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On the equality of generalized Bajraktarević means under first-order differentiability assumptions

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Dedicated to the 100th anniversary of the birth of Professor János Aczél.

Abstract. In this paper we consider the equality problem of generalized Bajraktarević means, i.e., we are going to solve the functional equation

$$f^{(-1)}\left(\frac{p_1(x_1)f(x_1)+\cdots+p_n(x_n)f(x_n)}{p_1(x_1)+\cdots+p_n(x_n)}\right)=g^{(-1)}\left(\frac{q_1(x_1)g(x_1)+\cdots+q_n(x_n)g(x_n)}{q_1(x_1)+\cdots+q_n(x_n)}\right), \tag{*}$$

which holds for all $x=(x_1,\dots,x_n)\in I^n$, where $n\geq 2$, I is a nonempty open real interval, the unknown functions $f,g:I\to\mathbb{R}$ are strictly monotone, $f^{(-1)}$ and $g^{(-1)}$ denote their generalized left inverses, respectively, and the vector-valued weight functions $p=(p_1,\dots,p_n):I\to\mathbb{R}_+^n$ and $q=(q_1,\dots,q_n):I\to\mathbb{R}_+^n$ are also unknown. This equality problem in the symmetric two-variable case (i.e., when $n=2$ and $p_1=p_2$, $q_1=q_2$) was solved under sixth-order regularity assumptions by Losonczi in 1999. The authors of this paper improved this result in 2023 by reaching the same conclusion assuming only first-order differentiability. In the nonsymmetric case, assuming the third-order differentiability of f, g and the first-order differentiability of at least three of the functions $p_1,\dots,p_n$, Grünwald and Páles proved that (*) holds if and only if there exist four constants $a,b,c,d\in\mathbb{R}$ with $ad\neq bc$ such that

$$cf+d>0,\qquad g=\frac{af+b}{cf+d},\qquad \text{and}\qquad q_\ell=(cf+d)p_\ell\qquad (\ell\in\{1,\dots,n\}).$$

The main goal of this paper is to establish the same conclusion under first-order differentiability.

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1. Introduction

Throughout this paper, the symbols $\mathbb{R}$ and $\mathbb{R}_+$ will stand for the sets of real and positive real numbers, respectively, and I will always denote a nonempty open real interval. In the theory of quasi-arithmetic means the characterization of the equality of means with different generators is a basic problem which was completely solved in the book [9]. Using this characterization, the homogeneous quasi-arithmetic means can also be found: they are exactly the power means and the geometric mean. In [3] (cf. also [4]) Bajraktarević introduced a new generalization of quasi-arithmetic means by adding a weight function to the formula of quasi-arithmetic means. He also characterized the equality of such means (called Bajraktarević means since then) in the at least 3-variable setting assuming three times differentiability. Daróczy and Losonczi in [5], later Daróczy and Páles in [6] arrived at the same conclusion with first-order differentiability and without differentiability, respectively, but assuming equality for all $n \in \mathbb{N}$. As an application of the characterization of the equality, Aczél and Daróczy in [1] determined the homogeneous Bajraktarević means that include the means which were introduced by Gini in [7]. Losonczi in [10] described the equality of two-variable Bajraktarević means under sixth-order regularity assumptions and an algebraic condition which was later removed in [11]. A new form of the result of Losonczi was established in the paper [14]. Using these results, homogeneous two-variable means were also determined by Losonczi [12,13].

A characterization of the equality of two-variable Bajraktarević means has been proved in [18, Theorem 7 and Corollary 8] by the authors assuming only first-order differentiability of the generating functions of the means.

The purpose of this paper is to consider an analogous problem for n variable nonsymmetric extension of Bajraktarević means (where $n \geq 2$), which was introduced in [8]. In Theorem 12 of this paper, the equality problem of the extended means was solved under three times differentiability conditions. Our main result is going to reach the same conclusion under first-order differentiability assumptions only.

We also mention that equality problems related to other generalizations of Bajraktarević means were investigated and solved in the papers [15–17].

2. Preliminary and auxiliary results

Given a subset $S \subseteq \mathbb{R}$, the smallest convex set containing S, which is identical to the smallest interval containing S, will be denoted by $\operatorname{conv}(S)$. For our definition of generalized Bajraktarević means, we shall need the following lemma about the existence and properties of the left inverse of strictly monotone (but not necessarily continuous) functions.

Lemma 1. ([8, Lemma 1]) *Let $f : I \to \mathbb{R}$ be a strictly monotone function. Then there exists a uniquely determined monotone function $f^{(-1)} : \operatorname{conv}(f(I)) \to I$ such that $f^{(-1)}$ is the left inverse of f, i.e.,*

$$(f^{(-1)} \circ f)(x) = x \qquad (x \in I). \tag{1}$$

Furthermore, $f^{(-1)}$ is monotone in the same sense as f, continuous,

$$(f \circ f^{(-1)})(y) = y \qquad (y \in f(I)), \tag{2}$$

and

$$\liminf_{x \to f^{(-1)}(y)} f(x) \le y \le \limsup_{x \to f^{(-1)}(y)} f(x) \qquad (y \in \operatorname{conv}(f(I))). \tag{3}$$

Thus, if f is lower (resp. upper) semicontinuous at $f^{(-1)}(y)$, then $f \circ f^{(-1)}(y) \le y$ (resp. $y \le f \circ f^{(-1)}(y)$).

It is clear from (1) and (2) that the restriction of $f^{(-1)}$ to $f(I)$ is the inverse of f in the standard sense. Therefore, $f^{(-1)}$ is the continuous and monotone extension of the inverse of f to the smallest interval containing the range of f.

Given a strictly monotone function $f : I \to \mathbb{R}$ and an n-tuple of positive valued functions $p = (p_1, \dots, p_n) : I \to \mathbb{R}_+^n$, we introduce the *$n$-variable generalized Bajraktarević mean* $A_{f,p} : I^n \to I$ by the following formula:

$$A_{f,p}(x) := f^{(-1)}\left(\frac{p_1(x_1)f(x_1) + \cdots + p_n(x_n)f(x_n)}{p_1(x_1) + \cdots + p_n(x_n)}\right) \quad (x = (x_1, \dots, x_n) \in I^n), \tag{4}$$

and, to simplify the notations, we will use the following definition:

$$R_{f,p}(x) := \frac{p_1(x_1)f(x_1) + \cdots + p_n(x_n)f(x_n)}{p_1(x_1) + \cdots + p_n(x_n)}. \tag{5}$$

Theorem 2. ([8, Theorem 2]) *Let $f : I \to \mathbb{R}$ be strictly monotone, $n \in \mathbb{N}$, and $p = (p_1, \dots, p_n) : I \to \mathbb{R}_+^n$. Then the function $A_{f,p} : I^n \to I$ given by (4) is well-defined and it is a mean, that is,*

$$\min(x) \le A_{f,p}(x) \le \max(x) \qquad (x = (x_1, \dots, x_n) \in I^n). \tag{6}$$

Theorem 3. ([8, Theorem 3]) *Let $f : I \to \mathbb{R}$ be strictly increasing, $n \in \mathbb{N}$, and $p = (p_1, \dots, p_n) : I \to \mathbb{R}_+^n$. Then, for all $x = (x_1, \dots, x_n) \in I^n$, the equality $y = A_{f,p}(x)$ holds if and only if*

$$\sum_{i=1}^{n} p_i(x_i)(f(z) - f(x_i)) \begin{cases} < 0 & \text{for } z \in I,\ z < y, \\ > 0 & \text{for } z \in I,\ z > y. \end{cases} \tag{7}$$

If f is strictly decreasing, then the inequalities (7) hold with a reversed inequality sign.

Corollary 4. ([8, Corollary 4]) *Let $f : I \to \mathbb{R}$ be continuous, strictly monotone, $n \in \mathbb{N}$, and $p = (p_1, \dots, p_n) : I \to \mathbb{R}_+^n$. Then, for all $x = (x_1, \dots, x_n) \in I^n$, the value $y = A_{f,p}(x)$ is the unique solution of the equation*

$$\sum_{i=1}^{n} p_i(x_i)(f(y) - f(x_i)) = 0. \tag{8}$$

For the formulation of the subsequent results, we define the *diagonal* $\operatorname{diag}(I^n)$ of I^n by

$$\operatorname{diag}(I^n) := \{(x, \dots, x) \in \mathbb{R}^n \mid x \in I\}$$

and, for a function $F : I^n \to \mathbb{R}$, its *diagonalization* $F^\Delta : I \to \mathbb{R}$ is given as

$$F^\Delta(x) := F(x, \dots, x) \qquad (x \in I).$$

Given two functions $F, G : I^n \to \mathbb{R}$, we say that *$F$ and G are locally equal on I^n* (in other words, *F and G are equal near the diagonal of I^n*) if there exists an open set $U \subseteq I^n$ containing $\operatorname{diag}(I^n)$ such that $F(x) = G(x)$ is valid for all $x \in U$. If the equality $F(x) = G(x)$ is valid for all $x \in I^n$, then we say that *F and G are globally equal on I^n*.

Given $p = (p_1, \dots, p_n) : I \to \mathbb{R}_+^n$ and $q = (q_1, \dots, q_n) : I \to \mathbb{R}_+^n$, we will also use the following notations:

$$p_0 := p_1 + \cdots + p_n, \qquad q_0 := q_1 + \cdots + q_n, \qquad \text{and} \qquad r_0 := \frac{q_0}{p_0}.$$

Theorem 5. ([8, Assertion (1) of Theorem 8]) *Let $f : I \to \mathbb{R}$ be a differentiable function on I with a nonvanishing first derivative, $n \in \mathbb{N}$, let $p = (p_1, \dots, p_n) : I \to \mathbb{R}_+^n$ and let $i \in \{1, \dots, n\}$. If p_i is continuous on I, then the first-order partial derivative $\partial_i A_{f,p}$ exists on $\operatorname{diag}(I^n)$ and*

$$\big(\partial_i A_{f,p}\big)^\Delta = \frac{p_i}{p_0}.$$

The next result establishes a sufficient condition for the equality of n-variable generalized Bajraktarević means. We will call this situation the *canonical case of the equality.*

Theorem 6. ([8, Theorem 5]) *Let $f, g : I \to \mathbb{R}$ be strictly monotone, $n \in \mathbb{N}$, and $p, q : I \to \mathbb{R}_+^n$. If there exist $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ such that*

$$cf + d > 0, \qquad g = \frac{af + b}{cf + d}, \qquad \text{and} \qquad q = (cf + d)p \tag{9}$$

hold on I, then the n-variable generalized Bajraktarević means $A_{f,p}$ and $A_{g,q}$ are identical on I^n, i.e., globally equal on I^n.

With the aid of the following lemma, we can reduce the regularity assumptions in our statements.

Lemma 7. ([8, Lemma 6]) *Let $f, g : I \to \mathbb{R}$ be continuous strictly monotone functions, $n \geq 2$, and $p = (p_1, \dots, p_n) : I \to \mathbb{R}_+^n$, $q = (q_1, \dots, q_n) : I \to \mathbb{R}_+^n$. Assume that $A_{f,p}$ and $A_{g,q}$ are locally equal on I^n. Then the following two assertions hold.*

(i) For all $i \in \{1, \dots, n\}$, the function p_i is continuous on I if and only if the function q_i is continuous on I.
(ii) Assume that $f, g : I \to \mathbb{R}$ are differentiable (resp. continuously differentiable) functions on I with nonvanishing first derivatives. Then, for all $i \in \{1, \dots, n\}$, the function p_i is differentiable (resp. continuously differentiable) on I if and only if q_i is differentiable (resp. continuously differentiable) on I.

The following theorem is of basic importance for our investigations.

Theorem 8. ([8, Theorem 7]) *Let $f, g : I \to \mathbb{R}$ be continuous, strictly monotone, $n \geq 2$ and $p : I \to \mathbb{R}_+^n$ be a continuous function on I. Let further $q : I \to \mathbb{R}_+^n$. Assume that $A_{f,p}$ and $A_{g,q}$ are locally equal on I^n and that there exist $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ and a nonempty open subinterval I_0 of I such that (9) holds on I_0. Then (9) is also valid on I.*

In the next result we point out that the local equality of n-variable Bajraktarević means implies the local equality of k-variable descendant Bajraktarević means for $k \leq n$.

Theorem 9. *Let $f, g : I \to \mathbb{R}$ be strictly monotone, $n \geq 2$, and $p = (p_1, \dots, p_n)$, $q = (q_1, \dots, q_n) : I \to \mathbb{R}_+^n$. Assume that the n-variable $A_{f,p}$ and $A_{g,q}$ are locally (globally) equal on I^n. Then, for any $k \in \mathbb{N}$ and $i_1, \dots, i_k \in \{1, \dots, n\}$ with $i_1 < \dots < i_k$, we have that the k-variable means $A_{f,(p_{i_1}, \dots, p_{i_k})}$ and $A_{g,(q_{i_1}, \dots, q_{i_k})}$ are locally (globally) equal on I^k.*

Proof. Without loss of generality, we may assume that f and g are strictly increasing functions. Then $f^{(-1)}$ and $g^{(-1)}$ are continuous increasing functions.

By the local equality of $A_{f,p}$ and $A_{g,q}$ on I^n, there exists an open set $U \subseteq I^n$ containing the diagonal of I^n such that, for all $x \in U$, we have $A_{f,p}(x) = A_{g,q}(x)$.

In what follows, for all $i \in \{1, \dots, n\}$, let $e_i \in \mathbb{R}^n$ denote the ith vector of the standard base of $\mathbb{R}^n$, i.e., let $e_i := (\delta_{i,j})_{j=1}^n$, where $\delta_{i,j}$ stands for the Kronecker symbol.

Let us now fix $k \in \mathbb{N}$ and $i_1, \dots, i_k \in \{1, \dots, n\}$ with $i_1 < \dots < i_k$ and define the set $V \subseteq I^k$ by

$$V := \left\{ (y_1, \dots, y_k) \in I^k \,\middle|\, \forall\, t \in \Big[\min_{1 \leq \alpha \leq k} y_\alpha, \max_{1 \leq \alpha \leq k} y_\alpha \Big] : \sum_{j=1}^n \Big[t + \sum_{\alpha=1}^k \delta_{j,i_\alpha} (y_\alpha - t) \Big] e_j \in U \right\}.$$

The interior of V (denoted as V°) is an open subset of I^k. To see that V° contains the diagonal of I^k, let $y \in I$ be fixed. Then the vector

$$\sum_{j=1}^{n} \big[t + \delta_{j,i_1}(y_1 - t) + \cdots + \delta_{j,i_k}(y_k - t)\big] e_j$$

depends continuously on $(y_1, \ldots, y_k, t)$ and it tends to the diagonal vector $(y, \ldots, y) \in U \subseteq I^n$ if the variables $y_1, \ldots, y_k$, and t tend to y. In other words, due to the openness of U, there exists a positive number r such that

$$\sum_{j=1}^{n} \big[t + \delta_{j,i_1}(y_1 - t) + \cdots + \delta_{j,i_k}(y_k - t)\big] e_j \in U \quad \text{for all} \quad (y_1, \ldots, y_k, t) \in (y-r, y+r)^{k+1}.$$

In particular, this inclusion holds if $y-r < \min(y_1, \ldots, y_k) \le t \le \max(y_1, \ldots, y_k) < y + r$, hence, in this case $(y_1, \ldots, y_k) \in V$. This shows that $(y, \ldots, y) \in I^k$ is an interior point of V and hence V° is an open subset of I^k which contains the diagonal of I^k. In addition, if $A_{f,p}$ and $A_{g,q}$ are globally equal on I^n, then $U = I^n$ and $V = V^\circ = I^k$.

We are now going to show that the means $A_{f,(p_{i_1},\ldots,p_{i_k})}$ and $A_{g,(q_{i_1},\ldots,q_{i_k})}$ are equal on V°. To the contrary, assume that, for some $(y_1, \ldots, y_k) \in V^\circ$, we have

$$f^{(-1)}\bigg(\frac{p_{i_1}(y_1)f(y_1) + \cdots + p_{i_k}(y_k)f(y_k)}{p_{i_1}(y_1) + \cdots + p_{i_k}(y_k)}\bigg) \neq g^{(-1)}\bigg(\frac{q_{i_1}(y_1)g(y_1) + \cdots + q_{i_k}(y_k)g(y_k)}{q_{i_1}(y_1) + \cdots + q_{i_k}(y_k)}\bigg).$$

We may assume that

$$f^{(-1)}\bigg(\frac{p_{i_1}(y_1)f(y_1) + \cdots + p_{i_k}(y_k)f(y_k)}{p_{i_1}(y_1) + \cdots + p_{i_k}(y_k)}\bigg) < g^{(-1)}\bigg(\frac{q_{i_1}(y_1)g(y_1) + \cdots + q_{i_k}(y_k)g(y_k)}{q_{i_1}(y_1) + \cdots + q_{i_k}(y_k)}\bigg).$$

The set of points where f or g is discontinuous is a countable subset of I, therefore, there exists a point $t \in I$, where f and g are continuous and

$$f^{(-1)}\bigg(\frac{p_{i_1}(y_1)f(y_1) + \cdots + p_{i_k}(y_k)f(y_k)}{p_{i_1}(y_1) + \cdots + p_{i_k}(y_k)}\bigg) < t < g^{(-1)}\bigg(\frac{q_{i_1}(y_1)g(y_1) + \cdots + q_{i_k}(y_k)g(y_k)}{q_{i_1}(y_1) + \cdots + q_{i_k}(y_k)}\bigg). \tag{10}$$

Due to the mean value property of the means (established in Theorem 2), it follows that

$$\min(y_1, \ldots, y_k) < t < \max(y_1, \ldots, y_k),$$

therefore,

$$x = (x_1, \ldots, x_n) := \sum_{j=1}^{n} \Big[t + \sum_{\alpha=1}^{k} \delta_{j,i_\alpha}(y_\alpha - t)\Big] e_j \in U,$$

and hence $A_{f,p}(x) = A_{g,q}(x)$. On the other hand, the inequalities in (10) imply that

$$\frac{p_{i_1}(y_1)f(y_1) + \cdots + p_{i_k}(y_k)f(y_k)}{p_{i_1}(y_1) + \cdots + p_{i_k}(y_k)} < f(t), \quad g(t) < \frac{q_{i_1}(y_1)g(y_1) + \cdots + q_{i_k}(y_k)g(y_k)}{q_{i_1}(y_1) + \cdots + q_{i_k}(y_k)}.$$

Rearranging these inequalities, we get

$$\sum_{\alpha=1}^{k} p_{i_\alpha}(y_\alpha)(f(y_\alpha)-f(t))<0, \qquad \sum_{\alpha=1}^{k} q_{i_\alpha}(y_\alpha)(g(y_\alpha)-g(t))>0.$$

Now taking into account the equalities

$$x_j := \begin{cases} t & \text{if } j \notin \{i_1,\dots,i_k\}, \\ y_\alpha & \text{if } j = i_\alpha \quad (\alpha \in \{1,\dots,k\}), \end{cases} \tag{11}$$

it follows that

$$\sum_{i=1}^{n} p_i(x_i)(f(x_i)-f(t))<0, \qquad \sum_{i=1}^{n} q_i(x_i)(g(x_i)-g(t))>0.$$

From these inequalities we obtain that

$$\frac{p_1(x_1)f(x_1)+\cdots+p_n(x_n)f(x_n)}{p_1(x_1)+\cdots+p_n(x_n)} < f(t), \quad g(t) < \frac{q_1(x_1)g(x_1)+\cdots+q_n(x_n)g(x_n)}{q_1(x_1)+\cdots+q_n(x_n)}.$$

Finally, using again that f and g are continuous at the point t, we can conclude that

$$f^{(-1)}\left(\frac{p_1(x_1)f(x_1)+\cdots+p_n(x_n)f(x_n)}{p_1(x_1)+\cdots+p_n(x_n)}\right) < t < g^{(-1)}\left(\frac{q_1(x_1)g(x_1)+\cdots+q_n(x_n)g(x_n)}{q_1(x_1)+\cdots+q_n(x_n)}\right),$$

which contradicts the equality $A_{f,p}(x) = A_{g,q}(x)$. □

Corollary 10. *Let $n \geq 2$, $f, g : I \to \mathbb{R}$ be strictly monotone and $p = (p_1,\dots,p_n)$, $q = (q_1,\dots,q_n) : I \to \mathbb{R}_+^n$. Assume that the n-variable means $A_{f,p}$ and $A_{g,q}$ are locally (globally) equal on I^n. Then, for all $i, j \in \{1,\dots,n\}$ with $i < j$, the two-variable means $A_{f,(p_i,p_j)}$ and $A_{g,(q_i,q_j)}$ are locally (globally) equal on I^2.*

3. Reduction of the equality problem

In our first result, with a certain substitution, we reduce the number of unknown functions in the local (global) equality problem of n-variable generalized Bajraktarević means from $2n+2$ to $2n+1$.

Theorem 11. *Let $f, g : I \to \mathbb{R}$ be continuous strictly monotone functions, $n \geq 2$ and $p = (p_1,\dots,p_n) : I \to \mathbb{R}_+^n$, $q = (q_1,\dots,q_n) : I \to \mathbb{R}_+^n$. Then the means $A_{f,p}$ and $A_{g,q}$ are locally (globally) equal on I^n if and only if, the means $A_{\mathrm{id},P}$ and $A_{h,Q}$ are locally (globally) equal on J^n, where the interval J, the functions $h : J \to \mathbb{R}$, and $P = (P_1,\dots,P_n), Q = (Q_1,\dots,Q_n)\colon J \to \mathbb{R}_+^n$ are defined by*

$$J := f(I), \qquad h := g \circ f^{-1}, \qquad P := p \circ f^{-1}, \qquad \text{and} \qquad Q := q \circ f^{-1}. \tag{12}$$

Proof. Assume that the means $A_{f,p}$ and $A_{g,q}$ are locally equal on I^n. Then the equality

$$f^{-1}\left(\frac{p_1(x_1)f(x_1)+\cdots+p_n(x_n)f(x_n)}{p_1(x_1)+\cdots+p_n(x_n)}\right)=g^{-1}\left(\frac{q_1(x_1)g(x_1)+\cdots+q_n(x_n)g(x_n)}{q_1(x_1)+\cdots+q_n(x_n)}\right) \tag{13}$$

is satisfied for all $(x_1,\dots,x_n)\in V$, where V is an open subset of I^n containing its diagonal. Consider the map $F: I^n\to J^n$ defined by $F(x_1,\dots.x_n):=(f(x_1),\dots,f(x_n))$. Then F is a continuous bijection between I^n and J^n whose inverse is also continuous. Therefore, $U:=F(V)$ is an open subset of J^n containing its diagonal. For $(u_1,\dots,u_n)\in U$ we have that $F^{-1}(u_1,\dots,u_n)=(f^{-1}(u_1),\dots,f^{-1}(u_n))\in V$, therefore, applying (13) with $(x_1,\dots,x_n):=(f^{-1}(u_1),\dots,f^{-1}(u_n))$, we get that

$$\begin{aligned}&\frac{(p_1\circ f^{-1})(u_1)u_1+\cdots+(p_n\circ f^{-1})(u_n)u_n}{(p_1\circ f^{-1})(u_1)+\cdots+(p_n\circ f^{-1})(u_n)}\\&=(g\circ f^{-1})^{-1}\left(\frac{(q_1\circ f^{-1})(u_1)(g\circ f^{-1})(u_1)+\cdots+(q_n\circ f^{-1})(u_n)(g\circ f^{-1})(u_n)}{(q_1\circ f^{-1})(u_1)+\cdots+(q_n\circ f^{-1})(u_n)}\right),\end{aligned}$$

holds for all $(u_1,\dots,u_n)\in U$. With the notations introduced in (12), the above equality can be rewritten, for all $(u_1,\dots,u_n)\in U$, as

$$\frac{P_1(u_1)u_1+\cdots+P_n(u_n)u_n}{P_1(u_1)+\cdots+P_n(u_n)}=h^{-1}\left(\frac{Q_1(u_1)h(u_1)+\cdots+Q_n(u_n)h(u_n)}{Q_1(u_1)+\cdots+Q_n(u_n)}\right), \tag{14}$$

which shows that the means $A_{\mathrm{id},P}$ and $A_{h,Q}$ are locally equal on J^n. If, in addition, the means $A_{f,p}$ and $A_{g,q}$ are globally equal on I^n, i.e., the equality (13) holds on $V=I^n$, then $U=J^n$, which shows that the means $A_{\mathrm{id},P}$ and $A_{h,Q}$ are globally equal on J^n.

The reversed implication can checked similarly, therefore its proof is omitted. □

In what follows, we deduce the first-order necessary condition for the equality of n-variable generalized Bajraktarević means.

Theorem 12. *Let $h: J\to\mathbb{R}$ be a differentiable function with a nonvanishing first derivative, $n\ge 2$, let $P=(P_1,\dots,P_n), Q=(Q_1,\dots,Q_n)\colon J\to\mathbb{R}_+^n$, and let $i\in\{1,\dots,n\}$ such that P_i is continuous on J. If the means $A_{\mathrm{id},P}$ and $A_{h,Q}$ are locally equal on J^n, then*

$$\frac{P_i}{P_0}=\frac{Q_i}{Q_0}, \tag{15}$$

where

$$P_0:=P_1+\cdots+P_n \qquad \text{and} \qquad Q_0:=Q_1+\cdots+Q_n. \tag{16}$$

Proof. By Lemma 7, the continuity of P_i implies the continuity of Q_i. On the other hand, by Theorem 5, it follows that the first-order partial derivatives $\partial_i A_{\mathrm{id},P}$ and $\partial_i A_{h,Q}$ exist at the diagonal points of J^n, and due to the local validity of the equality $A_{\mathrm{id},P} = A_{h,Q}$, they are equal to each other, i.e., (15) holds. □

Remark 13. Without assuming the differentiability of h, for fixed $i \in \{1, \dots, n\}$, $u \in J$, we have that

$$\frac{P_i(u)}{P_0(u)} = (\partial_i A_{h,Q})^{\Delta}(u)$$
$$= \lim_{v \to u} \frac{1}{v-u}\Bigg(h^{-1}\Bigg(\frac{(h \cdot Q_0)(u) - (h \cdot Q_i)(u) + (h \cdot Q_i)(v)}{Q_0(u) - Q_i(u) + Q_i(v)}\Bigg) - u\Bigg).$$

It is an open problem if one could derive the differentiability of h from here at a.e. point with a nonvanishing derivative?

Corollary 14. *Let $h : J \to \mathbb{R}$ be a differentiable function with a nonvanishing first derivative, $n \geq 2$, let $P = (P_1, \dots, P_n), Q = (Q_1, \dots, Q_n) \colon J \to \mathbb{R}_+^n$ and assume that P is continuous on J. Suppose that the means $A_{\mathrm{id},P}$ and $A_{h,Q}$ are locally equal on J^n. Then Q is also continuous on J and there exists a continuous function $r : J \to \mathbb{R}_+$ such that*

$$Q = rP. \tag{17}$$

Furthermore, the equality

$$h\Bigg(\frac{P_1(u_1)u_1 + \cdots + P_n(u_n)u_n}{P_1(u_1) + \cdots + P_n(u_n)}\Bigg) = \frac{(rhP_1)(u_1) + \cdots + (rhP_n)(u_n)}{(rP_1)(u_1) + \cdots + (rP_n)(u_n)}. \tag{18}$$

holds locally on J^n for $u = (u_1, \dots, u_n) \in J^n$.

Proof. Define the functions P_0 and Q_0 by (16) and let $r := Q_0/P_0$. Then, by Theorem 12, for all $i \in \{1, \dots, n\}$, the equality (15) holds and Q_i is continuous. Therefore the equation (17) is valid and r is also continuous.

Assume that the means $A_{\mathrm{id},P}$ and $A_{h,Q}$ are equal over an open subset $U \subseteq J^n$ which contains the diagonal of J^n. Then (14) is valid for all $u = (u_1, \dots, u_n) \in U$. Replacing $Q_i(u_i)$ by $r(u_i)P_i(u_i)$ and applying the function h to this equality side by side, we can conclude that (18) is valid for all $u = (u_1, \dots, u_n) \in U$, i.e., it holds locally on J^n. □

First we establish a sufficient condition for the equality (18).

Theorem 15. *If there exist four real constants a, b, c, d such that, for all $u \in J$,*

$$r(u) = cu + d > 0, \qquad h(u) = \frac{au + b}{cu + d} \tag{19}$$

and $P = (P_1, \dots, P_n) \colon J \to \mathbb{R}_+^n$ is arbitrary, then (18) holds for all $u = (u_1, \dots, u_n) \in J^n$.

Proof. Let $u = (u_1, \dots, u_n) \in J^n$. Then

$$h\left(\frac{P_1(u_1)u_1 + \cdots + P_n(u_n)u_n}{P_1(u_1) + \cdots + P_n(u_n)}\right) = \frac{a\dfrac{P_1(u_1)u_1 + \cdots + P_n(u_n)u_n}{P_1(u_1) + \cdots + P_n(u_n)} + b}{c\dfrac{P_1(u_1)u_1 + \cdots + P_n(u_n)u_n}{P_1(u_1) + \cdots + P_n(u_n)} + d}$$
$$= \frac{a(P_1(u_1)u_1 + \cdots + P_n(u_n)u_n) + b(P_1(u_1) + \cdots + P_n(u_n))}{c(P_1(u_1)u_1 + \cdots + P_n(u_n)u_n) + d(P_1(u_1) + \cdots + P_n(u_n))}$$
$$= \frac{(au_1 + b)P_1(u_1) + \cdots + (au_n + b)P_n(u_n)}{(cu_1 + d)P_1(u_1) + \cdots + (cu_n + d)P_n(u_n)} = \frac{(rhP_1)(u_1) + \cdots + (rhP_n)(u_n)}{(rP_1)(u_1) + \cdots + (rP_n)(u_n)}.$$

□

The first main goal of this paper is to show that if the equality (18) holds locally in J^n, then, under some regularity assumptions, the functions r and h are of the form (19) for some real constants a, b, c, d. To accomplish this goal, we are going to show that r and h are twice differentiable functions and

$$r'' = 0 \qquad \text{and} \qquad (rh)'' = 0.$$

The proof of this property will be split in several propositions.

Proposition 16. *Let $h : J \to \mathbb{R}$ be continuously differentiable with a nonvanishing first derivative and let $r : J \to \mathbb{R}_+$. Let $P = (P_1, \dots, P_n)\colon J \to \mathbb{R}_+^n$. Let $i, j \in \{1, \dots, n\}$ be distinct indices such that P_i and P_j are continuously differentiable. Assume that (18) holds locally for $(u_1, \dots, u_n) \in J^n$. Then r is continuously differentiable and*

$$\begin{aligned}
&\Big(P_i(u)\cdot(P_i(u)+P_j(v)) + P_i'(u)\cdot P_j(v)\cdot(u-v)\Big)\\
&\qquad\times\Big((h'rP_j)(v)\cdot((rP_i)(u)+(rP_j)(v)) + (rP_j)'(v)\cdot(rP_i)(u)\cdot(h(v)-h(u))\Big)\\
&=\Big(P_j(v)\cdot(P_i(u)+P_j(v)) + P_j'(v)\cdot P_i(u)\cdot(v-u)\Big)\\
&\qquad\times\Big((h'rP_i)(u)\cdot((rP_i)(u)+(rP_j)(v)) + (rP_i)'(u)\cdot(rP_j)(v)\cdot(h(u)-h(v))\Big)
\end{aligned} \tag{20}$$

is valid locally for $(u, v) \in J^2$.

Proof. The local validity of the equality (18) implies that the n-variable means $A_{\mathrm{id},P}$ and $A_{h,rP}$ are locally equal on J^n. By Corollary 10, it follows that the two-variable means $A_{\mathrm{id},(P_i,P_j)}$ and $A_{h,(rP_i,rP_j)}$ are locally equal on J^2. In view of assertion (ii) of Lemma 7, the continuous differentiability of P_i implies the continuous differentiability of rP_i. Therefore, the function r must be continuously differentiable. By the local equality of these two-variable means, there exists an open subset $V \subseteq J^2$ containing the diagonal of J^2 such that, for all $(u, v) \in V$, we have

$$h\left(\frac{P_i(u)u + P_j(v)v}{P_i(u) + P_j(v)}\right) = \frac{(rhP_i)(u) + (rhP_j)(v)}{(rP_i)(u) + (rP_j)(v)}.$$

Computing the partial derivatives of this equation side by side with respect to u and v, for all $(u,v)\in V$, we get

$$h'\bigg(\frac{P_i(u)u+P_j(v)v}{P_i(u)+P_j(v)}\bigg)\times\frac{P_i(u)\cdot(P_i(u)+P_j(v))+P_j(v)\cdot P_i'(u)\cdot(u-v)}{(P_i(u)+P_j(v))^2}$$
$$=\frac{(h'rP_i)(u)\cdot((rP_i)(u)+(rP_j)(v))+(rP_i)'(u)\cdot(rP_j)(v)\cdot(h(u)-h(v))}{((rP_i)(u)+(rP_j)(v))^2}$$

and

$$h'\bigg(\frac{P_i(u)u+P_j(v)v}{P_i(u)+P_j(v)}\bigg)\times\frac{P_j(v)\cdot(P_i(u)+P_j(v))+P_i(u)\cdot P_j'(v)\cdot(v-u)}{(P_i(u)+P_j(v))^2}$$
$$=\frac{(h'rP_j)(v)\cdot((rP_i)(u)+(rP_j)(v))+(rP_j)'(v)\cdot(rP_i)(u)\cdot(h(v)-h(u))}{((rP_i)(u)+(rP_j)(v))^2}.$$

Multiplying the left and right hand sides of the first equality by the right and left hand sides of the second one, after cancellations, the stated equality (20) follows. □

Proposition 17. *Let $h:J\to\mathbb{R}$ be continuously differentiable with a nonvanishing first derivative and let $r,P_i,P_j:J\to\mathbb{R}_+$ be continuously differentiable (where $i,j\in\{1,\dots,n\}$ and $i\neq j$). Assume that (20) holds locally for $(u,v)\in J^2$. Then $h'r^2$ is a constant function on J, hence h is twice continuously differentiable on J, and the equality*

$$\begin{aligned}\frac{r(v)-r(u)}{r(u)r(v)(u-v)}=&\frac{(rP_j')(v)\cdot(P_i^2)(u)+(rP_i')(u)\cdot(P_j^2)(v)}{(rP_i)(u)\cdot(rP_j)(v)\cdot(P_i(u)+P_j(v))}\\&-\frac{(rP_i)'(u)\cdot(rP_j^2)(v)+(rP_j)'(v)\cdot(rP_i^2)(u)}{P_i(u)\cdot P_j(v)\cdot((rP_i)(u)+(rP_j)(v))}\cdot\frac{1}{u-v}\int_v^u\frac{1}{r^2}\\&+\frac{(rP_j')(v)\cdot(r'P_i)(u)-(rP_i')(u)\cdot(r'P_j)(v)}{(P_i(u)+P_j(v))\cdot((rP_i)(u)+(rP_j)(v))}\cdot\int_v^u\frac{1}{r^2}\end{aligned}\tag{21}$$

is valid locally for $(u,v)\in J^2$.

Proof. Let $V\subseteq J^2$ be an open set containing the diagonal of J^2 where (20) holds. Then, rearranging this equality, for all $(u,v)\in V$, we obtain

$$\begin{aligned}0=&P_i(u)\cdot P_j(v)\cdot(P_i(u)+P_j(v))\cdot((rP_i)(u)+(rP_j)(v))\cdot((h'r)(u)-(h'r)(v))\\&+(P_j'(v)\cdot(h'rP_i^2)(u)+P_i'(u)\cdot(h'rP_j^2)(v))\cdot((rP_i)(u)+(rP_j)(v))\cdot(v-u)\\&+((rP_i)'(u)\cdot(rP_j^2)(v)+(rP_j)'(v)\cdot(rP_i^2)(u))\cdot(P_i(u)+P_j(v))\cdot(h(u)-h(v))\\&+P_i(u)\cdot P_j(v)\cdot((rP_j')(v)\cdot(r'P_i)(u)-(rP_i')(u)\cdot(r'P_j)(v))\cdot(v-u)\cdot(h(u)-h(v)).\end{aligned}$$

Dividing both sides of this equation by $P_i(u)\cdot P_j(v)\cdot(P_i(u)+P_j(v))\cdot((rP_i)(u)+(rP_j)(v))\cdot(u-v)$, we get

$$\begin{aligned}\frac{(h'r)(u)-(h'r)(v)}{u-v} =&\frac{P_j'(v)\cdot(h'rP_i^2)(u)+P_i'(u)\cdot(h'rP_j^2)(v)}{P_i(u)\cdot P_j(v)\cdot(P_i(u)+P_j(v))}\\ &-\frac{(rP_i)'(u)\cdot(rP_j^2)(v)+(rP_j)'(v)\cdot(rP_i^2)(u)}{P_i(u)\cdot P_j(v)\cdot((rP_i)(u)+(rP_j)(v))}\cdot\frac{h(u)-h(v)}{u-v}\\ &+\frac{(rP_j')(v)\cdot(r'P_i)(u)-(rP_i')(u)\cdot(r'P_j)(v)}{(P_i(u)+P_j(v))\cdot((rP_i)(u)+(rP_j)(v))}\cdot(h(u)-h(v)).\end{aligned} \tag{22}$$

Let $u\in J$ be fixed. Observe that the limit of the right hand side as $v\to u$ exists. (Here we use that V is open and contains the point (u,u).) This shows that $h'r$ is differentiable and hence h' is differentiable, too. This proves that h is twice differentiable. Upon taking the limit as $v\to u$, we get that the equality

$$h''r+h'r'=(h'r)'=h'r\cdot\frac{P_j'P_i^2+P_i'P_j^2}{P_iP_j(P_i+P_j)}-h'\cdot\frac{(rP_i)'P_j^2+(rP_j)'P_i^2}{P_iP_j(P_i+P_j)}=-h'r'$$

is valid over J. Therefore,

$$h''r+2h'r'=0,$$

which implies that $h'r^2$ is a constant function on J. Since r is continuously differentiable, it follows that h' is also continuously differentiable and hence h is twice continuously differentiable.

Denote $\gamma:=h'r^2\neq 0$. Then, we get that $h'=\gamma/r^2$, thus the equality (22) can be rewritten as (21) for all $(u,v)\in V$. □

In what follows, we are going to prove that, for $w\in I$, the *symmetric derivative of* r' *at* w, i.e., the limit

$$\lim_{t\to 0}\frac{r'(w+t)-r'(w-t)}{2t}$$

exists and equals 0. Applying the quasi-mean value theorem of Aull [2, Theorem 1] (see also [19, Theorem 6.3]) from the theory of symmetric derivatives, it follows that r' is a constant function.

Proposition 18. *Let $r,P_i,P_j:J\to\mathbb{R}_+$ be continuously differentiable (where $i,j\in\{1,\dots,n\}$ and $i\neq j$) such that $P_i(u)\neq P_j(u)$ for all $u\in J$. Assume that* (21) *holds locally for $(u,v)\in J^2$. Then r is an affine function on J, i.e., there exists real constants c,d such that $r(u)=cu+d$ holds for all $u\in J$.*

Proof. Let $V\subseteq J^2$ be an open set containing the diagonal of J^2 where (21) holds. Multiplying this equality by $((rP_i)(u)+(rP_j)(v))$ side by side and then rearranging it, for all $(u,v)\in V$, we arrive at the following equation

$$\begin{aligned}\frac{P_i(u)}{r(v)}\bigg(\frac{r(u)-r(v)}{u-v}-r'(v)\bigg)&+\frac{P_j(v)}{r(u)}\bigg(\frac{r(u)-r(v)}{u-v}-r'(u)\bigg)\\ =\bigg(\frac{P_i(u)r'(v)}{r(v)}&+\frac{P_j(v)r'(u)}{r(u)}+\frac{P_i'(u)P_j^3(v)+P_j'(v)P_i^3(u)}{(P_i(u)+P_j(v))P_i(u)P_j(v)}\bigg)\\ &\times\bigg(\frac{r(u)r(v)}{u-v}\int_v^u\frac{1}{r^2}-1\bigg)\\ &+\frac{P_i'(u)P_j(v)r(u)}{(P_i(u)+P_j(v))r(v)}\bigg(\frac{(r'(v)(u-v)+r(v))r(v)}{u-v}\int_v^u\frac{1}{r^2}-1\bigg)\\ &+\frac{P_j'(v)P_i(u)r(v)}{(P_i(u)+P_j(v))r(u)}\bigg(\frac{(r'(u)(v-u)+r(u))r(u)}{u-v}\int_v^u\frac{1}{r^2}-1\bigg).\end{aligned}\tag{23}$$

Let $w\in J$ be fixed and substitute $u:=w+t$ and $v:=w-t$ into (23). First, compute the limit of both sides of this equality as $t\to 0$. By the continuity of the function r, we have that

$$\lim_{t\to 0}\frac{1}{2t}\int_{w-t}^{w+t}\frac{1}{r^2}=\frac{1}{r(w)^2},\tag{24}$$

hence both sides of the equality (23) tend to zero as $t\to 0$. In what follows, we show that the right hand side of the equation (23) multiplied by $\frac{1}{2t}$ tends to zero as $t\to 0$. For this purpose, we compute the following three limits using (24), the equality

$$\frac{d}{dt}\bigg(\int_{w-t}^{w+t}\frac{1}{r^2}\bigg)=\frac{d}{dt}\bigg(\int_{w}^{w+t}\frac{1}{r^2}-\int_{w}^{w-t}\frac{1}{r^2}\bigg)=\frac{1}{r(w+t)^2}+\frac{1}{r(w-t)^2},$$

and L'Hospital's Rule

$$\begin{aligned}\lim_{t\to 0}\frac{1}{2t}\bigg(\frac{r(w+t)r(w-t)}{2t}\int_{w-t}^{w+t}\frac{1}{r^2}-1\bigg)&=\lim_{t\to 0}\frac{1}{4t^2}\bigg(r(w+t)r(w-t)\int_{w-t}^{w+t}\frac{1}{r^2}-2t\bigg)\\ =\lim_{t\to 0}\frac{1}{8t}\bigg(\big(r'(w+t)r(w-t)&-r(w+t)r'(w-t)\big)\int_{w-t}^{w+t}\frac{1}{r^2}\\ &+r(w+t)r(w-t)\bigg(\frac{1}{r(w+t)^2}+\frac{1}{r(w-t)^2}\bigg)-2\bigg)\\ =\frac{r'(w)}{4r(w)}-\frac{r'(w)}{4r(w)}&+\lim_{t\to 0}\frac{1}{8t}\bigg(\frac{r(w-t)}{r(w+t)}+\frac{r(w+t)}{r(w-t)}-2\bigg)\\ =\lim_{t\to 0}\frac{(r(w+t)-r(w-t))^2}{8tr(w+t)r(w-t)}&=\frac{1}{r^2(w)}\lim_{t\to 0}\frac{(r(w+t)-r(w-t))^2}{8t}=0.\end{aligned}$$

In a similar fashion, we also get

$$\begin{aligned}
&\lim_{t\to 0}\frac{1}{2t}\biggl(\frac{(-2tr'(w+t)+r(w+t))r(w+t)}{2t}\int_{w-t}^{w+t}\frac{1}{r^2}-1\biggr)\\
&=-\lim_{t\to 0}\biggl(\frac{r'(w+t)r(w+t)}{2t}\int_{w-t}^{w+t}\frac{1}{r^2}\biggr)+\lim_{t\to 0}\frac{1}{4t^2}\biggl(r(w+t)^2\int_{w-t}^{w+t}\frac{1}{r^2}-2t\biggr)\\
&=-\frac{r'(w)}{r(w)}+\lim_{t\to 0}\frac{1}{8t}\biggl(2r'(w+t)r(w+t)\int_{w-t}^{w+t}\frac{1}{r^2}+r(w+t)^2\biggl(\frac{1}{r(w+t)^2}+\frac{1}{r(w-t)^2}\biggr)-2\biggr)\\
&=-\frac{r'(w)}{r(w)}+\frac{r'(w)}{2r(w)}+\lim_{t\to 0}\frac{r(w+t)-r(w-t)}{2t}\cdot\frac{r(w+t)+r(w-t)}{4r(w-t)^2}=-\frac{r'(w)}{2r(w)}+\frac{r'(w)}{2r(w)}=0
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{t\to 0}\frac{1}{2t}\biggl(\frac{(2tr'(w-t)+r(w-t))r(w-t)}{2t}\int_{w-t}^{w+t}\frac{1}{r^2}-1\biggr)\\
&=\lim_{t\to 0}\biggl(\frac{r'(w-t)r(w-t)}{2t}\int_{w-t}^{w+t}\frac{1}{r^2}\biggr)+\lim_{t\to 0}\frac{1}{4t^2}\biggl(r(w-t)^2\int_{w-t}^{w+t}\frac{1}{r^2}-2t\biggr)\\
&=\frac{r'(w)}{r(w)}+\lim_{t\to 0}\frac{1}{8t}\biggl(-2r'(w-t)r(w-t)\int_{w-t}^{w+t}\frac{1}{r^2}+r(w-t)^2\biggl(\frac{1}{r(w+t)^2}+\frac{1}{r(w-t)^2}\biggr)-2\biggr)\\
&=\frac{r'(w)}{r(w)}-\frac{r'(w)}{2r(w)}+\lim_{t\to 0}\frac{1}{8t}\frac{r(w-t)^2-r(w+t)^2}{r(w+t)^2}=\frac{r'(w)}{r(w)}-\frac{r'(w)}{2r(w)}-\frac{r'(w)}{2r(w)}=0.
\end{aligned}$$

From the above three equalities and the continuity of P_i, P_i' and r, it follows that the right hand side of the equation (23) at $u=w+t$ and $v:=w-t$ multiplied by $\frac{1}{2t}$ tends to zero as $t\to 0$. Therefore, the same property is valid for the left hand side of (23), i.e, for all $w\in J$,

$$\begin{aligned}
0=\lim_{t\to 0}\biggl(&\frac{P_i(w+t)}{r(w-t)}\biggl(\frac{r(w+t)-r(w-t)}{4t^2}-\frac{r'(w-t)}{2t}\biggr)\\
&+\frac{P_j(w-t)}{r(w+t)}\biggl(\frac{r(w+t)-r(w-t)}{4t^2}-\frac{r'(w+t)}{2t}\biggr)\biggr).
\end{aligned}$$

Replacing t by $-t$ and multiplying the equality so obtained by -1, we also get that

$$\begin{aligned}
0=\lim_{t\to 0}\biggl(&\frac{P_i(w-t)}{r(w+t)}\biggl(\frac{r(w+t)-r(w-t)}{4t^2}-\frac{r'(w+t)}{2t}\biggr)\\
&+\frac{P_j(w+t)}{r(w-t)}\biggl(\frac{r(w+t)-r(w-t)}{4t^2}-\frac{r'(w-t)}{2t}\biggr)\biggr).
\end{aligned}$$

These equalities, with obvious manipulations of the limits, imply that

$$\begin{aligned}
0=\lim_{t\to 0}\biggl(\frac{P_j(w-t)}{P_i(w+t)}\frac{r(w-t)}{r(w+t)}&\biggl(\frac{r(w+t)-r(w-t)}{4t^2}-\frac{r'(w+t)}{2t}\biggr)\\
&+\frac{r(w+t)-r(w-t)}{4t^2}-\frac{r'(w-t)}{2t}\biggr)
\end{aligned}$$

and

$$0=\lim_{t\to 0}\left(\frac{P_i(w-t)}{P_j(w+t)}\frac{r(w-t)}{r(w+t)}\left(\frac{r(w+t)-r(w-t)}{4t^2}-\frac{r'(w+t)}{2t}\right)\right.$$
$$\left.+\frac{r(w+t)-r(w-t)}{4t^2}-\frac{r'(w-t)}{2t}\right).$$

Subtracting the two equalities side by side, we arrive at

$$0=\lim_{t\to 0}\frac{r(w-t)}{r(w+t)}\left(\frac{P_j(w-t)}{P_i(w+t)}-\frac{P_i(w-t)}{P_j(w+t)}\right)\left(\frac{r(w+t)-r(w-t)}{4t^2}-\frac{r'(w+t)}{2t}\right),$$

which yields that

$$0=\lim_{t\to 0}\left(P_j(w+t)P_j(w-t)-P_i(w+t)P_i(w-t)\right)\left(\frac{r(w+t)-r(w-t)}{4t^2}-\frac{r'(w+t)}{2t}\right).$$

Replacing t by $-t$ and then multiplying the equality so obtained by -1, we also get that

$$0=\lim_{t\to 0}\left(P_j(w+t)P_j(w-t)-P_i(w+t)P_i(w-t)\right)\left(\frac{r(w+t)-r(w-t)}{4t^2}-\frac{r'(w-t)}{2t}\right).$$

Subtracting the second equality from the first one, we conclude that

$$0=\lim_{t\to 0}\big(P_j(w+t)P_j(w-t)-P_i(w+t)P_i(w-t)\big)\frac{r'(w+t)-r'(w-t)}{2t}. \tag{25}$$

By our assumption, $P_i(w)\neq P_j(w)$ holds for all $w\in J$. Thus we have that

$$\lim_{t\to 0}\big(P_j(w+t)P_j(w-t)-P_i(w+t)P_i(w-t)\big)=P_j^2(w)-P_i^2(w)\neq 0,$$

therefore, the limit of the second factor in (25) exists and equals zero. This shows that the symmetric derivative of r' vanishes on J. By the quasi-mean value theorem of Aull [2, Theorem 1], this implies that r' is constant on J, hence r is an affine function on J. □

Theorem 19. *Let $f,g:I\to\mathbb{R}$ be strictly monotone continuous functions. Let $n\geq 2$ and $p=(p_1,\dots,p_n),q=(q_1,\dots,q_n):I\to\mathbb{R}_+^n$. Assume that p is continuous on I and there exist indices $i,j\in\{1,\dots,n\}$, $i\neq j$ and a nonempty open subinterval I_0 of I such that*

- *(a) $p_i(x)\neq p_j(x)$ for all $x\in I_0$.*
- *(b) $p_i\circ f^{-1}$ and $p_j\circ f^{-1}$ are continuously differentiable on the interval $f(I_0)$.*
- *(c) $g\circ f^{-1}$ is continuously differentiable on the interval $f(I_0)$ with a nonvanishing first derivative.*

Then the following assertions are equivalent to each other:

- *(i) The means $A_{f,p}$ and $A_{g,q}$ are globally equal on I^n.*
- *(ii) The means $A_{f,p}$ and $A_{g,q}$ are locally equal on I^n.*
- *(iii) There exist $a,b,c,d\in\mathbb{R}$ with $ad\neq bc$ such that the conditions in (9) hold on I.*

Proof. The implication (i)⇒(ii) is obvious. The implication (iii)⇒(i) is a consequence of Theorem 6. Therefore, we may restrict our attention to the proof of implication (ii)⇒(iii).

Assume now that assertion (ii) holds, i.e., the n-variable means $A_{f,p}$ and $A_{g,q}$ are locally equal on I^n and hence, on I_0^n as well. Define the interval $J_0 := f(I_0)$, and the functions h, $P = (P_1, \ldots, P_n)$, and $Q = (Q_1, \ldots, Q_n)$ on the interval J_0 by (12). Then, applying Theorem 11, we get that the n-variable means $A_{\mathrm{id},P}$ and $A_{h,Q}$ are locally equal on J_0^n. By our assumption (c), h is continuously differentiable on J_0 with a nonvanishing first derivative. The continuity of p, implies that P is continuous on J_0. Thus, by the first assertion of Lemma 7, it follows that Q is also continuous on J_0 and there exists a continuous function $r : J_0 \to \mathbb{R}_+$ such that (17) holds on J_0. By assumption (b), we have that P_i and P_j are continuously differentiable on J_0. This and the second assertion of Lemma 7 imply that Q_i and Q_j are also continuously differentiable on J_0. Therefore, r is also continuously differentiable on J_0. In addition, by assumption (a), we also have that $P_i(u) \neq P_j(u)$ for all $u \in J_0$.

As a consequence of Corollary 10, we obtain that the two-variable means $A_{\mathrm{id},(P_i,P_j)}$ and $A_{h,(Q_i,Q_j)}$ are locally equal on J_0^2. From Corollary 14, it follows that

$$h\left(\frac{P_i(u)u + P_j(v)}{P_i(u) + P_j(v)}\right) = \frac{(rhP_i)(u) + (rhP_j)(v)}{(rP_i)(u) + (rP_j)(v)}$$

holds locally for $(u, v) \in J_0^2$.

Using now Proposition 16, we obtain that (20) is valid locally for $(u, v) \in J_0^2$. Then, in view of Proposition 17, we get that $h'r^2$ is a constant function, h is twice differentiable on J_0, and (21) is satisfied locally for $(u, v) \in J_0^2$. Finally, using Proposition 18, we conclude that, for some $c, d \in \mathbb{R}$, the function r is of the form $r(u) = cu + d$. Since $h'r^2$ is a constant on J_0, it follows that

$$(hr)'' = h''r + 2h'r' + hr'' = h''r + 2h'r' = \frac{1}{r}(h'r^2)' = 0,$$

which shows that hr is also an affine function, i.e., there exist constants $a, b \in \mathbb{R}$ such that $(hr)(u) = au + b$ for all $u \in J_0$. This proves that (19) is valid on J_0. Therefore, with the substitution $u := f^{-1}(x)$, using also that (17) is valid on J_0, we can obtain that (9) is valid on the interval I_0. Then, applying Theorem 8, we can see that the equality (9) is also valid on I.

This completes the proof of the implication (ii)⇒(iii). □

Theorem 20. *Let $f, g : I \to \mathbb{R}$ be strictly monotone continuous functions. Let $n \geq 3$ and $p = (p_1, \ldots, p_n) \to \mathbb{R}_+^n$, $q = (q_1, \ldots, q_n) \to \mathbb{R}_+^n$. Assume that p is continuous on I and there exist indices $i, j, k \in \{1, \ldots, n\}$ with $i < j < k$ and a nonempty open subinterval I_0 of I such that*

(a) $p_i \circ f^{-1}$, $p_j \circ f^{-1}$ and $p_k \circ f^{-1}$ are continuously differentiable on the interval $f(I_0)$.

(b) $g \circ f^{-1}$ *is continuously differentiable on the interval* $f(I_0)$ *with a nonvanishing first derivative.*

Then the following three assertions are equivalent to each other:

(i) *The means* $A_{f,p}$ *and* $A_{g,q}$ *are globally equal on* I^n*.*
(ii) *The means* $A_{f,p}$ *and* $A_{g,q}$ *are locally equal on* I^n*.*
(iii) *There exist* $a, b, c, d \in \mathbb{R}$ *with* $ad \neq bc$ *such that the conditions in* (9) *hold on* I*.*

Proof. The implications (i)⇒(ii) and (iii)⇒(i) follow in the same way as in the proof of the previous theorem.

Assume now that assertion (ii) holds, i.e., the n-variable means $A_{f,p}$ and $A_{g,q}$ are locally equal on I^n and hence, on I_0^n as well. Define the interval $J_0 := f(I_0)$, and the functions h, $P = (P_1, \dots, P_n)$, and $Q = (Q_1, \dots, Q_n)$ on the interval J_0 by (12). Then, using the same argument as in the proof of the previous theorem, we can establish that h is continuously differentiable on J_0 with a nonvanishing first derivative, P and Q are continuous on J_0, $P_i, P_j, P_k, Q_i, Q_j, Q_k$ are continuously differentiable and there exists a continuously differentiable function $r : J_0 \to \mathbb{R}_+$ such that (17) holds on J_0.

Then, in view of Theorem 9, it follows that the 3-variable means $A_{f,(p_i,p_j,p_k)}$ and $A_{g,(q_i,q_j,q_k)}$ are locally equal on I^3. Let $x_0 \in I_0$ be fixed. The equalities

$$p_i(x_0) = p_j(x_0) + p_k(x_0), \;\; p_j(x_0) = p_i(x_0) + p_k(x_0), \;\; p_k(x_0) = p_i(x_0) + p_j(x_0)$$

cannot hold simultaneously (because the sum of the left hand sides is strictly smaller than the sum of the right hand sides). By the symmetric role of these indices, we may assume that $p_i(x_0) \neq p_j(x_0) + p_k(x_0)$. Then, by the continuity of the functions p_i, p_j, p_k at x_0, the inequality $p_i(x) \neq p_j(x) + p_k(x)$ holds for all x belonging to a neighborhood of x_0. Without loss of generality, we may also assume that it holds for all $x \in I_0$.

The local equality of the 3-variable means $A_{f,(p_i,p_j,p_k)}$ and $A_{g,(q_i,q_j,q_k)}$ easily implies that the 2-variable means $A_{f,(p_i,p_j+p_k)}$ and $A_{g,(q_i,q_j+q_k)}$ are locally equal on I^2. Observe that we can apply the previous theorem to these 2-variable means and we have that assertion (ii) holds in this setting. Thus, the assertion (iii) in this setting shows that there exist four constants $a, b, c, d \in R$ such that

$$cf + d > 0, \qquad g = \frac{af + b}{cf + d}, \;\; q_i = (cf + d)p_i, \;\; q_j + q_k = (cf + d)(p_j + p_k)$$

hold on I_0. The equality $q_i = (cf + d)p_i$ implies that $Q_i(u) = (cu + d)P_i(u)$ for all $u \in J_0 = f(I_0)$. Therefore, $r(u) = cu + d$ for all $u \in J_0$. This equality together with (17) yield that, for all $u \in J_0$,

$$Q(u) = (cu + d)P(u).$$

Therefore, we can deduce that

$$q = (cf + d)p$$

is valid on I_0 and hence (9) holds on I_0. Now, applying Theorem 8, it follows that (9) also holds on I, i.e., assertion (iii) of our theorem is valid. □

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