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Using the fractional order calculus in the combination of the MIT and Lyapunov stability method

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ABSTRACT

The main idea of the current paper consists in introducing the fractional order calculus in a control system. To control the system, an adaptive control technique with reference model is used. The fractional order models for the plant and reference model are obtained. To achieve the performances imposed by the fractional order reference model, a fractional order adaptive control law is proposed, which is a combination of two methods (MIT and Lyapunov stability). The original contribution in this paper is the use of fractional order calculus in the combined MIT and Lyapunov stability method and showing the dynamic behavior of the system. Several simulations are used to emphasize the effectiveness and benefits of the proposed method.

KEYWORDS

Fractional Order Calculus, Fractional Order Reference Model, Fractional Order Plant, Fractional Order Adaptive Control Law

1. INTRODUCTION

In the specialized literature, there are many papers which present how the fractional order calculus can be used to control a dynamical system [1–4] (see the Model Reference Adaptive System (MRAS) [5–11]). Introducing the fractional order calculus in the adaptive control system leads to good performances. More precisely, they are given by the fractional order systems over integer order systems [8].

Starting from the general idea mentioned in [12], in a closed loop system, one of the following situations can be found: 1) the integer order plant and integer order controller; 2) the fractional order plant and fractional order controller; 3) the fractional order plant and integer order controller; 4) the integer order plant and fractional order controller. For the aforementioned cases, a comparison is done by applying the MRAC (Model Reference Adaptive Control) theory, which is the most known and easy to apply adaptive method, to show the benefits of using the fractional order calculus.

So, in some cases, the MRAS basic scheme (for more details, refer to [13]) is modified by a fractional order reference model, and by the usage of the fractional order plant and fractional derivatives [14]. The fractional order reference model and fractional order plant are introduced in the adaptive scheme because they offer better representation of the system's dynamics. Also, these models improve the system's performances, such as response time and overshoot. The fractional derivatives in the chosen models ensure the system's stability.

The proposed control law is characterized by several adjustable parameters. The adaptation mechanism used to adjust the parameters is a combination between the MIT and Lyapunov stability methods [15], which will be modified using the fractional order calculus [1].

The paper's structure looks for: basic definitions on fractional order dynamic systems (Section 2). Then, how the fractional order models for the plant and reference model are

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obtained (Section 3). In Section 4, the theory about the combining MIT and Lyapunov stability method is extended using the fractional order calculus. Basically, a new fractional order control law was developed, which is the main contribution of the paper. Different scenarios are simulated in Section 5 and, finally, a few remarks are taken in Section 6.

2. FRACTIONAL ORDER CALCULUS IN CONTROL SYSTEMS

The control system can be modeled using the transfer function approach. This function is obtained after applying the Laplace transform on differential equation with integer order. In a similar way, the control system's dynamics can be described through a fractional order differential equation, which leads to a fractional order transfer function (after applying the classical Laplace transform). So, in general, the fractional order differential equation has the following form [1, 4, 14]:

$$a_n D_t^{\alpha_n} y(t) + .. + a_0 D_t^{\alpha_0} y(t) = b_m D_t^{\beta_n} u(t) + .. + b_0 D_t^{\beta_0} u(t), \quad (1)$$

where a_i, b_j are constant coefficients and α_i, $\beta_j \in R_+$.

The above Equation form is equivalent to:

$$\sum_{k=0}^{n} a_k D_t^{\alpha_k} y(t) = \sum_{k=0}^{m} b_k D_t^{\beta_k} u(t). \quad (2)$$

By applying the Laplace transform on Eq. (2) with zero initial condition, the general fractional order transfer function form becomes:

$$G(s) = \frac{b_m s^{\beta_m} + .. + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + .. + a_0 s^{\alpha_0}} = \frac{\sum_{k=0}^{m} b_k (s^\alpha)^k}{\sum_{k=0}^{n} a_k (s^\alpha)^k} = \frac{\sum_{k=0}^{m} b_k (\lambda)^k}{\sum_{k=0}^{n} a_k (\lambda)^k}, \lambda = s^\alpha. \quad (3)$$

After obtaining a fractional transfer function it is important to check the system's stability. The fractional order system can be stable, including the case when it has roots in the right half of the plan [4, 12], if the following condition is respected:

$$\arg|\lambda_i| > \alpha \frac{\pi}{2}, \quad (4)$$

where λ_i are the roots of the characteristic polynomial with $\alpha \in (0,\ 1)$.

3. IDENTIFICATION OF THE FRACTIONAL ORDER MODEL FOR THE PLANT AND REFERENCE MODEL

In the MRAS, the desired system performances are imposed by a reference model, when the plant's parameters vary in time. Therefore, the first modification which is done in the MRAS scheme is to introduce a fractional order model for the reference model and plant. To get the models, identification in the time domain using the *fotfid* tool is needed [16]. The identification is carried out (both for the reference model and plant), starting from the following integer order models:

- for the plant, the nominal transfer function is:

$$G(s) = \frac{2}{s^2 + 1.6s + 4} = \frac{b_m}{s^2 + a_{m1}s + a_{m2}} = \frac{Y(s)}{U(s)}, \quad (5)$$

- for the reference model (the transfer function is according to module criterion, i.e. damping ratio equal to 0.7):

$$G_m(s) = \frac{9}{s^2 + 4.2s + 9} = \frac{b}{s^2 + a_1 s + a_2} = \frac{Y_m(s)}{R(s)}. \quad (6)$$

The procedure continues with storing the data for each model (a special data structure was created), after the output was obtained considering the input a step signal.

Before obtaining the fractional order models in *fotfid* tool, a first guess model must be set up: an integer order model is obtained by choosing the characteristics $q = 1$, $n = 2$ for the pole polynomial and 1 for the zero polynomial [16]:

$$G_{init}(s) = \frac{1}{s^2 + s + 1}. \quad (7)$$

Using a free identification method with some coefficient limits $c_{\lim} = [0; 3000]$ and exponent limits $e_{\lim} = [10^{-9}; 3]$, finally, the fractional order models are achieved:

$$G_f(s) = \frac{1}{0.4298s^{2.13} + 0.7697s^{1.05} + 2.0024s^{0.0001}} = \frac{B_f(s)}{A_f(s)} = \frac{b_f}{a_{f1}s^{\alpha 1} + a_{f2}s^{\alpha 2} + a_{f3}s^{\alpha 3}}, \quad (8)$$

for the plant which reach the steady state value at 0.5 (considering the input value 1) and

$$G_{mf}(s) = \frac{1}{0.089s^{2.16} + 0.4358s^{1.02} + 1.0008s^{0.0001}} = \frac{B_{mf}(s)}{A_{mf}(s)} = \frac{b_{mf}}{a_{mf1}s^{\alpha f1} + a_{mf2}s^{\alpha f2} + a_{mf3}s^{\alpha f3}} \quad (9)$$

for the reference model with the steady state value equal to 1 (at the same input).

The models introduced in (8) and (9) are a better representation of the imposed dynamics of the control system.

By conducting a comparison with a square wave as an input signal, the fractional order models are confirmed and as it can be seen (Figs. 1 and 2), the fractional order models are accurate enough. The new transfer functions must be stable, property checked in the time domain. If the relation from Eq. (4) is satisfied, then the system is stable [12]. The property is illustrated in Figs. 3 and 4 (for the plant and reference model too): the poles are outside of the shaded area, so the models are stable.

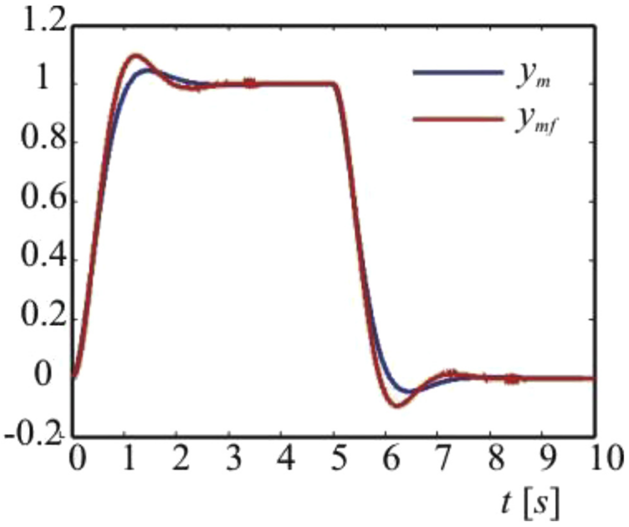

Fig. 1. Reference models' responses ($y_m(t)$ and $y_{mf}(t)$)

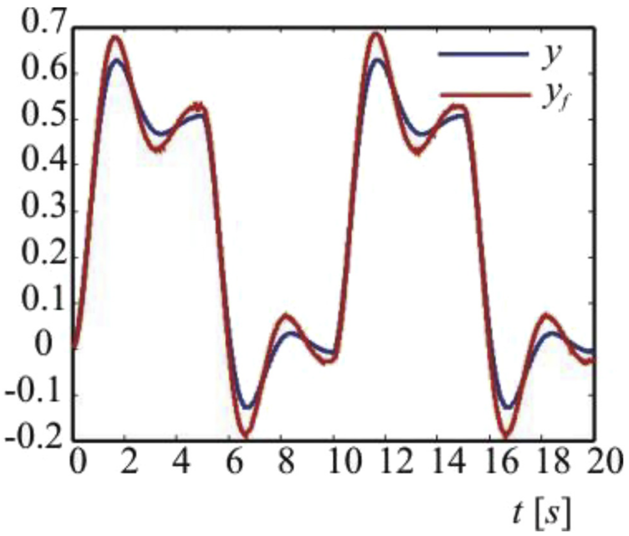

Fig. 2. Plants' responses ($y(t)$ and $y_f(t)$)

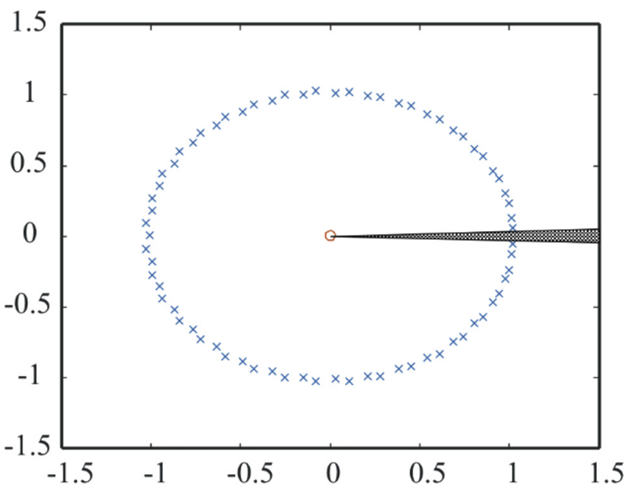

Fig. 3. Stability region for the fractional order plant

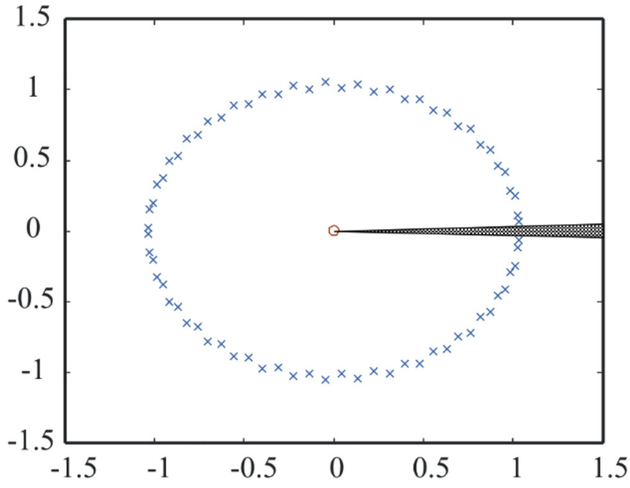

Fig. 4. Stability region for the fractional order reference model

4. OBTAINING THE FRACTIONAL ORDER ADJUSTMENT MECHANISM

To achieve the performances, imposed by the reference model, the controller's parameters must be adjusted/adapted (using an adjustment/adaptation mechanism), in such a way that the error between the plant output and reference model output is reduced to zero or exceedingly small. In literature, there are two approaches which can be used to derive an adaptation mechanism: the first one is the gradient method (or the MIT rule) and the second one is the Lyapunov stability method [13]. So, by modifying the parameter's adjustment mechanism using the fractional derivative [14], a second modification is introduced in the MRAS scheme.

As it is mentioned in the introduction chapter, in a closed loop system, several situations can be tackled. Thus, by choosing the proper control law, all these cases are going to be studied. The proposed control law (for the classical case when the plant and the controller are integer order) which offers good performances in a closed loop is:

$$u(t) = k_1 r(t) - k_2 y(t) - k_3 \frac{\mathrm{d}y(t)}{\mathrm{d}t}, \tag{10}$$

where the parameter k_1 ensures proper tracking for reference model in steady state, and the parameters k_2, k_3 correct the system dynamics. The adjustment mechanism used for these parameters is a combination between two methods: MIT rule and Lyapunov stability, with the purpose to use the same value for the adaptation gain [15]. So, to adapt the parameter k_1, the MIT rule is used in such a way that the overall DC gain for control system matches the reference model in steady state. In this context, in Eq. (10) the signal $y(t)$ is equalized to zero and the control law becomes $u(t) = k_1 r(t)$. The adjustment mechanism in this case is:

$$\frac{dk_1(t)}{dt} = -\gamma \left(\frac{1}{p^2 + a_{m1} p + a_{m2}} r(t) \right) e(t), \tag{11}$$

where γ is the adaptation gain and $e(t)$ the error.

For the parameters, k_2, k_3 the adjustment mechanism is obtained by using Lyapunov stability. Therefore, the input signal $r(t)$ is considered zero and the control law is: $u(t) = -k_2 y(t) - k_3 y(t)$. After the V function is chosen (which satisfies the selection condition [13]):

$$V(e, \dot{e}, k_2, k_3) = \frac{1}{2}\left(\dot{e}^2(t) + a_{m2}e^2(t) + \frac{1}{b\gamma}(a_{m1} - a_1 - bk_3)^2 + \frac{1}{b\gamma}(a_{m2} - a_2 - bk_2)^2\right), \quad (12)$$

it results the parameter's adjustment mechanism:

$$\frac{dk_2(t)}{dt} = \gamma y(t)\dot{e}(t), \qquad \frac{dk_3(t)}{dt} = \gamma \dot{y}(t)\dot{e}(t). \quad (13)$$

The whole above mentioned theory is applicable to integer order plant and integer order controller case. For other cases, when a fractional order controller is used, then a fractional order adaptive control law should be adopted. The fractional order adaptive control law is similar to the one presented in Eq. (10):

$$u(t) = k_1 r(t) - k_2 y(t) - k_3 \frac{d^\alpha y(t)}{dt^\alpha}, \quad (14)$$

where α is a real number which represents the fractional order derivative.

The parameter's adjustment mechanism is:

- in case of fractional order plant and fractional order controller:

$$\frac{d^\alpha k_1(t)}{dt^\alpha} = -\gamma\left(\frac{1}{A_{mf}(p)} r(t)\right)e(t), \qquad \frac{d^\alpha k_2(t)}{dt^\alpha} = \gamma y(t)\frac{d^\alpha e(t)}{dt^\alpha}, \qquad \frac{d^\alpha k_3(t)}{dt^\alpha} = \gamma \frac{d^\alpha y(t)}{dt^\alpha}\frac{d^\alpha e(t)}{dt^\alpha}. \quad (15)$$

- and for the integer order plant and fractional order controller case:

$$\frac{d^\alpha k_1(t)}{dt^\alpha} = -\gamma\left(\frac{1}{p^2 + a_{m1}p + a_{m2}} r(t)\right)e(t), \qquad \frac{d^\alpha k_2(t)}{dt^\alpha} = \gamma y(t)\frac{d^\alpha e(t)}{dt^\alpha}, \qquad \frac{d^\alpha k_3(t)}{dt^\alpha} = \gamma \frac{d^\alpha y(t)}{dt^\alpha}\frac{d^\alpha e(t)}{dt^\alpha}. \quad (16)$$

In the last case, in the MRAS scheme, a fractional order plant and an integer order controller are used. The parameter's adjustment mechanism is similar to the one used in Eqs. (11)–(13) (starting from the control law from Eq. (10)). A difference between them is due to the parameter k_1 modification:

$$\frac{dk_1(t)}{dt} = -\gamma\left(\frac{1}{A_{mf}(p)} r(t)\right)e(t). \quad (17)$$

It can be seen that when a fractional order plant is used, in the MRAS scheme a fractional order reference model must be used, too.

5. SIMULATIONS AND RESULTS

All scenarios already presented in the earlier section are simulated and analyzed, using different values such as 0.9 and 1.05 for alpha.

The simulations are done for $\gamma = 10$ and using a square wave as the reference signal $r(t)$. The system's response is examined in the following situations:

- Case 1: integer order plant and integer order controller (Fig. 5);
- Case 2: fractional order plant and fractional order controller (Fig. 6);
- Case 3: fractional order plant and integer order controller (Fig. 7);
- Case 4: integer order plant and fractional order controller (Fig. 8).

The simulations are presented in the same area (60–70s). By using a fractional order control with $\alpha = 0.9$ (for the

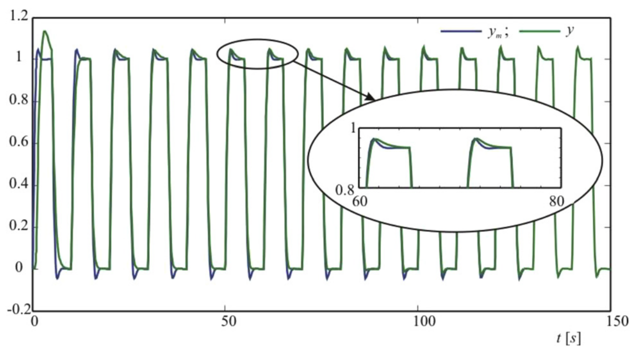

Fig. 5. Simulation results when using integer order plant and integer order controller

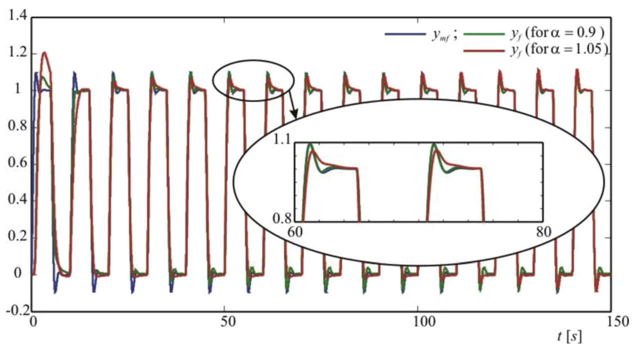

Fig. 6. Simulation results when using fractional order plant and fractional order controller

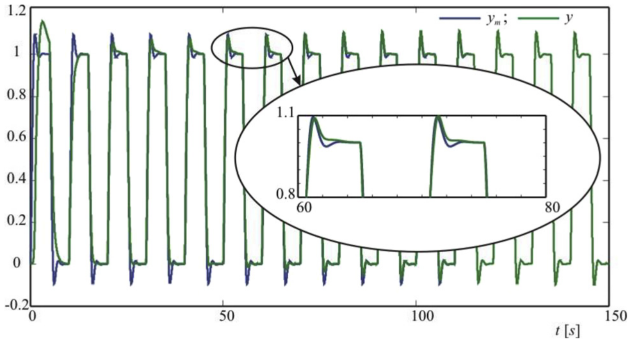

Fig. 7. Simulation results when using fractional order plant and integer order controller

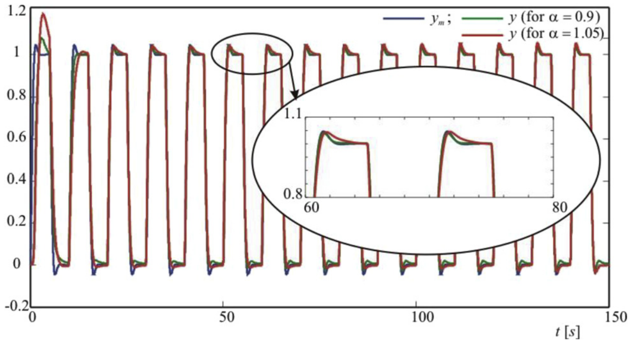

Fig. 8. Simulation results when using integer order plant and fractional order controller

adjustment mechanism derivatives), it can be observed that the performances are tracked much better, with a smaller overshoot, in case of using a fractional order reference model (Fig. 6) versus the case where a reference model is used (Fig. 8). If the value for α increases, the performances decrease (the overshoot and the error become bigger).

When comparing the resulted performances with the ones from the other two cases (see Figs. 5 and 7), where a classical adaptive control is used, it can be concluded that the MRAS has more benefits when using fractional order control.

Also, the error variation was examined and becomes exceedingly small much faster only for cases two and four.

To support the conclusions a performance index was defined (*IAE* – Integral of Absolute Error) [17]. All simulations were done over 100 seconds, using $Te = 0.001$ fixed-step size, so the output signals of the plant and reference models are saved as Matlab vectors with $N = 100.001$ samples. The performance index is computed from the saved samples $y_{ref}[t]$ and $y[t]$ as:

$$IAE = \int_0^N |e[t]|dt = \int_0^N \left| y_{ref}[t] - y[t] \right| dt \qquad (18)$$

Better performance means smaller value of the *IAE*. The results for all cases in Figs. 5–8 are presented in Table 1. Compared to the conventional control scheme with integer-order plant and reference model (Case 1), the fractional-order control law (Cases 2 and 4) led to approximately 10–20% improvement.

Table 1. The *IAE* values

Case 1	IAE = 3,325.6
Case 2, with alpha 0.9	IAE = 2,902.7
Case 2, with alpha 1.05	IAE = 5,085.0
Case 3	IAE = 4,137.6
Case 4, with alpha 0.9	IAE = 2,621.0
Case 4, with alpha 1.05	IAE = 4,031.2

As a remark, when using in the MRAS scheme a fractional order reference model, the tracking performances are carried out with fractional order plant.

6. CONCLUSIONS

In this paper, through examples, the benefits of the fractional order adaptive control are shown. The method used to adapt the parameters is a combination of two adaptive methods, MIT and Lyapunov stability, and implies a reference model.

By using the fractional order calculus, the combining MIT and Lyapunov stability method is extended. More precisely, the fractional order derivative and fractional order reference model are introduced in the parameter's adjustment mechanism. To achieve good performances, besides using a fractional reference model, a fractional order plant was used as well. All the cases supported by MRAS are studied and the benefits are highlighted.

Finally, it can be concluded that it is easy to apply this method on any second order plant.

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