TURÁN TYPE INEQUALITIES FOR SOME SPECIAL FUNCTIONS

Doktori (PhD) értekezés

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Tanusítom, hogy Baricz Árpád doktorjelölt 2004–2008 között a fent megnevezett Doktori Iskola Differenciálgeometria és alkalmazásai programjának keretében irányításommal végezte munkáját. Az értekezésben foglalt eredményekhez a jelölt önálló alkotó tevékenységével meghatározóan hozzájárult. Az értekezés elfogadását javaslom.

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Contents

Introduction

In 1941, while studying the zeros of Legendre polynomials

$$P_n(x) = \frac{\mathrm{d}^n}{\mathrm{dx}^n}\left[\frac{(x^2-1)^n}{n!2^n}\right] = \frac{1}{2^n}\sum_{k=0}^{[n/2]}(-1)^k\frac{(2n-2k)!}{k!(n-k)!(n-2k)!}x^{2n-k},$$

Turán [**Tu**] has discovered the famous inequality

$$[P_{n+1}(x)]^2 > P_n(x)P_{n+2}(x),$$

which holds for all $x \in (-1,1)$ and $n \in \{0,1,2,\ldots\}$. Even if Turán's paper [**Tu**] has been published just in 1950, Szegő [**Sze1**] in 1948 presented four different elegant proofs of the above inequality and extended the result to ultraspherical (or Gegenbauer), Laguerre and Hermite polynomials. Turán's inequality established for Legendre polynomials has generated considerable interest, and shortly after 1948, analogous results were obtained by several authors for other classical polynomials and special functions. Today there is an extensive literature dealing with Turán type inequalities, for example analogous inequalities to Turán's inequality has been found for:

1. Laguerre and Hermite polynomials [**Di**, **Sze1**]
2. ultraspherical polynomials [**BI1**, **BP**, **BuSa**, **Da**, **Sza1**, **Sze1**, **VL**]
3. Jacobi polynomials [**Ga1**, **Ga2**]
4. Appell polynomials [**CW**, **Si**]

5. Pollaczek and Lommel polynomials [**BI3**]

6. Askey-Wilson polynomials [**AbBu**]

7. Bessel functions of the first kind [**JB**, **Sza1**, **Sza2**]

8. modified Bessel functions of the first kind [**IL**, **JB**, **TN**]

9. modified Bessel functions of the second kind [**IL**, **IM**, **LN2**]

10. Galué's generalized modified Bessel functions of the first kind [**Ba1**]

11. polygamma function and Riemann zeta function [**CNV**, **IL**, **LN1**]

12. zeros of general Bessel functions [**Lo**]

13. zeros of ultraspherical, Laguerre and Hermite polynomials [**EL1**, **EL2**, **La**],

and this list is far from being complete. This classical inequality still attracts the attention of mathematicians and it is worth mentioning that recently the above Turán inequality was improved by Constantinescu [**Co**], and further by Alzer et al. [**AGKL**] (for more details see Remark 1.4.1). Moreover, it is important to note that even if the Turán type inequalities are interesting in their own right, there are many applications of these inequalities. For example, a necessary condition for the Riemann hypothesis can be written as a higher order Turán type inequality. The interested reader is referred to the papers [**CNV**, **Di**, **LN2**] and to the references therein. Another example of applications can be found in the recent paper of Krasikov [**Kr**], where among other things the author used some Turán type inequalities to give new non-asymptotic bounds on the extreme zeros of orthogonal polynomials. Finally, we note that recently Sun and Baricz [**SB**] conjectured that the generalized Marcum $Q-$function is log-concave with respect to its order, and consequently satisfies a Turán type inequality. This Marcum function is of frequent occurrence in radar signal processing and the above conjecture - which was proved to be affirmative by Sun et al. [**SBZ**] - is very useful in order to establish some new and very tight bounds for the generalized Marcum $Q-$function (see Remark 3.1.1 for more details).

This doctoral thesis is a further contribution to the subject and contains certain new Turán type inequalities for some special functions.

The thesis is divided into four chapters. In the first chapter our aim is to establish some Turán type inequalities for Gaussian hypergeometric functions and for generalized complete elliptic integrals. These results complete the earlier result of Turán proved for Legendre polynomials. Moreover, we show that there is a close connection between a Turán type inequality and a sharp lower bound for the generalized complete elliptic integral of the first kind. In section 1.3 we prove a recent conjecture of Sugawa and Vuorinen [**SV**] related to estimates of the hyperbolic distance of the twice punctured plane. The original results of the first three sections of this chapter were published by the author [**Ba4**] in *Mathematische Zeitschrift.* In section 1.4, in order to improve some results from section 1.1, our aim is to establish a Turán type inequality for Gaussian hypergeometric functions. This result completes the earlier result of Szegő [**Sze2**] and Gasper [**Ga2**] proved for Jacobi polynomials. Moreover, at the end of this section we present some open problems, which may be of interest for further research. The results of this section were taken from the author's paper [**Ba6**], which is in press in *Proceedings of the American Mathematical Society.*

In the second chapter we extend some known elementary trigonometric inequalities, and their hyperbolic analogues to Bessel and modified Bessel functions of the first kind. In order to generalize the Turán type inequalities established for Bessel and modified Bessel functions we present some monotonicity and convexity properties of some functions involving Bessel and modified Bessel functions of the first kind. For instance, we show that Elbert's result [**El**] on the concavity of the zeros of Bessel functions of the first kind can be used to improve the Turán type inequalities established for modified Bessel functions of the first kind. We also deduce some Turán type and Lazarević type inequalities for the confluent hypergeometric functions. The original results of this chapter may be found in Baricz's paper [**Ba8**], which was accepted for publication in *Expositiones Mathematicae.*

Chapter 3 is devoted to the study of some Turán type inequalities for the probability density function of the non-central chi-squared distribution, non-central chi distribution and Student distribution, respectively. Moreover, in this chapter we improve a result of Laforgia and Natalini [**LN2**] concerning a Turán type inequality for the modified Bessel functions of the second kind. The results of this chapter were taken from the author's paper [**Ba9**], which was submitted to *Studia Scientiarium Mathematicarum Hungarica*. As an application of some results deduced in sections 2.1 and 3.2, in section 3.3 we present a new very simple proof for the monotonicity of a product of two modified Bessel functions of different kind. This result complements and improves a recent result of Penfold et al. [**PVG**] which was motivated by a problem in biophysics. The original results of this section may be found in author's paper [**Ba7**], which was accepted for publication in *Proceedings of the American Mathematical Society*.

Finally, in chapter 4 we study the monotonicity properties of some functions involving the Mills' ratio of the standard normal law. From these we deduce some new functional inequalities involving the Mills' ratio, and we show that the Mills' ratio is strictly completely monotonic. At the end of this chapter we present some Turán type inequalities for Mills' ratio. The original results of this chapter may be found in author's paper [**Ba5**], published in *Journal of Mathematical Analysis and Applications*.

Chapter 1

Turán type inequalities for elliptic integrals

1.1 Turán type inequalities for elliptic integrals

Let $F(a,b,c,x)$ be the Gaussian hypergeometric series (function) [**AAR**, p. 64], which for real numbers a,b,c and $c \neq -1,-2,\ldots$ has the form

$$F(a,b,c,x) = \sum_{n\geq 0} \frac{(a)_n(b)_n}{(c)_n} \cdot \frac{x^n}{n!} \quad \text{for all} \quad x \in (-1,1), \tag{1.1.1}$$

where $(a)_n = a(a+1)(a+2)\ldots(a+n-1)$, $(a)_0 = 1$ denotes the Pochhammer (or Appell) symbol. It is known that the Legendre polynomials are particular cases of Gaussian hypergeometric functions, i.e. we have $P_n(1-2x) = F(-n,n+1,1,x)$ for all $x \in (0,1)$ and $n \in \{0,1,2,\ldots\}$. Let us consider the following notation $F_a(x) = F(a,1-a,1,x)$, where $x \in (0,1)$, $a = -n$ and $n \in \{0,1,2,\ldots\}$. In view of the above relation clearly Turán's inequality

$$[P_{n+1}(x)]^2 > P_n(x)P_{n+2}(x) \tag{1.1.2}$$

is equivalent with

$$\left[F_{\frac{a_1+a_2}{2}}(x)\right]^2 > F_{a_1}(x)F_{a_2}(x), \tag{1.1.3}$$

where $a_1 = a$ and $a_2 = a - 2$. Thus it is natural to ask when (1.1.3) or its reverse holds for other values of a. Motivated by this question, in this section we present some Turán type inequalities for the generalized complete elliptic integrals as well as for Gaussian hypergeometric functions. The key tools in our proofs are the classical integral inequalities, like Buniakowsky-Schwarz and Hölder-Rogers, and the simple fact that the shifted factorial $(a)_n(1-a)_n$ may be regarded as a sequence of functions of variable a. Moreover, in this section we formulate a conjecture related to a Turán type inequality involving zero-balanced hypergeometric functions. In section 1.2 the advantage of Chebyshev's integral inequality (1.2.22) is exploited. Moreover, we show that if the above mentioned conjecture is true, then we have sharp lower and upper bounds for the generalized complete elliptic integral of the first kind. In section 1.3 we prove a recent conjecture of Sugawa and Vuorinen [**SV**] related to estimates of the hyperbolic distance of the twice punctured plane, while at the end of this chapter (see section 1.4) we improve the results of the first section.

The generalized complete elliptic integrals [**BB**] for $x \in (0,1)$, $x' = \sqrt{1-x^2}$ and $a \in (0,1)$ are defined by

$$\begin{cases} \mathcal{K}_a(x) = \dfrac{\pi}{2} \cdot F(a, 1-a, 1, x^2), \\[2ex] \mathcal{K}'_a(x) = \mathcal{K}_a(x'), \\[2ex] \mathcal{K}_a(0) = \dfrac{\pi}{2}, \mathcal{K}_a(1) = \infty, \end{cases} \tag{1.1.4}$$

and

$$\begin{cases} \mathcal{E}_a(x) = \dfrac{\pi}{2} \cdot F(a-1, 1-a, 1, x^2), \\[2ex] \mathcal{E}'_a(x) = \mathcal{E}_a(x'), \\[2ex] \mathcal{E}_a(0) = \dfrac{\pi}{2}, \mathcal{E}_a(1) = \dfrac{\sin \pi a}{2(1-a)}. \end{cases} \tag{1.1.5}$$

In the particular case when $a = 1/2$, the functions $\mathcal{K}_a$ and $\mathcal{E}_a$ reduces to the elliptic integrals of the first and second kind $\mathcal{K}$ and $\mathcal{E}$, respectively [**Bo**].

Recently, Anderson et al. [**AQVV**, Corollary 7.3] proved that the function $a \mapsto \mathcal{K}_a(x)$ is increasing on $(0, 1/2]$ for each fixed $x \in (0, 1)$. Moreover Zhang et. al [**ZCW**] showed that the function $a \mapsto \mathcal{K}_a(x)/a$ is decreasing on $(0, 1/2]$ for each $x \in (0, 1)$. In this section our aim is to complete the above mentioned results, by showing that the function $a \mapsto \mathcal{K}_a(x)$ is strictly concave and sub-additive on $(0, 1)$ for each fixed $x \in (0, 1)$.

The following technical lemma is one of the crucial facts in the proof of our results.

Lemma 1.1.1. *Consider the sequence of functions* $f_n(a) = (a)_n(1-a)_n$, *where* $a \in [0, 1]$ *and* $n \in \{1, 2, 3, \dots\}$. *Then for each* $n \in \{1, 2, 3, \dots\}$ *the following assertions are true:*

1. f_n *is positive, is increasing on* $[0, 1/2]$ *and is decreasing on* $[1/2, 1]$;

2. $g_n = f_n/a$ *is strictly decreasing on* $(0, 1]$;

3. $g_1 = f_1/a$ *is concave and* $g_n = f_n/a$ *is strictly concave on* $(0, 1/2]$ *for each* $n \geq 2$;

4. f_n *is strictly concave on* $(0, 1)$.

Proof. Part **1** was proved by Anderson et al. [**AQVV**, Lemma 7.1] and part **2** for $a \in (0, 1/2]$ by Zhang et al. [**ZCW**]. However, we give here a different proof for part **1**.

1. Clearly the function $f_1(a) = a(1-a)$ is increasing on $[0, 1/2]$ and is decreasing on $[1/2, 1]$. Now suppose that for some $n \geq 2$ the function f_n has the same property. Since $(\alpha)_{n+1} = (\alpha)_n(\alpha + n)$, one has

$$f_{n+1}(a) = f_n(a)h_n(a), \quad \text{where} \quad h_n(a) = (a+n)(1-a+n). \tag{1.1.6}$$

for all $n \in \{1, 2, 3, \dots\}$. Clearly for each $n \in \{1, 2, 3, \dots\}$ the function h_n is increasing on $[0, 1/2]$ and decreasing on $[1/2, 1]$. Thus f_{n+1} has the same property, and hence by mathematical induction the required result follows.

2. Suppose that $a \in [1/2, 1]$. From part **1** f_n is decreasing and thus g_n is clearly strictly decreasing as a product of a decreasing and a strictly decreasing positive function.

3. Observe that the function $g_1(a) = f_1(a)/a = 1 - a$ is concave and $g_2(a) = f_2(a)/a = a^3 - 2a^2 - a + 2$ is clearly strictly concave on $(0, 1/2]$. Now suppose that g_n is strictly concave too for some $n \geq 3$. From (1.1.6) $g_{n+1}(a) = g_n(a)h_n(a)$, and thus

$$g''_{n+1}(a) = g''_n(a)h_n(a) + 2g'_n(a)h'_n(a) + g_n(a)h''_n(a) < 0,$$

because g_n is decreasing from part **2**, h_n is increasing and strictly concave on $(0, 1/2]$. Mathematical induction implies the strict concavity of g_n.

4. Since $f_n(a) = f_n(1 - a)$, it is enough to show the strict concavity of f_n for $a \in (0, 1/2]$. Because from part **3** g_n is strictly concave, one has for each $n \in \{1, 2, 3, \dots\}$ and $a \in (0, 1/2]$

$$g''_n(a) = \left[\frac{f_n(a)}{a}\right]'' = \frac{1}{a^3}\left[a^2 f''_n(a) - 2af'_n(a) + 2f_n(a)\right] < 0.$$

From this we have $a^2 f''_n(a) < 2af'_n(a) - 2f_n(a)$. Finally, since g_n from part **2** is decreasing on $(0, 1/2]$, we obtain that $\log g_n$ is decreasing too, and consequently $[\log[f_n(a)/a]]' \leq 0$. From this we obtain that $af'_n(a) \leq f_n(a)$, and hence $f''_n(a) < 0$. Thus the proof is complete. QED

Remark 1.1.1. From part **4** of Lemma 1.1.1 we have that for each $n \in \{1, 2, 3, \dots\}$ the functions φ_n, defined by $\varphi_n(a) = \log f_n(a)$, are concave on $(0, 1)$. We note that using a different argument to those given in the proof of Lemma 1.1.1, we may prove the followings:

1. If $a \in (0, 1)$ and $m = 2l$, where $l \in \mathbb{N}$, then $(-1)^{2l}\varphi_n^{(2l)}(a) < 0$.

2. If $a \in (0, 1/2)$ and $m = 2l+1$, where $l \in \mathbb{N}$, then $(-1)^{2l+1}\varphi_n^{(2l+1)}(a) < 0$. Moreover, when $a \in (1/2, 1)$ the above inequality is reversed.

For this let us recall the following formula for the digamma function [**WW**]:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + (x - 1)\sum_{k\geq 0} \frac{1}{(k+1)(x+k)}, \ x > 0,$$

where γ is the well-known Euler-Mascheroni constant. From this we obtain for all $x, y > 0$

$$\psi(x) - \psi(y) = \sum_{k \geq 0} \left(\frac{1}{y+k} - \frac{1}{x+k} \right).$$

In view of the above formula we have that

$$\varphi_n^{(m)}(a) = (m-1)! \sum_{k \geq 0} [q_{k,m}(n) - q_{k,m}(0)],$$

where for all $k \in \mathbb{N}$ and $m \in \{1, 2, 3, \dots\}$ the function $q_{k,m} : [0, \infty) \to \mathbb{R}$ is defined by

$$q_{k,m}(x) = \frac{1}{(1-a+x+k)^m} + \frac{(-1)^m}{(a+x+k)^m}.$$

Clearly when $a \in (0,1)$, the function $q_{k,2l}$ is strictly decreasing, i.e. we have $q_{k,2l}(n) < q_{k,2l}(0)$ and from this it follows the inequality $(-1)^{2l}\varphi_n^{(2l)}(a) < 0$. Now suppose that $m = 2l + 1$. In this case if $a \in (0, 1/2)$, then $q_{k,2l+1}$ is strictly increasing, i.e. $q_{k,2l+1}(n) > q_{k,2l+1}(0)$ holds, which implies the inequality $(-1)^{2l+1}\varphi_n^{(2l+1)}(a) < 0$. Finally, observe that when $a \in (1/2, 1)$, the function $q_{k,2l+1}$ is clearly strictly decreasing, thus the asserted result follows.

Our first main result reads as follows.

Theorem 1.1.1. *For $a, x \in (0,1)$ the function $a \mapsto \mathcal{K}_a(x)$ is strictly subadditive and strictly concave, consequently it is strictly log-concave. In particular, for all $a_1, a_2, x \in (0,1)$*

$$\sqrt{\mathcal{K}_{a_1}(x)\mathcal{K}_{a_2}(x)} \leq \frac{\mathcal{K}_{a_1}(x) + \mathcal{K}_{a_2}(x)}{2} \leq \mathcal{K}_{\frac{a_1+a_2}{2}}(x) \leq \mathcal{K}_{\frac{a_1}{2}}(x) + \mathcal{K}_{\frac{a_2}{2}}(x).$$

Proof. Using (1.1.1) and (1.1.4) clearly we have

$$\mathcal{K}_a(x) = \frac{\pi}{2} \sum_{n \geq 0} \frac{f_n(a)}{(n!)^2} x^{2n},$$

where $f_n(a) = (a)_n(1-a)_n$. From part **2** of Lemma 1.1.1 the function $a \mapsto f_n(a)/a$ is strictly decreasing on $(0,1)$ for each $n \in \{1,2,3,\dots\}$, thus clearly f_n is strictly sub-additive. From this we have that for all $a_3, a_4, x \in (0,1)$, $a_3 \neq a_4$

$$\begin{aligned}\mathcal{K}_{a_3+a_4}(x) &= \frac{\pi}{2}\sum_{n\geq 0}\frac{f_n(a_3+a_4)}{(n!)^2}x^{2n}\\ &< \frac{\pi}{2}\sum_{n\geq 0}\frac{f_n(a_3)+f_n(a_4)}{(n!)^2}x^{2n} = \mathcal{K}_{a_3}(x)+\mathcal{K}_{a_4}(x),\end{aligned}$$

i.e. the function $a \mapsto \mathcal{K}_a(x)$ is strictly sub-additive. Now from part **4** of Lemma 1.1.1 we know that $a \mapsto f_n(a)$ is strictly concave, thus for all $a_1, a_2, x \in (0,1)$, $a_1 \neq a_2$ and $\alpha \in (0,1)$ we have

$$\begin{aligned}\mathcal{K}_{\alpha a_1+(1-\alpha)a_2}(x) =&\frac{\pi}{2}\sum_{n\geq 0}\frac{f_n(\alpha a_1+(1-\alpha)a_2)}{(n!)^2}x^{2n}\\ >&\frac{\pi}{2}\sum_{n\geq 0}\frac{\alpha f_n(a_1)+(1-\alpha)f_n(a_2)}{(n!)^2}x^{2n}\\ =&\alpha\mathcal{K}_{a_1}(x)+(1-\alpha)\mathcal{K}_{a_2}(x),\end{aligned}$$

i.e. the function $a \mapsto \mathcal{K}_a(x)$ is strictly concave. Finally, since the concavity is stronger than the log-concavity, the proof is complete. QED

Recently, Heikkala et al. [**HVV**] introduced for $0 < a < \min\{c,1\}$, $x \in (0,1)$ and $x' = \sqrt{1-x^2}$ the following generalized complete elliptic integrals:

$$\begin{cases}\mathcal{K}_{a,c}(x) = \dfrac{B(a,c-a)}{2}F(a,c-a,c,x^2),\\[2ex] \mathcal{K}'_{a,c}(x) = \mathcal{K}_{a,c}(x'),\\[2ex] \mathcal{K}_{a,c}(0) = \dfrac{B(a,c-a)}{2}, \mathcal{K}_{a,c}(1) = \infty,\end{cases}$$

and

$$\begin{cases} \mathcal{E}_{a,c}(x) = \dfrac{B(a,c-a)}{2} F(a-1, c-a, c, x^2), \\ \\ \mathcal{E}'_{a,c}(x) = \mathcal{E}_{a,c}(x'), \\ \\ \mathcal{E}_{a,c}(0) = \dfrac{B(a,c-a)}{2}, \mathcal{E}_{a,c}(1) = \dfrac{B(a,c-a)B(c,1)}{2B(c+1-a,c)}, \end{cases}$$

where for α and β strictly positive real numbers $B(\alpha,\beta)$ stands for the Euler beta function. Observe that in the particular case when $c = 1$ from (1.1.4) and (1.1.5) we have that $\mathcal{K}_{a,1}(x) = \mathcal{K}_a(x)/\sin \pi a$ and $\mathcal{E}_{a,1}(x) = \mathcal{E}_a(x)/\sin \pi a$. The following result establishes a reverse Turán type inequality for these functions.

Theorem 1.1.2. *The functions $\mathcal{K}_{a,1}$ and $\mathcal{E}_{a,1}$ satisfy the reverse Turán type inequalities for all $a_1, a_2, x \in (0,1)$, that is, we have*

$$\left[\mathcal{K}_{\frac{a_1+a_2}{2},1}(x)\right]^2 \leq \mathcal{K}_{a_1,1}(x) \cdot \mathcal{K}_{a_2,1}(x) \tag{1.1.7}$$

and

$$\left[\mathcal{E}_{\frac{a_1+a_2}{2},1}(x)\right]^2 \leq \mathcal{E}_{a_1,1}(x) \cdot \mathcal{E}_{a_2,1}(x). \tag{1.1.8}$$

Moreover, the functions $a \mapsto \mathcal{K}_{a,1}(x)$ and $a \mapsto \mathcal{E}_{a,1}(x)$ are log-convex on $(0,1)$ for all $x \in (0,1)$ fixed.

Proof. First let us focus on the function $\mathcal{K}_{a,1}$. Since [**AS**, p. 558] for all $c > b > 0$ and $x \in (-1,1)$

$$F(a,b,c,x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\,\mathrm{dt}, \tag{1.1.9}$$

one has

$$\frac{2\mathcal{K}_a(x)}{\pi} = F(a,1-a,1,x^2) = \frac{1}{\Gamma(a)\Gamma(1-a)} \int_0^1 \left[\frac{1-t}{(1-tx^2)t}\right]^a \frac{1}{1-t}\,\mathrm{dt}\,.$$

From the reflection property [**AS**, p. 256] of the Euler gamma function, i.e.

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin \pi a},$$

we obtain that

$$2\mathcal{K}_{a,1}(x) = \frac{2\mathcal{K}_a(x)}{\sin \pi a} = \int_0^1 \left[\frac{1-t}{(1-tx^2)t}\right]^a \frac{1}{1-t}\,\mathrm{dt}\,. \tag{1.1.10}$$

For easy reference we record the following form of the Buniakowsky-Schwarz inequality

$$\int_a^b f(t)[g(t)]^\alpha\,\mathrm{dt} \cdot \int_a^b f(t)[g(t)]^\beta\,\mathrm{dt} \geq \left[\int_a^b f(t)[g(t)]^{\frac{\alpha+\beta}{2}}\,\mathrm{dt}\right]^2, \tag{1.1.11}$$

where f and g are two nonnegative functions of a real variable defined on $[a, b]$, α and β are real numbers such that the integrals exist. From (1.1.10) and (1.1.11) we easily obtain the inequality (1.1.7), i.e.

$$\int_0^1 \left[\frac{1-t}{(1-tx^2)t}\right]^{a_1} \frac{1}{1-t}\,\mathrm{dt} \int_0^1 \left[\frac{1-t}{(1-tx^2)t}\right]^{a_2} \frac{1}{1-t}\,\mathrm{dt}$$

$$\geq \left[\int_0^1 \left[\frac{1-t}{(1-tx^2)t}\right]^{\frac{a_1+a_2}{2}} \frac{1}{1-t}\,\mathrm{dt}\right]^2.$$

Now for the log-convexity of $a \mapsto \mathcal{K}_{a,1}(x)$ let us recall the Hölder-Rogers inequality [**Mi**, p. 54], that is,

$$\int_a^b |f(t)g(t)|\,\mathrm{dt} \leq \left[\int_a^b |f(t)|^p\,\mathrm{dt}\right]^{1/p} \left[\int_a^b |g(t)|^q\,\mathrm{dt}\right]^{1/q}, \tag{1.1.12}$$

where $p > 1$, $1/p + 1/q = 1$, f and g are real functions defined on $[a, b]$ and $|f|^p$, $|g|^q$ are integrable functions on $[a, b]$. From Hölder-Rogers' inequality

(1.1.12), for $a_1, a_2, x \in (0,1)$ and $\alpha \in [0,1]$, we easily get

$$\begin{aligned}
&2\mathcal{K}_{\alpha a_1+(1-\alpha)a_2,1}(x)\\
=&\int_0^1 \left[\left(\frac{1-t}{(1-tx^2)t}\right)^{a_1}\frac{1}{1-t}\right]^{\alpha}\left[\left(\frac{1-t}{(1-tx^2)t}\right)^{a_2}\frac{1}{1-t}\right]^{1-\alpha}\mathrm{dt}\\
\leq&\left[\int_0^1 \left(\frac{1-t}{(1-tx^2)t}\right)^{a_1}\frac{1}{1-t}\,\mathrm{dt}\right]^{\alpha}\left[\int_0^1 \left(\frac{1-t}{(1-tx^2)t}\right)^{a_2}\frac{1}{1-t}\,\mathrm{dt}\right]^{1-\alpha}\\
=&[2\mathcal{K}_{a_1,1}(x)]^{\alpha}[2\mathcal{K}_{a_2,1}(x)]^{1-\alpha},
\end{aligned}$$

and hence the required result follows.

For the function $\mathcal{E}_{a,1}$ we proceed exactly as in the case of the function $\mathcal{K}_{a,1}$. Using again (1.1.9) one has

$$\frac{2\mathcal{E}_a(x)}{\pi} = F(a-1,1-a,1,x^2) = \frac{1}{\Gamma(a)\Gamma(1-a)}\int_0^1\left[\frac{1-t}{(1-tx^2)t}\right]^a\frac{1-tx^2}{1-t}\,\mathrm{dt},$$

and from the reflection property of the Euler gamma function, we deduce that

$$2\mathcal{E}_{a,1}(x) = \frac{2\mathcal{E}_a(x)}{\sin \pi a} = \int_0^1\left[\frac{1-t}{(1-tx^2)t}\right]^a\frac{1-tx^2}{1-t}\,\mathrm{dt}\,. \tag{1.1.13}$$

From (1.1.13) and (1.1.11) we obtain the inequality (1.1.8).

Finally using again Hölder-Rogers' inequality (1.1.12), for $a_1, a_2, x \in (0,1)$ and $\alpha \in [0,1]$, we easily get

$$\begin{aligned}
&2\mathcal{E}_{\alpha a_1+(1-\alpha)a_2,1}(x)\\
=&\int_0^1 \left[\left(\frac{1-t}{(1-tx^2)t}\right)^{a_1}\frac{1-tx^2}{1-t}\right]^{\alpha}\left[\left(\frac{1-t}{(1-tx^2)t}\right)^{a_2}\frac{1-tx^2}{1-t}\right]^{1-\alpha}\mathrm{dt}\\
\leq&\left[\int_0^1 \left(\frac{1-t}{(1-tx^2)t}\right)^{a_1}\frac{1-tx^2}{1-t}\,\mathrm{dt}\right]^{\alpha}\left[\int_0^1 \left(\frac{1-t}{(1-tx^2)t}\right)^{a_2}\frac{1-tx^2}{1-t}\,\mathrm{dt}\right]^{1-\alpha}\\
=&[2\mathcal{E}_{a_1,1}(x)]^{\alpha}[2\mathcal{E}_{a_2,1}(x)]^{1-\alpha},
\end{aligned}$$

and hence the required result follows. QED

The decreasing homeomorphism $\mu_a : (0,1) \to (0,\infty)$, defined by

$$\mu_a(x) = \frac{\pi}{2\sin \pi a} \frac{\mathcal{K}'_a(x)}{\mathcal{K}_a(x)},$$

where $a \in (0,1)$, is the so-called generalized Grötzsch ring function, which appears in Ramanujan's generalized modular equations (see [**AQVV**]). This function and its particular form $\mu_{1/2} = \mu$ play an important role in various fields of mathematics, for example appear in quasiconformal theory, and in estimates of the hyperbolic distance. Related to this function Qiu and Vuorinen [**QV1**, Theorem 1.22] recently proved that $a \mapsto \mu_a(x)$ is decreasing on $(0, 1/2]$. The next result gives us more information about the dependence on a of $\mu_a(x)$.

Corollary 1.1.1. *For $x \in (0,1)$ the function $a \mapsto \mu_a(x)$ is strictly log-convex on $(0,1)$. In particular, $\mu_a(x)$ satisfies the reverse Turán type inequality, that is, for all $a_1, a_2, x \in (0,1)$ we have*

$$\left[\mu_{\frac{a_1+a_2}{2}}(x)\right]^2 \leq \mu_{a_1}(x)\mu_{a_2}(x).$$

Proof. Following the proof of Theorem 1.1.2 it is easy to see that $a \mapsto \mathcal{K}'_{a,1}(x) = \mathcal{K}_{a,1}(x') = \mathcal{K}_a(x')/\sin \pi a$ is also log-convex on $(0,1)$ for each fixed $x \in (0,1)$. Now from Theorem 1.1.1 we know that $a \mapsto [\mathcal{K}_a(x)]^{-1}$ is strictly log-convex on $(0,1)$. Hence $a \mapsto \mu_a(x)$ is strictly log-convex as a product of a strictly log-convex and log-convex functions. QED

Proceeding exactly as in the proof of Theorem 1.1.2, we may show the following results.

Theorem 1.1.3. *If $0 < a < \min\{c,1\}$ and $x \in (0,1)$, then the functions $a \mapsto \mathcal{K}_{a,c}(x)$, $c \mapsto \mathcal{K}_{a,c}(x)$, $a \mapsto \mathcal{E}_{a,c}(x)$ and $c \mapsto \mathcal{E}_{a,c}(x)$ are log-convex. In particular, if $c > 0$ is fixed, then for all $0 < a_1, a_2 < \min\{c,1\}$ and $x \in (0,1)$ we have*

$$\left[\mathcal{K}_{\frac{a_1+a_2}{2},c}(x)\right]^2 \leq \mathcal{K}_{a_1,c}(x) \cdot \mathcal{K}_{a_2,c}(x) \text{ and } \left[\mathcal{E}_{\frac{a_1+a_2}{2},c}(x)\right]^2 \leq \mathcal{E}_{a_1,c}(x) \cdot \mathcal{E}_{a_2,c}(x).$$

Moreover, if $a \in (0,1)$ *is fixed and* $c_1, c_2 > a$, $x \in (0,1)$, *then we get*

$$\left[\mathcal{K}_{a,\frac{c_1+c_2}{2}}(x)\right]^2 \le \mathcal{K}_{a,c_1}(x)\cdot\mathcal{K}_{a,c_2}(x) \text{ and } \left[\mathcal{E}_{a,\frac{c_1+c_2}{2}}(x)\right]^2 \le \mathcal{E}_{a,c_1}(x)\cdot\mathcal{E}_{a,c_2}(x).$$

Proof. From (1.1.9) we easily get

$$2\mathcal{K}_{a,c}(x) = \int_0^1 \left[\frac{1-t}{(1-tx^2)t}\right]^a \frac{t^c}{(1-t)t}\,\mathrm{dt}$$

and

$$2\mathcal{E}_{a,c}(x) = \int_0^1 \left[\frac{1-t}{(1-tx^2)t}\right]^a \frac{t^c(1-tx^2)}{(1-t)t}\,\mathrm{dt},$$

and thus the results follow from Hölder-Rogers' inequality (1.1.12). QED

Theorem 1.1.4. *If* $0 < a < \min\{c,1\}$ *and* $x \in (0,1)$, *then the functions* $f_1(a,c) = \mathcal{K}_{a,c}(x)$ *and* $f_2(a,c) = \mathcal{E}_{a,c}(x)$ *are log-convex as functions of two variable. In particular, for each* $x \in (0,1)$ *fixed and for each* $0 < a_1 < \min\{c_1,1\}$, $0 < a_2 < \min\{c_2,1\}$ *we have*

$$\left[\mathcal{K}_{\frac{a_1+a_2}{2},\frac{c_1+c_2}{2}}(x)\right]^2 \le \mathcal{K}_{a_1,c_1}(x)\cdot\mathcal{K}_{a_2,c_2}(x)$$

and

$$\left[\mathcal{E}_{\frac{a_1+a_2}{2},\frac{c_1+c_2}{2}}(x)\right]^2 \le \mathcal{E}_{a_1,c_1}(x)\cdot\mathcal{E}_{a_2,c_2}(x).$$

Proof. We prove the asserted result just for the function f_1. The other case is similar. Using (1.1.9) and Hölder-Rogers' inequality (1.1.12) we easily get for all $\alpha \in [0,1]$, $x \in (0,1)$, $0 < a_1 < \min\{c_1,1\}$ and $0 < a_2 < \min\{c_2,1\}$

that

$$
\begin{aligned}
&2\mathcal{K}_{\alpha a_1+(1-\alpha)a_2,\alpha c_1+(1-\alpha)c_2}(x)\\
=&\int_0^1 \left[\frac{1-t}{(1-tx^2)t}\right]^{\alpha a_1+(1-\alpha)a_2} \frac{t^{\alpha c_1+(1-\alpha)c_2}}{(1-t)t}\,\mathrm{dt}\\
=&\int_0^1 \left[\left[\frac{1-t}{(1-tx^2)t}\right]^{a_1}\frac{t^{c_1}}{(1-t)t}\right]^{\alpha}\left[\left[\frac{1-t}{(1-tx^2)t}\right]^{a_2}\frac{t^{c_2}}{(1-t)t}\right]^{1-\alpha}\mathrm{dt}\\
\leq&\left[\int_0^1 \left[\frac{1-t}{(1-tx^2)t}\right]^{a_1}\frac{t^{c_1}}{(1-t)t}\,\mathrm{dt}\right]^{\alpha}\left[\int_0^1 \left[\frac{1-t}{(1-tx^2)t}\right]^{a_2}\frac{t^{c_2}}{(1-t)t}\,\mathrm{dt}\right]^{1-\alpha}\\
=&[2\mathcal{K}_{a_1,c_1}(x)]^{\alpha}[2\mathcal{K}_{a_2,c_2}(x)]^{1-\alpha}.
\end{aligned}
$$

QED

The following result is an analogue of the reversed Turán type inequality for confluent hypergeometric functions **[Ba8]** (see also Theorem 2.1.2).

Theorem 1.1.5. *For $a, b, c > 0$ and $x \in (0,1)$ let us denote $f_{a,b,c}(x) = F(a,b,c,x)$. The following assertions are true:*

1. *If $a \geq c$, then*

$$[f_{a+1,b,c+1}(x)]^2 \leq f_{a,b,c}(x) \cdot f_{a+2,b,c+2}(x).$$

2. *If $(a+b+1)c \geq ab$, then*

$$\frac{ab}{c}[f_{a+1,b+1,c+1}(x)]^2 \leq \frac{(a+1)(b+1)}{c+1} f_{a,b,c}(x) \cdot f_{a+2,b+2,c+2}(x).$$

Proof. 1. Let us introduce the following notations

$$Q_n^{a,c}(x) = \sum_{k=0}^{n} \alpha_k(a,c)x^k, \text{ where } \alpha_k(a,c) = \frac{(a)_k(b)_k}{(c)_k k!}, \ k \in \{0,1,2,\ldots,n\}.$$

Using Cauchy-Buniakowsky-Schwarz's inequality [**Mi**, p. 41] one has

$$\begin{aligned}Q_n^{a,c}(x)Q_n^{a+2,c+2}(x) &= \sum_{k=0}^{n}\alpha_k(a,c)x^k\sum_{k=0}^{n}\alpha_k(a+2,c+2)x^k\\ &\geq \left[\sum_{k=0}^{n}\sqrt{\alpha_k(a,c)\alpha_k(a+2,c+2)}x^k\right]^2.\end{aligned}$$

In order to prove that

$$Q_n^{a,c}(x)Q_n^{a+2,c+2}(x) \geq \left[Q_n^{a+1,c+1}(x)\right]^2 \tag{1.1.14}$$

holds, we just need to show that

$$[\alpha_k(a,c)][\alpha_k(a+2,c+2)] \geq [\alpha_k(a+1,c+1)]^2 \tag{1.1.15}$$

holds for all $k \in \{0,1,2,\ldots,n\}$. Observe that (1.1.15) is equivalent with the inequality

$$\frac{a(a+k+1)}{c(c+k+1)} - \frac{(a+1)(a+k)}{(c+1)(c+k)} = \frac{(a-c)k(a+c+k+1)}{c(c+1)(c+k+1)(c+k)} \geq 0.$$

Thus, if n tends to infinity in (1.1.14), then we obtain the required inequality.

2. From the first part of Theorem 3.2 due to Anderson et al. [**AVV6**] it is known that $x \mapsto \log f_{a,b,c}(x) = \log F(a,b,c,x)$ is convex on $(0,1)$, where $a,b,c > 0$ and $c(a+b+1) \geq ab$. Using this result we get

$$f''_{a,b,c}(x)f_{a,b,c}(x) \geq [f'_{a,b,c}(x)]^2,$$

and thus from the derivative formulae [**AS**, p. 557]

$$\frac{\mathrm{d}}{\mathrm{dx}}[F(a,b,c,x)] = \frac{ab}{c}\cdot F(a+1,b+1,c+1,x), \tag{1.1.16}$$

$$\frac{\mathrm{d}^2}{\mathrm{dx}^2}[F(a,b,c,x)] = \frac{ab}{c}\cdot\frac{(a+1)(b+1)}{c+1}\cdot F(a+2,b+2,c+2,x)$$

yields the asserted result. QED

Recently, Anderson et al. [**AVV2**, Lemma 2.3] proved that if $a, d > 0$, $c \geq b > 0$ and $x \in (0,1)$, then $F(a,b,c,x) \leq F(a,b+d,c+d,x)$. Using this result it is clear that for all $a, b > 0$ and all $x \in (0,1)$ we have that $\partial F(a,b,a+b,x)/\partial b > 0$. Our numerical experiments show the validity of the following conjecture for the zero-balanced hypergeometric function $F(a,b,a+b,x)$.

Conjecture. *For each $a, b > 0$ and $x \in (0,1)$ we have*

$$\frac{\partial}{\partial b}\left[\frac{F(a+1,b+1,a+b+1,x)}{F(a,b,a+b,x)}\right] < 0.$$

In particular, for all $a, x \in (0,1)$ we have the following Turán type inequality

$$\frac{F(a+1,2-a,2,x)}{F(a,1-a,1,x)} > \frac{F(a+1,5/2-a,5/2,x)}{F(a,3/2-a,3/2,x)}. \tag{1.1.17}$$

1.2 Bounds for the generalized complete elliptic integral

In 1992 Anderson et al. [**AVV2**] proved that the Legendre complete elliptic integral of the first kind [**AS**, p. 591], i.e. $\mathcal{K}_{1/2} = \mathcal{K}$, defined by

$$\mathcal{K}(x) = \frac{\pi}{2} \cdot F\left(\frac{1}{2},\frac{1}{2},1,x^2\right) = \int_0^{\pi/2} \left(1 - x^2 \sin^2 t\right)^{-1/2} \mathrm{dt}, \tag{1.2.18}$$

where $x \in (0,1)$, can be approximated by the inverse hyperbolic tangent function [**AS**, p. 87] arth, i.e.

$$\operatorname{arth} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = x \cdot F\left(\frac{1}{2},1,\frac{3}{2},x^2\right),\ x \in (0,1).$$

For $x \in (0,1)$ we have [**AVV2**, Theorem 3.10]

$$\frac{\pi}{2} \cdot \left(\frac{\operatorname{arth} x}{x}\right)^{1/2} < \mathcal{K}(x) < \frac{\pi}{2} \cdot \frac{\operatorname{arth} x}{x}. \tag{1.2.19}$$

Very recently, in 2004 the left hand side of inequality (1.2.19) was improved by Alzer and Qiu [**AQ**]. They, among other things, proved that [**AQ**, Theorem 18] for all $x \in (0,1)$ we have

$$\frac{\pi}{2} \cdot \left(\frac{\operatorname{arth} x}{x}\right)^{\alpha} < \mathcal{K}(x) < \frac{\pi}{2} \cdot \left(\frac{\operatorname{arth} x}{x}\right)^{\beta} \tag{1.2.20}$$

with the best possible constants $\alpha = 3/4$ and $\beta = 1$. It is worth mentioning here that in 1998 Qi and Huang [**QH**] rediscovered the right hand side of (1.2.19) by using Chebyshev's integral inequality. They proved that

$$\frac{\pi}{2} \cdot \frac{\arcsin x}{x} < \mathcal{K}(x) < \frac{\pi}{2} \cdot \frac{\operatorname{arth} x}{x},$$

that is

$$\frac{\pi}{2} \cdot F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, x^2\right) < \mathcal{K}(x) < \frac{\pi}{2} \cdot F\left(\frac{1}{2}, 1, \frac{3}{2}, x^2\right). \tag{1.2.21}$$

Though from Neuman's result [**Ne2**] the left hand side of (1.2.21) is weaker than the left hand side of (1.2.19), motivated by the simplicity of the proof of inequality (1.2.21), in what follows we generalize (1.2.21) for the function $\mathcal{K}_a$. Before we state our first main result of this section, let us recall the Chebyshev's integral inequality [**Mi**, p. 40]: if $f, g : [a,b] \to \mathbb{R}$ are integrable functions, both increasing or both decreasing and $p : [a,b] \to \mathbb{R}$ is a positive integrable function, then

$$\int_a^b p(t)f(t)\,\mathrm{dt} \int_a^b p(t)g(t)\,\mathrm{dt} \le \int_a^b p(t)\,\mathrm{dt} \int_a^b p(t)f(t)g(t)\,\mathrm{dt}\,. \tag{1.2.22}$$

We note that if one of the functions f or g is decreasing and the other is increasing, then (1.2.22) is reversed.

Theorem 1.2.6. *If $a \in (0, 1/2]$ and $x \in (0,1)$, then the following inequalities hold*

$$B\left(1-a, \frac{1}{2}+a\right) \cdot F\left(a, 1-a, \frac{3}{2}, x^2\right) < \frac{4\mathcal{K}_a(x)}{\pi \sin \pi a} < 2\mathcal{K}_{a,3/2}(x), \tag{1.2.23}$$

where
$$2\mathcal{K}_{a,3/2}(x) = B\left(a, \frac{3}{2} - a\right) \cdot F\left(a, \frac{3}{2} - a, \frac{3}{2}, x^2\right).$$

Proof. Taking in (1.1.10) $t = \sin^2 u$, we obtain that

$$\frac{\mathcal{K}_a(x)}{\sin \pi a} = \int_0^{\pi/2} \frac{\tan^{1-2a} u}{(1 - x^2 \sin^2 u)^a} \,\mathrm{du},$$

thus if we choose the functions $p(u) = 1$, $f(u) = (\tan^{1-2a} u)(1-x^2 \sin^2 u)^{-a}$, $g(u) = \cos u$ or $\sin u$ and $[a, b] = [0, \pi/2]$ in (1.2.22), then

$$\frac{\mathcal{K}_a(x)}{\sin \pi a} \cdot \int_0^{\pi/2} \cos u \,\mathrm{du} > \int_0^{\pi/2} \mathrm{du} \cdot \int_0^{\pi/2} \frac{(\tan^{1-2a} u)(\cos u)}{(1 - x^2 \sin^2 u)^a} \,\mathrm{du} \tag{1.2.24}$$

or

$$\frac{\mathcal{K}_a(x)}{\sin \pi a} \cdot \int_0^{\pi/2} \sin u \,\mathrm{du} < \int_0^{\pi/2} \mathrm{du} \cdot \int_0^{\pi/2} \frac{(\tan^{1-2a} u)(\sin u)}{(1 - x^2 \sin^2 u)^a} \,\mathrm{du}\,. \tag{1.2.25}$$

Here we used the inequality

$$\frac{\mathrm{df}(u)}{\mathrm{du}} = \frac{\tan^{1-2a} u}{(1 - x^2 \sin^2 u)^a} \left[\frac{1 - 2a}{\sin u \cos u} + \frac{a x^2 \sin 2u}{1 - x^2 \sin^2 u}\right] \geq 0,$$

for all $u \in [0, \pi/2]$, $x \in (0, 1)$ and $a \in (0, 1/2]$. Now by direct calculations from (1.2.24) and (1.2.25) we obtain that

$$\int_0^1 \left[\frac{1 - t}{(1 - tx^2)t}\right]^a \frac{\sqrt{1 - t}}{1 - t} \,\mathrm{dt} < \frac{4\mathcal{K}_a(x)}{\pi \sin \pi a} < \int_0^1 \left[\frac{1 - t}{(1 - tx^2)t}\right]^a \frac{\sqrt{t}}{1 - t} \,\mathrm{dt},$$

which by (1.1.9) implies the estimates from (1.2.23) for the generalized elliptic integral $\mathcal{K}_a$. QED

Remark 1.2.2. a. First observe that when $a = 1/2$ the inequality (1.2.23) reduces to the inequality (1.2.21) and by definitions the right hand side of inequality (1.2.23) can be written as

$$\mathcal{K}_{a,1}(x) \leq \frac{\pi}{2} \mathcal{K}_{a,3/2}(x). \tag{1.2.26}$$

If we choose in the inequality

$$\left[\mathcal{K}_{a,\frac{c_1+c_2}{2}}(x)\right]^2 \le \mathcal{K}_{a,c_1}(x)\cdot\mathcal{K}_{a,c_2}(x)$$

$c_1 = 1$ and $c_2 = 2$, then in view of (1.2.26) we get

$$\frac{\pi}{2} \ge \frac{\mathcal{K}_{a,1}(x)}{\mathcal{K}_{a,3/2}(x)} \ge \frac{\mathcal{K}_{a,3/2}(x)}{\mathcal{K}_{a,2}(x)},$$

where $x \in (0,1)$ and $a \in (0,1/2]$. Moreover, it is clear that by induction we have that for all $a \in (0,1/2]$, $x \in (0,1)$ and $n \in \{1/2, 1, 3/2, \dots\}$ one has

$$\frac{\pi}{2} \ge \frac{\mathcal{K}_{a,1}(x)}{\mathcal{K}_{a,3/2}(x)} \ge \dots \ge \frac{\mathcal{K}_{a,n+1/2}(x)}{\mathcal{K}_{a,n+1}(x)}.$$

b. Secondly, we note that as we can see in the followings the right hand side of (1.2.23), i.e. (1.2.26) is not sharp. By Lemma 2.3 of Anderson et al. [**AVV2**] it is clear that for all $a, x \in (0,1)$ we have

$$\mathcal{K}_a(x) = \frac{\pi}{2}\cdot F(a, 1-a, 1, x^2) \le \frac{\pi}{2}\cdot F\left(a, \frac{3}{2}-a, \frac{3}{2}, x^2\right) \tag{1.2.27}$$

and for $a \in (0,1/2]$ and $x \in (0,1)$ this is better than the right hand side of (1.2.23), since

$$\frac{\pi}{2} \le \frac{\pi \sin \pi a}{4} B\left(a, \frac{3}{2}-a\right) \tag{1.2.28}$$

holds for all $a \in (0,1/2]$. To prove (1.2.28) observe that it is enough to show that

$$1 \le \frac{\sin \pi a}{2} B\left(a, \frac{3}{2}-a\right) = \frac{\Gamma(1/2)\Gamma(3/2-a)}{\Gamma(1-a)}.$$

Let us consider the function $f : [0,1/2] \to [1, \pi/2]$, defined by

$$f(a) = \frac{\Gamma(1/2)\Gamma(3/2-a)}{\Gamma(1-a)}.$$

This function is decreasing, since $f'(a) = f(a)\cdot[\psi(1-a) - \psi(3/2-a)] < 0$, where we used the fact that Euler's Γ function is log-convex on $(0,\infty)$, i.e. the digamma function $x \mapsto \psi(x) = \Gamma'(x)/\Gamma(x)$ is increasing on $(0,\infty)$. Hence $f(a) \geq f(1/2) = 1$ and thus (1.2.28) holds. Moreover inequality (1.2.27) is sharp as $x \to 0$, and is of the correct order as $x \to 1$ since for $x \in (0,1)$, we have the asymptotic formula (see [**Ev**]) due to Gauss: $F(a,b,a+b,x) \sim -\log(1-x)/B(a,b)$ as $x \to 1$.

Taking into account the above remarks we may prove the following result, which is a generalization of inequality (1.2.20). Note that our proof is based on the inequality (1.1.17), so if the above conjecture is true, then we have the next sharp estimates for the generalized complete elliptic integral $\mathcal{K}_a$. Another sharp lower and upper bounds for the generalized complete elliptic integral $\mathcal{K}_a$ has been deduced by András and Baricz [**AnBa2**] by using the classical Bernoulli inequality.

Theorem 1.2.7. *For all real numbers $a, x \in (0,1)$ the best possible constants α and β for which the inequalities*

$$\frac{\pi}{2}\left[\frac{2\mathcal{K}_{a,3/2}(x)}{B(a,3/2-a)}\right]^{\alpha} < \mathcal{K}_a(x) < \frac{\pi}{2}\left[\frac{2\mathcal{K}_{a,3/2}(x)}{B(a,3/2-a)}\right]^{\beta} \tag{1.2.29}$$

hold are $\alpha = 3(1-a)/(3-2a)$ and $\beta = 1$. Moreover, both inequalities are sharp as $x \to 0$ and the second inequality is of the correct order as $x \to 1$.

Proof. First let us focus on the second inequality of (1.2.29). This with $\beta = 1$ is equivalent with (1.2.27). Now the double-inequality (1.2.29) can be written as $\alpha < Q(x) < \beta$, where

$$Q(x) = \frac{\log\left[\frac{2}{\pi}\mathcal{K}_a(x)\right]}{\log\left[\frac{2\mathcal{K}_{a,3/2}(x)}{B(a,3/2-a)}\right]} = \frac{\log F(a,1-a,1,x^2)}{\log F(a,3/2-a,3/2,x^2)}.$$

Recall the behaviour of the function $x \mapsto F(a,b,c,x)$ near 1.

1. For $c > a+b$ (see [**Ra**, p. 49])

$$\lim_{x\to 1} F(a,b,c,x) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} < \infty.$$

2. For $c = a+b$ and $x \in (0,1)$, we have the following asymptotic formula (see [**Ev**]) due to Gauss: $F(a,b,a+b,x) \sim -\log(1-x)/B(a,b)$ as $x \to 1$.

3. For $c < a+b$, the corresponding asymptotic formula (see [**WW**, p. 299]) is the following: $F(a,b,c,x) \sim B(c,a+b-c)(1-x)^{c-a-b}/B(a,b)$ as $x \to 1$.

Using these properties (namely parts **2** and **3**), the derivative formula (1.1.16) and the l'Hospital rule, we obtain that $Q(1^-) = 1$ and $Q(0^+) = 3(1-a)/(3-2a) < 1$. Hence we conclude that the best possible constants in (1.2.29) are given by $\alpha = 3(1-a)/(3-2a)$ and $\beta = 1$. Actually the first inequality in (1.2.29) is equivalent with

$$[F(a,3/2-a,3/2,x^2)]^\alpha < F(a,1-a,1,x^2)$$

and for x in the neighborhood of the origin this can be written as

$$1 + \alpha \cdot [a(3-2a)/3]x^2 + \ldots < 1 + a(1-a)x^2 + \ldots.$$

Thus we infer that $\alpha = 3(1-a)/(3-2a)$ is the greatest value of α for which the first inequality in (1.2.29) holds. All that remains is to prove the first inequality in (1.2.29) for $\alpha = 3(1-a)/(3-2a)$. Observe that it suffices to show that the function $f : (0,1) \to \mathbb{R}$, defined by

$$f(x) = \frac{3-2a}{3(1-a)} \log[F(a,1-a,1,x^2)] - \log[F(a,3/2-a,3/2,x^2)]$$

is strictly increasing. Hence $f(x) > f(0)$ and the required inequality follows. Application of the derivative formula (1.1.16) and inequality (1.1.17) yields

$$\frac{3f'(x)}{2a(3-2a)x} = \frac{F(a+1,2-a,2,x)}{F(a,1-a,1,x)} - \frac{F(a+1,5/2-a,5/2,x)}{F(a,3/2-a,3/2,x)} > 0.$$

QED

1.3 Conjectures related to estimates of the hyperbolic distance

For z being in the open unit disk, i.e. $z \in \mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ the Poincaré metric $\rho_\Omega(z)|\,\mathrm{dz}\,|$ of the hyperbolic domain $\Omega \subset \mathbb{C}$ (a connected open set, with its boundary containing at least two points) usually is defined as

$$\rho_\Omega(f(z))|f'(z)| = \rho_{\mathbb{D}}(z) = \frac{1}{1-|z|^2},$$

where f is a holomorphic universal covering of Ω onto $\mathbb{D}$. The Poincaré (or hyperbolic) distance between two points $a, b \in \Omega$ is given by

$$d_\Omega(a,b) = \inf \int_\gamma \rho_\Omega(z)|\,\mathrm{dz}\,|,$$

where the infimum is taken over all rectifiable curves $\gamma \subset \Omega$ joining a and b. Due to Solynin and Vuorinen [**SV**, Lemma 3.10] it is known that the hyperbolic (or Poincaré) distance between $-x$ and $-y$ in $\mathbb{C} \setminus \{0,1\}$, where $x, y > 0$, is given by $d_{0,1}(-x,-y) = |\Phi(x) - \Phi(y)|$, where the function $\Phi : (0,\infty) \to (0,\infty)$ is defined by

$$\Phi(x) = \frac{1}{2}\log\frac{\mathcal{K}\left(\sqrt{x/(1+x)}\right)}{\mathcal{K}\left(\sqrt{1/(1+x)}\right)}.$$

Recently, Sugawa and Vuorinen [**SV**, Conjecture 5.9] conjectured that the function $t \mapsto \varphi(t) = 2\Phi(e^{t/2})$ has decreasing quotient $\varphi(t)/t$, and hence, it is sub-additive on $[0,\infty)$. The following result shows that the above conjecture is true. Another proof of the above conjecture has been found recently by Anderson et al. [**ATVV**].

Theorem 1.3.8. *The above conjecture is true, i.e.* $t \mapsto \varphi(t) = 2\Phi(e^{t/2})$ *is sub-additive on* $[0,\infty)$. *Moreover* $t \mapsto \varphi(t) = 2\Phi(e^{t/2})$ *is super-additive on* $(-\infty, 0]$. *This implies that we have for the function* Φ *the inequality* $\Phi(xy) \le \Phi(x) + \Phi(y)$, *where* $x, y \ge 1$ *(and its reverse if* $x, y \in (0,1]$*).*

Proof. Note, however, that our $\mathcal{K}(x)$, defined in (1.2.18) is $\mathcal{K}(x^2)$ in the notation of Sugawa and Vuorinen [**SV**]. So if we use the complete elliptic integral of the first kind $\mathcal{K}$ in the form defined by (1.2.18), then the function $t \mapsto \varphi(t) = 2\Phi(e^{t/2})$ defined by (5.5) in [**SV**] can be written as follows

$$\varphi(t) = \log \frac{\mathcal{K}\left(1/\sqrt{1+e^{-t/2}}\right)}{\mathcal{K}\left(1/\sqrt{1+e^{t/2}}\right)}.$$

Now let us consider $x = (1+e^{t/2})^{-1/2}$. It follows that

$$Q(t) = \frac{\varphi(t)}{t} = \frac{\log[\mathcal{K}'(x)/\mathcal{K}(x)]}{4\log(x'/x)} = \frac{\log[\mathcal{K}(x)/\mathcal{K}'(x)]}{4\log(x/x')} = \frac{1}{2g(x)}, \qquad (1.3.30)$$

where $t = 4\log(x'/x)$, $x' = \sqrt{1-x^2}$, $\mathcal{K}'(x) = \mathcal{K}(x')$ and

$$g(x) = \frac{2\log(x/x')}{\log[\mathcal{K}(x)/\mathcal{K}'(x)]}.$$

From the proof of Theorem 5.2 [**QVV**, p. 59] established by Qiu et al. we know that g is strictly increasing on $[1/\sqrt{2}, 1)$. From (1.3.30) it is clear that

$$Q'(t) = \left(\frac{\varphi(t)}{t}\right)' = -\frac{g'(x)}{2[g(x)]^2}\frac{\mathrm{dx}}{\mathrm{dt}} = \frac{x(1-x^2)g'(x)}{8[g(x)]^2} > 0 \ \text{ for all } \ t \leq 0,$$

because g is strictly increasing on $[1/\sqrt{2}, 1)$. Thus Q is strictly increasing on $(-\infty, 0]$. Finally, observe that Q is an even function (because φ is odd), hence Q is strictly decreasing on $[0, \infty)$. Thus φ is sub-additive on $[0, \infty)$ and super-additive on $(-\infty, 0]$, i.e. $\varphi(t_1 + t_2) \leq \varphi(t_1) + \varphi(t_2)$ holds for all $t_1, t_2 \geq 0$ (and its reverse if $t_1, t_2 \leq 0$). QED

Remark 1.3.3. a. It is worth mentioning that the function Φ may be expressed in terms of the modulus of Grötzsch ring, i.e. we have

$$e^{2\Phi(x)} = \frac{2}{\pi}\mu\left(\frac{1}{\sqrt{1+x}}\right),$$

where

$$\mu(x)=\mu_{1/2}(x)=\frac{\pi}{2}\frac{\mathcal{K}'(x)}{\mathcal{K}(x)}$$

denotes the modulus of Grötzsch ring $\mathbb{D}\setminus[0,x]$ (see for instance [**SV**] and [**OV**]). From Theorem 1.3.8 we immediately get

$$\mu\left(\frac{1}{\sqrt{1+x}}\right)\mu\left(\frac{1}{\sqrt{1+y}}\right)\geq\mu\left(\frac{1}{\sqrt{1+xy}}\right), \tag{1.3.31}$$

where $x,y\geq 1$. Moreover, inequality (1.3.31) is reversed when $x,y\in(0,1]$.

b. The hyperbolic (or Poincaré) metric of the twice punctured plane $\mathbb{C}\setminus\{a,b\}$ is usually denoted by $\lambda_{a,b}(z)|\,\mathrm{dz}\,|$. Set $\lambda=\lambda_{0,1}$ and $h(t)=e^t\lambda(-e^t)$ for $t\in\mathbb{R}$. In the same paper [**SV**, Conjecture 2.12] Sugawa and Vuorinen conjectured that the function $t\mapsto th(t)$ is increasing on $(0,\infty)$, where h can be expressed as follows

$$h(t)=\lambda_{1,1+e^{-t}}(0)=\frac{\pi}{8\mathcal{K}\left(1/\sqrt{1+e^t}\right)\mathcal{K}\left(1/\sqrt{1+e^{-t}}\right)}.$$

Letting $x=(1+e^t)^{-1/2}$ we have $h(t)=\pi/[8\mathcal{K}(x)\mathcal{K}'(x)]$ and thus using the notations $f_1(x)=\log(x'/x)$, $f_2(x)=(4/\pi)\mathcal{K}(x)\mathcal{K}'(x)$, one has $P(t)=th(t)=f_1(x)/f_2(x)$. It is known that if $x\in(0,1/\sqrt{2}]$, then f_2 is strictly decreasing [**AVV1**, Theorem 2.2 (8)]. Clearly in this case f_1 is decreasing and $f_1(x)\geq 0$. Thus

$$P'(t)=\frac{f_1'(x)f_2(x)-f_1(x)f_2'(x)}{[f_2(x)]^2}\frac{\mathrm{dx}}{\mathrm{dt}}$$

is not necessary positive.

However, we note that using these things above and the fact that f_2 is strictly increasing for $x\in[1/\sqrt{2},1)$ [**AVV1**, Theorem 2.2 (8)] we obtain that the function $x\mapsto f_1(x)f_2(x)$ is clearly decreasing on $(0,1)$. Consequently it is easy to see that the quotient $h(t)/t$ is decreasing on $\mathbb{R}$, since for all $t\in\mathbb{R}$ we have

$$4\left(\frac{h(t)}{t}\right)'=-\frac{[f_1(x)f_2(x)]'}{[f_1(x)f_2(x)]^2}\frac{\mathrm{dx}}{\mathrm{dt}}\leq 0.$$

This implies that h is sub-additive on $\mathbb{R}$, thus we have

$$\lambda_{1,1+xy}(0) \leq \lambda_{1,1+x}(0) + \lambda_{1,1+y}(0) \quad \text{for all} \quad x, y > 0.$$

Moreover, it is important to note here that recently the above conjecture has been settled by Anderson et al. [**ATVV**].

1.4 Turán type inequalities for hypergeometric functions

Karlin and Szegő in their mammoth work [**KS**] raised the question of determining the explicit range of parameters α and β for which the generalized Turán inequality

$$\left[R_{n+1}^{(\alpha,\beta)}(x)\right]^2 > R_n^{(\alpha,\beta)}(x) R_{n+2}^{(\alpha,\beta)}(x) \tag{1.4.1}$$

holds for all $x \in (-1, 1)$ and $n \in \{0, 1, 2, \ldots\}$, where

$$R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x) / P_n^{(\alpha,\beta)}(1)$$

is the normalized Jacobi polynomial and $P_n^{(\alpha,\beta)}$ is the Jacobi polynomial, that is

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} \cdot F\left(-n, n+\alpha+\beta+1, \alpha+1, \frac{1-x}{2}\right), \quad \alpha, \beta > -1.$$

Clearly we have $R_n^{(0,0)}(x) = P_n(x)$ for all $x \in (-1, 1)$ and

$$R_n^{(\alpha,\beta)}(x) = F\left(-n, n+\alpha+\beta+1, \alpha+1, \frac{1-x}{2}\right). \tag{1.4.2}$$

In 1962 Szegő [**Sze2**] proved that (1.4.1) holds for all $\beta \geq |\alpha|$ and $\alpha > -1$. Gasper [**Ga1**, **Ga2**] improved this result by showing that in fact (1.4.1) holds if and only if $\beta \geq \alpha > -1$.

Remark 1.4.1. Let P_n be the Legendre polynomial of degree n, and consider the Turánian

$$\Delta_n(x) = [P_n(x)]^2 - P_{n-1}(x)P_{n+1}(x),$$

where $n \in \{1, 2, 3, \dots\}$ and $x \in [-1, 1]$. Recently, in 2007, Alzer et al. [**AGKL**] improved the celebrated Turán inequality [**Tu**], i.e. $\Delta_n(x) \geq 0$, as follows:

$$a_n(1 - x^2) \leq \Delta_n(x) \leq b_n(1 - x^2), \tag{1.4.3}$$

where $x \in [-1, 1]$, $n \in \{1, 2, 3, \dots\}$ and the constants

$$a_n = \mu_{[n/2]}\mu_{[(n+1)/2]} \quad \text{and} \quad b_n = \frac{1}{2}$$

are the best possible. Moreover, for $n \in \{2, 3, 4, \dots\}$ the equality holds on the left hand side of (1.4.3) if and only $x \in \{-1, 0, 1\}$, and on the right hand side of (1.4.3) if and only if $x \in \{-1, 1\}$. Here $\mu_n = 2^{-2n}C_{2n}^n$ is the normalized binomial mid-coefficient. The key tool in the proof of (1.4.3) it was the fact that for each $n \in \{1, 2, 3, \dots\}$ fixed, the even function $f_n : (-1, 1) \to (0, \infty)$, defined by

$$f_n(x) = \frac{\Delta_n(x)}{1 - x^2},$$

is increasing on $[0, 1)$, and hence for all $n \in \{1, 2, 3, \dots\}$ and $x \in (-1, 1)$ we have

$$a_n = \Delta_n(0) = f_n(0) \leq f_n(x) \leq f_n(1^-) = -\frac{1}{2}\Delta_n'(1) = b_n.$$

It is worth mentioning that in the proof of the monotonicity of f_n the authors used a computer package to perform the induction. Moreover, the authors in [**AGKL**] conjectured that if $\alpha > -1/2$, $x \in [-1, 1]$ and $n \in \{1, 2, 3, \dots\}$, then the following generalization of (1.4.3) holds true:

$$a_n^{(\alpha)}(1 - x^2) \leq {}_1\Delta_{n,\alpha}(x) \leq b_n^{(\alpha)}(1 - x^2), \tag{1.4.4}$$

where

$${}_1\Delta_{n,\alpha}(x) = \left[R_n^{(\alpha,\alpha)}(x)\right]^2 - R_{n-1}^{(\alpha,\alpha)}(x)R_{n+1}^{(\alpha,\alpha)}(x),$$

$R_n^{(\alpha,\alpha)}$ is the normalized ultraspherical polynomial, defined by

$$R_n^{(\alpha,\alpha)}(x) = F\left(-n, n+2\alpha+1, \alpha+1, \frac{1-x}{2}\right), \tag{1.4.5}$$

and the constants

$$a_n^{(\alpha)} = \mu_{[n/2]}^{(\alpha)}\mu_{[(n+1)/2]}^{(\alpha)} \quad \text{and} \quad b_n^{(\alpha)} = \frac{1}{2(\alpha+1)}$$

are the best possible. Here $\mu_n^{(\alpha)} = \mu_n / C_{n+\alpha}^n$ is a binomial coefficient.

The purpose of this remark is to point out that the inequality (1.4.4) is in fact equivalent with a result of Venkatachaliengar and Lakshmana Rao [**VL**, Eq. 6.4], published in 1957. We note here that this fact has been observed first by Berg and Szwarc in [**BeSz**], where among other things, the inequality (1.4.4) has been extended to general symmetric orthogonal polynomials.

Now consider the Turánian

$${}_2\Delta_{n,\lambda}(x) = \left[\frac{P_n^{(\lambda)}(x)}{P_n^{(\lambda)}(1)}\right]^2 - \left[\frac{P_{n-1}^{(\lambda)}(x)}{P_{n-1}^{(\lambda)}(1)}\right]\left[\frac{P_{n+1}^{(\lambda)}(x)}{P_{n+1}^{(\lambda)}(1)}\right],$$

where $P_n^{(\lambda)}$ is the Gegenbauer polynomial, defined by

$$P_n^{(\lambda)}(x) = C_{n+2\lambda-1}^n \cdot F\left(-n, n+2\lambda, \lambda+\frac{1}{2}, \frac{1-x}{2}\right). \tag{1.4.6}$$

The authors in [**VL**], using an interesting method, proved that for each $n \in \{2, 3, 4, \dots\}$ fixed, the even function $f_{n,\lambda} : (-1, 1) \to (0, \infty)$, defined by

$$f_{n,\lambda}(x) = \frac{{}_2\Delta_{n,\lambda}(x)}{1-x^2},$$

is increasing on $[0, 1)$ and convex on $(-1, 1)$ for each $\lambda > 0$, decreasing on $[0, 1)$ and concave on $(-1, 1)$ for each $\lambda \in (-1/2, 0)$. Using the fact that $f_{n,\lambda}$ is increasing they deduced that for each $\lambda > 0$, $x \in (-1,1)$ and $n \in \{2,3,4,\dots\}$ the following inequalities hold

$$a_n^{(\lambda-1/2)} = {}_2\Delta_{n,\lambda}(0) = f_{n,\lambda}(0) \le f_{n,\lambda}(x),$$

$$f_{n,\lambda}(x) \le f_{n,\lambda}(1^-) = -\frac{1}{2} \cdot {}_2\Delta'_{n,\lambda}(1) = b_n^{(\lambda-1/2)},$$

which lead to the inequality

$$a_n^{(\lambda-1/2)}(1-x^2) \le {}_2\Delta_{n,\lambda}(x) \le b_n^{(\lambda-1/2)}(1-x^2), \tag{1.4.7}$$

where $\lambda > 0$, $x \in [-1,1]$ and $n \in \{2,3,4,\dots\}$, and the comment about the equality in (1.4.7) applies just as in (1.4.3). Now, replacing in (1.4.7) λ with $\alpha + 1/2$, in view of (1.4.5) and (1.4.6) it is clear that the inequalities (1.4.4) and (1.4.7) are equivalent. With other words, we have ${}_1\Delta_{n,\alpha} \equiv {}_2\Delta_{n,\alpha+1/2}$. Moreover, since $f_{n,\lambda}$ is decreasing for each $\lambda \in (-1/2,0)$, it follows that (1.4.7) is reversed for $\lambda \in (-1/2,0)$, and thus the inequality (1.4.4) is reversed too for $\alpha \in (-1,-1/2)$. It remains just to verify the case when $n = 1$. Since

$$P_0^{(\lambda)}(x) = 1, P_1^{(\lambda)}(x) = 2\lambda x \quad \text{and} \quad P_2^{(\lambda)}(x) = -\lambda + 2\lambda(1+\lambda)x^2,$$

we conclude that $f_{1,\lambda}(x) \equiv b_n^{(\lambda-1/2)}$ and thus inequalities (1.4.7) and (1.4.4) holds true for all $n \in \{1,2,3,\dots\}$. We notice that from (1.4.5), (1.4.6) and

$$P_n(x) = F\left(-n, n+1, 1, \frac{1-x}{2}\right)$$

one has $P_n^{(1/2)} \equiv R_n^{(0,0)} \equiv P_n$, and from this clearly we have

$${}_2\Delta_{n,1/2} \equiv {}_1\Delta_{n,0} \equiv \Delta_n, f_{n,1/2} \equiv f_n, a_n^{(0)} \equiv a_n \quad \text{and} \quad b_n^{(0)} \equiv b_n.$$

Thus, indeed the inequalities (1.4.4) and (1.4.7) are natural extensions of (1.4.3). Finally, we note that the right hand side of (1.4.7) can be improved as follows [**VL**, Eq. 6.4]:

$${}_2\Delta_{n,\lambda}(x) \le \left[a_n^{(\lambda-1/2)}(1-x) + b_n^{(\lambda-1/2)}x\right](1-x^2), \tag{1.4.8}$$

where $\lambda > 0$, $x \in [-1,1]$ and $n \in \{2,3,4,\dots\}$. This can be verified easily by taking into account that $f_{n,\lambda}$ is convex and thus the line segment joining the points $(0, f_{n,\lambda}(0))$ and $(c, f_{n,\lambda}(c))$ lies above the graph of $f_{n,\lambda}$ on $[0,c]$, where $c \in (0,1)$. Thus we have

$$f_{n,\lambda}(x) \leq (1 - x/c) f_{n,\lambda}(0) + (x/c) f_{n,\lambda}(c),$$

and if c tends to 1, then the required inequality (1.4.8) follows for $x \in [0,1]$. Suppose that $x \in [-1,0]$. Similarly, since $f_{n,\lambda}$ is convex, it follows that the line segment joining the points $(0, f_{n,\lambda}(0))$ and $(d, f_{n,\lambda}(d))$ lies above the graph of $f_{n,\lambda}$ on $[d,0]$, where $d \in (-1,0)$. Thus we have

$$f_{n,\lambda}(x) \leq (1 - x/d) f_{n,\lambda}(0) + (x/d) f_{n,\lambda}(d),$$

and if d tends to -1 it results that for all $\lambda > 0$, $x \in [-1,0]$ and all $n \in \{2,3,4,\dots\}$ we have

$${}_2\Delta_{n,\lambda}(x) \leq \left[a_n^{(\lambda-1/2)}(1+x) - b_n^{(\lambda-1/2)} x \right] (1-x^2). \tag{1.4.9}$$

Replacing x with $-x$ in (1.4.8) we obtain (1.4.9), and thus we can conclude that inequalities (1.4.8) and (1.4.9) holds for all $x \in [-1,1]$. Here we used that by the symmetry $f_{n,\lambda}(-1^+) = f_{n,\lambda}(1^-) = b_n^{(\lambda-1/2)}$. Moreover, since for $\lambda \in (-1/2, 0)$ the function $f_{n,\lambda}$ is concave, it follows that when $\lambda \in (-1/2, 0)$ the inequalities (1.4.8) and (1.4.9) are reversed.

Now suppose that $\beta = 0$ and consider the following notation $F_a(x) = F(a, c-a, c, x)$, where $x \in (0,1)$, $c = \alpha + 1 \in (0,1]$, $a = -n$ and as above $n \in \{0,1,2,\ldots\}$. Using Gasper's result and (1.4.2) for $\beta = 0$, we obtain that the inequality (1.4.1) is equivalent to

$$\left[F_{\frac{a_1+a_2}{2}}(x) \right]^2 > F_{a_1}(x) \cdot F_{a_2}(x), \tag{1.4.10}$$

where $a_1 = a$ and $a_2 = a - 2$. Thus it is natural to ask when (1.4.10) or its reverse holds for other values of a. Our aim in this section is to answer this question. It is worth mentioning here that the positive answer – in the particular case when $c = 1$ – to the above question was given in Theorem 1.1.1, which was motivated by the inequality (1.1.2). In fact, in Theorem 1.4.1 we prove a stronger statement, namely that the function $a \mapsto F_a(x)$ is strictly concave on $(0, c)$ for each fixed $x \in (0, 1)$ and $c \in (0, 1]$. This completes the result of Szegő and Gasper in the case of $\beta = 0$. At the end of this section we formulate some open problems which may be of interest for further research.

The following technical lemma improves Lemma 1.1.1 and is one of the crucial facts in the proof of our main result.

Lemma 1.4.1. *Let us consider the sequence of functions* $f_n(a) = (a)_n(c-a)_n$, *where* $n \in \{1, 2, 3, \dots\}$ *and* $0 \le a \le c \le 1$. *For each* $n \in \{1, 2, 3, \dots\}$ *the following assertions are true:*

1. f_n *is positive, is increasing on* $[0, c/2]$ *and is decreasing on* $[c/2, c]$;
2. $g_n = f_n/a$ *is strictly decreasing on* $(0, c]$;
3. $g_1 = f_1/a$ *is concave and* $g_n = f_n/a$ *is strictly concave on* $(0, c/2]$ *for each* $n \ge 2$;
4. f_n *is strictly concave on* $(0, c)$.

Proof. 1. Clearly the function $f_1(a) = a(c-a)$ is increasing on $[0, c/2]$ and is decreasing on $[c/2, c]$. Now suppose that for some $n \ge 2$ the function f_n has the same property. Since $(\alpha)_{n+1} = (\alpha)_n(\alpha + n)$, one has

$$f_{n+1}(a) = f_n(a)h_n(a), \quad \text{where} \quad h_n(a) = (a+n)(c-a+n) \qquad (1.4.11)$$

for all $n \in \{1, 2, 3, \dots\}$. Clearly the function h_n for each $n \in \{1, 2, 3, \dots\}$ is increasing on $[0, c/2]$ and is decreasing on $[c/2, c]$. Thus f_{n+1} has the same property, hence by mathematical induction the required result follows.

2. First suppose that $a \in (0, c/2]$ and let us write $g_n(a) = u(a)v_n(a)$, where

$$u(a) = [\Gamma(a+1)\Gamma(c-a)]^{-1} \quad \text{and} \quad v_n(a) = \Gamma(a+n)\Gamma(c-a+n).$$

Then clearly we have

$$[\log u(a)]' = \psi(c-a) - \psi(a+1) < 0, \quad a > 0 \geq (c-1)/2,$$

$$[\log v_n(a)]' = \psi(a+n) - \psi(c-a+n) \leq 0, \quad 0 < a \leq c/2,$$

where we have used that the digamma function $x \mapsto \psi(x) = \Gamma'(x)/\Gamma(x)$ is increasing on $(0, \infty)$, i.e. the gamma function is log-convex. Consequently the function u is strictly decreasing and v_n is decreasing. Thus the function g_n is strictly decreasing too.

Now assume that $a \in [c/2, c]$. From part **1** f_n is decreasing and thus g_n is clearly strictly decreasing as a product of a decreasing and a strictly decreasing functions.

3. The function $g_1(a) = f_1(a)/a = c - a$ is concave on $(0, c/2]$ and $g_2(a) = f_2(a)/a$ is strictly concave on $(0, c/2]$. Now suppose that g_n is strictly concave too for some $n \geq 3$. From (1.4.11) $g_{n+1}(a) = g_n(a)h_n(a)$, and thus

$$g''_{n+1}(a) = g''_n(a)h_n(a) + 2g'_n(a)h'_n(a) + g_n(a)h''_n(a) < 0,$$

because g_n is strictly decreasing from part **2**, h_n is increasing and strictly concave on $(0, c/2]$. Mathematical induction implies the strict concavity of g_n.

4. Since $f_n(a) = f_n(c-a)$, it is enough to show the strict concavity of f_n for $a \in (0, c/2]$. First suppose that $n = 1$. Then $f_1(a) = a(c-a)$ is clearly strictly concave on $(0, c/2]$. Now assume that $n \geq 2$. Because from part **3** g_n is strictly concave, one has for each $n \in \{2, 3, 4, \dots\}$ and $a \in (0, c/2]$

$$g''_n(a) = \left[\frac{f_n(a)}{a}\right]'' = \frac{1}{a^3}\left[a^2 f''_n(a) - 2af'_n(a) + 2f_n(a)\right] < 0.$$

From this we have $a^2 f''_n(a) < 2af'_n(a) - 2f_n(a)$. Finally, since g_n from part **2** is strictly decreasing on $(0, c/2]$, we obtain that $\log g_n$ is strictly decreasing too, and consequently $(\log[f_n(a)/a])' \leq 0$. From this we obtain that $af'_n(a) \leq f_n(a)$, and hence $f''_n(a) < 0$. Thus the proof is complete. QED

Our main result of this chapter improves Theorem 1.1.1.

Theorem 1.4.1. *For* $0 < a < c \leq 1$ *and* $x \in (0,1)$ *fixed the function* $a \mapsto F_a(x) = F(a, c-a, c, x)$ *is strictly sub-additive and strictly concave, consequently is strictly log-concave. In particular, for all* $a_1, a_2 \in (0,c)$ *and* $x \in (0,1)$, *we have*

$$\sqrt{F_{a_1}(x)F_{a_2}(x)} \leq \frac{F_{a_1}(r) + F_{a_2}(x)}{2} \leq F_{\frac{a_1+a_2}{2}}(x) \leq F_{\frac{a_1}{2}}(x) + F_{\frac{a_2}{2}}(x).$$

Proof. Using (1.1.1) clearly we have

$$F_a(x) = \sum_{n\geq 0} \frac{f_n(a)}{(c)_n n!} x^n,$$

where $f_n(a) = (a)_n (c-a)_n$. From part **2** of Lemma 1.4.1 the function $a \mapsto f_n(a)/a$ is strictly decreasing on $(0,c)$ for each $n \in \{1,2,3,\dots\}$, thus clearly f_n is strictly sub-additive. From this we have that for all $a_3, a_4 \in (0,c)$, $a_3 \neq a_4$ and $x \in (0,1)$

$$\begin{aligned} F_{a_3+a_4}(x) &= \sum_{n\geq 0} \frac{f_n(a_3+a_4)}{(c)_n n!} x^n \\ &< \sum_{n\geq 0} \frac{f_n(a_3) + f_n(a_4)}{(c)_n n!} x^n = F_{a_3}(x) + F_{a_4}(x), \end{aligned}$$

i.e. the function $a \mapsto F_a(x)$ is strictly sub-additive. Now from part **4** of Lemma 1.4.1 we know that $a \mapsto f_n(a)$ is strictly concave, thus for all $a_1, a_2 \in (0,c)$, $a_1 \neq a_2$, $x \in (0,1)$ and $\lambda \in (0,1)$ we have

$$\begin{aligned} F_{\lambda a_1 + (1-\lambda)a_2}(x) =& \sum_{n\geq 0} \frac{f_n(\lambda a_1 + (1-\lambda)a_2)}{(c)_n n!} x^n \\ >& \sum_{n\geq 0} \frac{\lambda f_n(a_1) + (1-\lambda) f_n(a_2)}{(c)_n n!} x^n \\ =& \lambda F_{a_1}(x) + (1-\lambda) F_{a_2}(x), \end{aligned}$$

i.e. the function $a \mapsto F_a(x)$ is strictly concave too. Finally, since the concavity is stronger than the log-concavity, the proof is complete. QED

We note that in fact the first and second inequalities in Theorem 1.4.1 may be confined as inequalities between geometric and arithmetic means. Namely, the first inequality is actually the arithmetic-geometric inequality between the values $F_{a_1}(x)$ and $F_{a_2}(x)$. These suggest the following interesting open problem:

Open Problem. *If m_1 and m_2 are two-variable means, i.e. for $i \in \{1,2\}$ and for all $x, y, \alpha > 0$ we have $m_i(x,y) = m_i(y,x)$, $m_i(x,x) = x$, $m_i(\alpha x, \alpha y) = \alpha m_i(x,y)$ and $x < m_i(x,y) < y$ whenever $x < y$, then find conditions on $a_1, a_2 > 0$ and $c > 0$ for which the inequality*

$$m_1(F_{a_1}(x), F_{a_2}(x)) \leq (\geq) F_{m_2(a_1,a_2)}(x)$$

holds true for all $x \in (0,1)$.

Recall that the decreasing homeomorphism $\mu_a : (0,1) \to (0,\infty)$, defined by

$$\mu_a(x) = \frac{\pi}{2\sin \pi a} \frac{F(a, 1-a, 1, 1-x^2)}{F(a, 1-a, 1, x^2)},$$

where $a \in (0,1)$, is the so-called generalized Grötzsch ring function, which appears in Ramanujan's generalized modular equations. Now, for $a \in (0,c)$ consider the decreasing homeomorphism $\mu_{a,c} : (0,1) \to (0,\infty)$, defined by

$$\mu_{a,c}(x) = \frac{\Gamma(a)\Gamma(c-a)}{2\Gamma(c)} \frac{F(a, c-a, c, 1-x^2)}{F(a, c-a, c, x^2)},$$

which is a natural extension of μ_a and is called the generalized modulus [**HVV**]. In Corollary 1.1.1 we proved that the function $a \mapsto \mu_a(x)$ is strictly log-convex on $(0,1)$. Since $\mu_{a,1} = \mu_a$, the next result improves Corollary 1.1.1.

Corollary 1.4.1. *For $c \in (0,1]$ and $x \in (0,1)$ fixed the function $a \mapsto \mu_{a,c}(x)$ is strictly log-convex on $(0,c)$. In particular, $\mu_{a,c}$ satisfies the reversed Turán type inequality, that is for all $a_1, a_2 \in (0,c)$ and $x \in (0,1)$ we have*

$$\left[\mu_{\frac{a_1+a_2}{2},c}(x)\right]^2 \leq \mu_{a_1,c}(x)\mu_{a_2,c}(x),$$

where equality holds if and only if $a_1 = a_2$.

Proof. In Theorem 1.1.3 we proved that the generalized complete elliptic integral [**HVV**]

$$a \mapsto \mathcal{K}_{a,c}(x) = \frac{\Gamma(a)\Gamma(c-a)}{2\Gamma(c)} F(a, c-a, c, 1-x^2)$$

is log-convex, whenever $0 < a < \min\{c, 1\}$ and $x \in (0,1)$. On the other hand, from Theorem 1.4.1 we know that the function $a \mapsto F(a, c-a, c, x^2)$ is strictly log-concave on $(0, c)$. Thus, the function $a \mapsto \mu_{a,c}(x)$ is strictly log-convex as a product of a strictly log-convex and log-convex functions. QED

Open Problems.

a. Recall that from Theorem 1.1.3 it is known that in fact the function $c \mapsto \mathcal{K}_{a,c}(x)$ is log-convex too whenever $0 < a < \min\{c, 1\}$ and $x \in (0, 1)$. This suggest the following question: *is the generalized modulus $\mu_{a,c}$ is strictly log-convex with respect to the parameter c?* Our numerical experiments suggest the validity of the following conjecture: *for each $n \geq 1$ and $0 < a < c$ the function $c \mapsto (c-a)_n/(c)_n$ is strictly concave.* If this result were true, then this would imply that the function $c \mapsto F(a, c-a, c, x^2)$ is strictly concave on (a, ∞) and this in turn would imply that $c \mapsto \mu_{a,c}(x)$ is strictly log-convex on (a, ∞).

b. It is known (see Theorem 1.1.4) that the generalized elliptic integral $\mathcal{K}_{a,c}$ is log-convex as a function of two variable with respect to variables (a, c). This naturally suggest the following question: *is it true that for each $n \geq 1$ and $c > a > 0$ the function $f_n(a, c) = (a)_n(c-a)_n/(c)_n$ is strictly concave as a function of two variable?* If this were true, then it would imply that $F(a, c-a, c, x^2)$ is strictly concave and $\mu_{a,c}(x)$ is strictly log-convex as functions of variables (a, c).

c. A function f with domain $(0, \infty)$ is said to be completely monotonic if it possesses derivatives $f^{(m)}(x)$ for each $m \in \{0, 1, 2, \dots\}$ and if $(-1)^m f^{(m)}(x) \geq 0$ for all $x > 0$. For more information on complete monotonicity the interested reader is referred to the papers [**AlBe**], [**Wi**] and to the references therein. Non-negative functions with a completely monotone derivative appear in literature as Bernstein functions [**Be**]. It is known

[**BI2**] that the function

$$x \mapsto \frac{\Gamma(x)\Gamma(x+a+b)}{\Gamma(x+a)\Gamma(x+b)}$$

is completely monotonic on $(0, \infty)$ for each $a, b > 0$. From this we have that the function

$$c \mapsto \frac{(c-a)_n}{(c)_n} = \frac{\Gamma(c)\Gamma(c-a+n)}{\Gamma(c-a)\Gamma(c+n)}$$

is completely monotonic too on $(0, \infty)$ for each $n \in \{1, 2, 3, \ldots\}$ and $a < 0$. On the other hand, it is easy to verify that the function $c \mapsto (c-a)_n/(c)_n$ is increasing on (a, ∞) for all $n \in \{1, 2, 3, \ldots\}$ and $a > 0$. Thus, in view of part **a** of these open problems we may ask the following: *is it true that the function*

$$x \mapsto \frac{\Gamma(x)\Gamma(x-a+b)}{\Gamma(x-a)\Gamma(x+b)}$$

is a Bernstein function on (a, ∞) *for each* $a, b > 0$*?*

Chapter 2

Turán and Lazarević type inequalities for Bessel and modified Bessel functions

2.1 Extension of Lazarević inequality to modified Bessel functions

The sine and cosine functions are particular cases of Bessel functions, while the hyperbolic sine and hyperbolic cosine functions are particular cases of modified Bessel functions. Thus it is natural to generalize some formulas and inequalities involving these elementary functions to Bessel functions and modified Bessel functions, respectively.

I. Lazarević [**Mi**, p. 270] proved that for all $x \neq 0$ the inequality

$$\cosh x < \left(\frac{\sinh x}{x}\right)^3 \tag{2.1.1}$$

holds and the exponent 3 is the least possible.

Our main motivation to write this chapter is the inequality (2.1.1) which we wish to extend to modified Bessel functions of the first kind. This chapter is organized as follows: in this section we deduce a known Turán-type inequality and using this we extend (2.1.1) to the function $\mathcal{I}_p$ defined below. Moreover, we present a generalization of the Turán type, Lazarević type and Wilker type inequalities, in order to improve the known results in the literature. For more details about the Turán type inequalities the interested reader is referred to the most recent papers [**AGKL**], [**Ba4**], [**IL**], [**LN2**] on this topic and to the references therein. At the end of this section we extend some of the main results to confluent hypergeometric functions and we improve a result of Ismail and Laforgia [**IL**]. In section 3.2 we extend the analogous of (2.1.1), Wilker's inequality (2.1.15) to Bessel functions, we deduce a known Turán type inequality for Bessel functions and we present some new inequalities involving the Bessel functions of the first kind.

For $p > -1$ let us consider the function $\mathcal{I}_p : \mathbb{R} \to [1, \infty)$, defined by

$$\mathcal{I}_p(x) = 2^p \Gamma(p+1) x^{-p} I_p(x) = \sum_{n \geq 0} \frac{(1/4)^n}{(p+1)_n n!} x^{2n}, \tag{2.1.2}$$

where $(p+1)_n = (p+1)(p+2)\ldots(p+n) = \Gamma(p+n+1)/\Gamma(p+1)$ is the well-known Pochhammer (or Appell) symbol defined in terms of Euler's gamma function, and I_p is the modified Bessel function of the first kind defined by [**Wa**, p. 77]

$$I_p(x) = \sum_{n \geq 0} \frac{(x/2)^{2n+p}}{n! \cdot \Gamma(p+n+1)} \quad \text{for all} \quad x \in \mathbb{R}. \tag{2.1.3}$$

It is worth mentioning that in particular we have

$$\mathcal{I}_{-1/2}(x) = \sqrt{\pi/2} \cdot x^{1/2} I_{-1/2}(x) = \cosh x, \tag{2.1.4}$$

$$\mathcal{I}_{1/2}(x) = \sqrt{\pi/2} \cdot x^{-1/2} I_{1/2}(x) = \frac{\sinh x}{x}, \tag{2.1.5}$$

$$\mathcal{I}_{3/2}(x) = 3\sqrt{\pi/2} \cdot x^{-3/2} I_{3/2}(x) = -3\left(\frac{\sinh x}{x^3} - \frac{\cosh x}{x^2}\right). \tag{2.1.6}$$

Thus the function $\mathcal{I}_p$ is of special interest in this section, because inequality (2.1.1) is actually equivalent with

$$[\mathcal{I}_{-1/2}(x)]^{-1/2+1} \leq [\mathcal{I}_{-1/2+1}(x)]^{-1/2+2}. \tag{2.1.7}$$

So in view of inequality (2.1.7) it is natural to ask: what is the analogue of this inequality for modified Bessel functions of the first kind? In order to answer this question we prove the following results.

Theorem 2.1.1. *Let $p > -1$ and $x \in \mathbb{R}$. Then the following assertions are true:*

1. *the function $p \mapsto \mathcal{I}_p(x)$ is decreasing and log-convex;*

2. *the functions $p \mapsto \mathcal{I}_{p+1}(x)/\mathcal{I}_p(x)$, $p \mapsto [\mathcal{I}_p(x)]^{p+1}$ are increasing;*

3. *the following inequalities*

$$[\mathcal{I}_{p+1}(x)]^2 \leq \mathcal{I}_p(x)\mathcal{I}_{p+2}(x), \tag{2.1.8}$$

$$[\mathcal{I}_p(x)]^{p+1} \leq [\mathcal{I}_{p+1}(x)]^{p+2}, \tag{2.1.9}$$

$$[\mathcal{I}_p(x)]^{\frac{p+1}{p+2}} \leq \mathcal{I}_{p+1}(x) \leq \mathcal{I}_p(x), \tag{2.1.10}$$

$$[\mathcal{I}_{p+1}(x)]^{1/(p+1)} + \frac{\mathcal{I}_{p+1}(x)}{\mathcal{I}_p(x)} \geq 2, \tag{2.1.11}$$

   hold true for all $p > -1$ and $x \in \mathbb{R}$. In (2.1.9) the exponent p is the best possible in the sense that $\tau = (p+2)/(p+1)$ is the smallest value of τ for which $\mathcal{I}_p(x) \leq [\mathcal{I}_{p+1}(x)]^\tau$ holds. Moreover, if $x > 0$ is fixed and $p \to \infty$, then $[I_p(x)]^2 \sim I_{p-1}(x)I_{p+1}(x)$;

4. *the inequality*

$$\frac{\mathcal{I}_p(x)}{\mathcal{I}_{p+1}(x)} - 1 \leq \log[\mathcal{I}_{p+1}(x)] \leq \log[\mathcal{I}_p(x)] \tag{2.1.12}$$

   holds true for all $p \geq 0$ and $x \in \mathbb{R}$.

Proof. 1. For convenience let us write

$$\mathcal{I}_p(x) = \sum_{n\geq 0} b_n(p)x^{2n}, \text{ where } b_n(p) = \frac{(1/4)^n}{(p+1)_n n!}, \ n \geq 0.$$

Clearly if $p \geq q > -1$, then $(p+1)_n \geq (q+1)_n$, and consequently $b_n(p) \leq b_n(q)$, for all $n \geq 0$. This implies that $\mathcal{I}_p(x) \leq \mathcal{I}_q(x)$ for all $x \in \mathbb{R}$, i.e. the function $p \mapsto \mathcal{I}_p(x)$ is decreasing. Now for log-convexity of $p \mapsto \mathcal{I}_p(x)$ we observe that it is enough to show the log-convexity of each individual term and to use the fact that sums of log-convex functions are log-convex too. Thus, we just need to show that for each $n \geq 0$ we have

$$\partial^2 \log[b_n(p)]/\partial p^2 = \psi'(p+1) - \psi'(p+n+1) \geq 0,$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the so-called digamma function. But ψ is concave, and consequently the function $p \mapsto b_n(p)$ is log-convex on $(-1, \infty)$.

We note that there is another proof of the log-convexity of $p \mapsto \mathcal{I}_p(x)$. Namely, if we consider the infinite product representation of the modified Bessel function of the first kind I_p, then we have **[Wa]**

$$\mathcal{I}_p(x) = \prod_{n \geq 1} \left(1 + \frac{x^2}{j_{p,n}^2}\right), \tag{2.1.13}$$

where $j_{p,n}$ is the $n-$th positive zero of the Bessel function J_p. Using (2.1.13) we have

$$\log[\mathcal{I}_p(x)] = \sum_{n \geq 1} \log\left(1 + \frac{x^2}{j_{p,n}^2}\right).$$

Owing to Elbert **[El]**, it is known that $p \mapsto j_{p,n}$ is concave on $(-n, \infty)$ for all $n \geq 1$. Consequently, we have that $p \mapsto j_{p,n}$ and $p \mapsto \log j_{p,n}$ are concave on $(-1, \infty)$ for all $n \geq 1$. Hence, $p \mapsto -2\log j_{p,n}$ is convex, i.e. $p \mapsto 1/j_{p,n}^2$ is log-convex on $(-1, \infty)$. But this implies that for all $n \geq 1$ the function $p \mapsto \log(1 + x^2/j_{p,n}^2)$ is convex on $(-1, \infty)$, and consequently the function $p \mapsto \log \mathcal{I}_p(x)$ is convex too on $(-1, \infty)$ as a sum of convex functions.

2. First we prove that the function $p \mapsto \mathcal{I}_{p+1}(x)/\mathcal{I}_p(x)$ is increasing. From part (a) of this theorem the function $p \mapsto \log[\mathcal{I}_p(x)]$ is convex, and hence it follows that $p \mapsto \log[\mathcal{I}_{p+a}(x)] - \log[\mathcal{I}_p(x)]$ is increasing for each $a > 0$. Thus choosing $a = 1$ we obtain that indeed the function $p \mapsto \mathcal{I}_{p+1}(x)/\mathcal{I}_p(x)$ is increasing.

Now suppose that $p \geq q > -1$ and define the function $\varphi_1 : \mathbb{R} \to \mathbb{R}$ with relation

$$\varphi_1(x) = \frac{p+1}{q+1} \log[\mathcal{I}_p(x)] - \log[\mathcal{I}_q(x)].$$

On the other hand

$$\varphi_1'(x) = \frac{p+1}{q+1} \left[\frac{\mathcal{I}_p'(x)}{\mathcal{I}_p(x)}\right] - \frac{\mathcal{I}_q'(x)}{\mathcal{I}_q(x)} = 2xb_1(q) \left[\frac{\mathcal{I}_{p+1}(x)}{\mathcal{I}_p(x)} - \frac{\mathcal{I}_{q+1}(x)}{\mathcal{I}_q(x)}\right],$$

where we used the differentiation formula $\mathcal{I}_p'(x) = 2xb_1(p)\mathcal{I}_{p+1}(x)$, which can be derived easily from (2.1.2). Since $p \geq q$ we have $\mathcal{I}_{p+1}(x)/\mathcal{I}_p(x) \geq \mathcal{I}_{q+1}(x)/\mathcal{I}_q(x)$ and from this conclude that the function φ_1 is increasing on $[0, \infty)$ and is decreasing on $(-\infty, 0]$. Consequently $\varphi_1(x) \geq \varphi_1(0) = 0$, i.e. $[\mathcal{I}_p(x)]^{p+1} \geq [\mathcal{I}_q(x)]^{q+1}$ holds for all $x \in \mathbb{R}$.

3. Since $p \mapsto \mathcal{I}_p(x)$ is log-convex, for all $p_1, p_2 > -1$, $x \in \mathbb{R}$ and $\alpha \in [0, 1]$ we have

$$\mathcal{I}_{\alpha p_1 + (1-\alpha)p_2}(x) \leq [\mathcal{I}_{p_1}(x)]^{\alpha}[\mathcal{I}_{p_2}(x)]^{1-\alpha}.$$

Now choosing $\alpha = 1/2$, $p_1 = p$ and $p_2 = p + 2$ we conclude that (2.1.8) holds. Inequality (2.1.9) follows from the monotonicity of $p \mapsto [\mathcal{I}_p(x)]^{p+1}$, while (2.1.10) is an immediate consequence of (2.1.9) and the monotonicity of $p \mapsto \mathcal{I}_p(x)$. Moreover, since in the neighborhood of the origin $\mathcal{I}_p(x) = 1 + b_1(p)x^2 + \ldots$ and $[\mathcal{I}_{p+1}(x)]^{\tau} = 1 + \tau \cdot b_1(p+1)x^2 + \ldots$, we infer that $\tau = b_1(p)/b_1(p+1) = (p+2)/(p+1)$ is the smallest value of τ such that $\mathcal{I}_p(x) \leq [\mathcal{I}_{p+1}(x)]^{\tau}$ holds. For inequality (2.1.11) we use the generalized Lazarević inequality (2.1.9) and the arithmetic-geometric mean inequality

$$\frac{1}{2}\left[[\mathcal{I}_{p+1}(x)]^{1/(p+1)} + \frac{\mathcal{I}_{p+1}(x)}{\mathcal{I}_p(x)}\right] \geq \sqrt{\frac{[\mathcal{I}_{p+1}(x)]^{(p+2)/(p+1)}}{\mathcal{I}_p(x)}} \geq 1.$$

It remains just to prove the asymptotic formula $[I_p(x)]^2 \sim I_{p-1}(x)I_{p+1}(x)$. In order to prove the asserted result, we show that for $p > 0$ and $x > 0$ we have

$$1 < \frac{[I_p(x)]^2}{I_{p-1}(x)I_{p+1}(x)} < 1 + \frac{1}{p}. \tag{2.1.14}$$

The left hand side of (2.1.14) is the well-known Amos inequality [**Am**, p. 243]. The right hand side of (2.1.14) can be deduced easily from (2.1.8) using the difference equation $\Gamma(a+1) = a\Gamma(a)$.

4. Using the the recurrence formula [**Wa**, p. 79] $xI_{p-1}(x) - xI_{p+1}(x) = 2pI_p(x)$ and the Mittag-Leffler expansion [**EMOT**, Eq. 7.9.3]

$$\frac{I_{p+1}(x)}{I_p(x)} = \sum_{n\geq 1} \frac{2x}{x^2 + j_{p,n}^2},$$

where $0 < j_{p,1} < j_{p,2} < \ldots < j_{p,n} < \ldots$ are the positive zeros of the Bessel function J_p, we obtain that

$$\frac{\mathcal{I}_p(x)}{\mathcal{I}_{p+1}(x)} - 1 = \frac{x}{2(p+1)} \frac{I_p(x)}{I_{p+1}(x)} - 1 = \frac{x}{2(p+1)} \frac{I_{p+2}(x)}{I_{p+1}(x)} = \frac{1}{p+1} \sum_{n\geq 1} \frac{x^2}{x^2 + j_{p+1,n}^2}.$$

On the other hand using the infinite product representation of the function $\mathcal{I}_p$, i.e.

$$\mathcal{I}_p(x) = \prod_{n\geq 1} \left(1 + \frac{x^2}{j_{p,n}^2}\right),$$

we have

$$\log[\mathcal{I}_{p+1}(x)] = \log\left[\prod_{n\geq 1} \left(1 + \frac{x^2}{j_{p+1,n}^2}\right)\right] = \sum_{n\geq 1} \log\left(1 + \frac{x^2}{j_{p+1,n}^2}\right).$$

Now using the equivalent form of inequality [**Mi**, p. 279] $x^x \geq e^{x-1}$, i.e. $\log x \geq 1 - 1/x$, which holds for all $x > 0$, we conclude that for all $p \geq 0$, $n \geq 1$ and $x \in \mathbb{R}$ we have

$$\log\left(1 + \frac{x^2}{j_{p+1,n}^2}\right) \geq \frac{x^2}{x^2 + j_{p+1,n}^2} \geq \frac{1}{p+1} \cdot \frac{x^2}{x^2 + j_{p+1,n}^2},$$

and consequently

$$\log[\mathcal{I}_{p+1}(x)] \geq \frac{\mathcal{I}_p(x)}{\mathcal{I}_{p+1}(x)} - 1.$$

Finally, because the function $p \mapsto \mathcal{I}_p(x)$ is decreasing we conclude that $\log[\mathcal{I}_{p+1}(x)] \leq \log[\mathcal{I}_p(x)]$, and with this the proof is complete.

QED

Remark 2.1.1. a. First we note that the Turán type inequality (2.1.8) was proved earlier in 1951 by Thiruvenkatachar and Nanjundiah [**TN**], while in 1991 Joshi and Bissu [**JB**] examined an alternate derivation of (2.1.8) and slightly extended this inequality. However, our proof is completely different, moreover, part **1** of the above theorem provides a generalization of (2.1.8). Recently, Ismail and Laforgia [**IL**, Remark 2.4] proved for all $p > -1/2$ and $x > 0$ the inequality

$$\mathcal{I}_p(x)\mathcal{I}_{p+2}(x) \geq \frac{(2p+1)(p+2)}{(2p+3)(p+1)} \cdot \mathcal{I}_{p+1}^2(x).$$

We note that, since $(2p+3)(p+1) > (2p+1)(p+2)$, the above Turán type inequality is weaker than (2.1.8).

b. On the other hand, observe that using (2.1.4), (2.1.5) and (2.1.6) in particular for $p = -1/2$ the Turán type inequality (2.1.8) becomes

$$x \sinh^2(x) \leq 3(\cosh x)(x \cosh x - \sinh x)$$

which holds for all $x \geq 0$. Moreover, when $x \leq 0$, the above inequality is reversed. We note here that using (2.1.4) and (2.1.5), from inequality (2.1.9) we get (2.1.7), while from $\mathcal{I}_{p+1}(x) < \mathcal{I}_p(x)$ we obtain the well-known inequality $\tanh x < x$, where $x > 0$.

c. Inequality (2.1.11) is a natural extension of the hyperbolic analogue of Wilker's inequality [**Wi**]

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2, \tag{2.1.15}$$

where $x \in (0, \pi/2)$. Namely, if we choose $p = -1/2$ in (2.1.11), then in view of (2.1.4) and (2.1.5) we have the hyperbolic analogue of (2.1.15)

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2,$$

where $x \neq 0$. This inequality was proved recently by Zhu [**Zh**].

d. Recently, Stolarsky [**St**], among other things, proved that the monotonicity of the Hölder mean is actually a consequence of a certain inequality for $x \mapsto \log \cosh x$. In this spirit he proved the following interesting inequalities:

$$\log\left(\frac{\sinh x}{x}\right) \leq \frac{\coth x}{x} - 1 \leq \log(\cosh x),$$

where $x > 0$ and the first inequality is in fact equivalent to the inequality between the logarithmic and identric means. Inequality (2.1.12) was motivated by the above result of Stolarsky and based on numerical experiments we conjecture the following: *for each* $p \in (-1, 0)$ *and* $x \in \mathbb{R}$ *we have*

$$\log[\mathcal{I}_{p+1}(x)] \leq \frac{\mathcal{I}_p(x)}{\mathcal{I}_{p+1}(x)} - 1 \leq \log[\mathcal{I}_p(x)].$$

e. Finally, we note that some of the results of Theorem 2.1.1 has been extended by the author [**Ba1**] to the Galué's generalized modified Bessel functions of the first kind.

By a confluent hypergeometric function, also known as a Kummer function, we mean the function

$$\Phi(a, c, x) = \sum_{n \geq 0} \frac{(a)_n}{(c)_n} \frac{x^n}{n!} \quad \text{for all } x \in \mathbb{R},$$

defined for $a, c \in \mathbb{R}$ with $c \neq 0, -1, -2, \ldots$. It is known that [**AS**, p. 509]

$$\mathcal{I}_p(x) = 2^p \Gamma(p+1) x^{-p} I_p(x) = e^{-x} \Phi(p + 1/2, 2p + 1, 2x).$$

Thus inequalities (2.1.8), (2.1.9) and (2.1.11) are equivalent with inequalities

$$[\Phi(a+1, 2a+2, x)]^2 \leq \Phi(a, 2a, x)\Phi(a+2, 2a+4, x),$$

$$[\Phi(a, 2a, x)]^{a+1/2} \leq e^{-x/2}[\Phi(a+1, 2a+2, x)]^{a+3/2},$$

$$\left[e^{-x/2}\Phi(a+1, 2a+2, x)\right]^{2/(2a+1)} + \frac{\Phi(a+1, 2a+2, x)}{\Phi(a, 2a, x)} \geq 2,$$

where $a > -1/2$ and $x \in \mathbb{R}$. In fact proceeding exactly as in the proof of Theorem 2.1.1, we obtain the followings, which complete the above results.

Theorem 2.1.2. *If $a \geq c > 0$ and $x \geq 0$, then $\mu \mapsto \Phi(a+\mu, c+\mu, x)$ is log-convex on $[0,\infty)$. Moreover, if $a, c > 0$ and $x \geq 0$, then the function $\mu \mapsto \Phi(a, c+\mu, x)$ is log-convex too on $[0,\infty)$. In particular, the following inequalities*

$$[\Phi(a+1, c+1, x)]^2 \leq \Phi(a, c, x)\Phi(a+2, c+2, x), \tag{2.1.16}$$

$$[\Phi(a, c, x)]^{(a+1)/(c+1)} \leq [\Phi(a+1, c+1, x)]^{a/c}, \tag{2.1.17}$$

$$[\Phi(a+1, c+1, x)]^{(a-c)/(c(a+1))} + \frac{\Phi(a+1, c+1, x)}{\Phi(a, c, x)} \geq 2 \tag{2.1.18}$$

hold for all $a \geq c > 0$ and $x \geq 0$, where the exponent $\tau = [a(c+1)]/[c(a+1)]$ is the smallest value of τ such that inequality $\Phi(a, c, x) \leq [\Phi(a+1, c+1, x)]^\tau$ holds. Moreover, for all $a, c > 0$ and $x \geq 0$, the following Turán type inequality holds true:

$$\Phi(a, c, x)\Phi(a, c+2, x) \geq \Phi^2(a, c+1, x). \tag{2.1.19}$$

Proof. As in the proof of Theorem 2.1.1, let us write

$$\Phi(a, c, x) = \sum_{n\geq 0} e_n(a, c)x^n, \text{ where } e_n(a, c) = \frac{(a)_n}{(c)_n n!}, \ n \geq 0.$$

Computations show that for each $n \geq 0$ we get

$$\partial^2 \log[e_n(a+\mu, c+\mu)]/\partial\mu^2 = f(a) - f(c),$$

where $f : (0,\infty) \to \mathbb{R}$ is defined by $f(x) = \psi'(x+\mu+n) - \psi'(x+\mu)$ and $\mu \geq 0$. It is well-known that the function $x \mapsto \psi''(x)$ is increasing on $(0,\infty)$, thus for all $x > 0, \mu, n \geq 0$ we have $f'(x) = \psi''(x+\mu+n) - \psi''(x+\mu) \geq 0$. Therefore f is increasing, i.e. $f(a) \geq f(c)$, and consequently the function $\mu \mapsto e_n(a+\mu, c+\mu)$ is log-convex on $[0,\infty)$. Thus $\mu \mapsto \Phi(a+\mu, c+\mu, x)$ is also log-convex on $[0,\infty)$, as we required. Similarly, we have

$$\partial^2 \log[e_n(a, c+\mu)]/\partial\mu^2 = \psi'(c+\mu) - \psi'(c+\mu+n) \geq 0$$

for all $a, c > 0$ and $n \geq 0$, since the digamma function $x \mapsto \psi(x)$ is concave, i.e. the trigamma function $x \mapsto \psi'(x)$ is decreasing. Consequently, the function $\mu \mapsto \Phi(a, c+\mu, x)$ is also log-convex on $[0, \infty)$, as we required.

Inequality (2.1.16) follows from the log-convexity of the function $\mu \mapsto \Phi(a+\mu, c+\mu, x)$. To prove the inequality (2.1.17) consider the function $\varphi_2 : [0, \infty) \to \mathbb{R}$, defined by

$$\varphi_2(x) = \frac{a}{c} \log \Phi(a+1, c+1, x) - \frac{a+1}{c+1} \log \Phi(a, c, x).$$

Then from (2.1.16) we have

$$\varphi_2'(x) = \frac{a(a+1)}{c(c+1)} \left[\frac{\Phi(a+2, c+2, x)}{\Phi(a+1, c+1, x)} - \frac{\Phi(a+1, c+1, x)}{\Phi(a, c, x)} \right] \geq 0,$$

where we used the differentiation formula $c\Phi'(a, c, x) = a\Phi(a+1, c+1, x)$. Thus φ_2 is increasing, and consequently $\varphi_2(x) \geq \varphi_2(0) = 0$. Since $\Phi(a, c, x) = 1 + e_1(a, c)x + \ldots$ and $[\Phi(a+1, c+1, x)]^\tau = 1 + \tau e_1(a+1, c+1)x + \ldots$ for x in the neighborhood of the origin, it follows that the smallest value of τ such that inequality $\Phi(a, c, x) \leq [\Phi(a+1, c+1, x)]^\tau$ holds true is $e_1(a, c)/e_1(a+1, c+1) = [a(c+1)]/[c(a+1)]$.

Finally, inequality (2.1.18) follows from (2.1.17) and the arithmetic-geometric mean inequality, while inequality (2.1.19) follows from the log-convexity of the function $\mu \mapsto \Phi(a, c+\mu, x)$. QED

Remark 2.1.2. Recently, Ismail and Laforgia [**IL**, Theorem 2.7] proved that if $c > a > 0$ and $x > 0$, then the following Turán type inequality holds true:

$$\Phi(a, c, x)\Phi(a, c+2, x) \geq \frac{(c-a)(c+1)}{(c+1-a)c} \cdot \Phi^2(a, c+1, x). \tag{2.1.20}$$

We note that our result from Theorem 2.1.2, i.e. the inequality (2.1.19) improves (2.1.20), because for all $c > a > 0$ we have $(c-a)(c+1) < (c-a+1)c$.

Remark 2.1.3. The following conjecture has been communicated to the author by the economist Michael Gordy: *if* $a \geq 0$, $c > a+2$ *and* $x \geq 0$, *then the following Turán type inequality holds*

$$[\Phi(a+1,c,x)]^2 \geq \Phi(a,c,x)\Phi(a+2,c,x).$$

This result would complement Theorem 2.1.2 and would have important applications in economic theory, as the proposer pointed out.

2.2 Extensions of trigonometric inequalities to Bessel functions

For $p > -1$ let us consider the function $\mathcal{J}_p : \mathbb{R} \to (-\infty, 1]$, defined by

$$\mathcal{J}_p(x) = 2^p\Gamma(p+1)x^{-p}J_p(x) = \sum_{n\geq 0} \frac{(-1/4)^n}{(p+1)_n n!}x^{2n}, \tag{2.2.1}$$

where

$$J_p(x) = \sum_{n\geq 0} \frac{(-1)^n (x/2)^{2n+p}}{n! \cdot \Gamma(p+n+1)} \quad \text{for all} \quad x \in \mathbb{R}$$

is the Bessel function of the first kind [**Wa**, p. 40]. It is worth mentioning that

$$\mathcal{J}_{-1/2}(x) = \sqrt{\pi/2} \cdot x^{1/2} J_{-1/2}(x) = \cos x, \tag{2.2.2}$$

$$\mathcal{J}_{1/2}(x) = \sqrt{\pi/2} \cdot x^{-1/2} J_{1/2}(x) = \frac{\sin x}{x}. \tag{2.2.3}$$

On the other hand, it is known that if $\tau \leq 3$ and $x \in (0, \pi/2)$, then the Lazarević-type inequality

$$\cos x < \left(\frac{\sin x}{x}\right)^{\tau} \tag{2.2.4}$$

holds [**Mi**, p. 238]. Moreover, here the exponent τ is not the least possible, i.e. if $\tau > 3$, then there exists $x_1 \in (0, \pi/2)$, depending on τ, such that

(2.2.4) holds for all $x \in (x_1, \pi/2)$. Observe that, using (2.2.2) and (2.2.3) inequality (2.2.4) for $\tau = 3$ can be rewritten as

$$[\mathcal{J}_{-1/2}(x)]^{-1/2+1} \leq [\mathcal{J}_{-1/2+1}(x)]^{-1/2+2}, \tag{2.2.5}$$

which is similar to (2.1.7). Note that because both members of (2.2.4) and (2.2.5) are even functions, we can deduce that both of inequalities hold for $|x| < \pi/2$. So in view of inequality (2.2.5), as in section 2.1, it is natural to ask: what is the analogue of this inequality for Bessel functions?

Our first main result of this section answer the above question. Moreover, we present some new inequalities for Bessel functions of the first kind.

Theorem 2.2.1. *Let $p > -1$ and let $j_{p,n}$ be the $n-$th positive zero of the Bessel function J_p. Further, consider the set $\Delta = \Delta_1 \cup \Delta_2$, where*

$$\Delta_1 = \bigcup_{n\geq 1} [-j_{p,2n}, -j_{p,2n-1}] \quad \text{and} \quad \Delta_2 = \bigcup_{n\geq 1} [j_{p,2n-1}, j_{p,2n}].$$

Then the following assertions are true:

1. *the function $x \mapsto \mathcal{J}_p(x)$ is negative on Δ and is strictly positive on $\mathbb{R} \setminus \Delta$;*
2. *the function $x \mapsto \mathcal{J}_p(x)$ is increasing on $(-j_{p,1}, 0]$ and is decreasing on $[0, j_{p,1})$;*
3. *the function $x \mapsto \mathcal{J}_p(x)$ is strictly log-concave on $\mathbb{R} \setminus \Delta$;*
4. *the function $x \mapsto J_p(x)$ is strictly log-concave on $(0, \infty) \setminus \Delta_2$, provided $p \geq 0$;*
5. *the function $p \mapsto \mathcal{J}_p(x)$ is increasing and log-concave for each fixed $x \in (-j_{p,1}, j_{p,1})$;*
6. *the function $p \mapsto J_p(x)$ is log-concave for each fixed $x \in (0, j_{p,1})$;*
7. *the function $p \mapsto \mathcal{J}_{p+1}(x)/\mathcal{J}_p(x)$ is decreasing for all fixed $x \in (-j_{p,1}, j_{p,1})$;*

8. *the function* $p \mapsto [\mathcal{J}_p(x)]^{p+1}$ *is increasing for each fixed* $x \in (-j_{p,1}, j_{p,1})$;

9. *the following inequalities hold for all* $\alpha \in (0,1)$ *and* $x, y \in (0,\infty) \setminus \Delta_2$, $x \neq y$

$$J_p(\alpha x) > \alpha^p J_p(x) [\mathcal{J}_p(x)]^{\alpha - 1}, \tag{2.2.6}$$

$$[x J_p'(x)]^2 > p\,[J_p(x)]^2 + x^2 J_p(x) J_p''(x), \tag{2.2.7}$$

$$J_p^2\left(\frac{x+y}{2}\right) > \left(\frac{x+y}{2\sqrt{xy}}\right)^{2p} J_p(x) J_p(y); \tag{2.2.8}$$

10. *the following inequalities hold for all* $x \in (-j_{p,1}, j_{p,1})$

$$[\mathcal{J}_{p+1}(x)]^2 \geq \mathcal{J}_p(x) \mathcal{J}_{p+2}(x), \tag{2.2.9}$$

$$[\mathcal{J}_p(x)]^{p+1} \leq [\mathcal{J}_{p+1}(x)]^{p+2}, \tag{2.2.10}$$

$$[\mathcal{J}_{p+1}(x)]^{1/(p+1)} + \frac{\mathcal{J}_{p+1}(x)}{\mathcal{J}_p(x)} \geq 2. \tag{2.2.11}$$

Proof. 1. It is known that [**Wa**, p. 498]

$$\mathcal{J}_p(x) = \prod_{n \geq 1} \left(1 - \frac{x^2}{j_{p,n}^2}\right). \tag{2.2.12}$$

Since $0 < j_{p,1} < j_{p,2} < \ldots < j_{p,n} < \ldots$ we have that if $x \in [j_{p,2n-1}, j_{p,2n}]$ or $x \in [-j_{p,2n}, -j_{p,2n-1}]$ then the first $(2n-1)$ terms of the above product is negative, and the remained terms are strictly positive. Hence $\mathcal{J}_p$ becomes negative on Δ. Now, if $x \in (-j_{p,1}, j_{p,1})$, then clearly each terms of the right hand side of (2.2.12) are strictly positive. Moreover, if $x \in (j_{p,2n}, j_{p,2n+1})$ or $x \in (-j_{p,2n+1}, -j_{p,2n})$, then the first $2n$ terms are strictly negative, while the rest is strictly positive. From this it follows that for the function $\mathcal{J}_p$ we have $\mathcal{J}_p(x) > 0$ for all $x \in \mathbb{R} \setminus \Delta$.

2. From part **1** the function $\mathcal{J}_p$ is strictly positive on $(-j_{p,1}, j_{p,1})$. Using the infinite product representation (2.2.12) we obtain

$$\frac{\mathrm{d}}{\mathrm{dx}} \log[\mathcal{J}_p(x)] = \frac{\mathcal{J}_p'(x)}{\mathcal{J}_p(x)} = -\sum_{n \geq 1} \frac{2x}{j_{p,n}^2 - x^2}. \tag{2.2.13}$$

From this we deduce that the function $x \mapsto \mathcal{J}_p(x)$ is increasing on $(-j_{p,1}, 0]$ and is decreasing on $[0, j_{p,1})$, as we required.

3. Using (2.2.13) and part **1** we conclude that

$$\frac{\mathrm{d}^2}{\mathrm{dx}^2} \log[\mathcal{J}_p(x)] = -2 \sum_{n \geq 1} \frac{j_{p,n}^2 + x^2}{(j_{p,n}^2 - x^2)^2} < 0$$

for all $x \in \mathbb{R} \setminus \Delta$, and consequently $\mathcal{J}_p$ is strictly log-concave on $\mathbb{R} \setminus \Delta$.

4. Rewriting (2.2.1) as

$$J_p(x) = \frac{x^p \mathcal{J}_p(x)}{2^p \Gamma(p+1)}, \tag{2.2.14}$$

the strict log-concavity of J_p follows from part **3**. Indeed, the function $x \mapsto x^p$ is log-concave on $(0, \infty)$ for all $p \geq 0$, which implies that J_p is strictly log-concave on $(0, \infty) \setminus \Delta_2$ as a product of a log-concave and a strictly log-concave function.

5. Using (2.2.12) we have

$$\log[\mathcal{J}_p(x)] = \sum_{n \geq 1} \log \left(1 - \frac{x^2}{j_{p,n}^2} \right).$$

On the other hand it is known [**Mu**, p. 317] that for each $n \geq 1$ integer the function $p \mapsto 1/j_{p,n}^2$ is decreasing and convex on $(-1, \infty)$. Consequently the functions $p \mapsto 1 - x^2/j_{p,n}^2$ are increasing and concave on $(-1, \infty)$, as well as the functions $p \mapsto \log(1 - x^2/j_{p,n}^2)$. Thus the function $p \mapsto \log[\mathcal{J}_p(x)]$ is increasing and concave for each fixed $x \in (-j_{p,1}, j_{p,1})$ as a sum of increasing and concave functions.

6. As in part **4** we use (2.2.14). Since the function $p \mapsto \Gamma(p+1)$ is log-convex and $p \mapsto \mathcal{J}_p(x)$ is log-concave, the function $p \mapsto J_p(x)$ is log-concave as a product of two log-concave functions.

7. Since $p \mapsto \log[\mathcal{J}_p(x)]$ is concave it follows that the function $p \mapsto \log[\mathcal{J}_{p+a}(x)] - \log[\mathcal{J}_p(x)]$ is decreasing for each $a > 0$. Choosing $a = 1$ we obtain that $p \mapsto \mathcal{J}_{p+1}(x)/\mathcal{J}_p(x)$ is decreasing, as we asserted.

8. To prove the required result consider the function $\varphi_3 : (-j_{p,1}, j_{p,1}) \to \mathbb{R}$, defined by

$$\varphi_3(x) = \frac{p+1}{q+1}\log[\mathcal{J}_p(x)] - \log[\mathcal{J}_q(x)],$$

where $q \geq p > -1$. On the other hand

$$\varphi_3'(x) = \frac{p+1}{q+1}\left[\frac{\mathcal{J}_p'(x)}{\mathcal{J}_p(x)}\right] - \frac{\mathcal{J}_q'(x)}{\mathcal{J}_q(x)} = -2xb_1(q)\left[\frac{\mathcal{J}_{p+1}(x)}{\mathcal{J}_p(x)} - \frac{\mathcal{J}_{q+1}(x)}{\mathcal{J}_q(x)}\right],$$

where we used the differentiation formula $\mathcal{J}_p'(x) = -2xb_1(p)\mathcal{J}_{p+1}(x)$, which can be derived easily from (2.2.1). Since $q \geq p$ we have $\mathcal{J}_{p+1}(x)/\mathcal{J}_p(x) \geq \mathcal{J}_{q+1}(x)/\mathcal{J}_q(x)$, we conclude that the function φ_3 is decreasing on $[0, j_{p,1})$ and is increasing on $(-j_{p,1}, 0]$. Consequently $\varphi_3(x) \leq \varphi_3(0) = 0$, i.e. the inequality $[\mathcal{J}_p(x)]^{p+1} \leq [\mathcal{J}_q(x)]^{q+1}$ holds for all $x \in (-j_{p,1}, j_{p,1})$.

9. Because from part **3** $\mathcal{J}_p$ is strictly log-concave, due to definition one has

$$\mathcal{J}_p(\alpha x + (1-\alpha)y) > [\mathcal{J}_p(x)]^{\alpha}[\mathcal{J}_p(y)]^{1-\alpha}, \tag{2.2.15}$$

where $p > -1$, $\alpha \in (0,1)$ and $x, y \in \mathbb{R} \setminus \Delta$, $x \neq y$. Choosing $y = 0$ in (2.2.15) and taking into account (2.2.1) we obtain (2.2.6). Moreover, taking in (2.2.15) $\alpha = 1/2$ from (2.2.1) yields (2.2.8). For (2.2.7) we use again the fact that $\mathcal{J}_p$ is strictly log-concave, that is $x \mapsto \mathcal{J}_p'(x)/\mathcal{J}_p(x) = J_p'(x)/J_p(x) - p/x$ is strictly decreasing.

10. Inequality (2.2.9) follows from the log-concavity of $p \mapsto \mathcal{J}_p(x)$, while inequality (2.2.10) follows from part **8**. Finally, the extension of Wilker's inequality, i.e. inequality (2.2.11) follows from (2.2.10) and the arithmetic-geometric mean inequality for the values $\mathcal{J}_{p+1}(x)]^{1/(p+1)}$ and $\mathcal{J}_{p+1}(x)/\mathcal{J}_p(x)$. With this the proof is complete. QED

Remark 2.2.1. a. Recently Giordano et al. **[GLP]** proved that the Bessel function $x \mapsto J_p(x)$ is log-concave on $(0, j_{p,1})$ for each $p > -1$. We note that part **4** of the above theorem states that this property for $p \geq 0$ remains true on $(0,\infty) \setminus \Delta_2$ too. Moreover, following the proof of part **4**, it is easy to see that the function $x \mapsto J_p(x)/x$ is also log-concave on $(0,\infty) \setminus \Delta_2$ for all $p \geq 1$. This was proved in [**GLP**] for $x \in (0, j_{p,1})$; see also [**Is**] for more details.

b. Part **6** was proved earlier by Muldoon [**Mu**] using a different argument. Moreover, Ismail and Muldoon [**IM**] showed that the function $p \mapsto J_{p+1}(x)/J_p(x)$ is decreasing when $p > -1, x > 0$ and $x \neq j_{p,n}$. We note that using part **7** and (2.2.1) we obtain that the function

$$p \mapsto \frac{J_{p+1}(x)}{J_p(x)} = \frac{x}{2(p+1)} \frac{\mathcal{J}_{p+1}(x)}{\mathcal{J}_p(x)}$$

is decreasing, but just for each $x \in (-j_{p,1}, j_{p,1})$.

c. It is worth mentioning that the analogous of (2.2.6), (2.2.7) and (2.2.8) for modified Bessel functions can be found in [**BN**] and [**Ne1**], while the Turán type inequality (2.2.9) was proved earlier for each $x \in \mathbb{R}$ by Szász [**Sza1**], and later by Joshi and Bissu [**JB**] using recursions. Finally, observe that inequality (2.2.10) in particular for $p = -1/2$ reduces to the Lazarević type inequality (2.2.5), while (2.2.11) reduces to Wilker's inequality (2.1.15). Here we used that $j_{-1/2,1} = \pi/2$, which can be verified using the infinite product representation of the cosine function [**AS**, p. 75] and formula (2.2.12).

It is also important to note that recently Wu and Srivastava [**WS**] have been improved significantly Wilker's inequality (2.1.15). Moreover, Baricz and Sándor [**BaSa**] have been extended the results from [**WS**] to the Bessel functions of the first kind by improving significantly the inequality (2.2.11). New researches, which are concerned with extensions of other trigonometric inequalities to Bessel functions of the first kind, are in active progress, readers can refer to the papers [**Ba2**, **BW**, **Ba10**].

Recently, Neuman [**Ne1**] proved that the function $x \mapsto \mathcal{I}_p(x)$ is strictly log-convex on $\mathbb{R}$ for all $p > -1/2$. We note that $x \mapsto \mathcal{I}_{-1/2}(x) = \cosh(x)$ is also strictly log-convex on $\mathbb{R}$, furthermore, we conjectured in [**BN**] that $x \mapsto \mathcal{I}_p(x)$ is strictly log-convex on $\mathbb{R}$ for each $p > -1$. The following result provides a partial positive answer to the above conjecture and is motivated by the proof of part **3** of Theorem 2.2.1.

Theorem 2.2.2. *If $p > -1$, then the function $x \mapsto \mathcal{I}_p(x)$ is strictly log-convex on $[-j_{p,1}, j_{p,1}]$, where $j_{p,1}$ is the first positive zero of the Bessel function J_p. Moreover, the function $x \mapsto \mathcal{I}_p(x)/\mathcal{J}_p(x) = I_p(x)/J_p(x)$ is strictly log-convex too on $(-j_{p,1}, j_{p,1})$. In particular, the following inequalities*

$$\frac{\left[I_p\left(\frac{x+y}{2}\right)\right]^2}{\left[J_p\left(\frac{x+y}{2}\right)\right]^2} \leq \frac{I_p(x)I_p(y)}{J_p(x)J_p(y)}, \quad x, y \in (-j_{p,1}, j_{p,1}),$$

$$\frac{\left[\cosh\left(\frac{x+y}{2}\right)\right]^2}{(\cosh x)(\cosh y)} \leq \frac{\left[\cos\left(\frac{x+y}{2}\right)\right]^2}{(\cos x)(\cos y)}, \quad x, y \in (-\pi/2, \pi/2),$$

$$\frac{\left[\sinh\left(\frac{x+y}{2}\right)\right]^2}{(\sinh x)(\sinh y)} \leq \frac{\left[\sin\left(\frac{x+y}{2}\right)\right]^2}{(\sin x)(\sin y)}, \quad x, y \in (-\pi, \pi)$$

hold true and equality hold in each of the above inequalities if and only if $x = y$.

Proof. It is known that, using (2.2.12), the function $\mathcal{I}_p$ may be represented as

$$\mathcal{I}_p(x) = \prod_{n \geq 1} \left(1 + \frac{x^2}{j_{p,n}^2}\right),$$

which implies that

$$\frac{\mathrm{d}^2}{\mathrm{dx}^2} \log[\mathcal{I}_p(x)] = 2 \sum_{n \geq 1} \frac{j_{p,n}^2 - x^2}{(j_{p,n}^2 + x^2)^2} > 0$$

for all $x \in [-j_{p,1}, j_{p,1}]$, and consequently $\mathcal{I}_p$ is strictly log-convex. Now using part **3** of Theorem 2.2.1 the function $x \mapsto 1/\mathcal{J}_p(x)$ is strictly log-convex on $(-j_{p,1}, j_{p,1})$. Hence, the function $x \mapsto \mathcal{I}_p(x)/\mathcal{J}_p(x) = I_p(x)/J_p(x)$ is strictly log-convex too on $(-j_{p,1}, j_{p,1})$, as a product of two strictly log-convex functions. Finally, the inequalities follows easily from the definition of log-convexity, taking into account that $j_{-1/2,1} = \pi/2$ and $j_{1/2,1} = \pi$. QED

Let us note the following trigonometric inequality which represent a partial answer to the problem E 1277 proposed by Oppenheim and solved by Carver in American Mathematical Monthly 65, 206-209 (1958): if $a \in (0, 1/2]$ and $|x| \leq \pi/2$, then [**Mi**, p. 238]

$$\frac{(a+1)\sin x}{1+a\cos x} \leq x \leq \frac{\pi}{2}\frac{\sin x}{1+a\cos x}. \tag{2.2.16}$$

The following result extends (2.2.16) to the function $\mathcal{J}_p$.

Theorem 2.2.3. *If* $0 < a \leq 1/2$, $p \geq -1/2$ *and* $x \in [-\pi/2, \pi/2]$, *then*

$$\frac{[a(2p+1)+(a+1)]\mathcal{J}_{p+1}(x)}{1+2a(p+1)\mathcal{J}_p(x)} \leq 1 \leq \frac{[a(2p+1)+\pi/2]\mathcal{J}_{p+1}(x)}{1+2a(p+1)\mathcal{J}_p(x)}. \tag{2.2.17}$$

Proof. Observe that when $p = -1/2$ from (2.2.2) and (2.2.3) it follows that (3.3.7) reduces to (2.2.16), which is equivalent to

$$(a+1)\mathcal{J}_{1/2}(x) \leq 1 + a\mathcal{J}_{-1/2}(x) \leq (\pi/2)\mathcal{J}_{1/2}(x). \tag{2.2.18}$$

Recall the Sonine integral formula [**Wa**, p. 373] for Bessel functions

$$J_{q+p+1}(x) = \frac{x^{p+1}}{2^p\Gamma(p+1)}\int_0^{\pi/2} J_q(x\sin\theta)\sin^{q+1}\theta\cos^{2p+1}\theta\,\mathrm{d}\theta,$$

where $p, q > -1$ and $x \in \mathbb{R}$. From this we obtain the following formula

$$\mathcal{J}_{q+p+1}(x) = \frac{2}{B(p+1,q+1)}\int_0^{\pi/2} \mathcal{J}_q(x\sin\theta)\sin^{2q+1}\theta\cos^{2p+1}\theta\,\mathrm{d}\theta, \tag{2.2.19}$$

which will be useful in the sequel. Here $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$ is the well-known Euler's beta function. Changing in (3.3.9) p with $p - 1/2$ and taking $q = -1/2$ ($q = 1/2$ respectively) one has for all $p > -1/2$ and $x \in \mathbb{R}$

$$\mathcal{J}_p(x) = \frac{2}{B(p+1/2,1/2)}\int_0^{\pi/2} \mathcal{J}_{-1/2}(x\sin\theta)\cos^{2p}\theta\,\mathrm{d}\theta, \tag{2.2.20}$$

$$\mathcal{J}_{p+1}(x) = \frac{2}{B(p+1/2,3/2)}\int_0^{\pi/2} \mathcal{J}_{1/2}(x\sin\theta)\sin^2\theta\cos^{2p}\theta\,\mathrm{d}\theta\,. \tag{2.2.21}$$

Thus, in view of (2.2.20) and (2.2.21), if we change x with $x\sin\theta$ in (3.3.8), and multiply (3.3.8) with $\sin^2\theta\cos^{2p}\theta$, then after integration it follows that the expression

$$\begin{aligned}E_p(x) &= \int_0^{\pi/2}\sin^2\theta\cos^{2p}\theta\,\mathrm{d}\theta + a\int_0^{\pi/2}\mathcal{J}_{-1/2}(x\sin\theta)(1-\cos^2\theta)\cos^{2p}\theta\,\mathrm{d}\theta\\ &= \frac{1}{2}B\left(p+\frac{1}{2},\frac{3}{2}\right)+\frac{a}{2}B\left(p+\frac{1}{2},\frac{1}{2}\right)\mathcal{J}_p(x)-\frac{a}{2}B\left(p+\frac{3}{2},\frac{1}{2}\right)\mathcal{J}_{p+1}(x)\end{aligned}$$

satisfies the following

$$\frac{a+1}{2}B\left(p+\frac{1}{2},\frac{3}{2}\right)\mathcal{J}_{p+1}(x)\le E_p(x)\le\frac{\pi}{4}B\left(p+\frac{1}{2},\frac{3}{2}\right)\mathcal{J}_{p+1}(x).$$

After simplifications we obtain that (3.3.7) holds. QED

We note that recently Baricz and Zhu [**Ba3**, **BZ**] improved the results of Theorem 2.2.3. Following the proof of the above theorem the next result is quite obvious.

Theorem 2.2.4. *For each $p\ge -1/2$ the function $\mathcal{J}_p$ is concave on* $[-\pi/2,\pi/2]$.

Proof. Since the cosine function is concave on $[-\pi/2,\pi/2]$, one has

$$\mathcal{J}_{-1/2}(\alpha x+(1-\alpha)y)\ge\alpha\mathcal{J}_{-1/2}(x)+(1-\alpha)\mathcal{J}_{-1/2}(y),$$

where $\alpha\in[0,1]$ and $x,y\in[-\pi/2,\pi/2]$. Changing x with $x\sin\theta$, y with $y\sin\theta$, from (2.2.20) it follows that $\mathcal{J}_p(\alpha x+(1-\alpha)y)\ge\alpha\mathcal{J}_p(x)+(1-\alpha)\mathcal{J}_p(y)$ holds for all $p\ge -1/2$, i.e. the function $\mathcal{J}_p$ is concave on $[-\pi/2,\pi/2]$. QED

Chapter 3

Turán type inequalities for probability density functions

3.1 Turán type inequalities for univariate distributions

In this chapter our aim is to present some Turán type inequalities for the probability density function (pdf) of the non-central chi-squared, non-central chi and Student distributions, in order to improve and complete some results from [**AnBa1**]. Moreover, at the end of this chapter we improve a known Turán type inequality for the modified Bessel function of the second kind, and we apply this result to deduce a short proof for the monotonicity of a product which involves modified Bessel functions of different kinds. Because the pdf-s of the non-central chi-squared and chi distribution are close connected to the modified Bessel function of the first kind, not surprisingly – as we will see below – these functions also satisfies some interesting Turán type inequalities. To achieve our goal first let us recall some basic things.

Let $X_1, X_2, \ldots, X_n$ be random variables that are normally distributed with unit variance and nonzero mean μ_i, where $i \in \{1, 2, \ldots, n\}$. It is known that the random variable $X_1^2 + X_2^2 + \cdots + X_n^2$ has the non-central chi-squared distribution with $n \in \{1, 2, 3, \dots\}$ degrees of freedom and non-centrality parameter $\lambda = \mu_1^2 + \mu_2^2 + \cdots + \mu_n^2$. The probability density function $\chi^2_{n,\lambda} : (0, \infty) \to (0, \infty)$ of the non-central chi-squared distribution [**JKB**] is defined as

$$\begin{aligned} \chi^2_{n,\lambda}(x) &= 2^{-n/2} e^{-(x+\lambda)/2} \sum_{k \geq 0} \frac{x^{n/2+k-1}(\lambda/4)^k}{\Gamma(n/2+k)k!} \\ &= \frac{e^{-(x+\lambda)/2}}{2} \left(\frac{x}{\lambda}\right)^{n/4-1/2} I_{n/2-1}(\sqrt{\lambda x}), \end{aligned}$$

where I_ν is the modified Bessel function of the first kind [**Wa**, p. 77]. Recall that when $\mu_1 = \cdots = \mu_n = 0$, i.e. $\lambda = 0$, the above distribution reduces to the classical (central) chi-squared distribution. The pdf $\chi^2_{n,0} : (0, \infty) \to (0, \infty)$ of this distribution is given by

$$\chi^2_n(x) = \chi^2_{n,0}(x) = \frac{x^{n/2-1} e^{-x/2}}{2^{n/2}\Gamma(n/2)}.$$

On the other hand it is known that if $X_1, X_2, \dots, X_n$ are random variables such as above, then $[X_1^2 + X_2^2 + \cdots + X_n^2]^{1/2}$ has the non-central chi distribution with $n \in \{1, 2, 3, \dots\}$ degrees of freedom and non-centrality parameter $\tau = [\mu_1^2 + \mu_2^2 + \cdots + \mu_n^2]^{1/2}$. The probability density function $\chi_{n,\tau} : (0, \infty) \to (0, \infty)$ of the non-central chi distribution [**JKB**] is defined as

$$\begin{aligned} \chi_{n,\tau}(x) &= 2^{-n/2+1} e^{-(x^2+\tau^2)/2} \sum_{k \geq 0} \frac{x^{n+2k-1}(\tau/2)^{2k}}{\Gamma(n/2+k)k!} \\ &= \tau e^{-(x^2+\tau^2)/2} \left(\frac{x}{\tau}\right)^{n/2} I_{n/2-1}(\tau x). \end{aligned}$$

Observe that when $\mu_1 = \cdots = \mu_n = 0$, i.e. $\tau = 0$, the above distribution reduces to the classical chi distribution with pdf $\chi_{n,0} : (0, \infty) \to (0, \infty)$ given by

$$\chi_n(x) = \chi_{n,0}(x) = \frac{x^{n-1} e^{-x^2/2}}{2^{n/2-1}\Gamma(n/2)}.$$

It is easy to verify, in view of the recurrence relation [**Wa**, p. 79]

$$xI_{\nu-1}(x) - xI_{\nu+1}(x) = 2\nu I_\nu(x),$$

that the pdf of the non-central chi-squared distribution satisfies the following recurrence formula

$$x\chi^2_{n,\lambda}(x) - \lambda\chi^2_{n+4,\lambda}(x) = n\chi^2_{n+2,\lambda}(x).$$

Recently, András and Baricz [**AnBa1**, Theorem 2.3] proved, among other things, that in fact the following Turán type inequality holds for all $\lambda, x \geq 0$ and $n \geq 1$ integer

$$\chi^2_{n,\lambda}(x)\chi^2_{n+4,\lambda}(x) < [\chi^2_{n+2,\lambda}(x)]^2. \tag{3.1.1}$$

This is an immediate consequence of the Turán type inequality

$$I_{\nu-1}(x)I_{\nu+1}(x) < I^2_\nu(x)$$

which is called in literature as Amos' inequality [**Am**, p. 243]. Moreover, using the Turán type inequality (2.1.8), i.e.

$$(\nu+1)I_{\nu-1}(x)I_{\nu+1}(x) \geq \nu I^2_\nu(x),$$

we deduced that the next reversed Turán-type inequality holds true

$$\left[\frac{\chi^2_{n+2,\lambda}(x)}{\chi^2_{n+2}(x)}\right]^2 \leq \left[\frac{\chi^2_{n,\lambda}(x)}{\chi^2_{n}(x)}\right]\left[\frac{\chi^2_{n+4,\lambda}(x)}{\chi^2_{n+4}(x)}\right]. \tag{3.1.2}$$

Before we state our first main result of this section we prove the following result for the modified Bessel function of the first kind, which provides a generalization of the above Amos' inequality. We note that the idea of the proof of the next lemma is taken from [**IM**, Lemma 2.3], where a similar result is proved for the Bessel function of the first kind.

Lemma 3.1.1. *The function* $\nu \mapsto I_\nu(x)$ *is log-concave on* $(-1, \infty)$ *for each fixed* $x > 0$.

Proof. First we prove that for each fixed $b \in (0,2]$ and each $x > 0$, the function $\nu \mapsto I_{\nu+b}(x)/I_\nu(x)$ is decreasing, where $\nu \geq -(b+1)/2$, $\nu > -1$. To show this, consider Neumann's formula [**Wa**, p. 441]

$$I_\mu(x)I_\nu(x) = \frac{2}{\pi}\int_0^{\pi/2} I_{\mu+\nu}(2x\cos\theta)\cos[(\mu-\nu)\theta]\,\mathrm{d}\theta,$$

which holds for all $\mu+\nu > -1$. Using this we find that for $2\nu+\varepsilon+b > -1$

$$I_\nu(x)I_{\nu+b+\varepsilon}(x) - I_{\nu+b}(x)I_{\nu+\varepsilon}(x) = -\frac{4}{\pi}\int_0^{\pi/2} I_{2\nu+b+\varepsilon}(2x\cos\theta)\sin(b\theta)\sin(\varepsilon\theta)\,\mathrm{d}\theta,$$

which is negative for all $b \in (0,2]$ and $\varepsilon \in (0,2]$. Consequently we obtain that the inequality

$$I_{\nu+b+\varepsilon}(x)/I_{\nu+\varepsilon}(x) < I_{\nu+b}(x)/I_\nu(x)$$

holds. Now, since $\nu \mapsto I_{\nu+b}(x)/I_\nu(x)$ is decreasing, it follows that $\nu \mapsto \log[I_{\nu+b}(x)] - \log[I_\nu(x)]$ is decreasing too. This implies that the function $\nu \mapsto \mathrm{d}\log[I_\nu(x)]/\mathrm{d}\nu$ is decreasing on $(-1,\infty)$, and with this the proof is complete.

QED

Our first main result of this section improves the above Turán type inequalities (3.1.1) and (3.1.2) and completes Theorem 2.3 from [**AnBa1**].

Theorem 3.1.1. *For $a > 0$ and $\lambda, \tau \geq 0$ consider the functions $\chi^2_{a,\lambda}, \chi_{a,\tau} : (0,\infty) \to (0,\infty)$, defined by the relations*

$$\chi^2_{a,\lambda}(x) = e^{-(x+\lambda)/2}\sum_{k\geq 0}\frac{(x/2)^{a/2}(\lambda/4)^k}{\Gamma(a/2+k)k!}x^{k-1},$$

$$\chi_{a,\tau}(x) = e^{-(x^2+\tau^2)/2}\sum_{k\geq 0}\frac{x^a(\tau/2)^{2k}}{2^{a/2-1}\Gamma(a/2+k)k!}x^{2k-1}.$$

Further, let us denote simply $\chi^2_{a,0}(x) = \chi^2_a(x)$ and $\chi_{a,0}(x) = \chi_a(x)$.

Then the functions

1. $x \mapsto \chi^2_{a,\lambda}(x)$ *and* $x \mapsto \chi_{a,\tau}(\sqrt{x})$ *are log-concave, provided* $a \geq 2$*;*

2. $\lambda \mapsto \chi^2_{a,\lambda}(x)$ *and* $\tau \mapsto \chi_{a,\sqrt{\tau}}(x)$ *are log-concave on* $[0, \infty)$*;*

3. $a \mapsto \chi^2_{a,\lambda}(x)$ *and* $a \mapsto \chi_{a,\tau}(x)$ *are log-concave on* $(0, \infty)$*;*

4. $a \mapsto \chi^2_{a,\lambda}(x)/\chi^2_a(x)$ *and* $a \mapsto \chi_{a,\tau}(x)/\chi_a(x)$ *are log-convex on* $(0, \infty)$*;*

5. $a \mapsto [\chi^2_{a+2,\lambda}(x)\chi^2_a(x)]/[\chi^2_{a+2}(x)\chi^2_{a,\lambda}(x)]$ *and* $a \mapsto [\chi_{a+2,\tau}(x)\chi_a(x)]/[\chi_{a+2}(x)\chi_{a,\tau}(x)]$ *are increasing on* $(0, \infty)$*.*

In particular, using parts **2**, **3**, **4**, **5**, *the following Turán type inequalities hold true for each* $x, a, a_1, a_2 > 0$ *and* $\lambda, \tau, \lambda_1, \lambda_2, \tau_1, \tau_2 \geq 0$

$$\left[\chi^2_{a,\frac{\lambda_1+\lambda_2}{2}}(x)\right]^2 \geq [\chi^2_{a,\lambda_1}(x)][\chi^2_{a,\lambda_2}(x)],$$

$$\left[\chi_{a,\sqrt{\frac{\tau_1+\tau_2}{2}}}(x)\right]^2 \geq [\chi_{a,\sqrt{\tau_1}}(x)][\chi_{a,\sqrt{\tau_2}}(x)],$$

$$\left[\chi^2_{\frac{a_1+a_2}{2},\lambda}(x)\right]^2 \geq [\chi^2_{a_1,\lambda}(x)][\chi^2_{a_2,\lambda}(x)],$$

$$\left[\chi_{\frac{a_1+a_2}{2},\sqrt{\tau}}(x)\right]^2 \geq [\chi_{a_1,\sqrt{\tau}}(x)][\chi_{a_2,\sqrt{\tau}}(x)],$$

$$\left[\frac{\chi^2_{\frac{a_1+a_2}{2},\lambda}(x)}{\chi^2_{\frac{a_1+a_2}{2}}(x)}\right]^2 \leq \left[\frac{\chi^2_{a_1,\lambda}(x)}{\chi^2_{a_1}(x)}\right]\left[\frac{\chi^2_{a_2,\lambda}(x)}{\chi^2_{a_2}(x)}\right],$$

$$\left[\frac{\chi_{\frac{a_1+a_2}{2},\tau}(x)}{\chi_{\frac{a_1+a_2}{2}}(x)}\right]^2 \leq \left[\frac{\chi_{a_1,\tau}(x)}{\chi_{a_1}(x)}\right]\left[\frac{\chi_{a_2,\tau}(x)}{\chi_{a_2}(x)}\right],$$

$$\left[\frac{\chi^2_{a_1+2,\lambda}(x)}{\chi^2_{a_2+2,\lambda}(x)}\right]\left[\frac{\chi^2_{a_2,\lambda}(x)}{\chi^2_{a_1,\lambda}(x)}\right] \geq \left[\frac{\chi^2_{a_1+2}(x)}{\chi^2_{a_2+2}(x)}\right]\left[\frac{\chi^2_{a_2}(x)}{\chi^2_{a_1}(x)}\right],\ a_1 \geq a_2,$$

$$\left[\frac{\chi_{a_1+2,\tau}(x)}{\chi_{a_2+2,\tau}(x)}\right]\left[\frac{\chi_{a_2,\tau}(x)}{\chi_{a_1,\tau}(x)}\right] \geq \left[\frac{\chi_{a_1+2}(x)}{\chi_{a_2+2}(x)}\right]\left[\frac{\chi_{a_2}(x)}{\chi_{a_1}(x)}\right],\ a_1 \geq a_2.$$

Proof. **1.** Let us consider the function $\gamma_\nu : (0,\infty) \to (1,\infty)$, defined by

$$\gamma_\nu(x) = 2^\nu \Gamma(\nu+1) x^{-\nu/2} I_\nu(\sqrt{x}),$$

which is called sometimes as the normalized modified Bessel function of the first kind of order ν. Taking into account the definition of γ_ν we have that

$$\chi^2_{a,\lambda}(x) = \left[\frac{x^{a/2-1}e^{-x/2}}{2^{a/2}\Gamma(a/2)}\right] e^{-\lambda/2}\gamma_{a/2-1}(\lambda x) = \chi^2_a(x) e^{-\lambda/2}\gamma_{a/2-1}(\lambda x). \tag{3.1.3}$$

It is easy to verify that the pdf χ^2_a is log-concave when $a \geq 2$. On the other hand [**BN**, Theorem 2.2] the function γ_ν is log-concave if $\nu > -1$. Thus the function $x \mapsto \gamma_{a/2-1}(\lambda x)$ is log-concave too for all $a > 0$. Hence in view of (3.1.3) the function $\chi^2_{a,\lambda}$ is log-concave as a product of two log-concave functions. Now observe that from definitions we easily have

$$\chi_{a,\tau}(\sqrt{x}) = \chi_{a,\sqrt{\lambda}}(\sqrt{x}) = 2\sqrt{x}\chi^2_{a,\lambda}(x). \tag{3.1.4}$$

Since the function $x \mapsto \sqrt{x}$ is log-concave on $(0,\infty)$, we conclude that the function $x \mapsto \chi_{a,\tau}(\sqrt{x})$ is log-concave as a product of two log-concave functions.

2. Using the same argument as in the previous part we have that the function $\lambda \mapsto \gamma_{a/2-1}(\lambda x)$ is log-concave too for all $a > 0$, and consequently $\lambda \mapsto \chi^2_{a,\lambda}(x)$ becomes also log-concave for all $a > 0$. Using the relation $\chi_{a,\sqrt{\tau}}(x) = 2\sqrt{x}\chi^2_{a,\tau}(x)$ the required log-concavity of the function $\tau \mapsto \chi_{a,\sqrt{\tau}}(x)$ follows.

3. First consider the case when $\lambda = \tau = 0$. Then clearly we have

$$\frac{\mathrm{d}^2}{\mathrm{d}a^2}\left[\log\left(\frac{x^{a/2-1}e^{-x/2}}{2^{a/2}\Gamma(a/2)}\right)\right] = \frac{\mathrm{d}^2}{\mathrm{d}a^2}\left[\log\left(\frac{x^{a-1}e^{-x^2/2}}{2^{a/2-1}\Gamma(a/2)}\right)\right] = -\frac{1}{4}\psi'\left(\frac{a}{2}\right) \leq 0,$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the so called digamma function, i.e. the logarithmic derivative of the gamma function. Here we used the known fact that the gamma function is log-convex on $(0,\infty)$, i.e. the digamma function

ψ is increasing on $(0,\infty)$. For this we refer the interested reader for example to [**DAB**] and to the references therein. Thus we have proved that the functions $a \mapsto \chi_a^2(x)$ and $a \mapsto \chi_a(x)$ are log-concave on $(0,\infty)$.

Now consider the case when $\lambda, \tau > 0$. Recall that

$$\chi_{a,\lambda}^2(x) = \frac{e^{-(x+\lambda)/2}}{2}\left(\frac{x}{\lambda}\right)^{a/4-1/2} I_{a/2-1}(\sqrt{\lambda x}),$$

$$\chi_{a,\tau}(x) = \tau e^{-(x^2+\tau^2)/2}\left(\frac{x}{\tau}\right)^{a/2} I_{a/2-1}(\tau x).$$

Since for each fixed $x, \lambda, \tau > 0$ the functions $a \mapsto (x/\lambda)^{a/4-1/2}$ and $a \mapsto (x/\tau)^{a/2}$ are log-concave, using Lemma 3.1.1 we conclude that the functions $a \mapsto \chi_{a,\lambda}^2(x)$ and $a \mapsto \chi_{a,\tau}(x)$ are log-concave too on $(0,\infty)$.

4. From (3.1.3) we get that

$$\chi_{a,\lambda}^2(x)/\chi_a^2(x) = e^{-\lambda/2}\gamma_{a/2-1}(\lambda x).$$

Observe that from part **1** of Theorem 2.1.1 clearly the function $\nu \mapsto \gamma_\nu(x^2)$ is log-convex on $(-1,\infty)$. This implies that the function $a \mapsto \gamma_{a/2-1}(\lambda x)$ is log-convex too on $(0,\infty)$ and consequently the function $a \mapsto \chi_{a,\lambda}^2(x)/\chi_a^2(x)$ is log-convex, as we required. Finally, application of (3.1.4) yields the log-convexity of $a \mapsto \chi_{a,\tau}(x)/\chi_a(x)$ on $(0,\infty)$.

5. It is known from part **2** of Theorem 2.1.1 that $\nu \mapsto \gamma_{\nu+1}(x)/\gamma_\nu(x)$ is increasing on $(-1,\infty)$ for all fixed $x > 0$. Hence we have that the function $a \mapsto \gamma_{a/2}(\lambda x)/\gamma_{a/2-1}(\lambda x)$ is increasing on $(0,\infty)$ for each fixed $x > 0$, $\lambda \geq 0$. Consequently in view of (3.1.3) and (3.1.4) the required result follows, and thus the proof is complete. QED

Remark 3.1.1. Let us consider the generalized Marcum $Q-$function of order $\nu > 0$ real, defined by

$$Q_\nu(a,b) = \int_b^\infty \chi_{2\nu,a}(t)\,\mathrm{d}t = \frac{1}{a^{\nu-1}}\int_b^\infty t^\nu e^{-\frac{t^2+a^2}{2}} I_{\nu-1}(at)\,\mathrm{d}t,$$

where $a, b \geq 0$, I_ν stands for the modified Bessel function of the first kind and the right hand side of the above equation is replaced by its limiting value when $a = 0$. This function is the survival (or reliability) function of the non-central chi distribution with shape parameter 2ν and non-centrality parameter a. From part **3** of Theorem 3.1.1 we know that the probability density function $\nu \mapsto \chi_{2\nu,a}(t)$ is log-concave. Recently, it was observed by Sun and Baricz [**SB**] that surprisingly for some values of a and b the function $Q_\nu(a, b)$ satisfies a Turán type inequality. Moreover, it was conjectured in [**SB**] that the function $\nu \mapsto Q_\nu(a, b)$ is strictly log-concave on $(0, \infty)$ for all $a \geq 0$ and $b > 0$. This conjecture was recently proved to be affirmative by Sun et al. [**SBZ**] and this result was used to deduce new and very tight bounds for the generalized Marcum $Q-$function. The problem of finding tight bounds for the generalized Marcum $Q-$function is useful in radar signal processing and in error performance analysis of multichannel dealing with partially coherent, differentially coherent, and non-coherent detections in digital communications. For more details we refer to the papers of the author [**Ba11**, **Ba12**].

It is also worth mentioning here that from the above inequalities can be deduced many similar interesting inequalities to those given in (3.1.1) and (3.1.2). For example, choosing in the fourth inequality of Theorem 3.1.1 $\tau = 0$ and $a_1 = n$, $a_2 = n+2$, respectively, we get the following Turán type inequality for the pdf of the chi distribution

$$[\chi_{n+1}(x)]^2 \geq \chi_n(x)\chi_{n+2}(x), \tag{3.1.5}$$

which holds for each $n \geq 1$ and $x > 0$. Now taking $n = 1$ in (3.1.5) for all $x > 0$ we obtain that

$$[\chi_2(x)]^2 \geq \chi_1(x)\chi_3(x).$$

In fact this last particular Turán type inequality establishes a relation between the pdf of the Rayleigh, half-normal and Maxwell distributions. Namely, χ_2 is the pdf of a particular Rayleigh distribution with parameter 1, χ_1 is the pdf of the well-known half-normal distribution, and χ_3 is the pdf of a particular Maxwell distribution which arises in many problems of physics and chemistry.

As we has seen in the previous theorem in particular for $\lambda = \tau = 0$ the pdf-s of the chi and the chi-squared distribution are log-concave with respect to their parameter. In fact many other univariate distributions has the same property. Namely, for example the exponential, power, Weibull, gamma and Pareto distributions has the property that their pdf is log-concave with respect to their parameter. Indeed, it is easy to verify that the functions

$$a \mapsto ae^{-ax}, \ a \mapsto ax^{a-1}e^{-x^a}, \ a \mapsto \frac{x^{a-1}e^{-x}}{\Gamma(a)}, \ a \mapsto ax^{-a-1}$$

are log-concave on $(0, \infty)$ for each fixed x belonging to the interval which support the distribution in the question.

In what follows we are mainly interested on the pdf of the Student distribution [**JKB**] of Gosset. Suppose that $X_1, X_2, \ldots, X_n$ are independent random variables that are normally distributed with expected value μ and variance 1. Then the random variable $\sqrt{n}(\overline{X} - \mu)/S$ – where $\overline{X}$ is the arithmetic mean of $X_1, X_2, \ldots, X_n$ and $(n-1)S^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2$ – has a Student distribution with $n-1$ degrees of freedom and the pdf $S_n : \mathbb{R} \to (0, \infty)$ of this distribution is defined as

$$S_n(x) = \frac{1}{\sqrt{\pi n}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}$$

The next result establishes two Turán type inequalities for this function.

Theorem 3.1.2. *For $a > 0$ consider the function $S_a : \mathbb{R} \to (0, \infty)$, defined by*

$$S_a(x) = \frac{1}{\sqrt{\pi a}} \frac{\Gamma\left(\frac{a+1}{2}\right)}{\Gamma\left(\frac{a}{2}\right)} \left(1 + \frac{x^2}{a}\right)^{-\frac{a+1}{2}}.$$

Then the function $a \mapsto S_a(x)$ is strictly log-concave on $(0, 1]$ for each fixed $x \in \mathbb{R}$, and is strictly log-concave too on $(1, \infty)$ for each fixed $|x| \leq \sqrt{2a/(a-1)}$. In particular, for each $n \geq 1$, $n \geq 2$ respectively, the next Turán type inequalities hold true

$$[S_{\frac{n+1}{n(n+2)}}(x)]^2 > S_{\frac{1}{n}}(x) S_{\frac{1}{n+2}}(x), \ x \in \mathbb{R},$$

$$[S_{n+1}(x)]^2 > S_n(x)S_{n+2}(x),\ |x| \leq \sqrt{\frac{2(n+2)}{n+1}}.$$

Proof. Simple computations show that

$$2\frac{\mathrm{d}}{\mathrm{da}}\left[\log S_a(x)\right] = \left[\psi\left(\frac{a+1}{2}\right) - \psi\left(\frac{a}{2}\right) - \frac{1}{a}\right] + \left[\frac{a+1}{a}\cdot\frac{x^2}{x^2+a} - \log\left(1+\frac{x^2}{a}\right)\right].$$

Observe that to prove the strict log-concavity of $a \mapsto S_a(x)$ it is enough to show that the function $a \mapsto \partial \log S_a(x)/\partial a$ is strictly decreasing. To prove this for convenience we consider the functions $f, g : (0,\infty) \to \mathbb{R}$, defined by

$$f(a) = \psi\left(a+\frac{1}{2}\right) - \psi\left(a\right) - \frac{1}{2a}$$

and

$$g(a) = \frac{a+1}{a}\cdot\frac{x^2}{x^2+a} - \log\left(1+\frac{x^2}{a}\right).$$

Then clearly we have $2\partial \log S_a(x)/\partial a = f(a/2) + g(a)$. Due to Qiu and Vuorinen [**QV2**, Theorem 2.1] it is known that the function f is strictly decreasing and convex from $(0,\infty)$ onto $(0,\infty)$. On the other hand

$$\frac{\mathrm{d}\, g(a)}{\mathrm{da}} = x^2\frac{(a-1)(x^2+a) - a(a+1)}{a^2(x^2+a)^2},$$

and this negative if $a \in (0,1]$ and $x \in \mathbb{R}$ or if $a > 1$ and $|x| \leq \sqrt{2a/(a-1)}$. Thus the function $a \mapsto \mathrm{d}\log S_a(x)/\,\mathrm{da}$ is decreasing, and consequently the strict log-concavity of $a \mapsto S_a(x)$ follows. Namely, if $a_1, a_2 \in (0,1]$ and $\alpha \in [0,1]$, then for each $x \in \mathbb{R}$ we have

$$S_{\alpha a_1+(1-\alpha)a_2}(x) \geq \left[S_{a_1}(x)\right]^{\alpha}\left[S_{a_2}(x)\right]^{1-\alpha}. \qquad (3.1.6)$$

Thus taking in (3.1.6) $\alpha = 1/2$, $a_1 = 1/n$ and $a_2 = 1/(n+2)$, we obtain the first inequality in Theorem 3.1.2. Moreover, (3.1.6) holds also for each $a_1, a_2 > 1$ and

$$|x| \leq \min\left\{\sqrt{\frac{2a_1}{a_1-1}}, \sqrt{\frac{2a_2}{a_2-1}}\right\}.$$

Finally, choosing in (3.1.6) $\alpha = 1/2$, $a_1 = n$ and $a_2 = n + 2$ we get the second Turán type inequality, and thus the proof is complete. QED

As far as we know the previous results are new. However, there is an extensive literature for similar inequalities for discrete random variables, see for example [**De**, **JG**] and the references therein. Let X be a random variable and for each $n \in \{1, 2, 3, \dots\}$ consider the probabilities $p_n = P(X = x_n)$, where $x_1 < x_2 < \dots$. By definition the discrete random variable X is log-concave if the (Turán type inequality) $p_{n+1}^2 \geq p_n p_{n+2}$ holds for all $n \geq 1$. This inequality is sometimes referred to the quadratic Newton inequality as Niculescu pointed out in [**Ni**]. For example, the Poisson distribution and the geometric distribution has the above property [**JG**]. Moreover, as Devroye [**De**] observed, the discrete Bessel distribution has the same property. Namely, b_{n+1}/b_n is decreasing in n and consequently $b_{n+1}^2 \geq b_n b_{n+2}$ holds true, where for each $\nu > -1$, $a > 0$ and $n \geq 0$ by definition

$$b_n := P(X = n) = \frac{(a/2)^{2n+\nu}}{I_\nu(a) n! \Gamma(n + \nu + 1)}.$$

3.2 Turán type inequalities for modified Bessel functions

Consider the modified Bessel function of the second kind K_ν (called sometimes as the MacDonald function), defined [**Wa**, p. 78] as follows:

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi},$$

where the right of this equation is replaced by its limiting value if ν is an integer or zero. Recently Laforgia and Natalini [**LN2**, Theorem 2.4] proved that for each $\nu_1, \nu_2 > -1/2$ and $x > 0$ the following reversed Turán type inequality holds true

$$K_{\nu_1}(x) K_{\nu_2}(x) \geq \left[K_{\frac{\nu_1+\nu_2}{2}}(x) \right]^2. \tag{3.2.1}$$

Observe that in particular, for $\nu_1 = \nu$ and $\nu_2 = \nu+2$, we get for all $\nu > -1/2$ and $x > 0$ the inequality [**LN2**, Eq. 2.18]

$$K_\nu(x)K_{\nu+2}(x) \geq [K_{\nu+1}(x)]^2 . \tag{3.2.2}$$

It is worth mentioning here that (3.2.2) holds true for $\nu = -1/2$, and consequently for $\nu = -3/2$ too. To prove these first recall the following formulas [**Wa**, p. 79]

$$x[K_{\nu-1}(x) - K_{\nu+1}(x)] = -2\nu K_\nu(x), \; K_{-\nu}(x) = K_\nu(x),$$

which hold for each ν and x unrestricted real numbers. From this clearly $K_{-1/2}(x) = K_{1/2}(x)$ and $K_{-3/2}(x) = K_{3/2}(x) = (1 + 1/x)K_{1/2}(x)$. These imply that the inequality

$$\begin{aligned} K_{-3/2}(x)K_{1/2}(x) - \left[K_{-1/2}(x)\right]^2 &= K_{-1/2}(x)K_{3/2}(x) - \left[K_{1/2}(x)\right]^2 \\ &= \frac{1}{x}\left[K_{1/2}(x)\right]^2 > 0 \end{aligned}$$

holds for all $x > 0$, as we required. Now, this suggest the following result, which improves Theorem 2.4 from [**LN2**]. We note here that a similar argument to those presented below in the proof of the next result can be used, as Giordano et al. [**GLP**, Remark 3.2] pointed out, in order to prove that the function K_ν is log-convex on $(0, \infty)$. For more details see also the paper of Sun and Baricz [**SB**].

Theorem 3.2.1. *The function $\nu \mapsto K_\nu(x)$ is strictly log-convex on $\mathbb{R}$ for each fixed $x > 0$. In particular, the reversed Turán type inequalities* (3.2.1) *and* (3.2.2) *hold for all $x > 0$, ν_1, ν_2 and ν arbitrary real numbers. Moreover – due to the strict log-convexity – in* (3.2.1) *equality holds if and only if $\nu_1 = \nu_2$ and the inequality* (3.2.2) *is strict.*

Proof. It is known the following integral representation [**Wa**, p. 181] of the modified Bessel function of the second kind

$$K_\nu(x) = \int_0^\infty e^{-x\cosh t}\cosh(\nu t)\,\mathrm{dt}, \tag{3.2.3}$$

which holds for each $x > 0$ and $\nu \in \mathbb{R}$. Using the Hölder-Rogers inequality (1.1.12), the integral formula (3.2.3) and the fact that the function $\nu \mapsto \cosh(\nu t)$ is strictly log-convex on $\mathbb{R}$ for all fixed $t \geq 0$, we conclude that

$$\begin{aligned}
K_{\alpha\nu_1+(1-\alpha)\nu_2}(x) &= \int_0^\infty e^{-x\cosh t}\cosh(\alpha\nu_1 t + (1-\alpha)\nu_2 t)\,\mathrm{dt} \\
&< \int_0^\infty e^{-x\cosh t}[\cosh(\nu_1 t)]^\alpha[\cosh(\nu_2 t)]^{1-\alpha}\,\mathrm{dt} \\
&= \int_0^\infty \left[e^{-x\cosh t}\cosh(\nu_1 t)\right]^\alpha \left[e^{-x\cosh t}\cosh(\nu_2 t)\right]^{1-\alpha}\mathrm{dt} \\
&\leq \left[\int_0^\infty e^{-x\cosh t}\cosh(\nu_1 t)\,\mathrm{dt}\right]^\alpha \left[\int_0^\infty e^{-x\cosh t}\cosh(\nu_2 t)\,\mathrm{dt}\right]^{1-\alpha} \\
&= [K_{\nu_1}(x)]^\alpha\,[K_{\nu_2}(x)]^{1-\alpha}
\end{aligned}$$

holds for all $\alpha \in (0,1)$, $\nu_1, \nu_2 \in \mathbb{R}$, $\nu_1 \neq \nu_2$ and $x > 0$. Consequently by definition $\nu \mapsto K_\nu(x)$ is strictly log-convex on $\mathbb{R}$, which completes the proof. QED

We note that after the first draft of the manuscript [**Ba9**] had been completed we found that Ismail and Muldoon [**IM**, Lemma 2.2] proved that for each fixed $x > 0$ and each fixed $b > 0$ the function $\nu \mapsto K_{\nu+b}(x)/K_\nu(x)$ is increasing on $\mathbb{R}$. Clearly this implies [**Mu**, p. 318] that $\nu \mapsto K_\nu(x)$ is log-convex on $\mathbb{R}$.

3.3 On a product of modified Bessel functions

Let I_ν and K_ν denote, as usual, the modified Bessel functions of the first and second kinds of order ν. Recently, motivated by a problem which arises in biophysics, in 2007 Penfold et al. [**PVG**, Theorem 3.1] proved, in a complicated way, that the product of the modified Bessel functions of the first and second kinds, i.e. $u \mapsto P_\nu(u) = I_\nu(u)K_\nu(u)$ is strictly decreasing on $(0, \infty)$ for all $\nu \geq 0$. It is worth mentioning that this result for $\nu = n \geq 0$ positive integer has been verified already in 1950 by Phillips and Malin

[**PM**, Corollary 2.2]. In this section our aim is to show that using the idea of Phillips and Malin the monotonicity of $u \mapsto P_\nu(u)$ for $\nu \geq -1/2$ can be verified easily by using the corresponding Turán type inequalities for modified Bessel functions. Moreover, we show that the function $u \mapsto I_\nu(u)K_\nu(u)$ is strictly completely monotonic on $(0, \infty)$ for all $\nu \in [-1/2, 1/2]$, i.e. for all $u > 0$, $\nu \in [-1/2, 1/2]$ and $m \in \{0, 1, 2, \dots\}$ we have

$$(-1)^m \left[I_\nu(u)K_\nu(u)\right]^{(m)} > 0.$$

In order to achieve our goal we improve some of the results of Phillips and Malin [**PM**, Eq. 1] concerning bounds for the logarithmic derivatives of the modified Bessel and Hankel functions.

Our main result of this section reads as follows:

Theorem 3.3.1. *The following assertions are true:*

1. P_ν *is strictly decreasing on* $(0, \infty)$ *for all* $\nu \geq -1/2$;
2. P_ν *is strictly completely monotonic on* $(0, \infty)$ *for all* $\nu \in [-1/2, 1/2]$;
3. $u \mapsto u^{1/2-\nu} P_\nu(u)$ *is strictly completely monotonic on* $(0, \infty)$ *for all* $\nu \geq 1/2$.

Proof. In what follows we prove the assertions stated in Theorem 3.3.1 and for the sake of completeness we point out also some historical facts concerning the corresponding Turán type inequalities.

1. Our strategy is as in the paper [**PM**]. Namely, we first show that for all $u > 0$ the following inequalities hold true:

$$uI'_\nu(u)/I_\nu(u) < \sqrt{u^2 + \nu^2}, \quad \text{where} \quad \nu \geq -1/2, \tag{3.3.1}$$

$$uK'_\nu(u)/K_\nu(u) < -\sqrt{u^2 + \nu^2}, \quad \text{where} \quad \nu \in \mathbb{R}. \tag{3.3.2}$$

Inequality (3.3.1) for $\nu > 0$ has been proved first by Gronwall [**Gron**, Eq. 16] in 1932, motivated by a problem in wave mechanics, and after than, in 1950 appears also in Phillips' and Malin's paper [**PM**, Eq. 1] for $\nu = n \geq 1$ positive integer. However, an equivalent form of (3.3.1) is known in literature as Amos' inequality. More precisely, observe that in view of the reccurence relations

$$I_{\nu-1}(u) = (\nu/u)I_\nu(u) + I'_\nu(u) \quad \text{and} \quad I_{\nu+1}(u) = I'_\nu(u) - (\nu/u)I_\nu(u),$$

the Turán type inequality

$$I_{\nu-1}(u)I_{\nu+1}(u) - [I_\nu(u)]^2 < 0 \tag{3.3.3}$$

is in fact equivalent with (3.3.1). Here we used the known fact that the function $u \mapsto I_\nu(u)$ is increasing on $(0,\infty)$ for all $\nu > -1$. The Turán type inequality (3.3.3) for $\nu \geq 0$ was proved first in 1951 by Thiruvenkatachar and Nanjundiah [**TN**] and later in 1974 by Amos [**Am**, p. 243], and in 1991 by Joshi and Bissu [**JB**, p. 339]. It is worth mentioning that due to Lemma 3.1.1 in fact the function $\nu \mapsto I_\nu(u)$ is log-concave on $(-1,\infty)$ for each fixed $u > 0$ (see also [**Lo**, Theorem 3]). Finally, let us note that in 1994 Lorch [**Lo**, p. 79] proved that (3.3.3) in fact holds for all $\nu \geq -1/2$ and $u > 0$. From this we conclude that (3.3.1) holds too for all $\nu \geq -1/2$ and $u > 0$.

Now, let us focus on the inequality (3.3.2). This inequality was proved for $\nu = n \geq 1$ positive integer by Phillips and Malin [**PM**, Eq. 1] in 1950. Now, for $\nu \in \mathbb{R}$ and $u > 0$ consider the following Turán type inequality

$$K_{\nu-1}(u)K_{\nu+1}(u) - [K_\nu(u)]^2 > 0, \tag{3.3.4}$$

which was proved in 1978 by Ismail and Muldoon [**IM**, Lemma 2.2] and recently by the author [**Ba9**, Theorem 1.13] (see also Theorem 3.2.1) using a completely different argument. Note that for $\nu > 1/2$ the Turán type inequality (3.3.4) appears also on Laforgia's and Natalini's paper [**LN2**, Eq. 2.18]. Observe that using the recurrence relations

$$K_{\nu-1}(u) = -(\nu/u)K_\nu(u) - K'_\nu(u) \quad \text{and} \quad K_{\nu+1}(u) = -K'_\nu(u) + (\nu/u)K_\nu(u),$$

we obtain that the Turán type inequality (3.3.4) is equivalent with the inequality

$$\left[uK'_\nu(u)/K_\nu(u) - \sqrt{u^2+\nu^2}\right]\left[uK'_\nu(u)/K_\nu(u) + \sqrt{u^2+\nu^2}\right] > 0,$$

which holds for all $\nu \in \mathbb{R}$ and $u > 0$. Now, since the function $u \mapsto K_\nu(u)$ is decreasing on $(0,\infty)$ for all $\nu \in \mathbb{R}$, it follows that (3.3.4) is equivalent with (3.3.2).

Finally, by using the inequalities (3.3.1) and (3.3.2), we conclude that

$$u\,[\log(P_\nu(u))]' = u\,[\log(I_\nu(u)K_\nu(u))]' < 0,$$

i.e. the function $u \mapsto P_\nu(u)$ is strictly decreasing on $(0,\infty)$ for all $\nu \geq -1/2$, as we required.

2. Recall that in 1978 Nåsell [**Na**, p. 2] proved that for all $u > 0$, $\nu > -1/2$ and $m \in \{0,1,2,\dots\}$ the inequality

$$(-1)^m \left[u^{-\nu} I_\nu(u) e^{-u}\right]^{(m)} > 0 \tag{3.3.5}$$

holds true, i.e. the function $u \mapsto u^{-\nu} I_\nu(u) e^{-u}$ is strictly completely monotonic on $(0,\infty)$ for each $\nu > -1/2$. Since $I_{-1/2}(u) = \sqrt{2/\pi} \cdot u^{-1/2} \cosh u$, it is easy to verify that for all $m \in \{0,1,2,\dots\}$ and $u > 0$ we have

$$(-1)^m [e^{-u} \cosh u]^{(m)} = \frac{1}{2}(-1)^m [e^{-2u} + 1]^{(m)} = 2^{m-1} e^{-2u} > 0.$$

Thus, in fact the function $u \mapsto u^{-\nu} I_\nu(u) e^{-u}$ is strictly completely monotonic on $(0,\infty)$ for each $\nu \geq -1/2$. On the other hand, due to Miller and Samko [**MS**, Theorem 5] it is known that the function $u \mapsto u^{\min\{\nu,1/2\}} e^u K_\nu(u)$ is completely monotonic on $(0,\infty)$ for each $\nu \geq 0$. Clearly, from this it follows that the function $u \mapsto u^\nu e^u K_\nu(u)$ is completely monotonic on $(0,\infty)$ for each $\nu \in [0,1/2]$. Now, since from Leibniz formula for derivatives the product of a strictly completely monotonic and a completely monotonic functions is strictly completely monotonic, we conclude that the function $u \mapsto P_\nu(u)$ is strictly completely monotonic on $(0,\infty)$ for all $\nu \in [0,1/2]$. In [**PVG**] Penfold et al. used a similar argument to those presented before in order to study the monotonicity of the function $u \mapsto P_\nu(u)$ for $\nu \in [0,1/2]$. However, it seems that the authors of [**PVG**] have overlooked the fact that inequality (3.3.5) is strict. Now, observe that since $K_\nu(u) = K_{-\nu}(u)$, we obtain from Miller's and Samko's result that the function $u \mapsto u^{-\nu} e^u K_\nu(u)$ is completely monotonic on $(0,\infty)$ for each $\nu \in [-1/2,0]$. Combining this with Nåsell's result, it follows that the function $u \mapsto u^{-2\nu} P_\nu(u)$ is strictly completely monotonic on $(0,\infty)$ for each $\nu \in [-1/2, 0]$. Finally, since $u \mapsto u^{2\nu}$ is strictly completely monotonic on $(0,\infty)$ for each $\nu \in [-1/2, 0]$, we conclude that the function $u \mapsto P_\nu(u)$ is strictly completely monotonic on $(0,\infty)$ for each $\nu \in [-1/2, 0]$, and hence for each $\nu \in [-1/2, 1/2]$.

3. Since the function $u \mapsto u^{-\nu} I_\nu(u) e^{-u}$ is strictly completely monotonic on $(0,\infty)$ for each $\nu \geq -1/2$ and the function $u \mapsto u^{1/2} e^u K_\nu(u)$ is completely monotonic on $(0,\infty)$ for each $\nu \geq 1/2$, the result follows. QED

In this section we have shown that the monotonicity of the function $u \mapsto I_\nu(u) K_\nu(u)$ can be deduced easily by using the Turán type inequalities (3.3.3) and (3.3.4). In order to deduce this result we have improved the range of validity for the inequalities (3.3.1) and (3.3.2), which are in fact equivalent with the inequalities (3.3.3) and (3.3.4). We note that our approach is much more simpler than the methods used in [**PVG**] and [**PM**].

Motivated, by the inequalities (3.3.1) and (3.3.2), in what follows we are mainly interested on the counterparts of inequalities (3.3.1) and (3.3.2). Phillips and Malin [**PM**, Eq. 1] proved that for each $u > 0$ and $\nu = n \geq 1$ positive integer the inequality

$$u I_\nu'(u)/I_\nu(u) > \sqrt{u^2 \nu/(\nu+1) + \nu^2} \qquad (3.3.6)$$

holds true. We note that in fact (3.3.6) holds true for all $\nu > 0$. To prove this consider again the Turán type inequality (2.1.8), i.e.

$$(\nu+1) I_{\nu-1}(u) I_{\nu+1}(u) - \nu \left[I_\nu(u)\right]^2 > 0, \qquad (3.3.7)$$

which was proved using completely different arguments in 1951 by Thiruvenkatachar and Nanjundiah [**TN**], in 1991 by Joshi and Bissu [**JB**, p. 339], and recently by the author [**Ba8**, Theorem 1]. Using again the corresponding recurrence relations, as in the proof of (3.3.1), it is easy to see that in fact (3.3.6) is equivalent with (3.3.7), thus the result of Phillips and Malin holds true in fact for all $\nu > 0$.

In the same paper Phillips and Malin [**PM**, Eq. 1] showed that for each $u > 0$ and $\nu = n > 1$ positive integer the inequality

$$u K_\nu'(u)/K_\nu(u) > -\sqrt{u^2 \nu/(\nu-1) + \nu^2}$$

holds true. From this we obtain that for each $\nu = n > 1$ positive integer and $u > 0$

$$\left[uK_\nu'(u)/K_\nu(u) + \sqrt{u^2\frac{\nu}{\nu-1} + \nu^2}\right]\left[uK_\nu'(u)/K_\nu(u) - \sqrt{u^2\frac{\nu}{\nu-1} + \nu^2}\right] < 0,$$

and using the corresponding recurrence relations, as in the proof of (3.3.2), we obtain that the Turán type inequality

$$(\nu-1)K_{\nu-1}(u)K_{\nu+1}(u) - (2\nu-1)\left[K_\nu(u)\right]^2 < 0 \tag{3.3.8}$$

holds true for all $\nu = n \geq 1$ positive integer and $u > 0$. Moreover, using again the recurrence relation $K_\nu(u) = K_{-\nu}(u)$, it follows from (3.3.8) that

$$(\nu+1)K_{\nu-1}(u)K_{\nu+1}(u) - (2\nu+1)\left[K_\nu(u)\right]^2 > 0, \tag{3.3.9}$$

holds true for all $\nu = n \leq -1$ negative integer and $u > 0$. As far as we know these Turán type inequalities are new and based on numerical experiments we conjecture the followings:

Conjecture. *The inequality* (3.3.8) *holds true for all real* $\nu \geq 0$, *while* (3.3.9) *holds true for all real* $\nu \leq 0$.

Chapter 4

Monotonicity patterns for Mills' ratio

4.1 Functional inequalities for Mills' ratio

Let us consider the probability density function $\varphi : \mathbb{R} \to [1/\sqrt{2\pi}, \infty)$ and the reliability function $\overline{\Phi} : \mathbb{R} \to (0, 1)$ of the standard normal law, defined by

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad \text{and} \qquad \overline{\Phi}(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \varphi(t)\, \mathrm{dt}\, .$$

The function $r : \mathbb{R} \to (0, \infty)$, defined by

$$r(x) = \frac{\overline{\Phi}(x)}{\varphi(x)} = e^{x^2/2} \int_x^\infty e^{-t^2/2}\, \mathrm{dt},$$

is known in literature as Mills' ratio [**Mi**, Sect. 2.26] of the standard normal law, while its reciprocal $1/r$, defined by $1/r(x) = \varphi(x)/\overline{\Phi}(x)$ for all $x \in \mathbb{R}$, is the so-called failure (hazard) rate. It is well-known that Mills' ratio is convex and strictly decreasing on $\mathbb{R}$, at the origin takes on the value $r(0) = \sqrt{\pi/2}$, and has tails described by the asymptotic expansion as $x \to \infty$

$$r(x) \sim \frac{1}{x} - \frac{1}{x^3} + \frac{1 \cdot 3}{x^5} - \frac{1 \cdot 3 \cdot 5}{x^7} + \dots .$$

This ratio is frequently used in mathematical statistics, see for example the paper of Feuerverger et al. [**FMAC**]. We note that various lower and upper bounds are known for this ratio, see [**Mi**, Sect. 2.26] and the references therein for results which refer to the problem of finding some simple functions which approximate r. The most known inequalities proved by Gordon [**Go**] in 1941 are the followings

$$\frac{x}{x^2+1} < r(x) < \frac{1}{x}, \tag{4.1.1}$$

for all $x > 0$. Recently Pinelis [**Pi4**] using a version of the monotone form of l'Hospital rule, i.e. Lemma 4.1.1 improved the inequality $r(x) < 1/x$, showing that in fact the function $x \mapsto xr(x)$ is strictly increasing on $(0, \infty)$. We note that in view of the derivative formula $r'(x) = xr(x) - 1$ this implies that the first inequality in (4.1.1) holds. Motivated by Pinelis' work in this section we show that using Pinelis' idea we can deduce some other known bounds for Mills' ratio, and in Lemma 4.1.2 we present a simple method to find other new functions which approximate r. Moreover, using the Pinelis version of the monotone l'Hospital's rule we study the monotonicity of some functions involving the Mills' ratio. As an application, at the end of section 4.1 we present an interesting chain of inequalities for Mills' ratio. Finally, in section 4.2 we prove that the Mills' ratio is strictly completely monotonic on $\mathbb{R}$ and using Buniakowsky-Schwarz's inequality for integrals we deduce some Turán-type inequalities for $n-$th derivative of Mills' ratio.

Before we state our main results of this section let us enounce the following version of the monotone form of l'Hospital rule due to Pinelis [**Pi2**].

Lemma 4.1.1. *Let $-\infty \leq a < b \leq \infty$ and let f and g be differentiable functions on (a, b). Assume that either $g' > 0$ everywhere on (a, b) or $g' < 0$ on (a, b). Furthermore, suppose that $f(a^+) = g(a^+) = 0$ or $f(b^-) = g(b^-) = 0$ and f'/g' is (strictly) increasing (decreasing) on (a, b). Then the ratio f/g is (strictly) increasing (decreasing) too on (a, b).*

We note that another version of monotone form of l'Hospital rule was proved by Anderson et al. [**AVV3**, **AVV4**]. Various versions of this monotone form of l'Hospital rule was used since 1982 in different areas of mathematics, for example in

1. differential geometry [**CGT**, **Grom**]

2. quasiconformal analysis [**AVV3**, **AVV4**]

3. statistics and probability [**Pi1**, **Pi3**, **Pi4**]

4. analytic inequalities [**AVV5**, **Pi5**].

The following result is an immediate application of Lemma 4.1.1 and provides a generalization of Proposition 1.2 due to Pinelis [**Pi4**].

Lemma 4.1.2. *Let us consider the differentiable function* $h : [a, \infty) \to (0, \infty)$, *where* $a \in \mathbb{R}$. *Assume that for all* $x \geq a$ *the product* $h(x)\varphi(x)$ *is not constant and* $[h\varphi]'$ *does not change sign, further* $\lim_{x\to\infty} h(x)\varphi(x) = 0$. *If the function* $g : [a, \infty) \to \mathbb{R}$, *defined by*

$$g(x) = [xh(x) - h'(x)]^{-1},$$

has the limit 1 *at infinity and is (strictly) increasing on* $[a, \infty)$, *then for all* $x \geq a$ *we have* $r(x) < h(x)$. *Moreover when* g *is (strictly) decreasing the above inequality is reversed, i.e. for all* $x \geq a$, *we have* $r(x) > h(x)$.

Proof. Since the product $h(x)\varphi(x)$ is not constant on $[a, \infty)$ it follows that $xh(x) - h'(x) \neq 0$ for all $x \in [a, \infty)$, thus the function g makes sense. Suppose that g is (strictly) increasing. But $\lim_{x\to\infty} h(x)\varphi(x) = \lim_{x\to\infty} \overline{\Phi}(x) = 0$ and $\varphi'(x) = -x\varphi(x)$, thus from the hypothesis it follows that the function

$$x \mapsto \frac{\overline{\Phi}'(x)}{[h(x)\varphi(x)]'} = \frac{-\varphi(x)}{h'(x)\varphi(x) + h(x)\varphi'(x)} = g(x)$$

is (strictly) increasing. Then applying the monotone form of l'Hospital rule, i.e. Lemma 4.1.1 we conclude that the ratio

$$\frac{r(x)}{h(x)} = \frac{\overline{\Phi}(x)}{h(x)\varphi(x)}$$

is (strictly) increasing too on $[a,\infty)$. Now using the l'Hospital rule for limits we obtain that

$$\lim_{x\to\infty}\frac{r(x)}{h(x)}=\lim_{x\to\infty}\frac{\overline{\Phi}(x)}{h(x)\varphi(x)}=\lim_{x\to\infty}\frac{\overline{\Phi}'(x)}{[h(x)\varphi(x)]'}=\lim_{x\to\infty}g(x)=1,$$

which implies that for all $x\geq a$ the inequality $r(x)<h(x)$ holds. Analogously, when the function g is (strictly) decreasing, we have that the ratio r/h is (strictly) decreasing too, thus for all $x\geq a$ we have the inequality $r(x)>h(x)$. QED

An interesting application of Lemma 4.1.2 is the following result.

Theorem 4.1.1. *Let us consider the Mills' ratio r of the standard normal law. Then for all $x\geq 0$ the following inequalities hold:*

$$\frac{2}{\sqrt{x^2+4}+x}<r(x)<\frac{4}{\sqrt{x^2+8}+3x}. \tag{4.1.2}$$

Proof. We note that the above lower and upper bounds improve the bounds from (4.1.1) of Gordon [**Go**]. The first inequality in (4.1.2) was proved by Birnbaum [**Bi**] and Komatsu [**Kom**], while the second inequality in (4.1.2) is due to Sampford [**Sa**]. However, we give here a different proof for these bounds. For this let us consider the functions $h_1,h_2:[0,\infty)\to(0,\infty)$, defined by

$$h_1(x)=\frac{2}{\sqrt{x^2+4}+x}\quad\text{and}\quad h_2(x)=\frac{4}{\sqrt{x^2+8}+3x}.$$

Clearly we have that the functions $x\mapsto h_1(x)\varphi(x)$ and $x\mapsto h_2(x)\varphi(x)$ are strictly decreasing on $[0,\infty)$. Moreover, it is easy to verify that

$$\lim_{x\to\infty}h_1(x)\varphi(x)=\lim_{x\to\infty}h_2(x)\varphi(x)=0.$$

Now consider the functions $g_1,g_2:[0,\infty)\to\mathbb{R}$, defined by

$$g_i(x)=[xh_i(x)-h_i'(x)]^{-1},$$

where $i \in \{1, 2\}$. Easy computations show that

$$g_1(x) = \frac{x\sqrt{x^2+4} + x^2 + 4}{2(x\sqrt{x^2+4} + 1)}$$

and

$$g_2(x) = \frac{\sqrt{x^2+8}(\sqrt{x^2+8} + 3x)^2}{4\left[3(x^2+1)\sqrt{x^2+8} + x^3 + 9x\right]}.$$

Moreover, $\lim_{x\to\infty} g_1(x) = \lim_{x\to\infty} g_2(x) = 1$ and for all $x \geq 0$ we have

$$\frac{\mathrm{d}\, g_1(x)}{\mathrm{dx}} = \frac{x\sqrt{x^2+4} - x^2 - 6}{\sqrt{x^2+4}(x\sqrt{x^2+4} + 1)^2} < 0,$$

$$\frac{\mathrm{d}\, g_2(x)}{\mathrm{dx}} = 6\frac{x(x^2+4)\sqrt{x^2+8} - x^4 - 8x^2 + 24}{\sqrt{x^2+8}\left[3(x^2+1)\sqrt{x^2+8} + x^3 + 9x\right]^2} > 0.$$

Here we used that the function $q : [0, \infty) \to \mathbb{R}$, defined by

$$q(x) = x^4 + 8x^2 - 24 - x(x^2+4)\sqrt{x^2+8},$$

is strictly decreasing, i.e. for all $x \geq 0$ we have

$$\frac{\mathrm{d}\, q(x)}{\mathrm{dx}} = 4\frac{x(x^2+4)\sqrt{x^2+8} - x^4 - 8x^2 - 8}{\sqrt{x^2+8}} < 0.$$

Consequently $q(x) \leq q(0) = -24 < 0$, and hence the required monotonicity follows. Finally, using that g_1 is strictly decreasing and g_2 is strictly increasing from Lemma 4.1.2 it follows that

$$h_1(x) < r(x) < h_2(x)$$

for all $x \geq 0$. Thus the proof is complete. QED

Another application of Lemma 4.1.1 is the following result.

Theorem 4.1.2. *The following assertions are true:*

1. *the Mills' ratio r is strictly log-convex on $\mathbb{R}$;*

2. *the function $x \mapsto xr'(x)/r(x)$ is strictly decreasing on $(0, \infty)$;*

3. *the function $x \mapsto xr'(x)$ is strictly decreasing on $(0, x_0)$ and is strictly increasing on (x_0, ∞), where $x_0 = 1.161527889\ldots$ is the unique positive root of the transcendent equation $x(x^2+2)\overline{\Phi}(x) = (x^2+1)\varphi(x)$;*

4. *the function $x \mapsto x^2 r'(x)$ is strictly decreasing on $(0, \infty)$.*

Proof. 1. Due to Sampford **[Sa]** we know that for all $x \in \mathbb{R}$ the inequality $(1/r(x))' < 1$ holds. Using the derivative formula $r'(x) = xr(x) - 1$ we get $(r'(x)/r(x))' = 1 - (1/r(x))' > 0$, i.e. the Mills' ratio r is strictly log-convex on $\mathbb{R}$. It is worth mentioning here that this above result can be derived too using Lemma 4.1.1. For this let us write the quotient $r'(x)/r(x)$ as follows

$$\frac{r'(x)}{r(x)} = \frac{x\overline{\Phi}(x) - \varphi(x)}{\overline{\Phi}(x)}.$$

Since $\lim_{x\to\infty}[x\overline{\Phi}(x) - \varphi(x)] = \lim_{x\to\infty}\overline{\Phi}(x) = 0$, using Lemma 4.1.1 it is enough to show that

$$x \mapsto \frac{[x\overline{\Phi}(x) - \varphi(x)]'}{[\overline{\Phi}(x)]'} = -\frac{\overline{\Phi}(x)}{\varphi(x)} = -r(x)$$

is strictly increasing, which is clearly true.

2. To prove the required result let us write the quotient $xr'(x)/r(x)$ as follows

$$\frac{xr'(x)}{r(x)} = \frac{x^2\overline{\Phi}(x) - x\varphi(x)}{\overline{\Phi}(x)}.$$

Clearly we have $\lim_{x\to\infty}[x^2\overline{\Phi}(x) - x\varphi(x)] = \lim_{x\to\infty}\overline{\Phi}(x) = 0$, and the function

$$x \mapsto \frac{[x^2\overline{\Phi}(x) - x\varphi(x)]'}{[\overline{\Phi}(x)]'} = \frac{\varphi(x) - 2x\overline{\Phi}(x)}{\varphi(x)} = 1 - 2xr(x)$$

is strictly decreasing on $(0,\infty)$, because from Lemma 4.1.1 we have that [**Pi4**, Proposition 1.2] the function $x \mapsto xr(x)$ is strictly increasing on $(0,\infty)$. In view of Lemma 4.1.1 this shows that the function $x \mapsto xr'(x)/r(x)$ is strictly decreasing on $(0,\infty)$.

3. To prove the required result we need to use instead of Lemma 4.1.1 a more general monotone form of l'Hospital's rule due to Pinelis [**Pi1**, Theorem 1.16]; see also the recent work of Pinelis [**Pi6**]. For this let us write the product $xr'(x)$ as follows

$$s_1(x) = xr'(x) = \frac{xr(x)-1}{1/x} = \frac{\overline{\Phi}(x) - \varphi(x)/x}{\varphi(x)/x^2}.$$

Since for all $x>0$ we have

$$\frac{\varphi(x)}{x^2}\frac{\mathrm{d}}{\mathrm{dx}}\left[\frac{\varphi(x)}{x^2}\right] = -\frac{1}{x^3}\left(1+\frac{2}{x^2}\right)\varphi^2(x) < 0,$$

to prove that $x \mapsto xr'(x)$ is strictly decreasing on $(0,x_0)$ and is strictly increasing on (x_0,∞), in view of [**Pi1**, Remark 1.17] and [**Pi1**, Theorem 1.16], it suffices to show that the function

$$x \mapsto s_2(x) = \frac{[\overline{\Phi}(x) - \varphi(x)/x]'}{[\varphi(x)/x^2]'} = -\frac{x}{x^2+2}$$

is strictly decreasing on $(0,\sqrt{2})$ and is strictly increasing on $(\sqrt{2},\infty)$, which is clearly true. With other words we proved that the waves of s_1 follows the waves of s_2. We note that the behaviour of the roots, i.e. the inequality $x_0 < \sqrt{2}$ is necessary, because in view of [**Pi1**, Remark 1.15] s_1 may switch from decrease to increase only on intervals of decrease of s_2.

4. Let us write the product $x^2r'(x)$ as follows

$$x^2r'(x) = \frac{xr(x)-1}{1/x^2} = \frac{\overline{\Phi}(x) - \varphi(x)/x}{\varphi(x)/x^3}.$$

Since $\lim_{x\to\infty}[\overline{\Phi}(x) - \varphi(x)/x] = \lim_{x\to\infty}\varphi(x)/x^3 = 0$, from monotone form of l'Hospital rule, i.e. Lemma 4.1.1 it is enough to show that

$$x \mapsto \frac{[\overline{\Phi}(x) - \varphi(x)/x]'}{[\varphi(x)/x^3]'} = -\frac{x^4}{x^4+3x^2}$$

is strictly decreasing on $(0,\infty)$, which is true. With this the proof is complete. $\boxed{\text{QED}}$

An immediate application of Theorem 4.1.2 is the following result.

Corollary 4.1.1. *If $x, y > x_0$, then the following chain of inequalities holds*

$$\frac{2r(x)r(y)}{r(x)+r(y)} \leq r\left(\frac{x+y}{2}\right) \leq \sqrt{r(x)r(y)}$$

$$\leq r(\sqrt{xy}) \leq \frac{r(x)+r(y)}{2} \leq r\left(\frac{2xy}{x+y}\right). \tag{4.1.3}$$

Moreover, the first, second, third and fifth inequalities hold for all x, y strict positive real numbers, while the fourth inequality is reversed if $x, y \in (0, x_0)$. In each of the above inequalities equality holds if and only if $x = y$.

Proof. Observe that the first inequality in (4.1.3) follows from the strict convexity of the failure rate $1/r$ on $(0,\infty)$, which was proved by Sampford [**Sa**]. For the second inequality in (4.1.3) we use that from part **1** of Theorem 4.1.2 the function r is strictly log-convex on $(0,\infty)$. To prove the third inequality recall that from part **2** of Theorem 4.1.2 we know that the function $x \mapsto xr'(x)/r(x)$ is strictly decreasing on $(0,\infty)$. This implies that the Mills' ratio is strictly multiplicatively concave on $(0,\infty)$, i.e. for all $x, y > 0$ and $\lambda \in (0,1)$, the inequality $r(x^\lambda y^{1-\lambda}) \geq [r(x)]^\lambda [r(y)]^{1-\lambda}$ holds. Here we used part 5 of Corollary 2.5 due to Anderson et al. [**AVV6**]. We use again Corollary 2.5 from [**AVV6**], namely part 4, in order to prove the fourth inequality in (4.1.3). More precisely if the function $x \mapsto xr'(x)$ is strictly increasing (decreasing respectively) then we have that the fourth inequality in (4.1.3) (and its reverse respectively) holds. Now from part **3** of Theorem 4.1.2 the asserted monotonicity follows. Finally, observe that for the last inequality in (4.1.3) it is enough to show that $x \mapsto r(1/x)$ is strictly concave, i.e. using part 7 of Corollary 2.5 from [**AVV6**] the function $x \mapsto x^2 r'(x)$ is strictly decreasing, which is proved in part **4** of Theorem 4.1.2.

From the strict convexity of $1/r$, strict log-convexity, strict multiplicatively concavity of r, strict monotonicity of $x \mapsto xr'(x)$ and strict concavity of $x \mapsto r(1/x)$ we deduce that equality holds in (4.1.3) if and only if $x = y$. $\boxed{\text{QED}}$

4.2 Complete monotonicity of Mills' ratio

Let $r_0(x) := 0$ for all $x > 0$. Further for all $n \geq 1$ and $x > 0$ let us consider the $n-$th initial segment of the Laplace continuous fraction for Mills' ratio

$$r_n(x) = \cfrac{1}{x + \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{\ddots \, x + \cfrac{n-1}{x}}}}}.$$

In [**Pi4**] Pinelis using the monotone form of l'Hospital rule investigated monotonicity properties of the relative error $\delta_n(x) = (r(x) - r_n(x))/r(x)$ and among other things proved that [**Pi4**, Theorem 1.5] for all $n \geq 0$ and all $x > 0$ we have $(-1)^n\delta_n > 0$. Observe that this inequality is actually equivalent with the following result of Shenton [**Sh**, Eq. 19]

$$r_{2n}(x) < r(x) < r_{2n+1}(x), \tag{4.2.1}$$

where $n \geq 1$ and $x > 0$. In what follows we show that the inequality $(-1)^n\delta_n > 0$, or (4.2.1) is in fact equivalent with the strict complete monotonicity of Mills' ratio on $(0, \infty)$. Moreover, we show that Mills' ratio is strictly completely monotonic on $\mathbb{R}$, and we obtain some Turán type inequalities for $n-$th derivative of Mills' ratio.

Theorem 4.2.1. *Let x be a real number and $n \geq 1$ a natural number. Further let $r^{(n)}$ be the $n-$th derivative of Mills' ratio and let us consider the expression $\Delta_n = (-1)^n r^{(n)}(x)$. Then the Mills' ratio is strictly completely monotonic on $\mathbb{R}$, i.e. $\Delta_n > 0$, and satisfies the following reversed Turán type inequality*

$$\Delta_n \cdot \Delta_{n+2} \geq [\Delta_{n+1}]^2. \tag{4.2.2}$$

Moreover the function $x \mapsto \left|r^{(n)}(x)\right|$ is strictly log-convex on $\mathbb{R}$ and consequently in view of (4.2.2) the following Turán type inequalities hold:

$$\Delta_{2n-1} \cdot \Delta_{2n+1} \geq [\Delta_{2n}]^2 \geq \frac{2n}{2n+1}\Delta_{2n-1} \cdot \Delta_{2n+1}. \tag{4.2.3}$$

Proof. Let us consider the following integral

$$\gamma_n(x) = \int_x^\infty (x-t)^n \varphi(t)\,\mathrm{dt} = (-1)^n \int_x^\infty (t-x)^n \varphi(t)\,\mathrm{dt}, \qquad (4.2.4)$$

where $x \in \mathbb{R}$ and $n \in \{1,2,3,\dots\}$. Then it is known that for all $x \in \mathbb{R}$ one has [**Sh**, Eq. 21] $r^{(n)}(x) = \gamma_n(x)\varphi(x)$, and hence

$$0 < \Delta_n = (-1)^n r^{(n)}(x) = (-1)^n \gamma_n(x)\varphi(x) = \varphi(x) \int_x^\infty (t-x)^n \varphi(t)\,\mathrm{dt}\,.$$

We note here that actually

$$r(x) = e^{x^2/2} \int_x^\infty e^{-t^2/2}\,\mathrm{dt} = \int_0^\infty e^{-xt} e^{-t^2/2}\,\mathrm{dt}, \qquad (4.2.5)$$

i.e. Mills' ratio r is the Laplace transform of the function $t \mapsto e^{-t^2/2}$. Thus using the classical Bernstein theorem [**Fe**], which characterizes completely monotonic functions as Laplace transforms of positive measures whose support is contained in $[0,\infty)$, we conclude that r is indeed strictly completely monotonic.

Taking in the Buniakowsky-Schwarz inequality (1.1.11) $\alpha = n$, $\beta = n+2$, $a = x$, $b = \infty$, and for all $x \in \mathbb{R}$ fixed, $f(t) = \varphi(t)$ and $g(t) = t - x$, where $t \in [x,\infty]$, we obtain that

$$\int_x^\infty \varphi(t)(t-x)^n\,\mathrm{dt} \cdot \int_x^\infty \varphi(t)(t-x)^{n+2}\,\mathrm{dt} \geq \left[\int_x^\infty \varphi(t)(t-x)^{n+1}\,\mathrm{dt}\right]^2$$

holds, which is equivalent with inequality (4.2.2). Now changing n with $2n-1$ in (4.2.2) immediately we get the first inequality in (4.2.3). Observe that to prove that the function $x \mapsto \left|r^{(n)}(x)\right|$ is log-convex on $\mathbb{R}$ it is enough to show that $\Delta_n \cdot \Delta_{n+2} \geq [\Delta_{n+1}]^2$ holds, which is exactly (4.2.2). Now for the strict log-convexity of the function $x \mapsto \left|r^{(n)}(x)\right|$ observe that using mathematical induction from (4.2.5) we have

$$\left|r^{(n)}(x)\right| = (-1)^n r^{(n)}(x) = \int_0^\infty e^{-xt} t^n e^{-t^2/2}\,\mathrm{dt}\,.$$

Let us observe that the integrand in the above integral is log-convex in x. Thus $x \mapsto \left|r^{(n)}(x)\right|$ is strictly log-convex by Theorem B-6 in [**Ca**, p. 296].

On the other hand from (4.2.4) easily follows that for all $n \geq 1$ we have $\gamma_n' = n\gamma_{n-1}$. Thus from the log-convexity of $x \mapsto r^{(2n)}(x)$ we obtain that

$$\begin{aligned} 0 \leq & \frac{\mathrm{d}^2}{\mathrm{dx}^2}\left[\log r^{(n)}(x)\right] = \frac{\mathrm{d}}{\mathrm{dx}}\left[\frac{r^{(2n+1)}(x)}{r^{(2n)}(x)}\right] = \frac{\mathrm{d}}{\mathrm{dx}}\left[\frac{\gamma_{2n+1}(x)}{\gamma_{2n}(x)}\right] \\ = & \frac{(2n+1)[\gamma_{2n}(x)]^2}{[\gamma_{2n}(x)]^2} - \frac{(2n)\gamma_{2n-1}(x)\gamma_{2n+1}(x)}{[\gamma_{2n}(x)]^2}, \end{aligned}$$

which is equivalent with the second inequality in (4.2.3). Thus the proof is complete. QED

Remark 4.2.1. a. Consider the following polynomial expressions f_n and g_n defined by the following formulas:

$$g_0(x) = 1;\ f_0(x) = 0;\ g_1(x) = x;\ f_1(x) = 1;$$

$$f_n(x) = xf_{n-1}(x) + (n-1)f_{n-2}(x) \text{ and } g_n(x) = xg_{n-1}(x) + (n-1)g_{n-2}(x),$$

where $x \in \mathbb{R}$ and $n \in \{2, 3, 4, \dots\}$. It is easy to verify that [**Pi4**, Lemma 2.2] for all $n \in \{1, 2, 3, \dots\}$ and $x \in \mathbb{R}$

$$f_n'(x) = xf_n(x) + nf_{n-1}(x) - g_n(x) \text{ and } g_n'(x) = ng_{n-1}(x). \qquad (4.2.6)$$

Due to Shenton [**Sh**, Eq. 17] it is known that $r_n(x) = f_n(x)/g_n(x)$, moreover from general properties of continuous fractions [**Sh**, Eq. 19] we have that (4.2.1) holds. As we has seen in Theorem 4.1.1 some known approximations for Mills' ratio can be deduced easily using the monotone form of l'Hospital rule. We note that with the second inequality in (4.2.1) the situation is the same. In what follows we would like to present an alternative proof of this inequality using Lemma 4.1.2. For this let us consider the function $h_u : (0, \infty) \to (0, \infty)$, defined by

$$h_u(x) = r_{2n+1}(x) = \frac{f_{2n+1}(x)}{g_{2n+1}(x)},$$

where $n \geq 1$. Since for all $n \geq 1$ integer and $x \in \mathbb{R}$ we have [**Sh**, Eq. 18]

$$f_n(x)g_{n-1}(x) - g_n(x)f_{n-1}(x) = (-1)^{n-1}(n-1)!,$$

in view of (4.2.6) it is easy to verify that

$$[h_u(x)\varphi(x)]' = \varphi(x)[h_u'(x) - xh_u(x)] = -\varphi(x)\left[\frac{(2n+1)!}{[g_{2n+1}(x)]^2} + 1\right] < 0,$$

i.e. for all $x > 0$ the product $h_u(x)\varphi(x)$ is not constant and $[h_u\varphi]'$ does not change sign, further $\lim_{x\to\infty} h_u(x)\varphi(x) = 0$. Now consider the function $g_u : (0, \infty) \to (0, \infty)$, defined by

$$g_u(x) = \frac{1}{xh_u(x) - h_u'(x)} = \frac{[g_{2n+1}(x)]^2}{(2n+1)! + [g_{2n+1}(x)]^2}.$$

Then the function g_u has the limit 1 at infinity and simple computations shows that

$$g_u'(x) = \frac{2(2n+1)(2n+1)!g_{2n}(x)g_{2n+1}(x)}{[(2n+1)! + [g_{2n+1}(x)]^2]^2} > 0,$$

thus from Lemma 4.1.2 we have that $r(x) < h_u(x) = r_{2n+1}(x)$.

b. A positive function f is called strictly logarithmically completely monotonic [**QH**] on an interval I if f has derivatives of all orders on I and its logarithm $\log f$ satisfies $(-1)^n[\log f(x)]^{(n)} > 0$, for all $x \in I$ and $n \geq 1$. Note that a strictly logarithmically completely monotonic function is always strictly completely monotonic, but not conversely. Recently, Berg [**Be**] pointed out that the logarithmically complete monotonic functions in fact are the same as those studied by Horn [**Ho**] under the name infinitely divisible completely monotonic functions. It is worth mentioning here that using the results of Sampford [**Sa**], i.e. inequalities $(1/r(x))' < 1$ and $(1/r(x))'' > 0$, it is easy to verify that for all $x \in \mathbb{R}$ we have $(-1)^n[\log r(x)]^{(n)} > 0$, where $n = 1, 2, 3$. All the same, as Berg pointed out in private communication, the Mills' ratio is not logarithmically completely monotonic, because in [**SH**,

p. 126] it is proved that the "half-normal density" is not infinitely divisible and this means that the Mills' ratio r is not logaritmically completely monotonic.

Finally, we note that after the manuscript [**Ba5**] had been completed we found on the preprint server www.arxiv.org the paper of Kouba [**Kou**], where the complete monotonicity of r and the strict log-convexity of $x \mapsto \left|r^{(n)}(x)\right|$ were proved using a different approach.

Moreover, during the course of writing this thesis we have found the paper of Alzer [**Al**], where it is proved that the function V_q, defined by

$$V_q(x) = \frac{2e^{x^2}}{\Gamma(q+1)} \int_x^\infty e^{-t^2}(t^2 - x^2)^q \, \mathrm{d}t,$$

is completely monotonic on $(0, \infty)$ if and only if $q \in (-1, 0]$. This function arise naturally in the study of atoms in magnetic fields, more precisely, V_q is the one-dimensional regularization of the Coulomb potential. For more details the interested reader is referred to the paper of Ruskai and Werner [**RW**]. Since the above function in particular reduces to Mills' ratio, that is, for all $x \in \mathbb{R}$ we have $V_0(x) = \sqrt{2}r(x\sqrt{2})$, it is clear that Alzer's result implies the complete monotonicity of Mills' ratio on $(0, \infty)$. This yields actually the fourth proof for the complete monotonicity of r. We note that Alzer's method is completely different to those presented above and it would be of interesting to see if the other main results of this chapter can be extended to the function V_q. Furthermore, using the Hölder-Rogers inequality (1.1.12), it is easy to see that the function $q \mapsto \Gamma(q+1)V_q(x)$ is log-convex on $(-1, \infty)$ for all $x > 0$ fixed. This naturally suggest the following open problem:

Open Problem. *Is it the function $q \mapsto V_q(x)$ log-convex on $(-1, \infty)$ for all $x > 0$ fixed?*

If this result were be true then would improve Alzer's result [**Al**, Theorem 3], which states that the function $q \mapsto V_q(x)$ is convex on $(-1, \infty)$ for all $x > 0$ fixed. Moreover, this would imply in particular that V_q satisfies a reversed Turán-type inequality.

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Összefoglaló

Turán Pál 1941-ben Legendre polinomokra igazolt egyenlőtlenségére 1948-ban Szegő Gábor négy elegáns bizonyítást adott, és kiterjesztette az eredményt ultraszférikus, Laguerre, Hermite és Jacobi polinomokra. Ezt követően, a Turán Pál és a Szegő Gábor eredményeinek hatására az 50-es és a 60-as években matematikusok serege igazolta, hogy a legtöbb (ortogonális) polinom (mint például Appell, Bernstein-Szegő, Hermite, Jacobi, Jensen, Pollaczek, Lommel, Laguerre, Askey-Wilson, Gegenbauer), illetve speciális függvény (mint például Bessel, q-Bessel, módosított Bessel, Riemann zeta) teljesít bizonyos Turán típusú egyenlőtlenségeket. Napjainkban ismét előtérbe kerültek a Turán típusú egyenlőtlenségek, más speciális függvények kapcsán.

A disszertációban különböző speciális függvényekre vonatkozó Turán típusú egyenlőtlenségekkel foglalkoztunk. Az első fejezetben a következő témákat tárgyaltuk: az elliptikus integrálokra és a Gauss-féle hipergeometrikus sorra vonatkozó Turán típusú egyenlőtlenségek, elliptikus integrálokra vonatkozó alsó és felső korlátok, az általánosított Grötzsch gyűrű függvény paraméter szerinti viselkedése és a Poincaré metrika elliptikus integrálokkal való közelítése. Itt megjegyezzük, hogy ezek az eredmények kiegészítik a Legendre és Jacobi polinomokra igazolt klasszikus eredményeket. Pontosabban, az 1.1.1 Tétel kiegészíti Turán Pál Legendre polinomokra igazolt eredményét [**Tu**], míg az 1.4.1 Tétel kiegészíti Szegő Gábor [**Sze1**] és George Gasper [**Ga2**] Jacobi polinomokra igazolt eredményeit. A fejezet végén felsoroltunk néhány a kutatás során nyitva maradt kérdést a hipergeometrikus és a gamma függvényekkel kapcsolatban.

A második fejezetben Bessel és módosított Bessel függvényekre vonatkozó Turán típusú egyenlőtlenségekkel, és azok azonnali alkalmazásaival foglalkoztunk. Alkalmaztuk a megfelelő Turán típusú egyenlőtlenségeket trigonometrikus és hiperbolikus függvényekre felírt egyenlőtlenlenségek általánosításához. A fő eredmények a 2.1.1 Tétel és 2.2.1 Tétel, amelyekben a fent említett egyenlőtlenségek mellett a Bessel és a módosított Bessel függvényekre megadtunk néhány fontos monotonitási és konvexitási tulajdonságot. Itt fontos szerepet játszik Elbert Árpád [**El**] híres eredménye, miszerint az elsőfajú Bessel függvények gyökei konkávak a paraméterük szerint.

A harmadik fejezetben a fontosabb egyváltozós eloszlások sűrűségfüggvényeinek paramétereik szerinti konkavitását vizsgáltuk. Az első alfejezetben, az Edward Neumannal közösen igazolt eredmények [**BN**] segítségével, igazoltuk, hogy a nemcentrált khi és khi négyzet eloszlások sűrűségfüggvényeire is fennállnak bizonyos Turán típusú eredmények, akárcsak az elsőfajú módosított Bessel függvényekre. Ugyancsak igazak lesznek bizonyos Turán típusú egyenlőtlenségek a Student eloszlás sűrűségfüggvényére is. Mi több, a második alfejezetben igazoltuk, hogy a másodfajú módosított Bessel függvények logaritmikusan konvexek a paraméterükre nézve. Végül, a harmadik alfejezetben az előbb igazolt Turán típusú egyenlőtlenségeket használtuk arra, hogy egy lényeges egyszerűbb bizonyítást adjunk a különböző fajú módosított Bessel függvények szorzatának monotonitására. Ez a monotonitási tulajdonság, amely egy biofizikai probléma kapcsán merült fel, finomítja Robert Penfold és társai [**PVG**] eredményét. Ezenkívül érdemes megjegyezni, hogy a 3.1 alfejezet eredményeit Yin Sun és a szerző [**SB**] eredményesen használta a radarjelek vizsgálatánál használatos általánosított Marcum $Q-$függvény vizsgálatánál.

Végül, a negyedik fejezetben a matematikai statisztikában használatos standard normál eloszlás Mills arányára vonatkozó monotonitási tulajdonságokkal és Turán típusú egyenlőtlenségekkel foglalkoztunk. Itt fontos szerepet játszik a Pinelis-féle ún. monoton l'Hospital szabály [**Pi2**], amely napjainkban egy nagyon fontos alapeszköz, többek között a kvázikonformis analizísben és az analítikus egyenlőtlenségek vizsgálatánál.

Summary

Gábor Szegő [**Sze1**] in 1948 presented four different elegant proofs of the famous Turán inequality established for Legendre polynomials and extended the result to ultraspherical (or Gegenbauer), Laguerre and Hermite polynomials. Turán's inequality and Szegő's results have generated considerable interest, and shortly after 1948, analogous results were obtained by several authors for other classical orthogonal polynomials (for example Jacobi, Appell, Pollaczek, Lommel, Askey-Wilson) and special functions (for example Bessel, modified Bessel, q-Bessel, Riemann zeta functions). This classical inequality still attracts the attention of mathematicians and it is worth mentioning that recently the above Turán inequality was improved by Eugen Constantinescu [**Co**], and further by Alzer Horst et al. [**AGKL**] (for more details see Remark 1.4.1).

This doctoral thesis is a further contribution to the subject and contains certain new Turán type inequalities for some special functions.

The thesis is divided into four chapters. In the first chapter we have established some Turán type inequalities for Gaussian hypergeometric functions and for generalized complete elliptic integrals. These results completements the earlier result of Turán proved for Legendre polynomials. Moreover, we have shown that there is a close connection between a Turán type inequality and a sharp lower bound for the generalized complete elliptic integral of the first kind. In section 1.3 we proved a recent conjecture of Toshiyuki Sugawa and Matti Vuorinen [**SV**] related to estimates of the hyperbolic distance of the twice punctured plane, while in section 1.4, in order to improve some results from section 1.1, we have established a Turán type

inequality for Gaussian hypergeometric functions. This result complements the earlier result of Gábor Szegő [**Sze2**] and George Gasper [**Ga2**] proved for Jacobi polynomials. Moreover, at the end of this section we presented some open problems, which may be of interest for further research.

In the second chapter we extended some known elementary trigonometric inequalities (like Lazarević type, Wilker), and their hyperbolic analogues to Bessel and modified Bessel functions of the first kind. In order to generalize the Turán type inequalities established for Bessel and modified Bessel functions we presented some new monotonicity and convexity properties of some functions involving Bessel and modified Bessel functions of the first kind. For instance, we showed that the famous result of Árpád Elbert [**El**] on the concavity of the zeros of Bessel functions of the first kind (with respect to the order) can be used to improve the Turán type inequalities established for modified Bessel functions of the first kind. We also deduced some Turán type and Lazarević type inequalities for the confluent (or Kummer) hypergeometric functions.

Chapter 3 is devoted to the study of some Turán type inequalities for the probability density function of some univariate distributions, like the non-central chi-squared distribution, non-central chi distribution and Student distribution, respectively. Moreover, in this chapter we improved a result of Andrea Laforgia and Pierpaolo Natalini [**LN2**] concerning a Turán type inequality by showing that the modified Bessel function of the second kind is log-convex with respect to its order. As an application of some results deduced in sections 2.1 and 3.2, in section 3.3 we presented a new very simple proof for the monotonicity of a product of two modified Bessel functions of different kind. This result complements and improves a recent result of Robert Penfold and collaborators [**PVG**], which was motivated by a problem in biophysics.

Finally, in chapter 4 we studied the monotonicity properties of some functions involving the Mills' ratio of the standard normal law. From these monotonicity properties we deduced an interesting chain of inequalities for Mills' ratio, and we have shown that the Mills' ratio is strictly completely monotonic. At the end of this chapter we presented some Turán type inequalities for Mills' ratio.

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B. Accepted Journal Articles

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3. Turán type inequalities for hypergeometric functions. Proc. Amer. Math. Soc.
4. Functional inequalities involving Bessel and modified Bessel functions of the first kind. Expo. Math.
5. [with Y. Sun] Inequalities for the generalized Marcum $Q-$function. Appl. Math. Comput.

Talks given at research seminars

1. February 24, 2005

Generalized and normalized Bessel functions

Seminar of the Institute of Mathematics, University of Debrecen.

2. February 08, 2007

Inequalities for elliptic integrals and Bessel functions

Seminar of the Institute of Mathematics, University of Debrecen.

3. April 12, 2007

Turán–type inequalities for Bessel and hypergeometric functions

Approximation Theory Seminar of the Alfréd Rényi Institute of Mathematics.

4. May 21, 2007

Inequalities for Bessel functions

Analysis seminar of the University of Helsinki, Finland.

5. April 10, 2008

Turán type inequalities and the generalized Marcum $Q-$function

Seminar of the Institute of Mathematics, University of Debrecen.

Turán type inequalities for some special functions

Értekezés a doktori (PhD) fokozat megszerzése érdekében
a matematika tudományágban.

Írta: Baricz Árpád okleveles matematikus.

Készült a Debreceni Egyetem Matematika doktori programja
Differenciálgeometria és alkalmazásai alprogramja keretében.

Témavezető: Dr. Nagy Péter Tibor

A doktori szigorlati bizottság:

elnök:	Dr.	....
tagok:	Dr.	....
	Dr.	....

A doktori szigorlat időpontja: 2008. . . . .

Az értekezés bírálói:

Dr. ....
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elnök:	Dr.	....
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	Dr.	....
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	Dr.	....

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