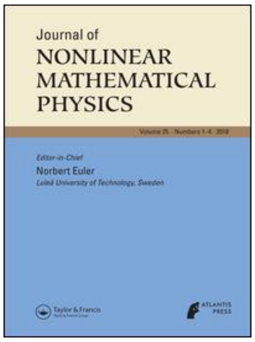

Journal of Nonlinear Mathematical Physics

ISSN: 1402-9251 (Print) 1776-0852 (Online) Journal homepage: https://www.tandfonline.com/loi/tnmp20

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To cite this article: Ágota Figula, Gábor Horváth, Tamás Milkovszki & Zoltán Muzsnay (2020) The Lie symmetry group of the general Liénard-type equation, Journal of Nonlinear Mathematical Physics, 27:2, 185-198, DOI: 10.1080/14029251.2020.1700623

To link to this article: https://doi.org/10.1080/14029251.2020.1700623

Journal of Nonlinear Mathematical Physics, Vol. 27, No. 2 (2020) 185–198

The Lie symmetry group of the general Liénard-type equation

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Received 8 June 2019

Accepted 13 August 2019

We consider the general Liénard-type equation $\ddot{u} = \sum_{k=0}^{n} f_k \dot{u}^k$ for $n \geq 4$. This equation naturally admits the Lie symmetry $\frac{\partial}{\partial t}$. We completely characterize when this equation admits another Lie symmetry, and give an easily verifiable condition for this on the functions $f_0, \dots, f_n$. Moreover, we give an equivalent characterization of this condition. Similar results have already been obtained previously in the cases $n = 1$ or $n = 2$. That is, this paper handles all remaining cases except for $n = 3$.

Keywords: second order ordinary differential equation; Liénard-type equation; Levinson–Smith-type equation; Lie group; symmetry analysis; Lie algebra of infinitesimal symmetries

2010 Mathematics Subject Classification: 34C14, 34A26, 34C20

1. Introduction

In this paper we consider the Liénard-type second order ordinary differential equation

$$\ddot{u} = \sum_{k=0}^{n} f_k(u)\dot{u}^k, \tag{1.1}$$

where the dot denotes differentiating by the independent variable t representing the time. Eq. (1.1) is a special case of the Levinson–Smith-type equation $\ddot{u} = g_1(u, \dot{u})\,\dot{u} + g_0(u)$ [1, G.3, p. 198–199], for which existence and uniqueness of a limit cycle have been established under certain conditions [2,3].

Eq. (1.1) is a common generalization of the Rayleigh-type equation $\ddot{u} + F(\dot{u}) + u = 0$ when F is a polynomial, the classical Liénard-type equation $\ddot{u} = f_1(u)\dot{u} + f_0(u)$, and the quadratic Liénard-type equation $\ddot{u} = f_2(u)\dot{u}^2 + f_1(u)\dot{u} + f_0(u)$. These equations come up quite often in Physics or Biology. Rayleigh-type systems play an important role in the theory of sound [4] or in the theory of non-linear oscillations [5, Chapter 2.2.4]. Classical Liénard-type equations arise in the model of the van der Pol oscillator applied in physical and biological sciences [6], but electric activity of the heart rate [7] or nerve impulses are modelled by a Liénard-type model, as well [8,9] or [10, Chapter 7]. In [11] the population of Easter Island is modelled, and the system of differential equations is then reduced to a second order quadratic Liénard-type equation. One can even find applications in economy [12–15].

Symmetry analysis is a very useful tool developed to understand and solve differential equations. Several examples come from Physics (see e.g. [16, 17] for comprehensive studies on the topic), and an increasing number of examples from Biology (see e.g. [11, 18–20]). Finding some symmetries for a differential equation can be used to derive an appropriate change of coordinates which then helps to eliminate the independent variables or to decrease the order of the system.

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In many cases mentioned above (e.g. the Fitzhugh–Nagumo model [8, 9] or the model for the population of Easter Island [11]) the model is based on a first order system of two equations equivalent to a second order Liénard equation. In such a case, one might benefit to consider the equivalent second order system, which would only admit a finite dimensional Lie symmetry group instead of an infinite one. If this Lie group is at least two-dimensional, then pulling the symmetries back to the original system could yield two independent symmetries of the original system, and solutions can be determined by quadratures. This method has been applied successfully in several situations in the past (see e.g. [11, 18, 20] for some recent examples in Biology). This motivates to study the Lie symmetries of (1.1).

Pandey, Bindu, Senthilvelan and Lakshmanan [21,22] considered the classical Liénard equation

$$\ddot{u} = f_1(u)\dot{u} + f_0(u), \tag{1.2}$$

where f_1 and f_0 are arbitrary, infinitely many times differentiable functions. They classified when (1.2) has a 1, 2, 3, or 8 dimensional Lie symmetry group depending on f_0 and f_1. Then Tiwari, Pandey, Senthilvelan and Lakshmanan [23, 24] classified the dimension of the Lie symmetry group of quadratic Liénard-type equations without $\dot{u}$ term, and then more generally [25] the mixed quadratic Liénard-type equation

$$\ddot{u} = f_2(u)\dot{u}^2 + f_1(u)\dot{u} + f_0(u), \tag{1.3}$$

where f_0, f_1 and f_2 are arbitrary, infinitely many times differentiable functions. Further, Paliathanasis and Leach [26] showed how one can simplify (1.3) by removing f_2 from (1.3) in the case $f_1 = 0$. The question naturally arises: what are the Lie symmetries if the right-hand side of (1.3) is a higher order polynomial in $\dot{u}$?

In this paper we consider (1.1) for $n \geq 4$ and for differentiable functions f_k depending only on u, and not on t. Note, that (1.1) is autonomous, therefore the tangential Lie algebra $\mathscr{L}$ of the Lie group of all its symmetries always contains the 1-dimensional subalgebra generated by the vector field $\frac{\partial}{\partial t}$. Determining another generator of $\mathscr{L}$ would then lead to a solution by quadratures of (1.1), and of any first order system equivalent to it. In Theorem 3.1 (see Section 3 for details) we completely characterize the case when (1.1) admits a more than 1 (in fact, 2) dimensional symmetry group. In particular, we give conditions (3.1–3.4) such that the symmetry group is 2-dimensional if and only if these conditions hold.

Here, conditions (3.1–3.3) are natural, but the meaning of the system (3.4) seems less intuitive, even though the system (3.4) is easily verifiable for a particular choice of F. In Theorem 4.1 (see Section 4 for details) we provide a necessary and sufficient condition for $f_0, \ldots, f_n$ to satisfy (3.4). It turns out that $f_0, \ldots, f_n$ satisfy (3.4) if and only if they are expressible by F and some constants.

2. The symmetry condition

We formulate the symmetry condition for (1.1) in this section. Consider (1.1) on the plane (t, u), where t is the independent variable, and u is the dependent variable. Further, the computations will be slightly easier if we consider the right-hand side as an infinite sum $\sum_k f_k \dot{u}^k = \sum_{k=-\infty}^{\infty} f_k \dot{u}^k$, where $f_k = 0$ if $k < 0$ or $k > n$.

The general form of an infinitesimal generator of a symmetry of (1.1) has the form

$$X = \xi(t,u)\frac{\partial}{\partial t} + \eta(t,u)\frac{\partial}{\partial u}. \tag{2.1}$$

Let D denote the total derivation by t, that is $D\xi = \xi_t + \dot{u}\xi_u$, $D\eta = \eta_t + \dot{u}\eta_u$. We use the convention of writing partial derivatives into the lower right index. Then the first prolongation of X is

$$X^1 = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + (D\eta - \dot{u}D\xi)\frac{\partial}{\partial \dot{u}} = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \Big(\eta_t + (\eta_u - \xi_t)\,\dot{u} - \xi_u\dot{u}^2\Big)\frac{\partial}{\partial \dot{u}}.$$

Further, let

$$S^1 = \frac{\partial}{\partial t} + \dot{u}\frac{\partial}{\partial u} + \left(\sum_k f_k\dot{u}^k\right)\frac{\partial}{\partial \dot{u}},$$

be the spray corresponding to the differential equation (1.1). The vector field (2.1) is an infinitesimal symmetry of (1.1) if and only if its first prolongation X^1 satisfies the Lie bracket condition

$$[X^1 - \xi S^1, S^1] = 0 \tag{2.2}$$

on the space $(t,u,\dot{u})$ (cf. [16, Chapter 4, §3]). Substituting X^1 and S^1 into (2.2) we obtain

$$\begin{aligned}
0 &= \left[X^1 - \xi S^1, S^1\right] \\
&= \Big((\eta - \xi\dot{u})\Big(\sum_k f_k\dot{u}^k\Big)_u + \Big(\eta_t + (\eta_u - \xi_t)\,\dot{u} - \xi_u\dot{u}^2 - \xi\big(\sum_k f_k\dot{u}^k\big)\Big)\Big(\sum_k f_k\dot{u}^k\Big)_{\dot{u}} \\
&\quad - \Big(\eta_t + (\eta_u - \xi_t)\,\dot{u} - \xi_u\dot{u}^2 - \xi\Big(\sum_k f_k\dot{u}^k\Big)\Big)_t - \dot{u}\Big(\eta_t + (\eta_u - \xi_t)\,\dot{u} - \xi_u\dot{u}^2 - \xi\Big(\sum_k f_k\dot{u}^k\Big)\Big)_u \\
&\quad - \Big(\sum_k f_k\dot{u}^k\Big)\Big(\eta_t + (\eta_u - \xi_t)\,\dot{u} - \xi_u\dot{u}^2 - \xi\Big(\sum_k f_k\dot{u}^k\Big)\Big)_{\dot{u}}\Big)\frac{\partial}{\partial \dot{u}} \\
&= \Big(-\eta_{tt} + (\xi_{tt} - 2\eta_{tu})\,\dot{u} + (2\xi_{ut} - \eta_{uu})\,\dot{u}^2 + \xi_{uu}\dot{u}^3 \\
&\qquad + \sum_k \left(f_k'\eta + (k+1)f_{k+1}\eta_t + (k-1)f_k\eta_u + (2-k)f_k\xi_t + (4-k)f_{k-1}\xi_u\right)\dot{u}^k\Big)\frac{\partial}{\partial \dot{u}},
\end{aligned}$$

therefore the symmetry condition is

$$\begin{aligned}
&-\eta_{tt} + f_0'\eta + f_1\eta_t - f_0\eta_u + 2f_0\xi_t \\
&+\left(\xi_{tt} - 2\eta_{tu} + f_1'\eta + 2f_2\eta_t + f_1\xi_t + 3f_0\xi_u\right)\cdot\dot{u} \\
&+\left(2\xi_{tu} - \eta_{uu} + f_2'\eta + 3f_3\eta_t + f_2\eta_u + 2f_1\xi_u\right)\cdot\dot{u}^2 \\
&+\left(\xi_{uu} + f_3'\eta + 4f_4\eta_t + 2f_3\eta_u - f_3\xi_t + f_2\xi_u\right)\cdot\dot{u}^3 \\
&+\textstyle\sum_{k=4}^{n-1}\left(f_k'\eta + (k+1)f_{k+1}\eta_t + (k-1)f_k\eta_u + (2-k)f_k\xi_t + (4-k)f_{k-1}\xi_u\right)\cdot\dot{u}^k \\
&+\left(f_n'\eta + (n-1)f_n\eta_u + (2-n)f_n\xi_t + (4-n)f_{n-1}\xi_u\right)\cdot\dot{u}^n \\
&+(3-n)f_n\xi_u\cdot\dot{u}^{n+1} = 0.
\end{aligned} \tag{2.3}$$

3. Lie symmetry algebra

We consider (1.1) for $n \geq 4$ and for differentiable functions f_k depending only on u, and not on t. In Theorem 3.1 we completely characterize the case when (1.1) admits more than 1 (in fact, 2) dimensional symmetry group. We prove following

Theorem 3.1. *Consider* (1.1) *for some $n \geq 4$ and for $f_0, \ldots, f_n \colon I \to \mathbb{R}$ being differentiable functions such that f_n is not constant zero on the open interval $I \subseteq \mathbb{R}$. Then the Lie symmetry algebra $\mathscr{L}$ of* (1.1) *is exactly* 1 *dimensional, unless there exists a constant $a \in \mathbb{R}$, an open interval $U \subseteq I$, and a three-times differentiable function $F \colon U \to \mathbb{R}$ such that for all $u \in U$ we have*

$$f_n(u) \neq 0. \tag{3.1}$$

$$F(u) \neq 0, \tag{3.2}$$

$$F'(u) = |f_n(u)|^{\frac{1}{n-1}}, \tag{3.3}$$

and with the notation $g(u) = \frac{(n-2)F(u)}{(n-1)F'(u)}$ the following hold:

$$-a^2 g + f_0' g + a f_1 g + (-1) f_0 g' + 2 f_0 = 0, \tag{3.4a}$$

$$a\left(1 - 2g'\right) + f_1' g + 2a f_2 g + f_1 = 0, \tag{3.4b}$$

$$-g'' + f_2' g + 3a f_3 g + f_2 g' = 0, \tag{3.4c}$$

$$f_k' g + (k+1) a f_{k+1} g + (k-1) f_k g' + (2-k) f_k = 0, \qquad 3 \leq k \leq n-1. \tag{3.4d}$$

Further, if both F_1 and F_2 satisfy conditions (3.1–3.4)*, then $F_1 = F_2$.*

Remark 3.1. (Generator of the symmetry algebra $\mathscr{L}$ of (1.1).) In case conditions (3.1–3.4)

1. do not hold, then the symmetry algebra $\mathscr{L}$ is generated by $\frac{\partial}{\partial t}$,
2. hold, then the 2 dimensional Lie symmetry algebra $\mathscr{L}$ is generated
   (a) by $\frac{\partial}{\partial t}$ and $t\frac{\partial}{\partial t} + g(u)\frac{\partial}{\partial u}$ if $a = 0$, or
   (b) by $\frac{\partial}{\partial t}$ and $e^{at}\frac{\partial}{\partial t} + a e^{at} g(u)\frac{\partial}{\partial u}$ if $a \neq 0$.

Proof. The left-hand side of (2.3) has to be zero for all $(t, u, \dot{u})$. As ξ, η, f_k ($0 \leq k \leq n$) do not depend on $\dot{u}$, (2.3) is a polynomial in $\dot{u}$. Thus, (2.3) holds if and only if each of its coefficients is zero. Since f_n is not constant 0 on the interval I, there exists an open interval $U' \subseteq I$ such that $f_n(u) \neq 0$ for $u \in U'$. Thus, from the coefficient of $\dot{u}^{n+1}$ by $n \geq 4$ we obtain

$$\xi_u = 0.$$

In particular, ξ only depends on t and not on u. Substituting $\xi_u = 0$ into (2.3) and considering the coefficients, we obtain that

$$-\eta_{tt} + f_0'\eta + f_1\eta_t + (-1) f_0 \eta_u + 2 f_0 \xi_t = 0, \tag{3.5a}$$

$$(\xi_{tt} - 2\eta_{tu}) + f_1'\eta + 2 f_2 \eta_t + f_1 \xi_t = 0, \tag{3.5b}$$

$$-\eta_{uu} + f_2'\eta + 3 f_3 \eta_t + f_2 \eta_u = 0, \tag{3.5c}$$

$$f_k'\eta + (k+1) f_{k+1}\eta_t + (k-1) f_k \eta_u + (2-k) f_k \xi_t = 0, \qquad 3 \leq k \leq n-1 \tag{3.5d}$$

$$f_n'\eta + (n-1) f_n \eta_u + (2-n) f_n \xi_t = 0. \tag{3.5e}$$

In the following we analyze the system (3.5). Note, that $\xi = c$, $\eta = 0$ (for any $c \in \mathbb{R}$) satisfies (3.5). Further, if $\eta = 0$ then from (3.5e) we have $\xi_t = 0$ and $\xi = c$ for some $c \in \mathbb{R}$. Thus, in the following we assume that η is not constant 0.

Consider (3.5e) first. Now, f_n is nonzero on the open interval U', therefore either $f_n(u) > 0$ for all $u \in U'$ or $f_n(u) < 0$ for all $u \in U'$. Choose $\varepsilon \in \{1, -1\}$ such that $\varepsilon f_n(u) > 0$ for all $u \in U'$, that is $|f_n| = \varepsilon f_n$. Then we have

$$|f_n|'\, \eta + (n-1)\,|f_n|\, \eta_u + (2-n)\,|f_n|\, \xi_t = 0.$$

Multiplying by $\frac{1}{n-1}\,|f_n|^{\frac{2-n}{n-1}}$ yields

$$\frac{|f_n|' \cdot |f_n|^{\frac{2-n}{n-1}}}{(n-1)}\eta + |f_n|^{\frac{1}{n-1}}\, \eta_u = \frac{n-2}{n-1}\,|f_n|^{\frac{1}{n-1}}\, \xi_t.$$

Here, the left-hand side is the u-derivative of $|f_n|^{\frac{1}{n-1}}\, \eta$, hence

$$\begin{aligned}\left(|f_n|^{\frac{1}{n-1}}\, \eta\right)_u &= \frac{n-2}{n-1}\,|f_n|^{\frac{1}{n-1}}\, \xi_t,\\ |f_n|^{\frac{1}{n-1}}\, \eta &= \frac{n-2}{n-1}\xi_t \int |f_n|^{\frac{1}{n-1}}\, du,\\ \eta &= \xi_t \frac{(n-2)\int |f_n|^{\frac{1}{n-1}}\, du}{(n-1)\,|f_n|^{\frac{1}{n-1}}}.\end{aligned}$$

Let $F\colon U' \to \mathbb{R}$ be a function defined as in (3.3), that is

$$F'(u) := |f_n(u)|^{\frac{1}{n-1}},$$

and let g be defined as in Theorem 3.1, that is

$$g(u) := \frac{(n-2)\,F(u)}{(n-1)\,F'(u)}.$$

Thus we obtained that

$$\eta = g(u) \cdot \xi_t(t). \tag{3.6}$$

Note, that $g(u)$ cannot be constant 0 on U' (otherwise both F and $F' = |f_n|^{\frac{1}{n-1}}$ were constant 0), thus there exists an open interval $U \subseteq U'$ such that $F(u) \neq 0$ for all $u \in U$, and hence $g(u) \neq 0$ $(u \in U)$. By substituting (3.6) into (3.5d) we have

$$-(k+1) f_{k+1} g \xi_{tt} = \left(f_k' g + (k-1) f_k g' + (2-k) f_k\right) \xi_t.$$

In particular, for $k = n-1$ we have $f_n(u) g(u) \neq 0$ for all $u \in U$, thus

$$\xi_{tt} = -\frac{f_{n-1}' g + (n-2) f_{n-1} g' + (3-n) f_{n-1}}{n f_n g}\xi_t.$$

Now, f_n, f_{n-1}, and g only depend on u, and ξ_{tt} and ξ_t only depend on t. Therefore there exists $a \in \mathbb{R}$ such that

$$a = -\frac{f_{n-1}' g + (n-2) f_{n-1} g' + (3-n) f_{n-1}}{n f_n g}, \tag{3.7}$$

$$\xi_{tt} = a \xi_t, \tag{3.8}$$

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or else $\xi_t = 0$ implying $\eta = 0$ by (3.6), a contradiction. Thus,

$$\xi_t = ce^{at}, \tag{3.9}$$
$$\eta = ce^{at} g(u) \tag{3.10}$$

for some $c \in \mathbb{R}$. Substituting (3.9) and (3.10) into (3.5), one obtains

$$\begin{aligned}
(-a^2 g + f_0' g + a f_1 g + (-1) f_0 g' + 2 f_0) \cdot ce^{at} &= 0, \\
(a\left(1-2g'\right) + f_1' g + 2a f_2 g + f_1) \cdot ce^{at} &= 0, \\
(-g'' + f_2' g + 3a f_3 g + f_2 g') \cdot ce^{at} &= 0, \\
(f_k' g + (k+1) a f_{k+1} g + (k-1) f_k g' + (2-k) f_k) \cdot ce^{at} &= 0, \qquad 3 \le k \le n-1.
\end{aligned}$$

Thus, either $f_0, f_1, f_2, \dots f_n, g, F$ satisfy the conditions (3.1–3.4), or else $c = 0$, implying $\xi_t = 0$ and $\eta = 0$, a contradiction.

Finally, assume that F_1 and F_2 both satisfy conditions (3.2–3.3) on U. Then let g_1 and g_2 be defined from F_1 and F_2, and let a_1 and a_2 be defined using (3.7). Let ξ_1, ξ_2 be such that $(\xi_i)_t = e^{a_i t}$, let $\eta_i = e^{a_i t} g_i(u)$, and let $X_i = \xi_i \frac{\partial}{\partial t} + \eta_i \frac{\partial}{\partial u}$ for $i = 1, 2$. Further, let $\xi = \xi_1 - \xi_2$, $\eta = \eta_1 - \eta_2$, and $X = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}$. Now, if F_1 and F_2 both satisfy conditions (3.2–3.3) and (3.4), then both X_1 and X_2 are elements of the Lie symmetry algebra $\mathscr{L}$, and thus $X = X_1 - X_2 = (\xi_1 - \xi_2) \frac{\partial}{\partial t} + (\eta_1 - \eta_2) \frac{\partial}{\partial u} \in \mathscr{L}$, as well. However, we have proved that if either $\xi_t \neq 0$ or $\eta \neq 0$, then they are of the form (3.9) and (3.10). Thus, $e^{a_1 t} - e^{a_2 t}$ is of the form ce^{at} for some $a, c \in \mathbb{R}$, that is $a_1 = a_2$ and $c = 0$. Then $\xi_t = 0$, implying $\eta = 0$ by (3.6), which yields $g_1 = g_2$. Finally, by $F_1' = F_2'$, the definition of g immediately implies

$$0 = g_1 - g_2 = \frac{n-2}{(n-1)F_1'} (F_1 - F_2),$$

and hence $F_1 = F_2$. This finishes the proof of Theorem 3.1. □

4. Equivalent description of the conditions

In this paragraph we provide a necessary and sufficient condition for $f_0, \dots, f_n$ to satisfy (3.4). We have the following

Theorem 4.1. *Let $U \subseteq \mathbb{R}$ be an open interval, $f_0, \dots, f_n, F \colon U \to \mathbb{R}$ be functions satisfying* (3.1–3.3)*, g as introduced in Theorem* 3.1 *and let $\varepsilon, \nu \in \{1, -1\}$, $a \in \mathbb{R}$ be real constants such that $|f_n| = \varepsilon f_n$, $|F| = \nu F$. Then $f_0, \dots, f_n, F, g$ satisfy* (3.4) *if and only if there exist real constants b_k and functions A_k, $B_k \colon U \to \mathbb{R}$ $(0 \le k \le n)$ where $b_n = \varepsilon \left(\frac{n-1}{n-2}\right)^{1-n}$, A_n is constant* 0*,*

$$A_k(u) = \sum_{i=1}^{n-k} (-\nu)^i \binom{k+i}{i} a^i b_{k+i} |F(u)|^{\frac{i(n-1)}{n-2}}, \qquad (0 \le k \le n-1), \tag{4.1}$$

and

$$\begin{aligned}
B_k(u) &= A_k(u), && (3 \le k \le n) && \text{(4.2a)}\\
B_2(u) &= \nu A_2(u), && && \text{(4.2b)}\\
B_1(u) &= A_1(u) - a\,|F(u)|^{\frac{n-1}{n-2}}\,(2b_2(1-\nu)+1), && && \text{(4.2c)}\\
B_0(u) &= A_0(u) + a^2\,(1+b_2(1-\nu))\,\nu\,|F(u)|^{\frac{2(n-1)}{n-2}}, && && \text{(4.2d)}
\end{aligned}$$

such that $f_0,\dots,f_n$ are of the form

$$f_k(u) = (b_k + B_k(u)) \cdot \left(\frac{n-1}{n-2}\right)^{k-1} \cdot |F(u)|^{\frac{k-n}{n-2}} \cdot \left(F'(u)\right)^{k-1}, \qquad (0 \le k \le n,\ k \ne 2) \qquad \text{(4.3a)}$$

$$f_2(u) = (b_2 + B_2(u)) \cdot \frac{n-1}{n-2} \cdot \frac{F'(u)}{F(u)} + \frac{F'(u)}{F(u)} - \frac{F''(u)}{F'(u)}. \qquad \text{(4.3b)}$$

In particular, if $a = 0$, then $B_k(u) = 0$ for all $u \in U$ $(0 \le k \le n)$, and thus $f_0,\dots,f_n,F,g$ satisfy (3.4) *if and only if there exist real constants b'_k $(0 \le k \le n)$ such that*

$$\begin{aligned}
f_k(u) &= b'_k \cdot |F(u)|^{\frac{k-n}{n-2}} \cdot \left(F'(u)\right)^{k-1}, && (0 \le k \le n, k \ne 2)\\
f_2(u) &= b'_2 \cdot \frac{F'(u)}{F(u)} - \frac{F''(u)}{F'(u)}.
\end{aligned}$$

Further, if F is positive on U, then

$$\begin{aligned}
B_k(u) &= A_k(u), && (2 \le k \le n)\\
B_1(u) &= A_1(u) - a\,(F(u))^{\frac{n-1}{n-2}},\\
B_0(u) &= A_0(u) + a^2\,(F(u))^{\frac{2(n-1)}{n-2}}.
\end{aligned}$$

First, in Section 4.1 we show that A_k $(0 \le k \le n)$ defined by (4.1) satisfy a recursive system of differential equations. The details are contained in Lemma 4.1. Then in Section 4.2 we consider the case $a = 0$, when (3.4) results in homogeneous equations for f_k. Finally, in Section 4.3 we prove Theorem 4.1 by considering the general case $a \ne 0$ and applying the method of variation of parameters.

4.1. *Auxiliary functions*

Let us use the notations of Theorem 4.1.

Lemma 4.1. *Let $A_n(u) = 0$. For all $0 \le k \le n-1$ a particular solution of the ordinary differential equation*

$$A_k(u)' = -(k+1)\,a\left(\frac{n-1}{n-2}\right)|F(u)|^{\frac{1}{n-2}}\,F(u)'\,(b_{k+1} + A_{k+1}(u)), \qquad \text{(4.6)}$$

where b_{k+1} is an arbitrary real constant, is the function A_k defined by (4.1) *in Theorem* 4.1.

Proof. We prove (4.1) by induction on $m = n-k$. For $m = 1$ we have

$$A'_{n-1} = -nab_n \left(\frac{n-1}{n-2}\right) |F|^{\frac{1}{n-2}} F',$$

and a particular solution is

$$A_{n-1} = -\nu nab_n |F|^{\frac{n-1}{n-2}}.$$

This proves (4.1) for $k = n-1$. Assume now, that (4.1) holds for an integer $m = n-k$, $1 \leq m \leq n$, that is

$$A_{n-m} = \left(\sum_{i=1}^{m} (-\nu)^i \binom{n-m+i}{i} a^i b_{n-m+i} |F|^{\frac{i(n-1)}{n-2}}\right).$$

Putting this into (4.6) for $k = n-(m+1)$ one obtains

$$A'_{n-(m+1)} = -(n-m)a\left(\frac{n-1}{n-2}\right)|F|^{\frac{1}{n-2}}F' \cdot \left(b_{n-m} + \left(\sum_{i=1}^{m}(-\nu)^i\binom{n-m+i}{i}a^i b_{n-m+i}|F|^{\frac{i(n-1)}{n-2}}\right)\right).$$

By integrating, one can obtain a particular solution as

$$\begin{aligned}
A_{n-(m+1)} &= -\nu(n-m)ab_{n-m}|F|^{\frac{n-1}{n-2}} - \nu\sum_{i=1}^{m}(-\nu)^i\frac{n-m}{i+1}\binom{n-m+i}{i}a^{i+1}b_{n-m+i}|F|^{\frac{(i+1)(n-1)}{n-2}} \\
&= -\nu(n-m)ab_{n-m}|F|^{\frac{n-1}{n-2}} + \sum_{j=2}^{m+1}(-\nu)^j\frac{n-m}{j}\binom{n-m-1+j}{j-1}a^j b_{n-m-1+j}|F|^{\frac{j(n-1)}{n-2}} \\
&= \sum_{i=1}^{m+1}(-\nu)^i\binom{n-(m+1)+i}{i}a^i b_{n-(m+1)+i}|F|^{\frac{i(n-1)}{n-2}}.
\end{aligned}$$

Hence, (4.1) holds for $k = n-(m+1)$ and by induction it holds for all integers $0 \leq k \leq n-1$. □

4.2. *The homogeneous case*

Assume $a = 0$. Now, (3.4) takes the form

$$f'_0 g + (-1)f_0 g' + 2f_0 = 0, \tag{4.7a}$$
$$f'_1 g + f_1 = 0, \tag{4.7b}$$
$$-g'' + f'_2 g + f_2 g' = 0, \tag{4.7c}$$
$$f'_k g + (k-1)f_k g' + (2-k)f_k = 0, \qquad (3 \leq k \leq n-1). \tag{4.7d}$$

Note, that (4.7d) for $k = 0,1$ gives (4.7a) and (4.7b). Now, $g(u) \neq 0$ for $u \in U$, hence the solution of (4.7d) is

$$\begin{aligned} f_k &= \exp\left(\int \frac{(1-k)g' + (k-2)}{g}\right) = \exp\left((1-k)\ln|g| + (k-2)\int \frac{1}{g}\right) \\ &= |g|^{1-k} \cdot \exp\left((k-2)\int \frac{(n-1)F'}{(n-2)F}\right) = |g|^{1-k} \cdot \exp\left(\frac{(k-2)(n-1)}{n-2}\int (\ln|F|)'\right) \\ &= b_k \cdot |g|^{1-k} \cdot |F|^{\frac{(k-2)(n-1)}{n-2}} = b_k \cdot \left(\frac{n-1}{n-2}\right)^{k-1} \cdot \left(F'\right)^{k-1} \cdot |F|^{\frac{k-n}{n-2}} \end{aligned} \tag{4.8}$$

for some $b_k \in \mathbb{R}$ $(0 \leq k \leq n-1,\ k \neq 2)$. For $k = 2$ eq. (4.7c) has the form $(-g' + f_2 g)' = 0$, thus

$$f_2 = \frac{g' + b_2}{g} = \left(1 + b_2 \frac{n-1}{n-2}\right) \cdot \frac{F'}{F} - \frac{F''}{F'} \tag{4.9}$$

for some $b_2 \in \mathbb{R}$. This proves Theorem 4.1 in the case $a = 0$ by selecting $b_k' = b_k \cdot \left(\frac{n-1}{n-2}\right)^{k-1}$ $(0 \leq k \leq n, k \neq 2)$ and $b_2' = 1 + b_2 \frac{n-1}{n-2}$.

4.3. *The general (inhomogeneous) case*

Proof. [Proof of Theorem 4.1] If $a \neq 0$, then (3.4d) is an inhomogeneous linear differential equation for f_k, and by (4.8) its general solution (by variation of parameters) is

$$f_k = (b_k + B_k) \cdot \left(\frac{n-1}{n-2}\right)^{k-1} \cdot |F|^{\frac{k-n}{n-2}} \cdot (F')^{k-1}, \tag{4.10}$$

for some function $B_k = B_k(u)$ and constant $b_k \in \mathbb{R}$. Write f_k in the form $f_k = (b_k + B_k)h_k$, where

$$h_k = \left(\frac{n-1}{n-2}\right)^{k-1} \cdot |F|^{\frac{k-n}{n-2}} \cdot (F')^{k-1}.$$

Putting (4.10) into (3.4d) we obtain

$$\left((b_k + B_k)\,h_k' g + (k-1)\,(b_k + B_k)\,h_k g' + (2-k)\,(b_k + B_k)\,h_k\right) + B_k' h_k g + (k+1) a f_{k+1} g = 0.$$

Now, h_k is a particular solution of the homogeneous differential equation (4.7d), thus

$$(b_k + B_k)h_k' g + (k-1)(b_k + B_k)h_k g' + (2-k)(b_k + B_k)h_k = 0.$$

Since $h_k(u) \neq 0$ for all $u \in U$, B_k is a particular solution of the differential equation

$$B_k' = -\frac{(k+1)af_{k+1}}{h_k} = -(k+1)af_{k+1}\left(\frac{n-1}{n-2}\right)^{1-k} |F|^{\frac{n-k}{n-2}} (F')^{1-k}, \qquad (3 \leq k \leq n-1). \tag{4.11}$$

We prove by induction on $m = n-k$ that $B_k = A_k$ $(3 \leq k \leq n-1)$ by showing that B_k satisfies the recursive system of differential equations (4.6) of Lemma 4.1. Let $m = 1$, that is $k = n-1$. From

(3.3) we have $f_n = \varepsilon(F')^{n-1}$. Applying (4.11) we obtain

$$B'_{n-1} = -naf_n \left(\frac{n-1}{n-2}\right)^{2-n} (F')^{2-n} |F|^{\frac{1}{n-2}} = -na\varepsilon \left(\frac{n-1}{n-2}\right)^{2-n} |F|^{\frac{1}{n-2}} F'. \tag{4.12}$$

Comparing (4.12) and (4.6) for $m=1$, we find $B'_{n-1} = A'_{n-1}$ by choosing $b_n = \varepsilon \left(\frac{n-1}{n-2}\right)^{1-n}$. Hence, a particular solution B_{n-1} of the differential equation $B'_{n-1} = A'_{n-1}$ is A_{n-1}. Therefore, for $k = n-1$ we have $B_k = A_k$.

Assume that for an integer $4 \le k \le n-1$ $B_k = A_k$ holds, thus from (4.10) for $k = n-m+1$ we have

$$f_{n-m+1} = (b_{n-m+1} + A_{n-m+1}) \cdot \left(\frac{n-1}{n-2}\right)^{n-m} \cdot |F|^{\frac{1-m}{n-2}} \cdot (F')^{n-m}.$$

Putting this into (4.11) for $k = n-m$ we obtain

$$\begin{aligned} B'_{n-m}(u) &= -(n-m+1)af_{n-m+1} \cdot \left(\frac{n-1}{n-2}\right)^{1-n+m} \cdot |F|^{\frac{m}{n-2}} \cdot (F')^{1-n+m} \\ &= -(n-m+1)a \cdot \frac{n-1}{n-2} \cdot |F|^{\frac{1}{n-2}} \cdot (F')(b_{n-m+1} + A_{n-m+1}). \end{aligned} \tag{4.13}$$

Comparing (4.6) for $k = n-m$ and (4.13) we see that $B'_{n-m} = A'_{n-m}$. Hence a particular solution B_{n-m} of the differential equation $B'_{n-m} = A'_{n-m}$ is A_{n-m}. Therefore for $k = n-m$ one has $B_k = A_k$. By induction, for all $3 \le k \le n-1$ one has $B_k = A_k$. Thus (4.3a) holds for $3 \le k \le n-1$, and (4.2a) is proved.

We continue by proving (4.3b) and (4.2b), that is we show the condition on f_k for $k=2$. Now, f_2 is the solution of the inhomogeneous linear differential equation (3.4c). The general solution of (3.4c) by (4.9) and by variation of parameters has the form

$$f_2 = \frac{g' + (b_2 + B_2)}{g} \tag{4.14}$$

for some function $B_2 = B_2(u)$ and constant $b_2 \in \mathbb{R}$. Putting (4.14) into (3.4c), then using $\left(\frac{1}{g}\right)' g = -\frac{g'}{g}$, and the fact that $\frac{g'+b_2}{g}$ is the solution to the homogeneous differential equation (4.7c), one has

$$\begin{aligned} 0 &= -g'' + \left(\frac{g'+b_2}{g}\right)' g + B'_2 + B_2 \left(\frac{1}{g}\right)' g + 3af_3 g + \frac{g'+b_2}{g} g' + \frac{B_2}{g} g' \\ &= -g'' + \left(\frac{g'+b_2}{g}\right)' g + \frac{g'+b_2}{g} g' + B_2 \left(\left(\frac{1}{g}\right)' g + \frac{g'}{g}\right) + B'_2 + 3af_3 g \\ &= B'_2 + 3af_3 g, \end{aligned}$$

that is $B'_2 = -3af_3 g$. Using the form of f_3 given by (4.10) and the definition of g we obtain

$$B'_2 = -3a(b_3 + B_3) \frac{n-1}{n-2} \nu |F|^{\frac{1}{n-2}} F'. \tag{4.15}$$

Now, $A_3 = B_3$ by (4.2a), thus comparing (4.15) and (4.6) for $k=2$ yields $B'_2 = \nu A'_2$. Hence, a particular solution B_2 of the differential equation $B'_2 = \nu A'_2$ has the form $B_2 = \nu A_2$. Thus, (4.3b) and (4.2b) hold.

Now, we obtain the condition on f_k for $k = 1$. The function f_1 is the solution of the inhomogeneous linear differential equation (3.4b). The general solution of (3.4b) by (4.8) and by variation of parameters is

$$f_1 = (b_1 + B_1)\,|F|^{\frac{1-n}{n-2}} \tag{4.16}$$

for some function $B_1 = B_1(u)$ and constant $b_1 \in \mathbb{R}$. Putting (4.16) into (3.4b), and using the fact that $|F|^{\frac{1-n}{n-2}}$ is the solution to the homogeneous differential equation (4.7b), one obtains

$$\begin{aligned}
0 &= a\left(1-2g'\right) + (b_1+B_1)\left(|F|^{\frac{1-n}{n-2}}\right)' g + B_1'\,|F|^{\frac{1-n}{n-2}}\,g + 2af_2g + (b_1+B_1)\,|F|^{\frac{1-n}{n-2}} \\
&= (b_1+B_1)\left(\left(|F|^{\frac{1-n}{n-2}}\right)' g + |F|^{\frac{1-n}{n-2}}\right) + B_1'\,|F|^{\frac{1-n}{n-2}}\,g + a\left(1-2g'\right) + 2af_2g \\
&= B_1'\,|F|^{\frac{1-n}{n-2}}\,g + a(1-2g') + 2agf_2.
\end{aligned}$$

As f_2 has the form (4.14) and $B_2 = \nu A_2$ by (4.2b), we obtain that

$$\begin{aligned}
B_1' &= \frac{|F|^{\frac{n-1}{n-2}}}{g}a\left(2g' - 1 - 2\left(g' + b_2 + B_2\right)\right) = -a\left(\frac{n-1}{n-2}\right)\nu\,|F|^{\frac{1}{n-2}}\,F'\left(1 + 2b_2 + 2\nu A_2\right) \\
&= -2a\left(b_2 + A_2\right)\left(\frac{n-1}{n-2}\right)|F|^{\frac{1}{n-2}}\,F' - a\left(2b_2\left(\nu-1\right) + \nu\right)\left(\frac{n-1}{n-2}\right)|F|^{\frac{1}{n-2}}\,F'.
\end{aligned} \tag{4.17}$$

Comparing (4.17) and (4.6) for $k = 1$ we obtain

$$B_1' = A_1' - a(2b_2(\nu-1)+\nu)\frac{n-1}{n-2}\,|F|^{\frac{1}{n-2}}\,F'. \tag{4.18}$$

Hence a particular solution B_1 of the differential equation (4.18) has the form

$$B_1 = A_1 - a(2b_2(1-\nu)+1)\,|F|^{\frac{n-1}{n-2}}. \tag{4.19}$$

Therefore, (4.3a) holds for $k = 1$, and (4.2c) is proved.

Finally, we prove that (4.3a) holds for $k = 0$. For $k = 0$ the function f_0 is the solution of the inhomogeneous linear differential equation (3.4a). The general solution of (3.4a) by (4.8) and by variation of parameters is

$$f_0 = (b_0 + B_0)\left(\frac{n-2}{n-1}\right)|F|^{\frac{-n}{n-2}}\left(F'\right)^{-1} \tag{4.20}$$

for some function $B_0 = B_0(u)$ and constant $b_0 \in \mathbb{R}$. Putting (4.20) into (3.4a), and using the fact that $\left(\frac{n-2}{n-1}\right)|F|^{\frac{-n}{n-2}}\left(F'\right)^{-1}$ is the solution to the homogeneous differential equation (4.7a), we obtain

$$\begin{aligned}
0 &= -a^2g + (b_0+B_0)\left(\frac{n-2}{n-1}\right)\left(|F|^{\frac{-n}{n-2}}\left(F'\right)^{-1}\right)' g + B_0'\left(\frac{n-2}{n-1}\right)|F|^{\frac{-n}{n-2}}\left(F'\right)^{-1} g \\
&\quad + af_1g - (b_0+B_0)\left(\frac{n-2}{n-1}\right)|F|^{\frac{-n}{n-2}}\left(F'\right)^{-1} g' + 2\left(b_0+B_0\right)\left(\frac{n-2}{n-1}\right)|F|^{\frac{-n}{n-2}}\left(F'\right)^{-1} \\
&= B_0'\left(\frac{n-2}{n-1}\right)|F|^{\frac{-n}{n-2}}\left(F'\right)^{-1} g - a^2g + af_1g.
\end{aligned}$$

As f_1 has the form (4.16), and B_1 has the form (4.19), we obtain

$$\begin{aligned}
B_0' &= \left(a^2 - af_1\right)\left(\frac{n-1}{n-2}\right)|F|^{\frac{n}{n-2}}F' = \left(a^2 - a\left(b_1 + B_1\right)|F|^{\frac{1-n}{n-2}}\right)\left(\frac{n-1}{n-2}\right)|F|^{\frac{n}{n-2}}F' \\
&= \left(a^2 - a\left(b_1 + A_1 - a\left(2b_2\left(1-\nu\right)+1\right)|F|^{\frac{n-1}{n-2}}\right)|F|^{\frac{1-n}{n-2}}\right)\left(\frac{n-1}{n-2}\right)|F|^{\frac{n}{n-2}}F' \\
&= -a\left(b_1 + A_1\right)\left(\frac{n-1}{n-2}\right)|F|^{\frac{1}{n-2}}F' + a^2\left(2 + 2b_2\left(1-\nu\right)\right)\left(\frac{n-1}{n-2}\right)|F|^{\frac{n}{n-2}}F' \\
&= -a\left(b_1 + A_1\right)\left(\frac{n-1}{n-2}\right)|F|^{\frac{1}{n-2}}F' + 2a^2\left(1 + b_2\left(1-\nu\right)\right)\left(\frac{n-1}{n-2}\right)|F|^{\frac{n}{n-2}}F'. \qquad (4.21)
\end{aligned}$$

Comparing (4.21) and (4.6) for $k = 0$ we have

$$B_0' = A_0' + 2a^2\left(1 + b_2\left(1-\nu\right)\right)\left(\frac{n-1}{n-2}\right)|F|^{\frac{n}{n-2}}F'. \qquad (4.22)$$

Therefore, a particular solution B_0 of (4.22) is

$$B_0 = A_0 + a^2(1 + b_2(1-\nu))\nu\,|F|^{\frac{2(n-1)}{n-2}}.$$

Hence, (4.3a) holds for $k = 0$, and (4.2d) is proved. This finishes the proof of Theorem 4.1 □

5. Open problems

Several questions arise after determining the symmetries of (1.1). Indeed, if the Lie group of symmetries is at least two dimensional, then one can apply the two-dimensional solvable Lie group to obtain the solutions of (1.1).

Problem 5.1. *Determine the solutions of* (1.1) *provided* f_k $(0 \le k \le n,\ n \ge 4)$ *satisfy the conditions of Theorem* 3.1.

The only remaining case for (1.1) not covered by Theorem 3.1 or by [21–26] is when $n = 3$. Then one cannot immediately conclude $\xi_u = 0$ from (2.3), because the $\dot{u}^4$ term of (2.3) is identically 0. In fact, for $n = 3$ the symmetry condition translates to

$$\begin{aligned}
-\eta_{tt} + f_0'\eta + f_1\eta_t - f_0\eta_u + 2f_0\xi_t &= 0, \\
(\xi_{tt} - 2\eta_{tu}) + f_1'\eta + 2f_2\eta_t + f_1\xi_t + 3f_0\xi_u &= 0, \\
(2\xi_{tu} - \eta_{uu}) + f_2'\eta + 3f_3\eta_t + f_2\eta_u + 2f_1\xi_u &= 0, \\
\xi_{uu} + f_3'\eta + 2f_3\eta_u - f_3\xi_t + f_2\xi_u &= 0. \qquad (5.1)
\end{aligned}$$

A potential simplification of the system (5.1) might be to eliminate f_2 from (5.1) by using a coordinate change $v = G(u)$, for some bijective, two-times differentiable G, for which $G''(u) = G'(u)f_2(u)$ is satisfied (see e.g. [26]). This, however, still does not give an immediate answer as to what the solutions of (5.1) are.

Problem 5.2. *Determine all symmetries* (*and solutions*) *of the autonomous differential equation*

$$\ddot{u} = f_0(u) + \dot{u}f_1(u) + \dot{u}^2 f_2(u) + \dot{u}^3 f_3(u),$$

where $f_0,\ f_1,\ f_2,\ f_3$ *are arbitrary continuous functions in u.*

Acknowledgements

The research was supported by the European Union's Seventh Framework Programme (FP7/2007-2013) under grant agreement no. 318202. The first, third and fourth authors were partially supported by the European Union's Seventh Framework Programme (FP7/2007-2013) under grant agreement no. 317721. The second author was partially supported by the Hungarian Scientific Research Fund (OTKA) grant no. K109185 and grant no. FK124814. The first and fourth authors were partially supported by the EFOP-3.6.1-16-2016-00022 project. These projects have been supported by the European Union, co-financed by the European Social Fund.

Bibliography

[1] R.E. Mickens, *An Introduction to Nonlinear Oscillations.* (Cambridge etc.: Cambridge University Press. XIV, 224 p., 1981).

[2] N. Levinson, On the existence of periodic solutions for second order differential equations with a forcing term., *J. Math. Phys., Mass. Inst. Techn.* **22** (1943) 41–48.

[3] N. Levinson and O.K. Smith, A general equation for relaxation oscillations, *Duke Math. J.* **9** (1942) 382–403.

[4] J.W. Strutt, *The Theory of Sound. 2nd ed.* (New York: Dover Publications. Two volumes in one. XIII, 480 p.; XII, 504 p., 1945).

[5] A.D. Polyanin and V.F. Zaitsev, *Handbook of Exact Solutions for Ordinary Differential Equations. 2nd ed.*, 2nd ed. edn. (Boca Raton, FL: CRC Press, 2003).

[6] B. van der Pol, On relaxation-oscillations, *The Lond, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* **2** (1927) 978–992.

[7] B. van der Pol and J. van der Mark, The heartbeat considered as a relaxation oscillation, and an electrical model of the heart, *The Lond, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* **6** (1928) 763–775.

[8] R. Fitzhugh, Impulses and physiological states in theoretical models of nerve membrane, *Biophys J.* **1** (6) (1961) 445–466.

[9] J. Nagumo, S. Arimoto and S. Yoshizawa, An active pulse transmission line simulating nerve axon, *Proc. IRE.* **50** (1962) 2061–2070.

[10] D.S. Jones, M. Plank and B.D. Sleeman, *Differential Equations and Mathematical Biology*, second edn. (Chapman and Hall/CRC, 2009).

[11] M.C. Nucci and G. Sanchini, Symmetries, Lagrangians and Conservation Laws of an Easter Island Population Model, *Symmetry* **7** (3) (2015) 1613–1632.

[12] J.R. Faria, J.C. Cuestas and L.A. Gil-Alana, *Unemployment and entrepreneurship: a cyclical relation?* (Discussion papers. Nottingham Trent University, Bottingham Business School, Economics Division, 2008/2).

[13] R.M. Goodwin, A growth cycle, in *Socialism, Capitalism and Economic Growth*, ed. C.H. Feinstein (Cambridge University Press, Cambridge, 1975), pp. 54–58.

[14] N. Kaldor, A model of the trade cycle, *The Economic Journal* **50**(197) (1940) 78–92.

[15] M. Kalecki, A theory of the business cycle, *The Review of Economic Studies* **4**(2) (1937) 77–97.

[16] A.V. Bocharov, V.N. Chetverikov, S.V. Duzhin, N.G. Khor′kova, I.S. Krasil′shchik, A.V. Samokhin, Y.N. Torkhov, A.M. Verbovetsky and A.M. Vinogradov, *Symmetries and conservation laws for differential equations of mathematical physics*, Translations of Mathematical Monographs, Vol. 182 (American Mathematical Society, Providence, RI, 1999). Edited and with a preface by Krasil′shchik and Vinogradov, Translated from the 1997 Russian original by Verbovetsky [A. M. Verbovetskiĭ] and Krasil′shchik.

[17] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Graduate Texts in Mathematics, Vol. 107, second edn. (Springer-Verlag, New York, 1993).

[18] M. Edwards and M.C. Nucci, Application of Lie group analysis to a core group model for sexually transmitted diseases, *J. Nonlinear Math. Phys.* **13** (2) (2006) 211–230.

[19] M.C. Nucci, The role of symmetries in solving differential equations, *Mathl. Comput. Modelling* **25** (8–9) (1997) 181–193.

[20] M.C. Nucci and P. G. L. Leach, Singularity and symmetry analyses of mathematical models of epidemics, *South African Journal of Science* **105** (2009) 136–146.

[21] S.N. Pandey, P.S. Bindu, M. Senthilvelan and M. Lakshmanan, A group theoretical identification of integrable cases of the Liénard-type equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$. I: Equations having nonmaximal number of Lie point symmetries., *J. Math. Phys.* **50** (8) (2009) 082702, 19.

[22] S.N. Pandey, P.S. Bindu, M. Senthilvelan and M. Lakshmanan, A group theoretical identification of integrable equations in the Liénard-type equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$. II: Equations having maximal Lie point symmetries., *J. Math. Phys.* **50** (10) (2009) 102701, 25.

[23] A.K. Tiwari, S.N. Pandey, M. Senthilvelan and M. Lakshmanan, Classification of Lie point symmetries for quadratic Liénard type equation $\ddot{x} + f(x)\dot{x}^2 + g(x) = 0$., *J. Math. Phys.* **54** (5) (2013) 053506, 19.

[24] A.K. Tiwari, S.N. Pandey, M. Senthilvelan and M. Lakshmanan, Erratum: "Classification of Lie point symmetries for quadratic Liénard type equation $\ddot{x} + f(x)\dot{x}^2 + g(x) = 0$"., *J. Math. Phys.* **55** (5) (2014) 059901, 2.

[25] A.K. Tiwari, S.N. Pandey, M. Senthilvelan and M. Lakshmanan, Lie point symmetries classification of the mixed Liénard-type equation, *Nonlinear Dynamics* **82** (4) (2015) 1953–1968.

[26] A. Paliathanasis and P.G.L. Leach, Comment on "Classification of Lie point symmetries for quadratic Liénard type equation $\ddot{x} + f(x)\dot{x}^2 + g(x) = 0$" [J. Math. Phys. 54, 053506 (2013)] and its erratum [J. Math. Phys. 55, 059901 (2014)]., *J. Math. Phys.* **57** (2) (2016) 024101, 2.