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Bernstein–Doetsch type results for s-convex functions

By PÁL BURAI (Debrecen), ATTILA HÁZY (Miskolc) and TIBOR JUHÁSZ (Eger)

Dedicated to 70th birthday of Professor Z. Daróczy

Abstract. As a possible generalization of the concept of s-convexity due to Breckner [2], we introduce the so-called (H, s)-convexity. Besides collecting some facts on this type of functions, the main goal of this paper is to prove some regularity properties of (H, s)-convex functions.

1. Introduction

Let D be a convex, open, nonempty subset of a real (complex) linear space X. Bernstein and Doetsch [1] (see [11] further references) proved that if a function $f : D \to \mathbb{R}$ is locally bounded from above at a point of D, then the Jensen-convexity of the function yields its local boundedness and continuity as well, which implies the convexity of the function f. This result has been generalized by several authors. The first such type results are due to Nikodem and Ng [13] for the approximately Jensen-convex functions (the so-called ε-Jensen-convexity), which was extended by Páles ([14], [15]) to approximately t-convex functions. Further generalizations can be found in papers of Mrowiec [12], Házy ([6], [7]), Házy and Páles ([8], [9]). In the paper of Gilányi, Nikodem and Páles [5] there are some Bernstein–Doetsch type results for quasiconvex functions.

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The concept of s-convexity was introduced by Breckner [2]. A real valued function $f : D \to \mathbb{R}$ is called *Breckner s-convex* (or shortly *s-convex*, in notation $f \in \mathcal{K}^s$), if

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1}$$

for every $x, y \in D$ and $\lambda \in [0,1]$, where $s \in]0,1]$ is a fixed number (see also [3], [10], [17]). The case $s = 1$ means the usual convexity of f.

Let $H \subseteq [0,1]$ be a nonempty set. A real valued function $f : D \to \mathbb{R}$ is called *Breckner (H,s)-convex*, (or shortly *(H,s)-convex*, in notation $f \in \mathcal{K}^s_H$), if it fulfills (1) for all $\lambda \in H$.

In the special cases when $H = \left\{\frac{1}{2}\right\}$, $H = \{\lambda\}$ or $H = \mathbb{Q} \cap [0,1]$, the corresponding Breckner (H,s)-convex functions are said to be *Breckner Jensen s-convex, Breckner (λ,s)-convex* and *Breckner rationally s-convex*, respectively (or shortly *Jensen s-convex, (λ,s)-convex* and *rationally s-convex*).

In [2] and [3] it was proved a Berstein–Doetsch type result on rationally s-convex functions, moreover, the s-Hölder property of s-convex functions. Pycia [17] gives a new proof of the latter statement, when f is defined on a nonempty, convex subset of a finite dimensional vector space. In [10] the authors collect some properties of s-convex functions defined on the nonnegative reals.

The main goal of this paper is to prove some regularity properties of (H,s)-convex functions, besides we also collect some facts on such functions.

2. Some elementary properties of s-convex functions

In this section we collect some interesting, easily-proved properties of Breckner s-convex functions.

Proposition 1. *If $\lambda, s \in]0,1[$ and $f : D \to \mathbb{R}$ is an (λ,s)-convex function, then f is nonnegative.*

Proof. Let x be an arbitrary element of D. Using (λ,s)-convexity of f

$$f(x) = f(\lambda x + (1-\lambda)x) \le \lambda^s f(x) + (1-\lambda)^s f(x) = (\lambda^s + (1-\lambda)^s) f(x),$$

which implies

$$0 \le (\lambda^s + (1-\lambda)^s - 1) f(x).$$

Since $\lambda^s + (1-\lambda)^s - 1 > 0$ for all $\lambda, s \in]0,1[$, we have that $f(x) \ge 0$, as desired. □

Remark 1. According to the previous proposition, (H, s)-convex functions are also nonnegative when $0 < s < 1$ and $H \setminus \{0, 1\} \neq \emptyset$. This is not true for $s = 1$.

Proposition 2. *Let $H \subseteq [0, 1]$. If $f, g \in \mathcal{K}^s$ (or $\mathcal{K}^s_H$), then $f + g$, cf (with $c > 0$), and $\max\{f, g\}$ are also in $\mathcal{K}^s$ (resp. $\mathcal{K}^s_H$).*

Proof. Easy calculation. □

The next two propositions imply that the set of s-convex functions is strictly increasing as s tends to zero.

Proposition 3. *Let $0 < s_2 \leq s_1 < 1$. If $f \in \mathcal{K}^{s_1}$ (or $\mathcal{K}^{s_1}_H$), then f is also in $\mathcal{K}^{s_2}$ (resp. $\mathcal{K}^{s_2}_H$).*

Proof. Assume that $f \in \mathcal{K}^{s_1}$, and let first $\lambda \in \,]\,0, 1\,[\,$. Then, by Proposition 1, $f(x)$ and $f(y)$ are nonnegative for all $x, y \in D$. Furthermore, $\lambda^{s_1} \leq \lambda^{s_2}$ and $(1-\lambda)^{s_1} \leq (1-\lambda)^{s_2}$, thus

$$f(\lambda x + (1-\lambda)y) \leq \lambda^{s_1} f(x) + (1-\lambda)^{s_1} f(y) \leq \lambda^{s_2} f(x) + (1-\lambda)^{s_2} f(y).$$

The above inequalities hold for $\lambda \in \{0, 1\}$, too, therefore $f \in \mathcal{K}^{s_2}$. □

Proposition 4. *Let $0 < s_1 < s_2 \leq 1$. Then there exists a function f such that $f \in \mathcal{K}^{s_1}_{\frac{1}{2}}$ but $f \notin \mathcal{K}^{s_2}_{\frac{1}{2}}$.*

Proof. Let the function $f : \,]\,0, \infty\,[\, \to \mathbb{R}$ be defined by $f(x) := x^{s_1}$. First we show that f is a Jensen s_1-convex function. To this we may assume that $x \leq y$ without loss of generality. Then the Jensen s_1-convexity of f is equivalent to the inequality

$$(u+1)^{s_1} \leq u^{s_1} + 1, \quad u \in \,]\,0, 1\,],$$

where $u := \frac{x}{y}$. The above inequality is equivalent to the nonnegativity of the function

$$g(u) = \log(u^{s_1} + 1) - s_1 \log(u + 1), \quad u \in [0, 1].$$

Because of $g(0) = 0$ and of g being monotone increasing on $[0, 1]$ (first derivative test), we get the Jensen s_1-convexity of f.

Now we prove $f \notin \mathcal{K}^{s_2}_{\frac{1}{2}}$. Assume to the contrary that $f \in \mathcal{K}^{s_2}_{\frac{1}{2}}$. Then

$$\left(\frac{x+y}{2}\right)^{s_1} \leq \frac{x^{s_1} + y^{s_1}}{2^{s_2}}, \quad x, y \in \,]\,0, \infty\,[\,.$$

We can assume again that $x \le y$. Divide by y^{s_1} both sides of the above inequality and substitute $u := \frac{x}{y}$. After some rearranging we get

$$1 \le 2^{s_1 - s_2} \frac{u^{s_1} + 1}{(u+1)^{s_1}}, \quad u \in \left]\, 0, 1\right].$$

Here the right-hand side tends to $2^{s_1 - s_2} < 1$ as u tends to zero, which is a contradiction. □

We give a simple characterization of s-convex functions, which is analogous to the characterization of convex functions.

Theorem 1. *Let $I \subset \mathbb{R}$ be a nonempty, open interval. A function $f : I \to \mathbb{R}$ is s-convex if and only if*

$$(z-x)^s f(y) \le (z-y)^s f(x) + (y-x)^s f(z)\,, \tag{2}$$

for every $x < y < z\,,\ x, y, z \in I$.

Proof. Assume that f is s-convex and let x, y and z be arbitrary element of I such that $x < y < z$. Then

$$f(y) = f\left(\frac{z-y}{z-x}x + \frac{y-x}{z-x}z\right) \le \left(\frac{z-y}{z-x}\right)^s f(x) + \left(\frac{y-x}{z-x}\right)^s f(z)\,,$$

which is equivalent to (2). One can prove the converse assertion in a similar manner. □

3. Regularity properties of (λ, s)-convex functions

In this section we assume that $(X, \|\cdot\|)$ is a real (complex) normed space instead of a real (complex) linear space. We recall that a function $f : D \to \mathbb{R}$ is called locally bounded from above on D if, for each point of $p \in D$, there exist $\varrho > 0$ and a neighborhood $U(p, \varrho) := \{x \in X : \|x - q\| < \varrho\}$ such that f is bounded from above on $U(p, \varrho)$.

Theorem 2. *Let $D \subset X$ be convex, open, nonempty and $f : D \to \mathbb{R}$. Let $\lambda \in \left]\, 0, 1\right[$ be fixed. If $f \in \mathcal{K}_\lambda^s$ is locally bounded from above at a point $p \in D$, then f is locally bounded at every point of D.*

PROOF. First we prove that f is locally bounded from above on D. Define the sequence of sets D_n by

$$D_0 := \{p\}, \qquad D_{n+1} := \lambda D_n + (1-\lambda)D.$$

Using induction on n, we prove that f is locally upper bounded at each point of D_n. By assumption, f is locally upper bounded at $p \in D_0$. Assume that f is locally upper bounded at each point of D_n. For $x \in D_{n+1}$, there exist $x_0 \in D_n$ and $y_0 \in D$ such that $x = \lambda x_0 + (1-\lambda)y_0$. By the inductive assumption, there exist $r > 0$ and a constant $M_0 \geq 0$ such that $f(x') \leq M_0$ for $\|x_0 - x'\| < r$. Then, by the (λ, s)-convexity of f, for $x' \in U_0 := U(x_0, r)$ we have

$$f(\lambda x' + (1-\lambda)y_0) \leq \lambda^s f(x') + (1-\lambda)^s f(y_0) \leq \lambda^s M_0 + (1-\lambda)^s f(y_0) =: M.$$

Therefore, for

$$y \in U := \lambda U_0 + (1-\lambda)y_0 = U(\lambda x_0 + (1-\lambda)y_0, \lambda r) = U(x, \lambda r),$$

we get that $f(y) \leq M$. Thus f is locally bounded from above on D_{n+1}.

On the other hand, we show that

$$D = \bigcup_{n=1}^{\infty} D_n.$$

From the definition of D_n, it follows by induction that $D_n = \lambda^n p + (1-\lambda^n)D$. For fixed $x \in D$, define the sequence x_n by

$$x_n := \frac{x - \lambda^n p}{1 - \lambda^n}.$$

Then $x_n \to x$ if $n \to \infty$. As D is open, $x_n \in D$ for some n. Therefore

$$x = \lambda^n p + (1-\lambda^n)x_n \in \lambda^n p + (1-\lambda^n)D = D_n.$$

Thus f is locally bounded from above on D.

Now, we prove that f is locally bounded from below. Let $q \in D$ be arbitrary. Since f is locally bounded from above at the point q, there exist $\varrho > 0$ and $M > 0$ such that

$$\sup_{U(q,\varrho)} f \leq M.$$

Let $x \in U(q, \lambda\varrho)$ and $y := \frac{q-(1-\lambda)x}{\lambda}$. Then y is in $U(q, \varrho)$. By (λ, s)-convexity,

$$f(q) \leq (1-\lambda)^s f(x) + \lambda^s f(y),$$

which implies

$$f(x) \geq \frac{f(q) - \lambda^s f(y)}{(1-\lambda)^s} \geq \frac{f(q) - \lambda^s M}{(1-\lambda)^s} =: M'.$$

Therefore f is locally bounded from below at any point of D. □

As an immediate consequence of the previous theorem we obtain:

Corollary 1. *Let $f : D \to \mathbb{R}$ be a Jensen s-convex function. If f is locally bounded from above at a point of D, then f is locally bounded at every point of D.*

The next theorem essentially weakens the local boundedness assumption if the underlying space is of finite dimension. It can be derived from Theorem 2 adopting the argument followed in [8] (that is based on Steinhaus' and Piccard's theorems (cf. [18], [16])).

Theorem 3. *Let D be an open convex subset of $\mathbb{R}^n$ and let $f : D \to \mathbb{R}$ be a (λ, s)-convex function with a fixed $0 < \lambda < 1$. Assume that there exist a Lebesgue-measurable set of positive measure (or a Baire-measurable set of second category) $S \subseteq D$ and a Lebesgue-measurable (resp. Baire-measurable) function $g : S \to \mathbb{R}$ such that $f \leq g$ on S. Then f is locally bounded on D.*

Proof. Let

$$S_{k,m} := \{x \in S \mid g(x) \leq k\} \cap U(0, m) \quad m, k \in \mathbb{N}.$$

Then

$$S = \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} S_{k,m},$$

therefore, for some k, m, the set $S_{k,m}$ is of positive measure. Therefore, f is bounded by k on $S_{k,m}$, which is a bounded set of positive measure (or a bounded set of second category).

Taking $x, y \in S_{k,m}$, we get that

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \leq (\lambda^s + (1-\lambda)^s)k \leq 2^{1-s}k.$$

That is, f is bounded on $\lambda S_{k,m} + (1-\lambda)S_{k,m}$, which, by the theorem of Steinhaus [18] (or the theorem of Piccard [16]) (cf. [11]), contains an interior point. Therefore, f is locally bounded from above at a point of D. As an immediate consequence of the previous theorem we obtain that f is locally bounded on D. □

Remark 2. It is a well-known fact that if a Jensen-convex function f is locally bounded above at a point of its domain (see [1], [11]), then it is continuous on its domain. This is not true for Jensen s-convex functions. Indeed, let $0 < s < 1$ be fixed and

$$f(x) := \begin{cases} 1, & \text{if } x \in \,](2^s-1)^{\frac{1}{s}}, 1[\, \setminus \mathbb{Q}; \\ x^s, & \text{if } x \in \,](2^s-1)^{\frac{1}{s}}, 1[\, \cap \mathbb{Q}, \end{cases}$$

Then f is Jensen s-convex, bounded and nowhere continuous.

Next theorem gives a sufficient condition for local boundedness to imply continuity.

Theorem 4. *Let the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ be such that $\lambda_n \in \,]0,1]$ and λ_n tends to 0 (when $n \to \infty$). If $f : D \to \mathbb{R}$ is in $\mathcal{K}^s_{\{\lambda_n\}_{n\in\mathbb{N}}}$ and f is locally bounded from above at a point $x_0 \in D$, then f is continuous at x_0.*

Proof. Since f is locally bounded from above at a point $x_0 \in D$, there exists a neighborhood U at x_0 and a constat $K \geq 0$ such that $f(x) \leq K$ for every $x \in U$. Let ε be an arbitrary nonnegative constant. Then there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then

$$\lambda_n^s K + [(1-\lambda_n)^s - 1]\, f(x_0) < \varepsilon,$$

whence

$$\frac{\lambda_n^s}{(1-\lambda_n)^s} K + \left[1 - \frac{1}{(1-\lambda_n)^s}\right] f(x_0) < \varepsilon.$$

Let V be a neighborhood of 0 such that $x_0 + V \subseteq U$, and let $U' = x_0 + \lambda_n V$. We prove that

$$|f(x) - f(x_0)| < \varepsilon \quad (x \in U').$$

For $x \in U'$ there exist $y, z \in x_0 + V$ such that

$$x = \lambda_n y + (1-\lambda_n)x_0, \qquad x_0 = \lambda_n z + (1-\lambda_n)x.$$

Indeed,

$$y - x_0 = \frac{1}{\lambda_n}(x - x_0) \in \frac{1}{\lambda_n}\lambda_n V = V,$$

and

$$z - x_0 = \frac{1-\lambda_n}{\lambda_n}(x_0 - x) \in \frac{1-\lambda_n}{\lambda_n}\lambda_n V = (1-\lambda_n)V \subseteq V.$$

According to (λ_n, s)-convexity of f,

$$f(x) \leq \lambda_n^s f(y) + (1-\lambda_n)^s f(x_0) \leq \lambda_n^s K + (1-\lambda_n)^s f(x_0),$$

$$f(x_0) \leq \lambda_n^s f(z) + (1-\lambda_n)^s f(x) \leq \lambda_n^s K + (1-\lambda_n)^s f(x).$$

We get

$$f(x) - f(x_0) \leq \lambda_n^s K + [(1-\lambda_n)^s - 1]\, f(x_0) < \varepsilon \tag{3}$$

and

$$f(x) \geq \frac{f(x_0) - \lambda_n^s K}{(1-\lambda_n)^s},$$

which implies

$$f(x) - f(x_0) \geq \left[\frac{1}{(1-\lambda_n)^s} - 1\right] f(x_0) - \frac{\lambda_n^s}{(1-\lambda_n)^s} K > -\varepsilon. \tag{4}$$

The inequalities (3) and (4) show that $|f(x) - f(x_0)| < \varepsilon$, that is f is continuous at x_0, which was to be proved. □

Corollary 2. *Let $H \subseteq [0,1]$ and assume that 0 or 1 is a limit point of H. If $f : D \to \mathbb{R}$ is (H,s)-convex, locally bounded at a point of D, then f continuous at that point.*

Proof. Since f is (H,s)-convex, it is also $(1-H,s)$-convex, so there exists a sequence in H or in $1-H$, which tends to zero. Now, we can apply the previous theorem. □

Theorem 5. *Let $H \subseteq [0,1]$ and assume that 0 or 1 is a limit point of H. If $f : D \to \mathbb{R}$ is (H,s)-convex and locally bounded at a point of D, then f continuous on D.*

Proof. According to Theorem 2, f is locally bounded at every point of D. So, we can use the previous corollary to get the continuity of f at every point of D. □

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PÁL BURAI
DEPARTMENT OF APPLIED MATHEMATICS AND PROBABILITY THEORY
UNIVERSITY OF DEBRECEN
H-4010 DEBRECEN, PF. 12
HUNGARY

E-mail: buraip@inf.unideb.hu

ATTILA HÁZY
DEPARTMENT OF APPLIED MATHEMATICS
UNIVERSITY OF MISKOLC
H-3515 MISKOLC-EGYETEMVÁROS
HUNGARY

E-mail: matha@uni-miskolc.hu

TIBOR JUHÁSZ
INSTITUTE OF MATHEMATICS AND INFORMATICS
ESZTERHÁZY KÁROLY COLLEGE
H-3300 EGER, LEÁNYKA ÚT 4
HUNGARY

E-mail: juhaszti@ektf.hu

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