E(\eta)}{E(\eta)^2\eta}\mathbf{z}(\mathbf{x},\mathbf{y}) =\\ &= (1-\eta\, G(0,\eta))\mathbf{x}+\mathbf{y}+\left(1-\frac{e^{\eta}G(0,\eta)}{E(\eta)}\right)\mathbf{z}(\mathbf{x},\mathbf{y}).\end{aligned}$$

If $(\xi,\eta)\neq(0,0)$, $\mathbf{z}(\mathbf{x},\mathbf{y})=\mathbf{0}$, then repeating the previous computation, according to Lemma 6.2, we get

$$\mathbf{x}\circ\mathbf{y} = (1-\eta\, G(\xi,\eta))\mathbf{x} + (1+\xi\, G(\xi,\eta))\,\mathbf{y},$$

giving (7.3) for any $(\xi,\eta)\neq(0,0)$.
In the case $(\xi,\eta)=(0,0)$ the same consideration as in (7.2) gives that

$$\mathbf{x}\circ\mathbf{y} = \mathbf{x}+\mathbf{y}+\frac{1}{2}\mathbf{z}(\mathbf{x},\mathbf{y}) = \mathbf{x}+\mathbf{y}+(1-G(0,0))\mathbf{z}(\mathbf{x},\mathbf{y}).$$

Since $1-G(0,0)=\frac{1}{2}$, we obtain (7.3). Equations (7.1) and (7.3) completely prove Theorem 7.1. □

8. Global analytic diassociative loop on $\mathfrak{bl}(\mu)$

The multiplication formula given in Theorem 7.1 can be extended from the Lie subalgebra $\mathfrak{h}(\mathbf{x},\mathbf{y})$ to the complete binary Lie algebra $\mathfrak{bl}(\mu)$ giving an analytic expression for $\mathbf{x}\circ\mathbf{y}=\exp^{-1}_{\mathfrak{h}(\mathbf{x},\mathbf{y})}\left(\exp_{\mathfrak{h}(\mathbf{x},\mathbf{y})}(\mathbf{x})\cdot\exp_{\mathfrak{h}(\mathbf{x},\mathbf{y})}(\mathbf{y})\right)$ for any 2-frame $(\mathbf{x},\mathbf{y})\in\mathfrak{bl}(\mu)$ (c.f. (2.1)). We obtain

Proposition 8.1. *If* $\mathbf{x} = (\xi, X_1, X_2, x)$, $\mathbf{y} = (\eta, Y_1, Y_2, y)$ *and* $\mathbf{z} = (\zeta, Z_1, Z_2, z)$, *then* $\mathbf{x} \circ \mathbf{y} = \mathbf{z}$ *holds if and only if*

$$\zeta = \xi + \eta,\ Z_1 = X_1 + Y_1 + G(\xi,\eta)(\xi Y_1 - \eta X_1),$$
$$Z_2 = X_2 + Y_2 + G(\xi,\eta)(\xi Y_2 - \eta X_2),\ z =$$
$$\begin{cases} x + y + G(\xi,\eta)\frac{1}{1-\mu}[X_1,Y_1] + \mu G(\mu\xi,\mu\eta)\big((\xi y - \eta x) - \frac{1}{1-\mu}[X_1,Y_1]\big), \mu \neq 1, \\ x + y + G(\xi,\eta)(\xi y - \eta x) + \left(1 + G(\xi,\eta)\left(\xi - \frac{e^{\xi+\eta}}{E(\xi+\eta)}\right)\right)[X_1,Y_1], \mu = 1. \end{cases}$$

Proof. In the case $\mu \neq 1$ we compute

$$\begin{aligned}\mathbf{x} \circ \mathbf{y} &= (1 - \eta\, G(\xi,\eta))(\xi, X_1, X_2, x) + (1 + \xi\, G(\xi,\eta))\,(\eta, Y_1, Y_2, y) + \\ &\quad + \frac{1}{1-\mu}\,(G(\xi,\eta) - \mu G(\mu\xi,\mu\eta))\,(0,0,0,(\mu-1)(\xi y - \eta x) + [X_1,Y_1]) = \\ &= (\xi + \eta, (1 - G(\xi,\eta)\eta)X_1 + (1 + G(\xi,\eta)\xi)Y_1, \\ &\quad (1 - G(\xi,\eta)\eta)X_2 + (1 + G(\xi,\eta)\xi)Y_2, (1 - G(\mu\xi,\mu\eta)\mu\eta)x + \\ &\quad + (1 + G(\mu\xi,\mu\eta)\mu\xi)y + \frac{1}{1-\mu}\,(G(\xi,\eta) - \mu G(\mu\xi,\mu\eta))\,[X_1,Y_1]).\end{aligned}$$

If $\mu = 1$, we get similarly,

$$\begin{aligned}\mathbf{x} \circ \mathbf{y} &= (1 - \eta\, G(\xi,\eta))(\xi, X_1, X_2, x) + (1 + \xi\, G(\xi,\eta))\,(\eta, Y_1, Y_2, y) + \\ &\quad + \left(1 + \xi\, G(\xi,\eta) - \frac{e^{\xi+\eta}G(\xi,\eta)}{E(\xi+\eta)}\right)(0,0,0,[X_1,Y_1]) = \\ &= (\xi + \eta, (1 - G(\xi,\eta)\eta)X_1 + (1 + G(\xi,\eta)\xi)Y_1, \\ &\quad (1 - G(\xi,\eta)\eta)X_2 + (1 + G(\xi,\eta)\xi)Y_2, (1 - G(\xi,\eta)\eta)x + \\ &\quad + (1 + G(\xi,\eta)\xi)y + \left(1 + G(\xi,\eta)\left(\xi - \frac{e^{\xi+\eta}}{E(\xi+\eta)}\right)\right)[X_1,Y_1]).\end{aligned}$$

Hence we get the assertion. □

Since the functions in the formulas expressing the multiplication are analytic, we obtain

Theorem 8.2. *The multiplication* $\exp^{-1}_{\mathfrak{h}(\mathbf{x},\mathbf{y})}\left(\exp_{\mathfrak{h}(\mathbf{x},\mathbf{y})}(\mathbf{x}) \cdot \exp_{\mathfrak{h}(\mathbf{x},\mathbf{y})}(\mathbf{y})\right)$ *determines a global analytic diassociative loop on the binary Lie algebra* $\mathfrak{bl}(\mu)$.

Example

In the following we construct a binary Lie algebra which differs from the algebra $\mathfrak{bl}(\mu)$ in such a way that the nilpotent ideal is the direct sum of two copies of the Heisenberg Lie algebra $\mathfrak{h}_3$. The algebra $\mathfrak{a}(\mu)$ defined on $\mathbb{R}^7 = \mathbb{R}^1 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ by the multiplication

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3,\ [\mathbf{e}_4, \mathbf{e}_5] = \mathbf{e}_6,\ [\mathbf{e}_0, \mathbf{e}_i] = \mathbf{e}_i,\ i = 1, 2, 4, 5,$$
$$[\mathbf{e}_0, \mathbf{e}_3] = \mu\mathbf{e}_3,\ [\mathbf{e}_0, \mathbf{e}_6] = \mu\mathbf{e}_6$$

for any $\mu \in \mathbb{R}$ is a binary Lie algebra, such that for $\mu \in \{-1, 2\}$ it is a Malcev algebra, whereas for $\mu = 2$ it is a Lie algebra. The algebra $\mathfrak{a}(\mu)$ contains the nilpotent ideal $\mathfrak{i} = \mathfrak{h}_3 \oplus \mathfrak{h}_3 = \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle \oplus \langle \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6 \rangle$. Let $\mathbf{x}$ and $\mathbf{y}$ be two linearly independent vectors in $\mathfrak{a}(\mu)$ such that the components of the vectors $\mathbf{x}, \mathbf{y}$ with respect to the decomposition $\mathbb{R}^7 = \mathbb{R}^1 \oplus \mathbb{R}^2 \oplus \mathbb{R}^1 \oplus \mathbb{R}^2 \oplus \mathbb{R}^1$ are $\xi, \eta, x_1, x_2, y_1, y_2 \in \mathbb{R}$ and $X_1, Y_1 \in \mathbb{R}^2$, $X_2, Y_2 \in \mathbb{R}^2$. The Lie subalgebra $\mathfrak{h}(\mathbf{x}, \mathbf{y})$ generated by the 2-frame $(\mathbf{x}, \mathbf{y})$ is isomorphic to

(a) the abelian algebra $\mathfrak{a}_2$: $[\mathbf{e}, \mathbf{f}] = \mathbf{0}$, if $\xi = \eta = [X_1, Y_1] = 0 = [X_2, Y_2]$,
(b) the Heisenberg algebra $\mathfrak{h}_3$: $[\mathbf{e}, \mathbf{f}] = \mathbf{g}$, if $\xi = \eta = 0$, $([X_1, Y_1], [X_2, Y_2]) \neq (0, 0)$,
(c) the Lie algebra $\mathfrak{aff}(\mathbb{R})$: $[\mathbf{e}, \mathbf{f}] = \mathbf{f}$, if $(\xi, \eta) \neq (0, 0)$, $(\mu - 1)(\xi y_1 - \eta x_1) + [X_1, Y_1] = (\mu - 1)(\xi y_2 - \eta x_2) + [X_2, Y_2] = 0$,
(d) the Lie algebra $\mathfrak{l}_\mu$: $[\mathbf{e}, \mathbf{f}] = \mathbf{f}$, $[\mathbf{e}, \mathbf{g}] = \mu\mathbf{g}$, for $\mu \neq 1$, and for $\mu = 1$ the Lie algebra $\mathfrak{l}_1 :$ $[\mathbf{e}, \mathbf{f}] = \mathbf{f} + \mathbf{g}$, $[\mathbf{e}, \mathbf{g}] = \mathbf{g}$, if $(\xi, \eta) \neq (0, 0)$, $((\mu - 1)(\xi y_1 - \eta x_1) + [X_1, Y_1], (\mu - 1)(\xi y_2 - \eta x_2) + [X_2, Y_2]) \neq (0, 0)$.

The Lie subalgebras $\mathfrak{h}(\mathbf{x}, \mathbf{y})$ of the algebra $\mathfrak{a}(\mu)$ are the same as those of $\mathfrak{bl}(\mu)$. Hence the corresponding local diassociative loop multiplication can be extended globally in the same way as in the algebra $\mathfrak{bl}(\mu)$.

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Declarations

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